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Unified Supersymmetric Description of Shape-Invariant Potentials Within and Beyond the Natanzon Class

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Abstract: The transformations of supersymmetric quantum mechanics are discussed within a formalism that employs a six-parameter function, from which the superpotential and the supersymmetric partner potentials $V_-(x)$ and $V_+(x)$ are constructed in a general form. By specific choice of the parameters, $V_-(x)$ and $V_+(x)$ are matched with the general form of PI class potentials and their rationally extended versions. The choice of the parameters also determines which of the four possible SUSY transformations T_i, $i = 1, \ldots 4$ is in effect. After this general discussion, the formulae are specified to the three members of this potential class, the Scarf I, Scarf II and generalized Pöschl–Teller potentials. Due to the different domains of definition and their consequences on the boundary conditions, the results turn out to be rather diverse for the three potentials, while the mathematical formalism and the network of the potentials interconnected by the SUSYQM transformations still remains common to a large extent. The general framework allows a unified and consistent interpretation of earlier isolated findings. It also helps to connect the results to further potential classes and to place them into a more general context within the zoo of exactly solvable potentials.

Keywords: supersymmetric quantum mechanics, exactly solvable potentials, special functions of mathematical physics

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1. Intoduction

Simple physical models have proven to be useful tools to understand the basic principles of the physical world. By increasing the complexity of these models, their range of applicability can be extended; however, these efforts are usually limited after some point by the increasing technical difficulties. Finding the point up to which the models improve without becoming overly complicated is a real challenge in general. An example for this scenario is studying exactly solvable quantum mechanical potentials. This field has been active since the introduction of modern quantum mechanics a century ago. The first simple models obtained by the exact solution of the Schrödinger equation were the harmonic oscillator and the Coulomb, Morse and Pöschl–Teller potentials, for example. Some of these problems have been incorporated into more developed models and have also become part of introductory quantum mechanical courses. The range of exactly solvable potentials still keeps extending, thanks to new innovative ideas and technical developments that appear from time to time (sometimes recycling or generalizing earlier ideas and developments). The technical difficulties one encounters in these efforts are usually of a mathematical nature. Fortunately, one can rely on the rich arsenal of mathematical physics, especially the theory of differential equations and special functions compiled well before the introduction of quantum mechanics. As the range of exactly solvable quantum mechanical potentials increased, the demand for finding principles along which they can be classified also appeared in a natural way.

The most straightforward and most widespread method of solving the Schrödinger equation (and also some further fundamental equations of physics) is mapping it to some second-order differential equation of mathematical physics by means of a well-chosen variable transformation. Many exactly solvable potentials have been found in this way. However, one cannot be sure whether these random and isolated results exhaust the set of all exactly solvable problems or not. This problem was handled by reversing the procedure: rather than setting out from a particular potential and arriving at a known differential equation, systematic procedures were formulated by which—in principle—all the exactly solvable problems with solutions containing a single special function of mathematical physics could be derived. These efforts resulted, for example, in the introduction of Natanzon potentials [1], for which the wave function is expressed in terms of a hypergeometric function. Natanzon confluent potentials [2] are related to the confluent hypergeometric function. The bound-state solutions of these problems are written in terms of Jacobi and generalized Laguerre polynomials, which are obtained from the named functions as special cases [3].

Applying this method also implied a natural classification of exactly solvable potentials in terms of the special function used in the particular case. However, this first level of classification leaves all the potentials within a potential class unidentified. As a second level of classification, the variable transformations applied in the mapping procedure proved to be a natural choice. Such an effort was proposed in 1962 by observing that a specific differential equation can be formulated for the function governing the variable transformation [4] by requiring that a constant term (corresponding to the energy eigenvalue) has to appear in the resulting transformed equation (the Schrödinger equation). The systematic identification of these secondary potential classes has been introduced [5] for potentials with bound-state wave functions containing Jacobi and generalized Laguerre polynomials (as well as Hermite polynomials, which can be considered special cases of the latter polynomials). The relation of these potential classes with other classification schemes of potentials has also been discussed in Ref. [5].

Another rather successful method of discussing exactly solvable potentials is supersymmetric quantum mechanics (SUSYQM) [6] (for a review, see Refs. [7,8], for example). In this approach, the focus is not on the structure of the solutions; rather, it is based on a symmetry, which implies a degeneracy of the energy levels of two potentials that are connected by a specific supersymmetric transformation that can be formulated by using a known particular (not necessarily physical) solution of one of the potentials. Supersymmetric quantum mechanics became a rather successful method of finding new exactly solvable potentials from an already known one. Not surprisingly, SUSYQM is also rooted in earlier theories, e.g., the Darboux transformation [9], going back well before the time of quantum mechanics.

SUSYQM can be used to generate new potentials both within and outside the range of exactly solvable potentials: it is equally applicable in numerical calculations. However, applying it to known exactly solvable potentials helped to gain further insight into their structure and classification. One such case was when the two potentials connected by SUSYQM turned out to have the same mathematical structure, with different parameters. This is how the concept of shape invariance was introduced [10]. The most well-known textbook examples for exactly solvable potentials (e.g., the harmonic oscillator, Coulomb, Morse, Pöschl–Teller, etc. potentials) were found to exhibit shape-invariance. In fact, these are the potentials that arise within the two-step classification procedure mentioned before as the most natural examples [5], and also in the factorization method [11] (introduced by E. Schrödinger [12,13]), which can be considered a precursor of the concept of shape-invariance.

Another notable case is when the two potentials connected by SUSYQM belong to two different classes in the variable transformation framework, i.e., their solutions are *different* special functions of mathematical physics. Examples of this scenario are the rationally extended versions of some of the shape-invariant potentials. The solutions of these potentials are exceptional orthogonal polynomials [14,15], in particular, the X_1-type Jacobi and

Laguerre polnomials. These differ from their classical counterparts in that the degree of the lowest polynomial in the sequence is one, rather than zero. The range of the potentials derived from these exceptional polynomials contain the rationally extended Scarf I [16], Pöschl–Teller [17] and Scarf II [18] potentials for the X_1-type Jacobi and the rationally extended harmonic oscillator [16] for the X_1-type Laguerre polynomials. As mentioned above, these new potentials can be obtained by SUSYQM transformation, with their classical counterparts as starting potentials. Furthermore, these rationally extended potentials are characterized by the shape-invariance property. Certainly, these potentials are outside the Natanzon class, because their solutions cannot be expressed in terms of the hypergeometric function, the confluent hypergeometric function or polynomials to which these functions can be reduced. However, the hypergeometric and the confluent hypergeometric functions can also be expressed as special cases of more general functions, e.g., the confluent Heun function [19]. It can be proven that this function can be specified in polynomial form in various ways, and both the X_1-type Jacobi and the X_1-type Laguerre polynomials can be obtained from the confluent Heun function. See, e.g., Refs. [20,21]. We also note that some of the potentials discussed here can also be obtained from the transformation of the general (rather than the confluent) Heun equation [22].

Here, we combine the two methods outlined above to discuss a number of potentials both within and outside the Natanzon class. We introduce a general formalism based on a rather general six-parameter function that satisfies the requirements of SUSYQM and introduce two potentials that are SUSYQM partners by construction. We then select the six parameters in such a way that the two potentials coincide with the Scarf I, Scarf II and generalized Pöschl–Teller potentials and/or their rational extension. The chosen parameters set the behavior of the original six-parameter function at the boundaries, which, in turn, defines the type of SUSYQM transformation and decides whether the transformation corresponds to broken or unbroken supersymmetry.

This project has been discussed already for the radial harmonic oscillator and its rational extension [23]. The connection between these potentials by different types of SUSYQM transformations has been analyzed, and their solutions, as well as their shape-invariance, were discussed in a unified and consistent way. In the present study, we implement this procedure to the Scarf I, Scarf II and the generalized Pöschl–Teller potentials and their rational extensions. Furthermore, the discussion is carried out on a higher level of abstraction in the sense that the formalism is developed in a general form valid for PI class potentials, to which all three mentioned potentials belong [5]. This potential class corresponds to type A potentials in the scheme of the factorization method [11], although it was not complete at the time of publication of the latter work. These potentials also belong to the same class in other classification schemes: (b) in Ref. [24] and A in Ref. [25]. The individual potentials are specified only at the stage where the domain of definition of the problems and the corresponding boundary conditions start to play a role: these differ in the case of the three potentials, while their general mathematical structure remains rather similar, as do also part of the conclusions. With this, a number of isolated earlier results appear in a general consistent framework. This general approach also helps to interpret the role of the exceptional polynomials; these are relatively new actors within the traditional formalisms that have been used previously to discuss exactly solvable potentials.

This paper is organized as follows. In Section 2, the basic properties of SUSYQM are reviewed, and the essential formulae for the general PI class potentials and their rational extension are collected for further reference. In Section 3, the $\chi(r)$ function generating the SUSYQM transformation is introduced, and the most general form of the two partner potentials is derived. The partner potentials are then matched with the general PI class potential and its rational extension. The general results are specified for the three particular potentials in Section 4. Finally, in Section 5, the results are summarized.

2. Preliminaries

Here, we discuss some general aspects of the solutions of the radial Schrödinger equation (using the units $\hbar = 2m = 1$)

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + [E - V(x)]\psi(x) = 0\,, \tag{1}$$

where $x \in [x_-, x_+]$, a domain which will be specified later. Normalizable solutions, i.e., bound-state wave functions, have to satisfy the boundary conditions $\psi(x_-) = \psi(x_+) = 0$.

2.1. The Basics of Supersymmetric Quantum Mechanics (SUSYQM)

Supersymmetric theories handle bosonic and fermionic quantities within a joint formalism and contain operators that satisfy a set of commutation and anticommutation relations. This is also the case in supersymmetric quantum mechanics, which was introduced as a toy model to study the breakdown of supersymmetry [6], but later evolved into a rather successful and robust theory in quantum mechanics. It was found that a handful of operators satisfying a few simple equations can lead to a rich variety of results concerning quantum mechanical potential problems [7,8]. This is also the case for $N = 2$ SUSYQM, which is based on the superalgebra

$$\{Q, Q^\dagger\} = \mathcal{H}\,, \qquad Q^2 = (Q^\dagger)^2 = 0\,, \qquad [Q, \mathcal{H}] = [Q^\dagger, \mathcal{H}] = 0\,. \tag{2}$$

Here $\mathcal{H}$ is the supersymmetric Hamiltonian, while Q and $Q^\dagger$ are called the supersymmetric charge operators. A simple realization of this superalgebra makes use of 2×2 matrices with operator entries

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \tag{3}$$

$$\mathcal{H} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}. \tag{4}$$

H_- and H_+ with eigenfunctions $\psi^{(-)}$ and $\psi^{(+)}$, respectively, are traditionally called the bosonic and fermionic Hamiltonians, respectively, and are interpreted as supersymmetric partners. The supersymmetric charge operators connect the bosonic and fermionic "sectors" via

$$Q\begin{pmatrix} \psi^{(-)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi^{(-)} \end{pmatrix}, \qquad Q^\dagger\begin{pmatrix} 0 \\ \psi^{(+)} \end{pmatrix} = \begin{pmatrix} A^\dagger\psi^{(+)} \\ 0 \end{pmatrix}. \tag{5}$$

A straightforward way to obtain the one-dimensional Schrödinger equation as a second-order linear differential equation is its factorization in terms of two first-order differential operators. The formalism of SUSYQM is suitable for this if one defines the A and $A^\dagger$ operators as

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \quad \text{and} \quad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\,. \tag{6}$$

Then, the bosonic and fermionic Hamiltonians take the form

$$H_\pm\psi^{(\pm)}(x) = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x)\right]\psi^{(\pm)}(x) = E^{(\pm)}\psi^{(\pm)}(x)\,. \tag{7}$$

The $V_-(x)$ and $V_+(x)$ potentials are called supersymmetric partner potentials and are obtained from the superpotential $W(x)$ as

$$V_\pm(x) = W^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}W(x). \tag{8}$$

Straightforward calculations reveal that H_- and H_+ have common energy eigenvalues. In particular, if $\psi^{(-)}(x)$ ($\psi^{(+)}(x)$) is the eigenfunction of H_- (H_+), then $A\psi^{(-)}(x)$ ($A^\dagger\psi^{(+)}(x)$) is the eigenfunction of H_+ (H_-).

It is customary to set the energy scale such that the ground state of H_- is at $E = 0$, i.e., $H_-\psi_0^{(-)}(x) = 0$. This result also follows from the relation

$$A\psi_0^{(-)} = 0\,. \tag{9}$$

It can be shown that there will be no eigenstate of H_+ at $E = 0$, while its other energy eigenvalues coincide with those of H_-, i.e.,

$$E_{n+1}^{(-)} = E_n^{(+)} \quad (n = 0, 1, 2, \ldots) \quad \text{with } E_0^{(-)} = 0. \tag{10}$$

Furthermore, Equation (9) allows expressing the superpotential in terms of the ground-state wave function of H_- as

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^{(-)}(x)\,. \tag{11}$$

In summary, in the knowledge of the bound-state wave functions of a potential $V_-(x)$, one can construct its supersymmetric partner potential $V_+(x)$ with (almost) the same energy spactrum, and its bound-state wave functions can also be easily obtained from those of $V_-(x)$.

Supersymmetric partner potentials can also be generated from further, not necessarily physical, solutions of H_- [26]. Taking the solution $\chi(x)$ with eigenvalue ϵ called the factorization energy [27],

$$H_-\chi(x) \equiv A^\dagger A\chi(x) = \epsilon\chi(x). \tag{12}$$

Equations (7) and (8) lead to the potential

$$\begin{aligned} V_-(x) &= \frac{\chi''(x)}{\chi(x)} + \epsilon \\ &= \left(-\frac{\chi'(x)}{\chi(x)}\right)^2 - \left(-\frac{\chi'(x)}{\chi(x)}\right)' + \epsilon\,, \end{aligned} \tag{13}$$

which will be a continuous function of x, provided that $\chi(x)$ is nodeless. The analogues of Equations (8) and (11) are then

$$V_\pm(x) = \tilde{W}^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}\tilde{W}(x) + \epsilon \tag{14}$$

and

$$\tilde{W}(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \chi(x)\,. \tag{15}$$

Note the presence of $\hat{W}(x) = -\chi'(x)/\chi(x)$ in Equation (13). It is often possible to factorize $\chi(x)$ as

$$\chi(x) = \psi_0^{(-)}(x)\xi(x)\,, \tag{16}$$

which leads to the relation connecting $W(x)$ in (11) and $\tilde{W}(x)$ in (15):

$$\tilde{W}(x) = W(x) - \frac{\mathrm{d}}{\mathrm{d}x} \ln \xi(x)\,. \tag{17}$$

The $\epsilon = 0$ choice recovers the special case $\chi(x) = \psi_0^{(-)}(x)$, which is interpreted as unbroken supersymmetry, while for $\epsilon \neq 0$ supersymmetry is said to be broken. For broken SUSY, H_+ will also possess a level at $E = E_0^{(-)}$.

The degeneracy of the energy levels of H_- and H_+ ultimately originates from the superalgebra (2), which contains relations implying that the supersymmetric Hamiltonian

$\mathcal{H}$ commutes with the supersymmetric charge operators. A Hamiltonian commuting with another operator indicates a symmetry, which generally implies a degeneracy. The origin of the usage of broken and unbroken supersymmetry is also related to these operators: the two-component ground state is the eigenfunction of $\mathcal{H}$, but it can also be the eigenfunction of Q if $A\psi^{(-)} = 0$ holds, i.e., for $\epsilon = 0$. For $A\psi^{(-)} \neq 0$, this does not occur, so supersymmetry is spontaneously broken.

The behavior of the possible nodeless $\chi(x)$ solutions (12) at the boundaries, i.e., their normalizability or non-normalizability, is also related to the question of whether SUSY is broken or not in the actual situation. In fact, depending on the boundary conditions satisfied by $\chi(x)$, four different SUSYQM transformations can be defined, as shown in Table 1 [26,28–30]. These have characteristic effects on the energy spectrum. The results are generally not affected by the domain of definition of the potential $[x_-, x_+]$, i.e., whether it is defined on a finite domain ($-\infty < x_-,\ x_+ < \infty$), on the full real x axis ($x_- \to -\infty,\ x_+ \to< \infty$) or on the positive real semi-axis ($x_=0,\ x_+ \to< \infty$). The last case contains radial potentials, where it is customary to denote x with r. When x_- or x_+ is finite, then it is advisable to inspect the possible singularities of the problem at these points. Let us assume that $x_- = 0$ and the solution behaves like $\chi(x) \sim x^\sigma$ there. It is straightforward to demonstrate by taking $\chi''(x)$ that the potential behaves like $V(x)\ \sim \sigma(\sigma-1)x^{-2}$ there. This is the case, for example, for radial potentials, when the centrifugal term is $l(l+1)r^{-2}$, while $\sigma = l+1$ for regular solutions and $\sigma = -l$ for irregular ones, where l is the orbital angular momentum with non-negative integer values. (Note that this applies to radial problem in three spatial dimensions.) The T_1 and T_3 transformations increase σ by one unit, while the T_2 and T_4 transformations decrease it by one unit. In the latter two cases, one has to consider $\sigma > 1$ in order to avoid attractive singularities in the SUSY partner potential $V_+(x)$ at $x = 0$.

The same prescription applies to potentials corresponding to non-integer values of σ. Furthermore, the singularities have to be handled in a similar way at both boundaries of problems defined on a finite domain (i.e., with finite x_- and x_+). In these cases, $x - x_-$ and $x_+ - x$ take the role of x. For the subtleties of singular potentials, see, e.g., Ref. [31].

When x_- and/or x_+ is infinite, the boundary conditions are handled in a different way. If the potentials are assumed to decay exponentially in the asymptotic limit(s), then the solutions also follow an exponential tail: $\chi(x)\ \sim \exp(\kappa x)$. Then, the sign of κ determines whether $\chi(x)$ converges or diverges asymptotically.

Table 1. SUSYQM transformations of a potential problem defined on the domain $x \in [x_-, x_+]$. These belong to four different types of solutions $\chi(x)$.

Transformation	T_1	T_2	T_3	T_4
ϵ	$\epsilon = E_0^{(-)}$	$\epsilon < E_0^{(-)}$	$\epsilon < E_0^{(-)}$	$\epsilon < E_0^{(-)}$
$\lim_{x\to x_-} \chi$	convergent	divergent	convergent	divergent
$\lim_{x\to x_+} \chi$	convergent	divergent	divergent	convergent
Spectrum modification	deletes g.s.	adds new g.s.	none	none

It has to be mentioned that for the transformations T_2, T_3 and T_4 ϵ, the factorization energy can take on any value below $E_0^{(-)}$, the ground-state energy of H_-. In these cases, it also appears in the spectrum of H_+, and it can also be considered a new parameter of $V_+(x)$. One finds that some ϵ values play a special role for a certain class of potentials: in these cases, the mathematical form of $V_+(x)$ and $V_-(x)$ turns out to be similar, and only the parameters differ. This is the case for shape-invariant potentials mentioned previously.

2.2. The General PI Class Potentials and Their Rational Extensions

Here, we collect the essential formulae for the potentials discussed in the present work, both in their general form as PI class potentials [5,32] and in their form specified for the three concrete cases, i.e., the generalized Pöschl–Teller, Scarf I and Scarf II potentials and their rationally extended version. The procedure of obtaining the general form of these potentials by transforming the differential equation of the Jacobi polynomials and that of the X_1-type exceptional Jacobi polynomials is presented in Appendix A.

The expression for the general PI type potentials [5,32] is

$$V(\alpha,\beta,x) = \frac{C}{1-z^2(x)}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2Cz(x)}{1-z^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) \tag{18}$$

while the bound-state energy wave functions and the corresponding energy eigenvalues are written, respectively, as

$$\psi_n(\alpha,\beta;x) = C_n^{(\alpha,\beta)}(1-z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(z(x)) \tag{19}$$

and

$$E_n = C\left(n + \frac{\alpha+\beta+1}{2}\right)^2. \tag{20}$$

Here $P_n^{(\alpha,\beta)}(z)$ is a Jacobi polynomial [3]. Conditions for normalizability and regularity at the boundaries follow from the boundary conditions, which can be defined only after specifying the $z(x)$ function and its domain of definition.

As discussed in Appendix A, there are three particular solutions belonging to the PI potential class, each with a different domain of definition. The generalized Pöschl–Teller potential is obtained for $z(x) = \cosh(ax)$ with $C = -a^2$, and it is defined on the positive real axis $x \in [0,\infty)$ corresponding to $z \in [1,\infty)$. The Scarf I potential is the trigonometric version of the generalized Pöschl–Teller potential defined on a finite domain: $z(x) = \cos(ax)$, $C = a^2$, $x \in [0,\pi/a]$, $z \in [-1,1]$. This potential is often derived from the $z(x) = \sin(ax)$ function defined on the $x \in [-\pi/(2a),\pi/(2a)]$ domain, but this is merely a shifted version of the original potential. Finally, the Scarf II potential can be obtained by the substitutions $z(x) = \mathrm{i}\sinh(ax)$, $C = -a^2$, $x \in (-\infty,\infty)$ and z running along the whole imaginary axis. Note that although $z(x)$ is an imaginary function, potential (18) will be real for an appropriate choice of α and β, as will also the bound-state wave functions (19), apart from an unimportant phase. We note that there are two more potentials, named Pöschl–Teller I and II, that are traditionally considered distinct entities; however, these can be derived from the Scarf I and the generalized Pöschl–Teller potentials by a trivial mapping $x \to x/2$ [7].

According to the results derived in Appendix A, the rationally extended versions of the same potentials are obtained for the same three choices of $z(x)$ and C, so their domain of definition will be the same as their classical correspondents. The general formulae corresponding to Equations (18)–(20) are as follows:

$$\begin{aligned}\hat{V}(\alpha,\beta,x) &= V(\alpha,\beta;x) \\ &+ \frac{2C(\alpha+\beta)}{(\alpha-\beta)z(x)+\alpha+\beta} + \frac{2C\left[(\alpha-\beta)^2-(\alpha+\beta)^2\right]}{[(\alpha-\beta)z(x)+\alpha+\beta]^2},\end{aligned} \tag{21}$$

$$\hat{\psi}_n(\alpha,\beta;x) = \hat{C}_n^{(\alpha,\beta)}(1-z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}+\frac{1}{4}}[(\alpha-\beta)z(x)+\alpha+\beta]^{-1}\hat{P}_n^{(\alpha,\beta)}(z(x)), \tag{22}$$

and

$$\hat{E}_n = C\left(n - 1 + \frac{\alpha+\beta+1}{2}\right)^2. \tag{23}$$

Here, $\hat{P}_n^{(\alpha,\beta)}(z)$ are the X_1-type exceptional Jacobi polynomials [14,15] discussed in Appendix A. Note that, of the three (five) linearly independent terms in Equation (A17) (Equation (A28)), only two (four) remain in Equation (19) (Equation (22)), because the third (fifth) constant term was incorporated into the energy E.

Before closing this section, we note that the $P_n^{(\alpha,\beta)}(z)$ Jacobi and $\hat{P}_n^{(\alpha,\beta)}(z)$ X_1-type exceptional Jacobi polynomials form an infinite orthonormal polynomial system only when $z \in [-1,1]$. For z outside this domain, they form a finite orthonormal system. See, e.g., Ref. [33].

3. Unified Discussion

In this section, we consider a particular form of the solution $\chi(x)$ in Equation (12). It is constructed to be nodeless and singularity-free within $[x_-, x_+]$, in accordance with the requirements outlined in Section 2.1. In the next step, we use Equation (15) to generate the corresponding superpotential $\tilde{W}(x)$ (from which we drop the tilde from now on, to ease the notation). Then, we express the $V_-(x)$ and $V_+(x)$ partner potentials using Equation (14). Next, we analyze the mathematical form of these potentials and attempt to set the parameters such that $V_-(x)$ and $V_+(x)$ take the form of either the general PI type potential (18) or its rationally extended version (21). These parameters also determine the behavior of $\chi(x)$ at the boundaries, also identifying the type of SUSYQM transformation in Table 1. We also calculate the factorization energy ϵ. This can be conducted up to a constant energy shift, unless one requires that the potentials be free from additive constants.

3.1. The General Results

The discussion starts with selecting a sufficiently general $\chi(x)$ function that reflects the requirements related to the boundary conditions specific to the possible T_i transformations. Let us consider the $\chi(x)$ function depending on six parameters

$$\chi(x) = (1 - z(x))^t (1 + z(x))^s \frac{(p + z(x))^k}{(q + z(x))^j} \, . \tag{24}$$

t and s (and especially their sign) determine the behavior of $\chi(x)$ at the boundaries $z = 1$ and $z = -1$, while $t + s + k - j$ sets its asymptotic properties for $z \to \pm\infty$ (whichever is applicable). This behavior also determines which of the four types of SUSY transformations $\chi(x)$ corresponds to from Table 1. Note that sometimes it is more natural to apply $(z(x) - 1)$ rather than $(1 - z(x))$; however, this change in sign does not have a significant effect on the results. We also choose k and j to be non-negative integers and assume that $p \neq q$: the $p = q$ case is included in the formula as the $j = 0$ and $k \to k - j$ choice. The $1/\chi(r)$ operation that reverses the behavior of the solution at the boundaries and changes $W(r)$ to $-W(r)$ can also be obtained by the $(t, s, k, j, p, q) \to (-t, -s, j, k, q, p)$ transformation. The natural requirement that $\chi(x)$ has to be nodeless and singularity-free leads to the prescriptions $z(x) \neq -p$ and $z(x) \neq -q$.

According to Equation (15), the superpotential is then given by

$$W(x) = -C^{1/2}(1 - z^2(x))^{1/2} \left[-\frac{t}{1 - z(x)} + \frac{s}{1 + z(x)} + \frac{k}{p + z(x)} - \frac{j}{q + z(x)} \right], \tag{25}$$

where the pre-factor originates from $z'(x)$ in Equation (A15). From this result and Equation (14), the partner potentials are obtained as

$$\begin{aligned} V_-(x) - \epsilon \;=\; & -C(t + s + k - j)^2 + \frac{C}{1 - z^2(x)}(2t^2 + 2s^2 - t - s) \\ & + \frac{Cz(x)}{1 - z^2(x)}(2t^2 - 2s^2 - t + s) + \frac{Ck(k-1)(1 - p^2)}{(p + z(x))^2} + \frac{Cj(j+1)(1 - q^2)}{(q + z(x))^2} \\ & + \frac{Ck}{p + z(x)} \left[p(2t + 2s + 2k - 1) - 2t + 2s - 2j \frac{1 - p^2}{q - p} \right] \\ & + \frac{Cj}{q + z(x)} \left[-q(2t + 2s - 2j - 1) + 2t - 2s + 2k \frac{1 - q^2}{q - p} \right] \end{aligned} \tag{26}$$

and

$$\begin{aligned}
V_+(x) - \epsilon \quad = \quad & -C(t+s+k-j)^2 + \frac{C}{1-z^2(x)}(2t^2+2s^2+t+s) \\
& + \frac{Cz(x)}{1-z^2(x)}(2t^2-2s^2+t-s) + \frac{Ck(k+1)(1-p^2)}{(p+z(x))^2} + \frac{Cj(j-1)(1-q^2)}{(q+z(x))^2} \\
& + \frac{Ck}{p+z(x)}\left[p(2t+2s+2k+1) - 2t + 2s - 2j\frac{1-p^2}{q-p}\right] \\
& + \frac{Cj}{q+z(x)}\left[-q(2t+2s-2j+1) + 2t - 2s + 2k\frac{1-q^2}{q-p}\right] . \qquad (27)
\end{aligned}$$

Note that the six terms of (26) and (27) contain terms with structure similar to the two and four terms appearing in (18) and (21), respectively, which allows us to link the parameters t, s, k, j, p and q with the α and β parameters of the Jacobi and the X_1-type exceptional Jacobi polynomials.

Note also that if we choose $V_-(x)$ and $V_+(x)$ to be free from an additive constant, e.g., in Equations (A17) and (A28) , then the factorization energy is found to be

$$\epsilon = C(t+s+k-j)^2 . \qquad (28)$$

In what follows, we focus on special cases that recover exactly solvable potentials.

3.2. Special Case: $j = 0, k = 0$

For $k = j = 0$, only the first two terms remain in Equations (26) and (27). Matching the coefficients of (26) with those of (18), one finds that

$$\begin{aligned}
\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4} \quad &= \quad 2t^2 + 2s^2 - t - s \\
2\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) \quad &= \quad 2t^2 - 2s^2 - t + s . \qquad (29)
\end{aligned}$$

Subtracting and adding the two equations, one obtains

$$\begin{aligned}
\left(\alpha+\frac{1}{2}\right)\left(\alpha-\frac{1}{2}\right) \quad &= \quad 2t(2t-1) \\
\left(\beta+\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right) \quad &= \quad 2s(2s-1) . \qquad (30)
\end{aligned}$$

There are four different solutions to this system of equations, corresponding to four distinct SUSY transformations. The actual values of t and s also determine the SUSY partner potential $V_+(x)$. The results are summarized in Table 2 for $j = k = 0$, where the results of the four different transformations are displayed. The table also contains the quantity $E_0^{(-)} - \epsilon$, where

$$\epsilon = C(t+s)^2 \qquad (31)$$

holds in this case. When this difference is zero, SUSY is unbroken, and we have the T_1 transformation from Table 1, while when it is negative, SUSY is broken and the T_2, T_3 and T_4 transformations may emerge, depending on the behavior of $\chi(x)$ at the boundaries. These latter findings can be analyzed only when the allowed $z(x)$ functions, and thus the potentials, are specified.

Table 2. SUSY transformations for the general form of PI class potentials. $V_-(x)$ and $V_+(x)$ are expressed in terms of potentials (18) or (21), respectively.

j,k	$V_-(x)$	t	s	p	q	$V_+(x)$	ϵ	$E_0^{(-)}-\epsilon$
0, 0	$V(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha+1,\beta+1,x)$	$C\left(\frac{\alpha+\beta+1}{2}\right)^2$	0
		$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha-1,\beta-1,x)$	$C\left(\frac{-\alpha-\beta+1}{2}\right)^2$	$C(\alpha+\beta)$
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta+1}{2}\right)^2$	$C(\alpha+1)\beta$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta+1}{2}\right)^2$	$C\alpha(\beta+1)$
0, 1	$V(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta}{\alpha-\beta+2}$		$\hat{V}(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta+3}{2}\right)^2$	$C(\alpha+2)(\beta-1)$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta}{\alpha-\beta-2}$		$\hat{V}(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta+3}{2}\right)^2$	$C(\alpha-1)(\beta+2)$
1, 0	$\hat{V}(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$		$\frac{\alpha+\beta}{\alpha-\beta}$	$V(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta-1}{2}\right)^2$	$C\alpha(\beta+1)$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$		$\frac{\alpha+\beta}{\alpha-\beta}$	$V(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta-1}{2}\right)^2$	$C\beta(\alpha+1)$
1, 1	$\hat{V}(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+2}{\alpha-\beta}$	$\frac{\alpha+\beta}{\alpha-\beta}$	$\hat{V}(\alpha+1,\beta+1,x)$	$C\left(\frac{\alpha+\beta+1}{2}\right)^2$	0
		$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta-2}{\alpha-\beta}$	$\frac{\alpha+\beta}{\alpha-\beta}$	$\hat{V}(\alpha-1,\beta-1,x)$	$C\left(\frac{-\alpha-\beta+1}{2}\right)^2$	$C(\alpha+\beta)$

3.3. Special Case: $j=0,k\neq 0$

Now, two more terms appear both in $V_-(x)$ in (26) and $V_+(x)$ in (27). One of these can be cancelled in the former with $k=1$, while to cancel the second in order to recover the general PI class potential (18), a specific choice of p has to be made:

$$p=\frac{2t-2s}{2t+2s+1}\,. \tag{32}$$

Then one obtains

$$V_-(x)=V(\alpha,\beta,x) \tag{33}$$

under the same conditions as in Section 3.2, i.e., for $t=\pm\alpha/2+1/4$ and $s=\pm\beta/2+1/4$. However, further conditions follow from matching $V_+(x)$ to $\hat{V}(\alpha,\beta,x)$ (in particular, to its last two terms) in Equation (21). These conditions rule out two possibilities in which α and β have the same sign in t and s. The two remaining transformations are displayed in Table 2 for $j=0,k=1$. The factorization energy is now

$$\epsilon=C(t+s+1)^2 \tag{34}$$

3.4. Special Case: $j\neq 0,k=0$

This case is the inverse of that discussed in Section 3.3 in the sense that now $V_+(x)$ in (27) will play the role of $V(\alpha,\beta,x)$ in (18), while $V_-(x)$ has to be matched with $\hat{V}(\alpha,\beta,x)$ in (21). For this, the choice

$$q=\frac{2t-2s}{2t+2s-1} \tag{35}$$

has to be made, while one finds that again, the possibilities with the same sign of α and β in t and s have to be ruled out. With the actual value of the factorization energy

$$\epsilon=C(t+s-1)^2, \tag{36}$$

the results are summarized in Table 2 for $j=1,k=0$.

3.5. Special Case: $j \neq 0, k \neq 0$

Now the six terms in Equations (26) and (27) do not vanish in general. However, for $j = 1$ and $k = 1$, they can be both reduced to five terms, while with the appropriate choice of p and q, both potentials can be reduced to four terms, and it becomes possible to match them with the rationally extended version of the general PI class potential (21). Again, all four possibilities $t = \pm\alpha/2 + 1/4, s = \pm\beta/2 + 1/4$ follow from matching the first two terms, corresponding to the general PI class potential, while the elimination of one term each by the choice of p and q rules out the possibilities with different signs of α and β in t and s, respectively. The supersymmetric partner of $V_-(x) = \hat{V}(\alpha, \beta, x)$ can be determined in both cases from the actual values of t and s. The results are summarized in Table 2 for $j = k = 1$, together with the consequences of the actual value of the factorization energy

$$\epsilon = C(t + s)^2. \tag{37}$$

4. Specializing to the Three PI Class Potentials

In this section, we substitute the three $z(x)$ functions into the general formulae and inspect the prescriptions imposed by the boundary conditions on the parameters α and β. Further conditions follow for the rationally extended potentials $\hat{V}(\alpha, \beta, x)$ from the requirement that the denominator of the last two potential terms not have zeros within the domain of definition of $z(x)$, i.e.,

$$(\alpha - \beta)z(x) + (\alpha + \beta) \neq 0 \qquad \text{for} \quad x \in (x_-, x_+). \tag{38}$$

All these prescriptions determine t and s, which, in turn, decide which of the four transformations in Table 1 the given parameter combinations correspond to. A summary of these conditions is presented in Table 3.

Table 3. The key values for the three potentials. The last column displays conditions for p and q.

$z(x)$	C	x_-	x_+	z_-	z_+	p, q
$\cosh(x)$	$-a^2$	0	∞	1	∞	$-1 < p, q$
$\cos(x)$	a^2	0	$\frac{\pi}{a}$	1	-1	$p, q < -1, 1 < p, q$
$\mathrm{i}\sinh(x)$	$-a^2$	$-\infty$	∞	$-\infty$	∞	n. a.

In order to simplify the notation, in what follows we introduce the expressions

$$\mathcal{A}(\alpha, \beta) \equiv \left(\frac{\alpha + \beta}{2}\right)^2 + \left(\frac{\alpha - \beta}{2}\right)^2 - \frac{1}{4}, \quad \mathcal{B}(\alpha, \beta) \equiv \left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right). \tag{39}$$

4.1. $z(x) = \cosh(ax)$: The Generalized Pöschl–Teller Potential and Its Rational Extension

For this choice, the actual PI class potential is the generalized Pöschl–Teller potential, which is

$$V_G(\alpha, \beta, x) = \frac{a^2}{\sinh^2(ax)}\mathcal{A}(\alpha, \beta) + \frac{2a^2\cosh(ax)}{\sinh^2(x)}\mathcal{B}(\alpha, \beta) \tag{40}$$

with bound-state eigenfunctions

$$\psi_n(\alpha, \beta; x) = C_n^{(\alpha,\beta)}(\cosh(ax) - 1)^{\frac{\alpha}{2}+\frac{1}{4}}(\cosh(ax) + 1)^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(\cosh(x)) \tag{41}$$

and energy eigenvalues

$$E_n = -a^2\left(n + \frac{\alpha + \beta + 1}{2}\right)^2. \tag{42}$$

For the rationally extended generalized Pöschl–Teller potential, the corresponding formulae are as follows:

$$\begin{aligned}\hat{V}_G(\alpha,\beta,x) &= V_G(\alpha,\beta,x) \\ &-\frac{2a^2(\alpha+\beta)}{(\alpha-\beta)\cosh(ax)+\alpha+\beta}-\frac{2a^2\left((\alpha+\beta)^2-(\alpha-\beta)^2\right)}{[(\alpha-\beta)\cosh(ax)+\alpha+\beta]^2}\end{aligned} \tag{43}$$

with bound-state eigenfunctions

$$\hat{\psi}_n(\alpha,\beta;x)=\hat{C}_n^{(\alpha,\beta)}(\cosh(ax)-1)^{\frac{\alpha}{2}+\frac{1}{4}}(\cosh(ax)+1)^{\frac{\beta}{2}+\frac{1}{4}}[(\alpha-\beta)\cosh(ax)+\alpha+\beta]^{-1}\hat{P}_n^{(\alpha,\beta)}(\cosh(x)) \tag{44}$$

and energy eigenvalues

$$\hat{E}_n=-a^2\left(n-1+\frac{\alpha+\beta+1}{2}\right)^2. \tag{45}$$

Here, n runs from $n=1$. These expressions recover the findings of Ref. [17] with $\alpha=A-B-1/2$ and $\beta=A+B-1/2$, noting also that the roles of $V_-(x)$ and $V_+(x)$ are reversed in that publication.

The behavior of both potentials at $x=0$ is of the type x^{-2}, while that of the bound-state wave functions is $x^{\alpha+1/2}$. For this reason, the prescription $\alpha>-1/2$ holds in general, while for transformations decreasing α by one unit, $\alpha>1/2$ has to hold in order to guarantee the proper behavior of the bound-state solutions of the partner potential. Asymptotically decaying solutions are obtained for $\alpha+\beta+1<-2n$ for the generalized Pöschl–Teller potential and $\alpha+\beta+1<-2(n+1)$ for its rationally extended version. In both cases, the condition for having at least one bound state is $\alpha+\beta+1<0$. In summary, the conditions valid for the α and β parameters for the potential $V_G(\alpha,\beta,x)$ are

$$\pm\frac{1}{2} < \alpha \tag{46}$$

$$\alpha+\beta+1 < 0 \tag{47}$$

These conditions also hold for the potential $\hat{V}(\alpha,\beta,x)$, but there is also an extra condition to guarantee its non-singularity for finite positive x values:

$$-1<\frac{\alpha+\beta}{\alpha-\beta}. \tag{48}$$

Elementary calculations show that combining conditions (46) and (47) leads to $0<-(\alpha+\beta+1)+1\pm1<\alpha-\beta$, which, in turn, leads to

$$1<\alpha \tag{49}$$

when combined with Equation (48). These conditions have to be considered for the generalized Pöschl–Teller potential $V_G(\alpha,\beta,x)$ and its rational extension $\hat{V}_G(\alpha,\beta,x)$ in the admissible transformation schemes, applying them to the actual parameter values.

The SUSY transformations summarized for the general PI class potential $V_G(\alpha,\beta,x)$ and its rational extension $\hat{V}_G(\alpha,\beta,x)$ are specified for the case of the generalized Pöschl–Teller potential in Table 4.

Table 4. SUSY transformations for the generalized Pöschl–Teller potential $V_G(\alpha,\beta,x)$ (40) and its rational extension $\hat{V}_G(\alpha,\beta,x)$ (43). These potentials play the role of $V_+(x)$, depending on the parameters t, s, j and k, while $V_-(x) = V_G(\alpha,\beta,x)$ for $j = 1$ and $V_-(x) = \hat{V}_G(\alpha,\beta,x)$ for $j = 1$. The type of transformation (T_i in Table 1 is determined by these parameters and the conditions prescribed for the potential parameters α and β in this subsection.)

j,k	$V_+(x)$	α Condition	t	s	$t+s+k-j$	$\frac{\alpha+\beta}{\alpha-\beta}$ in $V_-(x)$	$\frac{\alpha'+\beta'}{\alpha'-\beta'}$ in $V_+(x)$	$E_0^{(-)}-\epsilon$	T_i
0, 0									
	$V_G(\alpha+1,\beta+1,x)$	$>-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+1}{2}<0$			0	T_1
	$V_G(\alpha-1,\beta-1,x)$	$>\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$1-\frac{\alpha+\beta+1}{2}>0$			$-a^2(\alpha+\beta)>0$	T_2
	$V_G(\alpha+1,\beta-1,x)$	$>-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\alpha+1-\frac{\alpha+\beta+1}{2}>0$			$a^2(\alpha+1)(-\beta)>0$	T_3
	$V_G(\alpha-1,\beta+1,x)$	$>\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+1}{2}-\alpha<0$			$a^2(\beta+1)(-\alpha)>0$	T_4
0, 1									
	$\hat{V}_G(\alpha+1,\beta-1,x)$	$>-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\alpha+1-\frac{\alpha+\beta+1}{2}>0$		$\frac{\alpha+\beta}{\alpha-\beta+2}>-1$	$a^2(\alpha+2)(1-\beta)$	T_3
	$\hat{V}_G(\alpha-1,\beta+1,x)$	$>1>\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+1}{2}-\alpha<0$		$\frac{\alpha+\beta}{\alpha-\beta-2}>-1$	$a^2(1-\alpha)(\beta+2)>0$	T_4
1, 0									
	$V_G(\alpha+1,\beta-1,x)$	$>0>-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\alpha-\frac{\alpha+\beta+1}{2}>0$	$\frac{\alpha+\beta}{\alpha-\beta}>-1$		$a^2(-\alpha)(\beta+1)$	T_3
	$\hat{V}_G(\alpha-1,\beta+1,x)$	$>1>\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+1}{2}-\alpha<0$	$\frac{\alpha+\beta}{\alpha-\beta}>-1$		$a^2(\alpha+1)(-\beta)>0$	T_4
1, 1									
	$V_G(\alpha+1,\beta+1,x)$	$>0>-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+1}{2}<0$	$\frac{\alpha+\beta}{\alpha-\beta}>-1$	$\frac{\alpha+\beta+2}{\alpha-\beta}>-1$	0	T_1
	$\hat{V}_G(\alpha-1,\beta-1,x)$	$>1>\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$1-\frac{\alpha+\beta+1}{2}>0$	$\frac{\alpha+\beta}{\alpha-\beta}>-1$	$\frac{\alpha+\beta-2}{\alpha-\beta}>-1$	$a^2(\alpha+\beta)>0$	T_2

The first column indicates the relevant values of j and k, which set the actual form of the solution $\chi(x)$ in Equation (24) that generates the SUSY transformation. The second column displays $V_+(x)$, the supersymmetric partner of $V_-(x)$, the role of which is played by $V_G(\alpha, \beta, x)$ in (40) or $\hat{V}_G(\alpha, \beta, x)$ (43), depending on j and k. In the third column , the lower boundaries of α are presented, originating from Equations (46) and (49), whenever applicable. When both equations are relevant, both lower values are presented. The fourth and fifth columns contain t and s, while the sixth one is simply $t + s + k - j$, setting the asymptotic behavior of the $\chi(x)$ solution. The seventh and eighth columns contain the ratios that determine whether the potentials playing the role of $V_-(x)$ and $V_+(x)$ satisfy the condition of being non-singular at finite x values. These appear only when these potentials are of the type $\hat{V}_G(\alpha, \beta, x)$. In the ninth column, the difference in the ground state energy of $V_-(x)$ and the factorization energy ϵ is displayed. When this difference is zero, SUSY is unbroken, and the relevant transformation is of the type T_1, while when it is positive, SUSY is broken and the transformation can be of the type T_2, T_3 or T_4. Finally, the last column presents the actual type of transformation determined by the sign of t and $t + s + k - j$ in the fourth and sixth column, respectively, and confirmed by the ninth column. The values appearing in the inequalities in the third and in the sixth to ninth columns are calculated based on the conditions formulated in Equations (46)–(49).

It is seen that the SUSY transformation systematically changes α and β by one unit and connects the same type of potentials (for $j = k$) or different types (for $j \neq k$). The T_1- and T_2-type transformations confirm the shape-invariance of the generalized Pöschl–Teller potential $V_G(\alpha, \beta, x)$ (for $j = k = 0$) and of its rational extension $\hat{V}_G(\alpha, \beta, x)$ (for $j = k = 1$).

4.2. $z(x) = \cos(ax)$: *The Scarf I Potential and Its Rational Extension*

Now, the essential formulae for the Scarf I potential are the following:

$$V_I(\alpha, \beta, x) = \frac{a^2}{\sin^2(ax)}\mathcal{A}(\alpha, \beta) + \frac{2a^2\cos(ax)}{\sin^2(x)}\mathcal{B}(\alpha, \beta), \tag{50}$$

$$\psi_n(\alpha, \beta; x) = C_n^{(\alpha,\beta)}(1 - \cos(ax))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \cos(ax))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(\cos(x)), \tag{51}$$

$$E_n = a^2\left(n + \frac{\alpha + \beta + 1}{2}\right)^2. \tag{52}$$

Note that now the energy egenvalues are in the positive domain.

The corresponding results for the rationally extended Scarf I potential are

$$\begin{aligned}\hat{V}_I(\alpha, \beta, x) \quad &= \quad V_I(\alpha, \beta, x) \\ &+ \frac{2a^2(\alpha + \beta)}{(\alpha - \beta)\cos(ax) + \alpha + \beta} - \frac{2a^2\left((\alpha + \beta)^2 - (\alpha - \beta)^2\right)}{[(\alpha - \beta)\cos(ax) + \alpha + \beta]^2},\end{aligned} \tag{53}$$

$$\hat{\psi}_n(\alpha, \beta; x) = \hat{C}_n^{(\alpha,\beta)}(1 - \cos(ax) - 1)^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \cos(ax))^{\frac{\beta}{2}+\frac{1}{4}}[\alpha - \beta)\cos(ax) + \alpha + \beta]^{-1}\hat{P}_n^{(\alpha,\beta)}(\cos(x)), \tag{54}$$

$$\hat{E}_n = a^2\left(n - 1 + \frac{\alpha + \beta + 1}{2}\right)^2. \tag{55}$$

Again, n runs from $n = 1$ in accordance with the findings of Ref. [16], which correspond to the choice $\alpha = A - B - 1/2$ and $\beta = A + B - 1/2$.

In spite of the mathematical similarities to the generalized Pöschl–Teller potential and its rational extension, the boundary conditions differ significantly. Now both boundaries occur at finite real values, where the singularity of the potential is of the type ξ^{-2}, where $\xi = x$ and $\pi/a - x$. The strength of these singularities are set independently by α and β. The SUSY transformations summarized in Table 2 for the general case either increase or decrease α and β by one unit, which set the minimal values of these parameters to $-1/2$ or

1/2, respectively, for the given transformation. The actual requirements for α and β set by the boundary conditions of $V_I(\alpha, \beta, x)$ (and $\hat{V}_I(\alpha, \beta, x)$) are now

$$\pm\frac{1}{2} \quad < \quad \alpha \tag{56}$$

$$\pm\frac{1}{2} \quad < \quad \beta \tag{57}$$

depending of whether the actual parameter value is increased $(-)$ or decreased $(+)$ by one unit. Besides these prescriptions set by the boundary conditions, the non-singularity requirement of $\hat{V}_I(\alpha, \beta, x)$ introduces further conditions. These are now

$$\frac{\alpha+\beta}{\alpha-\beta} < -1 \tag{58}$$

or

$$1 < \frac{\alpha+\beta}{\alpha-\beta} \tag{59}$$

depending on whether $\alpha < \beta$ or $\beta < \alpha$ holds. Note that, since $\alpha + \beta > 0$ holds in most cases, the sign of the quantities in Equations (58) and (59) is determined by that of $\alpha - \beta$.

Table 5 summarizes the SUSY transformations specified for the Scarf I potential $V_I(\alpha, \beta, x)$ and its rational extension $\hat{V}_I(\alpha, \beta, x)$. The notation in Table 5 is essentially the same as in Table 4, with the difference in the sixth and seventh columns, where the quantity setting the behavior of $\chi(x)$ at the upper boundary ($x_+ = \pi/a$ here and $x_+ \to \infty$ in Table 4) is displayed.

The general pattern of the SUSY transformations is similar to that observed for the generalized Pöschl–Teller potential and its rational extension. The shape-invariance of the Scarf I potential $V_I(\alpha, \beta, x)$ and that of its rational extension $\hat{V}_I(\alpha, \beta, x)$ is confirmed by the T_1- and T_2-type transformations with $j = k = 0$ and $j = k = 1$, respectively.

Table 5. The same as Table 4 for the Scarf I potential $V_I(\alpha,\beta,x)$ (50) and its rational extension $\hat{I}_I(\alpha,\beta,x)$ (53). In the seventh and eighth columns, $1 < (A+B)/(A-B) < -1$ indicates that the first inequality holds for $A > B$, and the second one for $A < B$.

j,k	$V_+(x)$	α Condition	β Condition	t	s	$\frac{\alpha+\beta}{\alpha-\beta}$ in $V_-(x)$	$\frac{\alpha'+\beta'}{\alpha'-\beta'}$ in $V_+(x)$	$E_0^{(-)}-\epsilon$	T_i
0, 0									
	$V_I(\alpha+1,\beta+1,x)$	$> -\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			0	T_1
	$V_I(\alpha-1,\beta-1,x)$	$> \frac{1}{2}$	$> \frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$a^2(\alpha+\beta) > 0$	T_2
	$V_I(\alpha+1,\beta-1,x)$	$> -\frac{1}{2}$	$> \frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$a^2(\alpha+1)(\beta) > 0$	T_3
	$V_I(\alpha-1,\beta+1,x)$	$> \frac{1}{2}$	$> -\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			$a^2(\beta+1)(\alpha) > 0$	T_4
0, 1									
	$\hat{V}_I(\alpha+1,\beta-1,x)$	$> -\frac{1}{2}$	$> 1 > \frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$		$1 < \frac{\alpha+\beta}{\alpha-\beta+2} < -1$	$a^2(\alpha+2)(\beta-1) > 0$	T_3
	$\hat{V}_I(\alpha-1,\beta+1,x)$	$> 1 > \frac{1}{2}$	$> \frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$		$1 < \frac{\alpha+\beta}{\alpha-\beta-2} < -1$	$a^2(\alpha-1)(\beta+2) > 0$	T_4
1, 0									
	$V_I(\alpha+1,\beta-1,x)$	$> 0 > -\frac{1}{2}$	$> 1 > \frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$1 < \frac{\alpha+\beta}{\alpha-\beta} < -1$		$a^2\alpha(\beta+1) > 0$	T_3
	$V_I(\alpha-1,\beta+1,x)$	$> 1 > \frac{1}{2}$	$> 0 > -\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$1 < \frac{\alpha+\beta}{\alpha-\beta} < -1$		$a^2(\alpha+1)\beta > 0$	T_4
1, 1									
	$\hat{V}_I(\alpha+1,\beta+1,x)$	$> 0 > -\frac{1}{2}$	$> 0 > -\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$1 < \frac{\alpha+\beta}{\alpha-\beta} < -1$	$1 < \frac{\alpha+\beta+2}{\alpha-\beta} < -1$	0	T_1
	$\hat{V}_I(\alpha-1,\beta-1,x)$	$> 1 > \frac{1}{2}$	$> 1 > \frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$1 < \frac{\alpha+\beta}{\alpha-\beta} < -1$	$1 < \frac{\alpha+\beta-2}{\alpha-\beta} < -1$	$a^2(\alpha+\beta) > 0$	T_2

4.3. $z(x) = \mathrm{i}\sinh(ax)$: The Scarf II Potential—The Real Version and Its Rational Extension

The general form of the Scarf II potential is obtained from an imaginary $z(x)$ function; nevertheless, real potentials with real energy spectra follow from the formulae in a straightforward way. The potential obtained from the general PI class expressions is

$$V_{II}(\alpha,\beta,x) = -\frac{a^2}{\cosh^2(ax)}\mathcal{A}(\alpha,\beta) - \frac{2\mathrm{i}a^2\sinh(ax)}{\cosh^2(x)}\mathcal{B}(\alpha,\beta). \tag{60}$$

The bound-state eigenfunctions are

$$\psi_n(\alpha,\beta;x) = C_n^{(\alpha,\beta)}(1-\mathrm{i}\sinh(ax))^{\frac{\alpha}{2}+\frac{1}{4}}(1+\mathrm{i}\sinh(ax))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x)), \tag{61}$$

and the energy eigenvalues are

$$E_n = -a^2\left(n + \frac{\alpha+\beta+1}{2}\right)^2, \tag{62}$$

which are formally the same as those obtained for the generalized Pöschl–Teller potential (42).

This potential is real for $\alpha^* = \beta$. (Note that in this setting, the wave functions (61) can also be expressed in terms of Romanowski polynomials, which can be obtained from Jacobi polynomials with imaginary arguments [34]. See also the more recent Ref. [33].) In this case , $(\alpha+\beta)/2 = \alpha_R$ and $(\alpha-\beta)/2 = \mathrm{i}\alpha_I$. As another similarity to the generalized Pöschl–Teller potential, the regularity of the wave functions for $x \to \infty$ follows for

$$\alpha+\beta+1 < 0\,, \tag{63}$$

i.e.,

$$\alpha_R < -\frac{1}{2}\,. \tag{64}$$

However, now this is also the condition for the regularity at $x \to -\infty$, and no further conditions arise for the parameters α and β.

Another novelty is that the rational extension of the real Scarf II potential cannot be defined, because the $(\alpha-\beta)\mathrm{i}\sinh(ax) + \alpha + \beta = -2\alpha_I\sinh(ax) + 2\alpha_R$ function will take on a zero at a finite x for $\sinh(ax) = \alpha_R/\alpha_I$. For this reason, we discuss only the real Scarf II potential here, so Table 6 contains only the case with $j = k = 0$.

Table 6. The same as Table 4 for the real Scarf II potential $V_{II}(\alpha,\beta,x)$.

j,k	$V_+(x)$	α_R Condition	t	s	Re$(t+s)$	ϵ	T_i
0, 0							
	$V_{II}(\alpha+1,\beta+1,x)$	$< -\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\alpha_R+\frac{1}{2} < 0$	$-a^2\left(\alpha_R+\frac{1}{2}\right)^2 = E_0^{(-)}$	T_1
	$V_{II}(\alpha-1,\beta-1,x)$	$< -\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$-\alpha_R+\frac{1}{2} > 1$	$-a^2\left(\alpha_R-\frac{1}{2}\right)^2 < E_0^{(-)}$	T_2
	$V_{II}(\alpha+1,\beta-1,x)$	$< -\frac{1}{2}$	$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\frac{1}{2} > 0$	$-a^2\left(\mathrm{i}\alpha_I+\frac{1}{2}\right)^2$ complex	(T_2)
	$V_{II}(\alpha-1,\beta+1,x)$	$< -\frac{1}{2}$	$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{1}{2} > 0$	$-a^2\left(-\mathrm{i}\alpha_I+\frac{1}{2}\right)^2$ complex	(T_2)

The first line in Table 6 describes a standard T_1-type SUSYQM transformation associated with unbroken SUSY: it eliminates the state of $V_+(x)$ corresponding to the ground state of $V_-(x)$. The T_2-type transformation in the second line is its inverse: it introduces a new ground state of $V_+(x)$ at the energy corresponding to the factorization energy ϵ. The remaining two transformations are, in principle, T_2-type, because the $\chi(x)$ function that generates the SUSYQM transformation is unbound for $x \to \pm\infty$. However, the energy eigenvalues are complex (actually, the two energies are complex conjugates of each other), so they cannot be interpreted as valid transformations. These energy eigenvalues fit into the sequence of resonances that correspond to the irregular solutions of the real Scarf II potential, i.e., its resonances. Note that these can be obtained by the $\alpha \to -\alpha$ replacement, which leave potential (60) invariant, but converts the energy eigenvalues (62) into complex.

Before closing this section, we note that a much richer network of SUSYQM transformations can be generated for the $\mathcal{PT}$-symmetric Scarf II potential. (For a recent review of $\mathcal{PT}$-symmetric quantum mechanics, see the monograph [35]). $\mathcal{PT}$-symmetric potentials are special complex potentials that are invariant under the simultaneous space and time reflection, the latter represented by complex conjugation. In practical terms, for one-dimensional potentials $\mathcal{PT}$, symmetry corresponds to the requirement $V^*(-x) = V(x)$. The $\mathcal{PT}$-symmetric version of the Scarf II potential corresponds to real values of α and β for unbroken $\mathcal{PT}$ symmetry. With this choice, the denominators appearing in the formulae will be defined on the complex plane and avoid zero, cancelling any singularities along the real x axis. As another specific feature, the discrete spectrum of these potentials is real (for unbroken $\mathcal{PT}$ symmetry), which is secured by the real values of α and β. Furthermore, in many $\mathcal{PT}$-symmetric potentials, the bound-state (normalizable) solutions belong to two series discriminated by the quasi-parity quantum number Q, which takes on the values $+1$ and -1 or $+$ and $-$ in short. In the case of the $\mathcal{PT}$-symmetric Scarf II potential, this dual system is implemented by replacing α by $Q\alpha$. This means that the $\mathcal{PT}$-symmetric Scarf II potential has two sets of solutions, including two ground states, and consequently, two SUSYQM partner potentials discriminated by Q [36]. This result can be reproduced by adapting the four transformations for $j = k = 0$ in Table 2. Further findings will be discussed in detail elsewhere, leaving the present work focusing on real potentials only. For the general complex Scarf II potential, see, e.g., the recent study [37].

5. Summary and Conclusions

We presented a formalism in which the network of SUSYQM transformations between some shape-invariant potentials belonging to the PI class and their rationally extended versions can be discussed in a unified framework. We specified this formalism for the three members of this class, the generalized Pöschl–Teller, the Scarf I and the Scarf II potentials. The common element behind these potentials that allowed their unified treatment is that the variable transformation that maps the Schrödinger equation into the differential equation of the Jacobi polynomials and that of the X_1-type Jacobi polynomials satisfies the same differential equation.

The formalism was based on a general six-parameter function $\chi(x)$, from which the $W(x)$ superpotential and the supersymmetric partner potentials $V_-(x)$ and $V_+(x)$ could be constructed. With a careful choice of parameters, $V_-(x)$ could be brought to the form of the general PI class potential or its rational extension. The choice of parameters also determined the behavior of $\chi(x)$ at the boundaries, selecting one of the four possible types of SUSYQM transformations, corresponding to unbroken or broken supersymmetry. This construction led to a network of SUSYQM transformations within the PI class, between the PI class potential and its rationally extended version, as well as within the latter potential class.

We specified the general results for the members of the PI class and established the same network of SUSY transformations for the generalized Pöschl–Teller, Scarf I and Scarf II potentials, as well as for their rational extensions. At this point, the difference between the dissimilar domain of definition of the three potentials and the boundary conditions implied by them resulted in differences in the results, too. It was found that, although the generalized Pöschl–Teller and the Scarf I potential are defined on the positive real axis and on a finite domain, respectively, due to the fact that their mathematical formalism is based on related functions (hyperbolic and trigonometric), the general patterns of the basic results are rather similar. In contrast to these findings, the real Scarf II potential differed significantly from the other two: its rational extension could not be constructed, because no parameter sets could be chosen to guarantee that the potential is singularity-free on the real coordinate axis. In this case, SUSYQM transformations are restricted to the Scarf II potentials differing in their potential parameters. However, without presenting the details, we noted that the $\mathcal{PT}$-symmetric version of the Scarf II potential and its rational extension are associated with a rather rich network of SUSYQM transformations.

The present formalism was suitable to place the rationally extended PI class potentials into a unified framework of exactly solvable potentials. Furthermore, the fact that they fulfill the requirement of shape-invariance could be proven in a straightforward way. Their place beyond the Natanzon class could also be verified: the X_1-type Jacobi polynomials, which appear in the bound-state solutions, represent a special polynomial solution of the confluent Heun differential equation, functions that also contain the hypergeometric function, and thus, the Jacobi polynomials as special cases.

The procedure presented here is analogous to a recent one devised to discuss the relation of the radial harmonic oscillator and its rationally extended version [23]. There, the exceptional polynomial was the X_1-type Laguerre polynomial. A similar network of SUSYQM transformations between the named potentials was identified. The difference between that study and the present one is that the PI class potentials could be discussed on an abstract unified level, as there are several (three) members of a potential class, while the radial harmonic oscillator stands alone in its potential class, named LI [5].

The formalism presented here and in Ref. [23] also explains why some potentials have rational extensions, in contrast to others. The explanation lies in the key system of Equations (A18)–(A20) for the Jacobi polynomials and (A29)–(A33) for the X_1-type Jacobi polynomials (as well as the analogous set of equations in Ref. [23] for the X_1-type Laguerre polynomials), which have to be solved in order to obtain exactly solvable potentials. In all these cases, only one equation depends on n, the degree of the polynomial, which discriminates the individual energy levels. Setting only the corresponding p_i parameter to a non-zero value implies that the energy eigenvalues E_n are directly related to n, while the potential parameter s_i, as well as α and β, will be free from any dependence on n. In any other case, α and β necessarily depend on n, which complicates the formalism beyond the point that would allow n-independent (i.e., state-independent) potentials.

It has to be added that some of these results have been known in the literature in some form. The findings were usually isolated and typically covered only certain aspects of the problem. They also focused on different contexts and often used different formalisms. Reference [17] is an example where all three potentials were discussed in some detail, and some arguments exposing analogies between them were also formulated. We hope that the unified treatment of a class of potentials in a general context will help to arrange the fragments into a systematic and coherent framework.

The generalization of these results can also be envisaged. Here, the $\chi(x)$ function contained expressions $(p + z(x))$ corresponding to a first-order polynomial of z. This construction is based on potentials solved in terms of X_1-type Laguerre polynomials [14,15]. It would be a straightforward choice to consider $\chi(x)$ functions with higher (nodeless) polynomial forms. One can expect that this could be possible considering potentials generated from X_m- rather than X_1-type Jacobi and Laguerre polynomials [38].

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Appendix A

Here, we adapt a well-known transformation method [4,5,32] to the Jacobi and X_1-type exceptional Jacobi polynomials to generate the general PI class potentials and their rational extension. This transformation method maps the second-order differential equation of some $F(z)$ special function

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2}+Q(z)\frac{\mathrm{d}F}{\mathrm{d}z}+R(z)F(z)=0, \tag{A1}$$

into the one-dimesional Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+(E-V(x))\psi(x)=0, \tag{A2}$$

by applying the variable transformation $z(x)$. (For the sake of simplicity, the units $2m=1$ and $\hbar=1$ are used.) The domain of definition of $F(z)$ is not specified at this stage, while the Schrödinger equation is assumed to be defined on the positive real semi-axis ($x\in[0,\infty)$, for radial potentials), the full real axis ($x\in(-\infty,\infty)$) or a finite domain ($x\in[x_-,x_+]$) of it. The procedure is applicable to each of these situations. We assume that the solution of the Schrödinger equation is written as

$$\psi(x)=f(x)F(z(x)), \tag{A3}$$

Substituting this into Equation (A2) and comparing the appropriate terms, one finds that

$$\begin{aligned} E-V(x) \quad &= \quad \frac{z'''(x)}{2z'(x)}-\frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ &\quad +(zz'(x))^2\left(R(z(x))-\frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z}-\frac{1}{4}Q^2(z(x))\right). \end{aligned} \tag{A4}$$

Furthermore, the solutions of the Schrödinger equation can be expressed as

$$\psi(x)\sim(z'(x))^{-\frac{1}{2}}\exp\left(\frac{1}{2}\int^{z(x)}Q(z)\mathrm{d}z\right)F(z(x)). \tag{A5}$$

Up to this point, $Q(z)$ and $R(z)$ are defined by the choice of the special function $F(z)$, but the $z(x)$ function is unspecified. $z(x)$ defines the variable transformation, so one would expect that it satisfies certain criteria, e.g., that it is monontonous and single-valued. However, not any $z(x)$ function leads to an exactly solvable potential $V(x)$. This is because nothing guarantees that one finds the solutions of the *same* $V(x)$ potential for all energy eigenvalues E. This problem was handled by the proposal outlined in Ref. [4] after noticing that the constant E on the left-hand side of Equation (A4) should have a correspondent on the right-hand side among the various terms appearing there. Equating one (or more) term with a constant leads to a differential equation for $z(x)$. Furthermore, it is advisable to chose from the terms originating from the $Q(z)$ and $R(z)$ functions, because in this case, the differential equation will contain only $z'(x)$, the first derivative of z. Furthermore, these terms contain the parameters appearing in the special function $F(z)$. The two terms containing higher derivatives depend only on the parameters that appear in the $z(x)$ function, so they are less suitable to generate the E energy eigenvalues. These two terms together are called the Schwarzian derivative

$$\{z,x\}\equiv\frac{z'''(x)}{z'(x)}-\frac{3}{2}\left(\frac{z''(x)}{z'(x)}\right)^2. \tag{A6}$$

Let us prescribe the differential equation

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\Phi(z)=C. \tag{A7}$$

Its solution can be found by direct integration

$$\int \Phi^{1/2}(z)\mathrm{d}z = C^{1/2}x + x_0, \tag{A8}$$

although one obtains the $x(z)$ function this way, which is the inverse of $z(x)$. Nothing guarantees that one can also obtain $z(x)$ in closed form from $x(z)$. Nevertheless, both $V(x)$ and the solutions (A5) can be generated without any complication. In this case, the potential is called *implicit*. For the most well-known potentials, however, it is possible to obtain $z(x)$ in an explicit form. Note that an integration constant, x_0, appears in Equation (A8). It is a simple coordinate shift that usually does not play an important role. It has no effect on the energy spectrum, but it can be used to set $z(0) = 0$, for example. However, in $\mathcal{PT}$-symmetric quantum mechanics, it can be chosen to be imaginary, which allows one to avoid singularities appearing on the real x line.

Combining Equations (A4) and (A7), $z'(x)$ can be replaced in the former expression, leaving $V(x)$ and E as the function of $Q(z)$, $R(z)$, $\Phi(z)$, $z(x)$ and C:

$$\begin{aligned} E - V(x) &= \frac{1}{2}\{z, x\} \\ &\quad + \frac{C}{\Phi(z(x))}\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right). \end{aligned} \tag{A9}$$

when $F(z)$ is chosen to be the hypergeometric function ${}_2F_1(a,b;c;z)$, one obtains the Natanzon-class potentials [1]. The bound-state solutions of these potentials are written in terms of Jacobi polynomials, which are obtained from the hypergeometric function for $a = -n$ or $b = -n$. (Similarly, Natanzon confluent potentials [2] are obtained in an analogous way from the confluent hypergeometric function ${}_1F_1(a;c;z)$, which reduces to generalized Laguerre polynomials for $a = -n$.) Considering $F(z)$ functions that are different from or more general than the (confluent) hypergeometric function naturally leads to potentials beyond the Natanzon class.

Specifying the procedure for the Jacobi polynomial [32] $F(z) = P_n^{(\alpha,\beta)}(z)$ corresponds to the choice [3]

$$Q(z) = \frac{(\beta+\alpha+2)z + (\alpha-\beta)}{1-z^2}, \tag{A10}$$

$$R(z) = -\frac{n(n+\alpha+\beta+1)}{1-z^2}. \tag{A11}$$

With these choices and introducing for simplicity the definitions

$$\omega = \frac{\alpha+\beta}{2}, \qquad \rho = \frac{\alpha-\beta}{2}, \tag{A12}$$

the actual form of Equation (A4) becomes

$$\begin{aligned} E - V(x) &= \frac{1}{2}\{z, x\} + \frac{(z'(x))^2}{(1-z^2(x))^2}\left[\left(\left(n+\omega+\frac{1}{2}\right)^2 - \frac{1}{4}\right)(1-z^2(x)) \right. \\ &\quad \left. + (1-\omega^2-\rho^2) - 2\omega\rho z(x)\right], \end{aligned} \tag{A13}$$

while Equation (A7) turns into

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \Phi(z(x)) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \frac{\varphi(z(x))}{(z^2(x)-1)^2} = C, \tag{A14}$$

i.e.,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = C^{1/2}(z^2(x)-1)[\varphi(z(x))]^{-1/2}, \tag{A15}$$

where

$$\varphi(z(x)) = p_{\rm I}(1 - z^2(x)) + p_{\rm II} + p_{\rm III}z(x). \tag{A16}$$

The $x(z)$ function, and, whenever possible, its inverse, $z(x)$ is then determined from the integration of Equation (A15).

From the structure of (A13), one can conclude that the most general form of the derived potential is

$$V(x) = -\frac{1}{2}\{z, x\} + \frac{C}{\varphi(z(x))}\Big[s_{\rm I}(1 - z^2(x)) + s_{\rm II} + s_{\rm III}z(x)\Big]. \tag{A17}$$

Note that $\varphi(z(x))$ in Equation (A16) and the expressions in square brackets in Equations (A13) and (A17) contain the same linearly independent terms.

Substituting (A17) and $(z'(x))^2$ from (A14) into Equation (A13) and comparing the corresponding terms, one finds that the following three equations have to be satisfied simultaneously:

$$\left(n + \omega + \frac{1}{2}\right)^2 - \frac{1}{4} + s_{\rm I} - p_{\rm I}\frac{E}{C} = 0, \tag{A18}$$

$$1 - \omega^2 - \rho^2 + s_{\rm II} - p_{\rm II}\frac{E}{C} = 0, \tag{A19}$$

$$-2\omega\rho + s_{\rm III} - p_{\rm III}\frac{E}{C} = 0, \tag{A20}$$

This set of equations points out the intimate and subtle relation that connects the parameters p_i appearing in the $z(x)$ function, the coupling coefficients (s_i) of the potential (A17), the energy eigenvalue E and the parameters of the Jacobi polynomials (see Equation (A1) with Equations (A10) and (A11)).

The PI class potentials are obtained by setting $p_{\rm I} = 1$ and $p_{\rm II} = p_{\rm III} = 0$, corresponding to $\varphi(z) = 1 - z^2$. Note that this choice secures that E and n appear only in Equation (A18), so the dependence of E_n on n directly follows from this equation:

$$E_n = C\left(n + \omega + \frac{1}{2}\right)^2 = C\left(n + \frac{\alpha + \beta + 1}{2}\right)^2 \tag{A21}$$

if we set the energy scale such that $s_{\rm I} = \frac{1}{4}$. The remaining two equations directly set the coupling coefficients as $s_{\rm II} = \omega^2 + \rho^2 - 1$ and $s_{\rm III} = 2\omega\rho$. The general form of the PI class potential can be calculated after evaluating the contribution of the Schwarzian derivative (A6), which is found to be $3C/(1 - z^2(x)) - C/4$:

$$V(\alpha, \beta, x) = \frac{C}{1 - z^2(x)}\left[\left(\frac{\alpha + \beta}{2}\right)^2 + \left(\frac{\alpha - \beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2Cz(x)}{1 - z^2(x)} + \left(\frac{\alpha + \beta}{2}\right)\left(\frac{\alpha - \beta}{2}\right). \tag{A22}$$

The bound-state wave functions are

$$\psi_n(\alpha, \beta; x) = C_n^{(\alpha,\beta)}(1 - z(x))^{\frac{\alpha}{2} + \frac{1}{4}}(1 + z(x))^{\frac{\beta}{2} + \frac{1}{4}}P_n^{(\alpha,\beta)}(z(x))\,. \tag{A23}$$

Other choices of $\varphi(z)$ lead to further classes of potentials. Taking only $p_{\rm II} = 1$ to be non-zero, the PII potential class is obtained [5], which contains three potentials for three different $z(x)$ functions: the Rosen–Morse I and II and the Eckart potentials, while from $p_{\rm III} = 1$, the PIII potential follows [39], which is an implicit potential, meaning that $x(z)$ cannot be inverted into an explicit $z(x)$ function. Nevertheless, all the relevant quantities can be calculated exactly. Further potentials can be obtained by choosing two or three different p_i parameters to be non-zero. Examples are the Ginocchio [40], generalized Ginocchio [41] and DKV [42] potentials and a potential from which both the PI and PII class potentials can be obtained as special cases [43]. For a complete list of these potentials, see Ref. [32]. Alternative approaches to potentials [40,43] can be found in Ref. [44], while the relation of these studies is discussed in Ref. [45].

The analogous procedure for the X_1-type exceptional Jacobi polynomials $F(z) = \hat{P}_n^{(\alpha,\beta)}(z)$ follows from the choice [14,15]

$$Q(z) = \frac{(\beta+\alpha)z-(\beta-\alpha)}{z^2-1} - \frac{(\beta-\alpha)}{(\beta-\alpha)z-(\beta+\alpha)}, \tag{A24}$$

$$R(z) = \frac{(\beta-\alpha)z-(n-1)(n+\alpha+\beta)}{z^2-1} - \frac{(\beta-\alpha)^2}{(\beta-\alpha)z-(\beta+\alpha)}. \tag{A25}$$

With the notation in Equation (A12), Equation (A4) becomes

$$\begin{aligned} E-V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{(1-z^2(x))^2}\left[\left(\left(n+\omega-\frac{1}{2}\right)^2-\frac{1}{4}\right)(1-z^2(x))\right. \\ &\left. +(1-\omega^2-\rho^2) - 2\omega\rho z(x) + 2\omega\frac{z^2(x)-1}{\rho z(x)+\omega} + 2(\rho^2-\omega^2)\frac{z^2(x)-1}{(\rho z(x)+\omega)^2}\right]. \end{aligned} \tag{A26}$$

Now, the $\varphi(z)$ function contains two more linearly independent terms compared to that in Equation (A16):

$$\varphi(z(x)) = p_{\rm I}(1-z^2(x)) + p_{\rm II} + p_{\rm III}z(x) + p_{\rm IV}\frac{z^2(x)-1}{\rho z(x)+\omega} + p_{\rm V}\frac{(z^2(x)-1)^2}{(\rho z(x)+\omega)^2}. \tag{A27}$$

Two new terms also appear in the potential

$$\begin{aligned} \hat{V}(x) &= -\frac{1}{2}\{z,x\} + \frac{C}{\varphi(z(x))}\left[s_{\rm I}(1-z^2(x)) + s_{\rm II}\right. \\ &\left. + s_{\rm III}z(x) + s_{\rm IV}\frac{(z^2(x)-1)}{\rho z(x)+\omega} + s_{\rm V}\frac{(z^2(x)-1)}{(\rho z(x)+\omega)^2}\right]. \end{aligned} \tag{A28}$$

Substituting (A28) and $(z'(x))^2$ into Equation (A26), the set of key equations is enlarged:

$$\left(n-1+\omega+\frac{1}{2}\right)^2 - \frac{1}{4} + s_{\rm I} - p_{\rm I}\frac{E}{C} = 0, \tag{A29}$$

$$1-\omega^2-\rho^2 + s_{\rm II} - p_{\rm II}\frac{E}{C} = 0, \tag{A30}$$

$$-2\omega\rho + s_{\rm III} - p_{\rm III}\frac{E}{C} = 0, \tag{A31}$$

$$2\omega + s_{\rm IV} - p_{\rm IV}\frac{E}{C} = 0, \tag{A32}$$

$$2(\rho^2-\omega^2) + s_{\rm V} - p_{\rm V}\frac{E}{C} = 0. \tag{A33}$$

Note that the first three of these equations coincide with Equations (A18)–(A20), with the difference that in (A29) $n-1$ appears rather than n. This corresponds to the fact that the X_1-type exceptional Jacobi polynomials have $n-1$ zeros within the domain of integration. According to this, the energy eigenvalues of this potential class are written as

$$\hat{E}_n = C\left(n-1+\omega+\frac{1}{2}\right)^2 = C\left(n-1+\frac{\alpha+\beta+1}{2}\right)^2, \tag{A34}$$

where $n = 1, 2, \ldots$. The ground-state energy is $\hat{E}_1$. The corresponding equations for the potential and the bound-state wave functions are, respectively,

$$\hat{V}(\alpha,\beta,x) = \frac{C}{1-z^2(x)}\left[\left(\frac{\alpha+\beta}{2}\right)^2+\left(\frac{\alpha-\beta}{2}\right)^2-\frac{1}{4}\right]+\frac{2Cz(x)}{1-z^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) + \frac{2C(\alpha+\beta)}{(\alpha-\beta)z(x)+\alpha+\beta}+\frac{2C\big((\alpha-\beta)^2-(\alpha+\beta)^2\big)}{[(\alpha-\beta)z(x)+\alpha+\beta]^2}, \quad \text{(A35)}$$

$$\psi_n(\alpha,\beta;x) = \hat{C}_n^{(\alpha,\beta)}(1-z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}+\frac{1}{4}}(\alpha-\beta)z(x)+\alpha+\beta)^{-1}\hat{P}_n^{(\alpha,\beta)}(z(x)). \quad \text{(A36)}$$

As discussed above, the $\hat{P}_n^{(\alpha,\beta)}(z)$ X_1-type exceptional Jacobi polynomials [14,15], similarly to the X_1-type exceptional Laguerre polynomials, bear most characteristics of their classical counterparts except that the degree of the lowest polynomial is larger than zero: in the case of X_1-type exceptional orthogonal polynomials, the sequence starts with 1. This is because one of their zeros falls outside the domain of integration. It is thus customary to use $\nu = n-1$ as their index: in this case, $\nu = 0, 1, \ldots$ and ν indicates the degree of the polynomial within the domain of integration. The X_1-type exceptional Jacobi polynomials can be expressed in terms of two classical Jacobi polynomials as

$$\hat{P}_n^{(\alpha,\beta)}(z) = \left[-\frac{(\alpha-\beta)z+\alpha+\beta}{2(\alpha-\beta)}-\frac{\alpha+\beta}{(\alpha-\beta)(\alpha+\beta+2n-2)}\right]P_{n-1}^{(\alpha,\beta)}(z)+\frac{1}{\alpha+\beta+2n-2}P_{n-2}^{(\alpha,\beta)}(z) \quad \text{(A37)}$$

The X_1-type exceptional orthogonal polynomials have an even wider class, the X_m-type [46–49], which contains the X_1-type polynomials as a special case. Based on these, a more general rationally extended version of PI class potentials can be constructed; however, we do not consider this option here.

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