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ANALYTIC AND NUMERICAL CALCULATION OF COLLIDER LUMINOSTY WITH CRAB DYNAMICS*

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Abstract

For an integral part of electron-ion collider (EIC) design, the crab crossing scheme provides a head-on collision for beams with a nonzero crossing angle. Recently we provided a framework for accurate numerical simulations of beam-beam effects with crabbing crossing dynamics. The framework was implemented in a simulation code package named "CASA BeamBeam". We offer comprehensive formulas for calculation of collider luminosity for various cases in the code package. The luminosity calculation module of CASA Beam-Beam now includes the hourglass effect, the beam-tilt effects and the beam offset effect. The benchmarking results show good agreement between the numerical calculation and analytic solution.

INTRODUCTION

The Electron-Ion Collider (EIC) design relies upon short bunches and high repetition rates to achieve the desired high luminosity. the crab crossing is an integral part of the design. It features a non-zero crossing angle (25 mrad for the primary detector) for the colliding beams. It can accommodate short bunch spacing of the colliding beams such that the two beams can be rapidly separated before and after interaction point (IP) to avoid parasitic collisions. Such crossing angle affects the luminosity performance since the collider luminosity formula has a maximum value at zero crossing angle, namely, head-on. Without compensation of the crossing angle at the IP, the EIC would result in an unacceptable large loss of luminosity. In the EIC design, a local crabbing scheme is used and thus each beam is crabbed (tilted by half of the crossing angle) before collision and then de-crabbed after collision such that the two beams collide head-on in the center of momentum frame. Because the crab crossing scheme restores a head-on collision for beams with a nonzero crossing angle, EIC can achieve high luminosity to meeting the physics program requirements.

Beam-beam effects are one of the dominant effects limiting the luminosity in colliders. The simulations of crabbing dynamics for the current symplectic tracking codes such as Elegant [1] do not include beam-beam effects [2-4]. Based on Elegant, CASA BeamBeam is developed at the Center for Advanced Studies of Accelerators (CASA, Jefferson Lab). For the crabbing dynamics of CASA BeamBeam, the numerical calculation model [5, 6] is based on the Bassetti-Erskine analytic solution [7] of the beam-beam interaction and it is extended to finite-length bunches using a symplectic algorithm proposed by Hirata [2]. Hence, the luminosity calculation in CASA BeamBeam involves the beam-beam effect. CASA BeamBeam framework is as follows: The particle distribution is initiated at the start point of the tracking simulation. The beam is tracked through the crab cavity and is transported to the IP. At the IP, the particle information will be collected, the beam-beam interaction will be calculated by CASA BeamBeam and then all updated particle information will be fed back into Elegant for continued tracking through the rest of the collider ring optics. In this process, CASA BeamBeam will provide both the analytic and numerical results for the luminosity calculation. In this paper, we will address the model for the analytic solution and will demonstrate benchmark work in the following two sections.

ANALYTIC SOLUTION

For the luminosity calculation, we will demonstrate the analytic expressions for the luminosity in case of two bunched beams, (x_1, y_1, s_1) and (x_2, y_2, s_2), in terms of the beam parameters and the geometry here.

Excepting $s_0 = ct$, we define the following parameters:

- φ_x is the angle between the projection of the beam trajectory on the (x, s) plane and the s-axis.
- φ_y is the angle between the projection of the beam trajectory on the (y, s) plane and the s-axis.

For two beams in the general case, the coordinate-translation yields:

$$\begin{cases} x_1 = x \cdot \cos\varphi_x - s \cdot \sin\varphi_x + \Delta x_1 \\ y_1 = y \cdot \cos\varphi_y - s \cdot \sin\varphi_y + \Delta y_1 \\ s_1 = s\sqrt{1+\tan^2\varphi_x + \tan^2\varphi_y} + s_0 \end{cases} \tag{1}$$

$$\begin{cases} x_2 = x \cdot \cos\varphi_x + s \cdot \sin\varphi_x + \Delta x_2 \\ y_2 = y \cdot \cos\varphi_y + s \cdot \sin\varphi_y + \Delta y_2 \\ s_2 = s\sqrt{1+\tan^2\varphi_x + \tan^2\varphi_y} + s_0 \end{cases} \tag{2}$$

For a storage ring collider with bunch spacing S_B, bunches collide periodically with frequency $f_c = \beta c / S_B$, and $s_0 = ct$, excluding the dynamical effects, the luminosity is defined as

$$L = P_0 \int \iiint_{-\infty}^{\infty} dxdydsds_0 \, \rho_{1x}\rho_{1y}\rho_{1s}\rho_{2x}\rho_{2y}\rho_{2s} \tag{3}$$

where ρ_1 and ρ_2 are the time dependent distribution functions of the two beams, $P_0 = N_1 N_2 N_b f_c \frac{K}{c}$, N_1 and N_2 are the bunch intensities, and N_b is the number of colliding bunches, K is the kinematic factor defined as

$$K = \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2 / c^2} \tag{4}$$

and

$$\begin{cases} |v_1| = v_{s1}\sqrt{1+tan^2\,\varphi_x + tan^2\,\varphi_y} \\ |v_2| = -v_{s2}\sqrt{1+tan^2\,\varphi_x + tan^2\,\varphi_y} \end{cases} \tag{5}$$

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Assuming the beams are colliding at $s_0 = 0$, let

$$\begin{cases}\delta x = \Delta x_2 - \Delta x_1 \\ \delta y = \Delta y_2 - \Delta y_1\end{cases}, \begin{cases}\sigma_x^2 = \frac{1}{2}(\sigma_{1x}^2 + \sigma_{2x}^2) \\ \sigma_y^2 = \frac{1}{2}(\sigma_{1y}^2 + \sigma_{2y}^2) \\ \sigma_s^2 = \frac{1}{2}(\sigma_{1s}^2 + \sigma_{2s}^2)\end{cases} \quad (6)$$

and

$$\begin{cases}a = \frac{\sin^2\varphi_x}{\sigma_x^2} + \frac{\sin^2\varphi_y}{\sigma_y^2} + \frac{1+\tan^2\varphi_x+\tan^2\varphi_y}{\sigma_s^2} \\ b = \frac{(\delta x)\sin\varphi_x}{\sigma_x^2} + \frac{(\delta y)\sin\varphi_y}{\sigma_y^2} \\ c = \frac{(\delta x)^2}{4\sigma_x^2} + \frac{(\delta y)^2}{4\sigma_y^2}\end{cases} \quad (7)$$

Considering $\sigma(s) = \sigma^*\sqrt{1+\left(\frac{s}{\beta^*}\right)^2}$, the hourglass effect, we obtain

$$\sigma_x = \sigma_x^*\sqrt{As^2+1},\ \sigma_y = \sigma_y^*\sqrt{Bs^2+1} \quad (8)$$

where

$$\begin{cases}A = \left(\frac{{\sigma_{1x}^*}^2}{{\beta_{1x}^*}^2} + \frac{{\sigma_{2x}^*}^2}{{\beta_{2x}^*}^2}\right)\frac{1}{{\sigma_{1x}^*}^2+{\sigma_{2x}^*}^2} \\ B = \left(\frac{{\sigma_{1y}^*}^2}{{\beta_{1y}^*}^2} + \frac{{\sigma_{2y}^*}^2}{{\beta_{2y}^*}^2}\right)\frac{1}{{\sigma_{1y}^*}^2+{\sigma_{2y}^*}^2}\end{cases} \quad (9)$$

$$\begin{cases}\sigma_x^* = \sqrt{\frac{1}{2}\left({\sigma_{1x}^*}^2 + {\sigma_{2x}^*}^2\right)} \\ \sigma_y^* = \sqrt{\frac{1}{2}\left({\sigma_{1y}^*}^2 + {\sigma_{2y}^*}^2\right)}\end{cases} \quad (10)$$

then we have

$$\begin{cases}a = \frac{\sin^2\varphi_x}{{\sigma_x^*}^2(As^2+1)} + \frac{\sin^2\varphi_y}{{\sigma_y^*}^2(Bs^2+1)} + \frac{1+\tan^2\varphi_x+\tan^2\varphi_y}{\sigma_s^2} \\ b = \frac{(\delta x)\sin\varphi_x}{{\sigma_x^*}^2(As^2+1)} + \frac{(\delta y)\sin\varphi_y}{{\sigma_y^*}^2(Bs^2+1)} \\ c = \frac{(\delta x)^2}{4{\sigma_x^*}^2(As^2+1)} + \frac{(\delta y)^2}{4{\sigma_y^*}^2(Bs^2+1)}\end{cases} \quad (11)$$

Introduce the nominal luminosity,

$$L_0 = \frac{N_1N_2N_bf_c}{2\pi\sqrt{{\sigma_{1x}^*}^2+{\sigma_{1x}^*}^2}\sqrt{{\sigma_{1y}^*}^2+{\sigma_{1y}^*}^2}} = \frac{N_1N_2N_bf_c}{4\pi\sigma_x^*\sigma_y^*} \quad (12)$$

excluding the dynamical effects, then the luminosity is

$$L = L_0C_0\frac{1}{\sqrt{\pi}\sigma_s}\int_{-\infty}^{\infty}\frac{1}{\sqrt{As^2+1}\sqrt{Bs^2+1}}e^{-(as^2+bs+c)}ds \quad (13)$$

where $C_0 = \frac{\sqrt{2\cos^2\varphi_x+2\cos^2\varphi_y-3\cos^2\varphi_x\cos^2\varphi_y}}{\cos^2\varphi_x+\cos^2\varphi_y-\cos^2\varphi_x\cos^2\varphi_y}$.

NUMERICAL CALCULATIONS

For a real collider ring, the misalignments and magnetic im-perfections may yield orbit distortions and then lead to a combination of crossing angle and beam offset effects [2-6]. In CASA BeamBeam, for the numerical process, two colliding bunched beams are cut into many slices whose normal direction is parallel to the longitudinal direction. We implemented CASA BeamBeam to obtain the numerical results for the luminosity and then benching mark with our comprehensive analytic solution in the different cases of two bunched beams, (x_1, y_1, s_1) and (x_2, y_2, s_2), in terms of the beam parameters and the geometry.

For benchmarking the analytic solution with the numerical results for the special design, the following parameters were used: symmetric-electron-collider; flat beam; electron beam energy 10.0 GeV; collision frequency 1.18×10^8 Hz; number of electrons for each beam is 3.7×10^{10}. The luminosity is given in units of cm^{-2}s^{-1}.

Table 1: Parameters for Cases 1−3

Parameter	Case 1	Case 2	Case 3
σ_s (cm)	10.0	10.0	10.0
ε_{Nx} (mm mrad)	5.0	55.0	55.0
ε_{Ny} (mm mrad)	2.0	20.0	20.0
β_x^* (cm)	5.0	4.0	40.0
β_y^* (cm)	2.0	0.8	8.0
φ_x (mrad)	0~10	0~10	0.1
φ_y (mrad)	0	2	0
δ_x (unit: σ_x)	0	0.01	0~3
δ_y (unit: σ_y)	0	0.01	0.01

Case 1: $\varphi_y = 0$; $\delta x = 0$ and $\delta y = 0$; $\beta_{1x}^* \gg (\sigma_{1s} + \sigma_{2s})$ and $\sigma_{1x}^* \gg \sigma_{2x}^*$. For this case, since $s \sim (-3\sigma_{s,}\, 3\sigma_{s,})$ we have

$$As^2 = \left(\frac{{\sigma_{1x}^*}^2}{{\beta_{1x}^*}^2} + \frac{{\sigma_{2x}^*}^2}{{\beta_{2x}^*}^2}\right)\frac{s^2}{{\sigma_{1x}^*}^2+{\sigma_{2x}^*}^2} \sim 0, \quad (14)$$

then $a = \frac{\sin^2\varphi_x}{{\sigma_x^*}^2} + \frac{\sec^2\varphi_x}{\sigma_s^2}$, $b = 0$, $c = 0$, so the analytic solution of luminosity for Case 1 is

$$L = L_0\sqrt{2-\cos^2\varphi_x}\frac{1}{\sqrt{\pi}\sigma_s\sqrt{B}}e^{\frac{a}{2B}}K_0\left(\frac{a}{2B}\right) \quad (15)$$

where K_0 is the modified Bessel function of the second kind.

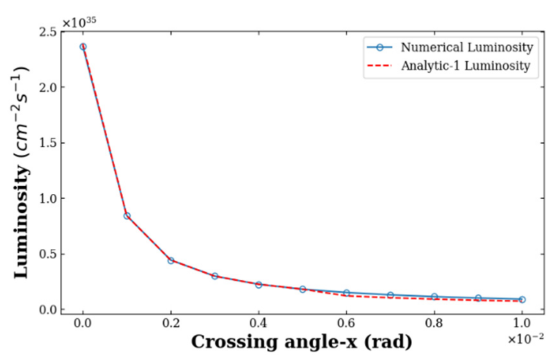

Figure 1: Numerical and analytic results for Case 1.

Figure 1 shows the numerical result matches the analytic result very well for Case 1 when one of the crossing angles is zero. It also shows that when "head-on" ($\varphi_x = 0$), the luminosity reduction factor is 1.0. We know that the major reduction of the luminosity comes from the crossing angle.

Case 2: $\varphi_y \neq 0$; $\delta x < \sigma_x^*$ and $\delta y < \sigma_y^*$ and $\delta y = 0$; $\beta_{1x}^* \gg (\sigma_{1s} + \sigma_{2s})$, $\beta_{1y}^* \gg (\sigma_{1s} + \sigma_{2s})$ and $\sigma_{1x}^* \gg \sigma_{2x}^*$, $\sigma_{1y}^* \gg \sigma_{2y}^*$. Considering $s \sim (-3\sigma_{s,} 3\sigma_{s,})$, the case 2 conditions yield

$$As^2 = \left(\frac{{\sigma_{1x}^*}^2}{{\beta_{1x}^*}^2} + \frac{{\sigma_{2x}^*}^2}{{\beta_{2x}^*}^2}\right)\frac{s^2}{{\sigma_{1x}^*}^2+{\sigma_{2x}^*}^2} \sim 0 \quad (16)$$

and

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$$Bs^2 = \left(\frac{{\sigma^*_{1y}}^2}{{\beta^*_{1y}}^2}+\frac{{\sigma^*_{2y}}^2}{{\beta^*_{2y}}^2}\right)\frac{s^2}{{\sigma^*_{1y}}^2+{\sigma^*_{2y}}^2}\sim 0 \quad (17)$$

We also have $a = \frac{\sin^2\varphi_x}{{\sigma^*_x}^2}+\frac{\sin^2\varphi_y}{{\sigma^*_y}^2}+\frac{1+\tan^2\varphi_x+\tan^2\varphi_y}{\sigma_s^2}$, $b = \frac{(\delta x)\sin\varphi_x}{{\sigma^*_x}^2}+\frac{(\delta y)\sin\varphi_y}{{\sigma^*_y}^2}$, $c = \frac{(\delta x)^2}{4{\sigma^*_x}^2}+\frac{(\delta y)^2}{4{\sigma^*_y}^2}$. So, the analytic solution of luminosity for the case 2 is

$$L = L_0\sqrt{2-cos^2\,\varphi_x}\,\frac{1}{\sqrt{a}\sigma_s}e^{\frac{b^2}{4a}-c} \quad (18)$$

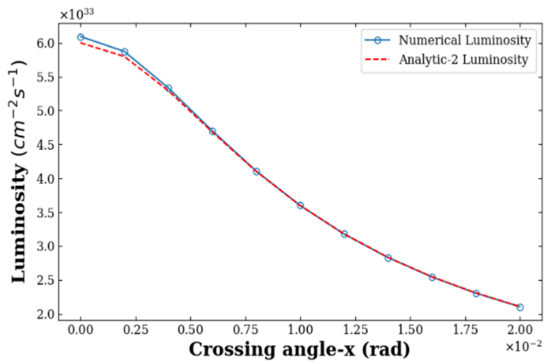

Figure 2: Numerical and analytic results for Case 2.

In Case 2, one of the crossing angles is not zero. If the two offsets are enough smaller than the transverse rms sizes, Figure 2 shows that the analytic solution matches the numerical result very well when increasing the other crossing angle. The maximum value of the luminosity reduction factor is 0.1437, this value is obviously introduced from the two crossing angles.

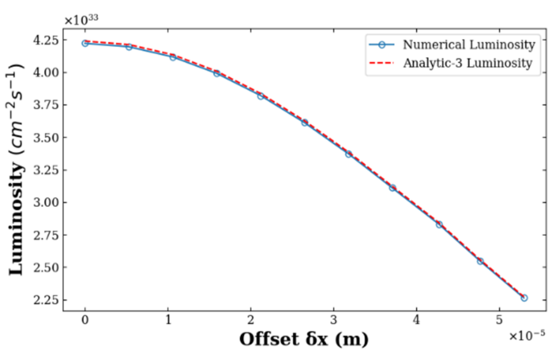

Figure 3: Numerical and analytic results for Case 3.

Case 3: $\varphi_x \neq 0$ and φ_x is very small, $\varphi_y = 0$; $\delta x < \sigma^*_{1x}$ or $\delta x < \sigma^*_{2x}$; $\beta^*_{1x} \gg (\sigma_{1s}+\sigma_{2s})$, $\beta^*_{1y} \gg (\sigma_{1s}+\sigma_{2s})$ and $\sigma^*_{1x} \gg \sigma^*_{2x}$, $\sigma^*_{1y} \gg \sigma^*_{2y}$. Since $s\sim(-3\sigma_{s,}3\sigma_{s,})$, again, we have

$$As^2 = \left(\frac{{\sigma^*_{1x}}^2}{{\beta^*_{1x}}^2}+\frac{{\sigma^*_{2x}}^2}{{\beta^*_{2x}}^2}\right)\frac{s^2}{{\sigma^*_{1x}}^2+{\sigma^*_{2x}}^2}\sim 0 \quad (19)$$

and

$$Bs^2 = \left(\frac{{\sigma^*_{1y}}^2}{{\beta^*_{1y}}^2}+\frac{{\sigma^*_{2y}}^2}{{\beta^*_{2y}}^2}\right)\frac{s^2}{{\sigma^*_{1y}}^2+{\sigma^*_{2y}}^2}\sim 0 \quad (20)$$

Now we obtain $a = \frac{\sin^2\varphi_x}{{\sigma^*_x}^2}+\frac{\sec^2\varphi_x}{\sigma_s^2}$, $b = \frac{(\delta x)\sin\varphi_x}{{\sigma^*_x}^2}$ and $c = \frac{(\delta x)^2}{4{\sigma^*_x}^2}+\frac{(\delta y)^2}{4{\sigma^*_y}^2}$. If φ_x is small enough, we have $\sin\varphi_x \sim \tan\varphi_x$ and $\sec\varphi_x \sim \cos\varphi_x \sim 1$, then

$$\sqrt{a} \approx \frac{1}{\sigma_s}\left(1+\frac{\sigma_s^2}{{\sigma^*_x}^2}\tan^2\varphi_x\right)^{1/2} \quad (21)$$

The reduction factor of luminosity can be expressed as

$$L/L_0 = \left(1+\frac{\sigma_s^2}{{\sigma^*_x}^2}\tan^2\varphi_x\right)^{-1/2}\exp\left(\frac{b^2}{4a}-c\right) \quad (22)$$

where

$$\frac{b^2}{4a}-c \approx -\left(\frac{\delta x}{2}\right)^2\left(\frac{1}{{\sigma^*_x}^2\cos^2\varphi_x+\sigma_s^2\sin^2\varphi_x}\right)-\left(\frac{\delta y}{2\sigma^*_y}\right)^2 \quad (23)$$

For Case 3, we can see in Fig. 3, when the crossing angle is small enough, the analytic solution will be consistent with the numerical simulation result.

Case 4: Head-on collision with Hourglass Effect for previous Jefferson Lab Electron-Ion Collider (JLEIC) design.

As a practical application, for the one of JLEIC design parameters (see Table 2) with the collision frequency 476×10^6 Hz, we did the calculations. The luminosity we obtained is 3.09×10^{33} cm^{-2}/s and the hourglass reduction factor is 0.85. These results are consistent with the results of other calculation methods and meet the original design requirements of JLEIC.

Table 2: Parameters in Case 4

Parameter	Proton	Electron
Beam Energy (GeV)	40	3
σ_z (cm)	2.5	1.0
ε_N , hor / ver, (mm, mrad)	0.5/0.2	18/3.6
β^* , hor / ver, (cm)	8.0/1.3	30/9.8
Particles/Bunch (10^{10})	0.59	3.9

CONCLUSION

The beam-beam effects on crabbing dynamics process in CASA BeamBeam was addressed in this paper. Especially, in the boosted frame, we addressed and provided the comprehensive analytic solutions of the luminosity calculation for different requirements. The benchmarks demonstrated that the analytic solutions are consistent with the numerical results. These analytic solutions will provide fast, valid and reliable estimates for different designs. For further studies, the development in CASA BeamBeam, the various aspects of the crabbing dynamics with beam-beam effect using luminosity evolution, will be one of the performance criteria in our accelerator physics projects.

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