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Algebraic and geometric reduction of multisymplectic manifolds

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Abstract. In this talk, we discuss an extension of the Marsden–Weinstein–Meyers symplectic reduction theorem to multisymplectic manifolds, and an adaptation of the Śniatycki–Weinstein, Dirac and Arms–Gotay–Cushman Poisson algebra reduction theorems to L_∞-algebras of multisymplectic observables. This is based on joint work with A Miti and L Ryvkin.

1. Introduction

Broadly stated, symplectic reduction begins with a symplectic manifold (M, ω), a group of symmetries $G \curvearrowright (M, \omega)$ and a means μ of realizing these symmetries in terms of the smooth functions on M, and, by a process of restriction and quotienting, returns a smaller symplectic manifold $(M_\lambda, \omega_\lambda)$ dependent on a parameter $\lambda \in \mathfrak{g}^*$. Even when the conditions necessary for performing this reduction fail to hold, it is possible to reduce the Poisson algebra of smooth functions $C^\infty(M)$ in such a way that coincides with $C^\infty(M_\lambda)$ whenever $(M_\lambda, \omega_\lambda)$ exists.

The aim of this brief exposition is to discuss our recent approach to adapting these ideas to the context of multisymplectic manifolds.

Definition 1. A *k-plectic structure* on M is a closed and nondegenerate $(k+1)$-form $\omega \in \Omega^{k+1}(M)$. A *multisymplectic structure* refers to an k-plectic structure for some $k \geq 1$.

Examples. (i) If (M^{2n}, σ) is a symplectic manifold, then σ^ℓ is a $(2\ell-1)$-plectic structure on M for $1 \leq \ell \leq n$. In particular, the 1-plectic structures on M are precisely the symplectic structures.

(ii) The volume forms on M are precisely the $(\dim M - 1)$-plectic structures on M.

(iii) Let $\pi_E : E \to \Sigma$ be a smooth fiber bundle. We will say that an element $\alpha \in \Lambda^k T^*_x E$ is *(1-)semihorizontal* if $\iota_u \iota_v \alpha = 0$ for all vertical tangent vectors $u, v \in \ker \mathrm{d}(\pi_E)_x$, and denote by $\pi : \Lambda^k_1 T^* E \to E$ the bundle of semihorizontal k-forms on E. The *canonical k-form* $\theta \in \Omega^k(\Lambda^k_1 T^* E)$ is given by

$$\theta_\alpha(v_1, \ldots, v_k) = \alpha(\pi_* v_1, \ldots, \pi_* v_k), \qquad v_i \in T_\alpha(\Lambda^k_1 T^* E),$$

and the *canonical k-plectic structure* on $\Lambda_1^k T^*E$ is defined to be $-\mathrm{d}\theta \in \Omega^k(\Lambda_1^k T^*E)$.

We refer to [6] for further details on mutlisymplectic structures in classical field theory.

2. Geometric reduction of k-plectic manifolds

Let us first recall certain relevant constructions from the symplectic setting. Fix a smooth action of a connected Lie group G on a symplectic manifold (M, ω).

Definition 2. The *canonical Poisson algebra structure* on $C^\infty(M)$ is given by

$$\{f, h\} = \omega(v_f, v_h),$$

where $v_f, v_h \in \mathfrak{X}(M)$ are the *Hamiltonian vector fields* associated to $f, h \in C^\infty(M)$, respectively. That is, $\mathrm{d}f = -\iota_{v_f}\omega$ and $\mathrm{d}h = -\iota_{v_h}\omega$.

The assignment of Hamiltonian vector fields in certain cases enables us to describe the action of G in terms of the members of $C^\infty(M)$. This motivates the following construction.

Definition 3. A *(symplectic) moment map* for the action $G \curvearrowright (M, \omega)$ is a smooth function $\mu : M \to \mathfrak{g}^*$ such that

(i) for every $\xi \in \mathfrak{g}$, the contraction $\langle \mu, \xi \rangle \in C^\infty(M)$ is a Hamiltonian function for the fundamental vector field $\underline{\xi} \in \mathfrak{X}(M)$, where $\underline{\xi}_p = \frac{\mathrm{d}}{\mathrm{d}t}\, e^{-t\xi} \cdot p\,|_{t=0}$, and

(ii) the assignment

$$\begin{aligned} \tilde{\mu} : \mathfrak{g} &\to C^\infty(M) \\ \xi &\mapsto \langle \mu, \xi \rangle \end{aligned}$$

is a homomorphism of Lie algebras.

Fix $\lambda \in \mathfrak{g}^*$ and write $G_\lambda \subseteq G$ for the λ-isotropy subgroup with respect to the coadjoint action $G \curvearrowright \mathfrak{g}^*$. We recall the Marsden–Weinstein–Meyer symplectic reduction theorem.

Theorem 1 ([9, 10])**.** *If $\mu^{-1}(\lambda) \subseteq M$ is smooth, and if $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ is free and proper, then there is a unique symplectic form $\omega_\lambda \in \Omega^2(M_\lambda)$ such that $\pi^*\omega_\lambda = i^*\omega$, where $i : \mu^{-1}(\lambda) \to M$ is the inclusion and $\pi : \mu^{-1}(\lambda) \to M_\lambda$ is the quotient map.*

We now turn to the multisymplectic setting. Fix a k-plectic manifold (M, ω) and an action of a connected Lie group G on M.

Definition 4. If a $(k-1)$-form $\alpha \in \Omega^{k-1}(M)$ and a vector field $v \in \mathfrak{X}(M)$ satisfy $\mathrm{d}\alpha = -\iota_v\omega$, then we say that α is a *Hamiltonian form* associated to v and that v is the *Hamiltonian vector field* associated to α. We write $\Omega^{k-1}_{\mathrm{ham}}(M)$ for the space of Hamiltonian forms on M.

In contrast with the symplectic case, the space of Hamiltonian forms $\Omega^{k-1}_{\text{ham}}(M)$ does not carry a natural Poisson structure. Instead, it exhibits both a natural Leibniz algebra structure and, when augmented with $\Omega^{\leq k-2}(M)$, a natural L_∞-algebra structure. We will return to the L_∞-algebra structure in Section 3. In this section, we consider only the former.

Definition 5. A *Leibniz algebra* is a vector space V equiped with an antisymmetric bilinear map $[\,,] : V \times V \to V$.

Informally, a Leibniz algebra is a Lie algebra in which the Jacobi condition has been omitted.

Definition 6. The *canonical Leibniz algebra structure* on $\Omega^{k-1}_{\text{ham}}(M)$ is given by

$$\{\alpha, \beta\} = \mathcal{L}_{v_\alpha}\beta,$$

where v_α is the Hamiltonian vector field associated to α.

In general, this bracket satisfies the Jacobi identity only up to exact terms.

Definition 7. A *(k-plectic) moment map* for the action $G \curvearrowright (M, \omega)$ is a smooth $\mathfrak{g}^*$-valued $(k-1)$-form $\mu \in \Omega^{k-1}(M, \mathfrak{g}^*)$ such that

(i) for every $\xi \in \mathfrak{g}$, the contraction $\langle \mu, \xi \rangle \in \Omega^{k-1}(M)$ is a Hamiltonian form for $\underline{\xi} \in \mathfrak{X}(M)$, and

(ii) the assignment

$$\begin{aligned} \tilde{\mu} : \mathfrak{g} &\to \Omega^{k-1}_{\text{ham}}(M) \\ \xi &\mapsto \langle \mu, \xi \rangle \end{aligned}$$

is a homomorphism of Leibniz algebras.

Given a closed $\mathfrak{g}^*$-valued $(k-1)$-form $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$, we denote the equalizer of μ and ϕ by

$$\mu^{-1}(\phi) = \{x \in M \mid \mu_x = \phi_x\},$$

we write $G_\phi \subseteq G$ for the ϕ-isotropy subgroup, and we denote the quotient of the two by $M_\phi = \mu^{-1}(\phi)/G_\phi$. Theorem 1 extends to the multisymplectic setting in the following manner.

Theorem 2 ([2] Theorem 4.1)**.** *If $\mu^{-1}(\phi) \subseteq M$ is an embedded submanifold and G acts freely on $\mu^{-1}(\phi)$, then there is a unique, closed $\omega_\phi \in \Omega^{k+1}(M_\phi)$ satisfying $i^*\omega = \pi^*\omega_\phi$, where $i : \mu^{-1}(\phi) \to M$ is the inclusion and $\pi : \mu^{-1}(\phi) \to M_\phi$ is the quotient map.*

$$\begin{tikzcd} \mu^{-1}(\phi) \arrow[r, "i"] \arrow[d, "\pi"'] & M \\ M_\phi & \end{tikzcd}$$

3. Algebraic reduction of k-plectic manifolds

As we recalled in Definition 2, the space of smooth functions on a symplectic manifold (M, ω) possesses a natural Poisson algebra structure. Even when the reduced space M_λ of Theorem 1 fails to exist, it is still possible to define a reduced Poisson algebra that, in the case that $(M_\lambda, \omega_\lambda)$ does exist, coincides with the natural Poisson structure on $C^\infty(M_\lambda)$. Here we present one such construction, due to Śniatycki and Weinstein [12].

Definition 8. The *momentum ideal* is the associative ideal $I_\mu \subseteq C^\infty(M)$ generated by the momenta $\mu_\xi = \langle \mu, \xi \rangle \in C^\infty(M)$ for any $\xi \in \mathfrak{g}$. That is,

$$I_\mu = \Big\langle \mu_\xi \Big\rangle_{\xi \in \mathfrak{g}} = \left\{ \sum_{i=1}^{n} f_i \; \mu_{\xi_i} \,\middle|\, f_i \in C^\infty(M),\ \xi_i \in \mathfrak{g},\ 1 \le i \le n \right\}$$

Theorem 3 ([12])**.** *The quotient* $(C^\infty(M)/I_\mu)^G$ *inherits a Poisson algebra structure from* $C^\infty(M)$*.*

There are distinct constructions for a reduced Poisson algebra due to Dirac [5] and to Arms, Gotay and Cushman [1]. Our aim here is to emulate these ideas in the multisymplectic case.

In Definition 6, we presented the Leibniz algebra structure on $\Omega^{k-1}_{\mathrm{ham}}(M)$. We now consider a natural L_∞-structure on $\widetilde{\Omega}^{k-1}_{\mathrm{ham}}(M) \oplus \Omega^{\le k-2}(M)$, where

$$\widetilde{\Omega}^{k-1}_{\mathrm{ham}}(M) = \{(v, \alpha) \mid \mathrm{d}\alpha = -\iota_v \omega,\ v \in \mathfrak{X}(M),\ \alpha \in \Omega^{n-1}(M)\}$$

is the space of Hamiltonian pairs. For an introduction to general L_∞-algebras, see, for example, [8]. Here we follow the conventions of [4].

Definition 9. The *L_∞-algebra of (classical) observables* $\mathrm{Ham}_\infty(M, \omega) = (\mathrm{Ham}, \{\tilde{l}_k\}_{k \ge 1})$ associated to the n-plectic manifold (M, ω) consists of

- the underlying graded vector space Ham, where

$$\begin{aligned} \mathrm{Ham}^0 &= \widetilde{\Omega}^{k-1}_{\mathrm{ham}}(M) \\ \mathrm{Ham}^i &= \Omega^{k-1+i}(M), \qquad 1-k \le i \le -1, \end{aligned}$$

  and $\mathrm{Ham}^i = 0$ otherwise;

- $n+1$ nontrivial multibrackets $\{\tilde{l}_i : \mathrm{Ham}^{\wedge i} \to \mathrm{Ham}\}_{1 \le i \le k+1}$, where

$$\begin{aligned} \tilde{l}_1(\alpha, v) &= 0, & (\alpha, v) &\in \mathrm{Ham}^0, \\ \tilde{l}_1(\alpha) &= (\mathrm{d}\alpha, 0), & \alpha &\in \mathrm{Ham}^{-1}, \\ \tilde{l}_1(\beta) &= \mathrm{d}\beta, & \beta &\in \mathrm{Ham}^{\le -2}, \end{aligned}$$

  where, for $(v_1, \alpha_1), \ldots, (v_i, \alpha_i) \in \mathrm{Ham}^0$,

$$\tilde{l}_2((v_1, \alpha_1), (v_2, \alpha_2)) = ([v_1, v_2], \varsigma(k)\iota(v_1 \wedge v_2)\ \omega)$$

  and

$$\tilde{l}_i((v_1, \alpha_1), \ldots, (v_i, \alpha_i)) = -(-1)^{\frac{i(i+1)}{2}} \iota(v_1 \wedge \ldots \wedge v_i)\ \omega, \qquad i \ge 3,$$

  and where, for $i \ge 2$, we have $\tilde{l}_i = 0$ on all other inputs.

Let $\mathfrak{g} \curvearrowright M$ be a Lie algebra action, and let $N \subseteq M$ be a $\mathfrak{g}$-preserved subset. In [3], by means of a series of intermediate constructions, we introduce an L_∞-subspace of *reducible observables* $\mathrm{Ham}_\infty(M,\omega)_{[N]} \subseteq \mathrm{Ham}_\infty(M,\omega)$ and a *vanishing ideal* $I_{\mathrm{Ham}_\infty}(N) \subseteq \mathrm{Ham}_\infty(M,\omega)_{[N]}$.

Theorem 4 ([3])**.** *The quotient*

$$\mathrm{Ham}_\infty(M,\omega)_N = \frac{\mathrm{Ham}_\infty(M,\omega)_{[N]}}{I_{\mathrm{Ham}_\infty}(N)}$$

possesses a natural L_∞-structure.

We call $\mathrm{Ham}_\infty(M,\omega)_N$ the *L_∞-reduction* of $\mathrm{Ham}_\infty(M,\omega)$ with respect to the action $\mathfrak{g} \curvearrowright (N \subseteq M)$. In particular, in the special case that this action admits a moment map, we obtain the following algebraic reduced space.

Definition 10. The *reduction* of $\mathrm{Ham}_\infty(M,\omega)$ with respect to the Hamiltonian action $\mathfrak{g} \curvearrowright (M,\omega)$, moment map $\mu \in \Omega(M,\mathfrak{g}^*)$ and level $\phi \in \Omega_{\mathrm{cl}}(M,\mathfrak{g}^*)$, is the reduction of $\mathrm{Ham}_\infty(M,\omega)$ with respect to $\mathfrak{g}_\phi \curvearrowright (\mu^{-1}(\phi) \subseteq M)$. We write $\mathrm{Ham}_\infty(M,\omega)_\phi = \mathrm{Ham}_\infty(M,\omega)_{\mu^{-1}(\phi)}$.

In the case that the quotient $M_N = N/G$ is a smooth manifold, and that ω descends to M_N, there is an inclusion $\bar{r}_N$ from the L_∞-reduced space of observables $\mathrm{Ham}_\infty(M,\omega)_N$ to the L_∞-algebra on the geometric reduced space $\mathrm{Ham}_\infty(M_N,\omega_N)$.

Theorem 5 ([3])**.** *The geometric reduction map*

$$\begin{aligned} r_N : \mathrm{Ham}_\infty(M,\omega)_{[N]} &\to \mathrm{Ham}_\infty(M_N,\omega_N) \\ (v,\alpha) &\mapsto (v_N,\alpha_N) \\ \alpha &\mapsto \alpha_N \end{aligned}$$

is a strict L_∞-morphism with kernel $I_{\mathrm{Ham}_\infty}(N)$. In particular, there is a natural inclusion of L_∞-algebras

$$\bar{r}_N : \mathrm{Ham}_\infty(M,\omega)_N = \frac{\mathrm{Ham}_\infty(M,\omega)_{[N]}}{I_{\mathrm{Ham}_\infty}(N)} \hookrightarrow \mathrm{Ham}_\infty(M_N,\omega_N).$$

Intriguingly, it is not generally the case that the map $\bar{r}_N$ is surjective. Informally, this means that there are more observables on the geometric reduced space (M_N,ω_N) than those that descend from (M,ω).

4. Outlook

We have introduced both a geometric and an algebraic multisymplectic reduction procedure. There are two clear paths for further development. First, it would be interesting to understand the potential phsyical applicability of our constructions, for example, to the multisymplectic approach to classical field theory advanced in [6]. Second, it would be interesting to relate our classical reduction procedures to recent work on the symmetries of prequantized 2-plectic manifolds [7, 11].

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