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RESEARCH ARTICLE

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Cross-ratio degrees and triangulations

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Abstract

The cross-ratio degree problem counts configurations of n points on $\mathbb{P}^1$ with $n-3$ prescribed cross-ratios. Cross-ratio degrees arise in many corners of combinatorics and geometry, but their structure is not well-understood in general. Interestingly, examining various special cases of the problem can yield combinatorial structures that are both diverse and rich. In this paper, we prove a simple closed formula for a class of cross-ratio degrees indexed by triangulations of an n-gon; these degrees are connected to the geometry of the real locus of $M_{0,n}$, and to positive geometry.

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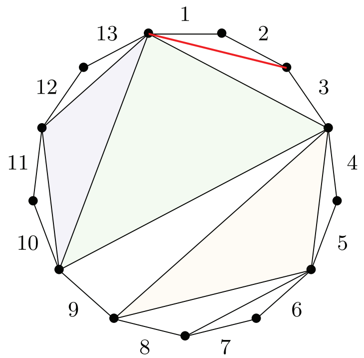
14N10, 14H10, 14H81 (primary)

1 | INTRODUCTION

Consider a regular n-gon X with edges labeled by $[n] = \{1, \ldots, n\}$, in order. To each diagonal D of X is naturally assigned a 4-element subset of $[n]$, consisting of the four edges that D touches. A triangulation T of X is a choice of $n-3$ diagonals in X that do not cross. Thus, we may associate to T a collection $\mathcal{U} = \{S_1, \ldots, S_{n-3}\}$ of 4-element subsets of $[n]$, see Figure 1.

Let $M_{0,n} = M_{0,[n]}$ denote the moduli space of configurations of n distinct points on the Riemann sphere, labeled by $[n]$, up to Möbius transformation. Recall that $M_{0,[n]}$ is a smooth $(n-3)$-dimensional affine variety, and that for $S \subseteq [n]$ with $|S| \geqslant 3$, there is a forgetful map $M_{0,[n]} \to M_{0,S}$. By the previous paragraph, a triangulation T of a regular n-gon defines a product of forgetful maps

$$\pi_T : M_{0,[n]} \to \prod_{S \in \mathcal{U}} M_{0,S}. \tag{1}$$



$\rightsquigarrow$

$\{1,2,3,13\}$
$\{1,3,4,13\}$
$\{1,9,10,13\}$
$\{1,11,12,13\}$
$\{3,4,5,6\}$
$\{3,4,8,9\}$
$\{3,4,9,10\}$
$\{5,6,7,8\}$
$\{5,6,8,9\}$
$\{9,10,11,12\}$

$\rightsquigarrow$

$$M_{0,[13]} \xrightarrow{\pi_T} M_{0,\{1,2,3,13\}} \times \cdots \times M_{0,\{9,10,11,12\}}$$

FIGURE 1 The subsets $S_1, \dots, S_{10} \subseteq [13]$ associated to a triangulation of a 13-gon, and the corresponding map of moduli spaces. For illustration, an edge and its corresponding subset/factor are colored red. The three "internal" triangles are shaded — Theorem 1.1 implies π_T has degree $d_T = 2^3 = 8$.

Here $M_{0,S} \cong \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ by taking the cross-ratio. Note that π_T is a map of $(n-3)$-dimensional varieties, hence a general fiber of π_T is zero-dimensional with some constant cardinality d_T — this cardinality is an example of a *cross-ratio degree* in the sense of [18]. The purpose of this brief paper is to prove:

Theorem 1.1. *$d_T = 2^{I(T)}$, where $I(T)$ is the number of triangles of T with no exterior edges.*

Motivation

The more general "cross-ratio degree problem" is as follows. Let $S_1, \dots, S_{n-3} \subseteq [n]$ with $|S_j| = 4$ for $1 \leqslant j \leqslant n-3$, and let $\mathcal{U} = \{S_1, \dots, S_{n-3}\}$. The map $\pi_{\mathcal{U}} : M_{0,[n]} \to \prod_{j=1}^{k} M_{0,S_j}$ is a map of $(n-3)$-dimensional varieties, hence is generically finite. The dependence of the degree of $\pi_{\mathcal{U}}$ on the combinatorial data $\mathcal{U}$ is not well-understood.

The particular type of cross-ratio degree appearing in Theorem 1.1 arises in connection with the *positive geometry* of $M_{0,n}$ [1], which has recently been applied to a range of important applications. Brown [7] studied coordinate systems arising from cross-ratios of this form under the name *dihedral coordinates*, and used them to give affine charts on $\overline{M}_{0,n}$ that are particularly well-behaved with respect to the cell decomposition of $\overline{M}_{0,n}(\mathbb{R})$, ultimately proving that all period integrals of $\overline{M}_{0,n}$ are multiple zeta values.

More recently, dihedral coordinates have been used extensively as a computational and conceptual tool in string theory, particularly in the study of *tree-level string scattering amplitudes*. Dihedral coordinates satisfy a remarkable collection of relations known as the *u-equations* that appear in the study of scattering amplitudes, and which exhibit $M_{0,n}$ as a *binary geometry* [3]. One can give natural expressions in terms of dihedral coordinates for the two key objects in the Cachazo–He–Yuan formalism for "gluon tree amplitudes in pure Yang-Mills theory" [4, 8, 10], the *scattering equations* and the *Parke–Taylor form*, a canonical top-degree differential form on $M_{0,n}$ that gives $M_{0,n}$ the structure of a *positive geometry* [1, 2, 5, 15]. Theorem 1.1 elicits a range of natural questions related to this subject — for example, writing the scattering equations and Parke–Taylor form in terms of dihedral coordinates relies on a choice of a triangulation of an n-gon with no internal triangles; Lam asked whether such expressions exist for general triangulations.

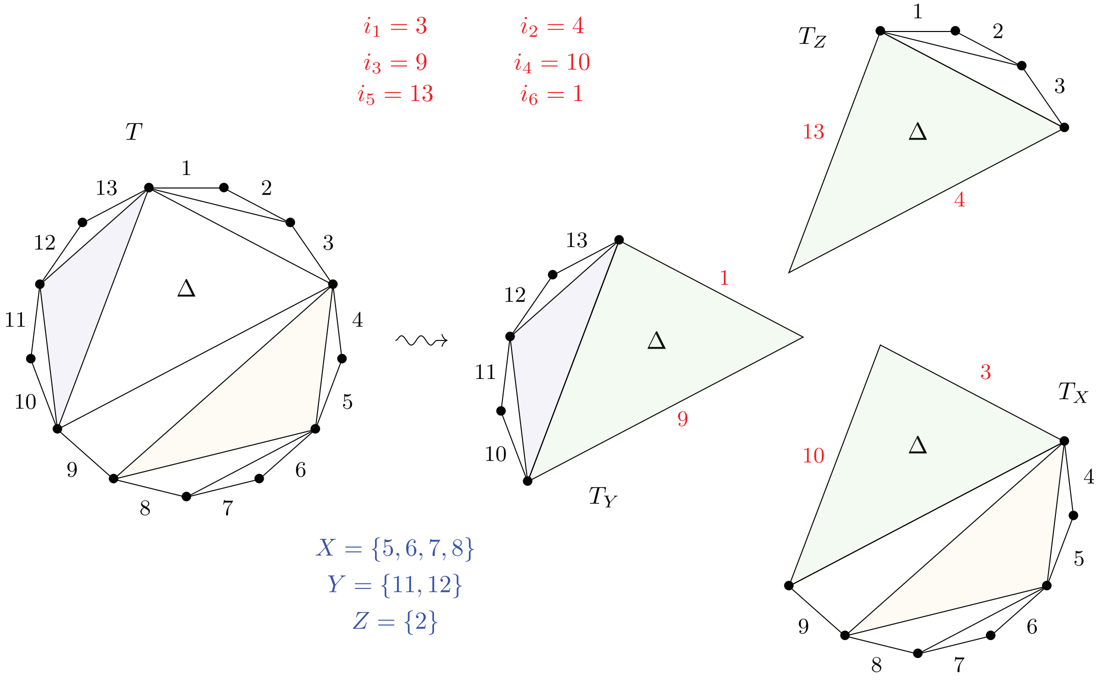


FIGURE 2 Decomposing the 13-gon as in the proof of Theorem 1.1.

The cross-ratio degree problem appears to be ubiquitous — special cases have been discovered and re-discovered, repeatedly and independently, by many people across a range of areas, including rigidity theory, polynomial root-finding algorithms, complex dynamics, combinatorics of matchings on bipartite graphs, birational geometry of $\overline{M}_{0,n}$, and Gromov–Witten theory [9, 11, 13, 14, 16–18].

Remark 1.2. Some of the lemmas stated in this paper apply more generally to certain natural generalizations of the cross-ratio degree problem, for example, considering arbitrary products of forgetful maps $M_{0,n} \to \prod_j M_{0,S_j}$ with $\sum_j (S_j - 3) = n - 3$, see also [6]. We do not include the generalizations here as we do not know of an application.

Idea of proof

We prove Theorem 1.1 by induction on $I(T)$, by cutting the n-gon into smaller polygons. Given an internal triangle Δ of T, we may produce three smaller polygons T_X, T_Y, T_Z as in Figure 2, satisfying $I(T) = I(T_X) + I(T_Y) + I(T_Z) + 1$. We prove Lemma 3.1, which comprises most of the work of the paper — this lemma applies more generally to cross-ratio degrees, and in this case implies $d_T = 2 \cdot d_{T_X} \cdot d_{T_Y} \cdot d_{T_Z}$, providing our inductive step. Our proof of Lemma 3.1 is somewhat technical, and involves analyzing the first three iterations of a recursive algorithm for computing cross-ratio degrees first used by Goldner, using some basic lemmas on cross-ratio degrees developed by myself and others.

Remark 1.3. A triangulation T has an $[n]$-marked trivalent dual tree τ, corresponding to a boundary point P_τ of $\overline{M}_{0,n}$, or alternatively a top-dimensional cone σ_τ of the tropical moduli space $M_{0,n}^{\mathrm{trop}}$.

One can easily show that the extension of π_T to $\overline{M}_{0,n}$ is a local isomorphism at P_τ, and correspondingly the tropicalized map $\pi_T^{\mathrm{trop}} : M_{0,[n]}^{\mathrm{trop}} \to \prod_{S\in\mathcal{U}} M_{0,S}^{\mathrm{trop}}$ maps σ_τ isomorphically (with determinant 1) onto a cone of the codomain. It would be interesting to understand this piece of the tropical picture better — for example, to classify combinatorially which *other* top-dimensional cones map to $\pi_T(\sigma_\tau)$.

An open problem

Let $C(n)$ denote the largest cross-ratio degree on $M_{0,n}$. Inscribing a $\lfloor n/2\rfloor$-gon inside the n-gon yields triangulations T with $I(T) = \lfloor n/2\rfloor - 2$, and Theorem 1.1 then implies $C(n) \geqslant 2^{\lfloor n/2\rfloor-2}$. One may also show $C(n) \leqslant 2^{n-5}$ for $n \geqslant 5$, see [18, Remark 4.3]. What are the asymptotics of $C(n)$?

The lower bound above is the best one I know of, but it is not sharp — here are some experimental data:

n	3	4	5	6	7	8	9	10	11	12	13	14
$C(n)$	1	1	1	2	≥2	≥4	≥6	≥10	≥13	≥20	≥28	≥41

It is likely that the bounds in the table are correct for $n \leqslant 10$. I assume it is a coincidence that the data are compatible with (a shift of) sequence A034406 from the Online Encyclopedia of Integer Sequences.

2 | BASIC LEMMAS

We will need several standard facts about cross-ratio degrees. The first is the following vanishing condition.

Lemma 2.1 ([18], Proposition 4.1). *Let $\mathcal{U} \in \binom{[n]}{4}^{n-3}$. If there exists a nonempty subset $\mathcal{U}' \subseteq \mathcal{U}$ such that $\bigcup_{S\in\mathcal{U}'} S < |\mathcal{U}'| + 3$, then $d_{[n],\mathcal{U}} = 0$.*

The following recursive algorithm for cross-ratio degrees is a straightforward application of standard facts[†] about the cohomology of $\overline{M}_{0,n}$. The algorithm seems to have first been written down (although somewhat implicitly) in [13, Corollary 3.2.22]. See also [11, 12].

Lemma 2.2 [13]. *Let $\mathcal{U} = \{S_1, \dots, S_{n-3}\} \in \binom{[n]}{4}^{n-3}$, and suppose $S_1 = \{i_1, i_2, i_3, i_4\}$. Then*

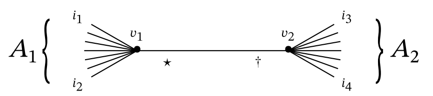
$$d_{[n],\mathcal{U}} = \sum_{\substack{[n]=A_1\sqcup A_2 \\ i_1,i_2\in A_1 \\ i_3,i_4\in A_2 \\ |S_j\cap A_1|\neq 2 \; for\; 2\leqslant j\leqslant n-3}} d_{A_1\cup\{\star\},\mathcal{U}_1} \cdot d_{A_2\cup\{\dagger\},\mathcal{U}_2}. \tag{2}$$

[†] Explicitly, if $\rho_S : \overline{M}_{0,[n]} \to \overline{M}_{0,S}$ is the forgetful map that remembers the marks in S, then (2) expresses the restriction of $\prod_{2\leqslant j\leqslant n-3} \rho_{S_j}^*[pt]$ to the sum of boundary divisors $\rho_{S_1}^*(D_{i_1i_2|i_3i_4})$, where $D_{i_1i_2|i_3i_4}$ is a boundary point in $\overline{M}_{0,S_1} \cong \overline{M}_{0,4}$.

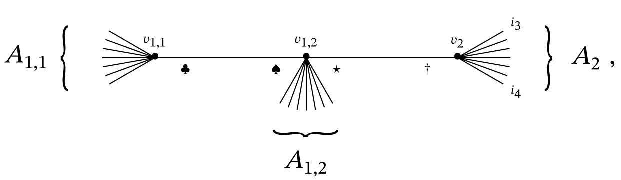
Here $\mathcal{U}_1$ is obtained by taking all $S \in \mathcal{U}$ such that $|S \cap A_1| \geqslant 3$, and in each such S, replacing any element (unique if it exists) of A_2 with $\star$. Similarly $\mathcal{U}_2$ is obtained by taking all $S \in \mathcal{U}$ with $|S \cap A_2| \geqslant 3$, and replacing elements of A_1 with $\dagger$. In the summand, $d_{[n],\mathcal{U}}$ is taken to be zero if $|\mathcal{U}| \neq n-3$ — in other words, we only need to consider terms where $|\mathcal{U}_1| \in \binom{A_1 \cup \{\star\}}{4}^{|A_1|-2}$ and, correspondingly, $|\mathcal{U}_2| \in \binom{A_2 \cup \{\dagger\}}{4}^{|A_2|-2}$.

Note that $\mathcal{U}_1, \mathcal{U}_2$ are naturally identified with disjoint subsets of $\mathcal{U}$, whose union is $\mathcal{U} \setminus \{S_1\}$.

Remark 2.3. It is helpful to keep in mind the following pictorial interpretation of Lemma 2.2. We may interpret the index set of the sum in (2) as the set of $[n]$-marked graphs Γ (i.e., graphs together with additional “half-edges” labeled by $[n]$) of the form



where we require that for each S_j with $2 \leqslant j \leqslant n$, there is a unique vertex v of Γ such that the four paths from v to the elements of S_j start along four *distinct* (half-)edges incident to v. We then say S_j is supported on v. Note that if S_j is supported on v_1 and $|S_j \cap A_1| = 3$, then one of the four half-edges in question will be $\star$ — this explains the “renaming” process in the lemma. This associates to each vertex a collection of 4-element subsets of the half-edges incident to that vertex, and the summand in (2) corresponding to Γ is the product of the two associated cross-ratio degrees. (We have the right *number* of 4-element subsets by the assumption $|\mathcal{U}_1| = |A_1| - 2$.) Note that these two cross-ratio degrees are defined on underlying sets of cardinality strictly less than n. The reason for this interpretation is as follows. Suppose that for some Γ, we apply Lemma 2.2 *again* to expand the factor $d_{A_1 \cup \{\star\},\mathcal{U}_1}$ in $d_{A_1 \cup \{\star\},\mathcal{U}_1} \cdot d_{A_2 \cup \{\dagger\},\mathcal{U}_2}$. The answer will be a sum over graphs of the form



where again the summand is a product of cross-ratio degrees, each consisting of subsets supported on a vertex. Iterating this process of “splitting a vertex” gives one strategy of computing cross-ratio degrees — the eventual output will be a collection of trivalent trees with no nontrivial cross-ratio degrees, and the number of these trees will be $d_{[n],\mathcal{U}}$.

Remark 2.4. The marked trees of Remark 2.3 can be thought of as tropical genus-zero curves — this connection is given a full geometric explanation in [12].

We will also need the following multiplicativity property.

Lemma 2.5. *Let $\mathcal{U} \in \binom{[n]}{4}^{n-3}$. Suppose there exists a partition $[n] = \{i_1, i_2, i_3\} \sqcup X \sqcup Y$ such that for all $S \in \mathcal{U}$, we have either $S \subseteq \{i_1, i_2, i_3\} \cup X$ or $S \subseteq \{i_1, i_2, i_3\} \cup Y$. Define $\mathcal{U}_X = \{S \in \mathcal{U} : S \subseteq$*

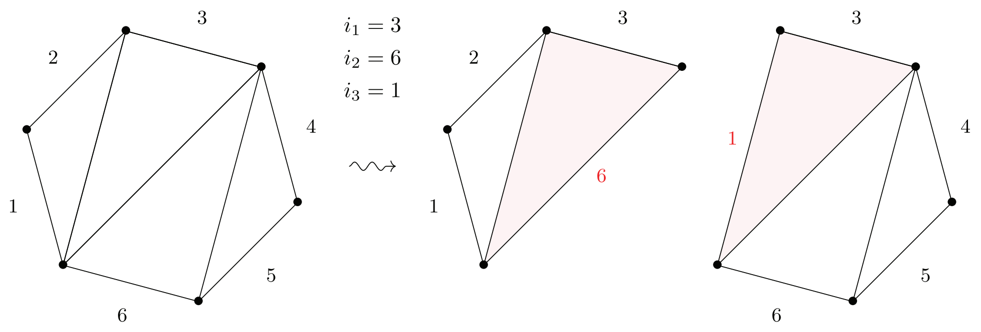


FIGURE 3 Decomposing along a triangle as in the proof of Corollary 2.6.

$\{i_1, i_2, i_3\} \cup X\}$, *and similarly* $\mathcal{U}_Y$. *Then*

$$d_{[n],\mathcal{U}} = \begin{cases} d_{\{i_1,i_2,i_3\}\cup X,\mathcal{U}_X} \cdot d_{\{i_1,i_2,i_3\}\cup Y,\mathcal{U}_Y} & |\mathcal{U}_X| = |X| \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The map $\pi_{\mathcal{U}} : M_{0,[n]} \to \prod_{S\in\mathcal{U}} M_{0,S}$ factors as:

$$M_{0,[n]} \longrightarrow M_{0,\{i_1,i_2,i_3\}\cup X} \times M_{0,\{i_1,i_2,i_3\}\cup Y} \xrightarrow{\pi_{\mathcal{U}_X}\times\pi_{\mathcal{U}_Y}} \prod_{S\in\mathcal{U}_X} M_{0,S} \times \prod_{S\in\mathcal{U}_Y} M_{0,S} = \prod_{S\in\mathcal{U}} M_{0,S}. \tag{3}$$

If $|\mathcal{U}_X| \neq |X|$, then either $\pi_{\mathcal{U}_X}$ or $\pi_{\mathcal{U}_Y}$ has positive relative dimension, hence so does $\pi_{\mathcal{U}}$. Thus, $d_{[n],\mathcal{U}} = 0$.

Suppose $|\mathcal{U}| = |X|$. Since Möbius transformations act simply 3-transitively on $\mathbb{P}^1$, we may uniquely choose coordinates on $\mathbb{P}^1$ so that $p_{i_1} = \infty$, $p_{i_2} = 0$, and $p_{i_3} = 1$. With this choice, the distinct complex numbers $p_{i_4}, \dots, p_{i_n} \in \mathbb{P}^1 \setminus \{\infty, 0, 1\}$ define global coordinates on $M_{0,[n]}$. The left-most map in (3), in these coordinates, is simply the inclusion of the open subset where none of the coordinates p_i coincide, hence is birational. Thus, the degree of $\pi_{\mathcal{U}}$ is equal to the degree of $\pi_{\mathcal{U}_X} \times \pi_{\mathcal{U}_Y}$, and the statement follows. □

Corollary 2.6 (Cf. [7, Corollary 2.19]). *If T has no internal triangles, then $d_T = 1$.*

Proof. Let T be a triangulation of the n-gon with no internal triangles. We induct on n, with base case $n = 4$. Both triangulations of a square induce the identity map $\pi_T : \overline{M}_{0,[4]} \to \overline{M}_{0,[4]}$, which has degree 1.

Suppose $n \geqslant 5$. On average, a triangle of T contains $\frac{3(n-2)-n}{n-2} > 1$ diagonals. Thus, there exists a triangle Δ containing at least two diagonals. Since T has no internal triangles, Δ contains an edge i_1 of the n-gon. The vertex of Δ opposite i_1 touches two edges i_2, i_3, with $i_1 < i_2 < i_3 < i_1$ in clockwise cyclic order. We construct two triangulations T_1, T_2 of smaller polygons as in Figure 3. Specifically, let

$$X = \{i \in [n] : i_1 < i < i_2\} \qquad Y = \{i \in [n] : i_3 < i < i_1\}$$

using clockwise cyclic order, and let T_1 be the triangulation of a $(|X| + 3)$-gon obtained by cutting T along the diagonal touching edges $i_1 - 1, i_1, i_2, i_3$, discarding the piece not containing Δ, and renaming the cut edge by i_2. Similarly, we have a triangulation T_2 of a $(|Y|+3)$-gon by cutting along the diagonal touching i_1, i_1+1, i_2, i_3. Since diagonals cannot cross, every diagonal of T either touches four sides in $\{i_1, i_2, i_3\} \cup X$ or touches four sides in $\{i_1, i_2, i_3\} \cup Y$. There are precisely $|X|$ of the first type since they triangulate a $(|X|+3)$-gon. Thus, by Lemma 2.5, we have

$$d_T = d_{T_1} \cdot d_{T_2}.$$

Note T_1 and T_2 have fewer than n sides and no internal triangles. Thus by induction, $d_T = 1$. $\square$

3 | KEY LEMMA AND PROOF OF THEOREM 1.1

Our key lemma is another (double-)multiplicativity property, which may have applications to computing other classes of cross-ratio degrees.

Lemma 3.1. *Let $\mathcal{U} = \{S_1, \dots, S_{n-3}\} \in \binom{[n]}{4}^{n-3}$. Suppose there are distinct elements $i_1, \dots, i_6 \in [n]$ with $S_1 = \{i_1, i_2, i_3, i_4\}$, $S_2 = \{i_3, i_4, i_5, i_6\}$, and $S_3 = \{i_1, i_2, i_5, i_6\}$. Suppose further that there exists a partition $[n] = \{i_1, \dots, i_6\} \sqcup X \sqcup Y \sqcup Z$ such that for each $S \in \mathcal{U}$, we have either $S \subseteq X \cup S_1$, $S \subseteq Y \cup S_2$, or $S \subseteq Z \cup S_3$. Define $\mathcal{U}_X = \{S \in \mathcal{U} : S \subseteq X \cup S_1\}$, and similarly $\mathcal{U}_Y, \mathcal{U}_Z$. Then,*

$$d_{[n],\mathcal{U}} = 2 \cdot d_{X\cup S_1,\mathcal{U}_X} \cdot d_{Y\cup S_2,\mathcal{U}_Y} \cdot d_{Z\cup S_3,\mathcal{U}_Z}. \tag{4}$$

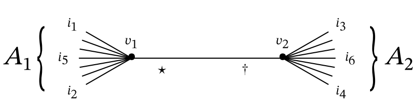
Proof. By Lemma 2.2,

$$d_{[n],\mathcal{U}} = \sum_{\substack{[n]=A_1\sqcup A_2 \\ i_1,i_2\in A_1 \\ i_3,i_4\in A_2 \\ |S_j\cap A_1|\neq 2 \; for \; 2\leqslant j\leqslant n-3}} d_{A_1\cup\{\star\};\mathcal{U}_1} \cdot d_{A_2\cup\{\dagger\};\mathcal{U}_2}. \tag{5}$$

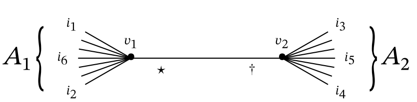
Note that the sum in (5) *excludes* partitions $[n] = A_1 \sqcup A_2$ where $\{i_5, i_6\} \subseteq A_1$ (since we would then have $|S_2 \cap A_1| = 2$) or where $\{i_5, i_6\} \subseteq A_2$ (since we would then have $|S_3 \cap A_2| = 2$). Thus:

$$d_{[n],\mathcal{U}} = \sum_{\substack{[n]=A_1\sqcup A_2 \\ i_1,i_2,i_5\in A_1 \\ i_3,i_4,i_6\in A_2 \\ |S_j\cap A_1|\neq 2 \; for \; 2\leqslant j\leqslant n-3}} d_{A_1\cup\{\star\};\mathcal{U}_1} \cdot d_{A_2\cup\{\dagger\};\mathcal{U}_2} + \sum_{\substack{[n]=A_1\sqcup A_2 \\ i_1,i_2,i_6\in A_1 \\ i_3,i_4,i_5\in A_2 \\ |S_j\cap A_1|\neq 2 \; for \; 2\leqslant j\leqslant n-3}} d_{A_1\cup\{\star\};\mathcal{U}_1} \cdot d_{A_2\cup\{\dagger\};\mathcal{U}_2}. \tag{6}$$

Using the interpretation from Remark 2.3, we have expressed $d_{[n],\mathcal{U}}$ as a sum, over graphs of the two types



and



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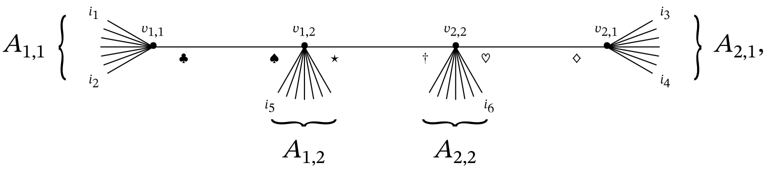
such that for each $2 \leqslant j \leqslant n-3$, S_j is supported on v_1 or v_2. We now apply Lemma 2.2 to all four of the cross-ratio degrees appearing in (6), for example,

$$d_{A_1 \cup \{\star\}, \mathcal{U}_1} = \sum_{\substack{A_1 \cup \{\star\} = A_{1,1} \sqcup (A_{1,2} \cup \{\star\}) \\ i_1, i_2 \in A_{1,1} \\ i_5 \in A_{1,2} \\ |S_j \cap A_{1,1}| \neq 2 \; for \; S_j \in \mathcal{U}_1 \text{ if } j > 2}} d_{A_{1,1} \cup \{\clubsuit\}; \mathcal{U}_{1,1}} \cdot d_{A_{1,2} \cup \{\spadesuit, \star\}; \mathcal{U}_{1,2}}.$$

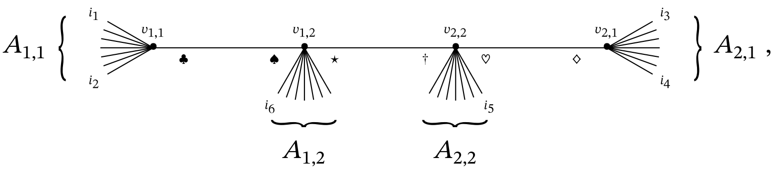
Here $\mathcal{U}_{1,1}$ is obtained by taking all $S \in \mathcal{U}_1$ with $|S \cap A_{1,1}| \geqslant 3$, and in each such S, replacing any element of $A_{1,2}$ with $\clubsuit$. Similarly $\mathcal{U}_{1,2}$ is obtained by taking all $S \in \mathcal{U}_1$ with $|S \cap A_{1,2}| \geqslant 3$, and replacing elements of $A_{1,1}$ with $\spadesuit$. Repeating this process for all cross-ratio degrees in (6), we see (again cf. Remark 2.3) that

$$d_{[n], \mathcal{U}} = \sum_{\Gamma} d_{A_{1,1} \cup \{\clubsuit\}, \mathcal{U}_{1,1}} \cdot d_{A_{1,2} \cup \{\spadesuit, \star\}, \mathcal{U}_{1,2}} \cdot d_{A_{2,1} \cup \{\dagger, \heartsuit\}, \mathcal{U}_{2,1}} \cdot d_{A_{2,2} \cup \{\diamond\}, \mathcal{U}_{2,2}}, \tag{7}$$

where Γ ranges over marked trees of the two types (note the positions of i_5 and i_6)



(I)



(II)

such that for each $4 \leqslant j \leqslant n-3$, there is a (unique) vertex on which S_j is supported. As in Lemma 2.2, we have nonzero contributions only from graphs satisfying

$$|\mathcal{U}_{1,1}| = |A_{1,1}| - 2 \quad |\mathcal{U}_{1,2}| = |A_{1,2}| - 1 \quad |\mathcal{U}_{2,1}| = |A_{2,1}| - 1 \quad \text{and} \quad |\mathcal{U}_{2,2}| = |A_{2,2}| - 2. \tag{8}$$

We claim types (I) and (II) each contribute $d_{X \cup S_1, \mathcal{U}_X} \cdot d_{Y \cup S_2, \mathcal{U}_Y} \cdot d_{Z \cup S_3, \mathcal{U}_Z}$ to $d_{[n]; \mathcal{U}}$; this clearly implies (4).

Fix a marked graph Γ of type (I). Recall that there is a natural injective map $\mathcal{U}_{1,1} \hookrightarrow \mathcal{U}$, and let $\mathcal{U}_{1,1;X} \subseteq \mathcal{U}_{1,1}$ consist of elements mapping to elements of $\mathcal{U}_X$. Similarly define $U_{1,1;Y}, U_{1,1;Z}, U_{1,2;X}, \ldots, U_{2,2;Z}$. Let $A_{1,1;X} = A_{1,1} \cap X$, and similarly define $A_{1,1;Y}, A_{1,1;Z}, A_{1,2;X}, \ldots, A_{2,2;Z}$. Note the following equalities; we have similarly statements for $\mathcal{U}_{1,2}, A_{1,2}, \mathcal{U}_{2,2}, A_{2,2}, \mathcal{U}_{2,1}, A_{2,1}$.

$$\mathcal{U}_{1,1} = \mathcal{U}_{1,1;X} \sqcup \mathcal{U}_{1,1;Y} \sqcup \mathcal{U}_{1,1;Z} \qquad A_{1,1} = A_{1,1;X} \sqcup A_{1,1;Y} \sqcup A_{1,1;Z} \sqcup \{i_1, i_2\}. \tag{9}$$

In order for Γ to contribute to $d_{[n], \mathcal{U}}$, the four cross-ratio degrees supported at the vertices $v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}$ must all be nonzero. Lemma 2.1 implies the following list of bounds:

(i) $\mathcal{U}_{1,1;X} = \emptyset$ or $\left| \bigcup_{S \in \mathcal{U}_{1,1;X}} S \right| \geqslant |\mathcal{U}_{1,1;X}| + 3$,

(ii) $\mathcal{U}_{1,1;Y} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{1,1;Y}} S\right| \geqslant |\mathcal{U}_{1,1;Y}| + 3$,
(iii) $\mathcal{U}_{1,1;Z} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{1,1;Z}} S\right| \geqslant |\mathcal{U}_{1,1;Z}| + 3$,
(iv) $\mathcal{U}_{1,2;X} \cup \mathcal{U}_{1,2;Y} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{1,2;X}\cup\mathcal{U}_{1,2;Y}} S\right| \geqslant |\mathcal{U}_{1,2;X} \cup \mathcal{U}_{1,2;Y}| + 3$,
(v) $\mathcal{U}_{1,2;Z} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{1,2;Z}} S\right| \geqslant |\mathcal{U}_{1,2;Z}| + 3$,
(vi) $\mathcal{U}_{2,2;X} \cup \mathcal{U}_{2,2;Z} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{2,2;X}\cup\mathcal{U}_{2,2;Z}} S\right| \geqslant |\mathcal{U}_{2,2;X} \cup \mathcal{U}_{2,2;Z}| + 3$,
(vii) $\mathcal{U}_{2,2;Y} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{2,2;Y}} S\right| \geqslant |\mathcal{U}_{2,2;Y}| + 3$,
(viii) $\mathcal{U}_{2,1;X} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{2,1;X}} S\right| \geqslant |\mathcal{U}_{2,1;X}| + 3$,
(ix) $\mathcal{U}_{2,1;Y} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{2,1;Y}} S\right| \geqslant |\mathcal{U}_{2,1;Y}| + 3$,
(x) $\mathcal{U}_{2,1;Z} = \emptyset$ or $\left|\bigcup_{S\in\mathcal{U}_{2,1;Z}} S\right| \geqslant |\mathcal{U}_{2,1;Z}| + 3$.

By definition, $\bigcup_{S\in\mathcal{U}_{1,1;X}} S \subseteq A_{1,1;X} \cup \{i_1, i_2, \clubsuit\}$, and $\bigcup_{S\in\mathcal{U}_{1,2;X}\cup\mathcal{U}_{1,2;Y}} S \subseteq (A_{1,2;X} \cup A_{1,2;Y}) \cup \{\spadesuit, \star\}$, and so on, so the conditions above imply:

(i) $|A_{1,1;X}| \geqslant |\mathcal{U}_{1,1;X}|$ (also holds if $\mathcal{U}_{1,1;X} = \emptyset$),
(ii) $|A_{1,1;Y}| \geqslant |\mathcal{U}_{1,1;Y}| + 2$ or $\mathcal{U}_{1,1;Y} = \emptyset$,
(iii) $|A_{1,1;Z}| \geqslant |\mathcal{U}_{1,1;Z}|$ (also holds if $\mathcal{U}_{1,1;X} = \emptyset$),
(iv) $|A_{1,2;X}| + |A_{1,2;Y}| \geqslant |\mathcal{U}_{1,2;X}| + |\mathcal{U}_{1,2;Y}| + 1$ or $\mathcal{U}_{1,2;X} \cup \mathcal{U}_{1,2;Y} = \emptyset$,
(v) $|A_{1,2;Z}| \geqslant |U_{1,2;Z}|$ (also holds if $\mathcal{U}_{1,2;Z} = \emptyset$),
(vi) $|A_{2,2;X}| + |A_{2,2;Z}| \geqslant |\mathcal{U}_{2,2;X}| + |\mathcal{U}_{2,2;Z}| + 1$ or $\mathcal{U}_{2,2;X} \cup \mathcal{U}_{2,2;Z} = \emptyset$,
(vii) $|A_{2,2;Y}| \geqslant |U_{2,2;Y}|$ (also holds if $\mathcal{U}_{2,2;Y} = \emptyset$),
(viii) $|A_{2,1;X}| \geqslant |U_{2,1;X}|$ (also holds if $\mathcal{U}_{2,1;X} = \emptyset$),
(ix) $|A_{2,1;Y}| \geqslant |U_{2,1;Y}|$ (also holds if $\mathcal{U}_{2,1;Y} = \emptyset$),
(x) $|A_{2,1;Z}| \geqslant |U_{2,1;Z}| + 2$ or $\mathcal{U}_{2,1;Z} = \emptyset$.

Using (8) and (9), we have

$$|\mathcal{U}_{1,1;X}| + |\mathcal{U}_{1,1;Y}| + |\mathcal{U}_{1,1;Z}| = |\mathcal{U}_{1,1}| = |A_{1,1}| - 2 = |A_{1,1} \cap X| + |A_{1,1} \cap Y| + |A_{1,1} \cap Z|, \quad (10)$$

and similarly

$$|\mathcal{U}_{1,2;X}| + |\mathcal{U}_{1,2;Y}| + |\mathcal{U}_{1,2;Z}| = |\mathcal{U}_{1,2}| = |A_{1,2}| - 1 = |A_{1,2} \cap X| + |A_{1,2} \cap Y| + |A_{1,2} \cap Z|. \quad (11)$$

$$|\mathcal{U}_{2,2;X}| + |\mathcal{U}_{2,2;Y}| + |\mathcal{U}_{2,2;Z}| = |\mathcal{U}_{2,2}| = |A_{2,2}| - 1 = |A_{2,2} \cap X| + |A_{2,2} \cap Y| + |A_{2,2} \cap Z|. \quad (12)$$

$$|\mathcal{U}_{2,1;X}| + |\mathcal{U}_{2,1;Y}| + |\mathcal{U}_{2,1;Z}| = |\mathcal{U}_{2,1}| = |A_{2,1}| - 2 = |A_{2,1} \cap X| + |A_{2,1} \cap Y| + |A_{2,1} \cap Z|. \quad (13)$$

Comparing (10) with (i)–(iii) above, we see that we must have

$$|\mathcal{U}_{1,1;X}| = |A_{1,1;X}|, \qquad \mathcal{U}_{1,1;Y} = \emptyset, \qquad \text{and} \qquad |\mathcal{U}_{1,1;Z}| = |A_{1,1;Z}|.$$

Similarly comparing (11) with (iv)–(v), (12) with (vi)–(vii), and (13) with (viii)–(x), we find:

$$\mathcal{U}_{1,2;X} = \emptyset, \qquad \mathcal{U}_{1,2;Y} = \emptyset, \qquad \text{and} \qquad |\mathcal{U}_{1,2;Z}| = |A_{1,2;Z}|,$$

$$\mathcal{U}_{2,2;X} = \emptyset, \qquad |\mathcal{U}_{2,2;Y}| = |A_{2,2;Y}|, \qquad \text{and} \qquad \mathcal{U}_{2,2;Z} = \emptyset,$$

$$|\mathcal{U}_{2,1;X}| = |A_{2,1;X}|, \qquad |\mathcal{U}_{2,1;Y}| = |A_{2,1;Y}|, \qquad \text{and} \qquad \mathcal{U}_{2,1;Z} = \emptyset.$$

The contribution to (7) from Γ is therefore

$$d_{A_{1,1}\cup\{\clubsuit\},\mathcal{U}_{1,1;X}\cup\mathcal{U}_{1,1;Z}} \cdot d_{A_{1,2}\cup\{\spadesuit,\star\},\mathcal{U}_{1,2;Z}} \cdot d_{A_{2,2}\cup\{\dagger,\heartsuit\},\mathcal{U}_{2,2;Y}} \cdot d_{A_{2,1}\cup\{\diamond\},\mathcal{U}_{2,1;X}\cup\mathcal{U}_{2,1;Y}}.$$

We may break this up further. Note that (8) and (9) also imply

$$A_{1,1;Y} = A_{1,2;X} = A_{1,2;Y} = A_{2,2;X} = A_{2,2;Z} = A_{2,1;Z} = \emptyset.$$

We thus have the decomposition $A_{1,1} \cup \{\clubsuit\} = \{i_1, i_2, \clubsuit\} \sqcup A_{1,1;X} \sqcup A_{1,1;Z}$, and any element of $\mathcal{U}_{1,1} = \mathcal{U}_{1,1;X} \sqcup \mathcal{U}_{1,1;Z}$ is contained by either $\{i_1, i_2, \clubsuit\} \cup A_{1,1;X}$ or $\{i_1, i_2, \clubsuit\} \cup A_{1,1;Z}$. Thus, by Lemma 2.5, we have

$$d_{A_{1,1}\cup\{\clubsuit\},\mathcal{U}_{1,1;X}\cup\mathcal{U}_{1,1;Z}} = d_{A_{1,1;X}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;X}} \cdot d_{A_{1,1;Z}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;Z}},$$

and similarly

$$d_{A_{2,1}\cup\{\diamond\},\mathcal{U}_{2,1;X}\cup\mathcal{U}_{2,1;Y}} = d_{A_{2,1;X}\cup\{i_1,i_2,\diamond\},\mathcal{U}_{2,1;X}} \cdot d_{A_{2,1;Y}\cup\{i_1,i_2,\diamond\},\mathcal{U}_{2,1;Y}}.$$

Now, consider the contribution to (7) from *all* graphs of type (I):

$$\sum_{\Gamma \text{ type (I)}} d_{A_{1,1;X}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;X}} \cdot d_{A_{1,1;Z}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;Z}} \cdot d_{A_{1,2;Z}\cup\{i_5,\spadesuit,\star\},\mathcal{U}_{1,2;Z}}$$
$$\cdot\, d_{A_{2,2;Y}\cup\{i_6,\dagger,\heartsuit\},\mathcal{U}_{2,2;Y}} \cdot d_{A_{2,1;X}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;X}} \cdot d_{A_{2,1;Y}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;Y}}$$

$$= \sum_{\substack{X=A_{1,1;X}\sqcup A_{2,1;X}\\ Y=A_{2,2;Y}\sqcup A_{2,1;Y}\\ Z=A_{1,1;Z}\sqcup A_{1,2;Z}}} d_{A_{1,1;X}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;X}} \cdot d_{A_{1,1;Z}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;Z}} \cdot d_{A_{1,2;Z}\cup\{i_5,\spadesuit,\star\},\mathcal{U}_{1,2;Z}}$$
$$\cdot\, d_{A_{2,2;Y}\cup\{i_6,\dagger,\heartsuit\},\mathcal{U}_{2,2;Y}} \cdot d_{A_{2,1;X}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;X}} \cdot d_{A_{2,1;Y}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;Y}}$$

$$= \left(\sum_{X=A_{1,1;X}\sqcup A_{2,1;X}} d_{A_{1,1;X}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;X}} \cdot d_{A_{2,1;X}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;X}}\right)$$
$$\cdot \left(\sum_{Y=A_{2,2;Y}\sqcup A_{2,1;Y}} d_{A_{2,2;Y}\cup\{i_6,\dagger,\heartsuit\},\mathcal{U}_{2,2;Y}} d_{A_{2,1;Y}\cup\{i_3,i_4,\diamond\},\mathcal{U}_{2,1;Y}}\cdot\right)$$
$$\cdot \left(\sum_{Z=A_{1,1;Z}\sqcup A_{1,2;Z}} d_{A_{1,1;Z}\cup\{i_1,i_2,\clubsuit\},\mathcal{U}_{1,1;Z}} \cdot d_{A_{1,2;Z}\cup\{i_5,\spadesuit,\star\},\mathcal{U}_{1,2;Z}}\right),$$

where the sums are, as usual, over partitions $X = A_{1,1;X} \sqcup A_{2,1;X}$ such that $S \cap A_{1,1;X} \neq 2$ for all $S \in \mathcal{U}_X$, and so forth. By Lemma 2.2, the last expression is equal to

$$d_{X\cup S_1,\mathcal{U}_X} \cdot d_{Y\cup S_2,\mathcal{U}_Y} \cdot d_{Z\cup S_3,\mathcal{U}_Z}$$

as desired. By symmetry, the contribution from graphs of type (II) is the same, completing the proof of the lemma. □

Proof of Theorem 1.1. We induct on the number $I(T)$ of internal triangles in T, with base case $I(T) = 0$ given by Corollary 2.6.

Fix a triangulation T with $I(T) > 0$. Let Δ be an internal triangle of T. Then Δ touches six sides of the n-gon, and these six sides are naturally split up into three pairs — each pair consists of the two sides adjacent to one of the vertices of Δ. Denote these pairs by $\{i_1, i_2\}$, $\{i_3, i_4\}$, and $\{i_5, i_6\}$, where $i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i_1$ in clockwise cyclic order. Let

$$X = \{i \in [n] : i_2 < i < i_3\} \qquad Y = \{i \in [n] : i_4 < i < i_5\} \qquad Z = \{i \in [n] : i_6 < i < i_1\},$$

using the clockwise cyclic order $1 \leqslant 2 \leqslant \cdots \leqslant n \leqslant 1$. By construction, the hypotheses of Lemma 3.1 are satisfied, and so

$$d_T = 2 \cdot d_{X\cup S_1;\mathcal{V}_X} \cdot d_{Y\cup S_2;\mathcal{V}_Y} \cdot d_{Z\cup S_3;\mathcal{V}_Z}.$$

In other words,

$$d_T = 2 \cdot d_{T_X} \cdot d_{T_Y} \cdot d_{T_Z}, \tag{14}$$

where T_X (resp. T_Y, T_Z) is the triangulation of an $(|X| + 4)$-gon (resp. $(|Y| + 4)$-gon, $(|Z| + 4)$-gon) formed by cutting T along the edges of Δ other than S_1 (resp. S_2, S_3) discarding the part not containing Δ, and renaming the cut edges with $\{i_1, i_4\}$ (resp. $\{i_3, i_6\}$, $\{i_5, i_2\}$), as in Figure 2. Note that $|X| + 4 < n$, since $[n] = \{i_1, \ldots, i_6\} \sqcup X \sqcup Y \sqcup Z$, and similar $|Y| + 4 < n$ and $|Z| + 4 < n$. Thus, we have $d_{T_X} = 2^{I(T_X)}$ by the inductive hypothesis, and similarly $d_{T_Y} = 2^{I(T_Y)}$ and $d_{T_Z} = 2^{I(T_Z)}$. Thus, (14) implies

$$d_T = 2^{I(T_X)+I(T_Y)+I(T_Z)+1}.$$

On the other hand, every internal triangle of T except Δ appears as an internal triangle in exactly one of T_X, T_Y, or T_Z, and all internal triangles of T_X, T_Y, and T_Z arise this way. That is, $I(T) = I(T_X) + I(T_Y) + I(T_Z) + 1$, so $d_T = 2^{I(T)}$. □

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