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Symmetric Spaces of Qubits and Gaussian Modes

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Abstract: The understanding of the properties of multipartite systems is a long-standing challenge in quantum theory that signals the need for new ideas and alternative frameworks that can shed light on the intricacies of quantum behavior. In this work, we argue that symmetric spaces provide a common language to describe two-qubit and two-mode Gaussian systems. Our approach relies on the use of equivalence classes that are defined by a subgroup of the maximal symmetry group of the system and involves an involution which enables the Cartan decomposition of the group elements. We work out the symmetric spaces of two qubits and two modes to identify classes which include an equal degree of mixing states, product states, and X states, among others. For three qubits and three modes, we point out how the framework can be generalized and report partial results about the physical interpretations of the symmetric spaces.

Keywords: Lie groups quantum system; Cartan decomposition; Gaussian states; mixed states; quantum correlations

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1. Introduction

Is there a framework that can systematically classify qubits and Gaussian modes systems? Since the second half of the 19th century, when Felix Klein initiated the so-called Erlangen program, the development of geometry has been intertwined with algebraic notions. The program included the proposal to study geometric structures in terms of symmetry and groups, and suggested that physicists could benefit from the resulting framework [1]. Modern Hamiltonian mechanics and Noether's theorem for continuous symmetries are two examples of the connection between geometry, groups, and physics.

At the root of Klein's proposal is the idea that different geometries are distinguished by a set of group invariants. For example, Euclidean geometry deals with areas and angles; hence, rotations and translations form the appropriate group. The concept of symmetric space (SS) is one outcome of the Erlangen program, and in this work, we use it to analyze the properties of quantum systems. To address this challenge, we exploit the close relationship between groups and quantum theory; examples of such a relationship are quantum states as representations of the associated Lie algebra, observables and generators, and time evolution modeled as a group action.

Applied to quantum mechanics, the Erlangen program is ambitious. Our work focuses on the study of some discrete-variable and continuous-variable quantum systems, hereafter referred to as DV and CV, respectively. Examples of the former are one or more qubits, while the latter include one or more Gaussian modes of the electromagnetic field. A common feature of DV and CV systems is the correlation between subsystems, as well as their type, for example, entanglement and quantum discord. Unfortunately, they also share difficulties in characterizing them. In the simplest case, when considering two parties, researchers approach entanglement using different tools; [2] uses quaternions and division algebras to parameterize two-qubit states, while [3] follows a differential-geometric approach, and ref. [4] recognizes hyperbolic spaces describing states of Gaussian modes. In this sense, we point out that SS has the tools required to address both qubits and modes, and more importantly, the formalism is general enough to be applied to systems of arbitrary dimension. The cases we present will allow the reader to identify the complexity that arises as the dimensionality of the system increases.

Researchers have dedicated considerable efforts to characterizing quantum states. The usual approach is the development of criteria that distinguish separable states from entangled ones. For two qubit, for instance, the concurrence $\mathcal{C}$ is employed not only to detect but also to quantify entanglement; $\mathcal{C}$ is a function that takes values in the range $0 \leq \mathcal{C} \leq 1$ and is monotonically increasing with the degree of entanglement [5]; and for two modes, it is also possible to determine if a state is separable or not [6]. On top of this, there exists a hierarchy of correlations whose quantitative measurements involve new criteria and quantifiers [7]. Although researchers have not presented them as symmetric spaces, there are publications that can be placed in this context. Thus, for example, the authors in [8] provide a detailed discussion on the identification of non-local operations using group decomposition, while [9] analyzes the case of Gaussian modes, establishing an invariant criterion for squeezing. Our work overlaps with some of these studies; however, from our standpoint, the approach of symmetric spaces enlarges the scope and provides a description of qubits and modes using a common framework, which to our knowledge has not been proposed. We note that, in a different context, the quantization of the symmetric spaces themselves has been considered and also in the context of spontaneous symmetry breaking [10,11].

The symmetry groups that we consider are $SU(2^n)$ for n qubits and the symplectic group $Sp(2n, \mathbb{R})$ for n modes. Both are part of the so-called *classical* Lie groups, which share properties and treatment methods, and are well suited for our purpose. Using for qubits $N = 2^n$ and $N = 2n$ for modes, the respective dimensions of the symmetry groups are $N^2 - 1$ and $N(N+1)/2$; we observe that the complexity of the analysis, in terms of the number of parameters involved, grows rapidly. Thus, 15 (10) parameters are required to describe the general state of two qubits (two modes). Since the group operations transform the states, then we can use group theory to classify the operations, and as far as the correlations are concerned, the full symmetry group involves too many parameters; therefore, we expect the identification of a subgroup that does not produce entangled states, i.e., a local subgroup. This is what we refer to as a physical interpretation of subgroups and SS [12–14].

Given the symmetry group G of a system, our work is based on the use of equivalence classes to classify states. Such classes are defined by a subgroup $K \subset G$; then, the meaning of the coset space G/K is that a state and all those that are obtained from the application of K are equivalent, i.e., the states are defined up to a K transformation. This sounds familiar—for example, in the case of one qubit, the Bloch sphere S^2 is the space of states and the states are defined up to a phase. In terms of cosets, this correspond to $SU(2)/U(1)$. Moreover, in the same sense that divisors of integer numbers $m = p/q$ allow the factorization $p = mq$, the SS of a Lie group are special quotients, G/K, that allow decomposing of the general element $g \in G$ as a product $g = KP$. The cosets are special in that an *involution* selects a limited list of subgroups among all the possibilities.

The subgroups we consider are those allowed by the definition of SS, which is formulated in terms of cosets and details of which will be presented in the following sections. There are three important advantages of using SS: (i) they have been classified for unitary and symplectic groups; (*ii*) as a byproduct, it gives rise to a procedure in building the general state of the coset; and (*iii*) it leads to a common treatment of DV and CV systems. For example, for special unitary groups $SU(N)$, there are three types of SS which are defined in terms of the cosets $SU(N)/SO(N)$, $SU(2n)/Sp(2n)$ and $SU(p+q)/S(U(p)\times U(q))$, with $p+q=N$. In the case of two qubits, $N=4$, and $n=2$, for the third coset, there are two options we must consider, namely, $SU(4)/S(U(3)\times U(1))$ and $SU(4)/S(U(2)\times U(2))$. For symplectic groups, we also consider three cosets $Sp(2n,\mathbb{R})/U(n)$, $Sp(p+q,\mathbb{R})/Sp(p,\mathbb{R})\times Sp(q,\mathbb{R})$ and $Sp(4,\mathbb{R})/Sp(2,\mathbb{C})$.

Besides the subgroup, each SS involves an involution θ that separates $\mathfrak{g}$ in even and odd generators, $\theta(\mathfrak{g}_i)=\pm\,\mathfrak{g}_i$. The even elements $\mathfrak{l}$ are the generators of the subgroup K while the odd elements $\mathfrak{p}$ generate the SS, so that we can write $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{p}$, with $(\mathfrak{l},\mathfrak{p})$ known as the Cartan pair of $\mathfrak{g}$. The Cartan pair is the basis for the Cartan decomposition (CD), which ensures that a general group element can be written as $U=KP$, where K and P are built as the exponential map of $\mathfrak{l}$ and $\mathfrak{p}$, respectively. When $\mathfrak{p}$ includes the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, then a further decomposition is possible $U=KAK$, where A is the exponential map of $\mathfrak{a}$. This is referred to as the KAK decomposition, which, in fact, coincides with Euler and Bloch–Messiah, and throughout the text, we will apply KP, PK, and KAK decomposition.

The relevance of CD is that it leads to a unique (up to permutations) parameterization of an arbitrary state of the coset with no redundancy in the parameters. Moreover, the process can be iterated to the subgroup K, such that $KP\to K'P'P$. Since the action of the subgroup defines an equivalence class, then the invariants associated to the subgroup portray the coset. Therefore, the characterization of the states in the SS is performed in terms of invariants under the action of the subgroup K. This subject is addressed in the main text where the relation to existing quantifiers, such as the purity and partial transpose criterion [15], is discussed. However, the challenge is non-trivial due to the large number of parameters and to the fact that subgroups are not disjoint, i.e., different cosets share some of the properties. The mixing of states is an example—it naturally appears in the formalism. In fact, it corresponds to the interpretation of one of the cosets, but it is not limited to that SS. Other cosets allow for the incorporation of mixing. This is accomplished by implementing a mechanism through the Cartan subalgebra.

Here, we present a systematic treatment of qubits and modes from which important results previously reported, scattered in the literature, can be reproduced within the framework of SS. Among these results we can mention, for example, the standard form of the two-modes Gaussian states [16], and the parameters of non-separable correlations in the two-qubits state. However, we go beyond this and obtain generalizations to higher-dimensional systems, the non-trivial results are a standard-like form of 3-modes, and a process of iteration is used in constructing three-qubit states. Moreover, we present cosets that have not been extensively considered in the literature.

The paper is organized as follows: In the first part of Section 2, we introduce the quantum states of interest and the groups involved, while the second part is used to introduce the notation and concepts related to SS and CD. We work out details of the group $SL(2,\mathbb{C})$ since it contains relevant information of both SU(2) and Sp(2,$\mathbb{R}$), so that the reader may be familiarized with the topics. Section 3 deals with the application of the SS formalism to qubits and Gaussian modes; it comprises three subsections dedicated to interpretation of the SS. Each subsection includes two families that share properties; Families AIII-CI, AI-CII and AII-CIII. In Section 4, with the help of a table, we present a summary of our findings and discuss the extension of the analysis to higher-dimensional systems, and point out the variety of options that open up, as well as the difficulties that arise for the interpretation. In the last section, we present our conclusion.

2. States and Symmetric Spaces

The theory of SS is explored across various disciplines; however, to the best of our knowledge, it has not been widely applied in the context of quantum systems from a comprehensive perspective. For that reason, in this section, we set the notation and conventions for SS following the standard reference [17]. Additionally, we extend the discussion to include pseudo-symmetric spaces (SS_p) [18], which are not considered in the aforementioned text. We start by considering the role of groups in the context of quantum systems.

2.1. Quantum States and Symmetries

A system of n qubits is described by the n-fold tensor product $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \mathbb{C}^2$ involving 2^n complex parameters, which can be organized, for pure states, as an array of 2^n complex numbers $|\psi\rangle = (z_1, z_2 \cdots z_{2^n})$ and as a $2^n \times 2^n$ density matrix which is Hermitian, positive-definite, and satisfies $Tr(\rho^2) \leq 1$, the equality holding for pure states and the strict inequality applying to mixed states. The group of unitary transformations $U(2^n) = SU(2^n) \times U(1)$ is identified as the maximum set of symmetries of the quantum mechanical system, maintaining the probability interpretation of quantum mechanics.

The general mixed-state density matrix, on an arbitrary basis, is obtained upon performing a unitary transformation with an element of the group $U(2^n)$, i.e., $\rho = U\rho_0 U^\dagger$. Below, we will show how to take advantage of the decomposition of the unitary transformation to perform this task.

One mode of the electromagnetic field is described in terms of one degree of freedom, a Hermitian operator $\hat{X}$ and its canonical conjugate $\hat{P}$, acting in a Hilbert space $\mathcal{H}$. For n-modes, it is customary to arrange the set of Hermitian operators in a 2n-quadrature vector $\hat{\chi}$ so that $\hat{\chi}_a = \{\hat{X}_1, \hat{X}_2 \cdots \hat{X}_n, \hat{P}_1, \hat{P}_2 \cdots \hat{P}_n\}^T$ (Canonical basis) or the standard basis $\hat{\xi} = \{\hat{X}_1, \hat{P}_1, \cdots \hat{X}_n, \hat{P}_n\}^T$, where T stands for transpose, and such that the commutation relations are as follows:

$$[\hat{\xi}_a, \hat{\xi}_b] = i\omega_{ab}, \qquad \omega = \bigoplus_1^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

We consider linear homogeneous transformations of $\hat{\xi}$, which preserve the commutation relation in Equation (1). These transformations can be implemented by a real $2n \times 2n$-dimensional matrix S (for modes, we follow the notation and conventions of [19]):

$$\hat{\xi}'_a = S_{ab}\hat{\xi}_b, \qquad S^t\omega S = \omega. \tag{2}$$

Such matrices form the group $Sp(2n, \mathbb{R})$, the symmetry group of the quantum mechanical description of n modes, which has dimension $n(2n+1)$, including $n^2 + n$ squeeze generators and n^2 rotations. It is convenient to introduce a block form for S involving four real $n \times n$ matrices:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3}$$

When dealing with modes of the radiation field, it is advantageous to work with the annihilation and creation operators defined as $\hat{a}_i = (\hat{X}_i + i\hat{P}_i)/\sqrt{2}$ and $\hat{a}_i^\dagger = (\hat{X}_i - i\hat{P}_i)/\sqrt{2}$. In this case, instead of the array $\hat{\xi}$, it is convenient to introduce $\hat{\zeta}$ so that $\hat{\zeta}_a = \{\hat{a}_1, \hat{a}_2 \cdots \hat{a}_n, \hat{a}_1^\dagger, \hat{a}_2^\dagger \cdots \hat{a}_n^\dagger\}^T$ for which the commutation relations are $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. In this case, the block form of the transformation matrix involves complex inputs, but is still a real group. Moreover, for each $S \in Sp(2n, \mathbb{R})$, it is possible to construct a unitary operator $\mathcal{U}(S)$ acting on $\mathcal{H}$ such that:

$$\hat{\xi}'_a = S_{ab}\hat{\xi}_b = \mathcal{U}(S)^{-1}\hat{\xi}_a\mathcal{U}(S) \qquad \text{and} \qquad \mathcal{U}(S)^\dagger\mathcal{U}(S) = 1. \tag{4}$$

It is important to note that there is a degree of arbitrariness in the phase of $\mathcal{U}$ and in the composition law since for $S_1, S_2 \in Sp(2n,\mathbb{R})$:

$$\mathcal{U}(S_1)\mathcal{U}(S_2) = e^{i\Phi(S_1,S_2)}\,\mathcal{U}(S_1S_2). \tag{5}$$

This is reminiscent of a double-cover relation, as with $SO(3) \to SU(2)$, and indeed can be paraphrased in these terms [19]. It goes under the name *metaplectic* representation of $Sp(2n,\mathbb{R})$; it leads to the simplest version of the composition law Equation (5) $\mathcal{U}(S_1)\mathcal{U}(S_2) = \pm\mathcal{U}(S_1S_2)$ and its generators are all Hermitian and bilinear in the $(\hat{a}, \hat{a}^\dagger)$ operators.

The Wigner distribution (WD) is a useful tool for the description of modes. The advantage of the WD is highlighted by the following result. For any quantum mechanical operator $\hat{\Gamma}$, in configuration space and specified in the Schrodinger representation by $\langle X|\hat{\Gamma}|X'\rangle$, the corresponding WD is obtained by a partial Fourier transform:

$$W(\xi) = \frac{1}{(2\pi)^n}\int d^nX'\,\langle X - \frac{1}{2}X'|\hat{\Gamma}|X + \frac{1}{2}X'\rangle\, e^{iX'\cdot P}. \tag{6}$$

The inverse transform makes it possible to recover the configuration space representation of the operator:

$$\langle X - \frac{1}{2}X'|\hat{\Gamma}|X + \frac{1}{2}X'\rangle = \int d^nP\, W(\frac{1}{2}(X + X'))e^{-iP\cdot(X-X')}, \tag{7}$$

where $W(\xi)$ is a function on the classical phase space, with arguments (X, P), which are classical c-numbers. The feature that makes the WD so special is that a metaplectic transformation $\mathcal{U}(S)$ in the Schrodinger representation has a simple realization in terms of the WD:

$$\hat{\Gamma}' = \mathcal{U}(S)^{-1}\hat{\Gamma}\,\mathcal{U}(S) \iff W'(\xi) = W(S\xi). \tag{8}$$

In words, a transformation of operators $(\hat{X}, \hat{P})$ in Hilbert space, realized with the infinite-dimensional representation $\mathcal{U}$, is implemented with the corresponding, finite-dimensional, symplectic representation in phase space (X, P). The WD of a Gaussian state is a Gaussian function; it can be shown that the complete description of a Gaussian state is encoded in σ, the covariance matrix (CM):

$$W_\rho(\xi) = \frac{e^{-\frac{1}{2}(\xi^T\sigma\,\xi)}}{\pi^n\sqrt{\det\sigma)}}, \tag{9}$$

where $\sigma_{i,j} = \langle\{X_i, X_j\}\rangle$, and all relevant quantities can be obtained from the CM, which is symmetric and positive-definite. In particular, in its diagonal form, the CM defines the symplectic eigenvalues ν_k in terms of which the purity of the state is expressed:

$$\mu_\rho = Tr(\rho^2) = \frac{1}{\sqrt{Det\sigma}} = \frac{1}{\Pi_k\,\nu_k}. \tag{10}$$

2.2. Symmetric Spaces and Cartan Decomposition

In this section, we use the simplest examples of SS to introduce the key concepts and definitions that allow dealing with n-qubits and n-modes. We will start by looking at the definition of the real form and the Cartan decomposition. The Lie algebras $\mathfrak{g}$ are vector spaces together with an extra operation called the Lie bracket, denoted $[X, Y] = XY - YX$, which is skew-symmetric and satisfies the Jacobi identity. Given elements X, Y of a real Lie algebra and a *complex structure* J which satisfies $J^* = -J$, $J^2 = -1$, an element of a complex Lie algebra is written as $Z = X + JY$. This process is called complexification. If $\mathbf{g} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$, then we say that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. Then, given a complex Lie algebra $\mathfrak{g}$ and a real form $\mathfrak{g}_0$, the decomposition $\mathfrak{g}_0 = \mathfrak{l}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition, if there exists a compact real form $\mathfrak{u}_0$ that leads to a complexification $\mathfrak{u}$ of $\mathfrak{g}$ and which satisfies:

$$\mathfrak{u}^* \subset \mathfrak{u}, \qquad \mathfrak{l}_0 = \mathfrak{g}_0 \cap \mathfrak{u}_0, \qquad \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{u}_0). \tag{11}$$

The following example illustrates the concepts introduced. $SL(2,\mathbb{C})$ is the group of two-by-two complex matrices with unit determinant. At the level of Lie algebra, this condition implies traceless matrices, and the general element can be expressed as:

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix} = z_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

The matrices appearing in this equation form a basis; they are usually denoted h, e, f and satisfy the commutation relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. Note that $e = \frac{1}{2}(\sigma_1 + i\sigma_2)$ and $f = \frac{1}{2}(\sigma_1 - i\sigma_2)$, σ_i being the Pauli matrices, which allows us to conclude that Z can be expressed in a linear combination, with complex coefficients, of the $su(2)$ generators, *i.e.*, $su(2)$ is indeed a real form of $sl(2,\mathbb{C}) = su(2) + i\,su(2)$. Alternatively, by writing $X = x_1 h + x_2 e + x_3 f$, we conclude that $sl(2,\mathbb{C}) = sl(2,\mathbb{R}) + i\,sl(2,\mathbb{R})$. Thus, $su(2)$ and $sl(2,\mathbb{R})$ are the compact and split real forms of $sl(2,\mathbb{C})$, respectively. Here, a compact real form is defined by having a negative-definite killing form, which for the classical families of Lie groups is given by $B(X,Y) = 4Tr(X,Y)$. Since $su(2)$ and $sl(2,\mathbb{R}) \equiv sp(2,\mathbb{R})$ are of signature $(0,3)$ and $(2,1)$, the corresponding Lie groups are compact and non-compact, respectively.

Additionally, we can verify the conditions in Equation (11) to obtain the Cartan pair [20], taking $\mathfrak{u}_0 = su(2)$ and $\mathfrak{g}_0 = sl(2,\mathbb{R})$. The first part is trivial, since any element in $sl(2,\mathbb{C})$ can be written as $X + iY$ with elements of $su(2)$; the second condition is seen in terms of the basis elements $l_0 = sl(2,\mathbb{R}) \cap su(2) = \{e - f = i\sigma_2\}$; similarly, the third condition is $p_0 = sl(2,\mathbb{R}) \cap (isu(2)) = \{h = \sigma_3, e + f = \sigma_1\}$. In this way, the Cartan decomposition both of $sp(2,\mathbb{R})$ and $su(2)$ is obtained and can be applied to a qubit and a Gaussian mode; however, this is not practical and for other groups, it is better to use the following facts.

Associated to the conjugation with respect to the real form $(u)^*$, there is an involutive automorphism θ of $\mathfrak{g}_o$ called the Cartan involution, and satisfies that $-B_g(Z_1,(Z_1)^*)$ is strictly positive-definite. Additionally, θ on $\mathfrak{g}_0$ has the property $\theta^2 = 1$ (since $(Z^*)^* = Z$), and the eigenspaces corresponding to the eigenvalues ± 1 are $\mathfrak{l}_0$, $\mathfrak{p}_0$. Moreover, since θ is an automorphism of the Lie algebra, it maintains the bracket operation so that:

$$[l_0, l_0] \subset l_0, \qquad [l_0, p_0] \subset p_0, \qquad [p_0, p_0] \subset l_0. \tag{13}$$

It follows that, if there exists a subalgebra $\mathfrak{a} \subset \mathfrak{p}$, it must be Abelian and when it is also maximal it is called Cartan subalgebra. In brief, Cartan involutions are related to real forms of a complex Lie algebra, and non-equivalent involutions lead to families of SS.

A significant advantage of using SS is the existence of a general classification that restricts the number of subgroups we have to consider. It provides a matrix representation of involutions and, moreover, it ensures the *transitivity* of the group action on the coset. According to the general classification, $su(n)$ has three different families denoted AI, AII, AIII that correspond to the subalgebras $so(n)$, $sp(n)$ and $s(u(p) \oplus u(q))$, $p + q = n$

and whose involutions are $\theta_I(X) = X^*$, $\theta_{II}(X) = J_n\, X^*\, J_n^{-1}$, and $\theta_{III}(X) = I_{p,q}\, X\, I_{p,q}$, respectively, where:

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \qquad K_{p.q} = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -I_q \end{pmatrix}.$$

We also consider three families of $Sp(2n, \mathbb{R})$ [18]. Since there is no standard notation for these cases, we will denote them as CI, CII, and CIII (Cartan involutions are stated in the canonical basis; when using the standard one, a change of basis is necessary to obtain the correct subgroup.). The subalgebras for the first two are $u(n)$, and $sp(2p, \mathbb{R}) \oplus sp(2q, \mathbb{R})$ and the involutions $\phi_I(X) = J_n(X)J_n^{-1}$, $\phi_{II}(X) = K_{p,q}(X)K_{p,q}$. Finally, for CIII, the subgroup is $Sp(n, \mathbb{C})$, when $n = 2$; this yields the double cover of the Lorentz group, i.e., the coset corresponds to an anti-de Sitter space (general involution unknown).

The general element of the group is written in terms of the Lie algebra by using the exponential map $exp : \mathfrak{g} \to G$, while the Cartan decomposition [21,22] guarantees that the map $K \times P \to G$ is a diffeomorphism and allows writing $G \ni g = K \cdot P, P \cdot K$ uniquely, with K and P expressed in terms of the Cartan pair, $K = exp(\mathfrak{l})$, $P = exp(\mathfrak{p})$. In terms of the group G and the subgroup K, the coset $P = G/K$ is defined as SS when G and K are both compact groups, or when only K is compact, while the space when neither G nor K are compact is called pseudo-symmetric (the notation G/H is commonly used to include both SS and SS_p; both are *homogeneous* spaces. We use G/K to refer to both of them) (SS_p).

An extra decomposition is possible $G = K_1 \cdot A \cdot K_2$ where $A = \exp(a)$ and a is the maximal abelian subalgebra in p. (Two points are worth remarking: (1) the decomposition is not unique, and (2) the map is not a diffeomorphism between manifolds, giving origin to topological conditions [23]). One way to follow Klein's philosophy is to use Cartan decomposition to obtain a general state $\rho = G^\dagger \rho_0 G$, where ρ_0 can be chosen in different ways. For example, by using the Cartan decomposition $G = KP$, it follows that $\rho = P^\dagger(K^\dagger \rho_0 K)P$. Clearly, it is advantageous that ρ_0 and $\rho_0 = K^\dagger \rho_0 K$ are in the same equivalence class; then, we can label the equivalence classes using the classical invariant theory of matrices [24] and relate them to physically meaningful quantities. For qubits, this program has been performed [25,26] and polynomial invariants of the local subgroup have been computed. In the case of modes, the role of the state is played by the general correlation matrix σ, which can be obtained through a symplectic transformation:

$$\sigma_s = S^t \sigma_0 S, \qquad S \in Sp(2n, \mathbb{R}), \tag{14}$$

Since the symplectic relation $S^t \omega S = \omega$ implies $S^t = \omega S^{-1} \omega^{-1}$, which together with $\omega^{-1} = -\omega$, leads to $\omega \sigma_s = S^{-1}(\omega\sigma)S$, then we observe that $\omega\sigma$ transforms by conjugation. Taking into account that $Sp(2n, \mathbb{R}) \subset SL(2n, \mathbb{R})$, we can use results valid for the latter group concerning invariant quantities, namely, the coefficients of the characteristic polynomial, and these can be computed from functions of $tr(X^k)$ [27]. Consider the case when $\Sigma = \omega\sigma$ is a four-by-four matrix with elements $[\Sigma_{i,j}]$, $i, j = 1 \cdots 4$. In this case, $Tr(\Sigma) = Tr(\Sigma^3) = 0$, whereas $Tr(\Sigma^4) \propto Det[\Sigma]$. Moreover, $tr(\Sigma^2)$ can be calculated from $Tr(\Sigma^{\alpha,\beta})$ and the six principal minors $D_2^{\alpha,\beta} = Det[\Sigma^{\alpha,\beta}]$, where $\Sigma^{\alpha,\beta}$ is a two-by-two matrix obtained form Σ by eliminating rows and columns other than α, β. The relevance of this result will be appreciated when discussing the physical interpretation of SS_p for two modes.

3. Qubits and Modes

In this section, we discuss the application of the SS formalism to qubits and modes. We start by presenting details for one qubit and one mode—this will help to grasp the ideas and methods that will be used when analyzing two qubits and two modes, and generalizations to n-parties. In particular, we emphasize the parallelism between qubits and Gaussian modes in terms of equivalence classes that involve mixing and correlations.

3.1. Symmetric Spaces AIII and CI

It is worth recalling the coset and involution of SS AIII. The subalgebra $s(u(n-1) \oplus u(1))$ is obtained by means of the involution $\theta_{III}(X) = I_{p,q}\, X\, I_{p,q}$. We start by considering one qubit. When discussing real forms, we have seen that the conditions in Equation (11) lead to the Cartan decomposition of $SU(2)$; however, for practical purposes, it is better to exploit the matrix representation of involutions. In the particular case of one qubit, we can discuss the three involutions; however, since $SU(2) \equiv Sp(2)$, we do not have to consider AII, so that only two involutions are relevant and both lead to the symmetric space $SU(2)/U(1)$ with either $l_0 = \{i\sigma_2\}$ and $p_0 = \{i\sigma_3, i\sigma_1\}$ or $l_0 = \{i\sigma_3\}$ and $p_0 = \{i\sigma_1, i\sigma_2\}$. A qubit mixed state of purity r is described by $\rho_m = \frac{1}{2}(\mathbb{I} + r\vec{\sigma}\cdot\vec{n})$, whence the diagonal mixed state ρ_d is a linear combination of the identity matrix and a diagonal generator $\rho_d = \frac{1}{2}(\mathbb{I} + r\sigma_3)$. In terms of ρ_d, the general mixed state is obtained using KAK decomposition with $l = i\sigma_3$, and $p = i\sigma_2$:

$$\rho_m = e^{-i\sigma_3 \frac{\phi}{2}} e^{-i\sigma_2 \frac{\theta}{2}} e^{-i\sigma_3 \frac{\chi}{2}}\, \rho_d\, e^{i\sigma_3 \frac{\chi}{2}} e^{i\sigma_2 \frac{\theta}{2}} e^{i\sigma_3 \frac{\phi}{2}} \tag{15}$$

Note that ρ_d commutes with K, the subgroup defining the equivalence class, $e^{-i\sigma_3 \frac{\chi}{2}} \rho_d\, e^{i\sigma_3 \frac{\chi}{2}} = \rho_d$ (by convention, we keep $U^\dagger$ at the left, to coincide with the adjoint action of a Lie group, which is not the standard action on kets in quantum mechanics) and the final mixed state involves the correct total number of parameters (θ, ϕ, r). We summarize this by saying that the χ parameter has been traded by the mixing parameter r. Geometrically, ρ_m describes a sphere of radius r (Bloch sphere) and every point on the surface of the two-dimensional sphere of radius r describes a state.

For two qubits, the symmetry group is $SU(4)$. In order to implement CD, a basis of the Lie algebra is required, for our purposes, it is advantageous to take the 15 traceless skew-Hermitian λ_{ij} generalization of the Gell-Mann matrices that schematically look like:

$$\lambda = \begin{pmatrix} su(3) & \vec{z} \\ (\vec{z})^\dagger & u(1) \end{pmatrix}. \tag{16}$$

Applying the involution, we obtain the subalgebra $\mathfrak{l} = \{\lambda_i, \lambda_{15},\ i = 1, 2, \cdots 8\}$, where the $\lambda_{1,2\cdots 8}$ denote the four-dimensional generalization of the Gell-Mann matrices, λ_{15} is a third traceless diagonal matrix, and the set $\mathfrak{p} = \{\lambda_i,\ i = 9, 10, \cdots 14\}$ contains the six generators that can be built from the complex vector $\vec{z}$ and $\vec{z}^\dagger$. The SS is the well-known $SU(4)/S(U(3) \times U(1)) = \mathbb{CP}^3$, the two-qubit generalization of the Bloch sphere. The form of a general density matrix describing two qubits was discussed in [28,29]. Their analysis is based on the recursive application of the CD, using that $SU(2) \subset SU(3) \subset SU(4)$. The bottom line of the work is that the general ρ is parameterized using twelve Euler angles and three populations and is written $\rho = U^\dagger \rho_d\, U$, where ρ_d is diagonal:

$$\rho_d = \frac{1}{4}(\mathbb{I}_4 + f_1(p_i)\lambda_3 + f_2(p_i)\lambda_8 + f_3(p_i)\lambda_{15}), \tag{17}$$

The $f_i = f_i(p_1, p_2, p_3)$ are given functions of the populations $p_i > 0,\ i = 1,2,3,\ \lambda_3, \lambda_8$, are the four-dimensional generalization of the Gell-Mann matrices, λ_{15} is the third $SU(4)$ diagonal matrix, and:

$$U = U_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)e^{i\lambda_{10}\alpha_6}U_2(\alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}\alpha_{12}), \tag{18}$$

where the $\alpha' s$ are parameters. An important characteristic of this result is that both U_1 and U_2 involve generators of the subgroup $S(U(3) \times U(1))$ so that, in this decomposition, the full operator U only involves one generator (λ_{10}) that does not belong to $S(U(3) \times U(1))$. From the perspective of SS, the following observations are incorporated in the analysis: (*i*) when considering the action of G on a state of the equivalence class, i.e., $\rho = U^\dagger \rho_d^0 U$ where ρ_d^0 and $\rho'^0 = K^\dagger \rho_d^0 K$ belong to the same equivalence class; therefore, the general state of this class of equivalence is $\rho = P^\dagger(\beta_i)\rho_d^0\, P(\beta_i)$, where the notation is intended to indicate that P depends on six parameters β_i; and (*ii*) the general state of this coset can be constructed taking as the starting point a diagonal ρ_d^0 including a single mixing parameter. In this way, we end up with the state of the coset P that includes mixed states and, as expected, depends on seven parameters:

$$\rho = P^\dagger(\beta_{1-6})\rho_d^0(p)\, P(\beta_{1-6}). \tag{19}$$

We now turn to one mode. To this end, consider $s \in sp(2, \mathbb{R})$, written as $X = x_1 s_1 + x_2 s_2 + x_3 s_3$, where:

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}$$

The involution $\phi_I(X)$ leads to $\mathfrak{l} = \{s_2\}$, $\mathfrak{p} = \{s_1, s_3\}$ and to the coset $Sp(2, \mathbb{R})/U(1)$, while the involution $\phi_{II}(X)$ yields $\mathfrak{l} = \{s_3\}$, $\mathfrak{p} = \{s_2, s_1\}$ and the coset $Sp(2, \mathbb{R})/SO(1, 1) \equiv SO(2, 1)/SO(1, 1)$. The latter coset belongs to the CII family and its appearance here follows from an *accidental* isomorphism and leads to an alternative expression for the general state in terms of two squeezing and one rotation. Modes of the electromagnetic field can be described in terms of Fock states $\{|n\rangle\}$ or Coherent states $\{|\alpha\rangle\}$, infinite-dimensional Hilbert spaces where the symmetry group generators are unitary operators. Alternatively, the phase space formulation in terms of the probability quasi-distribution, a finite-dimensional representation, although not unitary, can be achieved in terms of the creation-annihilation operators $\hat{a}, \hat{a}^\dagger$.

By analogy to the qubit case, our aim is to express the general state using group action on the covariance matrix. Consider the set of operators:

$$\tilde{s}_1 = \frac{1}{2}(\hat{a}^2 + (\hat{a}^\dagger)^2), \qquad \tilde{s}_2 = \frac{i}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2), \qquad \tilde{s}_3 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger),$$

They form a representation of the generators of the $sp(2, \mathbb{R})$ Lie algebra. The action of the group on the creation-annihilation operator is readily obtained, and from it, the action on the quadratures (X, P) (comparing the matrix representation of coset P with $\hat{S}(\zeta)$, the squeezing operator, we note that the latter defines the coset space).

$$e^{-is\tilde{s}_1}\begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix} e^{is\tilde{s}_1} \qquad \Longleftrightarrow \qquad e^{ss_1}\begin{pmatrix} X \\ P \end{pmatrix}, \tag{21}$$

In this context, it is relevant to recall that the identity correlation matrix corresponds to the vacuum state (and coherent states). The covariance matrix is symmetric and positive-definite; therefore, it can be written as a linear combination of $\{\mathbb{I}_2, s_3, s_1\}$, but not s_2 since it is skew-symmetric, coincides with the symplectic matrix ω, and remains invariant under symplectic transformations. Starting from the vacuum state correlation matrix ($\sigma_0 = \mathbb{I}_2$), the CM for the squeezed, typical mode state of the $Sp(2,\mathbb{R})/U(1)$ coset can be obtained, either using the unitary representation or the finite-dimensional form of the generators. Thus, using the KAK decomposition (Bloch–Messiah):

$$S = \exp\left(\frac{\phi}{2}s_2\right)\exp\left(\frac{s}{2}s_3\right)\exp\left(\frac{\theta}{2}s_2\right), \qquad \sigma_\zeta = S^t\sigma_0 S. \tag{22}$$

Note that $K = exp(\mathfrak{l})$ is the $U(1)$ subgroup that characterizes the equivalence class, and also in this case, we can argue that the ϕ parameter has been exchanged by the mixing parameter a, so that the general mixed state of this coset is $a\sigma_\zeta(\theta, s)$ (see Equation (22)). Since the parameter a determines the mixing degree, the coset $Sp(2,\mathbb{R})/U(1)$ may be interpreted as surfaces of equal mixing. In other words, $U(1)$ is the stabilizer of the mixing level for Gaussian states. An explicit calculation shows that $\sigma_\zeta = x_0\,\mathbb{I}_2 - x_1 s_1 - x_3 s_3$, with $x_0 = a\cosh(s)$, $x_1 = a\sin(\theta)\sinh(s)$, $x_3 = a\cos(\theta)\sinh(s)$, suggesting the interpretation of the x_α as coordinates and identifying states as points of a manifold (a hyperboloid) on a space of the signature (1,2) given by $|\sigma_\zeta| = a^2 = x_0^2 - x_1^2 - x_3^2$.

For two modes $n = 2$, the coset to consider is $sp(4,\mathbb{R})/U(2)$; as generators, we use the 10 symplectic generators (out of 16) given by $X_{ij} = s_i \otimes s_j$, where $s_4 = \mathbb{I}_2$:

$$\begin{array}{llll} X_1 = s_1 \otimes s_3 & X_4 = s_1 \otimes s_4 & X_7 = s_1 \otimes s_2 & \\ X_2 = s_3 \otimes s_3 & X_5 = s_3 \otimes s_4 & X_8 = s_3 \otimes s_2 & X_{10} = s_2 \otimes s_1 \ , \\ X_3 = s_4 \otimes s_3 & X_6 = s_4 \otimes s_4 & X_9 = s_4 \otimes s_2 & \end{array}$$

The subalgebra spanned by $\{X_7, X_8, X_9, X_{10}\}$ is isomorphic to $u(1) \oplus su(2) = u(2)$ and corresponds to the skew-symmetric matrices, while the p-space is given by the symmetric matrices $\{X_{1-6}\}$. Thus, in obtaining the SS $Sp(4,\mathbb{R})/U(2)$, the following points are worth making: (1) $Sp(4,\mathbb{R})$ is a rank two group, and the two diagonal matrices $\{X_1, X_2\}$ in $\mathfrak{p}$ can be chosen as the Cartan subalgebra. Therefore, we can implement K_1AK_2 decomposition, where each $K_{1,2}$ involves four parameters and the coset depends only on six parameters. (2) $K \in U(2)$ can be further decomposed as $SU(2) \times U(1)$, K=$e^{\theta_2 X_8}e^{\theta_1 X_9}e^{\phi_2 X_8}e^{\phi_1 X_7}$, where X_7 generates $U(1)$ since it commutes with the remaining rotations, (3) $\sigma_0 = \mathbb{I}_4$ belongs to the equivalence class of K:

$$\sigma = K_2^t A K_1^t\,\mathbb{I}_4\,K_1 A K_2 = K_2^t A\,\mathbb{I}_4\,A K_2. \tag{23}$$

Then, for pure states, we can obtain a parametrization for the geometric representation of two modes, by generalization of σ_ζ in Equation (22). It is convenient to introduce the linear combinations $Xl_{1,2} = \frac{1}{2}(X_1 \pm X_2)$ and $Xl_{7,8} = \frac{1}{2}(X_7 \pm X_8)$; they later generate $U(1) \times U(1) \subset U(2)$, such that $S_1 = e^{\frac{s_1}{2}Xl_1}e^{\frac{\theta}{2}Xl_7}$, $S_2 = e^{\frac{s_2}{2}Xl_2}e^{\frac{\phi}{2}Xl_8}$, are transformations acting in each mode separately and Equation (23) becomes:

$$\sigma = R_{X_8}^t R_{X_9}^t (S_1^t \mathbb{I}_2 S_1 \oplus S_2^t \mathbb{I}_2 S_2) R_{X_9} R_{X_8}, \tag{24}$$

where R_{X_i} is the rotation associated to generator X_i. This result shows that a pure state of two modes can be obtained from the vacuum CM transforming each mode separately (S_1, S_2) and then the rotations (R_{X_8}, R_{X_9}) produce the non-trivial combination of the two modes to obtain the general state.

Alternatively, instead of Equation (23), we start from $\sigma = K_2^t A K_1^t \sigma_0 K_1 A K_2$ and consider the $U(1) \times U(1)$ sitting in K_1 and take into account that it acts trivially on $\sigma_0 = a\mathbb{I}_2 \oplus b\mathbb{I}_2$ where a, b are population parameters. This argument shows that the two parameters of the $U(1) \times U(1)$ group are exchanged by the populations a, b, while the actions of the remaining operators are non-trivial and lead to the general ten parameters mixed state:

$$\sigma = K_2^t A R_{X_8}^t R_{X_9}^t (R_{Xl_7}^t a \mathbb{I}_2 R_{Xl_7} \oplus R_{Xl_8}^t b \mathbb{I}_2 R_{Xl_8}) R_{X_9} R_{X_8} A K_2. \quad (25)$$

Extension to the three-qubits and three-mode cases is direct; the involutions lead to the subgroups $S(U(7) \times U(1))$ and $U(3)$, respectively. We can select a basis such that these subgroups act trivially on ρ_d^0 and $\sigma_0 = \mathbb{I}_6$, yielding manifolds of dimension 14 and 12 for pure states. Also in this case, iterations of the CD on the subgroup allow exchange of the parameters associated to the abelian subalgebra by 7 and 3 populations. Generalization to n-parties is along the same lines. Using a different approach, the case of pure states of qubits and modes has been separately discussed in [30,31].

3.2. Symmetric Spaces AI and CII

For two qubits, we consider the coset $SU(4)/SU(2) \times SU(2)$. We consider $X_{\alpha\beta} = i\sigma_\alpha \otimes \sigma_\beta$, where $\alpha, \beta = 0, 1, 2, 3$ with $\sigma_0 = \mathbb{I}_2$, and the usual Pauli matrices. Except for $\mathbb{I}_4$, we use the remaining 15 Kronecker products as $SU(4)$ generators. The involution $\theta_I(X) = X^*$ identifies the subalgebra of the skew-symmetric matrices $\mathfrak{l} = \{X_{20}, X_{12}, X_{32}, X_{02}, X_{21}, X_{23}\}$. It generates the group $SO(4)$ that can be split into two sets $\mathfrak{l} = \mathfrak{l}_1 \bigcup \mathfrak{l}_2$ with $\mathfrak{l}_1 = \{X_{20}, X_{12}, X_{32}\}$ and $\mathfrak{l}_2 = \{X_{02}, X_{21}, X_{23}\}$. The generators in the two sets mutually commute, and within each set, the elements satisfy the angular momentum commutation relations. The Cartan subalgebra of $su(4)$ (the choice of a is not unique, $a_1 = \{X_{03}, X_{30}, X_{33}\}$, and $a_2 = \{X_{11}, X_{22}, X_{33}\}$, are two possibilities, but these are conjugate to each other by a change of basis) is fully contained in $p = \{X_{01}, X_{03}, X_{10}, X_{30}, X_{11}, X_{13}, X_{22}, X_{31}, X_{33}\}$ and enables $K_1 A K_2$ decomposition. A detailed analysis of this coset is presented in [8], leading to the identification of the subgroup K that characterizes the local transformations.

Up to this point, the 15 parameters that characterize $SU(4)$ are encoded; six in each K_i and three in A. A state ρ is obtained by applying the general element U of the group to an initial state ρ_0, and if we choose ρ_0 as a product state, then we obtain:

$$\rho = K_1^\dagger A^\dagger K_2^\dagger \rho_0 K_2 A K_1 = K_1^\dagger A^\dagger \rho_0 A K_1, \quad (26)$$

The simplification in the second equality follows from the fact that ρ_0 and $K_2^\dagger \rho_0 K_2$ belong to the same equivalence class; therefore, we can ignore the six parameters in K_2. Thus, this is a nine-parameters SS. When only pure states are considered, the elements in $A_1 = exp(a_1)$ are required to produce entanglement. According to [8], analysis of the invariants yields functions of the three parameters in $\mathfrak{a}$, which are used to classify and represent geometrically non-local operations. A key step is the change from the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to the Bell basis $\{|\psi^\pm\rangle, |\phi^\pm\rangle\}$ using:

$$Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}, \qquad Q_2(so(4))Q_2^\dagger \to su(2) \oplus su(2). \quad (27)$$

At this point, we highlight the value of our approach. We can consider mixtures since the three generators in $\mathfrak{a}_1$ are diagonal, and following the procedure used in the previous section, the expression $A\rho_0 A^\dagger$ may be used to introduce the population parameters (see Equation (17)). Since the population parameters introduce a hierarchy of correlations [7] and entanglement is no longer equal to non-locality, then a more detailed analysis of the regions of the correlations is needed. However, instead of pursuing that line, we describe another possibility. The SS $SU(4)/S(U(2) \times U(2))$ belongs to family AIII with $p = q = 2$, which we include here because of its close relation to the cosets of this family. The coset has eight parameters, and the subalgebra is $su(2) \oplus su(2) \oplus u(1)$, corresponding to $\mathfrak{l} = \{X_{01}, X_{02}, X_{03}, X_{31}, X_{32}, X_{33}, X_{30}\}$, where X_{30} is the generator of the $U(1)$ subgroup that decouples. Linear combinations of the remaining generators produce two sets of $SU(2)$ generators, $\mathfrak{l}_{1,2} = 1/2\{X_{01} \pm X_{31}, X_{02} \pm X_{32}, X_{03} \pm X_{33}\}$. The equivalence class associated to this subgroup conforms to X-states [32], which are known because the correlations, in particular the quantum discord, can be computed analytically, and in our approach, the quantifiers can be analyzed in terms of subgroup invariants.

We now outline some perspectives regarding generalizations for this SS. To analyze the case of 3-qubits, we consider the basis of $su(8)$ given by $X_{ijk} = i(\sigma_i \otimes \sigma_j \otimes \sigma_k)$ $i, j, k = 0, 1, 2, 3$, except the $\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2$. The involution $\theta_1(X)$ selects the skew-symmetric matrices that form a basis for $so(8)$ and the KAK decomposition is such that each K encodes 28 parameters while the abelian subalgebra in A includes the remaining 7 parameters. We propose that the subgroup $K_{ps} = SU(2) \otimes SU(2) \otimes SU(2)$ defines the equivalence class of the product states. Due to isomorphism between $su(2)$ and $so(3)$, we would expect to find a subgroup $(SO(3)\times)^3$ to be associated to K_{ps} and certainly there are various such subgroups in $SO(8)$. The SS approach is of help in this task, since by iterating the CD to the subgroups, a limited number of options are selected and the generators are identified through the involutions. The following is the list of options that arise from the iterations and that lead to the candidates for the $(SO(3)\times)^3$ we are looking for:

- T1: $so(8) \to so(7) \oplus p_1 \to (so(4) \oplus so(3)) \oplus p_2 \oplus p_1$,
- T2: $so(8) \to so(6) \oplus so(2) \oplus p_1$,
- T3: $so(8) \to so(5) \oplus so(3) \oplus p_1 \to so(4) \oplus p_2 \oplus so(3) \oplus p_1$,
- T4: $so(8) \to so(4) \oplus so(4) \oplus p_1$.

It is instructive to compare such options with the recognized classes of entangled states [33,34]:

$$|A, B, C\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle\,, \qquad |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$
$$|A, BC\rangle = |\psi_A\rangle \otimes |\psi_{BC}\rangle\,, \qquad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

At first sight, we could establish a correspondence between the iterations and the classes of states. For example, states of the type$|A, B, C\rangle$ are reminiscent of the group structure $T1$; however, as shown below, the group structure is not enough to establish the relation and each case must be discussed on its own. Consider the schematic matrix representation of the iteration $T4$:

$$so(8) = \begin{pmatrix} so(4) & p \\ -p^t & so(4) \end{pmatrix}, \tag{28}$$

Using the generalization of $Q_2 \to diag\{Q_2, Q_2\}$, we can map $so(4) \oplus so(4)$ into $(su(2) \oplus su(2)) \oplus (su(2) \oplus su(2))$ and find that these $su(2)$ can be arranged; for example, $\{X_{412} + X_{312}, X_{424} + X_{324}, X_{432} + X_{332}\}$, which are 4×4 block-diagonal matrices, and it can be verified that it conforms with the equivalence class of states $|A, BC\rangle$. The remaining subgroups of this SS must be analyzed along similar lines—for example, a schematic representation of iteration $T2$ is:

$$so(8) = \begin{pmatrix} so(4) & p_2 & p_{1i} \\ -p_2^t & so(3) & p_{1j} \\ -p_{1i} & -p_{1j} & 0 \end{pmatrix}, \tag{29}$$

And there exists a 3×3 matrix that maps $so(3) \rightarrow su(2)$. Unfortunately, so far, we can not make a strong statement regarding the origin of the subgroup of the local transformations corresponding to the states $|A, B, C\rangle$.

The coset $Sp(4,\mathbb{R})/Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})$ conduces to the derivation of the so-called standard form of two modes and to the local invariant quantities. The involution ϕ_{II} yields the subalgebra $\mathfrak{l} = \{X_1, X_4, X_7, X_2, X_5, X_8\}$ and the set $\mathfrak{p} = \{X_3, X_6, X_9, X_{10}\}$. This is the first case where an SS_p arises in the analysis, which is evidenced by the fact that $\mathfrak{l}$ contains 2 rotations and 4 boost (non-compact part). It is useful to introduce the linear combinations:

$$Xl_{1,2} = \frac{1}{2}(X_1 \pm X_2), \quad Xl_{4,5} = \frac{1}{2}(X_4 \pm X_5), \quad Xl_{7,8} = \frac{1}{2}(X_7 \pm X_8). \tag{30}$$

The two sets of generators, $\{Xl_1, Xl_4, Xl_7\}$, $\{Xl_2, Xl_5, Xl_8\}$, commute with each other $\{Xl_1, Xl_4, Xl_7\}$, $\{Xl_2, Xl_5, Xl_8\}$ and generate one $Sp(2,\mathbb{R})$. These subgroups are identified as local operations of each mode, which is clear from the block structure of the generators or the group elements [16]. The salient feature of this coset is its pseudo-symmetric nature and the fact that non-separable operations are associated with the P-subspace such that a general state can be written in terms of the $K \cdot P$ decomposition. We know that $\sigma_0 = \mathbb{I}$ and $\sigma_0 = diag\{a, a, b, b\}$ belong to the equivalence classes of pure and mixed states, respectively. The general form of the state in this equivalence class σ_s is obtained by taking $K = diag\{S_1, S_2\}$, and in terms of the Xl_i previously introduced, is given by:

$$\sigma_s = P^t \sigma_0 P = \begin{pmatrix} A & C \\ C^t & B \end{pmatrix}, \tag{31}$$

using the local subgroup K, we are able to perform a diagonalization that brings σ_s through the following procedure ($\sigma = K^t \sigma_s K$): A is transformed by S_1, $A \rightarrow S_1^t A S_1$, recalling the one-mode transformation Equation (22), the first rotation and the squeezing are enough to obtain $S_1^t A S_1 \rightarrow diag\{\lambda, \lambda\}$, and then the last rotation acts trivially. The same procedure can be applied to B using S_2, while the two free rotations are used to diagonalize C, $S_1^t C S_2$ so that we end up with:

$$\sigma = \begin{pmatrix} a\mathbb{I}_2 & C' \\ C'^t & b\mathbb{I}_2 \end{pmatrix}, \qquad C' = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{32}$$

For pure states $a = b$ and $c_2 = -c_1$, while for mixtures, $a \neq b$ and $c_2 \neq c_1$. The states belonging to this equivalence class are characterized by invariants under K and, according to the discussion in Section 2.2, the invariants are functions of traces and determinants of $\Sigma = \omega\sigma$. The following results will be required:

$$\begin{gathered} tr(\Sigma) = 0, \quad tr(\Sigma^2) = -2(a^2 + b^2 + 2c_1c_2), \quad tr(\Sigma^3) = 0 \\ tr(\Sigma^4) = ab(c_1^2 + c_2^2) + c_1c_2(a^2 + b^2) + 2(a^2 + c_1c_2)^2 + 2(c_1c_2 + b^2)^2, \end{gathered} \tag{33}$$

We further note that $\det A$, $\det B$, and $\det C$ are invariants under K, while $\Delta = \det A + \det B + 2\det C$ is a symplectic invariant, and $tr(A\omega C\omega B\omega C^t\omega) = ab(c_1^2 + c_2^2)$ can be expressed in terms of $tr(\Sigma^2)$ and $tr(\Sigma^4)$. Therefore, we have shown that all the invariants that characterize this coset can be obtained from the traces of powers of Σ. These results reproduce the invariants used in the formulation of the Peres–Horodecki criterion [16], with the advantage that our derivation can be extended to higher-dimensional systems.

We now comment on details regarding the generalization to the SS_p associated with the three-mode states. The coset is $Sp(6,\mathbb{R})/Sp(4,\mathbb{R}) \times Sp(2,\mathbb{R})$, and applying the CD to $Sp(4,\mathbb{R})$ leads to the identification of the local subgroup $K = (Sp(2,\mathbb{R})\times)^3$. This subgroup can be given the block form $S_l = diag\{S_1, S_2, S_3\}$, and the general form of the coset (including mixtures) is:

$$\sigma = \begin{pmatrix} A & D_1 & D_3 \\ D_1^t & B & D_2 \\ D_3^t & D_2^t & C \end{pmatrix}. \tag{34}$$

The subgroup of local transformations K can be used to obtain a standard-like form to the CM. Following the same procedure as for the two modes case, the matrices A, B, and C are diagonalized; however, there are not enough free parameters to diagonalize the three $D's$. There is some freedom in choosing which block to diagonalize; for example, we can fix rotations in $S_1^t D_1 S_2 \to diag\{\lambda_1, \lambda_2\}$, and we arrive to a standard-like form:

$$\sigma = \begin{pmatrix} a\mathbb{I}_2 & D_1' & D_3' \\ D_1' & b\mathbb{I}_2 & D_2' \\ (D_3')^t & (D_2')^t & c\mathbb{I}_2 \end{pmatrix} \qquad D_1' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{35}$$

Analysis of invariants also generalizes: the determinants of each sub-matrix are local invariants. These will be included in $tr(\Sigma^2)$; besides that, we have to consider now $tr(\Sigma^4)$ and $tr(\Sigma^6)$ to look for a complete set of local invariants. While the partial transpose criterion has no straightforward generalization, local invariant quantities can be used to formulate new criteria; however, we leave further analysis of this topic for future work.

3.3. Symmetric Spaces AII and CIII

We attempt to find a common interpretation of these families, considering that in both cases, the subgroup involved is of the symplectic type, $Sp(2n)$ and $Sp(2n, \mathbb{C})$, for qubits and modes, respectively. The isomorphism $Sp(4) \equiv SO(5)$ and $Sp(2, \mathbb{C}) \equiv SL(2, \mathbb{C})$ allows identifing that the geometry of the cosets are the 5-dimensional sphere S^5 and the anti-de Sitter space. In the case of 2-qubits, the involution $\theta_{II}(X) = J_n X^* J_n^{-1}$ leads to the subalgebra $sp(4)$, and $\mathfrak{p} = \{X_{01}, X_{03}, X_{12}, X_{22}, X_{32}\}$. According to the CD, the general state is expressed as $\rho = PK\rho_0 K^\dagger P^\dagger$, and the choice of ρ_0 plays a central role. We are interested in taking ρ_0 in the $Sp(4)$ equivalence class. We first consider pure states. In such a case, we can choose an eigenstate of the P-operator. The parameters then correspond to S^5:

$$\rho_s = \begin{pmatrix} \cos(r) + \frac{ix_1 \sin(r)}{r} & \frac{(ix_2+x_5)\sin(r)}{r} & 0 & \frac{(x_3-ix_4)\sin(r)}{r} \\ \frac{i(x_2+ix_5)\sin(r)}{r} & \cos(r) - \frac{ix_1 \sin(r)}{r} & -\frac{(x_3-ix_4)\sin(r)}{r} & 0 \\ 0 & \frac{(x_3+ix_4)\sin(r)}{r} & \cos(r) + \frac{ix_1 \sin(r)}{r} & \frac{i(x_2+ix_5)\sin(r)}{r} \\ -\frac{(x_3+ix_4)\sin(r)}{r} & 0 & \frac{(ix_2+x_5)\sin(r)}{r} & \cos(r) - \frac{ix_1 \sin(r)}{r} \end{pmatrix}, \tag{36}$$

where $r^2 = |(x_1, x_2, x_3, x_4, x_5)|^2$. The eigenstates include a vanishing component; for example, $|\psi\rangle = (\alpha, \beta, e^{i\theta}, 0)^T$ with $\alpha, \beta \in \mathbb{C}$. An interpretation in terms of a 4-level system is proposed in [12]. The other possibility is to introduce a population using the diagonal element X_1, to end up with a state given by:

$$\rho_s = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}, \tag{37}$$

Alternatively, an iteration is possible using the $Sp(4)$ decomposition in terms of $K = SU(2) \times SU(2)$, or $K = U(2)$ and the invariants of ρ_0 under K are analogous to those of two-mode states.

The case of two modes is considered with involution $\phi_{III}(X) = H X H^{-1}$, where $H = \mathbb{I}_2 \otimes s_2$ (we are unaware if matrix H generalizes to $Sp(2n, \mathbb{R})$), leads to the subalgebra $\mathfrak{l} = \{X_2, X_3, X_4, X_8, X_9, X_{10}\}$, and to the set $\mathfrak{p} = \{X_1, X_5, X_6, X_7\}$.

The coset is given by $\mathrm{P} = \exp(z_1 X_1 + z_5 X_5 + z_6 X_6 + z_7 X_7)$ where the z_i can be parameterized as $z_1 = z\cos(\theta)\cosh(z)$, $z_5 = z\sin(\theta)\cosh(z)\cos(\phi)$, $z_6 = z\sin(\theta)\cosh(z)\sin(\phi)$, $z_7 = z\sinh(z)$ and $z^2 = |(z_1, z_5, z_6, z_7)|^2$, which defines a hyperboloid shape in four dimensions. In fact, this coset is commonly studied in the context of Lorentz transformations [35]. The CM is of the form given in Equation (31), and below, we report the form of C for this coset, since the matrices A, B are not necessary for the argument:

$$C = \begin{pmatrix} \frac{z_6(a(z_5 - z_7) - b(z_5 + z_7))\sinh^2(z)}{z^2} & \frac{z_6 \sinh(z)((a+b)z\cosh(z) + (a-b)z_1 \sinh(z))}{z^2} \\ \frac{z_6 \sinh(z)((a+b)z\cosh(z) + (b-a)z_1 \sinh(z))}{z^2} & \frac{z_6(b(z_7 - z_5) + a(z_5 + z_7))\sinh^2(z)}{z^2} \end{pmatrix}. \quad (38)$$

Note that $\mathfrak{p}_{CII} \cap \mathfrak{p}_{CIII} = \{X_6\}$; therefore, $z_6 = 0$ reduces this coset to a subset of the local subgroup, while for $z_7 = 0$, it reduces to a subset of coset CII.

4. Discussion

In the last section, we presented a systematic treatment of qubits and modes. Important results scattered in the literature can be systematically reproduced starting from the concept of SS. The table included below is an attempt to summarize the results.

Table 1. (Pseudo) Symmetric spaces considered for qubits and modes. Blank spaces indicate that these cosets are not defined. Details of the analysis of each coset are included in Section 3.

SS	$n = 1$	$n = 2$	$n = 3$
AIII	$SU(2)/S(U(1) \times U(1))$	$SU(4)/S(U(p) \times U(q))$	$SU(8)/S(U(p) \times U(q))$
CI	$Sp(2,\mathbb{R})/U(1)$	$Sp(4,\mathbb{R})/U(2)$	$Sp(6,\mathbb{R})/U(3)$
AI	$SU(2)/SO(2)$	$SU(4)/SO(4)$	$SU(8)/SO(8)$
CII	$Sp(2,\mathbb{R})/SO(1,1)$	$Sp(4,\mathbb{R})/(Sp(2,\mathbb{R})\times)^2$	$Sp(6,\mathbb{R})/Sp(p,\mathbb{R}) \times Sp(q,\mathbb{R})$
AII		$SU(4)/Sp(4)$	$SU(8)/Sp(8)$
CIII		$Sp(4,\mathbb{R})/Sp(2,\mathbb{C})$	

However, the table 1 is not adequate to capture the diversity and advantages that the symmetric space formalism brings to the analysis of quantum states. We hope that the following points can partially address this shortcoming.

- Regardless of the number of qubits, there are only three SS denoted AI, AII, and AIII; for modes, we consider one SS denoted CI, and two SS_p, CII and CIII.
- In addition to traditional methods, the SS formalism is rich enough to include elements that allow generalization of concepts to higher-dimensional systems, examples of which are local invariant quantities and the standard form of CM, which may allow design of quantum gates on the lines of [8]. In this sense, the SS formalism serves as guidance to build states on demand with the certainty provided by the transitive action of the group.
- The number of equivalence classes (cosets) grows with n, the number of qubits and modes, since all possible combinations of p, q, so that $p + q = n$ must be considered and, besides that, possible iteration of subgroups increases the number of alternatives. For arbitrary n, it is possible to implement an algorithm that leads to all the CDs of a given system. The more labor-intensive part of the process is to find quantities which are invariant under the subgroup action and to obtain its physical interpretation.
- The SS AIII and CI share interpretation since both describe equivalence classes characterized by mixing parameters. This is true for an arbitrary number of qubits and Gaussian modes. Moreover, it is possible to use the Cartan subalgebra to introduce populations and the general states are constructed in each case.

- The SS AI and the SS_p CII also share interpretation—both of them encode information on the non-separable correlations, while the subgroup characterizes the product states. For two pure qubits, the SS AI reproduces the group structure found in [8], which serves as a basis for their analysis. The formalism we propose provides guidelines on how to generalize these results. In the main text, we discuss how to include mixtures. We show that for three pure qubits, there are different iterations of subgroups $SO(4) \times SO(4)$, $SO(4) \times SO(3)$, $SO(6) \times SO(2)$, which stabilize a subspace of states; therefore, they are considered local-like subgroups. We attempt an interpretation by comparing these subgroups with known types of non-equivalent entangled states.
  For two modes, the SS formalism identifies the subgroup of local transformations, and the group decomposition facilitates the derivation of local invariant quantities, as well as the separability criterion. Analogous results can be obtained for three modes, since the iteration on the subgroup $Sp(4, \mathbb{R}) \times Sp(2, \mathbb{R})$ enables the identification of local subgroup and a standard-like form of the correlation matrix. Similar results can be derived for $n \geq 3$ modes.
- For the SS AII and the SS_p CIII, the subgroup is of the symplectic-type, compact and non-compact, respectively. Therefore, the equivalence classes can be labeled by symplectic invariants. Possible iterations of $Sp(4)$ allow further decomposition analogous to those of two modes. The interpretation is elusive; however, limiting cases (for specific values of subgroup parameters) makes evident the overlap of this coset with CI and CII, so that this coset includes mixed and non-separable states. This is similar to what happens in family AIII with X states for qubits.

The results obtained are motivating and in the future we plan to perform an exhaustive analysis of three qubits and three modes as well as the case of qutrits.

5. Conclusions

The symmetric space formalism provides a unified framework for the treatment of qubit and Gaussian mode systems, addressing positively our initial question. By exploiting the structure of equivalence classes, symmetric spaces capture the intricacies and complexities observed in quantum states. Beyond systematically reproducing the known results of two qubits and two modes, this formalism extends naturally to higher dimensions. It yields information on three qubits and three modes that merits further study to extract its full interpretation. We thank one of the referees for calling to our attention that our framework could be used to improve Gaussian-based modes quantum computing schemes and to inspire quantum gravity models.

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References

1. Hawkins, T. The Erlanger Programm of Felix Klein: Reflections on its place in the history of mathematics. *Hist. Math.* **1984**, *11*, 442–470. [CrossRef]
2. Mosseri, R.; Dandoloff, R. Geometry of entangled states, Bloch spheres and Hopf fibrations. *J. Phys. A Math. Gen.* **2001**, *34*, 10243. [CrossRef]
3. Lévay, P. The geometry of entanglement: Metrics, connections and the geometric phase. *J. Phys. A Math. Gen.* **2004**, *37*, 1821. [CrossRef]
4. Hasebe, K. The split algebras and noncompact Hopf maps. *J. Math. Phys.* **2010**, *51*, 053524. [CrossRef]
5. Hill, S.A.; Wootters, W.K. Entanglement of a Pair of Quantum Bits. *Phys. Rev. Lett.* **1997**, *78*, 5022–5025. [CrossRef]
6. Duan, L.M.; Giedke, G.; Cirac, J.I.; Zoller, P. Inseparability Criterion for Continuous Variable Systems. *Phys. Rev. Lett.* **2000**, *84*, 2722–2725. [CrossRef]
7. Frérot, I.; Fadel, M.; Lewenstein, M. Probing quantum correlations in many-body systems: A review of scalable methods. *Rep. Prog. Phys.* **2023**, *86*, 114001. [CrossRef]
8. Zhang, J.; Vala, J.; Sastry, S.; Whaley, K.B. Geometric theory of nonlocal two-qubit operations. *Phys. Rev. A* **2003**, *67*, 042313. [CrossRef]
9. Arvind.; Dutta, B.; Mukunda, N.; Simon, R. Two-mode quantum systems: Invariant classification of squeezing transformations and squeezed states. *Phys. Rev. A* **1995**, *52*, 1609–1620. [CrossRef]
10. Berezin, F.A. Quantization in complex symmetric spaces. *Math. USSR-Izv.* **1975**, *9*, 341. [CrossRef]
11. Olshanetsky, M.A.; Perelomov, A.M. Quantum systems related to root systems, and radial parts of Laplace operators. *Funct Anal Its Appl.* **1978**, *12*, 121-128. [CrossRef]
12. Uskov, D.B.; Rau, A.R.P. Geometric phase for N-level systems through unitary integration. *Phys. Rev. A* **2006**, *74*, 030304. [CrossRef]
13. Rau, A.R.P. Symmetries and Geometries of Qubits, and Their Uses. *Symmetry* **2021**, *13*, 1732. [CrossRef]
14. Linden, N.; Popescu, S.; Popescu, S. On Multi-Particle Entanglement. *Fortschritte Der Phys.* **1998**, *46*, 567–578. [CrossRef]
15. Horodecki, M.; Horodecki, P.; Horodecki, R. Separability of mixed states: Necessary and sufficient conditions. *Phys. Lett. A* **1996**, *223*, 1–8. [CrossRef]
16. Simon, R. Peres-Horodecki Separability Criterion for Continuous Variable Systems. *Phys. Rev. Lett.* **2000**, *84*, 2726–2729. [CrossRef]
17. Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*; Elsevier Science: Amsterdam, The Netherlands, 1979.
18. Berger, M. Les espaces symétriques noncompacts. *Ann. Sci. L'École Norm. Supérieure* **1957**, *74*, 85–177. [CrossRef]
19. Arvind.; Dutta, B.; Mukunda, N.; Simon, R. The real symplectic groups in quantum mechanics and optics. *Pramana* **1995**, *45*, 471–497. [CrossRef]
20. Knapp, A. *Lie Groups Beyond an Introduction*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2002; Volume 140. [CrossRef]
21. Gilmore, R. *Lie Groups, Lie Algebras, and Some of Their Applications*; Dover Books on Mathematics; Dover Publications: Mineola, NY, USA, 2012.
22. Hermann, R. *Lie Groups for Physicists*; The mathematical physics monograph series; Benjamin: New York, NY, USA, 1966.
23. Brezov, D.S.; Mladenova, C.D.; Mladenov, I.M. New perspective on the gimbal lock problem. *AIP Conf. Proc.* **2013**, *1570*, 367–374. [CrossRef]
24. Procesi, C. The invariant theory of n × n matrices. *Adv. Math.* **1976**, *19*, 306–381. [CrossRef]
25. Makhlin, Y. Nonlocal properties of two-qubit gates and mixed states and optimization of quantum computations. *Quantum Inf. Process.* **2002**, *1*, 243–252. [CrossRef]
26. Grassl, M.; Rötteler, M.; Beth, T. Computing local invariants of quantum-bit systems. *Phys. Rev. A* **1998**, *58*, 1833–1839. [CrossRef]
27. Driss, B.; Bouzar, Z. Symplectic invariants and covariants of matrices. *Linear Multilinear Algebra* **2016**, *65*, 1–26. [CrossRef]
28. Byrd, M. Differential geometry on SU(3) with applications to three state systems. *J. Math. Phys.* **1998**, *39*, 6125–6136. [CrossRef]
29. Tilma, T.; Byrd, M.; Sudarshan, E.C.G. A parametrization of bipartite systems based onSU(4) Euler angles. *J. Phys. A Math. Gen.* **2002**, *35*, 10445–10465. [CrossRef]
30. Windt, B.; Jahn, A.; Eisert, J.; Hackl, L. Local optimization on pure Gaussian state manifolds. *SciPost Phys.* **2021**, *10*, 066. [CrossRef]
31. Dariusz, C.; Jamiolkowski, A. *Geometric Phases in Classical and Quantum Mechanics/D. Chruscinski, A. Jamiolkowski.*; Springer: Berlin/Heidelberg, Germany, 2004. [CrossRef]
32. Rau, A.R.P. Algebraic characterization of X-states in quantum information. *J. Phys. A Math. Theor.* **2009**, *42*, 412002. [CrossRef]
33. Dür, W.; Vidal, G.; Cirac, J.I. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A* **2000**, *62*, 062314. [CrossRef]

34. D'Hondt, E.; Panangaden, P. The computational power of the W And GHZ States. *Quantum Infom. Comput.* **2006**, *6*, 173–183. [CrossRef]
35. Balbinot, R.; Gradechi, A.M.E.; Gazeau, J.P.; Giorgini, B. Phase spaces for quantum elementary systems in anti-de Sitter and Minkowski spacetimes. *J. Phys. A Math. Gen.* **1992**, *25*, 1185. [CrossRef]

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