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# Spin–gravity coupling and gravity-induced quantum phases

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**Abstract** External gravitational fields induce phase factors in the wave functions of particles. The phases are exact to first order in the background gravitational field, are manifestly covariant and gauge invariant and provide a useful tool for the study of spin–gravity coupling and of the optics of particles in gravitational or inertial fields. We discuss the role that spin–gravity coupling plays in particular problems.

**Keywords** Spin–gravity interactions, Covariant wave equations, Quantum phases, Mashhoon effect, Gravitational effects

## 1 Introduction

The study of the interaction of spin with inertia and gravity has received a strong impulse from the work of Bahram Mashhoon [1; 2; 3; 4]. His work has stimulated the research on which we report below.

Covariant wave equations for scalar and vector bosons, for spin-1/2 fermions [5; 6; 7; 8; 9] and spin-2 [10] particles can be solved exactly to first order in the metric deviation. The background gravitational and inertial fields appear in the solutions as phase factors multiplying the wave function of the corresponding field-free equations. The phases can be calculated with ease for most metrics.

We summarize the solutions for vector, tensor bosons and fermions in Sects. 2, 3 and 4. In the same sections we also extract the spin–gravity interaction [11] from the gravity-induced phases. The optics of the particles is derived in Sect. 5. In Sects. 6, 7 and 8 we discuss the relevance of the Mashhoon coupling to muon

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$g-2$  experiments, discrete symmetries and neutrino helicity transitions. The conclusions are contained in Sect. 9.

## 2 Solution of the spin-1 wave equation

Photons in gravitational fields are described by the Maxwell–de Rahm equations [12]

$$\nabla_\alpha \nabla^\alpha A_\mu - R_{\mu\alpha} A^\alpha = 0, \quad (2.1)$$

which reduce to Maxwell equations

$$\nabla_\alpha \nabla^\alpha A_\mu = 0 \quad (2.2)$$

when  $R_{\mu\alpha} = 0$ , or when, as in lensing, the wavelength  $\lambda$  of  $A^\alpha$  is much smaller than the typical radius of curvature of the gravitational background.

In (2.1) and (2.2)  $\nabla_\alpha$  indicates covariant differentiation. We use units  $\hbar = c = 1$ .

In what follows we consider only (2.2) and its generalization to massive, charge-less, spin-1 particles

$$\nabla_\alpha \nabla^\alpha A_\mu + m^2 A_\mu = 0, \quad (2.3)$$

in the hope that among the atoms and molecules to be used in interferometry some indeed satisfy (2.3) [13; 14; 15; 16].

Following previous work [5; 6; 7], we show below that Eqs. (2.2) and (2.3) can be solved exactly to first order in the metric deviation  $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ , where  $|\gamma_{\mu\nu}| \ll 1$  and  $\eta_{\mu\nu}$  is the Minkowski metric of signature  $-2$ .

To first order in  $\gamma_{\mu\nu}$ , (2.2) and (2.3) become

$$\nabla_\nu \nabla^\nu A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha}) A_{\mu,\alpha\sigma} - (\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma}) A^{\sigma,\nu} - \frac{1}{2} \gamma_{\sigma\mu,\nu}^\nu A^\sigma = 0, \quad (2.4)$$

$$(\nabla_\nu \nabla^\nu + m^2) A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha}) A_{\mu,\alpha\sigma} - (\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma}) A^{\sigma,\nu} - \frac{1}{2} \gamma_{\sigma\mu,\nu}^\nu A^\sigma + m^2 A_\mu = 0, \quad (2.5)$$

where ordinary differentiation of a quantity  $\Phi$  is equivalently indicated by  $\Phi_{,\alpha}$  or  $\partial_\alpha \Phi$ . In deriving (2.4) and (2.5), we have used the Lanczos–De Donder gauge condition

$$\gamma_{\alpha\nu}^\nu - \frac{1}{2} \gamma_{\sigma,\alpha}^\sigma = 0. \quad (2.6)$$

In the massless case, the field  $A_\mu(x)$  satisfies the condition

$$\nabla_\mu A^\mu = 0. \quad (2.7)$$

It is convenient to impose (2.7) also in the case of a massive particle. Equations (2.2) and (2.3) can be handled simultaneously. Their solution is

$$\begin{aligned}
A_\mu(x) &\simeq e^{-i\xi} a_\mu(x) \approx (1 - i\xi) a_\mu(x) \\
&= a_\mu(x) - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\
&\quad \times \left[ (x^\alpha - z^\alpha) \partial^\beta a_\mu(x) - (x^\beta - z^\beta) \partial^\alpha a_\mu(x) \right] \\
&\quad + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha a_\mu(x) \\
&\quad - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\mu\lambda,\sigma}(z) - \gamma_{\sigma\mu,\lambda}(z) - \gamma_{\sigma\lambda,\mu}(z)) a^\sigma(x), \quad (2.8)
\end{aligned}$$

where  $a_\mu$  satisfies the equation  $\partial_\nu \partial^\nu a_\mu = 0$  in the case of (2.4), and  $(\partial_\nu \partial^\nu + m^2) a_\mu = 0$  when (2.8) is a solution of (2.5). In (2.8)  $P$  is a fixed reference point and  $x$  a generic point along the particle's worldline. We can prove that (2.8) is an exact solution to first order in  $\gamma_{\mu\nu}$  by straightforward differentiation.

The first two integrals in (2.8) represent by themselves a solution of the Klein-Gordon equation  $(\nabla_\mu \nabla^\mu + m^2) \phi = 0$ . The additional terms are related to spin. In fact (2.8) can be re-written in the form

$$\begin{aligned}
A_\mu &= a_\mu - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta} - \gamma_{\beta\lambda,\alpha}) (x^\alpha - z^\alpha) \partial^\beta a_\mu + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda} \partial^\alpha a_\mu \\
&\quad - \frac{i}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta} - \gamma_{\beta\lambda,\alpha}) S^{\alpha\beta} a_\mu + \frac{i}{2} \int_P^x dz^\lambda \gamma_{\alpha\beta,\lambda} T^{\alpha\beta} a_\mu(x), \quad (2.9)
\end{aligned}$$

where

$$(S^{\alpha\beta})_{\mu\nu} = -\frac{i}{2} (\delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha), \quad (T^{\alpha\beta})_{\mu\nu} = -\frac{i}{2} (\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha) \quad (2.10)$$

The rotation matrices  $S_i = 2i\varepsilon_{ijk} S^{jk}$  satisfy the commutation relation  $[S_i, S_j] = i\varepsilon_{ijk} S_k$ .

By applying Stokes theorem to the r.h.s. of (2.8) we find

$$A_\mu = \left( 1 - \frac{i}{4} \oint d\tau^{\sigma\delta} R_{\sigma\delta\alpha\beta} J^{\alpha\beta} \right) a_\mu, \quad (2.11)$$

where  $J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$  is the total angular momentum of the spin-1 particle,  $R_{\mu\nu\alpha\beta} = 1/2(\gamma_{\mu\beta,\nu\alpha} + \gamma_{\nu\alpha,\mu\beta} - \gamma_{\mu\alpha,\nu\beta} - \gamma_{\nu\beta,\mu\alpha})$  is the linearized Riemann tensor, and  $\tau$  is the surface bound by the closed path along which the integration is performed.

The weak field approximation  $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$  does not fix the reference frame completely. The transformations of coordinates  $x_\mu \rightarrow x_\mu + \xi_\mu$ , with  $\xi_\mu(x)$

also small of first order, are still allowed and lead to the ‘‘gauge’’ transformations  $\gamma_{\mu\nu} \rightarrow \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ . Equation (2.11) therefore indicates that solution (2.8) is covariant and also gauge invariant. It also follows from (2.11) that the term containing  $T^{\alpha\beta}$  in (2.9) does not contribute to integrations over closed paths, behaves as a gauge term and may therefore be dropped.

The spin–gravity coupling is contained in the third term on the r.h.s. of (2.9). Its time integral part is  $\xi_{sr} = -\frac{1}{4} \int_P^x dz^0 (\gamma_{\alpha 0, \beta}(z) - \gamma_{\beta 0, \alpha}(z)) S^{\alpha\beta}$ . Since for rotation  $\gamma_{0i} = (-\Omega y, \Omega x, 0)$ , one gets  $\xi_{sr} = -\frac{1}{2} \int dz^0 \gamma_{i0, j}(z) S^{ij} = \int dt \Omega S_z$ , where  $S_z = S^{12}$ . In general, one may write  $\int dt \boldsymbol{\Omega} \cdot \boldsymbol{S}$ , which must now be applied to a solution of the field free equations. One finds  $E'_\pm = E + \boldsymbol{\Omega} \cdot \boldsymbol{S}$  and, for particles polarized parallel or antiparallel to  $\boldsymbol{\Omega}$ ,  $E'_\pm = E \pm \Omega$ , as in [1].  $E'$  is the energy observed by the co-rotating observer.

### 3 Solution of the spin-2 wave equation

For spin-2 fields, the simplest equation of propagation is derived in [12] and is given by

$$\nabla_\alpha \nabla^\alpha \Phi_{\mu\nu} + 2R_{\alpha\mu\beta\nu} \Phi^{\alpha\beta} = 0. \quad (3.1)$$

In lensing, the second term in (3.1) may be neglected when the wavelength  $\lambda$  associated with  $\Phi_{\mu\nu}$  is smaller than the typical radius of curvature of the gravitational background [17].

We consider here massless or massive spin-2 particles described by the equation [10]

$$\nabla_\alpha \nabla^\alpha \Phi_{\mu\nu} + m^2 \Phi_{\mu\nu} = 0. \quad (3.2)$$

To first order in  $\gamma_{\mu\nu}$ , (3.2) can be written in the form

$$\begin{aligned} & \left( \eta^{\alpha\beta} - \gamma^{\alpha\beta} \right) \partial_\alpha \partial_\beta \Phi_{\mu\nu} + R_{\sigma\mu} \Phi_\nu^\sigma + R_{\sigma\nu} \Phi_\mu^\sigma - 2\Gamma_{\mu\alpha}^\sigma \partial^\alpha \Phi_{\nu\sigma} \\ & - 2\Gamma_{\nu\alpha}^\sigma \partial^\alpha \Phi_{\mu\sigma} + m^2 \Phi_{\mu\nu} = 0, \end{aligned} \quad (3.3)$$

where  $R_{\mu\beta} = -(1/2) \partial_\alpha \partial^\alpha \gamma_{\mu\beta}$  is the linearized Ricci tensor of the background metric and  $\Gamma_{\sigma\mu, \alpha} = 1/2 (\gamma_{\alpha\sigma, \mu} + \gamma_{\alpha\mu, \sigma} - \gamma_{\sigma\mu, \alpha})$  is the corresponding Christoffel symbol of the first kind.

It is easy to prove, by direct substitution, that a solution of (3.3), exact to first order in  $\gamma_{\mu\nu}$ , is represented by

$$\begin{aligned}
\Phi_{\mu\nu} &= \phi_{\mu\nu} - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\
&\quad \times \left[ (x^\alpha - z^\alpha) \partial^\beta \phi_{\mu\nu}(x) - (x^\beta - z^\beta) \partial^\alpha \phi_{\mu\nu}(x) \right] \\
&\quad + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha \phi_{\mu\nu}(x) + \int_P^x dz^\lambda \Gamma_{\mu\lambda,\sigma}(z) \phi_\nu^\sigma(x) \\
&\quad + \int_P^x dz^\lambda \Gamma_{\nu\lambda,\sigma}(z) \phi_\mu^\sigma(x), \tag{3.4}
\end{aligned}$$

where  $\phi_{\mu\nu}$  satisfies the field-free equation

$$(\partial_\alpha \partial^\alpha + m^2) \phi_{\mu\nu}(x) = 0, \tag{3.5}$$

and the gauge condition (2.6) has been used.

Equation (3.4) can be written in the form

$$\begin{aligned}
\Phi_{\mu\nu}(x) &= \phi_{\mu\nu}(x) + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha \phi_{\mu\nu}(x) \\
&\quad - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \left[ (x^\alpha - z^\alpha) \partial^\beta \right] \phi_{\mu\nu}(x) \\
&\quad - \frac{i}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) S^{\alpha\beta} \phi_{\mu\nu}(x) - \frac{i}{2} \int_P^x dz^\lambda \gamma_{\beta\sigma,\lambda}(z) T^{\beta\sigma} \phi_{\mu\nu}(x), \tag{3.6}
\end{aligned}$$

where

$$\begin{aligned}
S^{\alpha\beta} \phi_{\mu\nu} &\equiv \frac{i}{2} \left( \delta_\sigma^\alpha \delta_\mu^\beta \delta_\nu^\tau - \delta_\sigma^\beta \delta_\mu^\alpha \delta_\nu^\tau + \delta_\sigma^\alpha \delta_\nu^\beta \delta_\mu^\tau - \delta_\sigma^\beta \delta_\nu^\alpha \delta_\mu^\tau \right) \phi_\tau^\sigma \\
T^{\beta\sigma} \phi_{\mu\nu} &\equiv i \left( \delta_\mu^\beta \delta_\nu^\tau + \delta_\nu^\beta \delta_\mu^\tau \right) \phi_\tau^\sigma. \tag{3.7}
\end{aligned}$$

From  $S^{\alpha\beta}$  one constructs the rotation matrices  $S_i = -2i\varepsilon_{ijk} S^{jk}$  that satisfy the commutation relations  $[S_i, S_j] = i\varepsilon_{ijk} S_k$ . The spin-gravity interaction is therefore contained in the term

$$\begin{aligned}
\Phi'_{\mu\nu} &\equiv -\frac{i}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta} - \gamma_{\beta\lambda,\alpha}) S^{\alpha\beta} \phi_{\mu\nu}(x) \\
&= \frac{1}{2} \int_P^x dz^\lambda \left[ (\gamma_{\sigma\lambda,\mu} - \gamma_{\mu\lambda,\sigma}) \phi_\nu^\sigma + (\gamma_{\sigma\lambda,\nu} - \gamma_{\nu\lambda,\sigma}) \phi_\mu^\sigma \right]. \tag{3.8}
\end{aligned}$$

The solution (3.4) is invariant under the gauge transformations  $\gamma_{\mu\nu} \rightarrow \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ . If, in fact, we choose a closed integration path  $\Gamma$ , Stokes theorem transforms the first three integrals of (3.6) into the gauge invariant result

$$\Phi_{\mu\nu} = \left( 1 - \frac{i}{4} \int_{\Sigma} d\sigma^{\lambda\kappa} R_{\lambda\kappa\alpha\beta} J^{\alpha\beta} \right) \phi_{\mu\nu}, \quad (3.9)$$

where  $\Sigma$  is the surface bound by  $\Gamma$ , and  $J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$  is the total angular momentum of the particle. For the same path  $\Gamma$  the integral involving  $T^{\beta\sigma}$  in (3.6) vanishes. It behaves like a gauge term and is therefore dropped.

The helicity–rotation coupling for massless, or massive spin-2 particles follows immediately from the  $S^{\alpha\beta}$  term in (3.6). In fact, the particle energy is changed by virtue of its spin by an amount given by the time integral of this spin term

$$\xi^{hr} = -\frac{1}{2} \int_P^x dz^0 (\gamma_{\alpha 0, \beta} - \gamma_{\beta 0, \alpha}) S^{\alpha\beta}, \quad (3.10)$$

that must then be applied to a solution of (3.5). For rotation about the  $x^3$ -axis,  $\gamma_{0i} = \Omega(y, -x, 0)$ , we find  $\xi^{hr} = -\int_P^x dz^0 2\Omega S^3$  and the energy of the particle therefore changes by  $\pm 2\Omega$ , where the factor  $\pm 2$  refers to the particle's helicity, as discussed by Ramos and Mashhoon [18]. Equation (3.10) extends their result to any weak gravitational, or inertial field.

The effect of (3.8) on  $\phi_{\mu\nu}$  can be easily seen in the case of a gravitational wave propagating in the  $x$ -direction and represented by the components  $\phi_{22} = -\phi_{33} = \varepsilon_{22} \exp[ik(t-x)]$  and  $\phi_{23} = \varepsilon_{23} \exp[ik(t-x)]$ . For an observer rotating about the  $x$ -axis the metric is  $\gamma_{00} = -\Omega^2 r^2$ ,  $\gamma_{11} = \gamma_{22} = \gamma_{33} = -1$ ,  $\gamma_{0i} = \Omega(0, z, -y)$ . Then the two independent polarizations  $\phi_{23}$  and  $\phi_{22} - \phi_{33}$  are transformed by  $S_{\alpha\beta}$  into  $\Phi_{23} = -2\Omega(x^0 - x_P^0) (\phi_{22} - \phi_{33})/2$  and  $1/2 (\Phi_{22} - \Phi_{33}) = 2\Omega(x^0 - x_P^0) \phi_{23}$ .

For closed integration paths and vanishing spin, (3.6) coincides with the solution of a scalar particle in a gravitational field, as expected. This proves the frequently quoted statement [19] that gravitational radiation propagating in a gravitational background is affected by gravitation in the same way that electromagnetic radiation is (when the photon spin is neglected).

#### 4 The covariant Dirac equation

The behavior of spin-1/2 particles in the presence of a gravitational field  $g_{\mu\nu}$  is determined by the covariant Dirac equation

$$[i\gamma^\mu(x)\mathcal{D}_\mu - m]\Psi(x) = 0, \quad (4.1)$$

where  $\mathcal{D}_\mu = \nabla_\mu + i\Gamma_\mu(x)$ ,  $\Gamma_\mu(x)$  is the spin connection and the matrices  $\gamma^\mu(x)$  satisfy the relations  $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}$ . Both  $\Gamma_\mu(x)$  and  $\gamma^\mu(x)$  can be obtained

from the usual constant Dirac matrices by using the vierbein fields  $e_{\hat{\alpha}}^{\mu}$  and the relations

$$\gamma^{\mu}(x) = e_{\hat{\alpha}}^{\mu}(x)\gamma^{\hat{\alpha}}, \quad \Gamma_{\mu}(x) = -\frac{1}{4}\sigma^{\hat{\alpha}\hat{\beta}}e_{\hat{\alpha}}^{\nu}e_{\nu\hat{\beta};\mu}, \quad (4.2)$$

where  $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$ .

Equation (4.1) can be solved exactly to first order in  $\gamma_{\mu\nu}(x)$ . This is achieved by first transforming (4.1) into the equation [6; 7; 8; 9; 20]

$$[i\tilde{\gamma}^{\nu}(x)\nabla_{\nu} - m]\tilde{\Psi}(x) = 0, \quad (4.3)$$

where

$$\begin{aligned} \tilde{\Psi}(x) &= S^{-1}\Psi(x), \quad S(x) = e^{-i\Phi_s(x)}, \quad \Phi_s(x) = \mathcal{D} \int_P^x dz^{\lambda} \Gamma_{\lambda}(z), \\ \tilde{\gamma}^{\mu}(x) &= S^{-1}\gamma^{\mu}(x)S. \end{aligned} \quad (4.4)$$

By multiplying (4.3) on the left by  $(-i\tilde{\gamma}^{\nu}(x)\nabla_{\nu} - m)$ , we obtain the equation

$$(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + m^2)\tilde{\Psi}(x) = 0, \quad (4.5)$$

whose solution

$$\tilde{\Psi}(x) = e^{-i\Phi_G(x)}\Psi_0(x), \quad (4.6)$$

is exact to first order. The operator  $\hat{\Phi}_G(x)$  is defined as

$$\begin{aligned} \hat{\Phi}_G &= -\frac{1}{4} \int_P^x dz^{\lambda} [\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)] \hat{L}^{\alpha\beta}(z) + \frac{1}{2} \int_P^x dz^{\lambda} \gamma_{\alpha\lambda} \hat{k}^{\alpha}, \quad (4.7) \\ [\hat{L}^{\alpha\beta}(z), \Psi_0(x)] &= ((x^{\alpha} - z^{\alpha})\hat{k}^{\beta} - (x^{\beta} - z^{\beta})\hat{k}^{\alpha}) \Psi_0(x), \quad [\hat{k}^{\alpha}, \Psi_0(x)] = i\partial^{\alpha}\Psi_0, \end{aligned}$$

and  $\Psi_0(x)$  satisfies the usual flat space-time Dirac equation.  $\hat{L}_{\alpha\beta}$  and  $\hat{k}^{\alpha}$  are the angular and linear momentum operators of the particle. It follows from (4.6) and (4.4) that the solution of (4.1) can be written in the form

$$\Psi(x) = e^{-i\Phi_s} (-i\tilde{\gamma}^{\mu}(x)\nabla_{\mu} - m) e^{-i\Phi_G} \Psi_0(x), \quad (4.8)$$

and also as

$$\Psi(x) = -\frac{1}{2m} (-i\gamma^{\mu}(x)\mathcal{D}_{\mu} - m) e^{-i\Phi_T} \Psi_0(x), \quad (4.9)$$

where  $\Phi_T = \Phi_s + \Phi_G$  is of first order in  $\gamma_{\alpha\beta}(x)$ . The factor  $-1/2m$  on the r.h.s. of (4.9) is required by the condition that both sides of the equation agree when the gravitational field vanishes.

It is useful to re-derive some known results from the covariant Dirac equation. On multiplying (4.1) on the left by  $(-i\gamma^{\nu}(x)\mathcal{D}_{\nu} - m)$  and using the relations

$$\nabla_{\mu}\Gamma_{\nu}(x) - \nabla_{\nu}\Gamma_{\mu}(x) + i[\Gamma_{\mu}(x), \Gamma_{\nu}(x)] = -\frac{1}{4}\sigma^{\alpha\beta}(x)R_{\alpha\beta\mu\nu}, \quad (4.10)$$

and

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\frac{i}{4} \sigma^{\alpha\beta}(x) R_{\alpha\beta\mu\nu}, \quad (4.11)$$

we obtain the equation

$$\left( g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu - \frac{R}{4} + m^2 \right) \Psi(x) = 0. \quad (4.12)$$

In (4.11) and (4.12)  $\sigma^{\alpha\beta}(x) = (i/2)[\gamma^\alpha(x), \gamma^\beta(x)]$  and  $R$  is the Ricci scalar.

On applying Stokes theorem to a closed space–time path and using (4.10), we find that  $\Phi_T$  changes by

$$\Delta \Phi_T = -\frac{i}{4} \oint d\tau^{\mu\nu} J^{\alpha\beta} R_{\mu\nu\alpha\beta}, \quad (4.13)$$

where  $J^{\alpha\beta}$  is the total momentum of the particle. Equation (4.13) shows that (4.9) is gauge invariant.

The spin–rotation coupling derived by Mashhoon by extending the hypothesis of locality can be now derived rigorously from the solution found.

Choose cylindrical coordinates  $(t, r, \theta, z)$  for an inertial frame  $F_0$ . An observer at rest in a frame  $F'$  rotating with a constant angular velocity  $\Omega$  relative to  $F_0$  will follow the world line  $(r = \text{const.}, \theta = \text{const.} + \Omega t, z = \text{const.})$ . Consider an orthogonal tetrad consisting of the observer's four-velocity  $\lambda_{(0)}^\mu = dx^\mu/ds$  and the triad  $\lambda_{(i)}^\mu$  ( $i = 1, 2, 3$ ) normal to the world line. By using the local tetrad [6; 7]  $\lambda_{(0)}^\mu = (\gamma, 0, \frac{\gamma\Omega}{c}, 0)$ ,  $\lambda_{(1)}^\mu = (0, 1, 0, 0)$ ,  $\lambda_{(2)}^\mu = (\frac{\gamma\Omega r}{c}, 0, \frac{\gamma}{r}, 0)$ ,  $\lambda_{(3)}^\mu = (0, 0, 0, 1)$ , where  $\gamma \equiv (1 - r^2\Omega^2/c^2)^{-1/2}$ , one can construct a vierbein field  $h^\mu{}_a(x)$  along the world line of the observer  $h^\mu{}_{(0)} = (\gamma, 0, \frac{\gamma\Omega}{c}, 0)$ ,  $h^\mu{}_{(1)} = (-\frac{\gamma\Omega r}{c} \sin \gamma\Omega t, \cos \gamma\Omega t, \frac{\gamma}{r} \sin \gamma\Omega t, 0)$ ,  $h^\mu{}_{(2)} = (\frac{\gamma\Omega r}{c} \cos \gamma\Omega t, \sin \gamma\Omega t, \frac{\gamma}{r} \cos \gamma\Omega t, 0)$ ,  $h^\mu{}_{(3)} = (0, 0, 0, 1)$ . It is then easy to calculate the spinor connection  $\Gamma_\mu$ . In calculating the energy, only the component  $\Gamma_0$  is necessary. By using the Dirac representation for the  $\gamma$ -matrices, one obtains  $\Gamma_0 = \frac{\gamma\Omega}{2c} \sigma_z$  and from  $\Phi_s$  also  $\exp(-i \int \Gamma_0 dz^0) \Psi_0 = \exp(-\frac{1}{2} \int \gamma\Omega \sigma_z dt) \Psi_0$ , where  $\Psi_0$  has the usual plane wave form. Besides the contribution due to the coupling of the orbital angular momentum to rotation, which gives the Sagnac effect [5], one obtains the spin–rotation coupling  $E' = E + \frac{1}{2} \Omega \sigma_z$ , and also, for spin polarizations parallel or antiparallel to the direction of rotation, one

obtains,  $E' = E \pm \frac{1}{2} \Omega$ , as shown by Mashhoon. The present result is exact and follows from the general form of the solution (4.9). It also agrees with those of [8] and of Hehl and Ni [21].

The Mashhoon effect is obviously a prime candidate for experiments with accelerators and will be discussed at length below.

According to (4.13), both angular momentum and spin couple to a weak gravitational field in the same way. This confirms that, unlike the electromagnetic case, the gyro-gravitational ratio of a spin-1/2 particle is 1, as shown in [22; 23; 24]. A classical charge  $e$  moving in a circle with angular momentum  $\mathcal{L}$  forms a current

loop of magnetic moment  $\vec{M} = -\frac{e\vec{L}}{2m}$ , which gives the gyromagnetic factor  $g = 1$ . The magnetic moment of a charged particle depends therefore on the ratio  $e/m$  and, for a rotating object, on the space distributions of charge and mass. For a quantum particle, the Dirac equation indicates that  $g = 2$ . The corrections to  $g = 2$  come from quantum electrodynamics where the electron can be pictured at any instant as a bare particle in interaction with a cloud of virtual photons. Qualitatively, if the charge remains associated with the electron, part of the mass energy is carried by the photon cloud resulting in a slight increase for the value  $e/m$  of the electron itself.

In the gravitational case, however, the gyro-gravitational ratio of the spin-1/2 particle is  $g = 1$ . This suggests, according to [22], that the internal distributions of the gravitational mass, associated with the interaction, and of the inertial mass, associated with the angular momentum, equal each other.

## 5 Optics

### 5.1 Lensing

In the geometrical optics approximation, valid whenever  $|\partial_i \gamma_{\mu\nu}| \ll |k \gamma_{\mu\nu}|$ , where  $k$  is the momentum of the particle, the interaction between the angular momentum of the source and the particle's spin vanishes. This interaction is quantum mechanical in origin. Then the geometrical phase  $\Phi_G$  is sufficient to reproduce the classical angle of deflection [25], as it should, because (4.7) coincides with the first two integrals in (2.8) and (3.6). We can therefore treat photons, gravitons and fermions simultaneously when spin is neglected.

More detailed calculations involving neutrinos are given in [9].

If, e.g., we choose a gravitational background represented by the Lense–Thirring metric [26],  $\gamma_{00} = 2\phi$ ,  $\gamma_{ij} = 2\phi \delta_{ij}$ ,  $\phi = -GM/r$ , and  $\gamma_{0i} \equiv h_i = 2GJ_{ij}x^j/r^3$ , with  $x^j = (x, y, z)$ ,  $r = \sqrt{x^2 + y^2 + z^2}$  and  $J_{ij}$  is related to the angular momentum of the gravitational source. In particular, if the source rotates with angular velocity  $\omega = (0, 0, \omega)$ , then  $h_1 = 4GMR^2 \omega y/5r^3$ ,  $h_2 = -4GMR^2 \omega x/5r^3$ .

Without loss of generality, we assume that the particles are massless and propagate along the  $z$ -direction, hence  $k^\alpha \simeq (k, 0, 0, k)$ . Using plane waves for the field free solution, the phase of the wave equation becomes

$$\begin{aligned}\chi &= k_\alpha x^\alpha - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) [(x^\alpha - z^\alpha)k^\beta - (x^\beta - z^\beta)k^\alpha] \\ &\quad + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) k^\alpha.\end{aligned}\quad (5.1)$$

We can define the particle momentum as

$$\tilde{k}_\alpha = \frac{\partial \chi}{\partial x^\alpha}.\quad (5.2)$$

It is easy to show that  $\chi$  satisfies the eikonal equation  $g^{\alpha\beta} \chi_{,\alpha} \chi_{,\beta} = 0$ .

For the Lense–Thirring metric,  $\chi$  is given by

$$\begin{aligned}\chi &\simeq -\frac{k}{2} \int_P^Q [(x-x')\phi_{,z'} dx' + (y-y')\phi_{,z'} dy' - 2[(x-x')\phi_{,x'} + (y-y')\phi_{,y'}] dz'] \\ &\quad + k \int_P^Q dz' \phi\end{aligned}\quad (5.3)$$

$$\begin{aligned}& -\frac{k}{2} \int_P^Q [((x-x')(h_{1,z'} - 2h_{3,x'}) + (y-y')(h_{2,z'} - 2h_{3,y'})) dz' \\ &\quad - ((x-x')h_{1,x'} + (y-y')h_{1,y'}) dx' - ((x-x')h_{2,x'} + (y-y')h_{2,y'}) dy'] \\ &\quad + \frac{k}{2} \int_P^Q [2h_3 dz' + h_1 dx' + h_2 dy'],\end{aligned}\quad (5.4)$$

where  $P$  is the point at which the particles are generated, and  $Q$  is a generic point along their space–time trajectory. The components of the momentum are therefore

$$\tilde{k}_1 = 2k \int_P^Q \left( -\frac{1}{2} \frac{\partial \phi}{\partial z} dx - \frac{1}{2} \frac{\partial h_2}{\partial x} dy + \frac{\partial(\phi+h_3)}{\partial x} dz \right) - \frac{k}{2} (h_1(Q) - h_1(P)),\quad (5.5)$$

$$\tilde{k}_2 = 2k \int_P^Q \left( -\frac{1}{2} \frac{\partial \phi}{\partial z} dy + \frac{1}{2} \frac{\partial h_1}{\partial y} dx + \frac{\partial(\phi+h_3)}{\partial y} dz \right) + \frac{k}{2} (h_2(Q) - h_2(P)),\quad (5.6)$$

$$\tilde{k}_3 = k(1 + \phi + h_3).\quad (5.7)$$

We then have

$$\tilde{\mathbf{k}} = \tilde{\mathbf{k}}_\perp + \tilde{k}_3 \mathbf{e}_3, \quad \tilde{\mathbf{k}}_\perp = \tilde{k}_1 \mathbf{e}_1 + \tilde{k}_2 \mathbf{e}_2,\quad (5.8)$$

where  $\tilde{\mathbf{k}}_\perp$  is the component of the momentum orthogonal to the direction of propagation of the particles.

**Fig. 1** Geometry of a two-image gravitational lens. The *solid lines* represent the particle paths between the particle source at  $S$  and the observer at  $O$ .  $M$  is the spherically symmetric gravitational lens.  $S, M, O$  and the particle paths lie in the same plane

Since only phase differences are physical, it is convenient to choose the space-time path by placing the particle source at distances that are very large relative to the dimensions of the lens, and the generic point is located along the  $z$ -direction. We therefore replace  $Q$  with  $z$ , where  $z \gg x, y$ . Using the expression for  $h_{1,2}$  we find that their contribution is negligible and (5.5)–(5.7) simplify to

$$\tilde{k}_1 = 2k \int_{-\infty}^z \frac{\partial(\phi + h_3)}{\partial x} dz, \quad (5.9)$$

$$\tilde{k}_2 = 2k \int_{-\infty}^z \frac{\partial(\phi + h_3)}{\partial y} dz, \quad (5.10)$$

$$\tilde{k}_3 = k(1 + \phi + h_3). \quad (5.11)$$

From (5.9) to (5.11) we can determine the deflection angle  $\theta$ . Let us analyze the case of non-rotating lenses, i.e.,  $h_3 = 0$ . We get

$$\tilde{k}_1 \sim k \frac{2GM}{R^2} x \left(1 + \frac{z}{r}\right), \quad (5.12)$$

$$\tilde{k}_2 \sim k \frac{2GM}{R^2} y \left(1 + \frac{z}{r}\right), \quad (5.13)$$

$$\tilde{k}_3 = k(1 + \phi + h_3), \quad (5.14)$$

where  $R = \sqrt{x^2 + y^2}$ . By defining the deflection angle as

$$\tan \theta = \frac{\tilde{k}_\perp}{\tilde{k}_3}, \quad (5.15)$$

it follows that

$$\tan \theta \sim \theta \sim \frac{2GM}{R} \left(1 + \frac{z}{r}\right). \quad (5.16)$$

In the limit  $z \rightarrow \infty$  we obtain the usual Einstein result

$$\theta_M \sim \frac{4GM}{R}. \quad (5.17)$$

A general expression for the index of refraction  $n$  can also be derived from (5.1), (5.2) and  $n = \tilde{k}/\tilde{k}_0$ .

## 5.2 Wave effects in gravitational lensing

We now consider the propagation of light and gravitational waves in a background metric represented by  $\gamma_{00} = 2U(r)$ ,  $\gamma_{ij} = 2U(r)\delta_{ij}$ , where  $U(r) = -GM/r$  is the gravitational potential of the lens and  $r$  the distance from  $M$  to the particle. Wave optics effects can be seen by using the type of double slit arrangement indicated in Fig. 1. We will also use a solution  $\partial^\gamma \partial_\gamma a_\alpha = 0$  in the form of a plane wave  $a_\mu = a_\mu^0 e^{-ik_\sigma x^\sigma}$  and neglect spin effects. This limits the calculation [10] of the phase difference to the first two terms in (7.11) and (7.12). We also assume for simplicity that  $k^1 = 0$ , so that propagation is entirely in the  $(x^2, x^3)$ -plane and the set-up is planar.

The corresponding wave amplitude  $\phi$  is therefore

$$\begin{aligned} \phi(x) = \frac{e^{-ik_\sigma x^\sigma}}{r} & \left\{ 1 - \frac{1}{2} \left[ \int_S^x dz^0 \gamma_{0,2}(x^0 - z^0) \Pi^2 + \int_S^x dz^0 \gamma_{0,3}(x^0 - z^0) \Pi^3 \right. \right. \\ & - \int_S^x dz^0 \gamma_{0,2}(x^2 - z^2) \Pi^0 - \int_S^x dz^0 \gamma_{0,3}(x^3 - z^3) \Pi^0 + \int_S^x dz^2 \gamma_{2,3}(x^2 - z^2) \Pi^3 \\ & + \int_S^x dz^3 \gamma_{3,2}(x^3 - z^3) \Pi^2 - \int_S^x dz^2 \gamma_{2,3}(x^3 - z^3) \Pi^2 \\ & \left. \left. - \int_S^x dz^3 \gamma_{3,2}(x^2 - z^2) \Pi^3 \right] \right. \\ & \left. + \frac{1}{2} \left[ \int_S^x dz^0 \gamma_{00} \Pi^0 + \int_S^x dz^2 \gamma_{22} \Pi^2 + \int_S^x dz^3 \gamma_{33} \Pi^3 \right] \right\}, \end{aligned} \quad (5.18)$$

where  $\Pi^0 = -ik$ ,  $\Pi^i = -ik^i$ , and we have taken into account the fact that  $\gamma_{11}$  plays no role in the planar arrangement chosen. The phase must now be calculated along the different paths SP + PO and SL + LO taking into account the values of  $\Pi^i$  in the various intervals.

The total change in phase is [10]

$$\Delta \tilde{\phi} = \Delta \tilde{\phi}_{SL} + \Delta \tilde{\phi}_{LO} - \Delta \tilde{\phi}_{SP} - \Delta \tilde{\phi}_{PO}.$$

All integrations in (5.18) can be performed exactly and the results can be expressed in terms of physical variables  $r_s$ ,  $r_0$ ,  $b^+$ ,  $b^-$ , and  $s$  or lensing variables  $D_s$ ,

$D_{ds}, D_d, \theta^+, \theta^-$ , and  $\beta$ . We find

$$\begin{aligned} \Delta\tilde{\phi} = \tilde{y} \left\{ \ln \left( -\sqrt{D_{ds}^2 + (s+b^-)^2} + b^- \cos \gamma + r_S \right) - \ln \left( b^- (1 + \cos \gamma) \right) \right. \\ + \ln \left( b^+ (1 - \cos \varphi^+) \right) - \ln \left( r_S - r_L - b^+ \cos \varphi^+ \right) \\ + \ln \left( b^- + r_0 \cos \theta^- - \sqrt{b^{-2} + r_0^2} \right) - \ln \left( r_0 (1 + \cos \theta^-) \right) \\ \left. + \ln \left( r_0 (1 + \cos \theta^+) \right) - \ln \left( b^+ + r_0 \cos \theta^+ - \sqrt{b^{+2} + r_0^2} \right) \right\}, \quad (5.19) \end{aligned}$$

where  $r_S^2 = b^{+2} + r_L^2 + 2b^+ r_L \cos \varphi^+$ ,  $r_L^2 = D_{ds}^2 + (s - b^+)^2$ ,  $\varphi^+ + \alpha^+ + \alpha^- + \gamma - \theta^+ - \theta^- = \pi$  and  $\tilde{y} = 2GMk$ .

As an example, let us consider the simpler case  $b^+ = b^- \equiv b$ ,  $\theta^+ = \theta^- \equiv \theta$ ,  $s = 0$ ,  $r_L = \sqrt{b^2 + r_0^2}$ , from which we obtain (see Fig. 1)  $r_S = r_0$ ,  $\cos(\pi - \varphi^+) = \cos \gamma = b/\sqrt{r_0^2 + b^2} \tan \theta = b/r_0$ .

A simple calculation shows that the probability density of finding a photon, or graviton, at  $O$  is typically

$$\phi\phi^* \propto \cos^2 \frac{\Delta\tilde{\phi}}{2}. \quad (5.20)$$

## 6 Muon $g - 2$ experiments

Measurements of the interaction of spin with rotation have been carried out in the case of photons using signals from global positioning system satellites [27] and data published in [28] can be re-interpreted [29] as due to the coupling of Earth's rotation to the nuclear spins in mercury. The spin-rotation effect is also consistent with a small depolarization of electrons in storage rings [30]. We show below that the same coupling is of particular interest in experiments with storage rings. It is essential in getting the correct  $g - 2$  dependence in  $g - 2$  experiments [31] and eliminates the need of ad hoc, phenomenological arguments.

The effect is conceptually important. It extends to the quantum level the classical coupling of rotation to the intrinsic angular momentum of a body and simplifies the treatment of rotational inertia.

It also yields different potentials for different particles and for different spin states and can not, therefore, be considered universal [32].

Before discussing its connection with  $g - 2$  experiments, it is useful to briefly recall the usual experimental setup.

The experiment [33; 34] involves muons in a storage ring a few meters in diameter, in a uniform vertical magnetic field. Muons on equilibrium orbits within a small fraction of the maximum momentum are almost completely polarized with spin vectors pointing in the direction of motion. As the muons decay, those electrons projected forward in the muon rest frame are detected around the ring. Their angular distribution thence reflects the precession of the muon spin along the cyclotron orbits.

The calculations are performed in the rotating frame of the muon and do not therefore require a relativistic treatment of inertial spin effects [35]. Then the vierbein formalism yields  $\Gamma_i = 0$  and

$$\Gamma_0 = -\frac{i}{2} a_i \sigma^{0i} - \frac{1}{2} \omega_i \sigma^i, \quad (6.1)$$

where  $a_i$  and  $\omega_i$  are the three-acceleration and three-rotation of the observer and, in the chiral representation of the usual Dirac matrices,

$$\sigma^{0i} \equiv \frac{i}{2} [\gamma^0, \gamma^i] = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}.$$

The second term in (6.1) represents the Mashhoon term. The first term drops out. In fact, by symmetrization we obtain

$$(\vec{a} \cdot \vec{\mathcal{K}}) (\vec{\mathcal{E}} \cdot \vec{\mathcal{P}}) = \frac{1}{2} [(\vec{a} \cdot \vec{\mathcal{K}}) (\vec{\mathcal{E}} \cdot \vec{\mathcal{P}}) + (\vec{\mathcal{E}} \cdot \vec{\mathcal{P}}) (\vec{a} \cdot \vec{\mathcal{K}})] + \frac{i}{2} (\vec{a} \cdot \vec{\mathcal{E}}). \quad (6.2)$$

The last term in (6.2) and the first term in (6.1) therefore cancel each other. The remaining first order contributions in  $a_i$  and  $\omega_i$  to the Dirac Hamiltonian add up to [8; 21]

$$H \approx \vec{\mathcal{E}} \cdot \vec{\mathcal{P}} + m\beta + \frac{1}{2} [(\vec{a} \cdot \vec{\mathcal{K}}) (\vec{\mathcal{P}} \cdot \vec{\mathcal{E}}) + (\vec{\mathcal{P}} \cdot \vec{\mathcal{E}}) (\vec{a} \cdot \vec{\mathcal{K}})] - \vec{\mathcal{E}} \cdot \left( \vec{\mathcal{L}} + \frac{\vec{\mathcal{G}}}{2} \right). \quad (6.3)$$

For simplicity all quantities in  $H$  are taken to be time-independent. They are referred to a left-handed tern of axes rotating about the  $x_2$ -axis in the clockwise direction of motion of the muons. The  $x_3$ -axis is tangent to the orbits and in the direction of motion of the muons. The magnetic field is  $B_2 = -B$ . Only the Mashhoon term and the magnetic moment interaction then couple the helicity states of the muon. The remaining terms contribute to the overall energy  $E$  of the states, and we indicate by  $H_0$  the corresponding part of the Hamiltonian.

Before decay the muon states can be represented as

$$|\psi(t)\rangle = a(t)|\psi_+\rangle + b(t)|\psi_-\rangle, \quad (6.4)$$

where  $|\psi_+\rangle$  and  $|\psi_-\rangle$  are the right and left helicity states of the Hamiltonian  $H_0$  and satisfy the equation

$$H_0 |\psi_{+,-}\rangle = E |\psi_{+,-}\rangle.$$

The total Hamiltonian reduces effectively to  $H_{\text{eff}} = H_0 + H'$ , where

$$H' = -\frac{1}{2} \omega_2 \sigma^2 + \mu B \sigma^2. \quad (6.5)$$

$\mu = \left(1 + \frac{g-2}{2}\right) \mu_0$  represents the total magnetic moment of the muon and  $\mu_0$  is the Bohr magneton. Electric fields used to stabilize the orbits and stray radial electric fields can also affect the muon spin. Their effects can however be cancelled by

choosing an appropriate muon momentum [34] and will not be considered in what follows.

The coefficients  $a(t)$  and  $b(t)$  in (6.4) evolve in time according to

$$i \frac{\partial}{\partial t} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad (6.6)$$

where  $M$  is the matrix

$$M = \begin{bmatrix} E - i\frac{\Gamma}{2} & i(\frac{\omega_2}{2} - \mu B) \\ -i(\frac{\omega_2}{2} - \mu B) & E - i\frac{\Gamma}{2} \end{bmatrix} \quad (6.7)$$

and  $\Gamma$  represents the width of the muon. The non-diagonal form of  $M$  (when  $B = 0$ ) implies that rotation does not couple universally to matter.

$M$  has eigenvalues

$$h_1 = E - i\frac{\Gamma}{2} + \frac{\omega_2}{2} - \mu B,$$

$$h_2 = E - i\frac{\Gamma}{2} - \frac{\omega_2}{2} + \mu B,$$

and eigenstates

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} [i|\psi_+\rangle + |\psi_-\rangle],$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} [-i|\psi_+\rangle + |\psi_-\rangle].$$

The muon states that satisfy (6.6), and the condition  $|\psi(0)\rangle = |\psi_-\rangle$  at  $t = 0$ , are

$$|\psi(t)\rangle = \frac{e^{-\Gamma t/2}}{2} e^{-iEt} \left\{ i [e^{-i\tilde{\omega}t} - e^{i\tilde{\omega}t}] |\psi_+\rangle + [e^{-i\tilde{\omega}t} + e^{i\tilde{\omega}t}] |\psi_-\rangle \right\}, \quad (6.8)$$

where

$$\tilde{\omega} \equiv \frac{\omega_2}{2} - \mu B.$$

The spin-flip probability therefore is

$$P_{\psi_-\rightarrow\psi_+} = |\langle \psi_+ | \psi(t) \rangle|^2$$

$$= \frac{e^{-\Gamma t}}{2} [1 - \cos(2\mu B - \omega_2)t]. \quad (6.9)$$

The  $\Gamma$ -term in (6.9) accounts for the observed exponential decrease in electron counts due to the loss of muons by radioactive decay [34].

The spin-rotation contribution to  $P_{\psi_- \rightarrow \psi_+}$  is represented by  $\omega_2$  which is the cyclotron angular velocity  $\frac{eB}{m}$  [34]. The spin-flip angular frequency is then

$$\begin{aligned}\Omega &= 2\mu B - \omega_2 \\ &= \left(1 + \frac{g-2}{2}\right) \frac{eB}{m} - \frac{eB}{m} \\ &= \frac{g-2}{2} \frac{eB}{m},\end{aligned}\tag{6.10}$$

which is precisely the observed modulation frequency of the electron counts [36]. This result is independent of the value of the anomalous magnetic moment of the particle. The cancellation of the Dirac value of the magnetic moment contribution by the Mashhoon term must therefore take place for all spin-1/2 particles in a similar physical set-up. Hence, it is the spin-rotation coupling that generates the correct  $g-2$  factor in  $\Omega$  by exactly cancelling, in  $2\mu B$ , the much larger contribution  $\mu_0$  that fermions with no anomalous magnetic moment produce. The cancellation is made possible by the non-diagonal form of  $M$  and is therefore a direct consequence of the violation of the equivalence principle. It has, of course, been argued that this principle does not hold true in the quantum world [37].

## 7 Constraints on the C and P symmetries

Recently, discrepancies between the experimental and standard model values of  $a_\mu$  have been observed with very high accuracy [38]. The most precise data yet give  $b = a_\mu(\text{exp}) - a_\mu(\text{SM}) = 26 \times 10^{-10}$  for the negative muon [39; 40], and  $d = a_\mu(\text{exp}) - a_\mu(\text{SM}) = 33 \times 10^{-10}$  for the positive muon [41; 42]. This discrepancy can be used to set upper limits on  $P$  and  $T$  invariance violations in spin-rotation coupling [43].

The possibility that discrete symmetries in gravitation be not conserved has been discussed in the literature. Attention has in general focused on the potential

$$U(\mathcal{E}) = \frac{GM}{r} [\alpha_1 \mathcal{E} \cdot \hat{r} + \alpha_2 \mathcal{E} \cdot \mathcal{E} + \alpha_3 \hat{r} \cdot (\mathcal{E} \times \mathcal{E})],\tag{7.1}$$

which applies to a particle of generic spin  $\mathcal{E}$ . The first term, introduced by Leitner and Okubo [44], violates the conservation of  $P$  and  $T$ . The same authors determined the upper limit  $\alpha_1 \leq 10^{-11}$  from the hyperfine splitting of the ground state of hydrogen. The upper limit  $\alpha_2 \leq 10^{-3}$  was determined in [45] from SN 1987A data. The corresponding potential violates the conservation of  $P$  and  $C$ . Conservation of  $C$  and  $T$  is violated by the last term, while (7.1), as a whole, conserves  $CPT$ . There is, as yet, no upper limit on  $\alpha_3$ . These studies are extended here to the Mashhoon term.

Before decay, the muon states can be represented as in (6.4) where  $|\psi_+\rangle$  and  $|\psi_-\rangle$  again are the right and left helicity states of the Hamiltonian  $H_0$  defined in the previous section.

Assume now that the coupling of rotation to  $|\psi_+\rangle$  differs in strength from that to  $|\psi_-\rangle$ . Then the Mashhoon term can be modified by means of a matrix

$A = \begin{pmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{pmatrix}$  that reflects the different coupling of rotation to the two helicity states. The total Hamiltonian now is  $H_{\text{eff}} = H_0 + H'$ , where

$$H' = -\frac{1}{2}A\omega_2\sigma_2 + \mu B\sigma_2. \quad (7.2)$$

A violation of  $P$  and  $T$  in (7.2) would arise through  $\kappa_+ - \kappa_- \neq 0$ . The constants  $\kappa_+$  and  $\kappa_-$  are assumed to differ from unity by small amounts  $\varepsilon_+$  and  $\varepsilon_-$ .

The coefficients  $a(t)$  and  $b(t)$  in (6.4) evolve in time according to

$$i\frac{\partial}{\partial t} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad (7.3)$$

where

$$M = \begin{pmatrix} E - i\frac{\Gamma}{2} & i(\kappa_+ \frac{\omega_2}{2} - \mu B) \\ -i(\kappa_- \frac{\omega_2}{2} - \mu B) & E - i\frac{\Gamma}{2} \end{pmatrix}, \quad (7.4)$$

and  $\Gamma$  represents, as before, the width of the muon. The spin-rotation term is off-diagonal in (7.4) and does not therefore couple to matter universally. It violates Hermiticity as shown in [46] and, in a general way, by Sclarici and Solombrino [47]. It also violates  $T$ ,  $P$  and  $PT$ , while nothing can be said about  $CPT$  conservation which requires  $H_{\text{eff}}$  to be Hermitian. Because of the non-Hermitian nature of (7.2), one expects  $\Gamma$  itself to be non-Hermitian. The resulting corrections to the width of the muon are, however, of second order in the  $\varepsilon$ 's and are neglected.  $M$  has eigenvalues

$$\begin{aligned} h_1 &= E - i\frac{\Gamma}{2} + R \\ h_2 &= E - i\frac{\Gamma}{2} - R, \end{aligned} \quad (7.5)$$

where

$$R = \sqrt{\left(\kappa_+ \frac{\omega_2}{2} - \mu B\right) \left(\kappa_- \frac{\omega_2}{2} - \mu B\right)}, \quad (7.6)$$

and eigenstates

$$\begin{aligned} |\psi_1\rangle &= b_1 [\eta_1 |\psi_+\rangle + |\psi_-\rangle], \\ |\psi_2\rangle &= b_2 [\eta_2 |\psi_+\rangle + |\psi_-\rangle]. \end{aligned} \quad (7.7)$$

One also finds

$$\begin{aligned} |b_1|^2 &= \frac{1}{1 + |\eta_1|^2} \\ |b_2|^2 &= \frac{1}{1 + |\eta_2|^2} \end{aligned} \quad (7.8)$$

and

$$\eta_1 = -\eta_2 = \frac{i}{R} \left( \kappa_+ \frac{\omega_2}{2} - \mu B \right). \quad (7.9)$$

Then the muon states (6.4) are

$$|\psi(t)\rangle = \frac{1}{2} e^{-iEt - \frac{\Gamma t}{2}} [-2i\eta_1 \sin Rt |\psi_+\rangle + 2 \cos Rt |\psi_-\rangle], \quad (7.10)$$

where the condition  $|\psi(0)\rangle = |\psi_-\rangle$  has been applied. The spin-flip probability is therefore

$$\begin{aligned} P_{\psi_-\rightarrow\psi_+} &= |\langle \psi_+ | \psi(t) \rangle|^2 \\ &= \frac{e^{-\Gamma t}}{2(1 + \kappa_- \omega_2 - 2\mu B)} [1 - \cos(2Rt)]. \end{aligned} \quad (7.11)$$

When  $\kappa_+ = \kappa_- = 1$ , (7.11) yields [43]

$$P_{\psi_-\rightarrow\psi_+} = \frac{e^{-\Gamma t}}{2} \left[ 1 - \cos \left( a_\mu \frac{eB}{m} t \right) \right], \quad (7.12)$$

that provides the appropriate description of the spin-rotation contribution to the spin-flip transition probability. Notice that the case  $\kappa_+ = \kappa_- = 0$  (no spin-rotation coupling) yields

$$P_{\psi_-\rightarrow\psi_+} = \frac{e^{-\Gamma t}}{2} \left[ 1 - \cos \left( 1 + a_\mu \right) \frac{eB}{m} t \right] \quad (7.13)$$

and does not therefore agree with the results of the  $g - 2$  experiments. Hence the necessity of accounting for spin-rotation coupling whose contribution cancels the factor  $\frac{eB}{m}$  in (7.13).

Substituting  $\kappa_+ = 1 + \varepsilon_+$ ,  $\kappa_- = 1 + \varepsilon_-$  into (7.11), one finds

$$P_{\psi_-\rightarrow\psi_+} = \frac{e^{-\Gamma t}}{2} \frac{2(\varepsilon_+ - a_\mu)}{\varepsilon_+ + \varepsilon_- - 2a_\mu} \left[ 1 - \cos \left( t \frac{eB}{m} \sqrt{(\varepsilon_+ - a_\mu)(\varepsilon_- - a_\mu)} \right) \right]. \quad (7.14)$$

One may attribute the discrepancy between  $a_\mu(\text{exp})$  and  $a_\mu(\text{SM})$  to a violation of the conservation of the discrete symmetries by the spin-rotation coupling term in (6.6). The upper limit on the violation of  $P, T$  and  $PT$  is derived from (7.14) assuming that the deviation from the current value of  $a_\mu(\text{SM})$  is wholly due to  $\varepsilon_\pm$ . The upper limit is therefore  $26 \times 10^{-10}$  in the case of negative muons [39; 40] and of  $33 \times 10^{-10}$  for positive muons [41; 42]. At the same time the two values of  $a_\mu(\text{exp}) - a_\mu(\text{SM})$  can be thought of as due to a different coupling strength between rotational inertia and the two helicity states of the muon. Then the values of  $\varepsilon_+$  and  $\varepsilon_-$  can be determined from  $\cos(2Rt)$  in (7.11) according to the equations

$$(a_{\mu_+} - \varepsilon_+) (a_{\mu_+} - \varepsilon_-) = b^2 \quad (7.15)$$

and

$$(a_{\mu_-} - \varepsilon_+) (a_{\mu_-} - \varepsilon_-) = d^2. \quad (7.16)$$

Equations (7.15) and (7.16) have the approximate solutions

$$\varepsilon_+ \simeq \frac{a_{\mu_+} + a_{\mu_-}}{2} - \frac{d^2 - b^2}{2(a_{\mu_-} - a_{\mu_+})} \quad (7.17)$$

and

$$\varepsilon_- \simeq a_{\mu_+} + \frac{2b^2(a_{\mu_-} - a_{\mu_+})}{(a_{\mu_-} - a_{\mu_+})^2 + (d^2 - b^2)}. \quad (7.18)$$

More precise, numerical solutions give  $\varepsilon_+ \simeq 11659189 \times 10^{-10}$ ,  $\varepsilon_- \simeq 11659152 \times 10^{-10}$  and  $\Delta\varepsilon \equiv \varepsilon_+ - \varepsilon_- \simeq 37.65878 \times 10^{-10}$ . These values are significant in view of the precision with which  $a_{\mu_{\pm}}, b, d$  have been determined. It then follows that the coupling of rotation to positive helicity is larger than that to negative helicity, which agrees with the value  $\xi < 1$  for both  $a_{\mu} = a_{\mu_+}$  and  $a_{\mu} = a_{\mu_-}$ . This also means that the violation of  $P$  and  $T$  is relatively stronger for positive helicity at a level  $\Delta\varepsilon \simeq 3.7 \times 10^{-9}$  and that the spin-rotation interaction is an inherent source of  $P$  and  $T$  violation.

## 8 Neutrino helicity transitions

In this section, it is convenient to write the left and right neutrino wave functions in the form

$$\Psi_0(x) = v_{0L,R} e^{-ik_{\alpha}x^{\alpha}} = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} v_{L,R} \\ \frac{\sigma \cdot \mathbf{k}}{E+m} v_{L,R} \end{pmatrix} e^{-ik_{\alpha}x^{\alpha}}, \quad (8.1)$$

where  $\sigma = (\sigma^1, \sigma^2, \sigma^3)$  represents the Pauli matrices,  $v_{L,R}$  are eigenvectors of  $\sigma \cdot \mathbf{k}$  corresponding to negative and positive helicity and  $\bar{v}_{0L,R}(k) \equiv v_{0L,R}^{\dagger}(k) \gamma^0$ ,  $v_{0L,R}^{\dagger}(k) v_{0L,R}(k) = 1$ . This notation already takes into account the fact that if  $v_{\pm}$  are the helicity states, then we have  $v_L \simeq v_-$ ,  $v_R \simeq v_+$  for relativistic neutrinos.

In general, the spin precesses during the motion of the neutrino. This can be seen, for instance, from the contribution  $\Phi_s$  in  $\Phi_T$ . The expectation value of the contribution of  $\Gamma_0$  to the effective mechanical momentum can in fact be re-written in the form

$$\frac{1}{2} \Psi_0^{\dagger} \mathcal{E} \cdot \mathcal{S} \Psi_0, \quad (8.2)$$

where  $\mathcal{E} \equiv \frac{GM R^2}{5r^3} \left(1 - \frac{3z^2}{r^2}\right) \mathcal{S}$ . Equation (8.2) represents the spin-rotation coupling for the Lense-Thirring metric. Here rotation is provided by the gravitational source, rather than by the particles themselves.

We now study the helicity flip of one flavor neutrinos as they propagate in the gravitational field produced by a rotating mass [9]. The neutrino state vector can be written as

$$|\psi(\lambda)\rangle = \alpha(\lambda)|\nu_R\rangle + \beta(\lambda)|\nu_L\rangle, \quad (8.3)$$

where  $|\alpha|^2 + |\beta|^2 = 1$  and  $\lambda$  is an affine parameter along the world-line. In order to determine  $\alpha$  and  $\beta$ , we can write (4.8) as

$$|\psi(\lambda)\rangle = \hat{T}(\lambda)|\psi_0(\lambda)\rangle, \quad (8.4)$$

where

$$\hat{T} = -\frac{1}{2m} (-i\gamma^\mu(x)\mathcal{D}_\mu - m) e^{-i\Phi_T}, \quad (8.5)$$

and  $|\psi_0(\lambda)\rangle$  is the corresponding solution in Minkowski space-time. The latter can be written as

$$|\psi_0(\lambda)\rangle = e^{-ik\cdot x} [\alpha(0)|\nu_R\rangle + \beta(0)|\nu_L\rangle]. \quad (8.6)$$

Strictly speaking,  $|\psi(\lambda)\rangle$  should also be normalized. However, it can be shown [9] that  $\alpha(\lambda)$  is already of  $\mathcal{O}(\gamma_{\mu\nu})$ , can only produce higher order terms and is therefore unnecessary in this calculation. From (8.3), (8.4) and (8.6) we obtain

$$\alpha(\lambda) = \langle \nu_R | \psi(\lambda) \rangle = \alpha(0) \langle \nu_R | \hat{T} | \nu_R \rangle + \beta(0) \langle \nu_R | \hat{T} | \nu_L \rangle. \quad (8.7)$$

An equation for  $\beta$  can be derived in an entirely similar way.

If we consider neutrinos which are created in the left-handed state, then  $|\alpha(0)|^2 = 0$ ,  $|\beta(0)|^2 = 1$ , and we obtain

$$P_{L \rightarrow R} = |\alpha(\lambda)|^2 = |\langle \nu_R | \hat{T} | \nu_L \rangle|^2 = \left| \int_{\lambda_0}^{\lambda} \langle \nu_R | \dot{x}^\mu \partial_\mu \hat{T} | \nu_L \rangle d\lambda \right|^2, \quad (8.8)$$

where  $\dot{x}^\mu = k^\mu/m$ . As remarked in [48],  $\dot{x}^\mu$  need not be a null vector if we assume that the neutrino moves along an ‘‘average’’ trajectory. We also find, to lowest order,

$$\begin{aligned} \partial_\mu \hat{T} &= \frac{1}{2m} \left( -i2m\Phi_{G,\mu} - i(\gamma^{\hat{\alpha}}k_\alpha + m)\Phi_{s,\mu} + \gamma^{\hat{\alpha}}(h_{\hat{\alpha},\mu}^\beta k_\beta + \Phi_{G,\alpha\mu}) \right) \\ \Phi_{s,\lambda} &= \Gamma_\lambda, \quad \Phi_{G,\alpha\mu} = k_\beta \Gamma_{\alpha\mu}^\beta, \quad \mathbf{v}_0^\dagger (\gamma^{\hat{\alpha}}k_\alpha + m) = 2E \mathbf{v}_0^\dagger \gamma^{\hat{0}}, \end{aligned} \quad (8.9)$$

where  $\Gamma_{\alpha\mu}^\beta$  are the usual Christoffel symbols, and

$$\langle \nu_R | \dot{x}^\mu \partial_\mu \hat{T} | \nu_L \rangle = \frac{E}{m} \left[ -i \frac{k^\lambda}{m} \bar{\nu}_R \Gamma_\lambda \nu_L + \frac{k^\lambda k_\mu}{2mE} (h_{\hat{\alpha},\lambda}^\mu + \Gamma_{\alpha\lambda}^\mu) \mathbf{v}_R^\dagger \gamma^{\hat{\alpha}} \nu_L \right]. \quad (8.10)$$

In what follows, we compute the probability amplitude (8.10) for neutrinos propagating along the  $z$  and the  $x$ -directions explicitly.

### 8.1 Propagation along $z$

For propagation along the  $z$ -axis, we have  $k^0 = E$  and  $k^3 \equiv k \simeq E(1 - m^2/2E^2)$ . As in Sect. 3, we choose  $y = 0$ ,  $x = b$ . We get

$$\begin{aligned} -i \frac{k^\lambda}{m} \bar{v}_R \Gamma_\lambda v_L &= \frac{k}{m} \phi_{,1} + i \frac{m}{4E} h_{2,3}, \\ \frac{k^\lambda k_\mu}{2mE} (h_{\hat{\alpha},\lambda}^\mu + \Gamma_{\alpha\lambda}^\mu) v_R^\dagger \gamma^{\hat{\alpha}} v_L &= -\frac{k}{2m} \left( 1 + \frac{k^2}{E^2} \right). \end{aligned} \quad (8.11)$$

Summing up, and neglecting terms of  $\mathcal{O}(m/E)^2$ , (8.10) becomes

$$\langle v_R | \dot{x}^\mu \partial_\mu \hat{T} | v_L \rangle = \frac{1}{2} \phi_{,1} + \frac{i}{4} h_{2,3}. \quad (8.12)$$

The contributions to  $\mathcal{O}((E/m)^2)$  vanish. As a consequence

$$\frac{d\alpha}{dz} \simeq \frac{m}{E} \frac{d\alpha}{d\lambda} = \frac{m}{E} \left( \frac{1}{2} \phi_{,1} + \frac{i}{4} h_{2,3} \right), \quad (8.13)$$

and the probability amplitude for the  $v_L \rightarrow v_R$  transition is of  $\mathcal{O}(m/E)$ , as expected.

Integrating (8.13) from  $-\infty$  to  $z$ , yields

$$\begin{aligned} \alpha &\simeq \frac{m}{E} \left[ \frac{1}{2} \int_{-\infty}^z dz \phi_{,1} + \frac{i}{4} h_2(z) \right] \\ &= \frac{m}{E} \frac{GM}{2b} \left[ 1 + \frac{z}{r} - i \frac{2\omega R^2 b^2}{5r^3} \right]. \end{aligned} \quad (8.14)$$

It also follows that

$$P_{L \rightarrow R}(-\infty, z) \simeq \left( \frac{m}{E} \right)^2 \left( \frac{GM}{2b} \right)^2 \left[ \left( 1 + \frac{z}{r} \right)^2 + \left( \frac{2\omega b^2 R^2}{5r^3} \right)^2 \right]. \quad (8.15)$$

The first of the two terms in (8.15) comes from the mass of the gravitational source. The second from the source's angular momentum and vanishes for  $r \rightarrow \infty$  because the contribution from  $-\infty$  to 0 exactly cancels that from 0 to  $+\infty$ . In fact, if we consider neutrinos propagating from 0 to  $+\infty$ , we obtain

$$P_{L \rightarrow R}(0, +\infty) \simeq \left( \frac{m}{E} \right)^2 \left( \frac{GM}{2b} \right)^2 \left[ 1 + \left( \frac{2\omega R^2}{5b} \right)^2 \right]. \quad (8.16)$$

According to semiclassical spin precession equations [49], there should be no spin motion when spin and  $\mathcal{G}$  are parallel as in the present case. This is a hint that rotation of the source, rather than of the particles, should produce a similar effect. The probabilities (8.15) and (8.16) mark therefore a departure from expected results. They are however small of second order. Both expressions vanish for  $m \rightarrow 0$ , as it should for a stationary metric. In this case, in fact, helicity is conserved [50]. It is

interesting to observe that spin precession also occurs when  $\omega$  vanishes [51; 52]. In the case of (8.15) the mass contribution is larger when  $b < (r/R)\sqrt{\frac{5r}{2\omega}}$ , which, close to the source, with  $b \sim r \sim R$ , becomes  $R\omega < 5/2$  and is always satisfied. In the case described by (8.16), the rotational contribution is larger if  $b/R < 2\omega R/5$  which effectively restricts the region of dominance to a narrow strip about the  $z$ -axis in the equatorial plane, if the source is compact and  $\omega$  is relatively large.

## 8.2 Propagation along $x$

In this case, we take  $k^0 = E$ ,  $k^1 \equiv k \simeq E(1 - m^2/2E^2)$ . As in Sect. 3, the calculation can be simplified by assuming that the motion is in the equatorial plane with  $z = 0$ ,  $y = b$ . We then have

$$\begin{aligned} -i\frac{k^\lambda}{m}\bar{\nu}_R\Gamma_\lambda\nu_L &= i\frac{k}{m}\phi_{,2} + i\frac{E^2+k^2}{4mE}h_{1,2} - i\frac{E^2-k^2}{4mE}h_{2,1}, \\ \frac{k^\lambda k_\mu}{2mE}(h_{\alpha,\lambda}^\mu + \Gamma_{\alpha\lambda}^\mu)\nu_R^\dagger\gamma^{\hat{\alpha}}\nu_L &= -i\frac{k}{2m}\left(1 + \frac{k^2}{E^2}\right)\phi_{,2} - i\frac{k^2}{2mE}h_{1,2}. \end{aligned} \quad (8.17)$$

Summing up, and neglecting terms of  $\mathcal{O}(m/E)^2$ , (8.10) becomes

$$\langle\nu_R|\hat{x}^\mu\partial_\mu\hat{T}|\nu_L\rangle = \frac{i}{2}\phi_{,2} + \frac{i}{4}(h_{1,2} - h_{2,1}). \quad (8.18)$$

The contributions to  $\mathcal{O}((E/m)^2)$  again vanish and we get

$$\frac{d\alpha}{dx} \simeq \frac{m}{E}\frac{d\alpha}{d\lambda} = \frac{m}{E}\left[\frac{i}{2}\phi_{,2} + \frac{i}{4}(h_{1,2} - h_{2,1})\right] \sim \mathcal{O}(m/E). \quad (8.19)$$

Integrating (8.19) from  $-\infty$  to  $x$ , we obtain

$$\alpha \simeq i\frac{m}{E}\frac{GM}{2b}\left(1 - \frac{2\omega R^2}{5b}\right)\left(1 + \frac{x}{r}\right) \quad (8.20)$$

and

$$P_{L\rightarrow R}(-\infty, x) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left(1 - \frac{2\omega R^2}{5b}\right)^2\left(1 + \frac{x}{r}\right)^2. \quad (8.21)$$

Obviously, the mass contribution is the same as for propagation along the  $z$ -axis. However, the two cases differ substantially in the behavior of the angular momentum term. In this case, in fact, this term is even, so it does not vanish for  $r \rightarrow \infty$ . If we consider neutrinos generated at  $x = 0$  and propagating to  $x = +\infty$ , we find

$$P_{L\rightarrow R}(0, +\infty) \simeq \left(\frac{m}{E}\right)^2\left(\frac{GM}{2b}\right)^2\left(1 - \frac{2\omega R^2}{5b}\right)^2. \quad (8.22)$$

The mass term is larger when  $\frac{2\omega R^2}{5b} < 1$ . At the poles  $b \sim R$  and the mass term dominates because the condition  $\omega R < 5/2$  is always satisfied. The angular momentum contribution prevails in proximity of the equatorial plane. The transition probability vanishes at  $b = 2\omega R^2/5$ .

## 9 Conclusions

Covariant wave equations for massless and massive particles can be solved exactly to first order in  $\gamma_{\mu\nu}$ . The solutions are covariant and invariant with respect to the gauge transformations of the electromagnetic field and of  $\gamma_{\mu\nu}$  and are known when a solution of the free wave equation is known. The external gravitational field only appears in the phase of the wave function.

We have shown that the coupling of spin to inertia and gravitation follow from the solutions given. This allows a unified treatment of the interaction of gravity with spin and angular momentum without requiring ad hoc procedures.

According to Eqs. (2.9) and (3.6), the spin term  $S_{\alpha\beta}$  finds its origin in the skew-symmetric part of the space-time connection. In the case of fermions,  $S_{\alpha\beta}$  is accounted for by the spinorial connection. The terms that contain  $S_{\alpha\beta}$  gives rise to the Skrotskii effect for both electromagnetic [53; 54] and gravitational waves [18].

From the phases we have derived the geometrical optics of the particles and verified that their deflection is that predicted by general relativity. In addition, the background gravitational field acts as a medium whose index of refraction can be calculated for any metric from (5.1), (5.2) and  $n = \tilde{k}/\tilde{k}_0$ .

Because spin does not enter the examples given, the same results can be equally applied to the gravitational lensing of gravitational waves [17].

A more detailed treatment of the geometrical optics of single flavor neutrinos can be found in [9] where we also calculate corrections due to the neutrino mass. For propagation parallel to the axis of rotation of the source, the rotation corrections vanish at infinity. Not so for propagation perpendicular to the axis of rotation.

We can finally conclude that the validity of covariant wave equations in an inertial-gravitational context finds support in experimental verifications of some of the effects they predict [55; 56; 57; 58], in tests of the general relativistic deflection of light rays and also in the phase wrap-up in global position system measurements [27].

We have then asked ourselves the question whether, beside the phase wrap-up in GPS, there is a wider role for spin-gravity coupling in physics. In particular, we have considered muon  $g - 2$  experiments and helicity transitions in neutrino physics. We have found that spin-rotation coupling is largely responsible for producing the correct  $g - 2$  factor in the spin-flip angular frequency  $\Omega$ . Measurements of this factor already provide the most stringent test yet of Einstein's time dilation formula [59; 60]. However, muons in storage rings are also rotating quantum gyroscopes and inertia must be an essential ingredient of their description in experiments of high and ever increasing sensitivity. Possibly these experiments also concern problems like the violation of the equivalence principle in quantum mechanics and the conservation of discrete symmetries in inertia-gravitation. We have, in fact, shown that a slightly anomalous inertial contribution to (7.14) at a level of the values  $b = 26 \times 10^{-10}$  and  $d = 33 \times 10^{-10}$  of Sect. 7 can produce violations of the discrete symmetries at the same level. The upper limits on the violations of  $P$  and  $T$  that can be reached by  $g - 2$  experiments are in fact as sensitive as those obtained by other means [44; 45; 61; 62; 63], but with an important

difference. The  $g - 2$  measurements are performed in strictly controlled laboratory conditions rather than in astrophysical situations.

In derivations based on the covariant Dirac equation, the coupling of inertia and gravitation to spin is identical to that for orbital angular momentum. A suggestive

interpretation of this result is that the internal distributions of the gravitational mass, associated with the interaction, and of inertial mass, associated with the angular

momentum, equal each other. This is no longer so when  $\varepsilon_{\pm} \neq 0$ . There is almost a similarity, here, with the electromagnetic case where  $g = 2$  is required by the Dirac equation, but not by quantum electrodynamics. The deviations of  $\kappa_+$  and  $\kappa_-$  from unity that are consistent with  $g - 2$  experiments are both of the order of  $a_{\mu}$ , or  $\simeq 10^{-3}$ , and differ from each other by  $\Delta\varepsilon \simeq 3.7 \times 10^{-9}$ . While small values of  $\varepsilon_{\pm}$  do not give rise to measurable mass differences in macroscopic objects [46], violations of the discrete symmetries can have interesting astrophysical and cosmological implications.

Next, we have calculated the helicity transition amplitudes of ultra-relativistic, single flavor neutrinos as they propagate in a Lense–Thirring field. These transitions are interesting because at high energies chirality states are predominantly helicity states and right-handed neutrinos do not interact [6; 7; 64]. The transition probabilities are of  $\mathcal{O}(\gamma_{\mu\nu}^2)$ . Two directions of propagation have again been selected and the results contain contributions from both mass and angular momentum of the source. The transitions also occur in the absence of rotation or with spin parallel to rotation, which is unexpected on semiclassical grounds. The mass contributions predominate when the neutrinos propagate from  $r = 0$  to  $r = \infty$  (and matter effects are neglected), provided the impact parameter  $b > 2\omega R^2/5$ . There is, however, a narrow region about the axis of propagation in the equatorial plane where the  $\omega$  contribution is larger. The rotational contribution behaves differently in the two cases. It vanishes as  $z \rightarrow \mp\infty$  for propagation along  $z$ , but not so as  $x \rightarrow \infty$  in the second case. In addition, when the neutrinos propagate from  $x = 0$  to  $x = \infty$ , the mass term dominates in the neighborhood of the poles, while the contribution of  $\omega$  is larger close to the equator, with no attenuation at  $b = 2\omega R^2/5$ .

In [9] we have also calculated gravity induced, two-flavor oscillations and derived the relative equation and effective Hamiltonian. The transition probabilities do indeed oscillate for the Lense–Thirring metric, and the curvature of space–time enters the oscillation probability through the gravitational red-shift of the local energy  $E_l$  and the proper distance  $dl$ .

The results presented in this paper can be applied to a number of problems in astroparticle physics and cosmology [65]. For instance, an interesting question is whether gravity induced helicity and flavor transitions could effect changes in the ratio  $\nu_e : \nu_{\mu} : \nu_{\tau}$  of the expected fluxes at Earth.

Lepton asymmetry in the Universe [65] also is an interesting problem. It is known that the active–sterile oscillation of neutrinos can generate a discrepancy in the neutrino and antineutrino number densities. The lepton number of a neutrino of flavor  $f$  is defined by  $L_f = (n_{\nu_f} - n_{\bar{\nu}_f})/n_{\gamma}(T)$ , where  $n_{\nu_f}(n_{\bar{\nu}_f})$  is the number density of neutrinos (antineutrinos) and  $n_{\gamma}(T)$  is the number density of photons at temperature  $T$ . As noted above, the gravitational field generates transitions from

left-handed (active) neutrinos to right-handed (sterile) neutrinos. If, in primordial conditions, (8.16) and (8.22) become larger, then helicity transitions may contribute in some measure to lepton asymmetry.

Finally, we have recently re-examined the behavior of the spin-gravity interaction and found that gravity can distinguish between chirality and helicity [50]. We have also found that the spin-gravity interaction can distinguish between Dirac and Majorana wave packets [66; 67]. A spin-flip does in fact change a Majorana neutrino into an antineutrino and behaves like a charge conjugation operation.

A few words of caution must now be added.

The spin-gravity couplings discussed in our work make use of the weak field approximation in which gravity enters as a non-dynamical field. There are, of course, physical situations in which this approach can be trusted and the general agreement between the quasi-classical and the quantum mechanical approaches has been established [9; 68]. One could then be tempted to extend our approximation procedure to any order in the metric deviation, as suggested by Eqs. (2.11), (3.9) and (4.13). This would however lead to inconsistencies that can only be removed, as shown by Deser [69], by making use of the full non-linear apparatus of general relativity.

Furthermore, it is assumed, in calculating the phases induced by gravity, that all possible particle paths reduce, in the average, to the phase integration paths. This approximation worsens the more “quantum mechanical” particles and gravity become.

An additional point concerns the use of the locality hypothesis in replacing non-inertial frames with inertial ones. As shown by Mashhoon [2; 3; 70], this hypothesis has limitations and important consequences for the measuring process. For standard accelerated measuring devices, for instance, it entails the introduction of a maximal acceleration [71; 72; 73; 74]. In the absence of inertial frames, of a complete formulation of spin-gravity couplings in curved space-time and of direct experimental observations, assigning a role to curvature in spin related problems may then prove difficult, though worthy of investigation.

A final question regards the validity of the equivalence principle in spin-gravity interactions. In  $g - 2$  experiments the interaction of spin with gravity depends on the relative direction of spin and rotation of the source. As such it is not universal and may be regarded as violating those formulations of the equivalence principle that hinge on universality. But a more fundamental violation has been introduced in Sect. 7 where the strength of the coupling itself depends on the helicity of the particle [43; 75]. This is a violation of the post-Newtonian equivalence principle, recently discussed by Silenko and Teryaev [76], by which a particle’s spin and angular momenta precession frequencies coincide. This principle must be the object of rigorous experimental verifications and the authors themselves suggest a number of tests to find even more precise upper limits than those determined by using muon  $g - 2$  experiments [43]. In a closely related paper [68], the same authors show that the spin-gravity dipole coupling term found by Obukhov [77] using the Eriksen-Korlsrud transformation [78] does not lead to observable effects.

Recently, impressive technical developments in the field of masers [79] have succeeded in placing an upper limit of  $10^{-27}$  GeV on violations of Lorentz and  $CPT$  symmetries. Higher sensitivities are expected to be reached in the near future.

Since the interaction energy for a spin-1/2 particle in the rotation field of Earth is of the same order of magnitude, these developments bode well for the physics of spin-gravity interactions.

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