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How to (path-) integrate by differentiating

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Abstract. Path integrals are at the heart of quantum field theory. In spite of their covariance and seeming simplicity, they are hard to define and evaluate. In contrast, functional differentiation, as it is used, for example, in variational problems, is relatively straightforward. This has motivated the development of new techniques that allow one to express functional integration in terms of functional differentiation. In fact, the new techniques allow one to express integrals in general through differentiation. These techniques therefore add to the general toolbox for integration and for integral transforms such as the Fourier and Laplace transforms. Here, we review some of these results, we give simpler proofs and we add new results, for example, on expressing the Laplace transform and its inverse in terms of derivatives, results that may be of use in quantum field theory, e.g., in the context of heat traces.

1. Introduction

Path integrals over fields, such as that for a scalar field,

$$Z[J] = \int e^{iS[\phi]+i\int d^4x \; J(x)\phi(x)} D[\phi], \tag{1}$$

are at the very heart of quantum field theory. Notice that Equ.1 can also be viewed as the Fourier transform of $e^{iS[\phi]}$. In spite of their seeming simplicity, such functional integrals or functional Fourier transforms are notoriously difficult to define and evaluate, see, e.g., [1]-[6].

In comparison to functional integration, functional differentiation is relatively straightforward and it is of course widely used, for example, in extremization problems. This has motivated recent work, [7], in which it was shown that in certain circumstances functional integration can be expressed through functional differentiation. In fact, the results show that integrals in general can be expressed in terms of differentiations. This includes proper and improper integrals and also integral transforms such as the Fourier and Laplace transforms. The new methods of [7] are not meant to replace the existing integration methods. They merely add to the toolbox of techniques for integrals and integral transforms. The new tools have the advantage that when they apply they are often quick and simple to use, work naturally with distributions and can even sidestep the need for contour integrations.

Here, we will review the new tools, add simpler proofs and we also add some new results.

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2. Integration by differentiation

In [7], the following representations of integration through differentiation were shown:

$$\int_a^b f(x)\ dx = \lim_{\epsilon\to 0} f(\partial_\epsilon)\ \frac{e^{\epsilon b}-e^{\epsilon a}}{\epsilon} \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{\epsilon\to 0} 2\pi f(-i\partial_\epsilon)\ \delta(\epsilon) \tag{3}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi\delta(i\partial_\epsilon)\ f(\epsilon) \tag{4}$$

(Notice that, as was shown in [7], in Equ.4 it is not necessary to take ϵ to a fixed value because the ϵ-dependence drops out.) Here, $f(\nu\partial_\epsilon)$ with $\nu\in\{\pm 1,\pm i\}$ means in the simplest cases that when f is expanded in a MacLaurin or Laurent series, its argument is to be replaced by $\nu\partial_\epsilon$. More generally, f can be interpreted as a function on the spectrum of the operator ∂_ϵ. This means, in particular, that when $f(\nu\partial_\epsilon)$ acts on one of its eigenfunctions, $e^{\epsilon a}$, we have:

$$f(\nu\partial_\epsilon)\ e^{\epsilon a} = f(\nu a)e^{\epsilon a} \tag{5}$$

While we will give proofs below that apply to fairly large classes of functions f, the above equations also hold in a distributional sense. The conditions and exact size of the space of generalized functions for which the above equations hold is not yet known.

Let us now add the following equations which follow straightforwardly from Equ.2 by taking the limits $a\to 0, b\to\infty$ and $a\to-\infty, b\to 0$ respectively:

$$\int_0^{\infty} f(x)\ dx = \lim_{\epsilon\to 0^+} f(-\partial_\epsilon)\ \frac{1}{\epsilon} \tag{6}$$

$$\int_{-\infty}^{0} f(x)\ dx = \lim_{\epsilon\to 0^+} f(\partial_\epsilon)\ \frac{1}{\epsilon} \tag{7}$$

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{\epsilon\to 0} \left(f(\partial_\epsilon)+f(-\partial_\epsilon)\right)\ \frac{1}{\epsilon} \tag{8}$$

3. Examples of integration by differentiation

In [7], the aim was to derive new methods for integrals and integral transform for applications to quantum field theoretic path integrals. Here, we will look at these integration methods' general utility. To this end, in order to illustrate the above tools let us perform an integral that is usually considered nontrivial in the sense that it is usually done using contour integration. Namely, using the new Equ.6:

$$\int_0^{\infty}\frac{\sin(x)}{x}\ dx = \lim_{\epsilon\to 0^+}\frac{1}{2i}\left(e^{-i\partial_\epsilon}-e^{i\partial_\epsilon}\right)\frac{1}{-\partial_\epsilon}\ \frac{1}{\epsilon} \tag{9}$$

$$= \lim_{\epsilon\to 0^+}\frac{-1}{2i}\left(e^{-i\partial_\epsilon}-e^{i\partial_\epsilon}\right)(\ln(\epsilon)+c) \tag{10}$$

$$= \lim_{\epsilon\to 0^+}\frac{-1}{2i}\left(\ln(\epsilon-i)+c-\ln(\epsilon+i)-c\right) \tag{11}$$

$$= \frac{-1}{2i}\left(\frac{-i\pi}{2}-\frac{i\pi}{2}\right)=\frac{\pi}{2} \tag{12}$$

Here, we used that the antiderivative of $1/\epsilon$ is the logarithm with an integration constant c, and we used that the exponentiated derivate translates: $e^{a\partial_\epsilon}g(\epsilon)=g(\epsilon+a)$. For comparison, we now perform a similar integral using Equ.3:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = 2\pi \lim_{\epsilon\to 0} \frac{1}{2i}\left(e^{\partial_\epsilon} - e^{-\partial_\epsilon}\right)\frac{1}{-i\partial_\epsilon}\,\delta(\epsilon) \tag{13}$$

$$= \pi \lim_{\epsilon\to 0}\left(e^{\partial_\epsilon} - e^{-\partial_\epsilon}\right)\left(\Theta(\epsilon) + c'\right) \tag{14}$$

$$= \pi \lim_{\epsilon\to 0}\left(\Theta(\epsilon+1) + c' - \Theta(\epsilon-1) - c'\right) \tag{15}$$

$$= \pi \tag{16}$$

Here, Θ is the Heaviside function and c' is an integration constant. Let us close this section with a quick derivation of Equ.2:

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^b f(x)e^{-\epsilon x}\,dx \tag{17}$$

$$= \lim_{\epsilon\to 0} f(-\partial_\epsilon)\int_a^b e^{-\epsilon x}\,dx \tag{18}$$

$$= \lim_{\epsilon\to 0} f(-\partial_\epsilon)\,\frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon} \tag{19}$$

4. Forward and inverse Laplace transforms through differentiation

As was shown in [7], the representation of integration over the real line given in Equ.3 is a special case of a new representation of Fourier transformation by differentiation. Namely, with the following definition of the Fourier transform $F[f]$ of a function f,

$$F[f](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\,f(y)\,dy, \tag{20}$$

we have the new representation:

$$F[f](x) = \sqrt{2\pi}\,f(-i\partial_x)\,\delta(x) \tag{21}$$

It is straightforward to prove, see [7], that Equ.21 holds for the basis of plane waves, and they span the function space. This means that the new representation for the integral over the real line given in Equ.3 follows from Equ.21 because integration is the zero frequency limit of the Fourier transform.

Our aim now is to show that our new Equ.6 is, similarly, the "zero-frequency" special case of a new and very simple representation of the Laplace transform, expressed entirely in terms of differentiation. Consider the usual definition of the Laplace transform, $L[f]$, of a function f, for $x > 0$:

$$L[f](x) = \int_0^{\infty} f(y)\,e^{-xy}\,dy \tag{22}$$

We claim that:

$$L[f](x) = f(-\partial_x)\,\frac{1}{x} \tag{23}$$

$$L^{-1}[f](x) = f(\partial_x)\,\delta(x) \tag{24}$$

For example, using Equ.23, we can immediately read off the Laplace transforms of monomials, $f(x) = x^n$:

$$L[f](x) = (-\partial_x)^n\,\frac{1}{x} = \frac{n!}{x^{n+1}} \tag{25}$$

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The new representation, Equ.24, of the inverse Laplace transform may be particularly useful in practice. This is because, with the usual methods, the inverse Laplace transform is somewhat tedious as it involves analytic continuation. Equ.24 can be comparatively straightforward to evaluate. For example, let us calculate the inverse Laplace transform of the function $f(x) = 1/(x - a)$. Using Equ.24, we obtain:

$$\begin{aligned} L^{-1}[f](x) &= \frac{1}{\partial_x - a}\ \delta(x) && (26)\\ &= \int_0^\infty e^{-w(\partial_x - a)}\ dw\ \delta(x) && (27)\\ &= \int_0^\infty e^{aw}\ \delta(x-w)\ dw = e^{ax} && (28) \end{aligned}$$

Here, we used:

$$\frac{1}{k} = \int_0^\infty e^{-wk}\ dw \tag{29}$$

5. Proofs for the forward and inverse Laplace transforms

Equ.23 for the forward Laplace transform is easy to derive:

$$L[f](x) = \int_0^\infty e^{-xy}\ f(y)\ dy = f(-\partial_x) \int_0^\infty e^{-xy}\ dy = f(-\partial_x)\ \frac{1}{x} \tag{30}$$

As a consequence, we have also proven Equ.6, namely as the special case $x \to 0^+$. Let us now show that Equ.24 indeed inverts the action of L, i.e., that $L^{-1} \circ L = id$:

$$\begin{aligned} L^{-1} \circ L[f](x) &= L^{-1}\left[f(-\partial_x)\frac{1}{x}\right] = \lim_{\epsilon\to 0} L^{-1}\left[f(-\partial_\epsilon)\ \frac{1}{\epsilon + x}\right] && (31)\\ &= \lim_{\epsilon\to 0} f(-\partial_\epsilon)\ L^{-1}\left[\frac{1}{\epsilon + x}\right] = \lim_{\epsilon\to 0} f(-\partial_\epsilon)\ \frac{1}{\epsilon + \partial_x}\ \delta(x) && (32)\\ &= \lim_{\epsilon\to 0} f(-\partial_\epsilon)\ \int_0^\infty e^{-(\epsilon+\partial_x)w}\ dw\ \delta(x) && (33)\\ &= \lim_{\epsilon\to 0} \int_0^\infty f(-\partial_\epsilon)\ e^{-\epsilon w}\ \delta(x-w)\ dw && (34)\\ &= f(x) && (35) \end{aligned}$$

In Equ.33, we used Equ.29 and the final step used Equ.5.

6. Example: trace of the heat kernel

A prominent occurrence of the Laplace transform in quantum field theory is in the context of the trace of the heat kernel, see, e.g., [1, 10, 11]:

$$h(t) = \sum_n e^{-\lambda_n t} \tag{36}$$

Here, the λ_n are the (assumed positive) eigenvalues of the heat operator. Given the function $h(t)$ it is possible to recover the spectrum $\{\lambda_n\}$ that generated it. With the new methods, this is particularly simple to see. To this end, let us introduce the spectral comb function:

$$s(\lambda) = \sum_n \delta(\lambda - \lambda_n) \tag{37}$$

Clearly, the trace of the heat kernel, $h(t)$, is its Laplace transform:

$$h(t) = \sum_n e^{-\lambda_n t} = \int_0^\infty e^{-\lambda t}\ s(\lambda) = L[s](t) \tag{38}$$

Using the new representation for the inverse Laplace transform, Equ.24, we can invert this relation straightforwardly:

$$L^{-1}[h](\lambda) \quad = \quad L^{-1}\left[\sum_n e^{-\lambda_n t}\right](\lambda) = \sum_n e^{-\lambda_n \partial_\lambda}\ \delta(\lambda) = \sum_n \delta(\lambda - \lambda_n) = s(\lambda) \tag{39}$$

7. Outlook

The methods we have presented here are meant as additions to the toolbox for integration and for integral transforms such as the Fourier and Laplace transforms. Being general purpose methods, they may possess applications beyond physics, e.g., to communication engineering and signal processing, [7, 8, 9]. Within mathematics, there is the possibility that these methods may be extensible covariantly to integration over higher-dimensional curved manifolds. There, a new relationship between integration and differentiation could yield new insights into Stokes' theorem. Here, in one dimension, Stokes' theorem is simply the first fundamental theorem of calculus. We have for the derivative, $f'(x) = df(x)/dx$, as is straightforward to verify:

$$f'(\partial_\epsilon) = f(\partial_\epsilon)\epsilon - \epsilon f(\partial_\epsilon) \tag{40}$$

With this and Equ.2, the first fundamental theorem of calculus takes the form:

$$\int_a^b f'(x)\ dx \quad = \quad \lim_{\epsilon \to 0} \left(f(\partial_\epsilon)\epsilon - \epsilon f(\partial_\epsilon)\right)\ \frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon} \tag{41}$$

$$= \quad \lim_{\epsilon \to 0} \left(f(b)\ e^{\epsilon b} - f(a) e^{\epsilon a} - \epsilon f(\partial_\epsilon) \frac{e^{\epsilon b} - e^{\epsilon a}}{\epsilon} \right) \tag{42}$$

$$= \quad f(b) - f(a) - \lim_{\epsilon \to 0} \epsilon \int_a^b f(x)\ dx = f(b) - f(a) \tag{43}$$

The challenge is to generalize the present methods to higher dimensions so that, for example, a new perspective into the algebraic workings of boundary and coboundary operators may be gained. For the algebraic background for formal power series, see, for example, [12].

We started this paper by considering the analytical difficulties of quantum field theoretical path integrals. In this context, the new methods could inspire, for example, new approaches to perturbative expansions. Indeed, as was shown in [7], integration by differentiation methods yield not only the small coupling expansion but also naturally the strong coupling expansion. In general, the Dirac Deltas in integration-by-differentiation equations such as Equs.3,4, can be regularized in multiple ways, such as the Gaussian and sinc regularisations. Each choice yields another kind of perturbative expansion.

Also, the new integration methods presented here may provide a new perspective on anomalies in quantum field theories. This is because anomalies in quantum field theory originate in nontrivial transformation properties of the measure in the path integral, see e.g., [13].

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