Editors' Suggestion

Fate of stringy noninvertible symmetries

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Noninvertible symmetries in quantum field theory (QFT) generalize the familiar product rule of groups to a more general fusion rule. In many cases, gauged versions of these symmetries can be regarded as dual descriptions of invertible gauge symmetries. One may ask: are there any other types of noninvertible gauge symmetries? In theories with gravity we find a new form of noninvertible gauge symmetry that emerges in the limit of fundamental, tensionless strings. These stringy noninvertible gauge symmetries appear in standard examples such as non-Abelian orbifolds. Moving away from the tensionless limit always breaks these symmetries. We also find that both the conventional form of noninvertible gauge symmetries and these stringy generalizations are realized in AdS/CFT. Although generically broken, approximate noninvertible symmetries have implications for swampland constraints: in certain cases they can be used to prove the existence of towers of states related to the distance conjecture, and can sometimes explain the existence of slightly subextremal states which fill in the gaps in the sublattice weak gravity conjecture.

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I. INTRODUCTION

Symmetries play an important role in constraining the dynamics of quantum systems. This is especially true in the case of an unbroken symmetry, where we can derive exact selection rules, but it also applies in situations where the symmetry breaking is controlled by a small parameter.

Recently a number of investigations have suggested a generalization of global symmetries in quantum field theory (QFT) beyond the more familiar grouplike composition rule. In this broader setting of *noninvertible symmetry*, one defines a symmetry in terms of a topological operator that links with the charged object of interest [1]. The product of two such topological operators might end up realizing a more general fusion rule such as

$$\mathcal{N}_i \otimes \mathcal{N}_j = \sum_k \mathcal{T}_{ij}^k \mathcal{N}_k, \tag{1.1}$$

where the $\mathcal{T}_{ij}^k$ denote c-number coefficients.[1] There are by now many examples of this sort, such as in 2D rational conformal field theories (CFTs) with noninvertible Verlinde lines [2], in 4D gauge theories with a gauged charge conjugation symmetry [3], as well as in many other contexts.[2]

What becomes of these noninvertible symmetries in quantum gravity? On general grounds one expects that unbroken symmetries are "gauged," namely they instead specify a redundancy in physical configurations. In the context of QFT, where both the spacetime metric and

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[1]In general, $\mathcal{T}_{ij}^k$ is the partition function of a decoupled TQFT.
[2]The literature has substantially grown in the past few years. For reasonably up to date reviews, see e.g., the reviews [4–9] and references therein.

topology are not dynamical, there is a notion of "gauging a noninvertible symmetry" by inserting a mesh of topological operators (see e.g., [10–14]). We can view this as producing a notion of noninvertible gauge theory.[3] Based on this, it is natural to ask whether we can produce examples of this sort of noninvertible symmetries directly in quantum gravity. Here we find a few surprises, both from the point of view of world sheet constructions, and also from the perspective of the AdS/CFT correspondence.

To begin, recall that invertible global symmetries of a string world sheet theory correspond to gauge symmetries in the target space [16]. From this perspective, it would seem natural to expect that all we require to achieve a gauged, noninvertible spacetime symmetry is an example of a noninvertible global symmetry in the 2D world sheet CFT. Such noninvertible world sheet symmetries were recently discussed in the context of string theory in [17]. As a simple example, consider an orbifold as specified by a non-Abelian group Γ; the operation of gauging a global Γ symmetry on the world sheet results in an orbifold theory with Hilbert space sectors labeled by conjugacy classes of Γ (see Refs. [18,19]). In categorical terms, gauging the Γ symmetry results in a "magnetic" global zero-form categorical symmetry $\mathrm{Rep}(\Gamma)$.[4] In this case, symmetry operators are labeled by representations of Γ and we get a nontrivial fusion rule whenever Γ is non-Abelian (more than one summand can appear in the fusion of two irreducible representations of Γ). So, the appearance of this noninvertible *global* symmetry in the 2D CFT would seem to suggest the existence of a corresponding noninvertible *gauge* symmetry in the target space.

But string theory is more than just a 2D CFT; it also involves coupling this system to 2D gravity! Moreover, it is well-known that the selection rules arising from $\mathrm{Rep}(\Gamma)$ symmetry are violated at higher-loop order [23,24], leaving only the selection rules corresponding to representations $\mathrm{Rep}(\Gamma_{\mathrm{ab}})$ of the Abelianization, an invertible symmetry. So, while tree level string theory appears to enjoy a noninvertible $\mathrm{Rep}(\Gamma)$ symmetry, it is broken by $g_s \neq 0$ effects. This is an example of a more general phenomenon [25]: when a QFT in any dimension is coupled to semiclassical gravity, the appearance of nontrivial topologies in the gravitational path integral leads to a generic breaking of all noninvertible symmetries in the absence of extreme cancellations. We also present a similar construction of noninvertible symmetries broken at $g_s \neq 0$ for the case of string theory on toroidal orbifolds.

The lesson we draw is that the target space physics of noninvertible world sheet symmetries does not correspond to the QFT notion of noninvertible gauge symmetry coming from "summing over a mesh of topological operators", since the selection rules that would be exact in such a gauge theory are explicitly broken by string loops. Indeed, to see the noninvertible world sheet symmetry emerge we must take $g_s \to 0$ and consider the entire structure of perturbative string theory, taking us well outside the regime of local effective field theory. We will describe the target space physics of noninvertible world sheet symmetries as "stringy noninvertible gauge symmetry," which is generically Higgsed, but which can be restored in a limit where the string becomes tensionless in Planck units.

Given this state of affairs, it is natural to ask whether the breaking of this sort of noninvertible gauge symmetry at $g_s \neq 0$ is merely an artifact of this specific class of examples, or is something that holds more generally in quantum gravity. Along these lines, we consider another context where noninvertible gauge symmetries seem easy to realize in quantum gravity: examples from holography in which the CFT of an AdS/CFT pair enjoys a noninvertible symmetry. Indeed, recently there has been progress in realizing examples of noninvertible symmetries in a number of stringy and holographic constructions [26–37]. From this perspective, a noninvertible *global* symmetry of the boundary theory would seem to automatically imply the existence of a *gauged* noninvertible symmetry in the bulk.

In all examples with a semiclassical bulk, this is indeed the case, as has previously been discussed in the literature: the bulk contains a topological sector described by the symmetry topological field theory (SymTFT) (see, e.g., [10,38–53]). For a noninvertible global symmetry on the boundary, the corresponding SymTFT can be understood as a noninvertible bulk gauging of the boundary symmetry in the conventional sense.[5] However, it is also worth noting that in every example we consider, the sense in which the gauge symmetry in the bulk is noninvertible is rather benign: this topological sector admits a more conventional characterization as an invertible gauge theory, possibly after switching to a dual basis of fields. If the bulk quantum gravity theory admits a world sheet description, the invertible gauge theory presentation is more natural, as it is the one that acts most naturally on the string world sheet.

A particularly instructive example in this regard is the background $\mathrm{AdS}_3 \times S^3 \times T^4$. For tuned values of the T^4 moduli, we can have a non-Abelian symmetry Γ acting on T^4, leading to a Γ gauge theory propagating in the $\mathrm{AdS}_3 \times S^3$ factor. For suitable boundary conditions we get a $\mathrm{Rep}(\Gamma)$ symmetry in the boundary CFT_2, and the bulk Γ gauge theory in 6D could dually be viewed as a 3-form $\mathrm{Rep}(\Gamma)$

[3]See also [15] for a recent discussion of the sense in which this procedure can really be viewed as producing a "gauge theory."

[4]See Refs. [12,20], and for a complementary perspective see e.g., Refs. [21,22].

[5]This is always true in the sense of "summing over a mesh of topological operators," which are given by condensates of the gapped boundary condition (see, e.g., [10–12]). We expect further that the very recent work [15] for 3D SymTFTs generalizes to any dimension, so any SymTFT can be viewed as a gauge theory for the higher tube algebra [54–58] of the boundary symmetry.

gauge theory (reducing on the S^3 factor, this is a 0-form $\mathrm{Rep}(\Gamma)$ gauge theory in AdS_3). Nevertheless, if we consider the world sheet description of the bulk (or at least the T^4 factor), the symmetry that acts on the world sheet is still just Γ.

Moving beyond semiclassical bulks, we can also consider the limit of $\mathrm{AdS}_3 \times S^3 \times T^4$ with only a single unit of flux, such that the bulk is described by a tensionless string theory [59–61]. In this limit, we expect an enormous noninvertible symmetry to emerge in the CFT: the CFT dual is given by the symmetric orbifold CFT $\mathrm{Sym}^N(T^4)$.[6] Exactly at the symmetric orbifold point, this CFT_2 admits a noninvertible $\mathrm{Rep}(S_N)$ symmetry, where S_N is the symmetric group. We find that the bulk dual of this is not any conventional noninvertible gauge symmetry in $\mathrm{AdS}_3 \times S^3 \times T^4$. Instead, the large N limit $\mathrm{Rep}(S_\infty)$ is realized as a noninvertible global symmetry of the tensionless string theory world sheet, which remains unbroken as a result of the extreme cancellations that appear in the tensionless limit. We do not have a general characterization of which noninvertible symmetries of holographic CFTs are realized as conventional noninvertible bulk gauge symmetries or as stringy noninvertible gauge symmetries, but it is worth noting that the quantum dimensions of topological operators for $\mathrm{Rep}(S_N)$ symmetry scale with N, while the $\mathrm{Rep}(\Gamma)$ symmetry does not.[7]

The common theme in these examples is that in quantum gravity, less benign forms of gauged noninvertible symmetries are generically broken and only seem to emerge in special limits in field space (like the tensionless string limit $g_s \to 0$). This implies that the breaking effects become suppressed in these limits, so they appear as *approximate symmetries* in the effective field theory. Therefore, despite being broken, they can still have interesting applications in the string landscape. In particular, they fit rather well with a number of related swampland considerations connected with infinite distance limits (see Refs. [62–65] for reviews). The most important result is that the presence of a noninvertible symmetry in the world sheet combined with modular invariance implies the existence of an infinite tower of states which is charged under the noninvertible symmetry and becomes light at infinite distance, as predicted by the distance conjecture. In these cases this allows us to generalize the usual world sheet proof of the weak gravity conjecture [66–68] to the case in which the tower is not charged under a massless gauge field. We will also see that in these cases the approximate noninvertible symmetries provide a complementary perspective on a number of subtle examples for several swampland conjectures. In particular, they can sometimes explain the existence of slightly subextremal states which fill in the gaps in the sublattice weak gravity conjecture. Moreover, in certain cases, the existence of 4D $\mathcal{N} = 2$ theories with properties more analogous to theories with $\mathcal{N} = 4$ supersymmetry can be reinterpreted in terms of the existence of a noninvertible symmetry. Even though these features also have other more general explanations not related to noninvertible symmetries, it is satisfying to find a complementary explanation of these connected to symmetry principles, at least in certain cases.

The rest of this paper is organized as follows. In Sec. II we review the fact that noninvertible symmetries of the world sheet CFT are generically broken by string loop effects, and illustrate this effect in a few concrete examples. In Sec. III we discuss examples in AdS/CFT. In Sec. IV we prove the existence of an infinite tower of states charged under the noninvertible symmetry that becomes light at infinite field distance. We then explain the different applications of the weakly broken noninvertible symmetry for several swampland considerations. We present a broader discussion and some avenues for future investigation in Sec. V.

II. NONINVERTIBLE SYMMETRY BREAKING BY STRING LOOPS

In this section, we argue that any noninvertible symmetries of the world sheet CFT are generically broken by string loop effects, in the absence of conspiracies. More concretely, we review the well-known fact that the selection rules placed on sphere correlation functions by noninvertible symmetry fail to hold at higher genus. A simple example is the energy operator ε of the 2D Ising model: while ε is charged under the noninvertible Kramers-Wannier symmetry, it picks up a nonzero one-point function on Riemann surfaces of positive genus [69–72].

As a result of this breaking effect, if we use string perturbation theory to compute some scattering process forbidden at tree level by a noninvertible symmetry, we will generically pick up nonzero contributions at higher order in the string coupling g_s. From the perspective of target space physics, this means that the noninvertible gauge symmetry is only visible as an approximate symmetry for g_s small. In contrast, any invertible symmetry of the world sheet is preserved to all orders in g_s. Note that in Planck units, the limit $g_s \to 0$ in flat space corresponds to the tensionless limit of the string.

This section is organized as follows. First, we briefly review the general form of tree-level selection rules imposed by noninvertible symmetry derived in [73], and explain why these selection rules can fail to hold at higher genus. We then illustrate this effect in examples of

[6] If we choose boundary conditions that realize $\mathrm{Rep}(\Gamma)$ symmetry, we should strictly speaking discuss the Γ orbifold $\mathrm{Sym}^N(T^4)/\Gamma$.

[7] It is not enough to have the size of the symmetry scale with N: for example, even the invertible $\mathbb{Z}_N$ center 1-form symmetry of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills scales with N. Note also that in orbifolds based on the D_N-series of finite subgroups in $SU(2)$, the order of the group can be parametrically large, but the dimensions of irreducible representations remains small so that the fusion rule is still rather benign.

noninvertible symmetries in familiar string compactifications. Finally, we comment on the story for a general noninvertible symmetry of a 2D CFT.

A. Selection rules from topological operators

As stated in the Introduction, the modern understanding of symmetries in QFT is based around the notion of topological extended operators. In this section, we will focus on 0-form symmetries of 2D CFTs, which are generated by topological defect lines (TDLs). For a comprehensive discussion of TDLs in 2D CFTs, see Ref. [74].

Let us recall the standard derivation of selection rules for an invertible symmetry using the associated TDL $\mathcal{U}$. Consider a sphere correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ of local operators transforming as $\mathcal{O}_i \to e^{iq_i}\mathcal{O}_i$ under the action of $\mathcal{U}$. We can nucleate a small loop of $\mathcal{U}$, pass it through the various operators and then annihilate it "at infinity," leading to the same correlation function weighted by the sum of the charges:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = e^{i\sum_i q_i} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \tag{2.1}$$

If the sum of charges is nonzero, the correlator must vanish, and so we have derived a selection rule from the presence of an invertible symmetry.

What would happen if we tried to perform the same argument for a noninvertible TDL $\mathcal{N}$? First of all, when we nucleate a loop of $\mathcal{N}$, we would pick up a factor of the quantum dimension $\langle \mathcal{N} \rangle$; however, this factor will cancel when we annihilate $\mathcal{N}$ at infinity, so we ignore it. More interestingly, as we sweep $\mathcal{N}$ past any local operators, we might leave behind some network of TDLs, producing a correlation function of both local operators and disorder operators, i.e., point operators attached to topological lines (see Fig. 1). If we do produce such a network, then rather than deriving a constraint on a single correlation function, we might instead derive relationships between different correlation functions.

In fact, we could have run into a similar issue when deriving selection rules for invertible symmetries if we had not chosen our local operators to have definite charge, especially if our symmetry group were non-Abelian. In the invertible context, the solution is well known: we should organize our operators $\mathcal{O}_i$ into representations μ_i of our symmetry group. A correlation function can only be nonzero provided the fusion $\mu_1 \otimes \cdots \otimes \mu_n$ of the representations includes a copy of the trivial representation.

This motivates us to organize our operators into "representations" of the action of the noninvertible symmetry. Importantly, a given "representation" might involve both local operators and disorder operators. In general, if our TDLs form a fusion category $\mathcal{C}$, these charges for our noninvertible symmetry are given by representations of Ocneanu's tube algebra Tube($\mathcal{C}$) (see, e.g., [75]), or equivalently [76,77], by objects μ_i in the Drinfeld center $\mathcal{Z}(\mathcal{C})$. The most general selection rule for noninvertible symmetry tells us that a sphere correlation function involving local operators and disorder operators can only be nonzero if the fusion $\mu_1 \otimes \cdots \otimes \mu_n$ includes the trivial representation [73]. While this abstract characterization is very powerful, we will not use it directly in examples below, and instead describe selection rules on sphere correlation functions on a case-by-case basis.

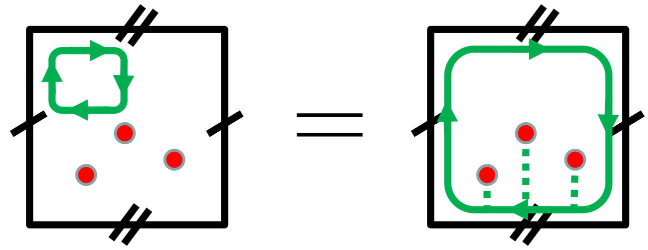

FIG. 1. Attempting to derive selection rules for a noninvertible symmetry on torus correlators. Starting with our correlator, we nucleate a topological line (green), and pass it through the various local operators (red) picking up weights corresponding to their charges. Since the operator is noninvertible, there may be a network of lines (green, dashed) which attach the operators back to the topological operator (see Ref. [74] for a comprehensive discussion). Moreover, once we have pushed our topological line past all the local operator, we still cannot annihilate it, and instead are left with the fusion $\mathcal{N} \otimes \mathcal{N}^\dagger$ wrapped on the two nontrivial cycles of $\mathbb{T}^2$.

What goes wrong with the argument when we consider correlation functions on a more general Riemann surface Σ, such as the torus (see Fig. 1)? Locally, we can proceed as before: we nucleate a loop of our noninvertible TDL $\mathcal{N}$ and pass it through our operators, possibly leaving behind a network of TDLs as before. The issue appears in the final step, where we attempt to annihilate $\mathcal{N}$ "at infinity." In addition to possibly getting caught on our local operators in the correlation function, $\mathcal{N}$ may also get caught on the nontrivial topology of our Riemann surface Σ. Thus, in addition to the network of TDLs connecting our local operators, we pick up a network of the fusion $\mathcal{N} \otimes \mathcal{N}^\dagger$ wrapping the noncontractible cycles of Σ.[8] Note that the fusion $\mathcal{N} \otimes \mathcal{N}^\dagger$ is the identity operator if and only if $\mathcal{N}$ is invertible.

We will return to the meaning of this particular network of TDLs in Sec. V. For now, let us note that the appearance of an additional network of TDLs spoils the derivation of

[8]To make this argument precise for general Σ, choose a Morse function on Σ, and sweep $\mathcal{N}$ down Σ according to the level sets of the Morse function. Each time we pass a Morse critical point of index one, $\mathcal{N}$ will get caught, leaving behind an insertion of the fusion $\mathcal{N} \otimes \mathcal{N}^\dagger$ on the descending manifold. This argument can be generalized to a noninvertible 0-form symmetry in any number of dimensions, where we will leave behind condensates built from $\mathcal{N}$ of various dimension on the descending manifolds of each Morse critical point of index $0 < i < n$.

selection rules, so that whatever selection rules hold for sphere correlation functions need not hold on a general Riemann surface Σ. If we were simply studying 2D CFT, then this effect could be viewed as a mixed gravitational anomaly of any noninvertible symmetry[9]: the selection rules that hold on the sphere are violated in the presence of a background topology. However, in the context of string theory, this violation of selection rules constitutes a genuine breaking of the symmetry, since we have made the world sheet topology dynamical.

B. Example: Non-Abelian orbifolds

Our first class of examples of noninvertible symmetry in the world sheet CFT are provided by non-Abelian orbifolds [18,19]. Suppose we have a 2D CFT with non-Abelian symmetry group Γ, and we form the orbifold CFT by gauging Γ. The twisted sectors are labeled by conjugacy classes $[g] \subset \Gamma$, whose fusion is defined as follows [24]: given conjugacy classes $[g], [h]$, choose representative elements $g \in [g], h \in [h]$,[10] and form the conjugacy class of their product $[gh]$. The fusion $[g] \otimes [h]$ is the sum of all conjugacy classes produced this way for different choices of g, h modulo simultaneous conjugation by Γ.

From this description, we can easily derive selection rules on sphere correlation functions of twisted sector operators. If we have a sphere correlation function

$$\langle \mathcal{O}_{[g_1]} \cdots \mathcal{O}_{[g_n]} \rangle_{\mathbb{S}^2} \tag{2.2}$$

of operators in twisted sectors $[g_i]$, this correlation function can only be nonzero if the fusion $[g_1] \otimes \cdots \otimes [g_n]$ contains the conjugacy class $[1]$ of the identity element. But this is true if and only if we can choose some representatives $g_i \in [g_i]$ such that the product $g_1 \ldots g_n = 1$ in the group Γ.

Let us now re-derive this selection rule using non-invertible topological operators (this is essentially the argument in [23]). In the orbifold theory, we have topological Wilson line operators $\mathcal{W}_\rho$ labeled by representations $\rho \in \mathrm{Rep}(\Gamma)$, which fuse according to the fusion of representations. Thus, the orbifold theory has $\mathrm{Rep}(\Gamma)$ noninvertible symmetry. Given a sphere correlation function (2.2) of twisted sector operators as before, for any representation ρ, we can insert $\mathcal{W}_\rho$ along a loop encircling each of the operator insertions in turn. By annihilating this insertion "at infinity," we learn that the holonomy around the loop must act trivially in ρ. Since ρ was arbitrary, we learn that the holonomy must be the identity element $1 \in \Gamma$. But this holonomy is also the product in Γ of the holonomies around each twisted sector operator, given by some representative elements $g_i \in [g_i]$ that multiply to the identity, so we have recovered the selection rule.

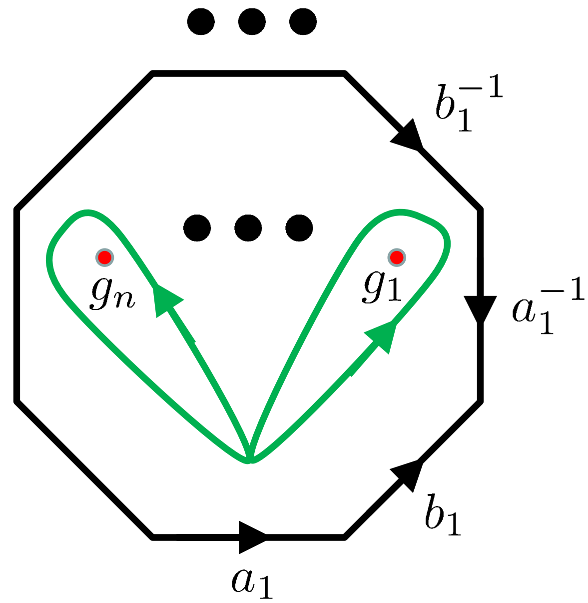

FIG. 2. We can construct a Riemann surface of genus g by gluing the edges of a $4g$-gon in the pattern specified in (2.3). When we encircle a collection of twisted-sector operator insertions (red) with a topological Wilson line (green), we conclude that the holonomy around our collection of operators is a product of g commutators in Γ. See also [23] [Fig. 11].

What happens on a nontrivial Riemann surface Σ of genus g?[11] Following [23], we note that a Riemann surface of genus g can be formed by gluing the edges of a $4g$-gon in the pattern

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \tag{2.3}$$

as depicted in Fig. 2. Now, when we push the topological Wilson line $\mathcal{W}_\rho$ "to infinity," we cannot simply annihilate $\mathcal{W}_\rho$. Instead, we pick up the action in ρ of an element in the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ formed from a product of g commutators $ghg^{-1}h^{-1}$. Thus, for a correlation function

[9]Not to be confused with two distinct notions of anomaly for noninvertible symmetry that have previously been considered. In the case of invertible symmetries, one can consider three equivalent notions: the obstruction to gauging, the obstruction to a trivially gapped phase, or the violation of Ward identities in the presence of a background field. In the case of noninvertible symmetry these notions are different, and the one we mean is the third, where we view the (pseudo)Riemannian manifold on which we place our CFT, possibly decorated with additional tangential structure (spin structure, etc) depending on the theory, as a generalized notion of background gravitational field. For further discussion see Refs. [78,79].

[10]This is a slight abuse of notation, where we use the same symbol g to denote the different possible representatives of its orbit $[g]$ under conjugation. We will continue to use this abuse of notation for representatives of orbits throughout this section for the purpose of readability.

[11]By another abuse of notation we shall use the same letter g to now refer also to a genus. It should be clear from the context which notion is meant.

$$\langle \mathcal{O}_{[g_1]} \cdots \mathcal{O}_{[g_n]} \rangle_\Sigma \tag{2.4}$$

of twisted sector operators to be nonzero, it is enough for the fusion $[g_1] \otimes \cdots \otimes [g_n]$ to contain the conjugacy class of a product of g commutators in G. Note that the conjugacy classes of commutators $[ghg^{-1}h^{-1}]$ are precisely those that appear in fusions $[g] \otimes [g^{-1}]$ of conjugacy classes with their inverses. See, e.g., [25] and references therein for further discussion.

What selection rules are preserved on all Riemann surfaces? In other words, how can we tell if a fusion product $[g_1] \otimes \cdots \otimes [g_n]$ does not contain a conjugacy class in the commutator subgroup $[\Gamma, \Gamma]$, so that (2.4) must vanish at any genus? The answer is straightforward: the fusion product $[g_1] \otimes \cdots \otimes [g_n]$ lands in the commutator subgroup if and only if images of $[g_i]$ in the Abelianization $\Gamma_{\text{ab}} = \Gamma/[\Gamma, \Gamma]$ multiply to the identity. In other words, the conjugacy class $[g_i]$ carries a charge valued in Γ_{ab} given by its image, and these charges must cancel on any Riemann surface.

These charges are, in fact, simply the charges of the operators $\mathcal{O}_i$ under an invertible symmetry [25]. While the $\text{Rep}(\Gamma)$ symmetry is, in general, noninvertible, it contains an invertible subsymmetry, generated by invertible Wilson lines $\mathcal{W}_\rho$ corresponding to one-dimensional representations ρ. These invertible Wilson lines can still be "annihilated at infinity" even on a nontrivial Riemann surface, and so they impose the same selection rules on every Riemann surface. Moreover, the commutator subgroup $[\Gamma, \Gamma]$ must act trivially in any one-dimensional representation, and we have that the set of one-dimensional Wilson lines forms a $\text{Rep}(\Gamma_{\text{ab}}) = \Gamma_{\text{ab}}^\vee$ invertible subsymmetry of our noninvertible $\text{Rep}(\Gamma)$ symmetry, which is the maximal invertible subsymmetry.

This example illustrates an important point: while we expect any noninvertible symmetry to be broken down to its maximal invertible subsymmetry by string loop effects, it is not true that the consequences of this breaking are always entirely visible at one loop. For example, suppose Γ were a group such that some element g_0 in the commutator subgroup could only be written as a product of commutators, but not as a single commutator (such Γ exist, see, e.g., [80] for a source of examples). Then a local operator in the twisted sector $[g_0]$ could not have a nonzero torus partition function, but could have a nonzero partition function on some higher-genus Riemann surface.

However, in general, the set of charges that can get a nonzero one-point function at some order in the string loop expansion is always generated under fusion by the charges that can get a nonzero torus one-point function. This can be seen by realizing that a one-point function on a Riemann surface of genus g can be built by sewing together g torus one-point functions with a single sphere $(g+1)$-point function (see Fig. 3). Each torus one-point function produces some charge, and these charges simply fuse in the sphere $(g+1)$-point function. Thus, whatever symmetry is broken by string loops must be entirely broken at one-loop, even if the nonzero one-point functions of certain charges do not show up until higher loop order.

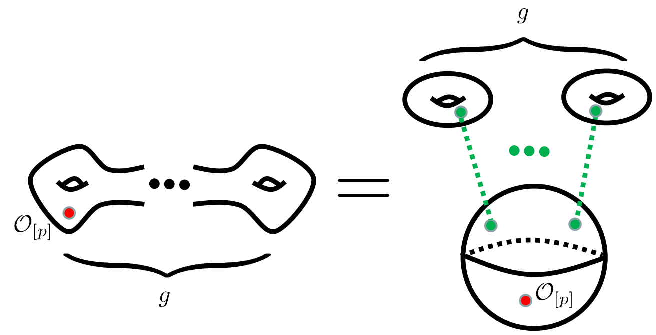

FIG. 3. A one-point function on a Riemann surface of genus g (left) can be built be sewing together g torus one-point functions and one sphere $(g+1)$-point function (right).

C. Example: Toroidal orbifolds

Our next class of examples of noninvertible symmetry in the world sheet CFT are given by toroidal orbifolds that break some part of the translation symmetry. These examples fall into the general class of noninvertible symmetries obtained by gauging a non-normal subgroup of a larger symmetry group (see, e.g., [3,81,82]); similar statements could be made for any of these more general examples, and in fact we will discuss one such generalization below in Sec. IV C. These noninvertible symmetries capture, in the case of toroidal orbifolds, the general perturbative string theory expectation that tree-level scattering amplitudes are independent of the choice of compactification for states whose existence is unchanged by the compactification.[12]

Suppose we have a world sheet CFT containing a T^n sigma model. We will denote the sigma model fields by X^μ, as is standard in string theory. Let us now orbifold by a finite group Γ of isometries of T^n (Γ can be Abelian or non-Abelian). Before orbifolding, the sigma model CFT has a continuous "momentum" symmetry which acts by translation (we could tell a completely analogous story for the "winding" symmetry). Let $\mathcal{U}_{\delta X}$ denote the invertible topological operator implementing a translation $X^\mu \to X^\mu + \delta X^\mu$. In general, $\mathcal{U}_{\delta X}$ will not be preserved by the Γ action, and will be taken to a different translation operator under the action of $g \in \Gamma$. Thus, if we gauge Γ, the operators $\mathcal{U}_{\delta X}$ will no longer be gauge-invariant.

However, while the operators $\mathcal{U}_{\delta X}$ are not individually gauge invariant, they can still be grouped into orbits $[\delta X]$ of

[12]For example the tree-level scattering of graviton amplitudes in 4D is the same as those in 10D at string tree-level, independent of the compactification.

the Γ action.[13] For each orbit $[\delta X]$, we can define a gauge-invariant topological operator by summing over the orbit

$$\mathcal{L}_{[\delta X]} = \bigoplus_{\delta X \in [\delta X]} \mathcal{U}_{\delta X}. \tag{2.5}$$

The quantum dimension $\langle \mathcal{L}_{[\delta X]} \rangle$ is given by the size of the orbit $[\delta X]$.

The collection of operators $\mathcal{L}_{[\delta X]}$ define a noninvertible "momentum" symmetry of the toroidal orbifold T^n/Γ which is the unbroken piece of the full translation symmetry of the unorbifolded theory T^n. The charged operators include (unnormalized) vertex operators:

$$\mathcal{O}_{[p]} = \sum_{p \in [p]} e^{i p_\mu X^\mu}, \tag{2.6}$$

defined by summing plane waves of definite momentum p over a Γ-orbit $[p]$ in order to form a Γ-invariant wave function.

What selection rules does the noninvertible symmetry place on correlation functions

$$\langle \mathcal{O}_{[p_1]} \cdots \mathcal{O}_{[p_n]} \rangle^{T^n/\Gamma}_{\mathbb{S}^2}, \tag{2.7}$$

in the orbifolded theory? The answer is simple: the associated selection rules are merely the selection rules coming from conservation of momentum before orbifolding, because the tree-level correlators of untwisted operators are exactly equal to those in the unorbifolded theory. In more detail, the correlation function (2.7) can only be nonzero if there are representatives $p_i \in [p_i]$ such that

$$p_1 + \cdots + p_n = 0, \tag{2.8}$$

i.e., such that momentum is conserved. In Appendix A, we explain in detail how to re-derive this selection rule using the topological operators (2.5) for the simple $c = 1$ orbifold $S^1/\mathbb{Z}_2$ (see also [83,84]).

Suppose we want to calculate the torus correlation function

$$\langle \mathcal{O}_{[p_1]} \cdots \mathcal{O}_{[p_n]} \rangle^{T^n/\Gamma}_{\mathbb{T}^2} \tag{2.9}$$

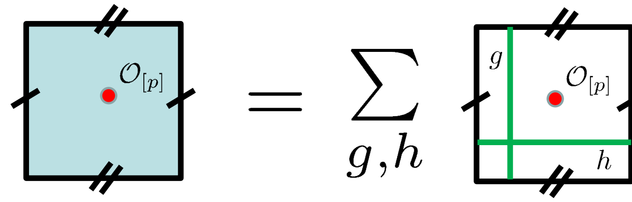

FIG. 4. The torus one-point function (2.10) in the orbifold theory (left, shaded) is computed by summing torus one-point functions in the unorbifolded theory (right) with insertions of Γ symmetry lines (green). The sum runs over all pairs $g, h \in \Gamma$ of commuting elements. Even if the contribution from $g = h = 1$ vanishes, the other terms with nontrivial line insertions may be nonzero. This illustrates that the difference between a theory and its orbifold is simply which topological line insertions are considered to contribute to "vacuum" correlation functions.

in the orbifold theory. For simplicity of the discussion, let us focus on the case of a torus 1-point function

$$\langle \mathcal{O}_{[p]} \rangle^{T^n/\Gamma}_{\mathbb{T}^2} \tag{2.10}$$

Because of the selection rule for the momentum symmetry "upstairs," the torus one-point function $\langle \mathcal{O}_{[p]} \rangle^{T^n}_{\mathbb{T}^2}$ in the unorbifolded theory must vanish if $p \neq 0$. However, in the orbifold theory, the torus one-point function (2.10) involves summing over insertions of commuting pairs of Γ symmetry lines on the two cycles of $\mathbb{T}^2$, as illustrated in Fig. 4. In contrast to the torus one-point function $\langle \mathcal{O}_{[p]} \rangle^{T^n}_{\mathbb{T}^2}$ in the vacuum of the unorbifolded theory, the torus one-point function of $\mathcal{O}_{[p]}$ in the presence of Γ symmetry lines may be nonzero.[14]

To see why this can happen, let us track momentum charge as it flows through the torus with the insertion of a symmetry line for $g \in \Gamma$ (see Fig. 5). Our insertion of $\mathcal{O}_{[p]}$ inserts some momentum $p \in [p]$, which can split into two parts k, $p - k$ running through the two sides of the torus. Before joining, one of the parts, say k, is acted on by the Γ symmetry line, transforming to some other momentum $g(k)$. Finally, the momentum charge running through the two sides meets, and must annihilate by the selection rules in the unorbifolded theory. Thus, we have

$$p - k + g(k) = 0, \quad \text{or,} \quad p = k + g(-k). \tag{2.11}$$

Thus, any operator $\mathcal{O}_{[p]}$ such that $p = k + g(-k)$ for some g and some k could acquire a nonzero torus one-point function. For example, in the $c = 1$ orbifold $S^1/\mathbb{Z}_2$ by $X \mapsto -X$ discussed in Appendix A, the condition (2.11) is equivalent to the condition that p be even. This remaining selection rule corresponds to an unbroken invertible translation symmetry, given by a π-rotation of S^1.

[13]Be careful: the action of Γ on the group of translations is not equal to the action on the sigma model target T^n. If $g \in \Gamma$ acts on sigma model fields as $X^\mu \mapsto \Lambda^\mu_\nu X^\nu + X^\mu_0$, then it acts on $\mathcal{U}_{\delta X}$ as $\mathcal{U}_{\delta X} \mapsto g\mathcal{U}_{\delta X} g^{-1} = \mathcal{U}_{\Lambda \cdot \delta X}$. Thus, even if Γ acts on the sigma model without fixed points, there may still be fixed points in its action on the symmetry operators $\mathcal{U}_{\delta X}$. For example, if we quotient a square T^2 by the $\mathbb{Z}_2$ action $(X, Y) \mapsto (X + \pi, -Y)$ in order to obtain a Klein bottle (as discussed in Appendix D), the action on translation operators is $\mathcal{U}_{(\delta X, \delta Y)} \mapsto \mathcal{U}_{(\delta X, -\delta Y)}$, and the space of orbits of symmetry operators is $S^1 \times (S^1/\mathbb{Z}_2)$, not a Klein bottle.

[14]This can be understood by saying that the Γ symmetry lines can carry momentum charge, due to their failure to commute with translation.

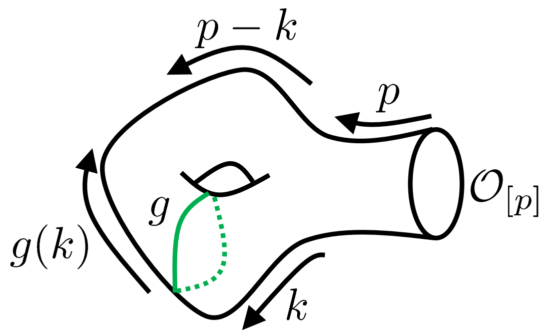

FIG. 5. Tracking the flow of momentum charge through a torus one-point function of $\mathcal{O}_{[p]}$ with the insertion of a symmetry line g (green). In the orbifold theory, this contributes to the torus one-point function of $\mathcal{O}_{[p]}$ in the orbifold vacuum.

An alternative way to make this argument is to build the torus one-point function (2.10) by sewing together sphere three-point functions (see Fig. 6)

$$\langle \mathcal{O}_{[p]}\mathcal{O}_{[k]}\mathcal{O}_{[-k]}\rangle_{\mathbb{S}^2}^{T^n/\Gamma}. \tag{2.12}$$

Because of the Γ orbifold, we have $\mathcal{O}_{[-k]} = \mathcal{O}_{[g(-k)]}$, and so this sphere three-point function can be nonzero if (2.11) is satisfied. Let us conclude this section by noting that if we define the fusion of orbits $[p_1] \otimes [p_2]$ analogously to the fusion of conjugacy classes (defined in the previous section), then (2.11) is precisely the condition that $[p]$ appears in a fusion $[k] \otimes [-k]$.

D. General story

So far, we have seen that in the example of the $\mathrm{Rep}(\Gamma)$ symmetry of non-Abelian orbifolds and in the example of the noninvertible momentum symmetry of toroidal orbifolds, the noninvertible symmetry present at tree level is broken at one loop. In both cases, we saw that the charges μ of the symmetry that could acquire nonzero one-point functions on the torus $\mathbb{T}^2$ were those that appeared in fusions $\rho \otimes \bar{\rho}$ of some charge with its dual. These charges generate what is known as the "adjoint subcategory" of the category of charges [85][Definition 4.14.5.] (see also [25]). A natural guess, then, would be that this is the general story.

However, this is certainly wrong, due to the possibility of non-Abelian symmetry. For example, rather than a non-Abelian orbifold with $\mathrm{Rep}(\Gamma)$ symmetry, consider the unorbifolded theory itself, with non-Abelian Γ symmetry. Then, the charges of local operators are given by representations μ of Γ. Even if μ appears in a fusion $\rho \otimes \bar{\rho}$, an operator $\mathcal{O}_\mu$ charged in μ cannot acquire a nonzero torus one-point function (or, indeed, a one-point function at any order in g_s), as it is charged under the invertible symmetry Γ, whose selection rules hold on any topology.

To see the issue, suppose we try to replicate the argument from the previous section depicted in Fig. 6 for an operator charged under a non-Abelian invertible symmetry. Thus, consider a torus one-point function

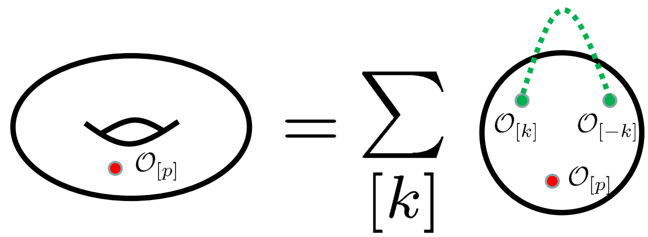

FIG. 6. A torus one-point function can be written as a sphere three-point function with two points sewn together. The sewing procedure involves a trace over the sewn operators' Hilbert space.

$$\langle \mathcal{O}_\mu^a\rangle_{\mathbb{T}^2}, \tag{2.13}$$

of an operator $\mathcal{O}_\mu^a$ charged in a representation $\mu \subset \rho \otimes \bar{\rho}$, where $a = 1, \ldots, \dim(\mu)$ is an internal index running over a basis for μ. If we try to build the one-point function (2.13) by sewing together sphere three-point functions

$$\langle \mathcal{O}_\mu^a \mathcal{O}_\rho^b \mathcal{O}_{\bar{\rho}}^{\bar{c}}\rangle_{\mathbb{S}^2}, \tag{2.14}$$

we are forced to trace over the internal indices b, c of the charged operators $\mathcal{O}_\rho^b$, $\mathcal{O}_{\bar{\rho}}^{\bar{c}}$. While (2.14) may be nonzero, it will be proportional to the Clebsch-Gordan coefficients $C^{ab\bar{c}}$ for the fusion channel $\rho \otimes \bar{\rho} \to \mu$. But if μ is a nontrivial irreducible representation, then the trace $\sum_b C^{ab\bar{b}}$ must vanish, so we do not produce a nonzero torus one-point function.

While the "adjoint subcategory" of charges appearing in fusions $\rho \otimes \bar{\rho}$ of charges with their duals does not generally describe the subset of charges that can acquire a torus one-point function, it does still have an important meaning: it described the set of charges that can acquire a torus one-point function possibly in the presence of topological line insertions on nontrivial cycles of $\mathbb{T}^2$ (see Appendix B). This set of charges is invariant under any possible orbifoldings, as the difference between a theory and its orbifold is merely which topological lines we consider "gauged," or "condensed," i.e., part of the vacuum.

However, for the purposes of string theory, where we only sum over world sheets without the insertion of topological lines, the more refined question of which charges can get a torus one-point function without any line insertions is essential. In Appendix B, we give a formal characterization of this set of charges. We strongly suspect that this set is always precisely the set of charges needed to break our noninvertible symmetry to its maximal invertible subsymmetry, but we were not able to give a full proof (outside of special cases such as the Verlinde lines of a diagonal RCFT, see Appendix B 1). As evidence for the generality of this claim, we verify it explicitly for the case of a truly exotic noninvertible symmetry (Haagerup) in Appendix C.

As a final note, strictly speaking, what we expect in general is that the selection rules of any noninvertible symmetry are not automatically imposed by the symmetry on correlation functions at higher genus. Thus, in the absence of a conspiracy, we expect the symmetry to be broken at $g_s \neq 0$. Of course, one could imagine there might be exceptional string theories where the selection rules of the noninvertible symmetry continue to hold at higher genus due to nontrivial cancellations, even though they did not have to. In fact, we will see precisely such a case below in Sec. III B on the world sheet of a tensionless (but not infinitely weakly coupled) string theory in AdS.

III. NONINVERTIBLE GAUGE SYMMETRIES IN ADS/CFT

In the previous section we argued that at least in perturbative string backgrounds, unbroken noninvertible gauge symmetries coupled to gravity can only arise in a suitable limit where a tower of light states enter the spectrum. In this section we explore more general backgrounds in quantum gravity such as anti–de Sitter (AdS) space using the AdS/CFT correspondence.

Indeed, there is a well-studied sense in which noninvertible gauge symmetries can easily arise in AdS backgrounds. To see why, suppose we have a D-dimensional CFT_D with a semiclassical gravity dual. Suppose also that the CFT_D has a noninvertible global symmetry. According to the "standard rules" of the $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$ correspondence, any global symmetry of the boundary theory ought to be gauged in the bulk. From this perspective, we can immediately generate examples of gauged noninvertible symmetries in the bulk.

To better understand this, it is helpful to briefly review some aspects of categorical symmetries for a general D-dimensional quantum field theory QFT_D. One way to capture the categorical symmetries of the QFT_D is in terms of a $(D+1)$-dimensional topological field theory (TFT) known as the symmetry TFT (SymTFT).[15] Working on a $(D+1)$-dimensional spacetime of the form $\mathbb{I} \times M_D$ with $\mathbb{I}$ an interval and M_D the original D-dimensional spacetime, we specify physical boundary conditions at one end of the interval and gapped (i.e., topological) boundary conditions at the other end. The theory on the physical boundary conditions is referred to as a "relative QFT" in the sense of [46]. The possible global forms of the QFT_D are specified by the choice of gapped boundary condition.[16] Contracting the interval then produces an absolute QFT_D. Switching from one choice of boundary conditions to another is interpreted in the boundary QFT_D as the gauging of a nonanomalous (possibly noninvertible) global symmetry.

For a holographic CFT_D with a semiclassical AdS_{D+1} dual, one can view the SymTFT_{D+1} as a topological subsector of the bulk gravitational theory, as has been explicitly verified in a number of top down constructions [26–34], as well as from a bottom up point of view in [35]. More precisely, the SymTFT_{D+1} should be viewed as a small sliver in the bulk AdS_{D+1}, where the physical boundary conditions of the relative QFT have now been "smeared out" over the rest of the $(D+1)$-dimensional bulk [35]. For our present purposes, it is enough to observe that the bulk AdS_{D+1} has a topological subsector given by the SymTFT_{D+1}.

As an illustrative example, consider 4D $\mathcal{N}=4$ super Yang-Mills (SYM) gauge theory with gauge algebra $\mathfrak{su}(N)$. The global form of the gauge group could, in general, be of the form $SU(N)/\mathbb{Z}_K$ for any K which divides N. The 1-form symmetries of this SYM theory are described by the 5D SymTFT with topological term of the form

$$S_{5D}^{\mathrm{top}} = \frac{N}{2\pi} \int B_2 \wedge dC_2, \tag{3.1}$$

where B_2 and C_2 are, in the 4D boundary, interpreted as the background fields for electric and magnetic 1-form symmetries of the theory. The global form of the SYM gauge group is specified by the boundary conditions for B, C: we can specify $B=0$ or $C=0$, or some more general admixture. This topological term naturally arises in type IIB supergravity via the 10D topological term $F_5 \wedge B_2 \wedge dC_2$ reduced over the S^5 factor of $\mathrm{AdS}_5 \times S^5$ in the presence of N units of F_5 flux. As explained in [86], the choice of boundary conditions for this doublet of 2-form potentials fixes the center of the gauge group on the boundary. More generally, there are now many known realizations of SymTFTs via string constructions (see, e.g., [48,51,87]). As a final comment on this example, observe that gauging the 1-form symmetry of the 4D CFT allows us to switch polarizations, i.e., this corresponds to changing the gapped boundary conditions of the boundary theory. For example, starting with $SU(N)$ gauge theory and an electric 1-form symmetry, gauging the 1-form symmetry produces the $SU(N)/\mathbb{Z}_N$ gauge theory with a magnetic 1-form symmetry.

This simple example describes the SymTFT for an invertible global symmetry of a CFT_D with a holographic dual, but many examples studied in the literature involve the SymTFT of a noninvertible global symmetry of the CFT_D [26–34]. For these cases, one could rightfully describe the bulk SymTFT sector as a noninvertible gauge theory in the conventional sense,[17] and so there are certainly many examples of unbroken noninvertible gauge

[15] For discussion of various aspects of SymTFTs, see, e.g., Refs. [10,35,38–53].

[16] There are some subtleties with imposing such boundary conditions in the case of continuous symmetries, and in the context of holography one ought not entertain a gapped bulk anyway. For further discussion on this in the context of SymTFTs and holography, see respectively [52,53] and [35].

[17] As noted in the Introduction, this is true in the sense of "summing over a mesh of topological operators," and likely also in the sense of a redundancy of the description assuming the results of [15] generalize.

symmetries in string theory. However, in spite of appearances, in all examples we know how to explicitly realize, these "noninvertible gauge symmetries" are of a rather mild type: they could alternatively be described by invertible gauge theories, either in a dual frame or with appropriate Chern-Simons terms.

So far, our discussion has focused on examples of AdS/CFT where the bulk dual is well described by semiclassical Einstein gravity. If we relax this assumption, we can look for examples of noninvertible symmetries in CFTs whose bulk duals are not semiclassical, and which are described by something like a tensionless string theory. We find, in the example of $\mathrm{AdS}_3 \times S^3 \times T^4$ with one unit of NS5 flux [59–61] that the CFT_2 admits noninvertible symmetries whose bulk dual is not described by the associated SymTFT. Instead, the bulk dual is the less-benign stringy noninvertible gauge symmetry described in the previous section, realized as noninvertible symmetry on the world sheet of the tensionless string.

The rest of this section is organized as follows. To illustrate some of the general issues, we first revisit the case of $\mathrm{Rep}(\Gamma)$ symmetries in the special case of the background $\mathrm{AdS}_3 \times S^3 \times T^4$. In this case, we argue that although the boundary theory admits a polarization with a global $\mathrm{Rep}(\Gamma)$ symmetry, the bulk theory is nevertheless captured by a conventional Γ gauge symmetry (i.e., an invertible theory), so in this sense in the bulk we have an invertible symmetry in disguise. We then turn to the limit captured by a tensionless string, where we see a large $\mathrm{Rep}(S_N)$ noninvertible symmetry, and argue that the bulk dual is a stringy noninvertible gauge symmetry, realized on the world sheet of the tensionless string. After this, we turn to a broader discussion of $\mathrm{AdS}_{D+1}/\mathrm{CFT}_D$ pairs for $D > 2$, where we typically encounter symmetries whose noninvertibility is of a very mild type. Consolidating these lessons, we put forward some conjectures on noninvertible symmetries motivated by gravity.

A. Example: Semiclassical $\mathrm{AdS}_3/\mathrm{CFT}_2$

To illustrate some of the general considerations just presented, we now turn to an explicit example. Consider the type IIB NS flux background $\mathrm{AdS}_3 \times S^3 \times T^4$, with its corresponding CFT_2 dual given by the F1/NS5 system. This configuration can be obtained from the near horizon limit of coincident N_1 F1-strings and N_5 NS5-branes on $\mathbb{R}^{1,1} \times \mathbb{R}^4 \times T^4$, where both stacks of branes fill the $\mathbb{R}^{1,1}$ factor, and the NS5-branes wrap the T^4 factor as well. In the near horizon limit, the string coupling is frozen via the attractor mechanism, and satisfies

$$g_s^2 \sim \frac{N_5}{N_1} \times \mathrm{Vol}(T^4), \tag{3.2}$$

to leading order in $1/N_1$.

We briefly note that this is S-dual to a D1/D5 system, although for our present purposes we will find the F1/NS5 description more convenient. Let us also note that in the special case of $N_5 = 1$ there is a tensionless world sheet description of the full 10D bulk gravity solution, given in [59–61]. This is not a semiclassical gravity theory, but it has the advantage of being a tractable example of an explicit world sheet description of the entire bulk. Let us note that instead of T^4 we can also consider a K3 surface, and a specific case of interest are limits of K3 realized as orbifolds of T^4.

We would now like to understand the presence/absence of noninvertible symmetries in this background, where we work in the large charge/supergravity limit. To begin, let us determine some of the discrete gauge symmetries in the bulk. Tuning the moduli of T^4 we can reach special points in moduli space where the tuned T^4_{tuned} admits a non-Abelian isometry Γ. So, in addition to the continuous gauge symmetries that arise generically (from translations of T^4) we see that the 6D spacetime $\mathrm{AdS}_3 \times S^3$ has a discrete Γ gauge symmetry (from discrete isometries of T^4_{tuned}). If we place the usual Dirichlet boundary conditions on $\mathrm{AdS}_3 \times S^3 \times T^4_{\mathrm{tuned}}$, this leads in the CFT_2 to a global, invertible 0-form Γ symmetry. Now, since this 0-form symmetry is nonanomalous in the 2D CFT, it is natural to ask what happens if we gauge it. This yields another 2D CFT which we denote as CFT_2/Γ. As explained in [12,20], and above in Sec. II B, the theory CFT_2/Γ has a 0-form noninvertible symmetry given by $\mathrm{Rep}(\Gamma)$. In this symmetry category the symmetry operators are labeled by finite-dimensional representations of Γ, and there is an accompanying fusion rule given by tensor products of such representations.

What is the bulk dual description of the CFT_2/Γ? From the perspective of the accompanying SymTFT_3, all we have done is modified the topological boundary conditions for the theory, changing them from Dirichlet to Neumann for the Γ gauge fields. Consequently, we conclude that in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ pair with 6D geometry $\mathrm{AdS}_3 \times S^3$ (after reduction on the T^4_{tuned} factor) we have changed from electric to magnetic boundary conditions for the bulk Γ gauge theory on $\mathrm{AdS}_3 \times S^3$. This example illustrates an important lesson: although one may certainly say that "the $\mathrm{Rep}(\Gamma)$ symmetry is gauged in the bulk," there is an alternative presentation of the bulk theory which is an invertible 0-form Γ gauge theory.

Do not confuse the bulk dual of CFT_2/Γ with the related background $\mathrm{AdS}_3 \times S^3 \times (T^4/\Gamma)$. This is related to $\mathrm{AdS}_3 \times S^3 \times T^4_{\mathrm{tuned}}$ example by gauging the global Γ symmetry of the *bulk world sheet theory* (which has $\hat{c} = 10$). We emphasize that this gauging operation is not happening in the boundary CFT_2 (which has $c \gg 1$), nor in the target space quantum gravity theory. In $\mathrm{AdS}_3 \times S^3 \times (T^4_{\mathrm{tuned}}/\Gamma)$, we generically expect the $\mathrm{Rep}(\Gamma)$ symmetry on the world sheet to be broken by string loops as described in Sec. II.

Summarizing, we have seen that in an explicit example with a noninvertible symmetry of the boundary CFT_2, the bulk description is rather benign: it is simply a question of how we choose boundary conditions for the bulk theory.

B. Example: Tensionless string in AdS_3

Let us consider more closely the special case $N_5 = 1$ and $N_1 = N$, considered extensively in [59–61]. Although there is no semiclassical gravity dual, this special case admits an explicit bulk world sheet description as a tensionless string theory and a characterization of the CFT_2 as the N-fold symmetric product orbifold of the T^4 sigma model, i.e., $\mathrm{Sym}^N(T^4) = (T^4)^N/S_N$ (where the symmetric group S_N acts by permutation). In this case the CFT_2 admits a $\mathrm{Rep}(S_N)$ noninvertible symmetry, which emerges precisely at the orbifold point. Because the bulk theory contains a tensionless string, it admits a higher spin gauge symmetry. For details of this Higgsing of this higher-spin symmetry as we move away from the symmetric orbifold point, see Ref. [88] (see also [89]).

What is the bulk dual of this enormous $\mathrm{Rep}(S_N)$ noninvertible symmetry? In [59,60], it was shown that, in the bulk world sheet theory, the vertex operators in the wth spectrally flowed sectors satisfy the same selection rules as the conjugacy classes of w-cycles in S_N, in the limit $N \to \infty$. In other words, the bulk world sheet theory admits a noninvertible $\mathrm{Rep}(S_\infty)$ symmetry,[18] described as the $N \to \infty$ limit of $\mathrm{Rep}(S_N)$ symmetry. This noninvertible symmetry on the world sheet of the bulk string is the holographic dual of the $\mathrm{Rep}(S_N)$ noninvertible global symmetry of the symmetric product orbifold $\mathrm{Sym}^N(T^4)$.

This identification raises a puzzle: the bulk string theory, while tensionless, is not infinitely weakly coupled; indeed, $g_s \sim \sqrt{\mathrm{Vol}(T^4)/N}$ to leading order in $1/N$, as in (3.2). So why is it not the case that the $\mathrm{Rep}(S_\infty)$ symmetry broken by string loops, as in our general story in Sec. II? There, we had noted a possible way to avoid the breaking effect: if the world sheet CFT correlation functions were subject to some highly nontrivial cancellations. This is exactly what happens on the tensionless string world sheet: as described in [61], world sheet correlation functions of the spectrally flowed vertex operators exactly localize on Riemann surfaces that admit a holomorphic branched cover of $\mathbb{S}^2 = \partial \mathrm{AdS}_3$, with branching specified by the charges w_i of the operator insertions. This localization exactly imposes the selection rules of $\mathrm{Rep}(S_\infty)$ noninvertible symmetry at any order in the string loop expansion, since $\pi_1(\mathbb{S}^2) = 0$ and the only interchange of sheets comes from operator insertions.[19] For a concrete example, no vertex operator with $w > 1$ can acquire a one-point function at any genus, since there are no holomorphic branched covers of $\mathbb{S}^2$ with only one branch point besides the identity map, which can be viewed as having "trivial branching" $w = 1$ (see e.g. [61], Equation 8.14]).

C. Weak invertibility

Recently a number of examples of noninvertible symmetries in holographic CFT_D for $D > 2$ have been discussed, along with their string theory realization in the bulk [26–36].[20] As discussed above, this means that there is a conventional noninvertible gauge symmetry in the bulk gravity dual, described by the SymTFT for the noninvertible symmetry.

However, an important caveat is that all of these examples are "weakly invertible," in the sense that the noninvertibility in their fusion rule only includes defects supported on lower-dimensional subspaces, i.e., a condensate (see, e.g., [90,91]). More explicitly, suppose we have a noninvertible symmetry defect $\mathcal{N}$ supported on a q-dimensional subspace. We say $\mathcal{N}$ is *weakly invertible*,[21] or *invertible up to condensates*, if the fusion of $\mathcal{N}$ with $\mathcal{N}^\dagger$ on any given q-manifold N satisfies

$$\mathcal{N}(N) \otimes \mathcal{N}(N)^\dagger = \sum_{\mathcal{M},M} \mathcal{M}(M), \tag{3.3}$$

where the sum runs over some topological operators $\mathcal{M}$ supported on submanifolds $M \subset N$ of strictly lower dimensions $\dim(M) < q$. If we have two such operators $\mathcal{N}_i, \mathcal{N}_j$, their fusion can be described as

$$\mathcal{N}_i(N) \otimes \mathcal{N}_j(N) = \mathcal{N}_{ij}(N) \otimes \sum_{\mathcal{M}_{ij},M} \mathcal{M}_{ij}(M), \tag{3.4}$$

where $\mathcal{N}_{ij}$ is another weakly invertible operator of the same dimension, and $\mathcal{M}_{ij}$ runs over some set of topological operators of lower dimension.[22] Note that if we ignore

[18]We would not expect to see the finiteness of $\mathrm{Rep}(S_N)$ symmetry in the bulk world sheet theory, since this finiteness is nonperturbative in $1/N$.

[19]This argument will fail if we consider a more general hyperbolic 3-manifold than AdS_3, for example a handlebody of higher genus. However, we do not expect the selection rules to hold even in the nongravitational CFT_2 when we place it on a boundary manifold of nontrivial topology.

[20]See, e.g., [4–9] for reviews discussing noninvertible symmetries in $D > 2$ in general.

[21]Note that a "weakly invertible" symmetry need not be invertible, following the convention that the "weak" version of a property does not imply the unmodified version.

[22]To prove this, it suffices to show that $\mathcal{N}_i \otimes \mathcal{N}_j$ is irreducible. Suppose it were not, so we had $\mathcal{N}_i \otimes \mathcal{N}_j = \mathcal{A} \oplus \mathcal{B}$. Then by fusing with $\mathcal{N}_j^\dagger$, we would have $\mathcal{N}_i \otimes (\text{condensates}) = (\mathcal{A} \otimes \mathcal{N}_j^\dagger) \oplus (\mathcal{B} \otimes \mathcal{N}_j^\dagger)$. Since reducibility is invariant under fusing with condensates, we would learn that $\mathcal{N}_i$ were reducible, which is incompatible with the weak invertibility of $\mathcal{N}_i$. By the same argument, the fusion of any irreducible operator with a weakly invertible operator is irreducible.

condensates, the fusion of weakly invertible operators defines a group law.[23]

Weakly invertible operators are to be contrasted with, e.g., the case of Verlinde lines of a 2D RCFT, where the fusion products involve multiple summands of topological defect lines of the same dimension. One of the general lessons from top down realizations of noninvertible symmetries is that bulk dual of weakly invertible symmetries is the well-known process of brane/antibrane annihilation [26,32], which produce lower-dimensional branes that were dissolved in the original brane/antibrane pair via the dielectric-brane effect [92].[24] The corresponding bulk gauge theory is an invertible gauge theory with triple Chern-Simons terms turned on (see also [93]), which capture the possibility of branes of different dimensions to dissolve into one another. Again, we see that the bulk dual of boundary noninvertible symmetry can be rephrased in terms of an invertible gauge theory, now with nontrivial topological couplings. Notably, in this case, we do not have to perform any electromagnetic duality in the bulk: the bulk gauge fields are invertible, but their electric symmetries are rendered noninvertible by the triple Chern-Simons terms [94–96].[25]

There may be more general notions of weak-invertibility beyond the definition (3.3), whose bulk duals correspond to invertible symmetries directly, without switching to a magnetic duality frame. For example, suppose we had a CFT_D with a global $O(2) = U(1) \rtimes \mathbb{Z}_2^{\mathrm{C}}$ symmetry. If we gauge $\mathbb{Z}_2^{\mathrm{C}}$, we obtain a continuous noninvertible symmetry, with topological operators $\mathcal{L}_\theta$ defined as in Appendix A (see also [3,82–84] for more discussion of this construction). In the bulk, where we have a dynamical $O(2)$ gauge theory, this corresponds to switching our boundary conditions for the $\mathbb{Z}_2^{\mathrm{C}}$ discrete gauge field, while leaving the Dirichlet boundary conditions for $U(1)$ unchanged. Thus, while we have switched to a magnetic duality frame for $\mathbb{Z}_2^{\mathrm{C}}$, the $U(1)$ gauge field is directly holographically dual to the noninvertible symmetry in the CFT_D, without any electromagnetic duality.[26]

From the perspective of the fusion algebra of symmetry operators in the CFT_D, what is happening is that we have a continuous family of topological operators $\mathcal{L}_\theta$ such that the limit $\theta \to 0$ of $\mathcal{L}_\theta$ is a condensate.[27] In particular, this means that these operators can be written as

$$\mathcal{L}_\theta(N) = \mathrm{Condensate}(N) \otimes \exp\left(i\theta \int_N \star J\right), \qquad (3.5)$$

where J is a 1-form conserved current operator that is only well-defined along the condensate (see also [83,99–101]). The formula (3.5) defines a more general sort of "weak invertibility" or "invertibility up to condensates" beyond (3.3), whose bulk dual involves a photon field which is itself charged under some other discrete gauge symmetry (in this case, $\mathbb{Z}_2^{\mathrm{C}}$). It would be very interesting to determine the most general notion of "weak invertibility" in QFT (see section V for further comments on this question).

D. Conjectures motivated by gravity

The general lesson from these examples is that while we do expect a gravity dual for noninvertible symmetries in theories with a semiclassical bulk, the bulk description is typically "benign," and can be rephrased as a more conventional invertible gauge theory description, either in an electromagnetically dual frame, or directly as is in the case of weak invertibility. Motivated by these considerations, it is natural to ask whether the SymTFT_{D+1} for any QFT_D (whether or not it has a semiclassical gravity dual) can always be presented as a more conventional invertible gauge theory with appropriate topological couplings (i.e., Chern-Simons terms). This is a broader QFT question, but the evidence we have from holographic examples suggests that this more general statement might be true.

Gravity also suggests that there may end up being an upper bound on the number of separate operators that can appear in the fusion products

$$\mathcal{N}_i \otimes \mathcal{N}_j = \sum_{k=1}^{k_{\max}} \mathcal{T}_{ij}^k \mathcal{N}_k. \qquad (3.6)$$

of topological operators in holographic CFTs with semiclassical bulk duals. It is tempting to conjecture that $k_{\max} \sim O(1)$ for noninvertible symmetries which are dual to conventional noninvertible gauge symmetries in actual

[23] A further comment here is that for nontopological operators, one can of course expect more general fusion rules. An interesting example in this regard is that of [32] where the brane fusion rule in the gravity dual produces multiple branes in the bulk (i.e., seemingly a stronger notion of noninvertibility). Even so, in all known cases where this occurs not all the bulk branes simultaneously specify topological operators in the dual CFT_D. That is to say, once one chooses a polarization/global structure, some of these bulk branes do not descend to topological operators of the CFT_D. See also the discussion to follow near Eq. (3.5) on trivialization up to condensation defects.

[24] For explicit examples in AdS/CFT, see, e.g., Refs. [26–30,32,35,36].

[25] This illustrates a general pattern: strictly speaking, the bulk dual of a global symmetry in the CFT is the approximate electric global symmetry of the bulk gauge fields (see Ref. [35] for a recent discussion of the approximate bulk symmetry operators arising from the boundary symmetries). This was referred to as "long range gauge symmetry" in [97]; see also [98] for the approximate noninvertible electric symmetry of a non-Abelian gauge theory in Maxwell phase.

[26] As in Footnote 25 what is really happening is that the approximate electric 1-form symmetry of $O(2)$ gauge theory in the bulk is noninvertible [3].

[27] More precisely, it is the condensate of the dual $(D-2)$-form symmetry generated by topological Wilson lines for $\mathbb{Z}_2^{\mathrm{C}}$ [3,82–84].

UV complete backgrounds.[28] Indeed, returning to the examples presented in Sec. III A, the collection of non-Abelian orbifolds Γ which can serve as isometries of an explicit tuned T^4 is rather small, and the resulting dimensions of irreducible representations is also quite limited. Indeed, the only case we saw with a possibly large number of fusion products involved S_N, whose bulk dual was realized as a stringy noninvertible gauge symmetry in a tensionless string theory. Observe that in other AdS backgrounds such as AdS $\times\, S^n/\Gamma$ (see Ref. [102]), the order of Γ can be arbitrarily large, but the dimensions of the irreducible representations (and thus the number k_{max} of fusion products) are far smaller. Clearly, it would be interesting to see whether there exist holographic CFTs having noninvertible symmetries with large k_{max}, or conversely, whether there is a nontrivial swampland constraint.

IV. APPROXIMATE NONINVERTIBLE SYMMETRIES IN THE STRING LANDSCAPE

Despite being broken in spacetime, in this section we will show that noninvertible symmetries on the world sheet can still have interesting implications for effective fields theories arising from string theory. First, we will explain the interplay of noninvertible symmetries with several swampland constrains, including the distance conjecture and the sub(lattice) weak gravity conjecture.

A. Existence of towers of states

In section II C we showed the generic presence of noninvertible symmetries at $g_s = 0$ whenever we have string theory compactified on toroidal orbifolds such as $S^1/\mathbb{Z}_2$. More generically, there are noninvertible symmetries present in any orbifold $\mathcal{M}/G$ where $\mathcal{M}$ is a smooth manifold with isometries broken by the G action. This includes many examples with fixed points (including toroidal orbifold Calabi-Yau manifolds); in these cases, there is a question of whether the noninvertible symmetries may be explicitly broken by turning on deformations corresponding to marginal twisted sector operators localized at the fixed points. All of the above arguments, however, also apply to the case when G is freely acting. In these cases, the noninvertible symmetry is exact classically for any choice of geometric moduli, and as discussed in the previous subsection, is only broken by quantum effects at $g_s \neq 0$. Examples of these manifolds include Riemmann and Ricci-flat examples, such as freely acting quotients of T^n and quotients of the form

$$\frac{K3 \times T^k}{\mathbb{Z}_n}, \tag{4.1}$$

where the $\mathbb{Z}_n$ is a common subgroup of isometries. Some of these constitute examples of Calabi-Yau manifolds with infinite fundamental group [103,104].

As we have seen, these noninvertible symmetries are only approximate unless we are in a decompactification or tensionless string limit. A natural question is then whether it plays any role in the physics close, but not exactly at the asymptotic/perturbative limits of effective field theory. We will now show that this noninvertible symmetry can be used to prove the existence of a tower of states that becomes light at infinite field distance, as predicted by the distance conjecture [105]. Using noninvertible symmetries we can therefore extend the range of asymptotic limits where a proof of the distance conjecture is available.[29] Importantly, for the first time, the argument does not use the existence of an unbroken gauge symmetry, as is the case in the usual perturbative string [107] and complex structure moduli space examples [108,109]. Since the noninvertible symmetry is not exact, we can expect small corrections to the mass and lifetime of the particles in the tower. However the existence of the tower itself in the asymptotic limit is guaranteed by the noninvertible symmetry.

The argument we have in mind is a minor modification of the proof in [66,67] of the sublattice weak gravity conjecture, which itself is a direct application of spectral flow.[30] We will now briefly review the argument in [66,67] (which was itself described inline in [106]), in the particular case of a single $U(1)$ gauge field and then explain how it gets modified for the case of a noninvertible symmetry.

Consider a 2D world sheet CFT with an invertible $U(1)$ symmetry generated by a holomorphic current j at level N. In other words,

$$j(z)j(0) \sim \frac{N}{z^2} \tag{4.2}$$

In these circumstances, one may consider the partition function with complex chemical potential

$$Z(\mu,\tau) \equiv \mathrm{Tr}(q^{L_0}\bar{q}^{\bar{L}_0}e^{2\pi i Q\mu}), \qquad q \equiv e^{2\pi i\tau}. \tag{4.3}$$

This partition function transforms covariantly under $SL(2,\mathbb{Z})$

$$Z\left(\frac{\mu}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = e^{i\pi N\frac{c\mu^2}{c\tau+d}}Z(\mu,\tau) \tag{4.4}$$

(the lack of exact invariance is due to the anomalous conservation of the holomorphic current, see, e.g., [110]) and it also satisfies

[28] This statement could possibly be extended to the general case by placing a bound on the bulk EFT cutoff.

[29] Of course another way to argue that in the large radius limit for arbitrary compactifications the light KK tower is related to gauge symmetry, as is anticipated by weak gravity conjecture [106], is to note that in this limit we get approximate translational symmetries which lead to gauge symmetries broken by $1/R$ effects.

[30] See also [68] for extremely recent progress in this direction.

$$Z(\mu,\tau) = Z(\mu+1,\tau) \tag{4.5}$$

since the charges are quantized. As shown in [66,67], imposing (4.5) and (4.4) together implies that the whole spectrum of CFT operators is invariant under a simultaneous shift

$$h \to h + \frac{Q^2}{2N}, \qquad Q \to Q + N, \tag{4.6}$$

known as a spectral flow automorphism. Equation (4.6) implies that the CFT spectrum arranges itself into towers of states where the $U(1)$ charge shifts by N and the dimensions increase accordingly. In particular, the tower associated to the identity is labeled by a parameter k and has charges $Q = kN$ and weight $h = Nk^2/2$. In a perturbative string context, after level-matching, these operators correspond to a tower of particles with mass and $U(1)$ charge

$$m \sim \sqrt{N}k, \qquad Q = kN \tag{4.7}$$

which exactly saturates or satisfies the sublattice weak gravity conjecture [66,67]. More to the point, this tower of states becomes light in the perturbative string limit: the proof of the sublattice WGC is also a particular case of the distance conjecture. Although we reviewed here the case of a single holomorphic current, the setup is general, applying to any number of Abelian currents of any chirality.

The basic point of this subsection is that the above argument goes through almost unchanged in the case where the symmetry is noninvertible. Imagine gauging a $\mathbb{Z}_2$ symmetry that sends j to $-j$, as would be the case when going from S^1 to the $S^1/\mathbb{Z}_2$ sigma model of section II C. The partition function (4.3) is no longer a well-defined object, but the quantity

$$\tilde{Z}(\mu,\tau) \equiv Z(\mu,\tau) + Z(-\mu,\tau) \tag{4.8}$$

is,[31] and clearly, it inherits the modular transformation properties (4.4), leading to the existence of a noninvertible version of spectral flow (notice that the weight h in (4.6) is invariant under sign flip of Q). Therefore, we obtain again a tower of particles in the spacetime, which are just the particles from the tower in the unorbifolded tower which were not projected out. Importantly, in the orbifold case they are not charged under any massless gauge field.

In the particular case of an $S^1/\mathbb{Z}_2$ sigma model, what the spectral flow automorphism predicts is precisely the tower of interval KK modes. More generally, for $\mathcal{M}/G$, the tower of states thus predicted is that of KK modes. Although the existence of these states was known directly from a bulk EFT analysis, it is interesting to see it arise purely from a noninvertible symmetry. The fact that one has a world sheet argument implies that similar towers can be obtained for winding modes and nongeometric models, too. Moreover, it supports the idea of interpreting the tower of states as a quantum gravity obstruction to restoring a global symmetry at infinite distance [98,108,109]. We have seen that these noninvertible symmetries are restored at weak string coupling. A preliminary argument in Appendix D suggests that they also seem to be restored at large radius, even without having a string world sheet description, and would become exact at infinite field distance in these decompactification limits. In all known examples in string theory, the towers of the distance conjecture are either KK modes or string modes [111], getting light and weakly coupled asymptotically. For the latter case, it seems we can sometimes identify a weakly broken symmetry that becomes exact as the string coupling vanishes. For the former case, any translational diffeomorphism of the higher-dimensional vacuum corresponds to a sort of approximate symmetry from the lower-dimensional perspective that gets restored upon decompactification. However, in most cases, these symmetries are already broken at classical level, unlike the noninvertible symmetry that is only broken by loop effects. If the compact manifold has some isometry, this yields a continuous gauge symmetry in the lower dimensional EFT (which would become global at infinite distance unless there is a KK tower of states signaling decompactification of extra dimensions). In the absence of an isometry, we can still have in certain cases an approximate noninvertible symmetry that is preserved at classical level and only broken by quantum corrections. It would be interesting to see whether this weakly broken noninvertible symmetry can be generalized to other decompactification limits beyond toroidal orbifolds.

B. Interplay with (sub)lattice WGC

The noninvertible symmetry also has implications for the sublattice WGC described in the previous subsection. As shown in Eq. (4.7), the states shown to exist via spectral flow only have charge given by a multiple of N, the level of the $U(1)$ current algebra. The charged states therefore only live in a sublattice. Since the value of N is unconstrained, this sublattice can be made arbitrarily sparse, in principle, and the swampland implications get correspondingly diluted; interesting swampland statements are about constraining the spectrum of light states, while for large enough N, the states predicted by spectral flow can become arbitrarily massive. This undesirable feature of the sublattice version of WGC is known as the "loophole" in the literature [112–115].[32]

[31] In the orbifold theory, it is the partition function in the untwisted sector with an insertion of a line for the quantum symmetry with complex potential. We are allowed to ignore the contribution of the twisted sector in the orbifold theory since it is charged under the quantum $\mathbb{Z}_2$ symmetry and we can consider only the uncharged states.

[32] This loophole and a related construction [116] was already noted in [106] and was the basis of the observation in the original paper that the WGC does not always hold for the minimally charged state in the theory.

Explicit examples realizing this loophole are known [66], though they all take the form of freely acting orbifolds of T^n, just like the ones discussed above. Although in the covering T^n there are n currents realized at level 1, after orbifolding it is possible to obtain currents that are realized at higher level, resulting in a sublattice of states. In those cases, though, we have seen that there is still a noninvertible global symmetry that survives at tree level in the world sheet. We will now explain how the interplay between this noninvertible and invertible currents can be used to improve on the sublattice WGC, to conclude that even for nonsuperextremal states there are light charged states (though they are not superextremal).

To illustrate this, let us consider for example the orbifold $T^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ discussed in [66], where the freely acting group acts as

$$\mathbb{Z}_2\colon\ \theta_w \to \theta_w + \pi, \theta_y \to \theta_y + \pi, \tag{4.9}$$

$$\mathbb{Z}_2'\colon \theta_w \to -\theta_w, \theta_z \to \theta_z + \pi. \tag{4.10}$$

The unorbifolded toroidal compactification contains three massless gauge fields, but the $\mathbb{Z}_2'$ projects out the first one associated to the direction w. Hence, the charge lattice is only given by KK charges (k_y, k_z). In what follows we will focus in the subspace with $k_z = 0$ for simplicity, but the lessons are general. Even in this subspace, we have KK modes with nonvanishing momentum k_w; Notice that the first $\mathbb{Z}_2$ implies that unprojected KK modes with odd values of k_y must also have an odd value of k_w. The corresponding charge lattice is represented in Fig. 7 as dots in a two dimensional slice. The KK masses and charges are given by

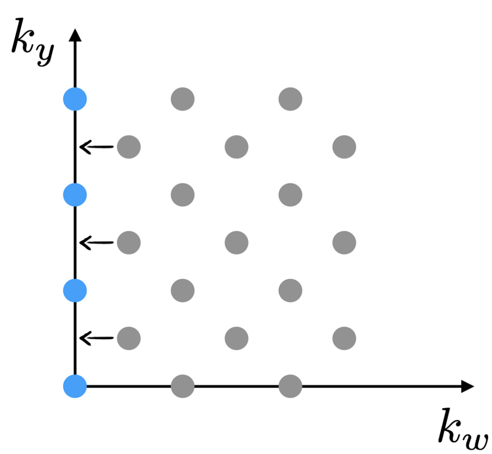

FIG. 7. Two dimensional slice of the lattice of KK momenta. Only the vertical axis is associated to a U(1) gauge charge. The blue dots are extremal states (saturating the WGC) while the gray ones are subextremal.

$$m^2 = \frac{k_w^2}{R_w^2} + \frac{k_y^2}{R_y^2} + \frac{k_z^2}{R_z^2}, \qquad gQ = (k_y, k_z)\frac{1}{R} \tag{4.11}$$

where we have also included the KK gauge coupling $1/R$.

As emphasized above, only the vertical direction in the figure corresponds to the charge direction of the massless gauge field that survives the orbifold action, so only k_y (and not k_w) is truly a gauge charge. Hence, we need to project all KK states over the vertical direction to obtain the set of charged states under the massless gauge field. The KK states with $k_w = 0$ and even k_y are extremal (i.e., saturate the WGC bound) since $|Q| = M$ (there are lattice sites on the vertical axis). However, states with odd k_y and odd k_w are subextremal (i.e., violate the WGC bound) since $|Q| < M$ due to the $\frac{k_w^2}{R_w^2}$ contribution on the mass. Therefore, there is only a sublattice of states satisfying the WGC, as noticed in [66].

Our main point here is that the modular flow argument for noninvertible symmetries of the previous section allows us to recover the existence of the full lattice of KK towers, including those with nonvanishing k_w that are not charged under any massless gauge field, even though they will be charged under some discrete gauge symmetries. Noninvertible spectral flow predicts the existence of states with masses and charges given by (4.11) for all allowed values of k_w, k_y, k_z. Moreover, it allows us to quantify how much these subextremal particles are violating the WGC. In this particular case we have that

$$\frac{m^2}{Q^2} = 1 + \frac{k_w^2}{k_y^2}\frac{R_y^2}{R_w^2} > 1. \tag{4.12}$$

We see that the violation of the lattice WGC becomes negligible for very large charges k_y; while it depends on the ratio of the radii for small charges. Hence, even if the first light state satisfying the WGC does not have the minimal possible value of the charge (the WGC states start with charge 2 in this example), we can still use spectral flow of the noninvertible symmetry to show the existence of light states of unit charge whose charge to mass ratio is constrained and which only mildly violate the WGC. For small charges, the charge-to-mass ratio is of order one, so in any case, the dangers of the sublattice WGC loophole for phenomenological applications in this example is significantly ameliorated; we still get a full lattice of light (in terms of the gauge coupling) charged states, even if they are not exactly superextremal.

However, part of the reason why one still gets light states is that in this example, the sublattice is of index 2 (half the lattice sites contain superextremal particles). It is possible to consider bottom-up constructions [66] where the index of the sublattice becomes very large; in these examples, the first values of the charge may still contain very heavy states.[33] It therefore remains an essential question to bound the index of the lattice in general. A related question suggested by the above analysis is whether all examples with an Abelian current at level N contain additional noninvertible symmetries that allow one to predict the masses of subextremal states.

C. Supersymmetric protection

The paper [117] discusses an interesting phenomenon in certain SUGRA theories, where observables which are generically nonvanishing were shown to vanish exactly, in violation of the naturalness principle. This violation could be avoided in the presence of higher SUSY, but this higher SUSY was not observed in these examples. This observation led to the "supersymmetric genericity conjecture," which states that such protections must be a result of the theory being related to a higher-SUSY theory in some indirect way. In practice, in all of the examples considered, the lower-SUSY theory was related to the higher-SUSY theory through the gauging of a discrete subgroup of the R-symmetry. The examples in the rest of this section follow from the discussions in [117].

Our discussion of noninvertible symmetries gives a new perspective on this observation. The construction in Sec. II C applies here as well, since certain supercharges appear to be projected out when we gauge a discrete subgroup of the R-symmetry, but instead they reappear as noninvertible symmetries.[34] Naturalness is thus not violated if we generalize its definition to include noninvertible symmetries.

For concreteness we discuss a specific example. In 4D $\mathcal{N} = 2$ SUGRA, the prepotential $\mathcal{F}$ for a vector multiplet contains a polynomial term in the superfields (of degree at most 3) and exponential terms generated by world sheet instanton effects,

$$\mathcal{F} = \mathcal{F}_{\text{polynomial}}(\Phi) + \sum_{n,i} B_n(\Phi) e^{-a_n^i \Phi_i}. \tag{4.13}$$

The coefficients B_n are generically nonzero, although if SUSY is enhanced to $\mathcal{N} = 4$ the coefficients B_n vanish for all n. We can now discuss an explicit example of an apparent violation of naturalness. Consider Type II string theory on orbifolds of T^6 or $K3 \times T^2$. If the orbifold preserves $\mathcal{N} \geq 4$ SUSY, then the coefficients B_n vanish automatically. However, these coefficients vanish also if the orbifold preserves only $\mathcal{N} = 2$ SUSY. The reason for this is that the prepotential is computed using only the genus-zero contribution on the world sheet. As a result, amplitudes of the orbifold theory restricted to untwisted operators are identical to that of the unorbifolded theory, and so restricting $\mathcal{F}$ to the untwisted fields gives only a cubic term. If the twisted sectors do not include massless fields, the prepotential will then be exactly cubic.

As discussed above, this violation of naturalness is avoided since the supercharges which are projected out under orbifolding still leave behind a noninvertible symmetry and its corresponding sphere selection rules. While these are broken at leading order in g_s, they are preserved at genus 0, and as a result they constrain tree-level amplitudes of untwisted operators just like their invertible versions in the unorbifolded theory.[35] Then the puzzling protection of $\mathcal{F}$ in this example can be attributed to noninvertible symmetries.

V. DISCUSSION AND FUTURE DIRECTIONS

In this paper we have studied potential realizations of gauged noninvertible symmetries in quantum gravity. In cases where we have a global noninvertible symmetry in the world sheet CFT, we found that higher-loop effects generically break the putative noninvertible symmetry to its maximal invertible subsymmetry, unless there are nontrivial cancellations as in the case of the tensionless string considered in Sec. III B. As a result, the target space physics of noninvertible world sheet symmetry, which we referred to as "stringy noninvertible gauge symmetry," is broken (Higgsed) away from the tensionless limit. Moreover, in CFTs with a semiclassical holographic dual, we found that the bulk dual of noninvertible symmetry was always "benign," in the sense that it could alternatively be characterized by an invertible gauge theory in one way or another, while in CFTs whose dual contained a tensionless string, we recovered an unbroken stringy noninvertible gauge symmetry.

How should we think about stringy noninvertible gauge symmetry? It is something beyond any gauge symmetry that can be described in effective field theory. One answer would be that it should be viewed as a noninvertible extension of the gauge symmetries of string field theory, which are generically Higgsed when the string acquires a tension. This would give a natural explanation for the fact that stringy noninvertible gauge symmetries are restored only in the limit of a tensionless string.

To support this picture, consider the continuous noninvertible symmetry in the simple orbifold $S^1/\mathbb{Z}^2$, studied in detail in Appendix A. What is the corresponding gauge boson? On the world sheet, the current for the invertible $U(1)$

[33]Indeed, there are open string examples where the index of the sublattice can be made very large, provided that a sufficiently long warped throat exists in a Calabi-Yau with fluxes.

[34]On the world sheet, the construction is analogous to (2.5) for toroidal orbifolds: one can take sums of the world sheet symmetry operators for target space SUSY over an orbit of the orbifold group. This definition is likely easier in the Green-Schwarz formalism, where target space SUSY is manifest on the world sheet.

[35]It is crucial that $\mathcal{F}$ is computed at genus 0, where the noninvertible symmetry is unbroken. In an analogous computation for the heterotic string on $\mathbb{T}^6/\Gamma$, the prepotential also receives one-loop corrections, and as a result the noninvertible SUSY is not enough to protect it, and indeed one finds nonvanishing B_n's.

symmetry of the S^1 sigma model has been projected out, and only survives as a disorder operator attached to a topological line. Thus, it cannot be used to create an on-shell state in the string theory spectrum. However, perhaps such disorder operators could be included in string field theory as a way to describe the Higgsed gauge bosons for stringy noninvertible gauge symmetry. A caveat is that these disorder operators appear for any value of g_s, so such an interpretation would also require understanding their role at nonzero g_s where the noninvertible symmetry is broken.

As a further comment, consider again the mesh of $\mathcal{N} \otimes \mathcal{N}^\dagger$ that appears on a Riemann surface when we sweep a noninvertible operator $\mathcal{N}$ across, as discussed in Sec. II A. This mesh has a very natural interpretation: the fusion $\mathcal{N} \otimes \mathcal{N}^\dagger$ has the structure of a Frobenius algebra, meaning it corresponds to a (possibly noninvertible) orbifolding of the world sheet theory, defined exactly by inserting it as a mesh (see, e.g., [12]). Moreover, $\mathcal{N} \otimes \mathcal{N}^\dagger$ is manifestly Morita trivial, meaning that the world sheet theory we obtain by orbifolding is the original world sheet theory! Thus, in general, a noninvertible operator tells us that the string theory at hand is a self-orbifold, possibly under a noninvertible orbifolding [83,118,119].

As a result, one way to interpret the results of this paper is that "the orbifold procedure" itself should be viewed as part of the gauge symmetries of string field theory. In a sense, it is then very odd that noninvertible symmetries are generically broken at $g_s \neq 0$, since the orbifold procedure (even by noninvertible symmetries) still makes sense order-by-order in string perturbation theory. One way to resolve the tension is to note that the orbifold procedure is acting on the string background: a fixed 2D CFT, viewed as a solution to the classical string theory equations of motion with $g_s = 0$. Thus, it makes sense to study string perturbations about this background to any loop order, even if those same loop effects lead to a breaking of the corresponding noninvertible symmetry at $g_s \neq 0$.

The low-energy limit of string theory is the effective field theory of supergravity. Therefore, whenever the relevant string states survive the field theory limit, it might be possible to reinterpret the one-loop effects that break noninvertible world sheet symmetries as a sum of ordinary field theory diagrams. For example, if we consider a toroidal orbifold as in Sec. II C with no fixed points, in the large-volume limit, the effects from twisted sectors running in the loop will be suppressed, so the breaking can be understood entirely from Kaluza-Klein modes running in the loop. In this sense, it may be possible to see noninvertible symmetries emerging not just in limits with a tensionless string, but also in decompactification limits, where the lower dimensional EFT breaks down to be replaced by a higher-dimensional one. We give some preliminary comments in this direction in Appendix D.

In this paper, we described how a stringy realization of noninvertible symmetries arises near the perturbative string limit where the symmetry gets approximately restored. It would be interesting to better understand the fate of these symmetries in other tuned backgrounds. For example, one might naively expect that M-theory compactified on a non-Abelian orbifold would have a noninvertible 1-form symmetry arising from the noninvertible 1-form symmetry on the M2-brane worldvolume theory. However, this symmetry seems to be badly broken,[36] unless we put the theory on a further circle, where in the limit of small radius it would lead to an approximate noninvertible 0-form symmetry as realized on the dual Type IIA string theory. It would be interesting to find further evidence for this expectation or to find evidence to the contrary.

As a practical comment, let us note that the breaking of noninvertible world sheet symmetries away from the tensionless limit does not mean they are useless: since they are still good approximate symmetries, they can be used to constrain the spectrum and interactions of the theory. We have shown how noninvertible symmetries are able to fill in the gaps left by the usual world sheet derivation of the sublattice WGC, and more generally, how they can be used to predict the existence of towers of states which are not charged under any continuous gauge symmetry, which is of interest for the Distance conjecture. Moreover, the existence of certain examples which exhibit properties as if they had higher supersymmetry can be attributed to the presence of a noninvertible fermionic symmetry (i.e., a $\mathbb{Z}_2$ odd internal symmetry) on the world sheet. The examples we considered here can also be understood via more elementary techniques; the role of noninvertible symmetries here is to provide a new perspective on an old physical phenomenon.

While our discussions have been phrased in the context of quantum gravity, there may be general lessons for noninvertible symmetries in D-dimensional QFTs without gravity. In particular, in Sec. III, we saw that the only example of a noninvertible symmetry which could not be viewed as "weakly invertible" in one sense or another[37] was the case of global categorical symmetries of a CFT_2. In this case, the bulk dual description in AdS_3 could be described as a non-Abelian gauge theory after performing electromagnetic duality. Notably, the magnetically charged objects under a discrete non-Abelian gauge symmetry have codimension 2. Now, in any dimension D, we can always find $(D-2)$-form global symmetries that cannot be viewed as "weakly invertible" simply by considering Γ gauge theory for a non-Abelian group Γ. However, it is possible that in any QFT_D, the only noninvertible symmetries which cannot be viewed as "weakly invertible" must be p-form

[36]This fits with the fact that there is no small parameter controlling any sum over M2-brane world volume topologies, which could have suppressed the symmetry breaking effect from topologically nontrivial configurations.

[37]Meaning that the bulk dual is directly an invertible gauge theory without switching duality frames, as discussed in Sec. III C. The general QFT definition of this property is still unclear.

symmetries for $p \geq D - 2$.[38] This statement must be restricted to local QFT, as we have discussed examples of 0-form stringy noninvertible gauge symmetries in dimensions $D > 3$ which cannot be viewed as "weakly invertible," such as the symmetries restored as $g_s \to 0$ in 6D from non-Abelian orbifolds T^4/Γ.

Lastly, let us mention that in the context of holography with a semiclassical bulk gravity, we did not find *any* examples of noninvertible symmetries that could not be regarded as invertible gauge symmetries in the bulk with appropriate choices of boundary conditions and topological terms. It would be interesting to study whether this pattern holds up in general, and whether there are nontrivial constraints on the noninvertible symmetries of holographic CFTs coming from constraints on UV complete quantum gravity.

Note added. As we were completing this work, we learned of [121] which we understand will also discuss noninvertible world sheet symmetries.

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APPENDIX A: SELECTION RULES FOR $S^1/\mathbb{Z}_2$

In this Appendix we illustrate the discussion in Sec. II C in the simple torodial orbifold $S^1/\mathbb{Z}_2$, and explicitly rederive the selection rules on sphere correlation functions from the noninvertible symmetries.

1. Noninvertible symmetries of $S^1/\mathbb{Z}_2$

We start with the unorbifolded theory S^1 with coordinate $X \sim X + 2\pi R$, which is the $c = 1$ compact boson CFT with radius R. We will discuss only the momentum symmetry, but there is a completely analogous discussion for the winding symmetry as well. We have local momentum vertex operators

$$\mathcal{O}_m(z, \bar{z}) = e^{imX(z,\bar{z})/R}. \tag{A1}$$

The $U(1)$ momentum symmetry is generated by the invertible TDLs

$$\mathcal{U}_\theta = \exp\left[\frac{i\theta R}{2\pi}\int \star dX\right], \tag{A2}$$

whose action on local operators is

$$\mathcal{U}_\theta\colon\ \mathcal{O}_m \mapsto e^{im\theta}\mathcal{O}_m. \tag{A3}$$

Now, we consider the orbifold by $X \to -X$, under which we have

$$\mathcal{O}_m \to \mathcal{O}_{-m}, \qquad \mathcal{U}_\theta \to \mathcal{U}_{-\theta}. \tag{A4}$$

In the orbifolded theory, the spectrum of local operators consists of twisted and untwisted sectors. In the untwisted sectors we have vertex operators

$$\mathcal{O}_m^+ = \frac{1}{\sqrt{2}}(\mathcal{O}_m + \mathcal{O}_{-m}), \tag{A5}$$

We also have a topological, invertible Wilson line for the gauged $\mathbb{Z}_2$ symmetry, which we will denote by η, and which implements the quantum $\mathbb{Z}_2$ symmetry. At the end of η, we have a sector of disorder operators, which includes the gauge-noninvariant operators

$$\mathcal{O}_m^- = \frac{i}{\sqrt{2}}(\mathcal{O}_m - \mathcal{O}_{-m}) \tag{A6}$$

[38] In fact, one could view [120] as establishing something like this in $D = 3$.

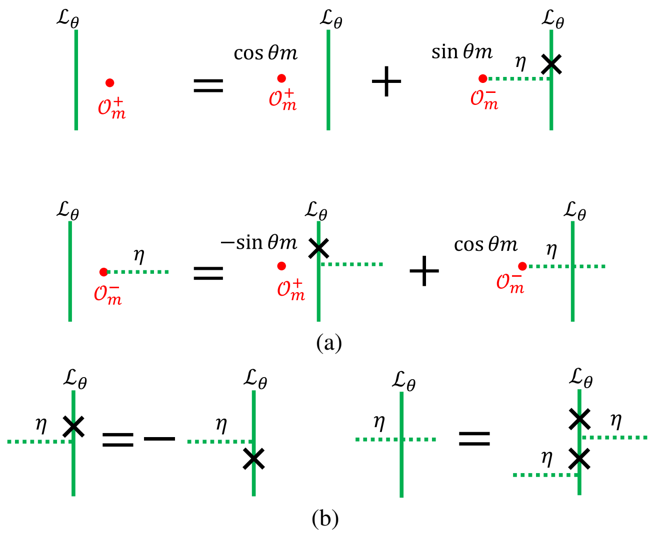


FIG. 8. (a) Rules for sweeping a TDL past vertex operators. A black "X" is used to denote the orientation of a junction (see e.g. [74]). (b) Conventions for the orientation of the junctions.

which have been projected out of the spectrum of local operators.[39]

In the orbifolded theory, the $U(1)$ momentum operators are generically projected out, leaving only the identity operator $\mathcal{U}_0$ and the half-shift operator $\mathcal{U}_\pi$, which corresponds to the invertible $\mathbb{Z}_2$ symmetry that flips the interval $S^1/\mathbb{Z}_2$. However, using the construction in equation (2.5), we have a continuum of noninvertible TDLs, labeled by $\theta \in (0,\pi)$ (see Refs. [3,83,84]):

$$\mathcal{L}_\theta = \mathcal{U}_\theta \oplus \mathcal{U}_{-\theta}. \tag{A7}$$

The action of these noninvertible lines on the physical degrees of freedom actually has a simple intuitive description: we first "unorbifold," then rotate S^1 by an angle θ (or $-\theta$, either will work), then "reorbifold," obtaining $S^1/\mathbb{Z}_2$ again, but folded along an axis rotated by θ.[40]

As described above, the action of noninvertible symmetries by sweeping generally maps local operators (or disorder operators) to linear combinations of local operators and disorder operators (see Fig. 8). We have

$$\begin{aligned}\mathcal{L}_\theta\colon\ \mathcal{O}_m^+ &\mapsto \cos(m\theta)\mathcal{O}_m^+ + \sin(m\theta)\mathcal{O}_m^-,\\ \mathcal{L}_\theta\ \colon\ \mathcal{O}_m^- &\mapsto -\sin(m\theta)\mathcal{O}_m^+ + \cos(m\theta)\mathcal{O}_m^-.\end{aligned} \tag{A8}$$

Their fusion rules are given by

$$\mathcal{L}_\theta\mathcal{L}_{\theta'} = \mathcal{L}_{\theta+\theta'} + \mathcal{L}_{\theta-\theta'}, \tag{A9}$$

where we have defined the reducible lines

$$\mathcal{L}_0 = 1+\eta, \qquad \mathcal{L}_\pi = (1+\eta)\otimes\mathcal{U}_\pi, \tag{A10}$$

to simplify notation.

2. Selection rules

We can now discuss selection rules on the sphere. Following Sec. II A, a naive attempt would be to consider some correlation function

$$\langle \mathcal{O}_{m_1}^+(x_1)\ldots\mathcal{O}_{m_n}^+(x_n)\rangle, \tag{A11}$$

and then generate a selection rule by nucleating a noninvertible line $\mathcal{L}_\theta$ and sweeping it past the operators (see Fig. 1). This fails, since the action of the noninvertible lines in (A8) inevitably maps this correlator to correlators involving $\mathcal{O}_m^-$. We are thus instead forced to consider all correlators of the form

$$\langle \mathcal{O}_{m_1}^\pm(x_1)\ldots\mathcal{O}_{m_n}^\pm(x_n)\rangle, \tag{A12}$$

and see how the noninvertible lines relate them.

For simplicity we start with the general two-point functions $\langle\mathcal{O}_m^+\mathcal{O}_{m'}^+\rangle, \langle\mathcal{O}_m^-\mathcal{O}_{m'}^-\rangle$, where we suppress the position dependence. Following the procedure in Fig. 1, one finds the following set of equations:

$$\begin{pmatrix} 1-\cos(\theta m)\cos(\theta m') & -\sin(\theta m)\sin(\theta m') \\ -\sin(\theta m)\sin(\theta m') & 1-\cos(\theta m)\cos(\theta m')\end{pmatrix} \times \begin{pmatrix}\langle\mathcal{O}_m^+\mathcal{O}_{m'}^+\rangle \\ \langle\mathcal{O}_m^-\mathcal{O}_{m'}^-\rangle\end{pmatrix} = 0. \tag{A13}$$

These equations have a nontrivial solution for generic θ only if the determinant of the matrix vanishes. Its eigenvalues are

$$\begin{aligned}E_\pm &= 1-\cos(\theta m)\cos(\theta m') \pm \sin(\theta m)\sin(\theta m')\\ &= 1-\cos(\theta(m\pm m')).\end{aligned} \tag{A14}$$

The nonzero correlators are thus only those with $m=\pm m'$, as expected.

Next we consider three-point functions. A similar analysis results in the set of equations

$$(1+M)\begin{pmatrix}\langle\mathcal{O}_m^+\mathcal{O}_{m'}^+\mathcal{O}_{m''}^+\rangle\\ \langle\mathcal{O}_m^+\mathcal{O}_{m'}^-\mathcal{O}_{m''}^-\rangle\\ \langle\mathcal{O}_m^-\mathcal{O}_{m'}^+\mathcal{O}_{m''}^-\rangle\\ \langle\mathcal{O}_m^-\mathcal{O}_{m'}^-\mathcal{O}_{m''}^+\rangle\end{pmatrix} = 0, \tag{A15}$$

[39]We have included a factor of i so that $\mathcal{O}_m^-$ descends from a Hermitian operator in the unorbifolded theory.

[40]This intuitive language can be made precise by noting that $\mathcal{L}_\theta = \mathcal{D}\otimes\mathcal{U}_\theta\otimes\mathcal{D}^\dagger$, where $\mathcal{D}$ is the topological gauging interface from S^1 to $S^1/\mathbb{Z}_2$, i.e., the Dirichlet boundary condition for the $\mathbb{Z}_2$ gauge fields.

where M is the matrix (using $c \equiv \cos$, $s \equiv \sin$)

$$\begin{pmatrix} -8c(\theta m)c(\theta m')c(\theta m'') & -2c(\theta m)s(\theta m')s(\theta m'') & -2c(\theta m')s(\theta m)s(\theta m'') & -2c(\theta m'')s(\theta m')s(\theta m) \\ -2c(\theta m)s(\theta m')s(\theta m'') & -8c(\theta m)c(\theta m')c(\theta m'') & 2c(\theta m'')s(\theta m')s(\theta m) & 2c(\theta m')s(\theta m)s(\theta m'') \\ -2c(\theta m')s(\theta m)s(\theta m'') & 2c(\theta m'')s(\theta m')s(\theta m) & -8c(\theta m)c(\theta m')c(\theta m'') & 2c(\theta m)s(\theta m')s(\theta m'') \\ -2c(\theta m'')s(\theta m')s(\theta m) & 2c(\theta m')s(\theta m)s(\theta m'') & 2c(\theta m)s(\theta m')s(\theta m'') & -8c(\theta m)c(\theta m')c(\theta m'') \end{pmatrix}. \tag{A16}$$

The eigenvalues of $1 + M$ are

$$E_{s_1,s_2} = 1 - \cos(\theta(m + s_1 m' + s_2 m'')), \qquad s_1, s_2 \in \{\pm 1\}. \tag{A17}$$

so we find that a nontrivial correlator must have $m + s_1 m' + s_2 m'' = 0$ for some choice of s_1, s_2, again as expected. The general result for longer correlators follows from repeated application of this method.

Finally, we explain why this procedure fails to generate selection rules on higher-genus surfaces. For simplicity we put the theory on the torus $\mathbb{T}^2$. The main caveat in the process appears in the last step of Fig. 1, where we annihilate the topological line "at infinity." On $\mathbb{T}^2$, this means taking the fusion of the two lines which meet from opposite ends of each cycle. For invertible lines, this fusion gives the identity and so annihilates the lines, leading to a selection rule. However, for noninvertible lines the result is more complicated. Using the fusion rule (A9) we find

$$\mathcal{L}_\theta \mathcal{L}_{-\theta} = 1 + \eta + \mathcal{L}_{2\theta}. \tag{A18}$$

Importantly, in addition to the contribution from the identity, noninvertibility forces other contributions to appear. As a result, the process for generating selection rules on the sphere fails to generate a selection rule on the torus (and similarly for higher-genus surfaces). Instead, this process relates a correlation function in the vacuum to the same correlation function but with the topological line $1 + \eta + \mathcal{L}_{2\theta}$ wrapping every 1-cycle of the manifold. We thus cannot extract a selection rule from this procedure.

APPENDIX B: GENERAL TORUS ONE-POINT FUNCTIONS

In this appendix, we give a general characterization of the set of charges described in Sec. II D that can acquire a torus one-point function, both with and without the insertion of topological lines wrapping nontrivial cycles. Recall from II A that if we have a 2D CFT with fusion category $\mathcal{C}$, the set of charges of local and disorder operators are described by representations of the tube algebra Tube($\mathcal{C}$), or equivalently, by objects in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ (for a physicist-friendly discussion, see Ref. [73]). For our discussion here, it will be helpful to recall the 3D (SymTFT) perspective on symmetries of 2D CFTs (again, see Ref. [73]). To any 2D CFT with symmetry $\mathcal{C}$, we can define a boundary condition $\mathcal{B}$ of the 3D Turaev-Viro (Levin-Wen) TQFT [122,123] associated to $\mathcal{C}$, which we will denote $\mathrm{TV}_\mathcal{C}$.[41] We can recover our original 2D CFT by dimensionally reducing $\mathrm{TV}_\mathcal{C}$ on an interval, with the physical boundary condition $\mathcal{B}$ on one end, and the topological Dirichlet boundary condition for $\mathcal{C}$ on the other. The category of bulk anyons is given by the Drinfeld center $\mathcal{Z}(\mathcal{C})$. We can form local and disorder operators in our 2D CFT by stretching anyon line operators across the interval, possibly attached to a topological line running along the Dirichlet boundary.

The Dirichlet boundary condition is associated to a Lagrangian algebra $\mathcal{A} \in \mathcal{Z}(\mathcal{C})$, which describes the anyons that are condensed on the boundary (can end on it).[42] More specifically, we have

$$\mathcal{A} = \sum_{\mu \in \mathcal{Z}(\mathcal{C})} V^\mu \cdot \mu, \tag{B1}$$

where the vector space V^μ is the space of topological junction operators between the anyon line μ and the Dirichlet boundary.[43] An anyon μ describes a charge that can carried by a local (not disorder) operator if and only if $\mu \in \mathcal{A}$, i.e., we have $\dim(V^\mu) > 0$. The algebra $\mathcal{A}$ is equipped with a multiplication map $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, defined by fusing anyon lines attached to the boundary. In components, the multiplication map is defined by a family of maps

$$(m^\rho_{\mu\nu})_i : V^\mu \otimes V^\nu \to V^\rho, \tag{B2}$$

where $i = 1, \ldots, N^\rho_{\mu\nu}$ runs over the distinct fusion channels $i: \mu \otimes \nu \to \rho$ (see Fig. 9).

We can now describe the set of charges that can acquire a torus one-point function, both with and without additional topological line insertions.

(i) The set of charges $\mu \in \mathcal{Z}(\mathcal{C})$ of local or disorder operators that can acquire a torus one-point function, possibly with other topological line insertions, are those that appear in fusions $\rho \otimes \bar{\rho}$. This set of charges

[41] $\mathrm{TV}_\mathcal{C}$ can be viewed as "Tube($\mathcal{C}$) gauge theory," due to very recent work [15].

[42] See, e.g., Ref. [124] for further discussion on Lagrangian algebras and their relationship to gapped boundary conditions.

[43] Formally, we have $V^\mu = \mathrm{Hom}_\mathcal{C}(F(\mu), 1_\mathcal{C})$, where $F: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor. In the notation of [73], we have $V^\mu = W^\mu_1$.

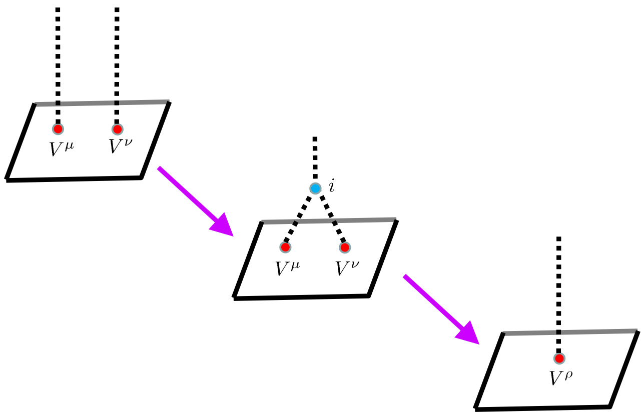


FIG. 9. Starting with two topological junctions in V^μ, V^ν and a fusion channel $i:\mu\otimes\nu\to\rho$, we can describe the algebra multiplication maps $(m^\rho_{\mu\nu})_i$ as follows. First, we fuse the anyon lines with the bulk junction operator corresponding to our chosen fusion channel. We then shrink the three junction operators (two boundary, one bulk) to a point, obtaining some topological junction in V^ρ.

generates the adjoint subcategory $\mathcal{Z}(\mathcal{C})_{\mathrm{ad}}$ of the Drinfeld center. Note that this set of charges is independent of the choice of boundary condition, and is manifestly invariant under any possible orbifolding.

(ii) The set of charges $\mu\in\mathcal{A}$ of local operators that can acquire a torus one-point function in the vacuum can be described as follows. First, according to the previous statement, there must exist an anyon ρ such that μ appears in the fusion $\rho\otimes\bar\rho$. For any fusion channel $i:\rho\otimes\bar\rho\to\mu$, we compute the state

$$(m^\mu_{\rho\bar\rho})_i\left(\sum_k|\psi_k\rangle\otimes|\bar\psi_k\rangle\right)\in V^\mu, \qquad \text{(B3)}$$

where $|\psi_k\rangle$ denotes an orthonormal basis for V^ρ. Then a local operator with charge μ can get only get a nonzero torus one-point function if the state (B3) in V^μ is nonzero. This answer is simply an abstract version of the operation illustrated above in Fig. 6, of sewing together sphere three-point functions by summing over a basis of local operators. The difference is that we have stripped off the physical boundary condition $\mathcal{B}$, obtaining a universal characterization for any 2D CFT with $\mathcal{C}$ symmetry.

We strongly suspect that the set of anyons described by (B3) are precisely those anyons uncharged under the maximal invertible subsymmetry $\mathcal{C}^\times$, i.e., the group of invertible objects of $\mathcal{C}$.[44] One direction is obvious: any

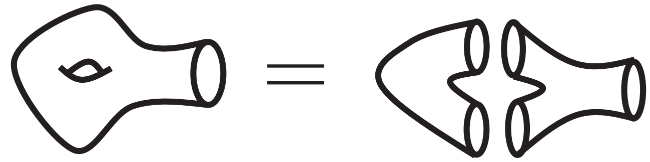


FIG. 10. We can decompose the punctured torus $\mathbb{T}^2_*:\varnothing\to\mathbb{S}^1$ as the composition of a bent cylinder $\mathrm{Cyl}:\varnothing\to\mathbb{S}^1\sqcup\mathbb{S}^1$ with a pair-of-pants $\mathrm{Pants}:\mathbb{S}^1\sqcup\mathbb{S}^1\to\mathbb{S}^1$.

anyon that gets a torus one-point function in the vacuum cannot carry charge under any invertible symmetry. While we were unable to provide a proof of the converse, this equivalence holds in every example we considered, including arbitrary modular symmetry categories (Appendix B 1) and exotic examples such as Haagerup symmetry (Appendix C).

Let us now derive this characterization using the tools of TQFT. Recall that the Turaev-Viro theory (equipped with Dirichlet boundary condition) is a fully extended TQFT with boundary (see, e.g., [125]):

$$\mathrm{TV}_{\mathcal{C}}:\ 3\text{-}\mathrm{Cob}_\partial\to 3\text{-}\mathrm{Vect}, \qquad \text{(B4)}$$

characterized through the cobordism hypothesis [126,127] by its value on the point,

$$\mathrm{TV}_{\mathcal{C}}(\mathrm{pt})=\mathcal{C}\text{-}\mathrm{Mod}, \qquad \text{(B5)}$$

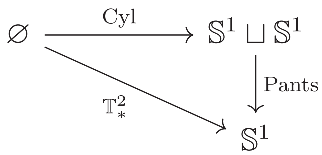
the 3-vector space of $\mathcal{C}$-module categories, and its value on the half-open interval $\mathbb{I}_*=[0,1)$ (viewed as a cobordism $\varnothing\to\mathrm{pt}$)

$$\mathrm{TV}_{\mathcal{C}}(\mathbb{I}_*)=\mathcal{C}, \qquad \text{(B6)}$$

viewed as a module category over itself. The category $\mathcal{Z}(\mathcal{C})$ of bulk anyons is given by the value $\mathrm{TV}_{\mathcal{C}}(\mathbb{S}^1)$ on the circle, and the algebra $\mathcal{A}$ of anyons condensable on the boundary is given by the value $\mathrm{TV}_{\mathcal{C}}(\mathrm{Ann})$ on the half-open annulus $\mathrm{Ann}=\mathbb{S}^1\times\mathbb{I}_*:\ \varnothing\to\mathbb{S}^1$.

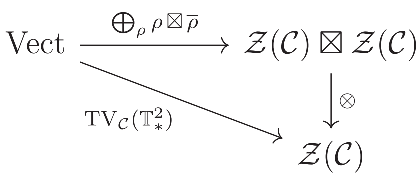
The set of anyons that can acquire a torus one-point function with possible insertions of arbitrary topological lines along the Dirichlet boundary are equivalent to the set of anyons which admit a nonzero state in the defect Hilbert space of the Turaev-Viro theory on the torus with one anyon insertion. To see this, note that any state in this Hilbert space can be prepared by adding arbitrary topological line insertions along the Dirichlet boundary. This defect Hilbert space is characterized by the value $\mathrm{TV}_{\mathcal{C}}(\mathbb{T}^2_*)$ on a punctured torus. To compute $\mathrm{TV}_{\mathcal{C}}(\mathbb{T}^2_*)$, note that the punctured torus can be decomposed (see Fig. 10) as a composition

[44]Formally, note that the invertible action of $\mathcal{C}^\times$ on topological junctions with the Dirichlet boundary defines a $\mathcal{C}^\times$ action on $\mathcal{A}$, i.e., a functor $B\mathcal{C}^\times\to\mathcal{Z}(\mathcal{C})$, whose value on the basepoint of $B\mathcal{C}^\times$ is given by $\mathcal{A}$. The desired statement is that the $\mathcal{C}^\times$ invariants in $\mathcal{A}$ generate the same fusion subcategory of $\mathcal{Z}(\mathcal{C})$ as the objects characterized by the map (B3) being nonzero.



(B7)

of two cobordisms: a bent cylinder $\mathrm{Cyl}:\varnothing \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ and a pair-of-pants $\mathrm{Pants}: \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$. The value of $\mathrm{TV}_{\mathcal{C}}$ on Cyl and Pants can be easily computed, so by applying $\mathrm{TV}_{\mathcal{C}}$ to (B7), we get a commutative diagram in 2-Vect:



(B8)

Thus, we have $\mathrm{TV}_{\mathcal{C}}(\mathbb{T}^2_*) = \bigoplus_\rho \rho \otimes \bar\rho$, and any anyon μ that appears in a fusion $\rho \otimes \bar\rho$ has a nonzero state in the associated $\mathbb{T}^2$ defect Hilbert space, as claimed.

Now, the set of anyons that can acquire a torus one-point function in the CFT vacuum is characterized by the boundary state associated to the Dirichlet boundary condition in the punctured torus Hilbert space. This boundary state is described by the value $\mathrm{TV}_{\mathcal{C}}(\mathbb{T}^2_* \times \mathbb{I}_*)$ on the manifold with corners $\mathbb{T}^2_* \times \mathbb{I}_*$, viewed as a cobordism $\mathbb{T}^2_* \to \mathbb{S}^1 \times \mathbb{I}_*$ from the punctured torus to the annulus (see Fig. 11). As before, we can view our cobordism as a composition of a bent cylinder and a pair of pants, now both multiplied by $\mathbb{I}_*$. Thus, we obtain a commutative diagram in the 2-category (1,2,3)-Cob_∂:

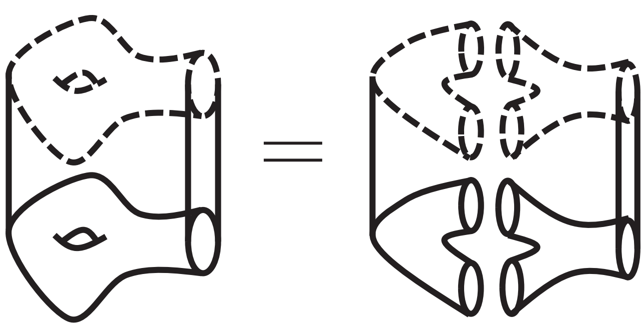
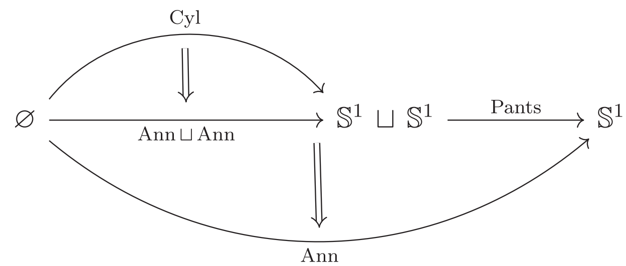
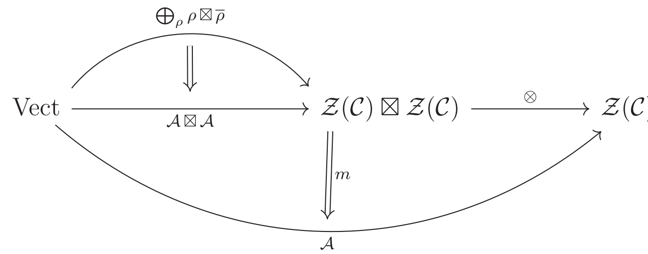


FIG. 11. The product $\mathbb{T}^2_* \times \mathbb{I}_*$ of the punctured torus with a half-open interval can be decomposed as a composition similarly to how we decomposed the punctured torus itself (Fig. 10). Each manifold with corners pictured here is viewed as a 2-morphism in (1,2,3)-Cob_∂ from the composition of its left edge with its top edge to its right edge. Note that the bottom edges, given by the Dirichlet boundary, does not count as a source or target from the perspective of (1,2,3)-Cob_∂. The corresponding commutative diagram in (1,2,3)-Cob is given by (B9).



(B9)

Applying $\mathrm{TV}_{\mathcal{C}}$, we get a commutative diagram in 2-Vect:



(B10)

The top 2-morphism is the map

$$\bigoplus_\rho \rho \boxtimes \bar\rho \to \mathcal{A} \boxtimes \mathcal{A}, \tag{B11}$$

defined on a component $\rho \boxtimes \bar\rho$ by the state

$$\sum_k |\psi_k\rangle \otimes |\bar\psi_k\rangle \in V^\rho \otimes V^{\bar\rho}, \tag{B12}$$

where $|\psi_k\rangle$ denotes an orthonormal basis for V^ρ as above. Horizontally composing with the 1-morphism $\mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C}) \xrightarrow{\otimes} \mathcal{Z}(\mathcal{C})$ and then vertically composing with the 2-morphism $\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$, we obtain the desired 2-morphism

$$\mathrm{TV}_{\mathcal{C}}(\mathbb{T}^2_* \times \mathbb{I}_*): \bigoplus_\rho \rho \otimes \bar\rho \to \mathcal{A}, \tag{B13}$$

which characterizes the set of charges $\mu \in \mathcal{A}$ of local operators in the 2D CFT that can acquire a torus one-point function in the vacuum. Expanding (B13) in components, we recover the desired formula (B3).

1. Diagonal RCFTs and modular symmetry categories

As an application, we can prove that all noninvertible symmetries are broken by string loops for the special case when the symmetry category $\mathcal{C}$ of our 2D CFT is itself a modular tensor category before passing to the Drinfeld center $\mathcal{Z}(\mathcal{C})$. This includes, in particular, the collection of Verlinde lines of any diagonal RCFT.

Suppose, then, we have a 2D CFT with a modular symmetry category $\mathcal{C}$.[45] By modularity, we have $\mathcal{Z}(\mathcal{C}) = \mathcal{C}\boxtimes\bar{\mathcal{C}}$ as modular tensor categories.[46] The algebra $\mathcal{A}$ associated to the Dirichlet boundary condition of the Turaev-Viro theory $\mathrm{TV}_{\mathcal{C}}$ is given by

$$\mathcal{A} = \bigoplus_{\rho} \rho\boxtimes\bar{\rho}. \tag{B14}$$

Note that all the vector spaces $V^{\rho\boxtimes\bar{\rho}}$ are one-dimensional. The multiplication map $m: \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ is defined via diagonal fusion. In components, for a fusion channel

$$(i,\bar{j}): (\mu\boxtimes\bar{\mu}) \otimes (\nu\boxtimes\bar{\nu}) \to \rho\boxtimes\bar{\rho}, \tag{B15}$$

we have

$$(m^{\rho}_{\mu\nu})_{(i,\bar{j})} = \delta_{ij}. \tag{B16}$$

Since each $V^{\rho\boxtimes\bar{\rho}}$ is one-dimensional, there is no sum in (B3), and so a charge $\mu\otimes\bar{\mu}$ can get a torus one-point function in the vacuum if and only if μ appears in a fusion $\rho\otimes\bar{\rho}$. This set of μ generate the adjoint subcategory $\mathcal{C}_{\mathrm{ad}}$ of $\mathcal{C}$ itself.

Thus, we need to show that having operators with charges in the adjoint subcategory $\mathcal{C}_{\mathrm{ad}}$ breaks the noninvertible symmetry down to the maximal invertible subsymmetry $\mathcal{C}^{\times}$. Now, the symmetry preserved once local operators of charges generating $\mathcal{C}_{\mathrm{ad}}$ acquire nonzero one-point functions is given [25] by the dual $U^{\vee}_{\mathcal{C}}$ of the universal grading group of $\mathcal{C}$ [[85], Definition 4.14.2]. The action of invertible lines on charges by linking defines a grading of $\mathcal{C}$ by the group of characters $(\mathcal{C}^{\times})^{\vee}$, which induces a map $U_{\mathcal{C}}\to(\mathcal{C}^{\times})^{\vee}$, or dually, a map $\mathcal{C}^{\times}\to U^{\vee}_{\mathcal{C}}$. But by modularity of $\mathcal{C}$, this map is an isomorphism [[128], Theorem 6.3], so we see that the nonzero torus one-point functions of operators generating $\mathcal{C}_{\mathrm{ad}}$ precisely breaks $\mathcal{C}$ to its maximal invertible subsymmetry $\mathcal{C}^{\times}$.

APPENDIX C: EXAMPLE: HAAGERUP SYMMETRY

As evidence that the breaking of all noninvertible symmetries by string loops happens generally, let us quickly verify this effect in the case of a truly exotic noninvertible symmetry in 2D: Haagerup symmetry, described by the Haagerup fusion category $\mathcal{H}_3$, with simple objects

$$1,\quad \alpha,\quad \alpha^2,\quad \rho,\quad \alpha\rho,\quad \alpha^2\rho, \tag{C1}$$

and fusion rules specified by

$$\alpha^3 = 1,\quad \alpha\rho = \rho\alpha^2,\quad \rho^2 = 1+\rho+\alpha\rho+\alpha^2\rho. \tag{C2}$$

If there were a counterexample to our general story, one might expect it to be something like Haagerup symmetry: a noninvertible, non-Abelian symmetry that cannot be obtained from group-like symmetry or Verlinde lines via discrete gaugings. Nevertheless, we will see that our general story holds true: at one-loop, Haagerup symmetry is broken to its $\mathbb{Z}_3$ invertible subsymmetry generated by α. While there is no formal construction of a 2D CFT with Haagerup symmetry (which could be used in a string compactification), recent numerical evidence favors its existence [129].[47]

What are the selection rules that Haagerup symmetry places on sphere correlation functions of local operators? The Drinfeld center $\mathcal{Z}(\mathcal{H}_3)$ of the Haagerup fusion category is described in [[130], section 8]. We will only need to consider two nontrivial anyons, given by π_1 and π_2. The Dirichlet boundary condition is specified by the Lagrangian algebra

$$\mathcal{A} = 1\oplus\pi_1\oplus(\mathbb{C}^2\cdot\pi_2). \tag{C3}$$

It will help to recall the analogy [131] between Haagerup symmetry and invertible S_3 symmetry; note that if we replaced the third equation in (C2) with $\rho^2 = 1$, we would recover a presentation of the symmetric group S_3. In this analogy, π_1 is analogous to the sign representation of S_3, while π_2 is analogous to the standard two-dimensional irreducible representation. Thus, it should not be surprising that local operators $\mathcal{O}_{\pi_1}$ of charge π_1 are uncharged under the invertible $\mathbb{Z}_3$ subsymmetry, while local operators $\mathcal{O}_{\pi_2}$ of charge π_2 come in a multiplet $\mathcal{O}^{\pm}_{\pi_2}$ of two local operators, with charges $\omega^{\pm1}$ under the $\mathbb{Z}_3$ symmetry, where ω is a primitive third root of unity. Moreover, the action of ρ on $\mathcal{O}_{\pi_1}$ is by a sign, while the action of ρ on $\mathcal{O}^{\pm}_{\pi_2}$ produces $\mathcal{O}^{\mp}_{\pi_2}$ together with a superposition of disorder operators.

Now, what charges can acquire a nonzero torus one-point function? Clearly, any operator charged in π_2 cannot, since it has nonzero charge under the invertible $\mathbb{Z}_3$ symmetry. What about operators of charge π_1? If we took the analogy with S_3 too seriously, we might guess that they could not, since there might be cancellations in (B3) similarly to those that appear for S_3 symmetry. If this were true, then Haagerup symmetry would be a counterexample to our general expectation, since the noninvertible symmetry ρ would remain unbroken to all orders in the string loop expansion.

[45] Note that $\mathcal{C}$ might not describe all the symmetries of our CFT (and certainly does not describe any non-Abelian symmetries, since $\mathcal{C}$ is assumed to be braided). For example, consider a WZW model based on a compact Lie group G. While the set of Verlinde lines describes all the symmetries that commute with the current algebra, it only includes the invertible symmetries corresponding to the center Z of G, and not the larger $(G_L\times G_R)/Z$ invertible symmetry of the WZW model.

[46] Here, $\bar{\mathcal{C}}$ denotes the category with the opposite braiding.

[47] In fact, [129] suggests that it might even exist in something as ordinary as a $\mathbb{Z}_3$ orbifold of T^2.

Let us check this guess in the 2D TQFT with Haagerup symmetry constructed in [132,133].[48] This TQFT has six (topological) local operators $1, v, u_1, \bar{u}_1, u_2, \bar{u}_2$. The operator v has charge π_1, while the operators $u_i, \bar{u}_i$ have charge π_2. We will compute the torus one-point function $\langle v\rangle_{\mathbb{T}^2}$ by sewing together sphere three point functions. For this, we need the fusion of the local operators with their conjugates, which are given by

$$v\times v = 1+3v, \quad u_1\times \bar{u}_1 = 1-\zeta^{-1}v, \quad u_2\times \bar{u}_2 = 1+\zeta v, \tag{C4}$$

where $\zeta = (3+\sqrt{13})/2 = \langle\rho\rangle$. We can now compute

$$\langle v\rangle_{\mathbb{T}^2} = \sum_{\mathcal{O}} \langle v\mathcal{O}\bar{\mathcal{O}}\rangle_{\mathbb{S}^2} = 3-2\zeta^{-1}+2\zeta = 9. \tag{C5}$$

So the torus one-point function $\langle v\rangle_{\mathbb{T}^2}$ is nonzero. The cancellation that would have happened for S_3 symmetry does not occur.[49] As a result, we see that the selection rules for Haagerup symmetry are violated at one loop, and even something as exotic as $\mathcal{H}_3$ symmetry on the world sheet would be broken to its maximal invertible subsymmetry after considering string loops.

APPENDIX D: EMERGENT NONINVERTIBLE SYMMETRIES BEYOND PERTURBATIVE STRINGS

One of the main points of this paper is that selection rules of noninvertible world sheet symmetries are generically broken by loop effects in string perturbation theory. When the states running in the loop survive the field theory limit, this string loop contribution can be rewritten as a sum of one-loop field theory diagrams, which suggests that these sorts of emergent symmetries may also appear in QFT or in quantum gravity away from the perturbative string limit. To illustrate this point, consider a perturbative string background with the Klein bottle as target space. Since the Klein bottle is a toroidal orbifold

$$T^2/\mathbb{Z}_2, \qquad (X,Y)\sim(X+\pi R, -Y), \tag{D1}$$

this sigma model has a noninvertible symmetry of the kind discussed in Sec. II C.

We can see echoes of this noninvertible symmetry in the low-energy approximation of string theory (supergravity), even if we keep the Klein bottle large in string units. Reducing any supergravity theory on the Klein bottle and keeping the full Kaluza-Klein (KK) tower, the interaction terms in the supergravity theory lead to couplings between KK modes that respect the selection rule for the noninvertible symmetry: i.e., if we label the KK modes by pairs (k_X, k_Y) of momenta, only defined up to $k_Y\to -k_Y$, interaction terms are only possible if

$$\sum_i k_{X,i} = 0, \qquad \sum_i \pm k_{Y,i} = 0, \tag{D2}$$

for some choice of $\pm$ signs. This symmetry is likely the low-energy EFT avatar of the one we found in the world sheet; the fact that the vertices satisfy the selection rules matches with the fact that the noninvertible symmetry is satisfied at tree level. Note that keeping the entire KK tower does not make sense as an EFT in the lower dimensional sense, since the lower-dimensional EFT cutoff is the KK scale. Instead, it is better understood as a way to organize the higher dimensional EFT when placed on a large Klein bottle; in any case, the KK states exist as physical excitations, and they are long-live enough that one can ask questions about their dynamics.

Breaking of a selection rule like (D2) at the quantum level is very natural; for instance, an external particle with Y momentum $k_{Y,1}$ can become a loop pair with momenta $k_{Y,2}$ and $k_{Y,1}-k_{Y,2}$ [respecting (D2)], and then recombine to form a particle with momentum $-k_{Y,2}+k_{Y,1}-k_{Y,2} = k_{Y,1}-2k_{Y,2}$ (analogously to the process illustrated in Fig. 5). The loop then mediates a transition of a particle with Y momentum $k_{Y,1}$ to another with momentum $k_{Y,1}-2k_{Y,2}$, and since $k_{Y,2}$ was arbitrary, the resulting process manifestly violates (D2). In this way ordinary, field theory loop effects can violate a noninvertible symmetry preserved by the couplings, just like in string theory. Again, we emphasize that the noninvertible symmetry does not act in any EFT of a fixed dimension: the states we are considering are above the cutoff of the lower-dimensional EFT, while the noninvertibility is coming from the Klein bottle reduction, and would not appear in the fully decompactified theory.

Because the symmetry is broken by quantum effects, one expects it to become a good approximate close to any classical limit, even away from the perturbative string. For instance, we could consider M-theory on the Klein bottle. Its low-dimensional expansion, 11D supergravity, is a power series expansion in powers of the 11D Planck mass M_{11}; in the decompactification limit, when the characteristic size of the Klein bottle R is very large in 11D Planck units, quantum corrections that break the noninvertible symmetry are naturally suppressed in powers of $1/(M_{11}R)$, and they should vanish in the limit, where the noninvertible symmetry becomes a subsymmetry of higher-dimensional Poincaré, which are exact. It would be interesting to check this example in detail; although we have not done so, we

[48]Even though the Haagerup symmetry is spontaneously broken in this TQFT, we should still expect its selection rules to hold if we define correlation functions in a direct sum over all of its distinct vacua, analogously to the selection rules for S_3 in the S_3 symmetry-breaking TQFT.

[49]We could have done an analogous calculation for S_3; the result can be obtained by taking $3\to 0$, $\zeta\to 1$ in (C4) and (C5), reproducing the cancellation in $\langle v\rangle_{\mathbb{T}^2}$.

have checked that corrections are suppressed in this way for a simple toy model (a Φ^3 theory) when a regularization preserving higher-dimensional Poincaré symmetry is used, as in [134,135].

From this point of view, it may be that the phenomenon of noninvertible selection rules of the classical action being weakly broken by quantum effects is not necessarily an intrinsically stringy phenomenon, but rather also appears as a general feature of compactification on manifolds with local isometries that fail to be well-defined globally. Clearly, it would be interesting to flesh this story out in more detail.

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