Bogoliubov-de Gennes equation on graphs: A model for tree-branched Majorana wire network

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Abstract. We consider Bogoliubov-de Gennes equation on a metric tree graph. Formulation of the problem for arbitrary graph topology is provided. Self-adjoint vertex boundary conditions are derived. Exact solutions of the problem is obtained for quantum tree graph. A quantum graph based model for tree-branched Majorana wire network is proposed.

1. Introduction

Low dimensional branched structures appear in many problems of modern condensed matter physics, as based functional part of, e.g. quantum materials. Describing the quantum dynamics in such structures require developing physically acceptable effective models providing solution for the tunable quantum transport problem. One of the powerful tools for modeling particles and waves in low-dimensional branched systes is using evolution equation on so-called metric graphs. These latter are the one-dimensional wires connected to each other according to some rule, which is called topology of a graph [1, 4].. A length is assigned to each wire. In case, when evolution equation is a quantum mechanical linear equation, such as Schrodinger, Dirac or Bogoliubov de Gennes equation, the graph is called a quantum graph. During past two decades, quantum graphs have attracted much attention in different contexts of contemporary physics (see, e.g., Refs.[1]-[6]). Dynamics of quantum particles and quasi-particles in such systems are described in terms of the one-dimensional quantum mechanical wave equations on metric graphs by imposing the boundary conditions at the branching points (vertices) and at the bond ends. Pioneering treatments of quantum mechanical motion in branched systems has been considered few decades ago in the Refs.[7]-[9]. However, strict treatment of quantum graphs was first presented by Exner and Seba to describe free quantum motion on branched wires [10]. Later Kostrykin and Schrader derived the general boundary conditions providing self-adjointness of the Schrödinger operator on graphs [11]. Bolte and Harrison extended such boundary conditions for the Dirac operator on metric graphs [12]. Hul *et al* considered experimental realization of quantum graphs in optical microwave networks [2]. An important topic related to quantum graphs was studied in the context of quantum chaos theory and spectral statistics [1, 12, 4, 13, 14]. Spectral properties and band structure of periodic quantum graphs also attracted much interest [15, 16]. Different aspects of the Schrödinger operators on graphs have been studied in the Refs.[6, 18, 17, 20, 21].

Despite the rapidly increasing interest to quantum graphs by mathematicians and physicists, within such approach one is restricted by modeling linear wave dynamics only. For modeling of nonlinear waves and soliton dynamics in branched structures one should consider nonlinear wave equations on metric graphs. During the past decade the studies of particle and wave dynamics in branched structures have been extended to nonlinear evolution equations by considering nonlinear Schrödinger and sine-Gordon equations on metric graphs [22]-[33]. For such equations, one should derive the vertex boundary conditions from fundamental conservation laws such as energy, momentum, charge and mass conservation [22, 31]. We note that because of the wide range applications of metric graphs based approach for wave dynamics in branched systems and networks, one can expect further extension of the studies to the case of other evolution equations, too.

In this paper we consider quantum graphs described by Bogoliubov de Gennes (BdG) equation. The latter can be used for modeling of quasiparticle dynamics in superconductors [34] and Majorana fermions in superconducting quantum wires [35]. Here we derive the vertex boundary conditions which provides the self-adjointness of BdG operator on metric graphs. We obtain the exact solutions for such boundary conditions. Motivation for the study of BdG equation on metric graphs comes from several practically important problems of conensed matter physics such as branched Kitaev chains, Majorana wire networks and nanoscale superconducting networks. Majorana fermions in condensed matter arise in some quantum materials in the form of quasipartile excitations which mimic Such approach can be powerful tool for modeling of quasiparticle dynamics in such structures.

The paper is organized as follows. In the next section we give brief description of the BdG equation on a line. Section 3 presents formulation of the problem and its solution for quantum graphs having arbitrary branching architecture. and present the vertex boundary conditions keeping the BdG operator as self-adjoint. Section 4 provides application of the result for modeling of tree-branched Majorana wire networks. Finally, section 6 presents some concluding remarks.

2. Bogoliubov-de Gennes equation on a line

BdG equation can be written as [32]

$$H_{\mathrm{BdG}}\Psi = E\Psi, \tag{1}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T$ and

$$H_{\mathrm{BdG}} = \begin{pmatrix} 0 & -i\frac{\partial}{\partial x} & \Delta_0 & 0 \\ -i\frac{\partial}{\partial x} & 0 & 0 & \Delta_0 \\ \Delta_0 & 0 & 0 & i\frac{\partial}{\partial x} \\ 0 & \Delta_0 & i\frac{\partial}{\partial x} & 0 \end{pmatrix} \tag{2}$$

Solution of Eq.(1) can be written as

$$\begin{aligned}
\Psi_1(x,E) &= C_{11}e^{i\kappa x} + C_{12}e^{-i\kappa x}, \\
\Psi_2(x,E) &= C_{21}e^{i\kappa x} + C_{22}e^{-i\kappa x}, \\
\Psi_3(x,E) &= \frac{E}{\Delta_0}C_{11}e^{i\kappa x} + \frac{E}{\Delta_0}C_{12}e^{-i\kappa x} - \frac{\kappa}{\Delta_0}C_{21}e^{i\kappa x} + \frac{\kappa}{\Delta_0}C_{22}e^{-i\kappa x}, \\
\Psi_4(x,E) &= -\frac{\kappa}{\Delta_0}C_{11}e^{i\kappa x} + \frac{\kappa}{\Delta_0}C_{12}e^{-i\kappa x} + \frac{E}{\Delta_0}C_{21}e^{i\kappa x} + \frac{E}{\Delta_0}C_{22}e^{-i\kappa x},
\end{aligned} \tag{3}$$

where $\kappa = \sqrt{E^2 - \Delta_0^2}$ and $C_{11}, C_{12}, C_{21}, C_{22}$ are constants which can be found, e.g. from the normalization and boundary conditions. Current for such system is determined as

$$J = \Psi^* \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \Psi, \tag{4}$$

where σ_x is the Pauli matrix.

3. BdG equation on arbitrary graph

In this section we consider the BdG equation on arbitrary graph topology. We will use notations and approach on quantum graphs provided in the Refs. [1, 4]. Graph consists of B bonds and connected at V vertices. The valency v_i of a vertex is number bonds meeting at ith vertex. We define the bond as $b = (\min(i,j), \max(i,j))$, if i and j vertices are connected.

Now we would like to introduce adjacency matrix as

$$C_{i,j} = C_{j,i} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are connected} \\ 0, & \text{otherwise} \end{cases}, \quad i,j = 1,...,V. \tag{5}$$

By using the adjacency matrix we can find valency and the number of bonds as

$$v_i = \sum_{j=1}^{V} C_{i,j}, \quad \text{and} \quad B = \sum_{i,j=1}^{V} C_{i,j}. \tag{6}$$

We assign a coordinate $x_{i,j}$ for bond $b = (i,j)$ as $(0, L_{i,j})$ when $i < j$.

Now we will extend BdG equation arbitrary graph topology. First of all, we need to write Eq. (1) on the each bond of the graph as [32]

$$H_{\text{BdG}} \Psi^{i,j} = E \Psi^{i,j}, \tag{7}$$

where $\Psi^{(i,j)} = \left(\Psi_1^{(i,j)}, \Psi_2^{(i,j)}, \Psi_3^{(i,j)}, \Psi_4^{(i,j)}\right)^T$. Using the notations

$$\begin{aligned}
\Psi_1^{(i)} := \Big(&\Psi_1^{(i,1)}(0); ...; \Psi_1^{(i,V)}(0); \Psi_3^{(i,1)}(0); ...; \Psi_3^{(i,V)}(0); \\
&\Psi_1^{(i,1)}(L_{i,1}); ...; \Psi_1^{(i,V)}(L_{i,V}); \Psi_3^{(i,1)}(L_{i,1}); ...; \Psi_3^{(i,V)}(L_{i,V})\Big)^T, \\
\Psi_2^{(i)} := \Big(&\Psi_2^{(i,1)}(0); ...; \Psi_2^{(i,V)}(0); -\Psi_4^{(i,1)}(0); ...; -\Psi_4^{(i,V)}(0); \\
&-\Psi_2^{(i,1)}(L_{i,1}); ...; -\Psi_2^{(i,V)}(L_{i,V}); \Psi_4^{(i,1)}(L_{i,1}); ...; \Psi_4^{(i,V)}(L_{i,V})\Big)^T,
\end{aligned}$$

and

$$\Psi_1 = \left(\Psi_1^{(1)}; ...; \Psi_1^{(V)}\right)^T, \quad \Psi_2 = \left(\Psi_2^{(1)}; ...; \Psi_2^{(V)}\right)^T, \tag{8}$$

we write vertex boundary condition as

$$\mathbf{A}\Psi_1 + \mathbf{B}\Psi_2 = 0, \tag{9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $4V^2 \times 4V^2$ matrices.

Solution of Eq. (7) for positive energy can be written as

$$\begin{aligned}
\Psi^{(i,j)}(x) = &C_{i,j}\mu_\alpha^{(i,j)} \begin{pmatrix} 1 \\ 0 \\ \frac{E}{\Delta_0} \\ -\frac{\kappa}{\Delta_0} \end{pmatrix} e^{i\kappa x} + C_{i,j}\mu_\beta^{(i,j)} \begin{pmatrix} 0 \\ 1 \\ -\frac{\kappa}{\Delta_0} \\ \frac{E}{\Delta_0} \end{pmatrix} e^{i\kappa x} \\
&+ C_{i,j}\hat{\mu}_\alpha^{(i,j)} \begin{pmatrix} 1 \\ 0 \\ \frac{E}{\Delta_0} \\ \frac{\kappa}{\Delta_0} \end{pmatrix} e^{-i\kappa x} + C_{i,j}\hat{\mu}_\beta^{(i,j)} \begin{pmatrix} 0 \\ 1 \\ \frac{\kappa}{\Delta_0} \\ \frac{E}{\Delta_0} \end{pmatrix} e^{-i\kappa x}.
\end{aligned} \tag{10}$$

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For the negative energy

$$\Psi^{(i,j)}(x) = C_{i,j}\mu_\alpha^{(i,j)}\begin{pmatrix} -\frac{E}{\Delta_0} \\ \frac{\kappa}{\Delta_0} \\ 1 \\ 0 \end{pmatrix} e^{i\kappa x} + C_{i,j}\mu_\beta^{(i,j)}\begin{pmatrix} \frac{\kappa}{\Delta_0} \\ -\frac{E}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{i\kappa x}$$
$$+ C_{i,j}\hat{\mu}_\alpha^{(i,j)}\begin{pmatrix} -\frac{E}{\Delta_0} \\ -\frac{\kappa}{\Delta_0} \\ 1 \\ 0 \end{pmatrix} e^{-i\kappa x} + C_{i,j}\hat{\mu}_\beta^{(i,j)}\begin{pmatrix} -\frac{\kappa}{\Delta_0} \\ -\frac{E}{\Delta_0} \\ 0 \\ 1 \end{pmatrix} e^{-i\kappa x}. \quad (11)$$

The vertex boundary conditions for these solutions can be written for positive and negative energies respectively as

$$(\mathbf{A}\Theta_1 + \mathbf{B}\Theta_2)\begin{pmatrix} \mu_\alpha \\ \mu_\beta \\ \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix} = 0, \quad (\mathbf{A}\Theta_3 + \mathbf{B}\Theta_4)\begin{pmatrix} \mu_\alpha \\ \mu_\beta \\ \hat{\mu}_\alpha \\ \hat{\mu}_\beta \end{pmatrix} = 0, \quad (12)$$

where

$$\Theta_m = \begin{pmatrix} \Theta_m^{(1)} & & 0 \\ & \ddots & \\ 0 & & \Theta_m^{(V)} \end{pmatrix}, \quad m = 1,2,3,4 \quad (13)$$

and

$$\Theta_1^{(i)} = \begin{pmatrix} I_V & 0 & I_V & 0 \\ \frac{E}{\Delta_0}I_V & -\frac{\kappa}{\Delta_0}I_V & \frac{E}{\Delta_0}I_V & \frac{\kappa}{\Delta_0}I_V \\ e^{i\kappa L_i} & 0 & e^{-i\kappa L_i} & 0 \\ \frac{E}{\Delta_0}e^{i\kappa L_i} & -\frac{\kappa}{\Delta_0}e^{i\kappa L_i} & \frac{E}{\Delta_0}e^{-i\kappa L_i} & \frac{\kappa}{\Delta_0}e^{-i\kappa L_i} \end{pmatrix},$$

$$\Theta_2^{(i)} = \begin{pmatrix} 0 & I_V & 0 & I_V \\ \frac{\kappa}{\Delta_0}I_V & -\frac{E}{\Delta_0}I_V & -\frac{\kappa}{\Delta_0}I_N & -\frac{E}{\Delta_0}I_N \\ 0 & -e^{i\kappa L} & 0 & e^{-i\kappa L} \\ -\frac{\kappa}{\Delta_0}e^{i\kappa L} & \frac{E}{\Delta_0}e^{i\kappa L} & \frac{\kappa}{\Delta_0}e^{-i\kappa L} & \frac{E}{\Delta_0}e^{-i\kappa L} \end{pmatrix},$$

$$\Theta_3^{(i)} = \begin{pmatrix} -\frac{E}{\Delta_0}I_V & \frac{\kappa}{\Delta_0}I_V & -\frac{E}{\Delta_0}I_V & -\frac{\kappa}{\Delta_0}I_V \\ I_V & 0 & I_V & 0 \\ -\frac{E}{\Delta_0}e^{i\kappa L_i} & \frac{\kappa}{\Delta_0}e^{i\kappa L_i} & -\frac{E}{\Delta_0}e^{-i\kappa L_i} & -\frac{\kappa}{\Delta_0}e^{-i\kappa L_i} \\ e^{i\kappa L_i} & 0 & e^{-i\kappa L_i} & 0 \end{pmatrix},$$

$$\Theta_4^{(i)} = \begin{pmatrix} \frac{\kappa}{\Delta_0}I_V & -\frac{E}{\Delta_0}I_V & -\frac{\kappa}{\Delta_0}I_V & -\frac{E}{\Delta_0}I_V \\ 0 & -I_V & 0 & -I_V \\ -\frac{\kappa}{\Delta_0}e^{i\kappa L_i} & \frac{E}{\Delta_0}e^{i\kappa L_i} & \frac{\kappa}{\Delta_0}e^{-i\kappa L_i} & \frac{E}{\Delta_0}e^{-i\kappa L_i} \\ 0 & e^{i\kappa L_i} & 0 & e^{-i\kappa L_i} \end{pmatrix}.$$

Eq. (12) leads to quantization conditions for finding the eigenvalues from the following secular equations for positive and negative energies, respectively as

$$\det(\mathbf{A}\Theta_1 + \mathbf{B}\Theta_2) = 0, \quad \det(\mathbf{A}\Theta_3 + \mathbf{B}\Theta_4) = 0 \quad (14)$$

Here $e^{i\kappa L_i} = \mathrm{diag}\{e^{i\kappa L_{i,1}}, \dots, e^{i\kappa L_{i,N}}\}$, $\mu_{\alpha,\beta} = \{\mu_{\alpha,\beta}^{(1)}, \dots, \mu_{\alpha,\beta}^{(N)}\}$, $\hat{\mu}_{\alpha,\beta} = \{\hat{\mu}_{\alpha,\beta}^{(1)}, \dots, \hat{\mu}_{\alpha,\beta}^{(N)}\}$, $\mu_{\alpha,\beta}^{(i)} = \{\mu_{\alpha,\beta}^{(i,1)}, \dots, \mu_{\alpha,\beta}^{(i,N)}\}$, $\hat{\mu}_{\alpha,\beta}^{(i)} = \{\hat{\mu}_{\alpha,\beta}^{(i,1)}, \dots, \hat{\mu}_{\alpha,\beta}^{(i,N)}\}$, I_V is the identity matrix with the Vth order.

4. Majorana wire networks

The above study deals non-zero energy ($E \neq 0$) solutions of BdG equation on metric graph. An important case having application in topological states of condensed matter is described by zero-energy solutions of BdG. Such solutions describe bound states of the Majorana fermions on a quantum wire (so-called Majorana wires) [35, 39, 40] which are localized at the ends of the wire. Majorana particles in quantum wires are considered as fixed (immobile), i.e. they do not carry any current [35, 39]. However, one can achieve current carrying regime by constructing Y-, or T-junctions of Majorana wires, or more complicated branching topologies [39]-[47]. It should be noted that previous studies do not use BdG equation and quantum graph based approach for describing of Majorana wire networks (see, e.g., [39]-[47] for review). One of the advantages of the quantum graph based approach is that one can apply vertex boundary conditions for "braiding" of Majorana wires. This allows to use different physical mechanisms for such braiding. Therefore, modeling the Majorana wire networks can be effectvely done in terms of BdG equation on metric graphs. Thus the problem we want to address is the BdG equation on quantum graphs by focusing on its zero-energy solutions. Earlier it was done for star branched Majorana network in [32]. Here we will briefly recall this approach, following the Ref.[32]. Consider BdG equation on quantum graph, which is given by

$$H_{\mathrm{BdG}}\Psi^{(i,j)} = 0, \tag{15}$$

where the spinor $\Psi^{(j)}$ has (for Majorana fermions) the Nambu structure which is given as [35, 39]

$$\Psi^{(i,j)}(x) = \left(\Psi_{\uparrow}^{(i,j)}, \Psi_{\downarrow}^{(i,j)}, \Psi_{\downarrow}^{(i,j)*}, -\Psi_{\uparrow}^{(i,j)*}\right)^T.$$

Eq.(15) describes Majorana wire networks. Such networks have attracted much attention during last few years (see, e.g., Refs.[41]-[47]).

We will focus on finding specific solutions of the BdG equation metric graphs for zero energy. Here we impose the vertex boundary conditions providing continuity and current conservation. General solution of Eq.(15) (for Nambu spinor) can be written as

$$\begin{aligned}\Psi^{(i,j)}(x) =& \mu_\alpha^{(i,j)} \begin{pmatrix} q^* \\ 0 \\ 0 \\ -q \end{pmatrix} e^{-\Delta_0 x} + \mu_\beta^{(i,j)} \begin{pmatrix} 0 \\ q \\ q^* \\ 0 \end{pmatrix} e^{-\Delta_0 x} \\ &+ \hat{\mu}_\alpha^{(i,j)} \begin{pmatrix} q \\ 0 \\ 0 \\ -q^* \end{pmatrix} e^{\Delta_0 x} + \hat{\mu}_\beta^{(i,j)} \begin{pmatrix} 0 \\ q^* \\ q \\ 0 \end{pmatrix} e^{\Delta_0 x}.\end{aligned} \tag{16}$$

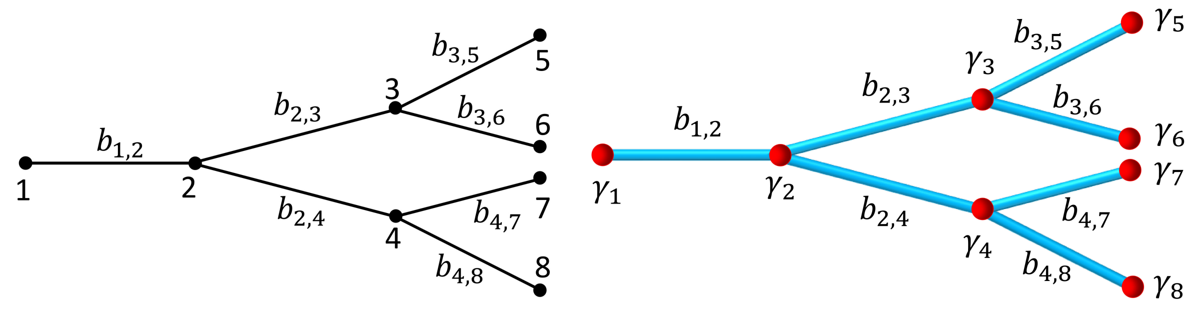
where $q = 1 + i$.

We choose the following vertex boundary conditions:

$$\Psi_k^{(i,j)}(x)|_{x=0} = \phi_k^{(i)}, \quad \Psi_k^{(i,j)}(x)|_{x=L_{i,j}} = \phi_k^{(j)}, \quad \text{for} \quad i<j \quad \text{and} \quad C_{i,j} \neq 0, \tag{17}$$

$$\sum_{j>i} C_{i,j} \frac{d}{dx}\Psi_k^{(i,j)}(x)|_{x=0} - \sum_{j<i} C_{i,j} \frac{d}{dx}\Psi_k^{(i,j)}(x)|_{x=L_{i,j}} = 0, \tag{18}$$

where $k = 1, 2, 3, 4$ and $\phi_k^{(i)}$ are the values of the wavefunction on the ith vertex. Let us consider the following relations: $\hat{\mu}_{\alpha,\beta}^{(i,j)} = \mu_{\alpha,\beta}^{(i,j)*}$, $\Delta_0 = q$. Substituting the Eq. (16) to Eq. (17) leads to the system of algebraic equation with respect to $\hat{\mu}_{\alpha,\beta}^{(i,j)}$ and $\mu_{\alpha,\beta}^{(i,j)}$. Since this system of equations has a non-trivial solution, we can find q or Δ_0.



(a) Tree graph (b) Majorana wire network

Figure 1: Tree graph based Majorana wire network

4.1. Tree branched Majorana wire networks

Here we will extend the model of Majorana wire network for the case of tree-like network. This can be done by considering BdG equation on a tree graph (see Fig. 1a). The graph has three branching points. We set the $x_{i,j}$ coordinates from left to right in this figure. Adjancency matrix for the above tree graph can be written as

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

Majorana wire network corresponding to such graph is presented in Fig.1b.
Consider BdG equation with zero energy on quantum tree graph, which is given by

$$H_{\rm BdG}\Psi^{(i,j)} = 0, \tag{20}$$

where $(i, j) = \{(1, 2), (2, 3), (2, 4), (3, 5), (3, 6), (4, 7), (4, 8)\}$. Boundary conditions at the branching points following from general conditions given by Eq. (17), can be written as

$$\begin{cases} \frac{d}{dx}\Psi_k^{(1,2)}|_{x=0} = 0, \quad \frac{d}{dx}\Psi_k^{(3,5)}|_{x=L_{3,5}} = 0, \\ \frac{d}{dx}\Psi_k^{(3,6)}|_{x=L_{3,6}} = 0, \quad \frac{d}{dx}\Psi_k^{(4,7)}|_{x=L_{4,7}} = 0, \quad \frac{d}{dx}\Psi_k^{(4,8)}|_{x=L_{4,8}} = 0, \end{cases} \tag{21}$$

$$\begin{cases} \Psi_k^{(1,2)}|_{x=L_{1,2}} = \Psi_k^{(2,3)}|_{x=0} = \Psi_k^{(2,4)}|_{x=0} = \phi_2, \\ \frac{d}{dx}\Psi_k^{(1,2)}|_{x=L_{1,2}} + \frac{d}{dx}\Psi_k^{(2,3)}|_{x=0} + \frac{d}{dx}\Psi_k^{(2,4)}|_{x=0} = \lambda_2\phi_2, \end{cases} \tag{22}$$

$$\begin{cases} \Psi_k^{(2,3)}|_{x=L_{2,3}} = \Psi_k^{(3,5)}|_{x=0} = \Psi_k^{(3,6)}|_{x=0} = \phi_3, \\ \frac{d}{dx}\Psi_k^{(2,3)}|_{x=L_{2,3}} + \frac{d}{dx}\Psi_k^{(3,5)}|_{x=0} + \frac{d}{dx}\Psi_k^{(3,6)}|_{x=0} = \lambda_3\phi_3, \end{cases} \tag{23}$$

$$\begin{cases} \Psi_k^{(2,4)}|_{x=L_{2,4}} = \Psi_k^{(4,7)}|_{x=0} = \Psi_k^{(4,8)}|_{x=0} = \phi_4, \\ \frac{d}{dx}\Psi_k^{(2,4)}|_{x=L_{2,4}} + \frac{d}{dx}\Psi_k^{(4,7)}|_{x=0} + \frac{d}{dx}\Psi_k^{(4,8)}|_{x=0} = \lambda_4\phi_4. \end{cases} \tag{24}$$

Zero-energy solution of BdG equation fulfilling these conditions can be written as

$$\Psi^{(i,j)}(x) = \mu_\alpha^{(i,j)} \begin{pmatrix} q^* \\ 0 \\ 0 \\ -q \end{pmatrix} e^{-\Delta_0 x} + \mu_\beta^{(i,j)} \begin{pmatrix} 0 \\ q \\ q^* \\ 0 \end{pmatrix} e^{-\Delta_0 x} + \hat{\mu}_\alpha^{(i,j)} \begin{pmatrix} q \\ 0 \\ 0 \\ -q^* \end{pmatrix} e^{\Delta_0 x} + \hat{\mu}_\beta^{(i,j)} \begin{pmatrix} 0 \\ q^* \\ q \\ 0 \end{pmatrix} e^{\Delta_0 x}. \quad (25)$$

We note that the above approach can be directly extended to other branching topologies that allows modeling of Majorana wire networks of different branching architectures.

5. Conclusions

In this paper we studied the problem of BdG equation on quantum graphs with the focus on tree-like graphs. Solutions of the problem providing self-adjointness of the BdG operator on graph of arbitrary graph branching topology is obtained. Application of the results for modelling of tree-branched Majorana wire networks is presented. The proposed approach can be used for describing Majorana wire networks and branched Kitaev chains having different branching architectures. Extension of the results for the case of discrete BdG equation on quantum graphs is under progress.

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