

## A derivation of the time-energy uncertainty relation

**John S Briggs**

Theoretical Quantum Dynamics, Institute of Physics, University of Freiburg,  
Hermann-Herder-Strasse 3, 79104 Freiburg, Germany.

E-mail: [briggs@physik.uni-freiburg.de](mailto:briggs@physik.uni-freiburg.de)

**Abstract.** Some of the difficulties that have occurred in the past associated with the derivation of a time-energy uncertainty relation are discussed. Then it is shown that, beginning with a time-independent quantum composite of system and environment, a time-energy uncertainty relation for the system alone can be derived in the limit that the environment becomes classical, to provide a classical clock with which to monitor the quantum system.

In his original paper [1], Heisenberg began by deriving the uncertainty relation for position and momentum on the basis of a supposed experiment in which an electron is observed using a  $\gamma$ -ray microscope. By consideration of the theory of the Compton effect he could argue that the precision of the determination of position  $x$  and momentum  $p$  are connected by the uncertainty relation

$$\Delta x \Delta p \sim \hbar \quad (1)$$

Later in the paper he considered specifically the case of a freely-moving Gaussian wavepacket in  $x$  space with a width  $\Delta x$ . Using the commutation relation

$$[\mathbf{x}, \mathbf{p}] = (\mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x}) = -i\hbar \quad (2)$$

of Max Born [2], Jordan [3] showed that a transformation between  $x$  and  $p$  representations is effected by the function  $\exp(ipx/\hbar)$  i.e. is a Fourier transformation. Using this result, Heisenberg could transform his wavepacket to momentum space and obtained a width  $\Delta p$  such that

$$\Delta x \Delta p = \hbar \quad (3)$$

It is important to note the equality in eq.(3), compared to the order of magnitude in eq.(1). Also it is clear that Heisenberg's derivation of the equality arises from the property of the Fourier transform relation between position and momentum free wavepackets and is not proved generally.

Heisenberg also discusses the classically conjugate variables of time and energy and defines a time operator through the, quote, "familiar relation"

$$[\mathbf{E}, \mathbf{t}] = (\mathbf{E}\mathbf{t} - \mathbf{t}\mathbf{E}) = -i\hbar \quad (4)$$

On the basis of this commutation relation, Heisenberg assumes a time-energy uncertainty relation (TEUR)

$$\Delta E \Delta t \sim \hbar \quad (5)$$

although the equality, as appears in eq.(3), was not claimed. The argument for the existence of this uncertainty relation was then based upon a consideration of the accuracy  $\Delta E$  of energy measurement in a Stern-Gerlach apparatus when the atoms are subject to a deflecting field for a time  $\Delta t$ . Again the relation (5) is recovered. Note however, that in this derivation  $\Delta t$  is the duration, not the accuracy, of the time measurement. Implicit is that the time can be measured infinitely accurately. Hence, even in the paper introducing uncertainty relations, that of time-energy appears on a different footing to that of position-momentum.

Pauli[4] gave a powerful and well-known argument against the existence of a time operator, based on considerations of the boundedness of the energy operator. Pauli writes " we conclude therefore that the introduction of a time operator  $\mathbf{t}$  must be abandoned fundamentally and that the time  $t$  in quantum mechanics has to be regarded as an ordinary real number." Despite this, beginning with the seminal paper of Aharonov and Bohm [5], there have been numerous attempts to define an operator for time ( see refs. [6] [7] for a summary of these approaches).

In 1929 the origin of uncertainty relations was clarified in a paper by H.P.Robertson [8] whose length is inversely proportional to its importance. Using the definition of expectation values of Hermitian Hilbert-space operators (observables) and the mathematical property of the Schwartz inequality, he showed that for any two operators

$$\Delta A \Delta B \geq \frac{1}{2} | \langle [\mathbf{A}, \mathbf{B}] \rangle | \quad (6)$$

Clearly in the case (2) one has

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad (7)$$

irrespective of the state in which the expectation value is taken. Generally, however, the r.h.s. of (6) depends upon this state, as do the uncertainties, which are defined as

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \quad (8)$$

This rigorous proof of a lower bound for the product of uncertainties in the simultaneous measurement of two non-commuting observables is now generally accepted. However, then the non-existence of a time operator and hence the commutation relation (4), precludes the derivation of a rigorous time-energy uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2} \quad (9)$$

Similarly, the interpretation of  $\Delta t$  as the duration of a measurement is precluded.

In the years ensuing the Heisenberg and Robertson papers, there have been many attempts to derive the TEUR, either in the indeterminate form (5) or the precise form (9). In these derivations,  $\Delta t$  has been variously attributed to the accuracy of, or to the duration of a measurement, or simply not specified at all. A summary of these methods is given in [9] and further discussion can be found in [5]. The more convincing of these methods concentrate upon the precise connection of the TEUR with the particular measurement process. This is true particularly in the Aharonov and Bohm paper [5], to be referred to as AB, which contains also a critique of earlier discussions of the TEUR. These authors were the first to examine the real content of uncertainty relations connecting time and energy measurement. Their main conclusion is to criticise the previous interpretations, going back to Heisenberg, of  $\Delta t$  as the duration of a measurement. They show that energy can be measured arbitrarily accurately in a finite time. This demonstration is done by carefully distinguishing the role of the measuring apparatus ("clock") from that of the measured system. That is, for the first time they pointed out that whilst  $x$  and  $p$  are variables of the same quantum system, the energy  $E$  is of the quantum

system but time is measured by an external clock. AB discuss also a TEUR of the standard form (9), as derived for example by Mandelstam and Tamm [10], in which  $\Delta t$  is an "inner time" as defined by system dynamical variables. Then for example,  $\Delta t$  is the lifetime of a decaying system state and  $\Delta E$  the uncertainty in its energy. Note that in this interpretation  $\Delta t$  is not an uncertainty but a property of a statistical distribution of decay times. Whilst below in our derivation an "inner" time cannot be defined since all system variables are quantum operators and time is a classical variable, nevertheless we show that a TEUR of the standard form (9) can be derived but in which  $\Delta t$  appears precisely as the accuracy of an "outer" time measurement.

In ref.[9] we criticised the many derivations of the TEUR on the grounds that they all begin by assuming the time-dependent Schrödinger equation (TDSE). In the present and previous papers [11][12][13] we adopt the standpoint that time only appears meaningfully in quantum mechanics from a Hamiltonian dependence upon an external **classical** time. From this standpoint none of the previous methods used to derive the TEUR have a firm quantum-mechanical basis because, by using time in a quantum context, they fail to recognise that time is always simply a *classical* parameter. That is, they fail to recognise that

- (i) Whenever time enters quantum mechanics it arises as the classical time parameter in the solution of classical equations of motion. The only origin of time dependence of the quantum Hamiltonian arises from classical fields external to the quantum system. That is, time arises from solutions of Newton's equations or classical wave equations, corresponding to particle beams or light or heat (photons or phonons) respectively. To my knowledge, such fields are the only source of time-dependence in quantum mechanics. A closed quantum system has a time-independent Hamiltonian and there is no reason to introduce time.
- (ii) Hence the TDSE is a mixed quantum- classical equation and derivations of the TEUR based upon it do not begin with a wholly quantum-mechanical formalism.
- (iii) Since the time is classical,  $\Delta t$  must be associated with a classical measurement of time. Hence the form and interpretation of the TEUR, as recognised by AB, depends upon the precise way in which the measurement is performed.

Here we proceed from a completely time-independent formalism involving time-independent operators in Hilbert space and the Robertson operator-based uncertainty principle (6). In previous papers we gave an alternative derivation of the TDSE to that found in all quantum mechanics text-books. It was shown that, beginning with a closed composite of a quantum system interacting with its quantum environment described by a time-independent Schrödinger equation (TISE), one can take the limit that the environment (field or particle) becomes classical to provide a time-dependent Hamiltonian for the remaining quantum system and hence dynamics described by the TDSE. In paper [11] a sketch was given of a derivation of the TEUR in the same limit but beginning with a wholly quantum mechanical Robertson operator-based uncertainty relation. In [13], another derivation of the TDSE was given. In this note we give a detailed derivation of the TEUR using the method of paper [13].

## 1. The derivation

To begin one considers a closed composite of quantum system  $S$ , with eigenfunctions  $\phi_n$  and quantum environment  $\varepsilon$ , with eigenfunctions  $\chi_n$ . Then an eigenstate  $\Psi$  of the closed composite with Hamiltonian  $H$  and eigenenergy  $E$ , connected by the TISE

$$H\Psi = E\Psi \tag{10}$$

can be written as the entangled state

$$\Psi = \sum_n \phi_n \chi_n. \tag{11}$$

The environment eigenstates  $\chi_n$  with eigenenergies  $E_\epsilon^n$  depend on environment variables. The system eigenstates  $\phi_n$  with eigenenergies  $E_s^n = E - E_\epsilon^n$  depend only on system variables. To emphasise the necessity to specify precisely what is measured, we consider a "collision" experiment in which the system remains localised but the environment not (this encompasses both particle beams and quantised fields as environment). However, almost all experiments in which external probes are used to induce transitions in quantum systems fall into this category. The formulation of the problem goes right back to Born's 1926 paper [2], in which continuum states are introduced, only a few months after Schrödinger's original papers on the quantisation of bound states. Born's paper is remarkable in that not only was he the first to write down an entangled state of the form of eq.(2), he also interpreted  $|\chi_n|^2$  taken in the limit of large separation of  $S$  and  $\epsilon$  to define the *probability* of the system  $S$  being in the state  $n$ .

In Born's fully time-independent theory of collision processes, one considers the transition between a composite state  $\phi_m\chi_m$  at fixed total energy  $E$  and with incoming wave  $\chi_m$  to a set of states  $\phi_n\chi_n$  with outgoing waves  $\chi_n$  and the same energy  $E$ . The short-range time-independent interaction between  $\epsilon$  and  $S$  results in the entangled state eq.(11) at small separation. The measurement selects one component of eq.(11) with a probability given by the asymptotic large separation value of  $|\chi_n|^2$ . Then the results of the measurement of energy and time relate only to the chosen component  $n$  and henceforth we can drop this subscript. That is we consider one system state  $\phi$ , and one environment state  $\chi$  such that  $E = E_\epsilon + E_s = \text{constant}$ .

Let us begin with a completely quantum Robertson uncertainty relation for the composite system i. e.

$$\Delta A \Delta B \geq \frac{1}{2} | \langle \{ [A, B] \} \rangle | \quad (12)$$

where  $\{ \dots \}$  denotes integration over system  $S$  variables and  $\langle \dots \rangle$  over environment  $\epsilon$  variables. Now we specify the measurement by taking  $A$  to be the environment Hamiltonian  $H_\epsilon$  and  $B$  to be one of the environment variables  $R$ . Since both operators are purely in the environment Hilbert space, integration over system variables is made trivially to give

$$\Delta H_\epsilon \Delta R \geq \frac{1}{2} | \langle [H_\epsilon, R] \rangle | \geq \frac{\hbar \langle P \rangle}{2 M} \quad (13)$$

where  $P$  is the momentum conjugate to  $R$  and we have taken for simplicity

$$H_\epsilon = \frac{P^2}{2M} + V_\epsilon(R). \quad (14)$$

Note however eq.(13) stands also when  $P$  and  $R$  in eq.(14) are multidimensional. The variance  $\Delta H_\epsilon$  in eq.(13) is the variance  $\Delta E_\epsilon$  of the environment energy. However, since the total energy  $E = E_\epsilon + E_s$  is constant (remember we are considering measurement at large separations so that there is no interaction), the uncertainty  $\Delta E_\epsilon$  is equal to the uncertainty  $\Delta E_s$ . Hence eq.(13) becomes

$$\Delta E_s \Delta R \geq \frac{\hbar \langle P \rangle}{2 M}. \quad (15)$$

Next we consider the state of the environment to be describable by a semi-classical approximation, whose deciding dynamics is of course, classical mechanics. This description is valid for any apparatus used to measure a classical time. Then the wavefunction  $\chi(R)$  of the environment is written

$$\chi(R) = A(R) \exp \left( \frac{i}{\hbar} W(R) \right), \quad (16)$$

where  $A$  is a real amplitude and  $W$  is the classical Hamilton characteristic action function, i. e.  $W(R) = \int^R [2m(E_\epsilon - V_\epsilon(R'))]^{1/2} dR'$ . Then, quantum mechanically

$$P\chi = -i\hbar \frac{\partial}{\partial R} \left( A \exp \left( \frac{i}{\hbar} W \right) \right) = \exp \left( \frac{i}{\hbar} W \right) \left[ A \frac{\partial W}{\partial R} - i\hbar \frac{\partial A}{\partial R} \right]. \quad (17)$$

Hence, to lowest order in  $\hbar$

$$P\chi = \chi \frac{\partial W}{\partial R} = \chi [2M(E - V_\varepsilon)]^{\frac{1}{2}}. \quad (18)$$

However, defining time as

$$t = M \int^R \left[ \frac{dR'}{2M(E_\varepsilon - V_\varepsilon)} \right]^{\frac{1}{2}} = M \int^R \frac{dR'}{P_{cl}(R')} \quad (19)$$

in terms of the classical momentum  $P_{cl} = M\dot{R}$ , where  $\dot{R}$  is the classical velocity. Then in eq.(6),  $\langle P \rangle = \langle \chi | P | \chi \rangle = M\dot{R} \langle \chi | \chi \rangle = M\dot{R}$ , so that one has

$$\Delta E_S \Delta R \geq \frac{\hbar}{2} \dot{R} \quad (20)$$

or,

$$\Delta E_S \Delta t \geq \frac{\hbar}{2} \quad (21)$$

with,

$$\Delta t \equiv \Delta R / \dot{R}. \quad (22)$$

This completes the derivation. Several comments are in order. First the TEUR in this derivation involves an exact inequality since it is derived from the general quantum uncertainty principle. The uncertainty in the energy of the quantum system reflects corresponding uncertainties in the energies of the environment. Since total energy is conserved, practically the quantum energy would be measured by measuring the environment energy e.g. the final energy of a particle beam. Since the fluctuations in this are related to the momentum fluctuations and, according to eq.(12), the time fluctuations to the accuracy of the distance measurement, one sees that the TEUR eq.(15) also represents a quantum limit for the accuracy of environment momentum and position. Such a "derivation" of the TEUR from the position-momentum uncertainty relation has often been given (see eq.(31) of [9]), but assuming that the quantities refer to the quantum system alone. Then it is necessary to define a time from *quantum* expectation values, which is unjustified. Here the TEUR is a mixed quantum-classical relationship and the uncertainties are related directly to the accuracy of the measurements made.

As far as energy measurement is concerned, there is no need to consider the environment in the classical limit because of  $E = E_\varepsilon + E_S$  and because the Hamiltonian is a Hilbert-space operator. However, this limit is crucial to the time measurement since this requires the introduction of a classical time parameter. Also the  $\Delta t$  is not the *duration* of the measurement but the *accuracy* of the time measurement.

## 2. Conclusions

Heisenberg[1] gave two derivations of his position-momentum uncertainty relations. The first was based on the theory of the Compton effect i.e. on localising a particle through the scattering of a high-energy photon. This led to the product of uncertainties being of the order of magnitude of  $\hbar$ . Then he considered a derivation based on the Fourier relationships of free wavepackets which gave a product of uncertainties equal to  $\hbar$ . Robertson[8] gave a mathematically more general derivation based on the non-commutivity of  $\mathbf{p}$  and  $\mathbf{x}$  and derived the result that the product of uncertainties is greater than or equal to  $\frac{\hbar}{2}$ . Although Heisenberg assumed the existence of a time operator and non-commutivity of  $\mathbf{E}$  and  $\mathbf{t}$  he only gave a derivation of the TEUR based upon the Stern-Gerlach experiment, leading to the result that the product of uncertainties is of

the order of  $\hbar$ . Here the standpoint is adopted that time is a classical parameter in quantum mechanics and the absence of a time operator precludes the direct derivation of a Robertson quantum uncertainty relation.

Nevertheless, by considering the measuring device (clock) initially as a quantum environment and then taking a classical limit, it has been shown that a TEUR arising from a Robertson uncertainty relation giving a product greater than or equal to  $\frac{\hbar}{2}$  can be derived. In this case the  $\Delta t$  is the uncertainty of time measurement by a classical clock. Of course, as in the case of position-momentum, this does not invalidate other forms of the TEUR derived by other means, for example a consideration of the Fourier relationship between energy and time wavepacket widths. As stressed by AB, it is important in the application of a given TEUR to state precisely what kind of measurement is being made and to specify accordingly the meaning of the  $\Delta t$  involved e.g. does it refer to the accuracy of measurement, to the duration of measurement or perhaps to the lifetime of a decaying state.

## References

- [1] Heisenberg W *Zeits. f. Phys.* **43** 172 (1927)
- [2] Born M *Zeits. f. Phys.* **38** 803 (1926)
- [3] Jordan P *Zeits. f. Phys.* **40** 809 (1926)
- [4] Pauli W *Die Allgemeinen Prinzipien der Wellenmechanik* (Springer, Berlin, 1990) p.84
- [5] Aharonov A and Bohm D *Phys. Rev.* **122**, 1649 (1961).
- [6] Gislason E A, Sabelli N H, and Wood J W *Phys. Rev. A* **31** 2078 (1985)
- [7] Kobe D H and Aguileva-Navarro V C *Phys. Rev. A* **50** 933 (1994)
- [8] Robertson H P *Phys. Rev.* **34** 163 (1929)
- [9] Briggs J S and Rost J M, *Found. of Phys.* **31**, 693 (2001)
- [10] Mandelstam L and Tamm I *J. Phys. (USSR)* **9** 249 (1945)
- [11] Briggs J S and Rost J M, *Eur. Phys. Jour. D* **10**, 311 (2000).
- [12] Braun L, Strunz W T, and Briggs J S, *Phys. Rev. A* **70**, 033814 (2004).
- [13] Briggs J S, Boonchui S, and Khemmani S *J. Phys. B* **40** 1289 (2007)