REDUCTION OF THE ALGEBRA OF FIELD OPERATORS FOR THE CASE OF SEVERAL VACUUM STATES

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With the word vacuum a state shall be meant which is invariant under the inhomogeneous restricted Lorentz-group.

The result of the following proof will be:

If there are several vacua, the algebra $\mathfrak{A}$ of field operators will be fully reducible. This means, speaking loosely, that you have as many invariant subspaces of the Hilbert space under $\mathfrak{A}$ as you have vacuum-states.

For simplicity, the following is written down only for a scalar field which may interact with itself. The proof is still correct if one has a manifold of fields, if one assumes that each of the fields commutes spacelike with itself and with each of the other fields of the manifold.

The assumptions we make are essentially those given in the axioms of Wightman:

(1) Lorentz covariance of the field;
(2) Existence of one or several vacua;
(3) The spectrum-condition for the momentum-operator;
(4) The vacuum expectation values of products of the field-operator are temperate distributions;
(5) Locality: $[A(x), A(y)] = 0$ for $(x - y)$ spacelike.
(6) Definite metric in the space of states.

The following proof uses a stronger spectrum condition:

(3a) $\Sigma p_\mu^2 \geq m^2$ with $m^2 > 0$ for all states except the vacuum-states; this means $p_\mu = 0$ is an isolated point in the momentum spectrum. The collection of all vacuum-states defines a subspace $\mathfrak{H}_\Omega$ of the Hilbert space $\mathfrak{H}$; we assume $\mathfrak{H}_\Omega$ to be separable and notate the projection-operator on to $\mathfrak{H}_\Omega$ by P_Ω:

$$P_\Omega \mathfrak{H} = \mathfrak{H}_\Omega .$$

With the algebra of field operators $\mathfrak{A}$ we mean the collection of linear combinations of products of field operators:

$$\{ \sum_m c_m A(f_1)A(f_2)\ldots A(f_{k_m})\} \quad \text{with } A(f_1) = \int A(x)f_1(x)dx .$$

$f_1(x)$ is a testfunction from D or S.

It is assumed that the elements of $\mathfrak{A}$ are defined everywhere in $\mathfrak{H}_\Omega$, and that $\mathfrak{H}_\Omega$ is a generating subspace for $\mathfrak{H}$ if we apply $\mathfrak{A}$ to $\mathfrak{H}_\Omega$. This means that the linear space built from $\{A_k\Omega_j\}$, $A \in \mathfrak{A}$, $\Omega_j \in \mathfrak{H}_\Omega$ is dense in $\mathfrak{H}$. Written symbolically:

$$\overline{\mathfrak{A}\mathfrak{H}_\Omega} = \mathfrak{H} .$$

After these preliminaries we can start with the proof for the reduction of $\mathfrak{A}$.

We use the property of cluster decomposition [1] which follows from the assumptions made above. We write it down for a special case and only in the weak form we need:

$$\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_n-\lambda\vec{a})[(\Omega, A(x_1)\dots A(x_n)A(y_1)\dots A(y_m)\Omega)$$
$$-(\Omega A(x_1)\dots A(x_n)\Omega)(\Omega A(y_1)\dots A(y_m)\Omega)]\,dxdy = 0 \qquad (1)$$

with λ real and $\vec{a}$ a space-like vector.

If one looks to the proofs of this cluster decomposition one sees easily the generalization of (1) for the case of several vacuum-states:

$$\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_m-\lambda\vec{a})(\Omega_k A(x_1)\dots$$
$$\dots A(x_n)(1-P_\Omega)A(y_1)\dots A(y_m)\Omega_\ell)dxdy = 0. \qquad (2)$$

Let us take $f_1,\dots f_n,\dots g_1,\dots g_m$ with compact supports that means from D. With the property of the locality of the field (assumption (5)) we get the equation:

$$\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_m-\lambda\vec{a})(\Omega_k A(x_1)\dots$$
$$\dots A(x_n)A(y_1)\dots A(y_m)\Omega_\ell)dxdy$$
$$=\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_m-\lambda\vec{a})(\Omega_k A(y_1)\dots$$
$$\dots A(y_m)A(x_1)\dots A(x_n)\Omega_\ell)dxdy$$

and applying (2):

$$\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_m-\lambda\vec{a})(\Omega_k A(x_1)\dots$$
$$\dots A(x_n)P_\Omega A(y_1)\dots A(y_m)\Omega_\ell)dxdy$$
$$=\lim_{\lambda\to\infty}\int f_1(x_1)\dots f_n(x_n)g_1(y_1-\lambda\vec{a})\dots g_m(y_m-\lambda\vec{a})(\Omega_k A(y_1)\dots$$
$$\dots A(y_m)P_\Omega A(x_1)\dots A(x_n)\Omega_\ell)dxdy\,.$$

The obtained equation does not depend on λ, so we get:

$$(\Omega_k A(f_1)\dots A(f_n)P_\Omega A(g_1)\dots A(g_m)\Omega_\ell) = (\Omega_k A(y_1)\dots A(g_m)P_\Omega A(f_1)\dots A(f_n)\Omega_\ell) \qquad (3)$$

with $A(f_1) = \int A(x_1)f_1(x_1)dx_1\dots,\ A(g_1) = \int A(y_1)g_1(y_1)dy_1,\dots$

Let us write down (3) in a shorter notation and more impressively with

$$A(f_1)A(f_2)\dots A(f_n) = A$$
$$A(g_1)A(g_2)\dots A(g_m) = B$$

$$(\Omega_k P_\Omega A P_\Omega P_\Omega B P_\Omega \Omega_\ell) = (\Omega_k P_\Omega B P_\Omega P_\Omega A P_\Omega \Omega_\ell). \tag{4}$$

Equation (4) is correct for each pair of vacuum-states. Therefore the operators $P_\Omega A P_\Omega$, $P_\Omega B P_\Omega$, ... now taken as operators in $\mathfrak{H}_\Omega$ are commutative. Or in short notation:

The algebra

$$\tilde{\mathfrak{A}} = P_\Omega \mathfrak{A} P_\Omega \quad \text{is}$$

(a) Commutative;
(b) It is involutive. This means with $\tilde{A}$ is also the adjoint $\tilde{A}^*$ in $\tilde{\mathfrak{A}}$;
(c) It consists of bounded operators. For one can write:

$$\tilde{A} = \tfrac{1}{2}(\tilde{A} + \tilde{A}^*) + \tfrac{1}{2i}(i\tilde{A} - i\tilde{A}^*).$$

The two operators on the right side are Hermitian operators defined everywhere in $\mathfrak{H}_\Omega$ and are therefore bounded, which follows from a theorem of Hellinger and Toeplitz [2]. From (a), (b), (c) and the separability of $\mathfrak{H}_\Omega$ one concludes [3], that all operators of $\tilde{\mathfrak{A}}$ can be expressed as functions of the same bounded self-adjoint operator which we will call S (S can be taken with a simple spectrum):

$$S = \int_b^c \lambda \, dE_\lambda, \qquad \tilde{A}_k = \int_b^c \tilde{a}_k(\lambda) \, dE_\lambda .$$

This means in the case that one has a n-dimensional $\mathfrak{H}_\Omega$: choosing an appropriate base in $\mathfrak{H}_\Omega$, the elements of $\tilde{\mathfrak{A}}$ are represented by n-dimensional diagonal matrices.

If we have an infinite-dimensional $\mathfrak{H}_\Omega$, the form of the decomposition depends on the spectrum of S. If it is a pure point spectrum, the operators of $\tilde{\mathfrak{A}}$ are represented by infinite-dimensional diagonal matrices if one chooses the appropriate base.

These two cases give the same structure of $\mathfrak{A}$ in $\mathfrak{H}$.

Let $\Omega'_1, \Omega'_2, \dots \Omega'_m \dots$ be the appropriate base (finite or infinite). Then $A_k \Omega'_\ell$ is orthogonal to $A_j \Omega'_m$ for $m \neq \ell$; for

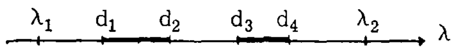
$$(A_k \Omega'_\ell, A_j \Omega'_m) = (\Omega'_\ell A_k^* A_j \Omega'_m) = (\Omega'_\ell A_r \Omega'_m) = (\Omega'_\ell P_\Omega A_r P_\Omega \Omega'_m) = 0$$

with A_k, A_j, $A_r \in \mathfrak{A}$.

In these cases we can split $\mathfrak{H}$ into a direct sum

$$\mathfrak{H} = \overline{\mathfrak{A}\Omega'_1} \oplus \overline{\mathfrak{A}\Omega'_2} \oplus \dots \oplus \overline{\mathfrak{A}\Omega'_n} \oplus \dots$$

$\theta_k = \mathfrak{A}\Omega'_k$ is invariant under the algebra $\mathfrak{A}$ and dense in $\mathfrak{H}_k = \overline{\mathfrak{A}\Omega'_k}$. If S has also a continuous spectrum one gets correspondingly also a direct integral for $\mathfrak{H}$. If for instance one has a continuous spectrum from λ_1 to λ_2



and takes as an example the two orthogonal projection-operators

$$\Delta_{(d_1,d_2)}E = \int_{d_1}^{d_2} dE_\lambda, \qquad \Delta_{(d_3,d_4)}E = \int_{d_3}^{d_4} dE_\lambda$$

$$\Delta_{(d_1,d_2)}E \cdot \Delta_{(d_3,d_4)}E = 0$$

one gets at first the two orthogonal subspaces

$$\mathfrak{H}_\Omega^{(1,2)} = \Delta_{(d_1,d_2)}E \cdot \mathfrak{H}_\Omega \text{ and } \mathfrak{H}_\Omega^{(3,4)} = \Delta_{(d_3,d_4)}E \cdot \mathfrak{H}_\Omega$$

of $\mathfrak{H}_\Omega$ and then the orthogonal subspaces

$$\mathfrak{H}^{(1,2)} = \mathfrak{A}\mathfrak{H}_\Omega^{(1,2)}, \quad \mathfrak{H}^{(3,4)} = \mathfrak{A}\mathfrak{H}_\Omega^{(3,4)}$$

of $\mathfrak{H}$. The form of the spectrum of S will depend on the manifold of fields, which operate in $\mathfrak{H}$.

Finally the following further remarks should be made:

(1) If there are also anticommutative fields in the manifold of fields, in the first step one can choose the greatest sub-algebra $\mathfrak{A}_c$ of $\mathfrak{A}$ with operators in $\mathfrak{A}_c$ which commute spacelike. $\mathfrak{A}_c$ is reduced in the way mentioned above. The remaining elements of $\mathfrak{A}$, if applied to a vacuum-state, produce a state with half-integer spin. If C is such an element $P_\Omega \ell P_\Omega$ must be zero. Therefore the proof is also correct in this more general case.

(2) The cluster decomposition property can be proved [1] without using locality. If one wishes to go from Eq. (2) to Eq. (3) one only needs the weaker condition

$$\lim_{\lambda \to \infty} [A(x + \lambda\vec{a}), A(y)]_- = 0$$

where λ real, $\vec{a}$ spacelike.
To get the results of the proof, therefore, only this weaker condition (instead of locality) is needed.

(3) The reduction of $\mathfrak{A}$ can also be proved, if one does not assume that $p_\mu = 0$ is isolated in the spectrum, but if one assumes CPT - invariance [4].

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