Eur. Phys. J. C (2025) 85:696
https://doi.org/10.1140/epjc/s10052-025-14249-z

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

DDF amplitudes are lightcone amplitudes and the naturalness of Mandelstam maps

Dripto Biswas[a], **Igor Pesando**[b]

Dipartimento di Fisica, Università di Torino, and I.N.F.N., Sezione di Torino, Via P. Giuria 1, 10125 Turin, Italy

Received: 9 January 2025 / Accepted: 29 April 2025

Abstract We show that on-shell DDF amplitudes are on-shell lightcone amplitudes and that Mandelstam maps emerge naturally with a precise normalization, and are intrinsic to the DDF states. Off-shell DDF and Mandelstam amplitudes à la Kaku–Kikkawa differ. In the process, we give a very explicit formula for the conformal transformation of a generic vertex in the form of a compact generating function for free theories.

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[a] e-mail: dripto.biswas@to.infn.it
[b] e-mail: ipesando@to.infn.it (corresponding author)

1 Introduction and summary

A hallmark of string theory is the presence of an infinite number of higher-mass spin excitations in its spectrum, which fill the higher-dimensional representations of the Poincaré group. String theory offers a consistent description of these massive modes. However, while the interactions of massless modes are well understood, the study of amplitudes involving higher-mass string modes remains relatively underexplored (see, however, [1–4] for a discussion of string form factors, [5–12] for the discussion on the stability of massive states which is relevant for the correspondence principle and [13] for an explanation of divergences in temporal string orbifolds due to massive states).

In the 70 s, the Del Giudice, Di Vecchia, and Fubini operators, commonly named DDF operators, were introduced to describe excited massive string states in bosonic string theory [14,15]. An important feature of these operators is that they commute with the generators of the Virasoro algebra. This property allows for the generation of the complete Hilbert space of non null (non BRST exact) physical states by applying an arbitrary combination of them to the ground state. Generic three point amplitudes involving such states of arbitrary string level were computed [15] and subsequently, the framework was extended to the fermionic Neveu–Schwarz model [16]. DDF operators play a crucial role in constructing string coherent states, which are proposed as the Conformal Field Theory (CFT) description of macroscopic objects, potentially identified with cosmic strings [17,18]. Recently, tree-level scattering amplitudes involving DDF- operators of exited strings have been explored [19–28].

In particular in [28] we have constructed the generating functional that provides the generating function of correlation functions for an arbitrary number N of DDF vertices, the N Reggeon in the old parlance [29–32]. This result has been obtained by constructing a Sciuto-Della Selva-Saito like (SDS) vertex [33,34] to describe the interaction of DDF coherent states, as introduced in [17,18] in the context of cosmic strings.

In the quoted paper we derived this generating function both in the standard formulation of the DDF operators and in their framed version, as recently proposed in [35]. Generic DDF correlation functions are subsequently derived by taking derivatives [17,36,37] with respect to the polarizations of the SDS vertex.

In this paper we rewrite the result obtained in [28] in a more conventional way, i.e. using auxiliary operators acting in auxiliary Fock spaces. While this may seem simply a question of taste it paves the way to a better understanding of the meaning of the DDF correlators. We show in facts that on-shell open string tree-level DDF correlators are exactly the same of on-shell open string tree-level lightcone correlators computed by Mandelstam [38–40]. Explicitly this relation is given in Eq. (2.4) (and Eq. (7.1) for the Reggeon) which we rewrite here

$$\begin{aligned}&\Big\langle V_{cov}(x_1;\ |DDF_1\rangle)\, V_{cov}(x_2;\ |DDF_2\rangle)\dots\\&V_{cov}(x_N;\ |DDF_N\rangle)\Big\rangle_{UHP}\langle c(x_1)\, c(x_2)\, c(x_N)\rangle\\&\quad=\mathcal{N}(\{x_r\})\Big\langle m_{[1]}\circ V_{lightcone}(0;\ \phi(|DDF_1\rangle))\ \dots\\&m_{[N]}\circ V_{lightcone}(0;\ \phi(|DDF_N\rangle))\Big\rangle_{UHP}\\&\qquad\times\delta\left(\sum_r k_{[r]-}\right)\delta\left(\sum_r k_{[r]+}\right).\end{aligned}\tag{1.1}$$

In the previous equation UHP stands for the Upper Half-Plane (plus boundary), i.e. $UHP = \{z \in \mathbb{C}|\Im z \geq 0\}$. All correlators $\langle\dots\rangle_{UHP}$ are then computed in UHP, i.e. with the UHP Green's function.

We have also introduced the DDF states $|DDF_r\rangle$ which are states in the $D = 26$ covariant Fock space $\mathcal{H}_{cov}$. Using the state-to-operator correspondence we can associate to $|DDF_r\rangle$ an operator $V_{cov}(x_r;\ |DDF_r\rangle)$ which we choose to insert at $x = x_r \in \partial UHP$. This operator can be computed using the Sciuto-Della Selva-Saito operator.

To any $|DDF_r\rangle$ we associate the state $\phi(|DDF\rangle)$ in lightcone Fock space $\mathcal{H}_{lightcone}$, i.e $\phi(|DDF\rangle)$ is the state in the lightcone formalism which corresponds naturally to the DDF state $|DDF\rangle$ and it is expressed using the lightcone transverse string coordinates.

In the previous formula $u = m_{[r]}(U_{[r]})$ $(r = 1,\dots N)$ are the Mandelstam maps (see Fig. 1) which are the local inverse of

$$U_{[r]}(u) = (u - x_r)\prod_{s=1}^{r-1}(x_s - u)^{\frac{\underline{k}^+_{[s]}}{\underline{k}^+_{[s]}}}\prod_{s=r+1}^{N}(u - x_s)^{\frac{\underline{k}^+_{[s]}}{\underline{k}^+_{[s]}}},\tag{1.2}$$

with $\underline{k}^+_{[r]}$ the framed DDF momentum which corresponds to $k \cdot p_{[r]}$ in the usual notation (k being the null momentum entering the DDF operator and $p_{[r]}$ the tachyonic momentum of the rth state).

These maps are used by Mandelstam to map charts (which corresponds to incoming or outgoing single strings and have coordinates $U_{[r]}$) of the upper half-plane with cuts (which has coordinate U) to the upper half-plane (which has coordinate u). The global mapping between the UHP with cuts and the UHP (without cuts) is given in Eq. (2.5).

Actually Mandelstam does not use the UHP with cuts but instead uses the strip with cuts described by the coordinate ρ. The strip has width α. There is however a one-to-one correspondence between the strip with cuts and the UHP with cuts given by $U = e^{\rho\pi/\alpha}$. The upper half-plane with cuts or the strip describes the lightcone open string interactions.

Using DDF amplitudes these maps arise in a natural way even if they are hidden. These results essentially but not exactly agree with what was obtained in [41–43] in the con-

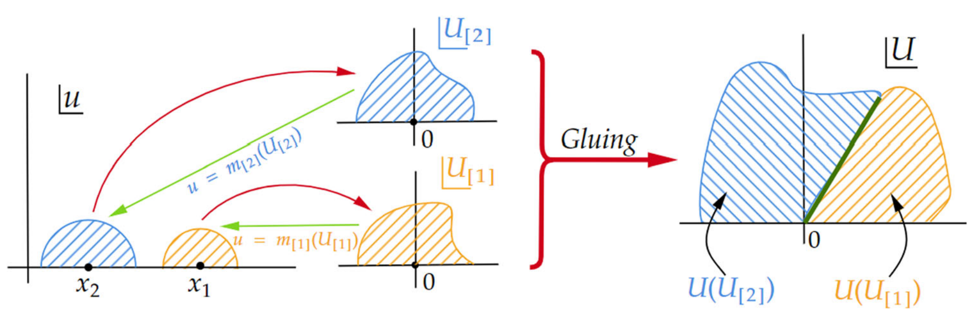

Fig. 1 In this picture we show the upper half-plane with coordinate u on which we compute the DDF correlators. We show also how $U_{[r]}$ and $m_{[r]}$ arise and how they are connected with the upper half-plane with cuts described by coordinate U (when the two shown states have $\underline{k}^+_{[1]}, \underline{k}^+_{[2]} > 0$ so that $U_{[r]} = 0$ is mapped to $U = 0$ otherwise it would be mapped to $U = \infty$)

text of string field theory. With a scaling difference in the Mandelstam maps. The maps used in [41–43] are such that the interactions are at $U_{[r]} = 1$, i.e. the unit half-disk which describes locally an ingoing or outgoing string is without any singularity in the interior of the half-disk. This is not true for the maps that emerge from the direct computation of the DDF amplitudes. However, these maps arise also from using Witten string field theory maps on DDF states and therefore the existence of singularities in the unit half-disk is intrinsic to DDF or lightcone states. We notice finally that the maps obtained here in a natural way do not give rise to the problematic "negative stubs" noted in [41]. Conceptually, there is a difference in the approaches. Here we start from off-shell DDF states as discussed in [35] and use them as (a subset of) a basis of string states. We then compute the off-shell amplitude for $N = 3$ states according to string field theory recipes. From this amplitude, we read the Mandelstam maps. Differently in [42,43] they assume a given expression for the Mandelstam maps.

While the mapping between on-shell DDF amplitudes and on-shell lightcone amplitudes works perfectly, it seems that there are some issues in the mapping between off-shell DDF amplitudes as given by some string field theory formulation and Kaku–Kikkawa lightcone string field theory. Nevertheless, it seems that these issues may be resolved if a slightly different vertex is used. However, this is strange, since the vertex used in Kaku–Kikkawa lightcone string field theory is the vertex used by Mandelstam to check the $N = 4$ tachyon amplitude factorization.

Another result is a very explicit formula for free theories. It gives the effect of a conformal transformation $u = f(U)$ on a generic vertex (both primary and non-primary) of a free theory. It is given in the form of a generating function[1] and follows from the SDS conformal transformation. It is given by Eq. (A.7) for the operatorial formulation or

$$
\begin{aligned}
& f\circ :\, e^{i\sqrt{\frac{\alpha'}{2}}\left[\lambda_0 L(X)+\sum_{n=1}^{\infty} \frac{\lambda_n}{(n-1)!}\, \partial_X^n L(X)\right]} := \\
& =:\, e^{i\sqrt{\frac{\alpha'}{2}}\left[\lambda_0 L(f(X))+\sum_{n=1}^{\infty} \frac{\lambda_n}{(n-1)!}\, \partial_X^n L(f(X))\right]} : \\
& \quad \times (f'(X))^{\frac{1}{2}\lambda_0^2}\, e^{-\frac{2}{\alpha'}\lambda_0 \sum_{n=1}^{\infty} \frac{\lambda_n}{(n-1)!}\, \partial_X^n \Delta(X,Y)|_{Y=0}} \\
& \quad e^{-\frac{1}{\alpha'} \sum_{n=1}^{\infty} \frac{\lambda_n}{(n-1)!} \sum_{m=1}^{\infty} \frac{\lambda_m}{(m-1)!}\, \partial_X^n \partial_Y^m \Delta(X,Y)|_{Y=0}},
\end{aligned}
\tag{1.3}
$$

for the coherent state formulation where $L(X)$ the chiral left moving part of the string coordinate and

$$
\Delta(X, Y) = -\frac{\alpha'}{2} \ln \frac{f(X) - f(Y)}{X - Y}. \tag{1.4}
$$

In the future, it would be interesting to check that this correspondence between DDF amplitudes and lightcone ones works at the loop level and to explore the maps when each DDF state has a different lightcone. The main reason is that in lightcone there seems to be a simple description of the moduli space of the Riemann surfaces while in the covariant formalism it is known only locally at the level of the measure but the global structure is not known.

The paper is organized as follows.

In Sect. 2 we give the general recipe of the mapping between DDF amplitudes and lightcone ones and we give also a simple example on how it works in Sect. 2.1.

We then proceed step by step to uncover the Mandelstam maps.

In order to do so we review the classical but by now not so used formalism of the operatorial Reggeon in Sect. 3. Then we rewrite the DDF Reggeon seen as a coherent state as an operatorial Reggeon in Sect. 4. This allows in Sect. 5 to reassemble the operatorial Reggeon in a way which lends itself to uncover the Mandelstam maps. In Sect. 6 we uncover the Mandelstam maps for the part of the Reggeon which describes the interaction between two different stings. The Reggeon has also a part which describes a string self-interaction. In Sect. 7 we show that these terms arise because of the conformal transformation performed by Mandelstam maps on lightcone vertices. In particular lightcone vertices are defined in a CFT with non-vanishing central charge and are generically non-primary. Also, the corresponding SDS vertices have non-trivial conformal transformations. We give a very explicit formula for free theories for the conformal transformation of a generic vertex in the form of a compact generating function as follows from the SDS conformal transformation. This is derived in Appendix A.

As a final step in Sect. 8 we compare the normalization factors on-shell and discuss how they match. Details are given in Appendix B.

Finally in Sect. 9 we discuss the differences between the DDF results and the Mandelstam–Kaku–Kikkawa–Cremmer–Gervais approach [44–46].

2 Mandelstam maps from DDF amplitudes

In this section we would like to give a more precise overview of how on-shell amplitudes with DDF states are lightcone amplitudes computed Mandelstam maps.

In particular given a covariant vertex for a DDF state we can associate a lightcone vertex as

$$
\begin{aligned}
& V_{cov}(x;\, |DDF\rangle) \to V_{lightcone}(x;\, \phi(|DDF\rangle)) \\
& e^{ik_- x^-_{lightcone0}},
\end{aligned}
\tag{2.1}
$$

[1] For example : $(\partial^3 L)^2\, \partial L\, e^{ikL} : (x) \Rightarrow \left.\frac{\partial^3}{\partial\lambda_3^2\, \partial\lambda_1}\right|_{\lambda_0=k,\, \lambda_{n>0}=}$.

where $\phi(|DDF\rangle)$ is the natural lightcone state associated to $|DDF\rangle$. The zero modes have to be treated carefully since on the lightcone , when the gauge fixing $X^+(\sigma,\tau)=\tau$ is chosen, there are only $x^i_{lightcone\,0}$, $p_{lightcone\,0\,i}$, $x^-_{lightcone\,0}$ and $p_{lightcone\,0\,-}$ but there is no independent $H_{lightcone}=P_+$ zero mode since $p_{lightcone\,0+}$ is the Hamiltonian. In particular $\phi(|DDF\rangle)$ depends on transverse momenta k_i only.

An important point is that $V_{cov}(x;\,|DDF\rangle)$ is a primary conformal operator of dimension $h=\alpha'\underline{k}_T^2$ (where $\underline{k}_T$ is the momentum of the associated tachyonic state) of a CFT with total charge $c=0$. Therefore, under a conformal transformation it transforms as

$$f\circ V_{cov}(x;\,|DDF\rangle)=\left(\frac{df}{dx}\right)^h\,V_{cov}(f(x);\,|DDF\rangle),\tag{2.2}$$

while $V_{lightcone}(x;\,\phi(|DDF\rangle))$ is generically not a primary conformal operator of a CFT with total charge $c=D-2$ and $h_{lightcone}$ dependent on the state $\phi(|DDF\rangle)$. It is generically very different from h, so its conformal transformation

$$\begin{aligned}&f\circ V_{lightcone}(x;\,\phi(|DDF\rangle))\\&\quad=\left(\frac{df}{dx}\right)^{h_{lightcone}}V_{lightcone}(f(x);\,\phi(|DDF\rangle))+\ldots,\end{aligned}\tag{2.3}$$

involves some "anomalous" terms hidden in $\ldots$.

The correspondence is then[2,3]

$$\begin{aligned}&\Big\langle V_{cov}(x_1;\,|DDF_1\rangle)\,V_{cov}(x_2;\,|DDF_2\rangle)\ldots\\&V_{cov}(x_N;\,|DDF_N\rangle)\Big\rangle_{UHP}\langle c(x_1)\,c(x_2)\,c(x_N)\rangle\\&\quad=\mathcal{N}(\{x_r\})\Big\langle m_{[1]}\circ V_{lightcone}(0;\,\phi(|DDF_1\rangle))\ldots m_{[N]}\circ\\&V_{lightcone}(0;\,\phi(|DDF_N\rangle))\Big\rangle_{UHP}\\&\quad\times\delta\left(\sum_r k_{[r]-}\right)\,\delta\left(\sum_r k_{[r]+}\right).\end{aligned}\tag{2.4}$$

In the previous expression $\mathcal{N}(\{x_r\})$ is the factor that in the path integral approach and in *critical dimension* arises from the functional determinant of the 2d Laplacian, and the Jacobian (due to the change of coordinates from $\tau_E+i\sigma$ to upper plane ones) (see [38,47] vol. 2 and [43] for a more recent approach).

We have also introduced the two $\delta\left(\sum_r k_{[r]-}\right)\,\delta\left(\sum_r k_{[r]+}\right)$. Although, from the covariant point of view this seems completely arbitrary and strange, this is not so from the lightcone point of view. In facts $\delta(\sum_r k_{[r]-})=\delta\left(\sum_r k^+_{[r]}\right)$ comes from the factor $e^{ik_{[r]-}x^-_{lightcone\,0}}$ in Eq. (2.1). And $\delta(\sum_r k_{[r]+})=\delta\left(\sum_r k^-_{[r]}\right)$ comes from computing the path integral over the infinite lightcone time interval, since the lightcone Hamiltonian is $H_{lightcone}=P_+$. One could wonder whether this δ along with the other factors come from the highly non-trivial, but natural piece $e^{ik_{[r]-}\hat{X}^-_{lightcone\,n.z.m.}}$ where $\hat{X}^-_{lightcone\,n.z.m.}$ is a composite operator.

Finally we have introduced the fundamental functions $u=m_{[r]}(U_{[r]})$[4] which are the Mandelstam maps. They map local coordinates $U_{[r]}$ of the lightcone rth vertex to the global upper plane coordinate u. The local coordinates $U_{[r]}$ are associated to an open covering of the Riemann surface with boundary $\Sigma_N(\{x_r,\underline{k}^+_{[r]}\})$ which is defined by the global equation

$$[U(u)]^{\frac{\phi_M-\phi_m}{\pi}}=e^{-i\phi_m}\prod_{s=1}^N(u-x_s)^{\sqrt{2\alpha'}\underline{k}^+_{[s]}}=e^{-i\phi_m}\,\hat{U},\tag{2.5}$$

where both u and U take values in the upper half-plane, while $\hat{U}$ is a partial multiple cover of the upper half-plane. In the previous equation we have defined

$$\begin{aligned}&\phi(x)=\pi\sum_{s=1}^N\theta(x_s-x)\,\sqrt{2\alpha'}\underline{k}^+_{[s]},\\&\phi_m=\min\phi(x),\quad\phi_M=\max\phi(x),\end{aligned}\tag{2.6}$$

so that $0\le arg(U)\le\pi$. See Fig. 2 for an example which explains the normalizations used.

Explicitly the local coordinates $U_{[r]}$ are

$$\begin{aligned}U_{[r]}(u)&=\left[e^{i\pi\sum_{s=1}^{r-1}\sqrt{2\alpha'}\underline{k}^+_{[r]}}U(u)\right]^{\frac{1}{\sqrt{2\alpha'}\underline{k}^+_{[r]}}}\\&=(u-x_r)\prod_{s=1}^{r-1}(x_s-u)^{\rho_{sr}}\prod_{s=r+1}^N(u-x_s)^{\rho_{sr}},\end{aligned}\tag{2.7}$$

where $\underline{k}^+_{[r]}=\underline{k}^+_{T[r]}$ are the $+$ component of the rth vertex momentum and

$$\rho_{sr}=\frac{\underline{k}^+_{T[s]}}{\underline{k}^+_{T[r]}}=\frac{\underline{k}^+_{[s]}}{\underline{k}^+_{[r]}}.\tag{2.8}$$

The definition is such that an open half-disk around $u=x_r$

$$|u-x_r|<min_{s\neq r}\{|x_s-x_r|\},\tag{2.9}$$

[2] This result is very similar to the one obtained in [41] but the local maps differ by a scaling factor. In our case, there is no ambiguity in getting the local maps.

[3] Here and in the following we use $\underline{k}_{[r]}$ for the framed DDF momenta and $k_{[r]}$ for the lightcone theory momenta.

[4] We use the notation $m_{[r]}$ even if it could be confused with a mass in order to stress that it is the Mandelstam map and that is the reason of the m and it takes value in the global coordinate u which is a lower letter.

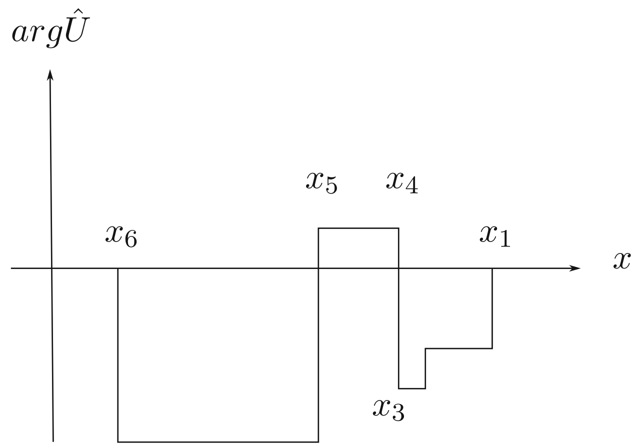

Fig. 2 An example of the argument of $\hat{U} = e^{i\phi_m} U^{\pi/(\phi_M-\phi_m)}$

is mapped to the upper half-plane of $U_{[r]}$. In particular, singular points may occur for $|U_{[r]}| < 1$. For example, for $N = 3$ the interaction point u_I defined by $\left.\frac{dU_{[r]}}{du}\right|_{u=u_I} = 0$ is such that,

$$|U_{[r]}(u_I)|^{\sqrt{2\alpha'}\underline{k}^{+}{}_{[r]}} = \left|\frac{\underline{k}^{+}{}_{[1]}}{\underline{k}^{+}{}_{[1]} + \underline{k}^{+}{}_{[2]}}\right|^{\sqrt{2\alpha'}\underline{k}^{+}{}_{[1]}} \left|\frac{\underline{k}^{+}{}_{[2]}}{\underline{k}^{+}{}_{[1]} + \underline{k}^{+}{}_{[2]}}\right|^{\sqrt{2\alpha'}\underline{k}^{+}{}_{[2]}} \tag{2.10}$$

For later use we record that

$$\left.\frac{dU_{[r]}(z)}{du}\right|_{u=x_r} = \prod_{s=1}^{r-1}(x_{sr})^{\rho_{sr}} \prod_{s=r+1}^{N}(x_{rs})^{\rho_{sr}}. \tag{2.11}$$

The function $u = m_{[r]}(U_{[r]})$ is the local inverse of Eq. (2.7) so that $x_r = m_{[r]}(U_{[r]} = 0)$.

2.1 An explicit example of the correspondence

Before describing how the previous correspondence works, we would like to give the simplest example of the same.

Let us consider the tachyons. In this case we have

$$\begin{aligned} V_{cov}(x; |k_{T\mu}\rangle) =: e^{i2k_{T\mu}\underline{L}^{\mu}(x)} : \quad \to \quad & V_{lightcone}(X; |k_{Ti}\rangle) \\ =: & e^{i2k_{Ti}\, L_{lc}{}^{i}(x)} :, \end{aligned} \tag{2.12}$$

where X is the lightcone local coordinate which differs from the DDF local coordinate x. It is worth noticing that the fields used in lightcone are only the transverse ones and therefore the CFT has non-trivial central charge.

In the previous case, it so happens that $V_{lightcone}(X; |k_{Ti}\rangle)$ is also a primary operator with conformal dimension $h_{lightcone} = \alpha' k_{Ti} k_T^i$ but this is an exception. It then follows that

$$m_{[r]} \circ V_{lightcone}(0;\ |k_{T[r]}\rangle)$$

$$= \left(\frac{dm_{[r]}}{dU_{[r]}}\right)^{\alpha' k_{T[r]i}k^i_{T[r]}}\Bigg|_{U_{[r]}=0} V_{lightcone}(x_r = m_{[r]}(0);\ |k_{[r]}\rangle). \tag{2.13}$$

Then the correspondence reads

$$\begin{aligned} &\prod_{r<s}(x_r - x_s)^{2\alpha' k_{T[r]\mu}k^{\mu}_{T[s]}}(x_1 - x_2)(x_1 - x_N)(x_2 - x_N)\delta^{D-2} \\ &\times q\left(\sum_r k_{T[r]i}\right)\delta\left(\sum_r k_{T[r]-}\right)\delta\left(\sum_r k_{T[r]+}\right) \\ &= \mathcal{M}(\{x_r\}) \times \prod_r \left(\frac{dm_{[r]}}{dU_{[r]}}\right)^{\alpha' k_{T[r]i}k^i_{T[r]}}\Bigg|_{U_{[r]}=0} \\ &\quad\times \prod_{r<s}(x_r - x_s)^{2\alpha' k_{T[r]i}k^i_{T[s]}}\delta^{D-2}\left(\sum_r k_{T[r]i}\right) \\ &\quad\times \delta\left(\sum_r k_{T[r]-}\right)\delta\left(\sum_r k_{T[r]+}\right). \end{aligned} \tag{2.14}$$

Now we can use Eq. (2.11) and evaluate

$$\begin{aligned} &\prod_r \left(\frac{dm_{[r]}}{dU_{[r]}}\right)^{\alpha' k_{T[r]i}k^i_{T[r]}}\Bigg|_{U_{[r]}=0} \\ &= \prod_r \left(\frac{dU_{[r]}}{du}\right)^{-\alpha'(k_{T[r]i}k^i_{T[r]}+m^2_{T[r]})}\Bigg|_{u=x_r} \\ &\quad\times \prod_r \left(\frac{dU_{[r]}}{du}\right)^{+\alpha' m^2_{T[r]}}\Bigg|_{u=x_r} \\ &= \prod_{r=1}^{N}\left(\prod_{s=1}^{r-1}(x_s - x_r)^{\frac{\underline{k}^{+}_{T[s]}}{\underline{k}^{+}_{T[r]}}}\right. \\ &\quad\times \left.\prod_{s=r+1}^{N}(x_r - x_s)^{\frac{\underline{k}^{+}_{T[s]}}{\underline{k}^{+}_{T[r]}}}\right)^{-\alpha'(k_{T[r]i}k^i_{T[r]}+m^2_{T[r]})} \\ &\quad\times \prod_r \left(\frac{dU_{[r]}}{du}\right)^{+\alpha' m^2_{T[r]}}\Bigg|_{u=x_r} \\ &= \prod_{r=1}^{N}\left(\prod_{s=1}^{r-1}(x_s - x_r)^{-2\alpha'\underline{k}^{+}_{T[s]}\underline{k}^{-}_{T[r]}}\right. \\ &\quad\times \left.\prod_{s=r+1}^{N}(x_r - x_s)^{-2\alpha'\underline{k}^{+}_{T[s]}\underline{k}^{-}_{T[r]}}\right) \\ &\quad\times \prod_r \left(\frac{dU_{[r]}}{du}\right)^{+\alpha' m^2_{T[r]}}\Bigg|_{u=x_r}, \end{aligned} \tag{2.15}$$

where we used $k^- = \frac{k_i k^i + m^2}{2k^+}$. The first factor in the last line completes the lightcone exponents to $2\alpha' k_{T[r]\mu}k^{\mu}_{T[s]} = -2\alpha' k_{T[r]+}k_{T[s]-} - 2\alpha' k_{T[r]-}k_{T[s]+} + 2\alpha' k_{T[r]i}k_{T[s]i}$ while the second one (when the amplitude is on-shell, i.e. with

$+\alpha' m^2_{T[r]} = -1$) along with $\mathcal{M}(\{x_r\})$ makes the ghost contribution.

3 Reminder on the usual operatorial formulation of the Reggeon vertex

Recall that the N-reggeon vertex can be obtained from the SDS vertex.

Given any state, physical or unphysical, $|\phi\rangle$ in a Hilbert space $\mathcal{H}$ the state-to-operator correspondence associates to it, the operator $V[X](x;\, |\phi\rangle)$ which is a functional of the string coordinates $X(u,\bar{u})$. The state is then recovered as $|\phi\rangle = \lim_{x\to 0} V[X](x;\, |\phi\rangle)|0\rangle$.

We can easily and explicitly compute this operator in a different but isomorphic Hilbert space $\widetilde{\mathcal{H}}$. The reason for two isomorphic Hilbert spaces is technical, and it is due to the way we construct the SDS vertex, the object which performs this mapping.

This vertex operator in the different (but isomorphic) Hilbert space $\widetilde{\mathcal{H}}$ is then $V[\tilde{X}](x;\, |\phi\rangle)$ and is computed using the SDS vertex as

$$\widetilde{\mathcal{S}}(x)|\phi\rangle = V[\tilde{X}](x;\, |\phi\rangle), \tag{3.1}$$

so $\widetilde{\mathcal{S}}(x) : \mathcal{H} \to Hom(\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}})$. Explicitly the Sciuto-Della Selva-Saito vertex for the covariant formulation is given by

$$\begin{aligned}\widetilde{\mathcal{S}}(x) &= \langle x^\mu = 0, 0_{a^\mu}| : e^{\frac{i}{\sqrt{2\alpha'}} g_{\mu\nu} \sum_{n=0}^\infty \frac{\alpha^\mu_n}{n!} \partial^n_x \tilde{X}^\nu(x,\bar{x})} : \\ &= \langle x^\mu = 0, 0_{a^\mu}| : e^{-\frac{2}{\alpha'} g_{\mu\nu} \oint_{z=0,\,|z|<|x|} \frac{dz}{2\pi i} \partial_z L^\mu(z) \tilde{L}^\nu(x+z)} : \\ &= \langle x^\mu = 0, 0_{a^\mu}| : e^{i\sqrt{\frac{2}{\alpha'}} g_{\mu\nu} \left(\alpha^\mu_0 \tilde{L}^\nu(x) + \sum_{n=1}^\infty \frac{\alpha^\mu_n}{n!} \partial^n_x \tilde{L}^\nu(x)\right)} : \quad \text{when } x>0,\end{aligned} \tag{3.2}$$

where the normal ordering is acting w.r.t. $\tilde{\alpha}$ operators since only for them there are creators and annihilators, the vacuum $\langle 0_a|$ is the one for the α operators acting in $\mathcal{H}$ and similarly $\langle x = 0|$ is the null eigenstate of x_0.

The N-Reggeon, i.e. the generating function for all the tree-level amplitudes is then

$$\langle V_N(x_1, \dots x_N)| = \langle \tilde{0}|\, \widetilde{\mathcal{S}}_{[1]}(x_1) \dots \widetilde{\mathcal{S}}_{[N]}(x_N)\, |\tilde{0}\rangle_{UHP}, \quad |x_1| > \dots > |x_N|, \tag{3.3}$$

and when all x_r are fixed it is a map

$$\langle V_N(x_1, \dots x_N)| :\ \mathcal{H}^{\otimes N} \ \to\ \mathbb{C}. \tag{3.4}$$

The Reggeon vertex is such that any correlator can be computed as

$$\begin{aligned}&\langle V_N(x_1, \dots x_N)| \left(|\phi_{[1]}\rangle \otimes \dots |\phi_{[N]}\rangle\right) \\ &= \left\langle \tilde{0} \right| V[\tilde{X}](x_1; \phi_{[1]}) \dots V[\tilde{X}](x_N; \phi_{[N]}) \left|\tilde{0}\right\rangle_{UHP},\end{aligned} \tag{3.5}$$

where $V[\tilde{X}](x_r;\, |\phi_{[r]}\rangle)$ is the vertex operator for the emission of the state $|\phi_{[r]}\rangle$ in the Fock space DB: $\tilde{\mathcal{H}}$ or $\mathcal{H}$? and the correlators are computed in the UHP.

An easy computation gives

$$\begin{aligned}&\mathcal{V}^{(cov)}_N(\{x_r\}) = \langle V_N(x_1, \dots x_N)| = \\ &\left(\otimes_{r=1}^N \langle x^\mu_{[r]0} = 0; 0_{a^\mu_{[r]}}|\right) \prod_{1\le r<s\le N} \\ &e^{-\frac{2}{\alpha'} \oint_{z_{[r]}=0} \oint_{z_{[s]}=0} \frac{dz_{[r]}}{2\pi i} \frac{dz_{[s]}}{2\pi i} \partial L^{(+)\mu}_{[r]}(z_{[r]}) \partial L^{(+)\mu}_{[s]}(z_{[s]}) \ln\left(z(z_{[r]}) - z(z_{[s]})\right)} \\ &\times \delta^D\left(\sum_{r=1}^N \alpha^\mu_{[r]0}\right),\end{aligned} \tag{3.6}$$

where we defined the map

$$z(Z_{[r]}) = Z_{[r]} + x_r, \tag{3.7}$$

for which $z(Z_{[r]} = 0) = x_r$. This can be interpreted as the map between the local doubled coordinate $Z_{[r]}$ associated with the covariant vertex and the doubled global coordinate z. It is the local inverse of the map

$$Z_{[r]}(z) = z - x_r. \tag{3.8}$$

We have written 'doubled' because the true map for the open string would read

$$u(U_{[r]}) = U_{[r]} + x_r, \tag{3.9}$$

where both u and $u_{[r]}$ are defined on the UHP but we need to 'double' their domain of definition to the whole complex plane in order to be able to perform the contour integrals. In this case the Riemann surface is simply

$$U = Z. \tag{3.10}$$

4 Recovering Mandelstam map 1: mapping coherent state parameters λ to operator

We need to compare the usual expression for the Reggeon using operators in (3.6) with the one using $\lambda^i_{[r]n}$:

$$\begin{aligned}&\mathcal{V}^{(DDF)}_N(\{x_r\}) = \left\langle \underline{p}^i = \underline{p}^- = 0 \right| \\ &\prod_{r=1}^N \left\{ \exp\left(\sum_{i=2}^{D-1} \sum_{n=1}^\infty \lambda^i_{[r]n} \oint_{z=0} \frac{dz}{2\pi i} \left(\sum_{t\ne r} \frac{\sqrt{2\alpha'}\underline{k}^i_{T[t]}}{z + x_{rt}} \right.\right.\right. \\ &\left.\left. + \sqrt{2\alpha'}\underline{k}^i_{T[r]} \frac{1}{z} \right) [Z_{[r]}(z + x_r)]^{-n}\, e^{-in\frac{x_0^+}{2\alpha' \underline{k}^+_{T[r]}}} \right) \\ &\times \exp\left(\frac{1}{2} \sum_{i=2}^{D-1} \sum_{n_1,n_2=1}^\infty \sum_{r,s=1}^N \lambda^i_{[r]n_1} \lambda^i_{[s]n_2} \oint_{z_1=0,\,|z_1|>|z_2|} \right.\end{aligned}$$

$$\frac{dz_1}{2\pi i}\oint_{z_2=0}\frac{dz_2}{2\pi i}\frac{e^{-i\frac{n_1\underline{x}_0^+}{2\alpha'\underline{k}^+_{T[r]}}-i\frac{n_2\underline{x}_0^+}{2\alpha'\underline{k}^+_{T[s]}}}}{(z_1-z_2+x_{rs})^2}$$

$$\left.\left.[Z_{[r]}(z_1+x_r)]^{-n_1}\,[Z_{[s]}(z_2+x_s)]^{-n_2}\right)\right\}$$

$$\times\prod_{r=1}^{N}e^{i\underline{k}^i_{T[r]}\underline{x}^i_0-i\underline{k}^-_{T[r]}\underline{x}^+_0}\left|\underline{p}^i=\underline{p}^-=0\right\rangle$$

$$\prod_{r=1}^{N-1}\prod_{s=r+1}^{N}x_{rt}^{2\alpha'\underline{k}_{T[r]}\cdot\underline{k}_{T[s]}}\,\delta\left(\sum_r^N\underline{k}^+_{[r]}\right), \tag{4.1}$$

where only $\underline{x}_0^+$ and $\underline{x}_0^i$ are operators and we use the notation

$$x_{rs}\equiv x_r-x_s,\qquad \rho_{sr}\equiv\frac{\underline{k}^+_{T[s]}}{\underline{k}^+_{T[r]}}. \tag{4.2}$$

We have also introduced the doubled version of Eq. (2.7) and defined the analytic functions in the cut complex plane z

$$Z_{[r]}(z)=(z-x_r)\prod_{s=1}^{r-1}(x_s-z)^{\rho_{sr}}\prod_{s=r+1}^{N}(-x_s+z)^{\rho_{sr}}. \tag{4.3}$$

They are analytic for $|z-x_r|<min_{s\neq r}\{|x_s-x_r|\}$. However it may well happen that $\frac{dZ_{[r]}}{dz}=0$ at one point closer and then we cannot invert the map anymore.

For later use, notice that Eq. (2.11) is in the doubled version

$$\left.\frac{dU_{[r]}(z)}{du}\right|_{u=x_r}=\left.\frac{dZ_{[r]}(z)}{dz}\right|_{z=x_r}=\prod_{s=1}^{r-1}(x_{sr})^{\rho_{sr}}\prod_{s=r+1}^{N}(x_{rs})^{\rho_{sr}}. \tag{4.4}$$

We want to map the previous Eq. (4.1) to an equation with αs similar to Eq. (3.6). In our case α^μ are actually $\underline{A}^i$ or better $\alpha_{lc}{}^i$. In order to do so we have to map λs which are associated with DDF operators to α_{lc}s. It is easy to realize that we have the mapping

$$\begin{cases}\lambda^i_{[r]n} \leftrightarrow +\frac{1}{n}\alpha_{lc}{}^i_{[r]n} & n\geq 1\\ \sqrt{2\alpha'}k^i_{[r]} \leftrightarrow -\alpha_{lc}{}^i_{[r]0} & \end{cases}, \tag{4.5}$$

where the $1/n$ comes from the α_{lc} algebra and the lightcone zero modes $\alpha_{lc}{}^i_{[r]0}$ have an index which spans only the transverse directions and not the whole spacetime. In fact k^+, k^- are treated in a different way because we are already thinking about the correspondence with the DDF case.

Finally, notice that we treat all momenta in a different way w.r.t. non-zero modes since there is an added -1 to get better equations afterwards. The reason is that we want to get Eq. (6.7) with the factor $-\frac{2}{\alpha'}$ in order to be able to write Eq. (6.11).

5 Recovering Mandelstam map 2: rewriting the Reggeon with operators

We are now ready to rewrite the N point DDF Reggeon using the operators.

As a first step, we eliminate all the $e^{\cdots\underline{x}_0^+}$ exponentials. To do so we notice that each time we take a derivative w.r.t. $\lambda^i_{[r]n}$ we bring down a factor $\left(e^{-i\frac{\underline{x}_0^+}{2\alpha'\underline{k}^+_{[r]}}}\right)^n$ which increases $\underline{k}^-_{T[r]}$ by $\frac{n}{2\alpha'\underline{k}^+_{[r]}}$. This means that when we have taken all derivatives, we have to replace the tachyonic $\underline{k}^-_{T[r]}$ with the state momentum $\underline{k}^-_{[r]}$.

So we eliminate the exponential simply by dropping them and making the substitution $\underline{k}^-_{T[r]}\to\underline{k}^-_{[r]}$ in the factor $e^{i\underline{k}^i_{T[r]}\underline{x}^i_0-i\underline{k}^-_{T[r]}\underline{x}^+_0}$.

Then we can compute the zero modes expectation value and get a conservation $\delta^{D-2}(\sum_r\underline{k}^i_r)$. Finally, we use the $\lambda\to\alpha_{lc}$ mapping to write $\delta^{D-2}(\sum_r\alpha_{lc}{}^i_{[r]0})$.

As a second step, we rewrite $\underline{k}^-_{T[r]}$ in the exponents of $x_{rt}^{2\alpha'\underline{k}_{T[r]}\cdot\underline{k}_{T[t]}}$ in terms of the true momenta $\underline{k}_{[r]}$.

Moreover, we want to eliminate the $\underline{k}_{[r]+}=-\underline{k}^-_{[r]}$ as they are dependent on the other momenta in the Hamiltonian lightcone formulation. To this end, we notice that

$$\begin{aligned}&\underline{k}^i_{T[r]}=\underline{k}^i_{[r]},\quad\underline{k}^+_{T[r]}=\underline{k}^+_{[r]},\\&\underline{k}^-_{T[r]}=\frac{\underline{k}^i_{[r]}\underline{k}^i_{[r]}+M^2_{T[r]}}{2\underline{k}^+_{[r]}},\end{aligned} \tag{5.1}$$

where we have introduced the free parameters $M^2_{T[r]}=-\underline{k}^2_{[r]}$ which may vary for each particle. Then we get

$$\begin{aligned}\prod_{r=1}^{N-1}\prod_{t=r+1}^{N}x_{rt}^{2\alpha'\underline{k}_{T[r]}\cdot\underline{k}_{T[t]}}&=\prod_{r=1}^{N-1}\prod_{t=r+1}^{N}x_{rt}^{\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[t]0}}\\&\times\prod_{r=1}^{N-1}\prod_{t=r+1}^{N}x_{rt}^{\rho_{rt}\left(\frac{1}{2}\alpha_{lc}{}^i_{[t]0}\alpha_{lc}{}^i_{[t]0}+\alpha'M^2_{T[t]}\right)+\rho_{rt}\left(\frac{1}{2}\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[r]0}+\alpha'M^2_{T[r]}\right)}\\&=\prod_{r=1}^{N-1}\prod_{t=r+1}^{N}x_{rt}^{\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[t]0}}\\&\times\prod_{r=1}^{N}\left[\left.\frac{dZ_{[r]}(z)}{dz}\right|_{z=x_r}\right]^{-\frac{1}{2}\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[r]0}-\alpha'M^2_{T[r]}}.\end{aligned} \tag{5.2}$$

The other terms are easier to rewrite. Keeping in mind our original goal of rewriting the Reggeon in term of the lightcone SDS vertexes we write

$$\langle V_N^{(DDF+ghost)}(\{x_r\})|=\left(\otimes_{r=1}^N\langle\underline{x}^i_{[r]0},0_{\alpha_{lc}{}^i_{[r]}}|\right)$$

$$\prod_{r=1}^{N}\left\{\left[\left.\frac{dZ_{[r]}(z)}{dz}\right|_{z=x_r}\right]^{+\alpha'M^2_{T[r]}}\right\}$$

$$
\begin{aligned}
&\times (x_1 - x_2)\,(x_1 - x_N)\,(x_2 - x_N) \\
&\times \prod_{r=1}^{N} \Bigg\{ \left[\frac{d Z_{[r]}(z)}{dz}\Big|_{z=x_r} \right]^{-\frac{1}{2}\sum_i \alpha_{lc}{}^{i}_{[r]0} \alpha_{lc}{}^{i}_{[r]0}} \\
&\exp\Bigg[-\sum_{i=2}^{D-1} \sum_{n=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n}}{n}\, \alpha_{lc}{}^{i}_{[r]0} \oint_{z=x_r} \frac{dz}{2\pi i} \frac{Z_{[r]}(z)^{-n}}{z - x_r} \Bigg] \\
&\exp\Bigg[\frac{1}{2} \sum_{i=2}^{D-1} \sum_{n_1,n_2=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n_1}}{n_1} \frac{\alpha_{lc}{}^{i}_{[r]n_2}}{n_2} \oint_{z_1=x_r} \\
&\frac{dz_1}{2\pi i} \oint_{z_2=x_r} \frac{dz_2}{2\pi i} \frac{Z_{[r]}(z_1)^{-n_1}\, Z_{[r]}(z_2)^{-n_2}}{(z_1 - z_2)^2} \Bigg] \Bigg\} \\
&\times \prod_{r=1}^{N-1} \prod_{t=r+1}^{N} x_{rt}^{\sum_{i=2}^{D-1} \alpha_{lc}{}^{i}_{[r]0} \alpha_{lc}{}^{i}_{[t]0}} \\
&\exp\Bigg[- \sum_{r,t=1,\, r\neq t}^{N} \sum_{i=2}^{D-1} \sum_{n=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n}}{n}\, \alpha_{lc}{}^{i}_{[t]0} \oint_{z=x_r} \\
&\frac{dz}{2\pi i} \frac{Z_{[r]}(z)^{-n}}{z - x_t} \Bigg] \\
&\exp\Bigg\{ \sum_{r=1}^{N-1} \sum_{t=r+1}^{N} \sum_{i=2}^{D-1} \sum_{n_1,n_2=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n_1}}{n_1} \frac{\alpha_{lc}{}^{i}_{[t]n_2}}{n_2} \oint_{z_1=x_r} \frac{dz_1}{2\pi i} \oint_{z_2=x_t} \\
&\frac{dz_2}{2\pi i} \frac{Z_{[r]}(z_1)^{-n_1}\, Z_{[t]}(z_2)^{-n_2}}{(z_1 - z_2)^2} \Bigg\} \\
&\times \delta^{D-2}\left(\sum_{r=1}^{N} \alpha_{lc}{}^{i}_{[r]0} \right) \delta\left(\sum_{r=1}^{N} \underline{k}^{+}_{[r]} \right) \delta\left(\sum_{r=1}^{N} \underline{k}^{-}_{[r]} \right), \qquad (5.3)
\end{aligned}
$$

where we have grouped the different factors keeping in mind the final result. Notice the special treatment for the $\underline{k}^{\pm}$ components.

6 Recovering Mandelstam map 3: Mandelstam maps from interactions between different strings

Let us now use the substitution rule

$$
\begin{aligned}
\lambda^{i}_{[r]n} \;\rightarrow\; & \frac{\alpha_{lc}{}^{i}_{[r]n}}{n} = -i\sqrt{\frac{2}{\alpha'}} \oint_{W_{[r]}=0} \\
& \frac{dZ_{[r]}}{2\pi i} L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]})\, Z_{[r]}{}^{n-1}, \quad n > 1 \qquad (6.1)
\end{aligned}
$$

where $Z_{[r]}$ is the local coordinate around the lightcone emission vertex and we defined as usual

$$
\begin{aligned}
L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]}) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n}}{n} Z_{[r]}{}^{-n}, \\
L_{lc}{}^{i}_{[r](0)}(Z_{[r]}) &= \frac{1}{2} x_{lc}{}^{i}_{[r]0} - i\alpha' p_{lc}{}^{i}_{[r]0} \ln(Z_{[r]}), \qquad (6.2)
\end{aligned}
$$

so that trivially

$$
\partial L_{lc}{}^{i}_{[r](0)}(Z_{[r]}) = -i\sqrt{\frac{\alpha'}{2}} \frac{\alpha_{lc}{}^{i}_{[r]0}}{Z_{[r]}},
$$

$$
\partial L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n}}{Z_{[r]}{}^{n+1}}. \qquad (6.3)
$$

We now perform all steps carefully for the first few terms, while for the other terms we quote the result since the steps are the same.

Explicitly we examine the terms

$$
\begin{aligned}
\alpha_{lc0}\alpha_{lcn} \text{ terms } \Rightarrow & -\sum_{r=1}^{N} \sum_{t=1,t\neq r}^{N} \sum_{i=2}^{D-1} \alpha_{lc}{}^{i}_{[t]0} \oint_{z=x_r} \\
& \frac{dz}{2\pi i} \sum_{n=1}^{\infty} \frac{\alpha_{lc}{}^{i}_{[r]n}}{n} Z_{[r]}(z)^{-n} \frac{1}{z - x_t}. \qquad (6.4)
\end{aligned}
$$

We can rewrite the expression of interest as

$$
\begin{aligned}
= & +\frac{2}{\alpha'} \sum_{r=1}^{N} \sum_{t=1,t\neq r}^{N} \sum_{i=2}^{D-1} \oint_{Z_{[t]}=0} \frac{dZ_{[t]}}{2\pi i} \oint_{z=x_r} \\
& \frac{dz}{2\pi i} \frac{L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]}(z))\, \partial L_{lc}{}^{(+)i}_{[t](0)}(Z_{[t]})}{z - x_t}. \qquad (6.5)
\end{aligned}
$$

Now we change the integration variable from z to $Z_{[r]}$ and use the fact that $z(Z_{[r]} = 0) = x_r$. This is possible since the mapping is analytic for $|z - x_r| < min_{s\neq r}\{|x_s - x_r|\}$ and invertible for a possibly smaller open set determined by the zeros of $\frac{dZ_{[r]}}{dz} = 0$. However, notice that this means that we may have singularities for $|Z_{[r]}| < 1$. This is the major difference with [41–43] and somewhat unusual for string field theory.

We get

$$
\begin{aligned}
= & +\frac{2}{\alpha'} \sum_{r=1}^{N} \sum_{t=1,t\neq r}^{N} \sum_{i=2}^{D-1} \oint_{Z_{[t]}=0} \frac{dZ_{[t]}}{2\pi i} \oint_{Z_{[r]}=0} \\
& \frac{dZ_{[r]}}{2\pi i}\, \partial L_{lc}{}^{(+)i}_{[t](0)}(Z_{[t]})\, L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]}) \frac{dz}{dZ_{[r]}} \frac{1}{z(Z_{[r]}) - x_t} \\
= & +\frac{2}{\alpha'} \sum_{r=1}^{N} \sum_{t=1,t\neq r}^{N} \sum_{i=2}^{D-1} \oint_{Z_{[t]}=0} \frac{dZ_{[t]}}{2\pi i} \oint_{Z_{[r]}=0} \\
& \frac{dZ_{[r]}}{2\pi i}\, L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]})\, \partial L_{lc}{}^{(+)i}_{[t](0)}(Z_{[t]}) \frac{d}{dZ_{[r]}} \ln\left(z(Z_{[r]}) - x_t\right). \qquad (6.6)
\end{aligned}
$$

Now we integrate by parts w.r.t. $Z_{[r]} = 0$. This can be done despite the logarithm since we are integrating around x_r and $x_t \neq 0$ so we do not cross the cut and get

$$
\begin{aligned}
= & -\frac{2}{\alpha'} \sum_{r=1}^{N} \sum_{t=1,t\neq r}^{N} \sum_{i=2}^{D-1} \oint_{Z_{[t]}=0} \frac{dZ_{[t]}}{2\pi i} \oint_{Z_{[r]}=0} \\
& \frac{dZ_{[r]}}{2\pi i} \partial L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]})\, \partial L_{lc}{}^{(+)i}_{[t](0)}(Z_{[t]}) \ln\left(z(Z_{[r]}) - x_t\right)
\end{aligned}
$$

$$= -\frac{2}{\alpha'}\sum_{r=1}^{N}\sum_{t=1,t\neq r}^{N}\sum_{i=2}^{D-1}\oint_{Z_{[t]}=0}\frac{dZ_{[t]}}{2\pi i}\oint_{Z_{[r]}=0}\frac{dZ_{[r]}}{2\pi i}\,\partial L_{lc}{}^{(+)i}_{[r](\neq)}(Z_{[r]})\,\partial L_{lc}{}^{(+)i}_{[t](0)}(Z_{[t]})\ln\left(z(Z_{[r]})-z(Z_{[t]})\right), \tag{6.7}$$

where the last step is possible since $\partial \underline{L}^{(+)i}_{[t](0)}(Z_{[t]})$ has only a single pole at $Z_{[t]}=0$ and $z(Z_{[t]}=0)=x_t$.

Along the same line we can consider $\alpha_{lc}{}^{i}_{[r]0}\alpha_{lc}{}^{i}_{[t]0}$ with $r\neq t$. We get

$$\alpha_{lc[r]0}\alpha_{lc[t]0}\text{ terms } \Rightarrow \prod_{r=1}^{N-1}\prod_{t=r+1}^{N} x_{rt}^{\alpha_{lc}{}^{i}_{[r]0}\alpha_{lc}{}^{i}_{[t]0}}$$
$$= \exp\{\sum_{r=1}^{N-1}\sum_{t=r+1}^{N}\sum_{i=2}^{D-1}\alpha_{lc}{}^{i}_{[r]0}\alpha_{lc}{}^{i}_{[t]0}\ln(z(Z_{[r]})-z(Z_{[t]})|_{Z_{[r]}=Z_{[t]}=0}\}$$
$$= \exp\left\{-\frac{2}{\alpha'}\sum_{r=1}^{N-1}\sum_{t=r+1}^{N}\sum_{i=2}^{D-1}\oint_{Z_{[r]}=0}\frac{dZ_{[r]}}{2\pi i}\oint_{Z_{[t]}=0}\frac{dZ_{[t]}}{2\pi i}\partial\underline{L}^{(+)i}_{[r](0)}(Z_{[r]})\,\partial\underline{L}^{(+)i}_{[t](0)}(Z_{[t]})\ln\left(z(Z_{[r]})-z(Z_{[t]})\right)\right\}. \tag{6.8}$$

Finally let us now consider $\alpha_{lc}{}^{i}_{[r]n_1}\alpha_{lc}{}^{i}_{[t]n_2}$ with $r\neq t$. We get

$$\alpha_{lc[r]n_1}\alpha_{lc[t]n_2}\text{ terms } \Rightarrow +\sum_{r=1}^{N-1}\sum_{t=r+1}^{N}\sum_{i=2}^{D-1}\sum_{n_1,n_2=1}^{\infty}\frac{\alpha_{lc}{}^{i}_{[r]n_1}}{n_1}\frac{\alpha_{lc}{}^{i}_{[t]n_2}}{n_2}\oint_{z_1=0}\frac{dz_1}{2\pi i}\oint_{z_2=0}\frac{dz_2}{2\pi i}\frac{Z_{[r]}(z_1)^{-n_1}\,Z_{[t]}(z_2)^{-n_2}}{(z_1-z_2+x_{rt})^2}$$
$$= -\frac{2}{\alpha'}\sum_{r=1}^{N-1}\sum_{t=r+1}^{N}\sum_{i=2}^{D-1}\oint_{Z_{[r]}=0}\frac{dZ_{[r]}}{2\pi i}\oint_{Z_{[t]}=0}\frac{dZ_{[t]}}{2\pi i}\partial\underline{L}^{(+)i}_{[r](\neq)}(Z_{[r]})\,\partial\underline{L}^{(+)i}_{[t](\neq)}(Z_{[t]})\ln\left(z(Z_{[r]})-z(Z_{[t]})\right). \tag{6.9}$$

Using the trivial fact that

$$\underline{L}^{(+)i}_{[r]}(Z_{[r]}) = \underline{L}^{(+)i}_{[r](\neq)}(Z_{[r]})+\underline{L}^{(+)i}_{[r](0)}(Z_{[r]}), \tag{6.10}$$

we can now assemble all interactions with $r\neq t$ given in Eqs. (6.7), (6.8) and (6.9) under a very simple expression

$$\sum_{r\neq t}\left[\alpha_{lc[r]n}\alpha_{lc[t]0}\text{ terms } + \alpha_{lc[r]0}\alpha_{lc[t]0}\text{ terms } + \alpha_{lc[r]n}\alpha_{lc[t]m}\text{ terms }\right] =$$
$$= -\frac{2}{\alpha'}\sum_{r=1}^{N-1}\sum_{t=r+1}^{N}\sum_{i=2}^{D-1}\oint_{Z_{[r]}=0}\frac{dZ_{[r]}}{2\pi i}\oint_{Z_{[t]}=0}\frac{dZ_{[t]}}{2\pi i}\partial\underline{L}^{(+)i}_{[r]}(Z_{[r]})\,\partial\underline{L}^{(+)i}_{[t]}(Z_{[t]})\ln\left(z(Z_{[r]})-z(Z_{[t]})\right). \tag{6.11}$$

The previous expression strongly suggests that hidden within DDF correlators are the Mandelstam maps. In fact, the previous equation (6.11) is very similar to the usual one in Eq. (3.6) up to use of different maps, i.e Eq. (3.8) for the usual case and Eq. (4.3) for the DDF case. Explicitly, we identify $z(Z_{[r]})=m_{[r]}(Z_{[r]})$.

Looking to the whole Reggeon we see that the DDF Reggeon (5.3) has further contributions, notably some self-interactions and a normalization term.

We show in the next section that the self-interaction terms arise from the conformal transformation of the lightcone SDS vertex (given in Eq. (7.2)) under the local Mandelstam maps suggested by the previous expression.

We are then left to explain the extra normalization factor in the first line of Eq. (5.3). We shall argue that this is the factor Mandelstam obtained from the functional determinant of 2*d* Laplacian and other normalization factors in the path integral. The underlying reason is simply that the DDF amplitudes are covariant, and the factors computed by Mandelstam are needed for ensuring covariance.

7 Recovering Mandelstam map 4: Mandelstam from self-interactions

As mentioned at the end of the previous section we need to explain the self-interactions. In particular, we would interpret Eq. (5.3) as a result of a computation similar to the usual one given in Eq. (3.3). Explicitly, we would like to show that (with $|x_1|>\cdots>|x_N|$)

$$\langle V_N^{(DDF+ghost)}(\{x_r\})| = \mathcal{N}(\{x_r\})\,\delta\left(\sum_{r=1}^{N}\underline{k}^{+}_{[r]}\right)\,\delta\left(\sum_{r=1}^{N}\underline{k}^{-}_{[r]}\right)\langle\tilde{0}_{lightcone}|\,m_{[1]}\circ\widetilde{\mathcal{S}}_{lightcone[1]}(0)\dots m_{[N]}\circ\widetilde{\mathcal{S}}_{lightcone[N]}(0)\,|\tilde{0}_{lightcone}\rangle_{\widetilde{UHP}}, \tag{7.1}$$

where the basic lightcone SDS is given in Eq. (7.2), $m_{[r]}(0)=x_r$ and the normalization factor $\mathcal{N}(\{x_r\})$ is discussed later. Notice that we have written $\widetilde{UHP}$ to clarify that the expectation value is w.r.t. the $\widetilde{L_{lc}}$ fields despite the fact that mathematically there is only one UHP.

Since the contribution from the $\widetilde{L_{lc}}$ has been considered in the previous section and is completely analogous to the usual one, because it gives $\ln\left(z(Z_{[r]})-z(Z_{[t]})\right)=\ln\left(m_{[r]}(Z_{[r]})-m_{[t]}(Z_{[t]})\right)$, we would now consider the contributions from self-interactions. In particular, we would like to show that self-interactions arise from regularizing the lightcone SDS vertex on the previous Riemann surface. In fact, the self-interaction is the left over of the regularization of the non-normal ordered lightcone SDS vertex or, more exactly, the effect of a non-trivial conformal transformation.

In order to do so we write the SDS vertex for the lightcone fields as

$$\begin{aligned}\widetilde{\mathcal{S}}_{lightcone}(X) &= \langle x^i=0, 0_{\alpha_{lc}{}^i}| \\ &\quad : e^{\frac{i}{\sqrt{2\alpha'}}\delta_{ij}\sum_{n=0}^{\infty}\frac{\alpha_{lc}{}^i_n}{n!}\partial_x^n \widetilde{L_{lc}}^j(X,\bar{X})} : \\ &= \langle x^i=0, 0_{\alpha_{lc}{}^i}| \\ &\quad : e^{-\frac{2}{\alpha'}\delta_{ij}\oint_{Z=0,\,|Z|<|X|}\frac{dZ}{2\pi i}\partial_Z L_{lc}{}^i(Z)\widetilde{L_{lc}}^j(X+Z)} :,\end{aligned} \tag{7.2}$$

and notice that this vertex has very different conformal properties as opposed to the apparently analogous vertex for DDF operators

$$\begin{aligned}\widetilde{\mathcal{S}}_{DDF}(x) &= \langle x^\mu=0, 0_{\underline{A}^i}| : e^{\frac{i}{\sqrt{2\alpha'}}\delta_{ij}\sum_{n=0}^{\infty}\frac{\underline{A}_n^i}{n!}\underline{\tilde{A}}_n^j(x)} \\ &\quad :: e^{i\underline{k}_{T\mu}\underline{\widetilde{L}}^\mu(x,\bar{x})} :,\end{aligned} \tag{7.3}$$

where $\underline{\tilde{A}}_n^j(x)$ is the DDF operator at x which contains both creators and annihilators of the covariant field. The reason is that $\underline{\tilde{A}}_n^i(u)$ (and $\underline{A}_n^i$) are good zero dimensional conformal operators, i.e. primary fields, while $\partial_x^n \widetilde{L_{lc}}^j(x,\bar{x})$ (i.e. $\widetilde{\alpha_{lc}}{}^i_n$ and $\alpha_{lc}{}^i_n$) are (for $n \neq 1$) not primary fields.

This implies that under a conformal transformation $u = f(U)$ (in the UHP) generated by an operator $\tilde{U}_f$ and acting on the tilded fields they transform as (see Appendix A)

$$\begin{aligned}&\tilde{U}_f\,\widetilde{\mathcal{S}}_{lightcone}(X)\,\tilde{U}_f^{-1} = f\circ\widetilde{\mathcal{S}}_{lightcone}(X) \\ &\quad = \widetilde{\mathcal{S}}_{lightcone}(f(X)) \\ &\quad\quad \times \lim_{\epsilon\to 0} e^{\frac{-2}{\alpha'^2}\oint_{W=\epsilon,}\frac{dW}{2\pi i}\oint_{Z=0,\,|Z|<\epsilon}\frac{dZ}{2\pi i}\partial_W L_{lc}{}^i(W-\epsilon)\,\partial_Z L_{lc}{}^i(Z)\,\Delta_f(X+W,X+Z)},\end{aligned} \tag{7.4}$$

where

$$\begin{aligned}\Delta_f(X+W,X+Z) &= G_0(f(X+W), \\ &\quad f(X+Z)) - G_0(X+W,X+Z) \\ &= -\frac{\alpha'}{2}\ln\frac{f(X+W)-f(X+Z)}{W-Z}.\end{aligned} \tag{7.5}$$

This transformation has to be compared with

$$\begin{aligned}\tilde{U}_f\,\widetilde{\mathcal{S}}_{DDF}(x)\,\tilde{U}_f^{-1} &= f\circ\widetilde{\mathcal{S}}_{DDF}(x) \\ &= \left(\frac{df}{dx}\right)^{\frac{1}{2}\underline{A}_0^i\underline{A}_0^i}\widetilde{\mathcal{S}}_{DDF}(f(x)).\end{aligned} \tag{7.6}$$

We can now apply the previous discussion to the case where $u = f(U)$ is given by $u - x_r = f_{[r]}(U_{[r]})$ and $X = 0$, and compare with Eq. (5.3). We can then use $U_{[r]} = F_{[r]}(u - x_r) = U_{[r]}(u)$ and $X = 0 \Rightarrow U_{[r]} = 0$, so that $u - x_r = 0$. Notice that in the text but not here we write $u = m_{[r]}(U_{[r]})$ instead of $u = u(U_{[r]})$, the reason being that here, it is more functional to use $u(U_{[r]})$ since this makes it clear that they are the inverse. However, in the text we want to stress that they are the local Mandelstam maps. On writing Eq. (7.4) in a more explicit way, we get

$$\begin{aligned}&u\circ\widetilde{\mathcal{S}}_{lightcone}(U_{[r]}) = \widetilde{\mathcal{S}}_{lightcone}(u(U_{[r]}))\lim_{\epsilon\to 0} \\ &\quad\times e^{\frac{2}{\alpha'^2}\oint_{W=\epsilon,}\frac{dW}{2\pi i}\oint_{Z=0,\,|Z|<\epsilon}\frac{dZ}{2\pi i}\partial_W L_{lc}{}^i(W-\epsilon)\,\partial_Z L_{lc}{}^i(Z)\,\Delta_u(X+W,X+Z)} \\ &= \widetilde{\mathcal{S}}_{lightcone}(u(U_{[r]}))\times e^{+\frac{1}{2}\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[r]0}\ln\left.\frac{du}{dU_{[r]}}\right|_{U_{[r]}=0}} \\ &\quad\times e^{\frac{-2}{\alpha'}\sum_{n=1}^{\infty}\alpha_{lc}{}^i_{[r]0}\alpha_{lc}{}^i_{[r]n}\frac{1}{n!}\partial_U^n\Delta_u(U,0)\big|_{U=0}} \\ &\quad\times e^{-\frac{1}{\alpha'}\sum_{n,m=1}^{\infty}\alpha_{lc}{}^i_{[r]n}\alpha_{lc}{}^i_{[r]m}\frac{1}{n!m!}\partial_{U_1}^n\partial_{U_2}^m\Delta_u(U_1,U_2)\big|_{U_{1,2}=0}}.\end{aligned} \tag{7.7}$$

We notice, immediately, that the expression in Eq. (5.3) is in terms of the inverse function $U = F(u)$. In Appendix A we give a proof of the required identity

$$\begin{aligned}&\lim_{U_i\to 0}\frac{1}{m!n!}\frac{-2}{\alpha'}\partial_{U_1}^m\partial_{U_2}^n\Delta_f(U_1,U_2) \\ &\quad = \frac{1}{mn}\oint_{z_1=0}\frac{dz_1}{2\pi i}\oint_{z_2=0,|z_1|>|z_2|}\frac{dz_2}{2\pi i} \\ &\quad\quad \frac{1}{(z_1-z_2)^2}\frac{1}{[F(z_1)]^m[F(z_2)]^n},\end{aligned} \tag{7.8}$$

for $m + n \geq 1$ which allows us to perform the desired map.

8 Recovering Mandelstam map 5: the normalization factor when on-shell

We would now discuss the normalization factor

$$\begin{aligned}\mathcal{N}(\{x_r\}) &= \prod_{r=1}^{N}\left\{\left[\left.\frac{dZ_{[r]}(z)}{dz}\right|_{z=x_r}\right]^{+\alpha' M^2_{T[r]}}\right\} \\ &\quad (x_1-x_2)(x_1-x_N)(x_2-x_N),\end{aligned} \tag{8.1}$$

in Eq. (5.3).

We would like to argue that in *critical dimension* and *on-shell* when $\alpha' M^2_{T[r]} = -1$, this is *essentially* the factor computed by Mandelstam in path integral approach which arises from the functional determinant of the 2d Laplacian, and the Jacobian due to the change of coordinates from $\tau_E + i\sigma$ to upper plane ones ([38] and [47] vol. 2 and [43] for a more recent approach).

The reason is simply that *on-shell* DDF amplitudes in *critical dimension* are covariant and the normalization factor $\mathcal{N}(\{x_r\})$ is needed to get a covariant amplitude exactly as the factor computed by Mandelstam is needed for covariance.

There is also another reason to stress on-shell. Mandelstam's results are in an Hamiltonian framework where there is no lightcone energy conservation unless one considers an infinite period of evolution, i.e. the S matrix. In this case, we have $\sum_r p_r^- = 0$ so the states are on-shell.

This is also the reason why we have to add by hand the $\delta\left(\sum_{r=1}^N \underline{k}_{[r]}^-\right)$ in (7.1).

On the other hand, $\delta\left(\sum_{r=1}^N \underline{k}_{[r]}^+\right)$ is implicit in Mandelstam's work because of the conservation of the strip width imposed by hand.

We would now like to comment on the adverb "essentially". Mandelstam's computations are performed by always choosing one vertex in the upper half-plane at $x = \infty$. The way in which we performed the computations with the DDF vertices does not allow us to send any coordinate to ∞. On the other hand, on-shell we know that the amplitude can be written in terms of anharmonic ratios which allows us to take one coordinate to infinity in a smooth fashion. Therefore, we take one coordinate to infinity and drop any divergent factor in the Reggeon, since it must be canceled when expanding it for computing an explicit amplitude. This argument is indirect, but can be checked directly in the $N = 3$ case. In particular, we can compare in detail the integrals, i.e. the Neumann coefficients in the case $N = 3$ and find a perfect match. Details are given in the Appendix B.

9 Going off-shell: a difference

Finally, we can consider what happens *off-shell*. Off-shell there is a difference. In this section, we keep the discussion at the conceptual level and we write the details in Appendix B.

Before we compare, we need to explain what we mean by 'going off-shell' in our case.

The simplest way is to let $\alpha' \underline{k}_{T\,[r]}^2$ which on-shell are the tachyon masses be free parameters. In doing so, we get amplitudes which depend on the Koba-Nielsen variables and are not unique. This is to be expected since off-shell physics is not unique. On the other hand, if we want an off-shell extension which is well-defined and consistent, we can consider the Witten three-vertex for the $N = 3$ amplitudes and then build the others from it.

Let us consider the Witten vertex. Instead of the usual basis of covariant states of ghost number -1 we can use a basis formed by DDF, improved Brower states and the dual to the improved Brower states. These states obviously do not include the zero momentum states. This basis is, however, interesting since, in critical string and on-shell, only DDF states are physical and non-null, while at the same time improved Brower states are BRST exact, i.e. physical and null. In critical dimension and off-shell, improved Brower states are physical since the improved Brower operators commute with the Virasoro generators but may have negative norm as shown in some examples in [35]. Moreover, correlators among DDF states, and improved Brower states do not vanish but are proportional to the mass shell conditions of the Brower states, as can be directly verified in some very simple examples.

In order to compare with the lightcone approach, we can restrict our attention to DDF states. We notice that the DDF states and operators are primary even when "the tachyon mass", more exactly $\alpha' \underline{k}_T^2$ is a free parameter. In fact, the framed DDF operators $\underline{A}_n^i(x)$ are conformal operators of dimension zero ([35]). (Framed DDF operators are slightly different from DDF operators stripped of $\underline{x}_0^+$ zero modes, which were used earlier and also shown to be good conformal operators in ([41])). It follows that the dimension of an off-shell DDF state is equal to the dimension of the associated tachyon vertex, i.e. $\alpha' \underline{k}_T^2$. The Witten vertex is then simply

$$\langle W|\Big|_{DDF} = \left(\frac{8}{3}\right)^{\alpha' \underline{k}_{T[1]}^2} \left(\frac{2}{3}\right)^{\alpha' \underline{k}_{T[2]}^2} \left(\frac{8}{3}\right)^{\alpha' \underline{k}_{T[3]}^2} \langle V_{N=3}^{(DDF+ghost)}(\{x_3 = -\sqrt{3}, x_2 = 0, x_1 = \sqrt{3}\})|. \quad (9.1)$$

Because DDF states are primary operators this discussion (see [41] and also [42,43]) can be extended to any consistent string field theory whose maps for the cubic order are $u = f_{3[r]}(U_{[r]})$ as ([41])

$$\langle V_3|\Big|_{DDF} = \left(f'_{3[1]}(0)\right)^{\alpha' \underline{k}_{T[1]}^2} \left(f'_{3[2]}(0)\right)^{\alpha' \underline{k}_{T[2]}^2} \left(f'_{3[3]}(0)\right)^{\alpha' \underline{k}_{T[3]}^2} \langle V_{N=3}^{(DDF+ghost)}(\{x_3 = f_{3[3]}(0), x_2 = f_{3[2]}(0), x_1 = f_{3[1]}(0)\})|. \quad (9.2)$$

Notice that conceptually there is also a difference in the approaches. Here we start from off-shell DDF states as discussed in [35] and we use them as (a subset of) a basis of string states. We then compute the off-shell amplitude for $N = 3$ states according to string field theory recipes. From this amplitude we read off the Mandelstam maps which match the result from first quantized sting theory because of Eq. (9.2). These maps are intrinsic to the DDF states and not imposed or chosen. Therefore, the eventual singularities in the local half-disk are also intrinsic. Differently in [42,43] they assume a given expression for the Mandelstam maps but then the expressions for the Mandelstam maps do not match. Comparison with [41] is less straightforward since in this paper they integrate out the Brower states (or at least states which should correspond to Brower states even if their expression does not match the usual expression for Brower states see eq. 3.80 of [41]).

Now we can compare with lightcone formalism. First of all we notice that Mandelstam worked in the Hamiltonian formalism and "off shell" means evolution for a finite time, which implies that lightcone energy is not conserved, i.e. there is *no* $\delta\left(\sum_{r=1}^{N} \underline{k}^{-}_{[r]}\right)$.

When considering finite time intervals Mandelstam Reggeon vertex (which is the cubic interaction term of the Hamiltonian) acquires a dependence on the interaction times, for the $N=3$ only one interaction time τ_0. The precise form of τ_0 seems to be fundamental in proving the lightcone factorization of the $N=4$ amplitudes (see paragraph 7 of [38]).

We cannot then compare directly with Mandelstam's work when off-shell, because we are computing an off-shell Green's function and he is computing a matrix $U(\tau_f, \tau_i)$ element. We can however compare with the formulation of lightcone String Field Theory proposed in [45,46,48]. In this paper Kaku and Kikkawa argue that the very same vertex of Mandelstam extended off-shell can be used for building the 3 string functional quantum *Hamiltonian* in interaction picture, i.e $H_{I\,3}(\tau)$, as

$$H_{I\,3}(\tau) = \sum_{\phi_1,\phi_2,\phi_3} \left(h_{3\,\phi_1,\phi_2,\phi_3}\, A^\dagger_{I\,\phi_1}(\tau)\, A_{I\,\phi_2}(\tau)\, A_{I\,\phi_3}(\tau) + h.c.\right)$$
$$= \sum_{\phi_1,\phi_2,\phi_3} \left(h_{3\,\phi_1,\phi_2,\phi_3}\, e^{-i\tau(\hat{P}_{[1]+}-\hat{P}_{[2]+}-\hat{P}_{[3]+})} A^\dagger_{\phi_1}\, A_{\phi_2}\, A_{\phi_3} + h.c.\right). \tag{9.3}$$

In this expression ϕ_r are "particles", i.e. second quantized lightcone string states. Each ϕ is in correspondence with a first quantized lightcone string and therefore is labeled by $|k_i,\, k_-,\, \{N_{i\,n}\}\rangle$. The creator $A^\dagger_\phi$ (which is in the Schroedinger picture while $A^\dagger_{I\,\phi}(\tau)$ is in the interaction picture) must not be confused with $\underline{A}^i_n$ since the latter is a first quantized DDF operator while the former is a second quantized creator which creates a first quantized string $\prod_{i,n}\left(\alpha_{lc}{}^{i}_{-n}\right)^{N_{in}} |k_i,\, k_-\rangle$. $A^\dagger_\phi$ creates a second quantized string $A^\dagger_\phi|\Omega\rangle$ [5] which is eigenstate of the second quantized free Hamiltonian H_2 with eigenvalue P_+, i.e.

$$H_2\, A^\dagger_\phi|\Omega\rangle = \hat{P}_+\, A^\dagger_\phi|\Omega\rangle, \quad \hat{P}_+ = \frac{\sum_i k_i^2 + \sum_{i,n} N_{in} - 1}{2k_-}. \tag{9.4}$$

[5] In particular $\prod_{r=1}^{M} A^\dagger_{\phi_{[r]}}|\Omega\rangle$ is a multi string state composed by M strings.

As a consequence the coefficient $h_{3\,\phi_1,\phi_2,\phi_3}$ can be written as

$$h_{3\,\phi_1,\phi_2,\phi_3} = c\left(\tau_0,\, \left\{k_{[r]\,i},\, k_{[r]\,-},\, \{N_{r\,in}\}\right\}_{r=1,2,3}\right) \delta^{D-2} \left(\sum_{r=1}^{N} \underline{k}_{[r]\,i}\right) \delta\left(\sum_{r=1}^{N} \underline{k}_{[r]-}\right). \tag{9.5}$$

Kaku and Kikkawa use explicitly Mandelstam vertex given in Eq. (B.1) as

$$\begin{aligned} h_{3\,\phi_1,\phi_2,\phi_3} = \langle V_3^{(M)}(0)|\, &\left|k_{[1]\,i},\, k_{[1]\,-},\, \{N_{[1]\,in}\}\right\rangle \\ &\otimes \left|k_{[2]\,i},\, k_{[2]\,-},\, \{N_{[2]\,in}\}\right\rangle \\ &\otimes \left|k_{[3]\,i},\, k_{[3]\,-},\, \{N_{[3]\,in}\}\right\rangle \\ &\times \delta\left(\sum_{r=1}^{N} \underline{k}_{[r]-}\right). \end{aligned} \tag{9.6}$$

The use of Mandelstam vertex extended off-shell was further researched in [49] without a clear final answer whether it is fully correct.

To compare with the modem point of view according to which tree-level string field theory is classical we have to "de-quantize" Eq. (9.3) and make the quantum operators $A_{I\phi}(\tau)$ classical modes of a classical lightcone field $\Phi_{\{N_{in}\}}(k_i,\, k_-,\, k_+)$ and then integrate over the lightcone time $\tau = X^+$ as $\int_{-\infty}^{+\infty} d\tau$ (see also [42,43]) to get finally the classical action

$$\begin{aligned} S_3^{(lc)} = \int \prod_{r=1}^{3} d^D k_{[r]} \sum_{\{N_{[r]\,in}\}} &\times \Big(\langle V_3^{(M)}(0)|\, \otimes_{r=1}^{3} \left|k_{[r]\,i},\, k_{[r]\,-},\, \{N_{[r]\,in}\}\right\rangle \\ &\times \delta^D(k_{[1]} - k_{[2]} - k_{[3]})\, \Phi^*_{\{N_{[1]in}\}}(k_{[1]\mu}) \\ &\times \Phi_{\{N_{[2]in}\}}(k_{[2]\mu})\, \Phi_{\{N_{[3]in}\}}(k_{[3]\mu}) + c.c.\Big). \end{aligned} \tag{9.7}$$

This classical action can then be compared with the classical action obtained from (9.2). Kaku and Kikkawa use the vertex $\langle V_3^{(M)}(0)|$ which is given in term of the Neumann coefficients $\bar{N}$ (see (B.2)). These Neumann coefficients $\bar{N}$ have an explicit τ_0 dependence which is well defined function of $\sqrt{2\alpha'}\underline{k}^{+}{}_{[r]}$ (see Eq. (B.2)). Now the classical action obtained from (9.2) has Neumann coefficients which do not depend on $\sqrt{2\alpha'}\underline{k}^{+}{}_{[r]}$ unless the maps $u = f_{3[r]}(U_{[r]})$ are chosen to depend explicitly on $\sqrt{2\alpha'}\underline{k}^{+}{}_{[r]}$ (see Eq. (B.6)) . This means that the results do not agree.

Would they use $\langle V_3^{(M)}(\tau_0)|$ the vertex would contain the Neumann coefficients N and this would match our results in the limit $x_3 \to \infty$ despite the fact that Mandelstam uses $\langle V_3^{(M)}(0)|$ to prove the factorization $N = 4$.

The reason for the difference cannot be traced to the existence of improved Brower states which are null, i.e. BRST exact only on-shell but can enter the internal off-shell leg since we are considering the $N = 3$ Green function.

After these considerations, we can ask where we are left. The answer is that it is not clear whether the String Field Theory construction by Kaku and Kikkawa is consistent as it stands. It may well be that they missed some higher order contact terms. On the other hand, the construction in [41] is well founded and has a clear framework.

Acknowledgements We would like to thank Raffaele Marotta for discussions. This research is partially supported by the MUR PRIN contract 2020KR4KN2 "String Theory as a bridge between Gauge Theories and Quantum Gravity" and by the INFN project ST&FI "String Theory & Fundamental Interactions".

Funding No funding.

Data Availability Statement This manuscript has associated data in a 547 data repository. [Author's comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement This manuscript has associated code/soft 549 ware in a data repository. [Author's comment: Code/Sotware sharing not applicable to this article as no code/sofware was generated or analysed during the current study.]

A Details on conformal transformation of lightcone SDS

We want to consider the transformation of the lightcone SDS under a generic conformal transformation. To this purpose we consider just one lightcone direction since they are independent. The non trivial transformation is the origin of the self-interaction integral appearing in the SDS correlator generating function.

We proceed naively and then we justify in a more rigorous way the result. We use capital coordinates like X, u since we think they are local coordinates. We start from the point splitted version of the SDS vertex

$$[\tilde{S}(X)]_{p.s} = \langle 0; 0_a| \exp\left[\frac{2}{\alpha'}\Big(\oint_{Z=0;|Z|<\epsilon} \frac{dZ}{2\pi i}\,\partial L^{(+)}(Z)\tilde{L}^{(-)}(X+Z) + \oint_{W=\epsilon} \frac{dW}{2\pi i}\,\partial L^{(+)}(W-\epsilon)\tilde{L}^{(+)}(X+W)\Big)\right], \tag{A.1}$$

and then we normal order it as

$$\begin{aligned}
&[\tilde{S}(X)]_{p.s} = \\
&=\langle 0; 0_a| : .. : \exp\left\{-\frac{1}{2}\frac{4}{\alpha'^2}\oint_{Z=0}\frac{dZ}{2\pi i}\oint_{W=\epsilon} \frac{dW}{2\pi i}\left[\partial L^{(+)}(Z)\tilde{L}^{(-)}(X+Z), \partial L^{(+)}(W-\epsilon)\tilde{L}^{(+)}(X+W)\right]\right\} \\
&=\langle 0; 0_a| : .. : \exp\left[\frac{1}{2}\frac{4}{\alpha'^2}\oint_{Z=0}\frac{dZ}{2\pi i}\oint_{W=\epsilon} \frac{dW}{2\pi i}\,\partial L^{(+)}(Z)\partial L^{(+)}(W-\epsilon)\,G_0(X+W, X+Z)\right],
\end{aligned} \tag{A.2}$$

where we have used $[\tilde{L}^{(+)}(W), \tilde{L}^{(-)}(Z)] = G_0(W, Z) = G_0(W-Z)$ in obtaining the last line.

The regularized SDS vertex, i.e. the normal ordered one is then obtained by removing the UV divergent extra factor above and taking the limit $\epsilon \to 0$,

$$[\tilde{S}(X)]_{reg} =: \tilde{S}(X) := \lim_{\epsilon\to 0}[\tilde{S}(X)]_{p.s}\,\mathcal{N}_0(X, \epsilon), \tag{A.3}$$

where

$$\mathcal{N}_0(X, \epsilon) = \exp\left[-\frac{1}{2}\frac{4}{\alpha'^2}\oint_{Z=0}\frac{dZ}{2\pi i}\oint_{W=\epsilon} \frac{dW}{2\pi i}\,\partial L^{(+)}(Z)\,\partial L^{(+)}(W-\epsilon)\,G_0(X+W, X+Z)\right]. \tag{A.4}$$

Notice that $\mathcal{N}_0$ is operatorial and must be written after $[\tilde{S}(X)]_{p.s}$ which contains a $\langle 0; 0_a|$ on which $\mathcal{N}_0$ acts.

We now apply the conformal transformation $u = f(U)$ generated by $\tilde{U}_f$ on the chiral tilded field $\tilde{L}(u)$. We proceed naively and apply the transformation on the regularized expression not taking into account that $\tilde{L}(u)$ is point splitted and not really a primary field. We get an expression that we verify in a different way.

We find naively

$$\tilde{U}_f\,[\tilde{S}(X)]_{p.s}\,\tilde{U}_f^{-1} = \langle 0; 0_a| \exp\left[\frac{2}{\alpha'}\Big(\oint_{Z=0} \frac{dZ}{2\pi i}\,\partial L^{(+)}(Z)\,\tilde{L}^{(-)}(f(X+Z)) + \oint_{W=\epsilon} \frac{dW}{2\pi i}\,\partial L^{(+)}(W-\epsilon)\tilde{L}^{(+)}(f(X+W))\Big)\right], \tag{A.5}$$

then we normal order and consider the regularized version

$$\tilde{U}_f\,[\tilde{S}(X)]_{reg}\,\tilde{U}_f^{-1} = f\circ[\tilde{S}(X)]_{reg} =$$

$$= \langle 0; 0_a| \,:\, \exp\left[\frac{2}{\alpha'}\Big(\oint_{Z=0}\frac{dZ}{2\pi i}\,\partial L^{(+)}(Z)\tilde{L}(f(X+Z))\Big)\right] :$$
$$\times \lim_{\epsilon\to 0}\exp\left[\frac{1}{2}\frac{4}{\alpha'^2}\oint_{Z=0}\frac{dZ}{2\pi i}\oint_{W=\epsilon}\frac{dW}{2\pi i}\right.$$
$$\left.\partial L^{(+)}(Z)\,\partial L^{(+)}(W-\epsilon)\,\Delta(X+W,X+Z)\right], \tag{A.6}$$

where the conformal transformation produces the non trivial (operatorial) factor

$$\mathcal{N}(X) = \lim_{\epsilon\to 0}\exp\left[\frac{1}{2}\frac{4}{\alpha'^2}\oint_{Z=0}\frac{dZ}{2\pi i}\oint_{W=\epsilon}\right.$$
$$\left.\frac{dW}{2\pi i}\,\partial L^{(+)}(Z)\,\partial L^{(+)}(W-\epsilon)\,\Delta(X+W,X+Z)\right], \tag{A.7}$$

in which we have defined

$$\Delta(X+W,X+Z) = G_0(f(X+W),$$
$$f(X+Z)) - G_0(X+W,X+Z)$$
$$= -\frac{\alpha'}{2}\ln\left(\frac{f(X+W)-f(X+Z)}{W-Z}\right), \tag{A.8}$$

which is well behaved as $W\to Z$ ($\epsilon\to 0$) since the UV singularity is canceled by the $G_0(X+W,X+Z)$ subtraction.

We notice that we can use Eq. (4.5) to rewrite the previous transformation in a coherent state approach as

$$f\circ :\, e^{i\sqrt{\frac{\alpha'}{2}}\left[\lambda_0\tilde{L}(X)+\sum_{n=1}^{\infty}\frac{\lambda_n}{(n-1)!}\partial_X^n\tilde{L}(X)\right]} :=$$
$$= :\, e^{i\sqrt{\frac{\alpha'}{2}}\left[\lambda_0\tilde{L}(f(X))+\sum_{n=1}^{\infty}\frac{\lambda_n}{(n-1)!}\partial_X^n\tilde{L}(f(X))\right]} :$$
$$\times (f'(X))^{\frac{1}{2}\lambda_0^2}\, e^{-\frac{2}{\alpha'}\lambda_0\sum_{n=1}^{\infty}\frac{\lambda_n}{(n-1)!}\partial_X^n\Delta(X,Y)|_{Y=0}}$$
$$e^{-\frac{1}{\alpha'}\sum_{n=1}^{\infty}\frac{\lambda_n}{(n-1)!}\sum_{m=1}^{\infty}\frac{\lambda_m}{(m-1)!}\partial_X^n\partial_Y^m\Delta(X,Y)|_{Y=0}}. \tag{A.9}$$

A.1 Checking the conformal transformation: an example

The goal of this section is to show that the regularization factor $\mathcal{N}(x)$ derived above is in fact the correct term despite the naive approach. We consider a simple operator $:(\partial L)^2:(X)$ with $|U_1|>|U_2|$

$$:(\partial L)^2:(X)$$
$$= \lim_{U_a\to X}\left[\partial L(U_1)\,\partial L(U_2) - \partial_{U_1}\partial_{U_2}G_0(U_1,U_2)\right]$$
$$\&\ f\circ(\partial L(U_1)\,\partial L(U_1))$$
$$= f'(U_1)\,\partial_{U_1}L(f(U_1))\,f'(U_2)\,\partial_{U_2}L(f(U_2))$$
$$\Longrightarrow f\circ :(\partial L)^2:(X)$$
$$= \lim_{U_i\to X}\Big\{\partial_{f(U_1)}L(f(U_1))\,\partial_{f(U_2)}L(f(U_2)) + \partial_{U_1}\partial_{U_2}$$
$$[G_0(f(U_1),f(U_2)) - G_0(U_1,U_2)]\Big\}, \tag{A.10}$$

which reproduces the previous naive result.

It is then immediate to generalize the previous example to $:(\partial L)^n\,(\partial L)^m:(X)$ by simply taking derivatives w.r.t. U_1 and U_2.

For the more general case $:\prod_{a=1}^{M}(\partial L)^{n_a}:(X)$ we can proceed as before starting from

$$\lim_{U_a\to X}\left[\partial^{n_1}L(U_1)\,\partial^{n_2}L(U_2)\ldots\partial^{n_M}L(U_M)\right.$$
$$\left.-\sum\text{possible contractions}\right], \tag{A.11}$$

with $|U_1|>|U_2|>\ldots|U_M|$.

A.2 Comparison with self-interaction integral - generic proof

In the main text in order to match the result from the conformal transformation of the lightcone SDS under a map $u=f(U)$ with the result from the DDF computation we have to rewrite the result for the conformal transformation in term of the inverse map $U=F(u)$.

Concretely, we need to prove the following generic relation,

$$\lim_{u_i\to 0}\frac{1}{m!n!}\frac{-2}{\alpha'}\partial_{U_1}^{m+1}\partial_{U_2}^{n+1}\Delta_f(U_1,U_2)$$
$$= \oint_{z_1=0}\frac{dz_1}{2\pi i}\oint_{z_2=0,|z_1|>|z_2|}\frac{dz_2}{2\pi i}\frac{1}{(z_1-z_2)^2}$$
$$\frac{1}{[F(z_1)]^{m+1}[F(z_2)]^{n+1}}, \tag{A.12}$$

for an arbitrary holomorphic function $u=f(U)$ such that $f(0)=0 \implies F(0)=0$. We work with the l.h.s. of (A.12) and also resolve the limit in a rigorous way as,[6]

$$\lim_{\epsilon\to 0}\left\{\frac{1}{m!n!}\partial_{U_1}^m\partial_{U_2}^n\left[\frac{f'(U_1)\,f'(U_2)}{(f(U_1)-f(U_2))^2}\right.\right.$$
$$\left.\left.-\frac{1}{(U_1-U_2)^2}\right]\right\}\Bigg|_{U_1=\epsilon,\ U_2=0}$$
$$= \lim_{\epsilon\to 0}\left\{\oint_{U_1=\epsilon}\frac{dU_1}{2\pi i}\oint_{U_2=0}\frac{dU_2}{2\pi i}\left[\frac{f'(U_1)\,f'(U_2)}{(f(U_1)-f(U_2))^2}\right.\right.$$
$$\left.\left.-\frac{1}{(U_1-U_2)^2}\right]\frac{1}{(U_1-\epsilon)^{m+1}U_2^{n+1}}\right\}$$
$$= \lim_{\epsilon\to 0}\left\{\oint_{z_1=f(\epsilon)}\frac{dz_1}{2\pi i}\oint_{z_2=0,|z_1|>|z_2|}\frac{dz_2}{2\pi i}\right.$$
$$\frac{1}{(z_1-z_2)^2}\frac{1}{[F(z_1)-\epsilon]^{m+1}[F(z_2)]^{n+1}} - \oint_{U_1=\epsilon}\frac{dU_1}{2\pi i}$$

[6] It is understood that the limit $\epsilon\to 0$ is taken after the substitution $u_2=0$, $u_1=\epsilon$.

$$\oint_{U_2=0} \frac{dU_2}{2\pi i} \frac{1}{(U_1-U_2)^2} \frac{1}{(U_1-\epsilon)^{m+1} U_2^{n+1}} \Bigg\}, \qquad (A.13)$$

where $z_i = f(U_i)$.

Although the individual ϵ contributions from the two terms in braces diverge, they exactly cancel each other in the combination in which they appear in (A.13). Thus, we can *naively* take the limit inside and get the self-interaction,

$$= \oint_{z_1=0} \frac{dz_1}{2\pi i} \oint_{z_2=0,|z_1|>|z_2|} \frac{dz_2}{2\pi i} \frac{1}{(z_1-z_2)^2} \frac{1}{[F(z_1)]^{m+1}[F(z_2)]^{n+1}}. \qquad (A.14)$$

We note that in writing the last equation, we have used the fact that the integral

$$\oint_{U_1=0} \frac{dU_1}{2\pi i} \oint_{U_2=0} \frac{dU_2}{2\pi i} \frac{1}{(U_1-U_2)^2} \frac{1}{(U_1)^{m+1} U_2^{n+1}} \qquad (A.15)$$

is zero based on the simple observation that the integrand is holomorphic for $U_2 \in B_\epsilon(0)$ (or equivalently for $|U_2| < |U_1|$).

Since the previous discussion is based on a not rigorous argument we demonstrate the claim for the lowest order terms. Explicitly, we compute the following term arising from the regularized SDS vertex,

$$\lim_{U_i \to 0} \frac{-2}{\alpha'} \partial_{U_1} \partial_{U_2} \Delta_f(U_1, U_2) = \frac{c_1 c_3 - c_2^2}{c_1^2}, \qquad (A.16)$$

where, $f(U) = \sum_{n=1}^{\infty} c_n U^n$ is a general holomorphic function (the inverse of the analytic map (4.3) in our specific case). We compare this with the corresponding term obtained from the self-interaction integral to get

$$\oint_{z_1=0} \frac{dz_1}{2\pi i} \oint_{z_2=0,|z_1|>|z_2|} \frac{dz_2}{2\pi i} \frac{1}{(z_1-z_2)^2} \frac{1}{[F(z_1)][F(z_2)]}$$
$$= \sum_{k=1}^{\infty} k \oint_{z_1=0} \frac{dz_1}{2\pi i} \frac{1}{[F(z_1)] z_1^{k+1}} \oint_{z_2=0} \frac{dz_2}{2\pi i} \frac{z_2^{k-1}}{F(z_2)]}. \qquad (A.17)$$

We can now write the series expansion of the inverse starting from,

$$w = f(z) = c_1 z + c_2 z^2 + \ldots$$
$$\implies z = \frac{1}{c_1}\left(w - c_2 z^2 - c_3 z^3 - ..\right)$$
$$\implies z = F(w) = \frac{w}{c_1} - \left(\frac{c_2}{c_1^3} w^2 + 2wz^2 \frac{c_2^2}{c_1^3} + \mathcal{O}(w^4)\right)$$
$$- \left(\frac{c_3}{c_1^4} w^3 + \mathcal{O}(w^4)\right)$$
$$\implies z = \frac{w}{c_1} - \left(\frac{c_2}{c_1^3}\right) w^2 + \left(\frac{2c_2^2 - c_1 c_3}{c_1^5}\right) w^3 + \mathcal{O}(w^4), \qquad (A.18)$$

and inserting it into the previous series we obtain

$$\sum_{k=1}^{\infty} k \oint_{z_1=0} \frac{dz_1}{2\pi i} \frac{1}{[F(z_1)] z_1^{k+1}} \oint_{z_2=0} \frac{dz_2}{2\pi i} \frac{z_2^{k-1}}{[F(z_2)]}$$
$$= \oint_{z_1=0} \frac{dz_1}{2\pi i} \frac{1}{[F(z_1)] z_1^2} \frac{1}{c_1}$$
$$= \frac{1}{c_1} \left(\frac{c_1 c_3 - c_2^2}{c_1}\right), \qquad (A.19)$$

which matches exactly with (A.16).

B Details on the comparison with Mandelstam result for $N = 3$

We would now give the details of the comparison with Mandelstam's result for $N = 3$.

Mandelstam writes the $N = 3$ Reggeon with interaction at $\tau = 0$ $|V_3^{(M)}(0)\rangle$ (see [47] vol. 2 section 11.4.5 where the result is clearer stated than in the original paper) as the time translated of the corresponding Reggeon computed with the path integral $|V_3^{(M)}(\tau_0)\rangle$ which comes naturally (with his conventions) with the interaction at time τ_0 as

$$\begin{aligned}
|V_3^{(M)}(0)\rangle =& e^{\frac{1}{2}\tau_0 \sum_{r=1}^3 \hat{P}_r^-} |V_3^{(M)}(\tau_0)\rangle \\
=& e^{\frac{1}{2}\tau_0 \sum_{r=1}^3 \hat{P}_r^-} \\
& e^{\sum_{r=1}^3 N_m^{[r]} \alpha^i_{[r]-m} \mathcal{P}^i + \frac{1}{2}\sum_{[r],[s]=1}^3 N_m^{rs} \alpha^i_{[r]-m} \alpha^i_{[s]-n}} |0\rangle \\
& \delta^{D-2}(\sum_{r=1}^3 p^i_{[r]}) \\
=& e^{-\tau_0 \sum_{r=1}^3 \frac{1}{\alpha_r} + \frac{1}{2}\tau_0 \sum_{r=1}^3 \frac{p^i_{[r]} p^i_{[r]}}{\alpha_{[r]}}} \\
& e^{\sum_{r=1}^3 \bar{N}_m^r \alpha^i_{[r]-m} \mathcal{P}^i + \frac{1}{2}\sum_{r,s=1}^3 \bar{N}_m^{rs} \alpha^i_{[r]-m} \alpha^i_{[s]-n}} |0\rangle \\
& \delta^{D-2}\left(\sum_{r=1}^3 p^i_{[r]}\right),
\end{aligned} \qquad (B.1)$$

where $\hat{P}_r^-$ is the lightcone Hamiltonian. The vertexes are obtained in an Hamiltonian framework where there is not lightcone energy conservation unless the consider an infinite period of evolution, i.e. the S matrix. When this happens

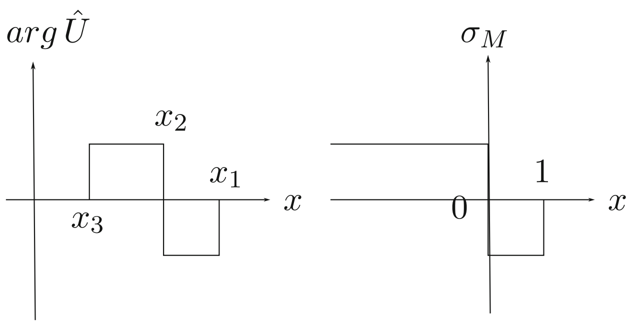

Fig. 3 The phases of $\hat{U}$ and the imaginary part of τ_0. In this way they can be easily compared where $\hat{U} = e^{i\phi_m} U^{\pi/(\phi_M-\phi_m)}$

we have $\sum_r \hat{P}^-_r = 0$ so the states are on-shell and the two vertices give the same S matrix.

In the previous expression we have introduced the following quantities

$$\begin{aligned}
\alpha_r =&2\sqrt{2\alpha'}\, {p^+}_r,\\
e^{\tau_0} =&|\alpha_1|^{\alpha_1}\, |\alpha_2|^{\alpha_2}\, |\alpha_3|^{\alpha_3},\\
\mathcal{P}^i =&\alpha_1\, p^i_{[2]} - \alpha_2\, p^i_{[1]},\\
N^r_m =&\frac{1}{\alpha_r}\, f_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right),\\
f_m(\gamma) =&\frac{1}{m!}\,(m\gamma-1)\ldots(m\gamma-m+1),\\
\bar{N}^r_m =&\left(e^{\tau_0}\right)^{\frac{m}{\alpha_r}}\; N^r_m.
\end{aligned} \tag{B.2}$$

The interaction time τ_0 is the real part of the following map

$$rho = \tau + i\sigma_M = \quad 2\sqrt{2\alpha'}\; {p^+}_{[1]}\,\log(u-1)\;\; \sqrt{2\alpha'}\; {p^+}_{[2]}\,\log(u), \tag{B.3}$$

when evaluated at stationary point $\left.\frac{d\rho}{du}\right|_{u_0} = 0$. The previous map from the upper half-plane to the strip means that the vertexes are at $x = 1, 0, \infty$. Notice moreover that we have written σ_M to signal that this variable has the usual meaning of worldsheet space coordinate but it has not the usual range as shown in Fig. 3.

If we consider the ranges we see that we have to compare $e^{\frac{1}{2}(\tau+i\sigma_M)}$ with $\hat{U} = (u-x_1)^{\sqrt{2\alpha'}{\underline{k}^+}_{[1]}}\,(u-x_2)^{\sqrt{2\alpha'}{\underline{k}^+}_{[2]}}\,(u-x_3)^{\sqrt{2\alpha'}{\underline{k}^+}_{[3]}}$ and send $x_3 \to \infty$ by analytic continuation

In order to compare with our expression we need

$$\begin{aligned}
\mathcal{P}^i =&2\sqrt{2\alpha'}{\underline{k}^+}_{[r]}\left(\alpha_{lc}{}^i_{[r+1]0} - \rho_{sr}\alpha_{lc}{}^i_{[r]0}\right),\\
f_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right) =&\frac{1}{m}\binom{-m\rho_{r-1\,r}-1}{m-1},
\end{aligned} \tag{B.4}$$

when we identify Mandelstam momenta with DDF momenta as $p^+ = \underline{k}^+$ and $\sqrt{2\alpha'}\,p^i_{[r]} = \alpha_{lc}{}^i_{[r]0}$.

Now we can exam the expression given in Eq. (5.3) considering the terms $\alpha_{lc}{}^i_{[r]n}\alpha_{lc}{}^i_{[s]0}$ and make use of the integrals computed in [28]

$$\begin{aligned}
&-\sum_{r,s=1}^{3}\sum_{i=2}^{D-1}\sum_{m=1}^{\infty}\frac{\alpha_{lc}{}^i_{[r]m}}{m}\,\alpha_{lc}{}^i_{[s]0}\oint_{z=x_s}\frac{dz}{2\pi i}\,\frac{Z_{[r]}(z)^{-m}}{z-x_s}\\
&\quad=-\sum_{i=2}^{D-1}\sum_{m=1}^{\infty}\sum_{r=1}^{3}\frac{\alpha_{lc}{}^i_{[r]m}}{n}\,I^i_{[r]\,m}\\
&\quad=-\sum_{i=2}^{D-1}\sum_{m=1}^{\infty}\sum_{r=1}^{3}\frac{\alpha_{lc}{}^i_{[r]n}}{m}\\
&\qquad\left(\alpha_{lc}{}^i_{[r+1]0} - \rho_{sr}\alpha_{lc}{}^i_{[r]0}\right)\,\hat{I}_{[r]\,m,\,r+1},
\end{aligned} \tag{B.5}$$

where the explicit evaluation gives

$$\begin{aligned}
&\hat{I}_{[r]\,m,\,[r+1]} =\\
&-\left(x_{12}^{\sqrt{2\alpha'}{\underline{k}^+}_{[3]}}\,x_{13}^{\sqrt{2\alpha'}{\underline{k}^+}_{[2]}}\,x_{23}^{\sqrt{2\alpha'}{\underline{k}^+}_{[1]}}\right)^{\frac{m}{\sqrt{2\alpha'}{\underline{k}^+}_{[r]}}}\\
&\binom{-m\rho_{r-1\,r}-1}{m-1}.
\end{aligned} \tag{B.6}$$

We can now compare with Mandelstam. If we compare with $\bar{N}^r_m$ we get

$$e^{\frac{1}{2}\tau_0} = x_{12}^{2\sqrt{2\alpha'}{k^+}_{[3]}}\,x_{13}^{2\sqrt{2\alpha'}{k^+}_{[2]}}\,x_{23}^{2\sqrt{2\alpha'}{k^+}_{[1]}}, \tag{B.7}$$

if we compare with N^r_m we get

$$1 = x_{12}^{2\sqrt{2\alpha'}{k^+}_{[3]}}\,x_{13}^{2\sqrt{2\alpha'}{k^+}_{[2]}}\,x_{23}^{2\sqrt{2\alpha'}{k^+}_{[1]}}. \tag{B.8}$$

This seems inconsistent but on-shell for $N = 3$ and considering the ghost contribution all x dependence cancels. This simply means that on-shell and for $N = 3$ we can effectively drop the x dependence in the Reggeon. This is also the same statement we wrote on the equivalence of $|V_3^{(M)}(\tau_0)\rangle$ and $|V_3^{(M)}(0)\rangle$ on-shell.

In doing so the two results matches perfectly.

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