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The nonlinear tails in black hole ringdown: the scattering perspective

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Abstract: Black holes regain their static configuration by emitting ringdown gravitational waves, whose amplitude decays in time following a power law at fixed spatial positions. We show that the nonlinear decay power law may be obtained by simple scattering calculations using the in-in formalism and argue that the nonperturbative law should be $t^{-2\ell-1}$, where ℓ is the multipole of the propagating spherical gravitational wave.

Keywords: GR black holes, gravitational waves / theory, Gravitational waves in GR and beyond: theory

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Contents

1 Introduction

The advent of gravitational wave (GW) astronomy has ushered in a new era for probing the strong-field regime of gravity with unprecedented precision [1–4]. Current ground-based detectors — such as LIGO, VIRGO, and KAGRA — and the upcoming space-based observatory LISA are rapidly achieving the sensitivity required to examine the post-merger, or ringdown, phase of black hole (BH) collisions in detail [5–9]. A central aspect of this effort is to analyze the behavior of BH perturbations at late times, when both linear and nonlinear effects imprint distinctive signatures on the outgoing gravitational radiation.

Traditionally, our understanding of BH ringdowns has been rooted in linear perturbation theory. According to this approach, BHs settle into equilibrium by emitting characteristic damped oscillations — known as quasinormal modes (QNMs) — which decay exponentially, followed at late times by a slower, power-law falloff (for a review, see ref. [10]). A classic example is Price's result: for massless fields on a static, spherically symmetric BH background, the late-time signal at a fixed radius diminishes as $\sim t^{-2\ell-3}$, where ℓ is the multipole order [11–20].

Considerable progress has been made in refining theoretical predictions for both QNM spectra and late-time decay tails [21–33]. These linear predictions have been corroborated by numerical relativity simulations and increasingly precise GW observations.

In recent years, research attention has shifted towards the nonlinear domain of BH perturbations [34–66]. This line of work has revealed phenomena that transcend the linear regime, such as the emergence of second-order QNMs and their corresponding power-law decay tails. These nonlinear features are of particular interest because they naturally emerge

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from the QNM structure governing BH ringdowns [67, 68], thus offering a unique window into the nonlinear dynamics of gravity. Numerical studies have recently identified distinct decay laws in the Weyl scalar linked to these effects, differing notably from the classic Price tail behavior [69]. Moreover, full 3+1 numerical relativity simulations of BH mergers have reported power-law decays inconsistent with linear theory predictions [70, 71]. These insights not only deepen our understanding of nonlinear dynamics in BH perturbation theory but also highlight the importance of enhancing GW modeling to capture their potential observational signatures.

One notable finding is that nonlinear interactions can generate a second-order $\ell = 4$ mode — sourced by quadratic coupling of $\ell = 2$ modes — which decays at late times as $\sim t^{-9}$, as found numerically in ref. [69].[1] This points towards a more general decay pattern of the form $\sim t^{-2\ell-1}$, attributable to the evolution of quadratic QNMs in asymptotically flat spacetimes [72–74].

In this work, we reconsider the nonlinear tails of the QNMs from a different perspective. Using the in-in formalism, we show: *i)* that the nonlinear QNM tails can be simply understood as the result of the scattering of the QNMs off the Newtonian potential in flat spacetime and *ii)* that at any order in perturbation theory the nonlinear tail decays as $\sim t^{-2\ell-1}$, reinforcing the argument that the nonlinearities in gravity are relevant and even potentially more important than the linear effects.

The paper is organized as follows. In section 2 we briefly remind the reader of the in-in formalism. In section 3, as a warm-up, we rederive Price's law for the linear tail. In section 4 we address the nonlinear tail power-law and conclude in section 5. The article is supplemented with two appendices.

2 The in-in formalism

Our goal is to calculate the expectation value, at a given time, of the gravitational wave. Obviously, the in-out formalism, commonly adopted in quantum field theory to calculate cross-sections from one asymptotic state to another, is not applicable: the initial and the final state are the same. The most appropriate extension of the field theory to deal such issue it to generalize the time contour of integration to a closed-time path. More precisely, the time integration contour is deformed to run from a given time t_0 to t and back to t_0 [75]. For this reason, the formalism is also dubbed in-in formalism. Given an operator $\mathcal{O}$, its expectation value reads

$$\begin{aligned}\langle\mathcal{O}(t)\rangle &= \left\langle\left(Te^{-i\int_{t_0}^{t}dt' H_{\rm int}(t')}\right)^{\dagger}\mathcal{O}(t)\left(Te^{-i\int_{t_0}^{t}dt'' H_{\rm int}(t'')}\right)\right\rangle \\ &= \sum_{N=0}^{\infty} i^N\int_{t_0}^{t}dt_N\int_{t_0}^{t_N}dt_{N-1}\dots\int_{t_0}^{t_2}dt_1 \\ &\quad\cdot\left\langle\Big[H_{\rm int}(t_1),\Big[H_{\rm int}(t_2),\dots\Big[H_{\rm int}(t_N),\mathcal{O}(t)\Big]\dots\Big]\Big]\right\rangle,\end{aligned} \tag{2.1}$$

where T is the time-ordering operator. Each term in the series can be identified, as we shall see, with a given Feynman diagram. In particular, starting from the Schwarzschild

[1] To avoid any further confusion, we point out that the nonlinear law quoted in ref. [69], $\sim t^{-10}$, refers to the RW gauge. One power less is obtained for the helicity two degrees of freedom in the TT gauge [72].

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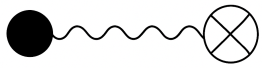

Figure 1. Tree-level graviton coupling directly to the matter source.

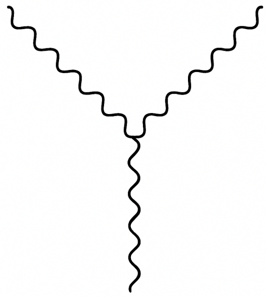

Figure 2. Three-graviton vertex.

metric of a spinless BH with mass M

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.2}$$

as long as we are far enough from the location of the BH, we can expand it for $r \gg M$. At first order in M/r, or at the tree-level, we get the classical metric in the Newtonian gauge

$$h^{\rm cl}_{\mu\nu} = \begin{pmatrix} 2M/r & 0 & 0 & 0 \\ 0 & 2M/r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

and the corresponding diagram in figure 1, where the black blob indicates the mass M sourcing a classical gravitational field. To study the nonlinear tails, we will also need the trilinear graviton vertex, for which we do not write the full matrix element here; we will do it in the following. The corresponding Feynman diagram is given in figure 2.

3 The linear tail

After an initial burst of radiation, the ringdown dominated by QNMs, the perturbation decays as a power-law tail rather than exponentially. This is the well-known Price's law [12]. The interpretation in terms of backscattering is known and is the following. The Schwarzschild geometry has a curved spacetime that modifies how waves propagate the spacetime curvature acts as an effective potential barrier. As a consequence, some part of the outgoing wave is backscattered off this potential. At late times, the dominant signal is not from waves trapped near the BH, the QNMs, but rather from backscattered waves far from the BH, scattered off the asymptotically decaying potential. This is why the tail decays as a power law.

The goal of this section is to formalize what described above in terms of Feynman diagrams, see figure 1. In this sense, it is preparatory to the next section where we will deal with the nonlinear tails. Our starting point is the definition of the graviton plane wave in Minkowski spacetime

$$h_{\mu\nu}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_s \left[\frac{e^{i\vec{k}\cdot\vec{x}-i\omega t}}{\sqrt{2\omega}} \epsilon^s_{\mu\nu} a^s(\vec{k}) + \text{h.c.} \right], \tag{3.1}$$

where $s = (+,\times)$ indicate the two graviton helicities with polarization vector $\epsilon^s_{\mu\nu}$.

We consider here on-shell propagating QNMs in flat spacetime which suffices to describe radiative modes of the metric at leading order in the $1/r$. We remark that a general off-shell treatment of the external graviton is required to draw firm conclusions about the late-time behaviour of higher order terms in $1/r$ in the metric perturbation (for a review, see ref. [76]). We point out as well that our procedure focusses on the ringdown phase and neglects backreaction effect, a procedure which is consistent in the soft limit which controls the large-time behaviour of the waveform in accordance with soft theorems [77].

Using the spherical harmonic expansion of the plane waves

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{r}) Y_{\ell m}(\hat{k}), \tag{3.2}$$

we can write

$$h_{\mu\nu}(t,\vec{x}) = 4\pi \int \frac{dk\, k^2}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_s \sum_{\ell m} \left[i^\ell j_\ell(kr) Y^*_{\ell m}(\theta,\phi) a^s_{\ell m}(k) \epsilon^s_{\mu\nu} e^{-i\omega t} + \text{h.c.} \right], \tag{3.3}$$

where we have defined the annihilation operator

$$a^s_{\ell m}(k) = \int d\Omega_k Y_{\ell m}(\hat{k}) a^s(\vec{k}). \tag{3.4}$$

Since the graviton is massless we can replace the momentum k with its frequency ω, so that the creation and annihilation operators obey the following commutator relation

$$\left[\hat{a}^s_{\ell m}(\omega), \hat{a}^{\dagger s'}_{\ell' m'}(\omega') \right] = (2\pi)^3 \frac{\delta(\omega-\omega')}{\omega^2} \delta_{\ell\ell'} \delta_{mm'} \delta^{ss'}, \tag{3.5}$$

where the ω^2 at the denominator will cancel the measure in the three-dimensional Fourier transform of the quantized graviton.

The next step is to consider the quadratic Pauli-Fierz action for the kinetic part of the graviton

$$S = \frac{1}{16\pi G} \int d^4x \Big[-\frac{1}{4}\partial_\rho h_{\mu\nu} \partial^\rho h^{\text{cl}\,\mu\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu} \partial^\nu h^{\text{cl}\,\rho\mu} + \frac{1}{4}\partial_\mu h \partial^\mu h_{\text{cl}} - \frac{1}{2}\partial_\nu h^{\mu\nu} \partial_\mu h_{\text{cl}} + \text{perm.} \Big], \tag{3.6}$$

where we treat $h_{\mu\nu}$ as a quantized graviton and $h = h^\mu_\mu$. Here $h^{\rm cl}_{\mu\nu}$ is the classical field which incorporates the information that we are not in a completely flat region of spacetime, but in the presence of a BH. The linear tail will be produced by the back reaction between the gravitational wave and the curved background.

To find the Hamiltonian, we derive the conjugate momentum from the quadratic action

$$\begin{aligned}\pi_{i,\alpha\beta} = &-\frac{1}{4}\partial^0 h_{j,\alpha\beta} + \frac{1}{2}\partial_\beta h^0_{j,\alpha} + \frac{1}{4}\partial^0 h_j \eta_{\alpha\beta} - \frac{1}{2}\partial_\beta h_j \eta^0_\alpha - \frac{1}{4}\partial^0 h_{j,\alpha\beta} \\ &+ \frac{1}{2}\partial_\alpha h^0_{j,\beta} + \frac{1}{4}\partial^0 h_j \eta_{\alpha\beta} - \frac{1}{2}\partial_\nu h^{0\nu}_j \eta_{\alpha\beta}, \end{aligned} \tag{3.7}$$

where $i \neq j$ can represent the graviton or, alternatively, its classical value. The corresponding Hamiltonian density reads

$$\mathcal{H}_{\rm kin}(t,\vec{x}) = \pi_{\alpha\beta}\partial_0 h^{\alpha\beta} + \pi_{{\rm cl}\,\alpha\beta}\partial_0 h^{{\rm cl}\,\alpha\beta} - \mathcal{L}_{\rm kin}(t,\vec{x}) \tag{3.8}$$

so that the final Hamiltonian for the quadratic vertex is the following

$$\begin{aligned} H_{\rm kin}(t) = 4\pi\sqrt{G}\int dr\, r^2 d\Omega\, \Bigg[\pi_{\alpha\beta}\partial_0 h^{\alpha\beta} + \pi_{{\rm cl}\,\alpha\beta}\partial_0 h^{{\rm cl}\,\alpha\beta} - \Bigg[-\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{{\rm cl}\,\mu\nu} \\ + \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{{\rm cl}\,\rho\mu} + \frac{1}{4}\partial_\mu h \partial^\mu h_{\rm cl} - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h_{\rm cl} - \frac{1}{4}\partial_\rho h_{{\rm cl}\,\mu\nu}\partial^\rho h^{\mu\nu} \\ + \frac{1}{2}\partial_\rho h_{{\rm cl}\,\mu\nu}\partial^\nu h^{\rho\mu} + \frac{1}{4}\partial_\mu h_{\rm cl}\partial^\mu h - \frac{1}{2}\partial_\nu h^{\mu\nu}_{\rm cl}\partial_\mu h\Bigg]\Bigg]. \end{aligned} \tag{3.9}$$

We take for simplicity a graviton that propagates along the x-axis in cartesian coordinates, with momentum

$$k^\mu = (\omega, \omega, 0, 0)\,. \tag{3.10}$$

The corresponding polarization tensors in TT gauge[2] are

$$\epsilon^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.11}$$

and

$$\epsilon^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.12}$$

In spherical polar coordinates, which we will use in the following, the four-momentum vector becomes

$$k^\mu = \left(\omega, \omega\cos\phi\sin\theta, \frac{\omega\cos\theta\cos\phi}{r}, \frac{-\omega\csc\theta\sin\phi}{r}\right) \tag{3.13}$$

[2]More correctly, we are making use of the outgoing radiation gauge (setting also the transverse-free and traceless conditions) which is the same gauge where the non-linearities are extracted numerically [44].

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so that

$$\epsilon^{+}_{\mu\nu}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & (\sin\theta\sin\phi)^2-\cos^2\theta & r\sin\theta\sin^2\phi\cos\theta+r\cos\theta\sin\theta & r\sin^2\theta\sin\phi\cos\phi\\ 0 & r\sin\theta\sin^2\phi\cos\theta+r\cos\theta\sin\theta & r^2\sin^2\phi\cos^2\theta-r^2\sin^2\theta & r^2\sin\theta\cos\theta\sin\phi\cos\phi\\ 0 & r\sin^2\theta\sin\phi\cos\phi & r^2\sin\theta\cos\theta\sin\phi\cos\phi & r^2\sin^2\theta\cos^2\phi\end{pmatrix} \tag{3.14}$$

and

$$\epsilon^{\times}_{\mu\nu}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 2\sin\theta\sin\phi\cos\theta & -r\sin\phi(\sin^2\theta-\cos^2\theta) & r\cos\theta\sin\theta\cos\phi\\ 0 & -r\sin\phi(\sin^2\theta-\cos^2\theta) & -2r^2\cos\theta\sin\theta\sin\phi & -r^2\sin^2\theta\cos\phi\\ 0 & r\cos\theta\sin\theta\cos\phi & -r^2\sin^2\theta\cos\phi & 0\end{pmatrix}. \tag{3.15}$$

The corresponding kinetic Hamiltonian reads

$$\begin{aligned}H_{\text{kin}}(t)=4\pi\sqrt{G}\int dr\, r^2 d\Omega\ \Bigg[\int\frac{d\omega\,\omega^2}{(2\pi)^{3/2}\sqrt{2\omega}}\sum_{\ell,m} i^\ell j_\ell(\omega r)Y^*_{\ell m}(\theta,\phi)e^{-i\omega t}\\ \cdot\left(a^+(\omega)\frac{M\omega}{r^2}f(\theta,\phi)+a^\times(\omega)\frac{M\omega}{r^2}g(\theta,\phi)\right)+\text{h.c.}\Bigg],\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}f(\theta,\phi)&=\frac{3\Big[3\cos 3\phi\sin\theta+\cos\phi[\sin\theta-2\left(-3+\cos 2\phi\right)\sin 3\theta]\Big]}{16},\\ g(\theta,\phi)&=-6\cos\theta\cos\phi\sin^2\theta\sin\phi.\end{aligned} \tag{3.17}$$

Substituting eq. (3.16) in eq. (2.1), we get (see figure 1)

$$\begin{aligned}\langle h_{\ell m}\rangle\equiv\sum_s\langle\epsilon^{s\mu\nu}h_{\mu\nu}\rangle&=-i\int_{t_0}^t dt'\left\langle\left[\frac{\sqrt{G}\,\omega^3 a^{s\dagger}_\omega}{\sqrt{2\omega}}j_\ell(\omega y)e^{i\omega t},H_{\text{kin}}(t')\right]\right\rangle\\ &\simeq -i\,G\int_{t_0}^t dt'\int^\infty dr\ r^2 d\Omega\int_0^\infty\frac{d\omega'\ \omega'^2}{\sqrt{2\omega'}}\frac{\omega^3}{\sqrt{2\omega}}\frac{\delta(\omega-\omega')}{\omega^2}j_\ell(\omega' r)j_\ell(\omega y)Y^*_{\ell m}(\theta,\phi)\\ &\quad\cdot e^{-i\omega' t'}e^{i\omega t}\frac{M\omega'}{r^2}f(\theta,\phi).\end{aligned} \tag{3.18}$$

From now on for simplicity, we just consider the + polarization. Expanding the spherical Bessel functions for small arguments $\omega r\ll 1$ and $\omega y\ll 1$, we get

$$\begin{aligned}\langle h_{\ell m}\rangle&=-iG\int_{t_0}^t dt'\ \int dr\ r^2 d\Omega\Big[j_\ell(\omega r)j_\ell(\omega y)e^{-i\omega(t'-t)}\Big]\omega^3\frac{f(\theta,\phi)M}{r^2}Y^*_{\ell m}(\theta,\phi)\\ &\simeq -iG\int_0^{t_0-t}d\tau\int dr\ r^2 d\Omega\ \omega^{2\ell+3}e^{-i\omega\tau}\frac{f(\theta,\phi)M}{r^2}(yr)^\ell Y^*_{\ell m}(\theta,\phi),\end{aligned} \tag{3.19}$$

where we have performed a change of variable $\tau=(t'-t)$. Performing the time integral we obtain

$$\langle h_{\ell m}\rangle=G\int\ dr\ r^2 d\Omega\ \omega^{2\ell+2}e^{-i\omega\tau}\frac{f(\theta,\phi)}{r^2}M(yr)^\ell Y^*_{\ell m}(\theta,\phi). \tag{3.20}$$

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Finally, we need to Fourier transform the frequency part of the integral in order to get the time behavior

$$\int dr\, r^2 d\Omega\, \frac{M(yr)^\ell}{r^2} f(\theta,\phi) Y^*_{\ell m}(\theta,\phi) \int_0^\infty \frac{d\omega}{2\pi}\omega^{2\ell+2} e^{-i\omega t}. \tag{3.21}$$

Since the function is analytical in the complex plane, we can rotate the integration path along the complex axis using the residual theorem and the Jordan lemma, to get

$$\langle h_{\ell m}\rangle \simeq \int dr\, r^2 d\Omega\, \frac{M(yr)^\ell}{r^2} f(\theta,\phi) Y^*_{\ell m}(\theta,\phi) \int_0^{-i\infty} \frac{d\omega}{2\pi}\omega^{2\ell+2} e^{-i\omega t} \simeq A(\theta,\phi)\frac{GMr'^{\ell+1}y^\ell}{t^{2\ell+3}}. \tag{3.22}$$

This is Price's law tail, where $A(\theta,\phi)$ encodes the angular dependence. This calculation shows that the Price's law can be understood by a simple scattering of the QNM off the classical gravitational Newtonian potential generated by the BH mass [12, 16].

4 The nonlinear tail

The aim of this section is to derive the nonlinear tail of a gravitational wave QNM which is generated by the interaction of two other QMNs giving rise to a third wave and several rescattering off the gravitational potential generated by the mass of the BH. For instance, we may think of an $\ell = 4$ mode generated by two $\ell = 2$ modes. Our starting point is the cubic Lagrangian obtained

$$\begin{aligned}\mathcal{L}^{(3)} = \sqrt{G}\,\Big[&\frac{1}{4}h^{\alpha\beta}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} - \frac{1}{4}h^{\alpha\beta}\partial_\alpha h\,\partial_\beta h + h^{\alpha\beta}\partial_\beta h\,\partial_\mu h^\mu{}_\alpha - \frac{1}{2}h^{\mu\nu}\partial_\alpha h\,\partial^\alpha h_{\mu\nu}\\ &+\frac{1}{8}h\,\partial_\mu h\,\partial^\mu h - h^{\mu\nu}\partial_\alpha h_\mu{}^\alpha\partial_\beta h_\nu{}^\beta - h^{\mu\nu}\partial_\nu\, h_\mu{}^\alpha\partial_\beta h_\alpha{}^\beta + \frac{1}{2}h\,\partial_\mu h^{\mu\nu}\,\partial_\alpha h_\nu{}^\alpha\\ &+\frac{1}{2}h^{\mu\nu}\partial^\alpha h_{\mu\nu}\,\partial_\beta h_\alpha{}^\beta - \frac{1}{4}h\,\partial_\alpha h\,\partial_\beta h^{\alpha\beta} + \frac{1}{2}h^{\mu\nu}\partial_\alpha h_{\nu\beta}\,\partial^\beta h_\mu{}^\alpha + \frac{1}{2}h^{\mu\nu}\partial_\beta h_{\nu\alpha}\,\partial^\beta h_\mu{}^\alpha\\ &-\frac{1}{4}h\,\partial_\alpha h_{\mu\nu}\,\partial^\nu h^{\mu\alpha} - \frac{1}{8}h\,\partial_\alpha h^{\mu\nu}\,\partial^\alpha h_{\mu\nu}\Big],\end{aligned} \tag{4.1}$$

where we have expanded the metric far away from the BH

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\,h_{\mu\nu} + \frac{G}{2}h^2_{\mu\nu} + \cdots . \tag{4.2}$$

Using the TT gauge, the action reduces to

$$\begin{aligned}S_3[h] = 4\pi\sqrt{G}\int dt\; dr\; r^2\; d\Omega\,\Big[&\frac{1}{4}h^{\alpha\beta}\partial_\alpha h^{\mu\nu}\partial_\beta h_{\mu\nu} - h^{\mu\nu}\partial_\alpha h_\mu{}^\alpha\partial_\beta h_\nu{}^\beta - h^{\mu\nu}\partial_\nu\, h_\mu{}^\alpha\partial_\beta h_\alpha{}^\beta\\ &+\frac{1}{2}h^{\mu\nu}\partial^\alpha h_{\mu\nu}\,\partial_\beta h_\alpha{}^\beta + \frac{1}{2}h^{\mu\nu}\partial_\alpha h_{\nu\beta}\,\partial^\beta h_\mu{}^\alpha + \frac{1}{2}h^{\mu\nu}\partial_\beta h_{\nu\alpha}\,\partial^\beta h_\mu{}^\alpha\Big].\end{aligned} \tag{4.3}$$

The integral in eq. (4.3) becomes

$$S_3[h]=\frac{\sqrt{G}}{4}\int dt\,dr\,r^2\,d\Omega\sum_{s_i,\ell_i,m_i}\int\frac{d\omega_1}{\sqrt{2\omega_1}}\frac{d\omega_2}{\sqrt{2\omega_2}}\frac{d\omega_3}{\sqrt{2\omega_3}}\omega_1^2\omega_2^2\omega_3^2 j_{\ell_1}(\omega_1 r)j_{\ell_2}(\omega_2 r)j_{\ell_3}(\omega_3 r)$$
$$\cdot Y_{\ell_1 m_1}(\theta,\phi)e^{-i\omega_1 t}Y^*_{\ell_2 m_2}(\theta,\phi)e^{i\omega_2 t}Y_{\ell_3 m_3}(\theta,\phi)e^{-i\omega_3 t}\cdot\Big[\frac{1}{4}\epsilon_1^{\alpha\beta}k_{2,\alpha}\epsilon_2^{\mu\nu}k_{3,\beta}\epsilon_{3,\mu\nu}$$
$$-\epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\nu}-\epsilon_1^{\mu\nu}k_{2,\nu}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha}+\frac{1}{2}\epsilon_1^{\mu\nu}k_2^{\alpha}\epsilon_{2,\mu\nu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha}$$
$$+\frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon_{2,\nu\beta}k_3^{\beta}\epsilon^{\alpha}_{3,\mu}+\frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\beta}\epsilon_{2,\nu\alpha}k_3^{\beta}\epsilon^{\alpha}_{3,\mu}+\text{perm}(1,2,3)\Big]a(\omega_1)a^\dagger(\omega_2)a(\omega_3)+\cdots. \tag{4.4}$$

Incidentally, we notice that the conjugate momenta from the interaction $\pi_{\alpha\beta}=\delta\mathcal{L}^{(3)}/\delta(\partial_0 h^{\alpha\beta})$ are vanishing in the TT gauge. From now on we also consider the case of two identical QNMs, such that $\omega_1=\omega_2$. The interaction Hamiltonian becomes

$$H^{\rm q\,q\,q}_{\rm int}(t)=\frac{\sqrt{G}}{4}\int dr\,r^2\,d\Omega\sum_{s_i,\ell_i,m_i}\int\frac{d\omega_1}{\sqrt{2\omega_1}}\frac{d\omega_2}{\sqrt{2\omega_2}}\frac{d\omega_3}{\sqrt{2\omega_3}}\omega_1^2\omega_2^2\omega_3^2 j_{\ell_1}(\omega_1 r)j_{\ell_2}(\omega_2 r)j_{\ell_3}(\omega_3 r)$$
$$\cdot Y^*_{\ell_1 m_1}(\theta,\phi)e^{-i\omega_1 t}Y_{\ell_2 m_2}(\theta,\phi)e^{i\omega_2 t}Y^*_{\ell_3 m_3}(\theta,\phi)e^{-i\omega_3 t}\,\mathcal{L}_{\rm tens}\,\delta(\omega_1-\omega_2)a(\omega_1)a^\dagger(\omega_2)a(\omega_3)+\cdots,$$
$$\mathcal{L}_{\rm tens}=\sqrt{G}\Big[\frac{1}{4}\epsilon_1^{\alpha\beta}k_{2,\alpha}\epsilon_2^{\mu\nu}k_{3,\beta}\epsilon_{3\mu\nu}-\epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\nu}-\epsilon_1^{\mu\nu}k_{2,\nu}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha} \tag{4.5}$$
$$+\frac{1}{2}\epsilon_1^{\mu\nu}k_2^{\alpha}\epsilon_{2,\mu\nu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha}+\frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon_{2,\nu\beta}k_3^{\beta}\epsilon^{\alpha}_{3,\mu}+\frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\beta}\epsilon_{2,\nu\alpha}k_3^{\beta}\epsilon^{\alpha}_{3,\mu}+\text{perm}(1,2,3)\Big]$$
$$=\frac{3\omega_1^2+6\omega_1\omega_3}{4}. \tag{4.6}$$

Here [q q q] in the interaction Hamiltonian means that we are inserting three quantized gravitons. The last passage is derived in appendix A. Notice that the cubic interaction introduces the Clebsch-Gordon coefficients in the action

$$\int d\Omega\;Y_{\ell_1 m_1}Y_{\ell_2 m_2}Y_{\ell_3 m_3}=\mathcal{C}_{\ell_1\ell_2\ell_3,m_1 m_2 m_3}, \tag{4.7}$$

which impose angular momentum conservation selecting $m_1+m_2=m_3$ and $|\ell_1-\ell_2|\le\ell_3\le\ell_1+\ell_2$. Specifically, in our case, we have

$$\int d\Omega\;Y^*_{\ell_1 m_1}Y_{\ell_2 m_2}Y^*_{\ell_3 m_3}=(-1)^{m_1+m_3}\mathcal{C}_{\ell_1\ell_2\ell_3,-m_1 m_2 -m_3}. \tag{4.8}$$

In the following we will consider the case in which two gravitons with two multipoles ℓ_1 and ℓ_2 generate a third graviton with multipole $\ell_3=(\ell_1+\ell_2)$. This choice is dictated by the discussions of the recent literature where mostly the case $\ell_3=2\ell_1=2\ell_2=4$ is discussed, both numerically and analytically.

4.1 No external sources: the tadpole

In the case of no external sources we start from

$$H^{\rm q\,q\,q}_{\rm int}(t)\simeq\frac{\sqrt{G}}{4}\int^{\infty}dr\,r^2\int_0^{\infty}\frac{d\omega_1\,\omega_1^4}{2\omega_1}\int_0^{\infty}\frac{d\omega_3}{\sqrt{2\omega_3}}\omega_3^2 e^{-i\omega_3 t}\,\mathcal{L}_{\rm tens}\,j_{\ell_1}(\omega_1 r)j_{\ell_2}(\omega_1 r)j_{\ell_3}(\omega_3 r)$$
$$\cdot\,a(\omega_1)a^\dagger(\omega_1)a(\omega_3)+\cdots \tag{4.9}$$

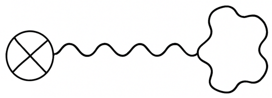

Figure 3. Tadpole diagram.

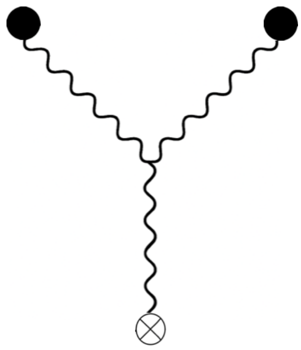

Figure 4. Two external sources.

and can easily calculate the commutator

$$\begin{aligned}
A = [H^{\mathrm{q\,q\,q}}_{\mathrm{int}}(t'), h(t)] &= \frac{G}{4}\int^{\infty} dr\; r^2 \int_0^{\infty} \frac{d\omega_1\;\omega_1^3}{2}\int_0^{\infty}\frac{d\omega_3}{\sqrt{2\omega_3}}\omega_3^2 e^{-i\omega_3 t'} e^{i\omega' t}\;\mathcal{L}_{\mathrm{tens}} \\
&\quad \cdot j_{\ell_1}(\omega_1 r) j_{\ell_2}(\omega_1 r) j_{\ell_3}(\omega_3 r) j_{\ell_3}(\omega' y)\frac{\omega'^3}{\sqrt{\omega'}}\frac{\delta(\omega'-\omega_3)}{\omega_3^2} a_{\omega_1} a^{\dagger}_{\omega_1} \\
&\simeq \int^{\infty} dr\; r^2 \quad \int_0^{\infty} d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t'-t)}\; j_{\ell_1}(\omega_1 r) j_{\ell_2}(\omega_1 r) j_{\ell_3}(\omega' r) j_{\ell_3}(\omega' y) a_{\omega_1} a^{\dagger}_{\omega_1},
\end{aligned} \tag{4.10}$$

such that, at first order in the Hamiltonian

$$\langle h_{\ell_3 m_3}\rangle = \int_{t_0}^{t} dt' \Big\langle\, [H^{\mathrm{q\,q\,q}}_{\mathrm{int}}(t'), h(t)]\,\Big\rangle. \tag{4.11}$$

Performing the integral, we obtain

$$\begin{aligned}
\langle h_{\ell_3 m_3}\rangle \simeq \frac{G}{4}\int_{t_0}^{t} dt' \int^{\infty} dr\; r^2 \int_0^{\infty}\frac{d\omega_1\;\omega_1^5}{2}\omega'^2 e^{-i\omega'(t'-t)}\; j_{\ell_1}(\omega_1 r) j_{\ell_2}(\omega_1 r) j_{\ell_3}(\omega' r) j_{\ell_3}(\omega' y) \\
\cdot \Big\langle a(\omega_1) a^{\dagger}(\omega_1)\Big\rangle.
\end{aligned} \tag{4.12}$$

This result corresponds diagrammatically to a tadpole, see figure 3, which is renormalized to zero by an appropriate counterterm. As expected, there is therefore no contribution to the nonlinear tail in pure flat space time.

4.2 Two external sources

In this subsection we provide the calculations for the diagram at order $(GM)^2$, see figure 4, and we take $\ell_1 = \ell_2 = 1$ and $\ell_3 = 2$ (this choice will be clear in the following), for which

$$\langle h_{2\,m_3}\rangle = \int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \Big\langle\, [H_{\mathrm{kin}}(t_1), [H_{\mathrm{kin}}(t_2), [H^{\mathrm{q\,q\,q}}_{\mathrm{int}}(t_3), h(t)]]]\,\Big\rangle. \tag{4.13}$$

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The first inner commutator has been calculated already, see eq. (4.10). The second commutator in eq. (4.13) will be

$$\begin{aligned}B=[H_{\rm kin}(t_2),A]\simeq&\int^{\infty}dr_3\;r_3^2\;\int_0^{\infty}d\omega_1\omega_1^5\omega'^2e^{-i\omega'(t_3-t)}j_1^2(\omega_1r_3)j_2(\omega'r_3)j_2(\omega'y)\\&\cdot\int dr_2\;r_2^2\;d\Omega_2\int_0^{\infty}\frac{d\omega_4\;\omega_4^2}{\sqrt{2\omega_4}}e^{-i\omega_4t_2}\frac{M\omega_4}{r_2^2}\sum_{\ell_4,m_4}i^{\ell_4}Y^*_{\ell_4m_4}(\theta_2,\phi_2)j_{\ell_4}(\omega_4r_2)f(\theta_2,\phi_2)\\&\cdot\left[a_{\omega_4},a_{\omega_1}a^\dagger_{\omega_1}\right].\end{aligned}\tag{4.14}$$

Since

$$\left[a_{\omega_4},a_{\omega_1}a^\dagger_{\omega_1}\right]=\frac{a_{\omega_1}\delta(\omega_4-\omega_1)}{\omega_1^2}\delta_{\ell_4 1}\tag{4.15}$$

we get

$$\begin{aligned}B=[H_{\rm kin}(t_2),A]\simeq&\int^{\infty}dr_3\;r_3^2\;\int_0^{\infty}d\omega_1\omega_1^5\omega'^2e^{-i\omega'(t_3-t)}j_1^2(\omega_1r_3)j_2(\omega'r_3)j_2(\omega'y)\\&\cdot\int^{\infty}dr_2\;r_2^2\;\frac{\omega_1^2}{\sqrt{2\omega_1}}j_2(\omega_1r_2)e^{-i\omega_1t_2}\frac{M\omega_1}{r_2^2}\frac{a_{\omega_1}}{\omega_1^2}.\end{aligned}\tag{4.16}$$

Consequently, the final commutator will be

$$\begin{aligned}C=[H_{\rm kin}(t_1),B]\simeq&\int^{\infty}dr_3\;r_3^2\;\int_0^{\infty}d\omega_1\omega_1^5\omega'^2e^{-i\omega'(t_3-t)}j_1^2(\omega_1r_3)j_2(\omega'r_3)j_2(\omega'y)\\&\cdot\int^{\infty}dr_2\;r_2^2\;\frac{j_2(\omega_1r_2)}{\sqrt{2\omega_1}}e^{-i\omega_1t_2}\frac{M\omega_1}{r_2^2}\int dr_1\;r_1^2\;d\Omega_1\\&\cdot\int_0^{\infty}\frac{d\omega_5\;\omega_5^2}{\sqrt{2\omega_5}}e^{i\omega_5t_1}\frac{M\omega_5}{r_1^2}\sum_{\ell_5,m_5}i^{\ell_5}Y_{\ell_5m_5}(\theta_1,\phi_1)j_{\ell_5}(\omega_5r_1)f(\theta_1,\phi_1)\left[a_{\omega_1},a^\dagger_{\omega_5}\right].\end{aligned}\tag{4.17}$$

Using

$$\left[a_{\omega_1},a^\dagger_{\omega_5}\right]=\frac{\delta(\omega_1-\omega_5)}{\omega_1^2}\delta_{\ell_5 1},\tag{4.18}$$

we get

$$\begin{aligned}C\simeq&\int^{\infty}dr_3\;r_3^2\;\int^{\infty}dr_2\;r_2^2\;\int^{\infty}dr_1\;r_1^2\;\int_0^{\infty}d\omega_1\omega_1^6\omega'^2j_2(\omega'r_3)j_2(\omega'y)\\&\cdot j_1^2(\omega_1r_3)j_1(\omega_1r_2)j_1(\omega_1r_1)\frac{M^2}{r_1^2r_2^2}e^{-i\omega'(t_3-t)}e^{-i\omega_1t_2}e^{i\omega_1t_1}.\end{aligned}\tag{4.19}$$

Finally we can write the vacuum expectation value of the graviton as

$$\begin{aligned}\langle h_{2\,m_3}\rangle\simeq&\int_{t_0}^{t}dt_3\;\int_{t_0}^{t_3}dt_2\int_{t_0}^{t_2}dt_1\int^{\infty}dr_3\;r_3^2\;\int^{\infty}dr_2\;\int^{\infty}dr_1\\&\cdot\int_0^{\infty}d\omega_1\omega_1^6j_1^2(\omega_1r_3)j_1(\omega_1r_2)j_1(\omega_1r_1)e^{i\omega_1t_1}e^{-i\omega_1t_2}\\&\cdot\int_0^{-i\infty}d\omega'\omega'^2j_2(\omega'r_3)j_2(\omega'y)e^{-i\omega'(t_3-t)}M^2\end{aligned}\tag{4.20}$$

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where we Fourier transformed the frequency part of the integral in order to get the time behavior. Performing the time integral we get

$$\begin{aligned}\langle h_{2\,m_3}\rangle \simeq &\int_0^{t_0-t} d\tau_3\ \tau_3 \int^\infty dr_3\ r_3^2 \int^\infty dr_2 \int^\infty dr_1 \\ &\cdot \int_0^\infty d\omega_1 \omega_1^5 j_1^2(\omega_1 r_3) j_1(\omega_1 r_2) j_1(\omega_1 r_1) \\ &\cdot \int_0^{-i\infty} d\omega' \omega'^2 j_2(\omega' r_3) j_2(\omega' y) e^{-i\omega' \tau_3} M^2 \end{aligned} \tag{4.21}$$

where $\tau_3 = (t_3 - t)$. Here we can recognize the usual Price Green function [78]

$$G(r_3, \tau_3, y) = \int_0^{-i\infty} d\omega' \omega'^2 j_\ell(\omega' r_3) j_\ell(\omega' y) e^{-i\omega' \tau_3}. \tag{4.22}$$

Inserting this result in eq. (4.21) one gets at $(GM)^2$ order

$$\langle h_{2\,m_3}\rangle \sim \frac{(GM)^2}{t^5}. \tag{4.23}$$

We now see that there is a specific selection rule which links the value of $\ell_3 \leq 2$ of the final state with respect to the perturbative order in GM. In the static limit $t_3 \gg r_3$ [74]

$$G(r_3, t_3, y) \sim \frac{P_{\ell_3}(\chi)}{t_3}, \tag{4.24}$$

where we have defined AdS_2 invariant distance

$$\chi = \frac{-(t-t_3)^2 + (r_3 - y)^2}{2 r_3 y}, \tag{4.25}$$

so that we can simplify the integrals (returning to the generic case $\ell_1 = \ell_2$ and $\ell_3 = (\ell_1 + \ell_2)$) as

$$\begin{aligned}\langle h_{\ell_3 m_3}\rangle \simeq M^2 &\int_0^{t_0-t} d\tau_3\ \tau_3 \int^\infty dr_3\ r_3^2 \int^\infty dr_2 \int^\infty dr_1 \\ &\cdot \int_0^\infty d\omega_1 \omega_1^5 j_{\ell_1}(\omega_1 r_3) j_{\ell_2}(\omega_1 r_3) j_{\ell_1}(\omega_1 r_2) j_{\ell_1}(\omega_1 r_1) \frac{P_{\ell_3}(\chi)}{\tau_3}. \end{aligned} \tag{4.26}$$

Far away from the BH we can confuse the positions of the three vertices, getting

$$\begin{aligned}\langle h_{\ell_3 m_3}\rangle \simeq \lim_{r_1,\, r_2,\, r_3 \to \infty} M^2 &\int_0^{t_0-t} d\tau_3 \int^\infty dr_3\ r_3^2 \int^\infty dr_2 \int^\infty dr_1\ \delta(r_1 - r_2)\delta(r_2 - r_3) \\ &\cdot \int_0^\infty d\omega_1 \omega_1^5 j_{\ell_1}(\omega_1 r_3) j_{\ell_2}(\omega_1 r_3) j_{\ell_1}(\omega_1 r_2) j_{\ell_1}(\omega_1 r_1) P_{\ell_3}(\chi) \end{aligned} \tag{4.27}$$

so that

$$\langle h_{\ell_3 m_3}\rangle \sim \int_0^{t_0-t} d\tau_3 \int^\infty dr\ P_{\ell_3}(\chi) \left(\frac{GM}{r}\right)^2, \tag{4.28}$$

where we have expanded the spherical Bessel function $j_{\ell_i}(\omega_1 r)$ for $\omega_1 r \gg 1$. Now, following the ref. [74] it is easy to show that from the orthogonality's properties of the Legendre polynomial, the first non-zero contribution will be given by $\ell_3 \leq 2$.

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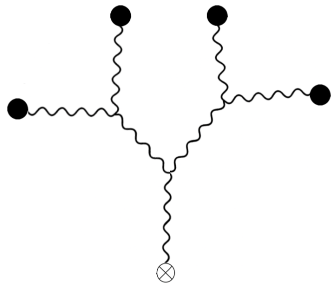

Figure 5. Four external sources.

4.3 Four external sources

In this section we provide the calculations for the diagram at order $(GM)^4$, see figure 5. We take $\ell_1 = \ell_2 = 2$ and $\ell_3 = 4$ (this choice, as did in the previous section, will be clear in the following). We need to calculate the following vacuum expectation value for the graviton

$$\langle h_{4\,m_3}\rangle = \int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \Big\langle \Big[H^{\mathrm{q\,cl\,cl}}_{\mathrm{int}}(t_1), \Big[H^{\mathrm{q\,cl\,cl}}_{\mathrm{int}}(t_2), [H^{\mathrm{q\,q\,q}}_{\mathrm{int}}(t_3), h(t)]\Big]\Big] \Big\rangle \tag{4.29}$$

where [q cl cl] in the interaction Hamiltonian means that we are inserting one quantized graviton and two classical sources. The first inner commutator has been calculated already, and it is eq. (4.10),

$$A = [H^{\mathrm{q\,q\,q}}_{\mathrm{int}}(t_3), h(t)] = \int^{\infty} dr_3\, r_3^2 \int_0^{\infty} d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t_3-t)}\, j_2^2(\omega_1 r_3) j_4(\omega' r_3) j_4(\omega' y) a_{\omega_1} a^\dagger_{\omega_1}. \tag{4.30}$$

The second commutator in eq. (4.29) reads

$$\begin{aligned} B = \Big[H^{\mathrm{q\,cl\,cl}}_{\mathrm{int}}(t_2), A\Big] &= \int^{\infty} dr_3\, r_3^2 \int_0^{\infty} d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t_3-t)} j_2^2(\omega_1 r_3) j_4(\omega' r_3) j_4(\omega' y) \\ &\cdot \int dr_2\, r_2^2\, d\Omega_2 \int_0^{\infty} \frac{d\omega_4\, \omega_4^2}{\sqrt{2\omega_4}} e^{-i\omega_4 t_2} \frac{M^2\omega_4}{r_2^3} \sum_{\ell_4, m_4} i^{\ell_4} Y^*_{\ell_4 m_4}(\theta_2,\phi_2) j_{\ell_4}(\omega_4 r_2) f(\theta_2,\phi_2) \\ &\cdot \Big[a_{\omega_4}, a_{\omega_1} a^\dagger_{\omega_1}\Big]\,. \end{aligned} \tag{4.31}$$

The calculation of $H^{\mathrm{q\,cl\,cl}}_{\mathrm{int}}(t_2)$ is analyzed in eq. (B.4). Since

$$\Big[a_{\omega_4}, a_{\omega_1} a^\dagger_{\omega_1}\Big] = \frac{a_{\omega_1}\delta(\omega_4-\omega_1)}{\omega_1^2}\delta_{\ell_4 2} \tag{4.32}$$

we get

$$\begin{aligned} B = \Big[H^{\mathrm{q\,cl\,cl}}_{\mathrm{int}}(t_2), A\Big] &\simeq \int^{\infty} dr_3\, r_3^2 \int_0^{\infty} d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t_3-t)} j_2^2(\omega_1 r_3) j_4(\omega' r_3) j_4(\omega' y) \\ &\cdot \int^{\infty} dr_2\, r_2^2\, \frac{\omega_1^2}{\sqrt{2\omega_1}} j_2(\omega_1 r_2) e^{-i\omega_1 t_2} \frac{M^2\omega_1}{r_2^3} \frac{a_{\omega_1}}{\omega_1^2}. \end{aligned} \tag{4.33}$$

Therefore the final commutator will be

$$C = \left[H^{\text{q cl cl}}_{\text{int}}(t_1), B\right] \simeq \int^\infty dr_3\, r_3^2 \int_0^\infty d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t_3-t)} j_2^2(\omega_1 r_3) j_4(\omega' r_3) j_4(\omega' y)$$
$$\cdot \int^\infty dr_2\, r_2^2\, \frac{j_2(\omega_1 r_2)}{\sqrt{2\omega_1}} e^{-i\omega_1 t_2} \frac{M^2\omega_1}{r_2^3} \int dr_1\, r_1^2\, d\Omega_1$$
$$\cdot \int_0^\infty \frac{d\omega_5\, \omega_5^2}{\sqrt{2\omega_5}} e^{i\omega_5 t_1} \frac{M^2\omega_5}{r_1^3} \sum_{\ell_5,m_5} i^{\ell_5} Y_{\ell_5 m_5}(\theta_1,\phi_1) j_{\ell_5}(\omega_5 r_1) f(\theta_1,\phi_1) \left[a_{\omega_1}, a^\dagger_{\omega_5}\right]. \tag{4.34}$$

Since now

$$\left[a_{\omega_1}, a^\dagger_{\omega_5}\right] = \frac{\delta(\omega_1 - \omega_5)}{\omega_1^2} \delta_{\ell_5 2}, \tag{4.35}$$

we get

$$C \simeq \int^\infty dr_3\, r_3^2 \int^\infty dr_2\, r_2^2 \int^\infty dr_1\, r_1^2 \int_0^\infty d\omega_1 \omega_1^6 \omega'^2 j_4(\omega' r_3) j_4(\omega' y)$$
$$\cdot\, j_2^2(\omega_1 r_3) j_2(\omega_1 r_2) j_2(\omega_1 r_1) \frac{M^4}{r_1^3 r_2^3} e^{-i\omega'(t_3-t)} e^{-i\omega_1 t_2} e^{i\omega_1 t_1}. \tag{4.36}$$

Finally, we can write the vacuum expectation value of the graviton as

$$\langle h_{4m_3}\rangle \simeq \int_{t_0}^t dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \int^\infty dr_3\, r_3^2 \int^\infty \frac{dr_2}{r_2} \int^\infty \frac{dr_1}{r_1}$$
$$\cdot \int_0^\infty d\omega_1 \omega_1^6 j_2^2(\omega_1 r_3) j_2(\omega_1 r_2) j_2(\omega_1 r_1) e^{i\omega_1 t_1} e^{-i\omega_1 t_2}$$
$$\cdot \int_0^{-i\infty} d\omega' \omega'^2 j_4(\omega' r_3) j_4(\omega' y) e^{-i\omega'(t_3-t)} M^4 \tag{4.37}$$

where we Fourier transformed the frequency part of the integral in order to get the time behavior. Performing the integrals over the times we get, at leading order in t,

$$\langle h_{4m_3}\rangle \simeq \int_0^{t_0-t} d\tau_3\, \tau_3 \int^\infty dr_3\, r_3^2 \int^\infty \frac{dr_2}{r_2} \int^\infty \frac{dr_1}{r_1} \int_0^\infty d\omega_1 \omega_1^5 j_2^2(\omega_1 r_3) j_2(\omega_1 r_2) j_2(\omega_1 r_1)$$
$$\cdot \int_0^{-i\infty} d\omega' \omega'^2 j_4(\omega' r_3) j_4(\omega' y) e^{-i\omega'\tau_3} M^4, \tag{4.38}$$

where $\tau_3 = (t_3 - t)$. We can easily see that the time dependence is

$$\langle h_{4\,m_3}\rangle \sim \frac{(GM)^4}{t^9}. \tag{4.39}$$

As done in the previous section, we find a specific selection rule which links the value of $\ell_3 \leq 4$ of the final state with respect to the perturbative order in GM. We can plug the expression (4.24) in eq. (4.38) and return to the generic case $\ell_1 = \ell_2$ and $\ell_3 = (\ell_1 + \ell_2)$

$$\langle h_{\ell_3 m_3}\rangle \simeq M^4 \int_0^{t_0-t} d\tau_3\, \tau_3 \int^\infty dr_3\, r_3^2 \int^\infty \frac{dr_2}{r_2} \int^\infty \frac{dr_1}{r_1}$$
$$\cdot \int_0^\infty d\omega_1 \omega_1^5 j_{\ell_1}(\omega_1 r_3) j_{\ell_2}(\omega_1 r_3) j_{\ell_1}(\omega_1 r_2) j_{\ell_1}(\omega_1 r_1) \frac{P_{\ell_3}(\chi)}{\tau_3}. \tag{4.40}$$

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Again, far away from the BH we can confuse the positions of the three vertices, getting

$$\langle h_{\ell_3 m_3}\rangle \sim \int_0^{t_0-t} d\tau_3 \int^{\infty} dr\ P_{\ell_3}(\chi)\left(\frac{GM}{r}\right)^4, \tag{4.41}$$

where we have expanded the spherical Bessel function $j_{\ell_i}(\omega_1 r)$ for $\omega_1 r \gg 1$. As in the previous subsection, the orthogonality's properties of the Legendre polynomial, selects a non zero result only for $\ell_3 \leq 4$. We infer the generic rule that the dominant contribution to the nonlinear tail of a QNM of multipole ℓ_3 generated by the annihilation of two QNMs with multipoles $\ell_1 = \ell_2$ reads

$$\langle h_{\ell_3 m_3}\rangle \sim \frac{(GM)^{\ell_3}}{t^{2\ell_3+1}}. \tag{4.42}$$

We stress again that we are working in TT gauge, so that we reproduce the same results of ref. [72].

4.4 Generalization to higher order

We can take a step forward and demonstrate that the non-linear behavior $t^{-2\ell-1}$ is a general solution that can be extended up to all the perturbative orders. Let us imagine, as an example, the Feynman diagram that describes the third perturbative order: in a vertex (t_1, r_1) we have the scattering between two gravitons, that will produce a new third graviton. Now this graviton will scatter with another free graviton in a vertex (t_3, r_3) (the one that we have studied before). The resulting new graviton will be contracted with the external field, becoming the observed tail.

Obviously, we need to attach to the external gravitons legs the usual classical legs, in order to avoid the formation of the tadpole. The diagram under consideration is described by the following integral

$$\begin{aligned}\langle h_{\ell_3 m_3}\rangle = &\int_{t_0}^{t} dt_5 \int_{t_0}^{t_5} dt_4 \int_{t_0}^{t_4} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \Big\langle \Big[H_{\rm int}^{\rm qclcl}(t_5), \Big[H_{\rm int}^{\rm qclcl}(t_4), \Big[H_{\rm int}^{\rm qclcl}(t_2), \Big[H_{\rm int}^{\rm qqq}(t_1),\\ &[H_{\rm int}^{\rm qqq}(t_3), h(t)]\Big]\Big]\Big]\Big]\Big\rangle.\end{aligned} \tag{4.43}$$

The first inner commutator is the same of the previous diagram in eq. (4.10)

$$\begin{aligned}A = [H_{\rm int}^{\rm qqq}(t_3), h(t)] \simeq &\int^{\infty} dr_3\, r_3^2 \int_0^{\infty} d\omega_1 \omega_1^5 \omega'^2 e^{-i\omega'(t_3-t)}\, j_{\ell_1}(\omega_1 r) j_{\ell_2}(\omega_1 r) j_{\ell_3}(\omega' r)\\ &\cdot j_{\ell_3}(\omega' y) a_{\omega_1} a^\dagger_{\omega_1}.\end{aligned} \tag{4.44}$$

Now we write again the usual Hamiltonian for a vertex with three gravitons

$$H_{\rm int}^{\rm qqq}(t) \simeq \int^{\infty} dr\, r^2 \int_0^{\infty} d\omega_1\, \omega_1^5 \int_0^{\infty} \frac{d\omega_3}{\sqrt{2\omega_3}} \omega_3^2 e^{-i\omega_3 t}\, j_{\ell_1}(\omega_1 r) j_{\ell_2}(\omega_1 r) j_{\ell_3}(\omega_3 r) a_{\omega_1} a^\dagger_{\omega_1} a_{\omega_3}. \tag{4.45}$$

This is the generic term that we need to attach to the inner commutator to compose the final vacuum expectation value. Now we notice that the only component that provide information on the non linear tail (so the only integral that contain information on the frequency ω' that we send to zero by definition of the observed tail) is the inner commutator. All the other possible external commutator (related to vertices with respect to cubic graviton interaction or cubic vertex associated to external classical leg) will not give any contribution to the tail behavior, just for the momentum conservation and the angular momentum conservation applied to all the vertices of the diagrams. We can see this just by looking to the inner commutator: this does not have any annihilation or creation operator related to the external frequency ω'. Therefore we can argue that, at each perturbative order,

$$\langle h_{\ell m}(t)\rangle \sim \frac{1}{t^{2\ell+1}}. \tag{4.46}$$

This conclusion holds beyond the cubic order. Since the full interaction Hamiltonian has always two derivatives in the gravitational field in each vertex order, the power of the frequency of the final state will not change and the power-law tail will not be modified.

There is another way to think of this result. Imagine we write the equation of motion of a graviton with a source written, say, at the 27th order. Such a source will be composed by terms of the form (first order)^{27}, (first-order)$^{25}\cdot$(second-order), $\cdots$, (first-order)$\cdot$(26th-order). Now, far from the source even the graviton calculated at the 26th-order will be a free propagating wave, like its first-order counterpart, with an amplitude decaying like $1/r$. From the point of view of the source, therefore, there will be always a term, at any order, decaying like $1/r^2$, the same which happens for the second-order source. We expect therefore the same power-law for the tail one obtains at second-order, but obviously with a much smaller overall amplitude.

5 Conclusions

Tails are a fundamental feature of the ringdown phase in gravitational wave signals from black holes — two cornerstones of general relativity. By employing the in-in formalism, we have rederived Price's law for the linear tail and provided a transparent explanation for the power-law behavior of the nonlinear tail. Our interpretation rests on a simple picture: the generated QNMs scatter off the external classical field produced by the black hole itself in the asymptotic region.

There remain several avenues for refinement. One is to consider that a given nonlinear mode may arise from interactions with other nonlinear QNMs, whose dynamics could modify the power-law, even though their amplitudes are necessarily subleading [73]. Another important question concerns the effects of interactions beyond the cubic level — in particular, whether the interaction with the external field can be resummed. We leave these questions for future work.

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A Calculation of $\mathcal{L}_{\text{tens}}$

We consider the following setup. A graviton that propagates along the x-axis with momentum k_1^μ scatters off a second graviton that is propagating along the y-axis, having momentum k_2^μ. The collision between the two will originate a new graviton that will propagate along the diagonal direction, with a momentum k_3^μ given by the momentum conservation at the vertex of the interaction. The three four-momenta in cartesian coordinates are therefore

$$\begin{aligned} k_1^\mu &= (\omega_1, \omega_1, 0, 0) \\ k_2^\mu &= (\omega_2, 0, \omega_2, 0) \\ k_3^\mu &= (\omega_3, \omega_3, \omega_3, 0). \end{aligned} \tag{A.1}$$

where we notice that the conjugate momenta from the interaction $\pi_{\alpha\beta} = \delta\mathcal{L}^{(3)}/\delta(\partial_0 h^{\alpha\beta})$ are vanishing in the TT gauge. In polar coordinates the polarization tensors along the y- and d-axis (identified by the momentum conservation) are

$$\epsilon^+_{\mu\nu,y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\cos^2\theta + \cos^2\phi\sin^2\theta & r\cos\theta\sin\theta + r\cos\theta\cos^2\phi\sin\theta & -r\cos^2\phi\sin^2\theta\sin\phi \\ 0 & r\cos\theta\sin\theta + r\cos\theta\cos^2\phi\sin\theta & r^2\cos^2\theta\cos^2\phi - r^2\sin^2\theta & -r^2\cos\theta\cos\phi\sin\phi\sin\theta \\ 0 & -r\cos\phi\sin^2\theta\sin\phi & -r^2\cos\theta\cos\phi\sin\phi\sin\theta & r^2\sin^2\theta\sin^2\phi \end{pmatrix}, \tag{A.2}$$

$$\epsilon^\times_{\mu\nu,y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\cos\theta\cos\phi\sin\theta & r\cos^2\theta\cos\phi - r\cos\phi\sin^2\theta & -r\cos\theta\sin\theta\sin\phi \\ 0 & r\cos^2\theta\cos\phi - r\cos\theta\sin^2\theta & -2r^2\cos\theta\cos\phi\sin\theta & r^2\sin^2\theta\sin\phi \\ 0 & -r\cos\theta\sin\theta\sin\phi & r^2\sin^2\theta\sin\phi & 0 \end{pmatrix}, \tag{A.3}$$

$$\epsilon^+_{\mu\nu,d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}(-1-3\cos 2\theta - 2\sin^2\theta\sin 2\phi) & -\frac{r}{4}\sin 2\theta(-3+\sin 2\phi) & -\frac{r}{2}\cos 2\phi\sin^2\theta \\ 0 & -\frac{r}{4}\sin 2\theta(-3+\sin 2\phi) & \frac{r^2}{4}(-1+3\cos 2\theta - 2\cos^2\theta\sin 2\phi) & -\frac{r^2}{4}\cos 2\phi\sin 2\theta \\ 0 & -\frac{r}{2}\cos 2\phi\sin^2\theta & -\frac{r^2}{4}\cos 2\phi\sin 2\theta & \frac{r^2}{2}\sin^2\theta(1+\sin 2\phi) \end{pmatrix}, \tag{A.4}$$

$$\epsilon^\times_{\mu\nu,d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}\cos\theta\sin\theta(\cos\phi - \sin\phi) & \frac{r\cos 2\theta(\cos\phi - \sin\phi)}{\sqrt{2}} & -\frac{r\cos\theta\sin\theta(\cos\phi+\sin\phi)}{\sqrt{2}} \\ 0 & \frac{r\cos 2\theta(\cos\phi - \sin\phi)}{\sqrt{2}} & \sqrt{2}r^2\cos\theta\sin\theta(\cos\phi+\sin\phi) & \frac{r^2\sin^2\theta(\cos\phi+\sin\phi)}{\sqrt{2}} \\ 0 & -\frac{r\cos\theta\sin\theta(\cos\phi+\sin\phi)}{\sqrt{2}} & \frac{r^2\sin^2\theta(\cos\phi+\sin\phi)}{\sqrt{2}} & 0 \end{pmatrix}. \tag{A.5}$$

We can now write the momentum vectors in polar coordinates

$$k_1^\mu = \omega_1\left(1, \cos\phi\sin\theta, \frac{\cos\theta\cos\phi}{r}, -\frac{\csc\theta\sin\phi}{r}\right), \tag{A.6}$$

and

$$k_2^\mu = \omega_2\left(1, \sin\phi\sin\theta, \frac{\cos\theta\sin\phi}{r}, -\frac{\csc\theta\cos\phi}{r}\right), \tag{A.7}$$

and

$$k_3^\mu = \omega_3\left(2, (\sin\phi + \cos\phi)\sin\theta, \frac{\cos\theta(\sin\phi + \cos\phi)}{r}, \frac{\csc\theta(\cos\phi - \sin\phi)}{r}\right). \tag{A.8}$$

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We obtain

$$\mathcal{L}_{\rm tens} = \sqrt{G}\left[\frac{1}{4}\epsilon_1^{\alpha\beta}k_{2,\alpha}\epsilon_2^{\mu\nu}k_{3,\beta}\epsilon_{3\,\mu\nu} - \epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\nu} - \epsilon_1^{\mu\nu}k_{2,\nu}\epsilon^{\alpha}_{2,\mu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha}\right. \tag{A.9}$$

$$\left. + \frac{1}{2}\epsilon_1^{\mu\nu}k_2^{\alpha}\epsilon_{2,\mu\nu}k_{3,\beta}\epsilon^{\beta}_{3,\alpha} + \frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\alpha}\epsilon_{2,\nu\beta}k_3^{\beta}\epsilon^{\alpha}_{3,\mu} + \frac{1}{2}\epsilon_1^{\mu\nu}k_{2,\beta}\epsilon_{2,\nu\alpha}k_3^{\beta}\epsilon^{\alpha}_{3,\mu} + \mathrm{perm}(1,2,3)\right]$$

$$= \frac{3\omega_1\omega_2 + 3\omega_1\omega_3 + 3\omega_2\omega_3}{4}. \tag{A.10}$$

In our case, we always consider $\omega_1 = \omega_2$, and once the contraction with the external field is accounted for, we set $\omega_3 \to 0$. In such a case we have

$$\mathcal{L}_{\rm tens} \simeq \frac{3\omega_1^2}{4}. \tag{A.11}$$

B One graviton and two classical source vertices

In this section we want to provide the calculation of the cubic vertex with one quantized graviton that scatters two time with the background perturbation metric, far away from the BH. We can write the cubic coupling in the following way

$$S_3[h] = 4\pi\sqrt{G}\int dt\, dr\, r^2\, d\Omega\left[\frac{1}{4}h_1^{\alpha\beta}\partial_\alpha h_2^{\mu\nu}\partial_\beta h_{3,\mu\nu} - h_1^{\mu\nu}\partial_\alpha h^{\alpha}_{2,\mu}\partial_\beta h^{\beta}_{3,\nu} - h_1^{\mu\nu}\partial_\nu h^{\alpha}_{2,\mu}\partial_\beta h^{\beta}_{3,\alpha}\right.$$

$$+ \frac{1}{2}h_1^{\mu\nu}\partial^{\alpha}h_{2,\mu\nu}\,\partial_\beta h^{\beta}_{3,\alpha} + \frac{1}{2}h_1^{\mu\nu}\partial_\alpha h_{2,\nu\beta}\,\partial^{\beta}h^{\alpha}_{3,\mu}$$

$$\left. + \frac{1}{2}h_1^{\mu\nu}\partial_\beta h_{2,\nu\alpha}\,\partial^{\beta}h^{\alpha}_{3,\mu} + \mathrm{perms}(1,2,3)\right], \tag{B.1}$$

in which we consider just as a quantized graviton h_1, and h_2, h_3 as classical sources. Those fields will, indeed, incorporate the information that we are not in a completely flat region of spacetime, but we are in presence of a BH.

We can extract the first order metric perturbation as

$$h_{2,\mu\nu} = h_{3,\mu\nu} = \begin{pmatrix} 2M/r & 0 & 0 & 0 \\ 0 & 2M/r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.2}$$

Doing the contraction, as we did for the linear Price case, we get (for the + polarization)

$$\mathcal{L}_{\rm tens} \simeq \frac{5M^2\omega\left(3\cos 3\phi\sin\theta + \cos\phi\left(\sin\theta - 2(-3+\cos 2\phi)\sin 3\theta\right)\right)}{2r^3}$$

$$+ \frac{24M^2\left(\cos^2\theta - \sin^2\theta\,\sin^2\phi\right)}{r^4}. \tag{B.3}$$

Now since we are working far away from the BH, so $r \gg 2M$ we can just take the first leading order in r (since the frequency ω will not be contracted with the external field it will be non zero). We can finally write the Hamiltonian of interaction (for the + polarization)

$$H_{\rm int}^{\rm q\,cl\,cl} \simeq 4\pi\sqrt{G}\int dr\, r^2\, d\Omega \left[\int \frac{d\omega\ \omega^2}{(2\pi)^{3/2}\sqrt{2\omega}} \sum_{\ell,m} i^\ell j_\ell(\omega r) Y^*_{\ell m}(\theta,\phi) e^{-i\omega t} \cdot \left[a^+(\omega)\frac{M^2\omega}{r^3} f(\theta,\phi)\right] + \text{h.c}\right] \tag{B.4}$$

where

$$f(\theta,\phi) \simeq \frac{5\,(3\cos 3\phi \sin\theta + \cos\phi\,(\sin\theta - 2(-3+\cos 2\phi)\sin 3\theta))}{2}. \tag{B.5}$$

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