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Conserved non-Noether charge in general relativity: Physical definition versus Noether's second theorem

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In this paper, we make a close comparison of a covariant definition of an energy/entropy in general relativity, recently proposed by a collaboration including the present authors, with existing definitions of energies such as the one from the pseudo-tensor and the quasi-local energy. We show that existing definitions of energies in general relativity are conserved charges from Noether's second theorem for the general coordinate transformation, whose conservations are merely identities implied by the local symmetry and always hold without using equations of motion. Thus, none of the existing definitions in general relativity reflects the dynamical properties of the system, and the need for a physical definition of an energy. In contrast, our new definition of the energy/entropy in general relativity is generically a conserved non-Noether charge and gives physically sensible results for various cases such as the black hole mass, the gravitational collapse and the expanding universe, while existing definitions sometimes lead to unphysical ones including zero and infinity. We conclude that our proposal is more physical than existing definitions of energies. Our proposal makes it possible to define almost uniquely the covariant and conserved energy/entropy in general relativity, which brings some implications to future investigations.

Keywords: Conserved charge; general relativity; Noether's second theorem; energy and entropy; black hole; gravitational collapse; expanding Universe.

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1. Introduction

Since Einstein proposed general relativity as a theory for gravity,[1] a proper definition of an energy, more generally a conserved charge from an energy–momentum tensor (EMT), has been looked for. A main obstruction comes from a fact that a covariant conservation law with a covariant derivative ∇_a for an EMT of matters $T^a{}_b$ in general relativity

$$\nabla_a T^a{}_b = 0 \tag{1}$$

is different from the standard conservation law

$$\partial_a\left(\sqrt{-g}T^a{}_b\right) = 0, \quad g := \det g_{ab}, \tag{2}$$

which is required to construct a conserved energy but is not covariant under the general coordinate transformation, the most fundamental symmetry of general relativity. Einstein himself modified a definition of the EMT as $\tilde{T}^a{}_b = T^a{}_b + t^a{}_b$ to satisfy (2). Since $t^a{}_b$ is not a tensor under the general coordinate transformation except the affine transformation, $\tilde{T}^a{}_b$ is called Einstein's energy–momentum *pseudo-tensor*. A more modern way is to define a total energy of a system by a surface integral of gravitational fields in its asymptotic region, called a quasi-local energy, for an asymptotically flat space–time.[2–4] This approach has been extended further for more general asymptotic behaviors by properly incorporating extra surface terms.[5–8] See Ref. 9 for a recent summary of the problem including historical perspectives.

Recently, the present authors and their collaborator have proposed a different definition for conserved charges such as the energy and its generalization in a curve space–time including general relativity,[10,11] directly from the EMT of matters but still keeping its covariance under the general coordinate transformation. Advantages of this definition, however, have not been fully recognized, partly because our previous papers focused on the idea and the quick report of the results without detailed comparisons to existing definitions. Thus, in this paper, we make detailed comparisons between our proposal and other definitions for conserved charges in general relativity, showing that our definition is much more natural and physical than others, in order to establish that our definition of the energy and its generalization solves the long-standing issue for the definition of the energy in general relativity.

In Sec. 2, we demonstrate that (almost) all existing definitions of the energy in general relativity can be regarded as a conserved charge implied by Noether's second theorem for local symmetries.[12] We show that definitions of the energy as charges from Noether's second theorem are categorized either as Einstein's pseudo-tensor type or as the Komar energy[15] type, the later of which includes the Arnowitt-Deser-Misner (ADM) mass,[2] and the energy in the asymptotically flat space–time[3,4] as well as in the asymptotically de Sitter (dS)/anti-de Sitter (AdS) space–time.[5–8] Since both types of definitions allow quasi-local expressions, we can easily change their definitions of the energy by adding an arbitrary total divergent term to the Einstein–Hilbert action. Even worse, the energy from these two types of definitions is conserved *without* using equations of motion. Thus, the conservation of the energy is merely identity implied by the general coordinate transformation rather than a consequence of a time evolution, so that it cannot represent dynamics of the system. We conclude that none of existing definitions from Noether's second theorem can provide a physical definition of an energy in general relativity. Indeed Noether herself referred the charge from the second theorem *improper* by citing the word from Hilbert and Klein.[12]

In Sec. 3, we instead explain our proposal for a covariant definition of the energy and its generalization in general relativity, which requires equations of motion,[10, 11] and thus is not a charge from the second theorem. After reviewing our proposal, we discuss three cases, (1) energy conservation by a global symmetry, (2) energy conservation without symmetry, (3) conserved charge in the absence of energy conservation, together with explicit examples, where we also compare results from our proposal with those from Noether's second theorem. In the case (1), our definition gives the finite energy of the Schwarzschild black hole even for nonzero cosmological constant Λ, while definitions from Noether's second theorem require a subtraction of the infinite vacuum energy to obtain the finite black hole energy for $\Lambda \neq 0$ cases, which agrees with the one from our definition only at $d = 4$. We have a similar comparison for the energy during a gravitational collapse in the case (2). In the case (3), the homogeneous and isotropic expanding universe is analyzed. While the energy in our covariant definition is not conserved, we show that our definition allows a conserved charge as the generalization of the energy, which we identify the entropy. On the other hand, the conservation of the energy for definitions from Noether's second theorem implies the vanishing total energy, which is physically meaningless.

Our conclusion and discussion are given in Sec. 4. For the sake of readers, Noether's second theorem is explained for general cases in App. A.

2. Noether's Second Theorem and Conserved Charges in General Relativity

In this section, we derive conservation equations using Noether's second theorem in general relativity. We then show that these conservation equations lead to a pseudo-tensor as well as charges associated with asymptotic symmetry including the ADM mass.

2.1. *Noether's second theorem in general relativity*

We apply Noether's second theorem to general relativity. Noether's second theorem is given in Ref. 12, and its application to general relativity is discussed in Ref. 13, but these considerations, except the famous Noether's first theorem, have not been recognized well or have been sometimes misunderstood in the community. Thus, for the sake of readers, we explain the second theorem here in the case of general relativity, and the derivation of the theorem is presented for a general case in App. A.

To make our argument concrete, we take a scalar field theory coupled to the Einstein gravity, whose Lagrangian density is given by

$$L = L_G + L_M, \tag{3}$$

where

$$L_G = \frac{1}{2\kappa}\sqrt{-g}(R-2\Lambda), \quad \kappa := 4\pi G, \tag{4}$$

$$L_M = \sqrt{-g}\left[-\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right], \tag{5}$$

and consider the integral of L over an arbitrary d-dimensional region Ω in the d-dimensional space–time as

$$S_\Omega := \int_\Omega d^dx\, L. \tag{6}$$

We first derive an equation of motion by considering an arbitrary variation δ_v as

$$\begin{aligned} 2\kappa\delta_g S_\Omega &= \int_\Omega d^dx\,\sqrt{-g} \\ &\quad\times\left[\left(\frac{1}{2}g^{ab}(R-2\Lambda) - R^{ab} + 2\kappa T^{ab}\right)\delta_v g_{ab} + \nabla_a\left(g^{bc}\delta_v\Gamma^a_{bc} - g^{ab}\delta_v\Gamma^c_{bc}\right)\right], \\ \delta_\phi S_\Omega &= \int_\Omega d^dx\,\left[\sqrt{-g}(\nabla_a\nabla^a\phi - V'(\phi))\delta_v\phi - \partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\delta_v\phi\right)\right], \end{aligned} \tag{7}$$

where

$$T^{ab} := \frac{1}{\sqrt{-g}}\frac{\partial L_M}{\partial g_{ab}} = \frac{1}{2}\left[\partial^a\phi\partial^b\phi - \frac{1}{2}g^{ab}(\partial^c\phi\partial_c\phi + 2V(\phi))\right], \tag{8}$$

and we use a fact that $\delta_v\Gamma^a_{bc}$ can be regarded as a mixed tensor. Since we can take arbitrary variations which, together with their derivatives, vanishes at the boundary of Ω, we obtain equations of motion as

$$E_G^{ab} := -\frac{\sqrt{-g}}{2\kappa}\left(R^{ab} - \frac{1}{2}g^{ab}(R-2\Lambda) - 2\kappa T^{ab}\right) = 0, \tag{9}$$

$$E_\phi := \sqrt{-g}(\nabla_a\nabla^a\phi - V'(\phi)) = 0. \tag{10}$$

Note that we can add the total derivative term $\partial_a(\sqrt{-g}K^a)$ to the Lagrangian density L without changing equations of motion. Thus, there is an ambiguity for a choice of the Lagrangian density from which we can derive the above equations of motion. In our analysis, we exclusively use the above L, keeping this ambiguity in mind. In particular, we take the Einstein–Hilbert type for L_G.

We now consider a general coordinate transformation generated by ξ^a as

$$\begin{aligned} \delta x^a &:= (x')^a - x^a = \xi^a(x), \quad \delta\phi := \phi'(x') - \phi(x) = 0, \\ \delta g_{ab} &:= g'_{ab}(x') - g_{ab}(x) = -\xi^c{}_{,a}(x)g_{cb}(x) - \xi^c{}_{,b}(x)g_{ac}(x). \end{aligned} \tag{11}$$

Since δ does not commute with derivatives, we introduce the Lie derivative by ξ as

$$\bar{\delta}g_{ab} := \delta g_{ab} - g_{ab,c}\xi^c = -\nabla_a\xi_b - \nabla_b\xi_a, \quad \bar{\delta}\phi := \delta\phi - \xi^c\phi_{,c} = -\xi^c\nabla_c\phi, \tag{12}$$

which satisfies

$$\bar{\delta}(g_{ab,c\cdots}) = (\bar{\delta}g_{ab})_{,c\cdots}, \quad \bar{\delta}(\phi_{,c\cdots}) = (\bar{\delta}\phi)_{,c\cdots}. \tag{13}$$

A fact that an integration of the Lagrangian density over a d-dimensional domain Ω is invariant under the general coordinate transformation leads to

$$\begin{aligned}\delta S_\Omega &= \int_\Omega d^dx\,[\delta(L_G + L_M) + (L_G + L_M)\xi^a_{,a}] \\ &= \int_\Omega d^dx\,[\bar{\delta}(L_G + L_M) + \partial_a\{(L_G + L_M)\xi^a\}] = 0,\end{aligned} \tag{14}$$

where we employ

$$d^d(x + \delta x) = \det[\delta^a_b + (\delta x^a)_{,b}]d^dx \simeq (1 + \mathrm{tr}\,\xi^a_{,b})d^dx = (1 + \xi^a_{,a})d^dx, \tag{15}$$

$$\delta(L_G + L_M) = \bar{\delta}(L_G + L_M) + \xi^a\partial_a(L_G + L_M). \tag{16}$$

Using

$$\begin{aligned}\bar{\delta}(L_G + L_M) &= (E^{ab}_G\bar{\delta}g_{ab} + E_\phi\bar{\delta}\phi) + \partial_a \\ &\quad \times \left\{\frac{\sqrt{-g}}{2\kappa}(g^{bc}\bar{\delta}\Gamma^a_{bc} - g^{ab}\bar{\delta}\Gamma^c_{bc} - 2\kappa g^{ab}\partial_b\phi\bar{\delta}\phi)\right\}\end{aligned} \tag{17}$$

and

$$E^{ab}_G\bar{\delta}g_{ab} = \xi^c[2\partial_a(E^{ab}_G g_{bc}) - E^{ab}_G g_{ab,c}] - 2\partial_a(E^{ab}_G g_{bc}\xi^c), \tag{18}$$

we have

$$\delta S_\Omega = \int_\Omega d^dx\,\xi^c\left[2\partial_a(E^{ab}_G g_{bc}) - E^{ab}_G g_{ab,c} - E_\phi\nabla_c\phi\right] + \int_\Omega d^dx\,\partial_a J^a[\xi] = 0, \tag{19}$$

where

$$\begin{aligned}J^a[\xi] &= (L_G + L_M)\xi^a - 2E^{ab}_G g_{bc}\xi^c + \frac{\sqrt{-g}}{2\kappa}(g^{bc}\bar{\delta}\Gamma^a_{bc} - g^{ab}\bar{\delta}\Gamma^c_{bc} - 2\kappa g^{ab}\partial_b\phi\bar{\delta}\phi) \\ &= \frac{1}{2\kappa}\sqrt{-g}[2R^a{}_b\xi^b + g^{bc}\bar{\delta}\Gamma^a_{bc} - g^{ab}\bar{\delta}\Gamma^c_{bc}] = \frac{1}{2\kappa}\sqrt{-g}\nabla_b[\nabla^{[a}\xi^{b]}].\end{aligned} \tag{20}$$

To obtain the last line, we use $R^a{}_b\xi^b = -g^{ac}[\nabla_c, \nabla_b]\xi^b$,

$$g^{bc}\bar{\delta}\Gamma^a_{bc} = -g^{bc}\nabla_b\nabla_c\xi^a + g^{ac}[\nabla_c, \nabla_b]\xi^b, \quad g^{ab}\bar{\delta}\Gamma^c_{bc} = -g^{ab}\nabla_b\nabla_c\xi^c. \tag{21}$$

Since we can take an arbitrary vector field $\xi_a(x)$ which satisfies $\xi_a = \xi_{a,b} = \xi_{a,bc} = 0$ at $\partial\Omega$ (the boundary of the region Ω) as a general coordinate transformation, (19) implies

$$2\partial_a(E^{ab}_G g_{bc}) - E^{ab}_G g_{ab,c} - E_\phi\nabla_c\phi = 0, \tag{22}$$

for *off-shell* g_{ab} and ϕ, which give d constraints among the quantities E_G^{ab} and E_ϕ, which would vanish at on-shell, so that solutions to the equation of motion contain d undetermined free functions. In other words, (22) identically holds. Thus, a number of independent components for the symmetric tensor g_{ab} become $d(d+1)/2 - d = d(d-1)/2$, as is well known.

Furthermore, taking an arbitrary $\xi_a(x)$ without constraints on $\partial\Omega$, (19) with (22) leads to

$$\partial_a J^a[\xi] = 0, \tag{23}$$

where $J^a[\xi]$ includes the arbitrary vector ξ^a. Indeed, we can confirm that $\partial_a J^a[\xi] = 0$ holds identically using an explicit form of $J^a[\xi]$ in the last line of (20).

The current $J^a[\xi]$ is expanded as

$$J^a[\xi] = A^a{}_b \xi^b + B^a{}_b{}^c \xi^b_{,c} + C^a{}_b{}^{cd} \xi^b_{,cd}, \tag{24}$$

where

$$\begin{aligned} A^a{}_b &= \frac{\sqrt{-g}}{2\kappa}(2R^a{}_b + g^{ca}\Gamma^d_{db,c} - g^{cd}\Gamma^a_{cd,b}) \\ &= \frac{\sqrt{-g}}{2\kappa}[\partial_c(g^{d[a}\Gamma^{c]}_{db}) + \Gamma^e_{ec} g^{d[a}\Gamma^{c]}_{db}], \end{aligned} \tag{25}$$

$$B^a{}_b{}^c = \frac{\sqrt{-g}}{2\kappa}(g^{ac}\Gamma^d_{db} - 2g^{dc}\Gamma^a_{db} + g^{de}\delta^a_b\Gamma^c_{de}), \tag{26}$$

$$C^a{}_b{}^{cd} = \frac{\sqrt{-g}}{4\kappa}(g^{ac}\delta^d_b + g^{ad}\delta^c_b - 2g^{cd}\delta^a_b) = C^a{}_b{}^{dc}, \tag{27}$$

and (23) for an arbitrary ξ^a implies

$$\partial_a A^a{}_b = 0, \tag{28}$$

$$A^a{}_b + \partial_c B^c{}_b{}^a = 0, \tag{29}$$

$$B^a{}_b{}^c + B^c{}_b{}^a + 2\partial_d C^d{}_b{}^{ac} = 0, \tag{30}$$

$$C^a{}_b{}^{cd} + C^d{}_b{}^{ac} + C^c{}_b{}^{da} = 0. \tag{31}$$

Combining (29)–(31), we can generally write

$$A^a{}_b = -\frac{1}{2}\partial_c B^{[c}{}_b{}^{a]} - \frac{1}{2}\partial_c B^{\{c}{}_b{}^{a\}} = -\frac{1}{2}\partial_c B^{[c}{}_b{}^{a]} + \partial_c\partial_d C^d{}_b{}^{ac} = -\partial_c \tilde{B}^c{}_b{}^a, \tag{32}$$

where

$$\tilde{B}^c{}_b{}^a := \frac{1}{2}B^{[c}{}_b{}^{a]} - \frac{1}{3}\partial_d C^{[c}{}_b{}^{a]d}, \tag{33}$$

which is anti-symmetric under $a \leftrightarrow c$.[a]

[a]In classical particle mechanics, A, B, C and J all vanish. Therefore, these equations are trivially satisfied.[14]

We fully utilize the fact that the general coordinate transformation is generated by an arbitrary vector field $\xi^a(x)$ to obtain (22), (23) and (28)–(31), which are the consequence of Noether's second theorem.

There are two remarks. First of all, if we add the total derivative term $X := \partial_a(\sqrt{-g}K^a)$ to L, its variation under δ becomes (see (14))

$$\int_\Omega d^d x\, [\bar{\delta} X + \partial_a(X\xi^a)] = \int_\Omega d^d x\, \partial_a[\bar{\delta}(\sqrt{-g}K^a) + X\xi^a], \tag{34}$$

which leads to a shift of $J^a[\xi]$ as

$$J^a[\xi] \to J^a[\xi] + \sqrt{-g}[\xi^{[a}\nabla_b K^{b]} - K^{[a}\nabla_b \xi^{b]}], \tag{35}$$

where we use

$$\bar{\delta} K^a = K^b \nabla_b \xi^a - \xi^b \nabla_b K^a, \quad \bar{\delta}\sqrt{-g} = -\sqrt{-g}\nabla_b \xi^b. \tag{36}$$

Second, even though we can take $\xi^a(x) = \xi_0^a$ with a constant vector ξ_0^a, we still have Noether's second theorem, so that the current associated with this symmetry is always conserved without using equations of motion.

Using (23) and (28), we can define two types of conserved charges, one is covariant, the other is noncovariant, which will be explained below. Their conservation, however, is an *identity* implied by the general coordinate transformation, and holds without using equations of motion.

2.2. *Noncovariant conserved charge from Noether's second theorem: Pseudo-tensor*

The noncovariant *off-shell* conserved current density is given by

$$A^a{}_b := \frac{\sqrt{-g}}{2\kappa}(2R^a{}_b + g^{ac}\Gamma^d_{db,c} - g^{cd}\Gamma^a_{cd,b}), \tag{37}$$

and the conservation law $\partial_a A^a{}_b = 0$ implies

$$\begin{aligned} 0 &= \int_M d^d x\, \partial_a A^a{}_b \\ &= \int_{\Sigma_2} (d^{d-1}x)_a\, A^a{}_b - \int_{\Sigma_1} (d^{d-1}x)_a\, A^a{}_b + \int_{\partial M_s} (d^{d-1}x)_a\, A^a{}_b, \end{aligned} \tag{38}$$

where M is the d-dimensional space–time whose boundary consists of $\partial M = \Sigma_1 \oplus \partial M_s \oplus \Sigma_2$. Here, Σ_1 and Σ_2 are past- and future-directed space-like surfaces, respectively, and ∂M_s is a time-like boundary of M. If $\int_{\partial M_s} (d^{d-1}x)_a\, A^a{}_b = 0$, we can define a conserved charge as

$$Q_{\mathrm{pseudo},b} = \int_\Sigma (d^{d-1}x)_a\, A^a{}_b, \tag{39}$$

since it does not depend on a choice of space-like surfaces $\Sigma_{1,2}$. We call $Q_{\mathrm{pseudo},b}$ the noncovariant conserved charge, since $A^a{}_b$ is not covariant under the general coordinate transformation.[b] Furthermore, (32) leads to a quasi-local expression of

[b]It is only covariant under affine transformation that $\xi^a(x) := m^a{}_b x^b - b^a$.

$Q_{\text{pseudo},b}$ as

$$Q_{\text{pseudo},b} = -\int_{\partial\Sigma} (d^{d-2}x)_{ac}\, \tilde{B}^c{}_b{}^a, \tag{40}$$

where the boundary of Σ is denoted by a spatial surface $\partial\Sigma$.

As already noted before, the conservation of $Q_{\text{pseudo},b}$ is an identity, which is not a consequence from the dynamics of general relativity, since equations of motion are not required to show it. In addition, if the equation of motion for g_{ab} ($E_G^{ab} = 0$) is used, $A^a{}_b$ becomes

$$\begin{aligned} A^a{}_b &= \sqrt{-g}(T^a{}_b + t^a{}_b), \\ t^a{}_b &:= \frac{1}{2\kappa}\left[R^a{}_b + \frac{R - 2\Lambda}{2}\delta^a_b + g^{ca}\Gamma^d_{db,c} - g^{cd}\Gamma^a_{cd,b}\right], \end{aligned} \tag{41}$$

where $t^a{}_b$ is not covariant due to the last two terms. In the case of the vanishing cosmological constant, by adding an appropriate total divergent term $\partial_\mu(\sqrt{-g}K^a)$ to the total Lagrangian density, $t^a{}_b$ can be transformed to Einstein's gravitational pseudo-tensor, which was claimed to represent the gravitational contribution. A distinction between matter and gravitational field, however, seems ambiguous, since $R^a{}_b$ and R in $t^a{}_b$ are also expressed in terms of T and $T^a{}_b$.

Using (41) for $b = 0$, one may define the conserved energy as

$$E_{\text{pseudo}} = -\int_\Sigma [d^{d-1}x]_a \sqrt{-g}(T^a{}_0 + t^a{}_0) \left(= \int_{\partial\Sigma} [d^{d-2}x]_{ac}\, \tilde{B}^c{}_0{}^a\right), \tag{42}$$

where a minus sign is introduced for E_{pseudo} to match the standard definition of the energy. While Einstein interpreted the second contribution from his pseudo-tensor $t^0{}_0$ as the energy of the gravitational field, it depends on a choice of the coordinates due to its noncovariance, and it sometimes diverges.

2.3. *Covariant conserved charge from Noether's second theorem: Komar integral*

The second type of the conserved current is given by J^a itself as

$$J^a[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_b[\nabla^{[a}\xi^{b]}], \tag{43}$$

which satisfies $\partial_a J^a[\xi] = 0$ for an arbitrary vector ξ^b. Then one may define the covariantly conserved charge as

$$Q_{\text{Komar}}[\xi] := \int_\Sigma [d^{d-1}x]_a\, J^a[\xi] = \frac{1}{2\kappa}\int_\Sigma [d^{d-1}x]_a \sqrt{-g}\nabla_b[\nabla^{[a}\xi^{b]}] \tag{44}$$

$$= \frac{1}{2\kappa}\int_{\partial\Sigma} [d^{d-2}x]_{ab} \sqrt{-g}\nabla^{[a}\xi^{b]}, \tag{45}$$

where the second line is a quasi-local expression. We call this charge the Komar integral, since the expression is identical to the one introduced by Komar.[15] This charge is conserved not only for an arbitrary metric g_{ab} but also for an arbitrary vector ξ^b. Thus, one may define various different charges depending on a choice of ξ^b. We introduce several such charges used in literature.

2.3.1. *Komar energy*

If the space–time allows a time-like Killing vector ξ_K^a, one may define the energy as a charge associated with the Killing vector as $E_{\text{Komar}} = Q_{\text{Komar}}[\xi_K]$, which we call Komar "energy". Explicitly

$$E_{\text{Komar}} = \frac{1}{\kappa}\int_\Sigma [d^{d-1}x]_a \sqrt{-g} R^a{}_b \xi_K^b \tag{46}$$

$$= \frac{1}{\kappa}\int_\Sigma [d^{d-1}x]_a \sqrt{-g}\left[2\kappa\left(T^a{}_b\xi_K^b - \frac{T\xi_K^a}{d-2}\right) + \frac{2\Lambda\xi_K^a}{d-2}\right], \tag{47}$$

where we use the equations of motion to obtain the second line, which shows that the Komar "energy" does not lead to the standard definition of the energy in the limit of the flat space–time. A time-like Killing vector is given by $\xi_K^a = -\delta_0^a$ for the stationary space–time, for example, where the metric g_{ab} does not depend on the time coordinate x^0. Since $\xi_K^a = -\delta_0^a$ is constant, the Komar energy coincides with the energy from the pseudo-tensor by definition: $E_{\text{Komar}} = E_{\text{pseudo}}$. Note that the Komar "energy" E_{Komar} is always conserved as a consequence of Noether's second theorem, even though ξ_K^a is not a Killing vector for a generic (nonstationary) space–time.

2.3.2. *Wald entropy*

It has been proposed to define the black hole entropy,[16] by choosing $\xi^a = t^a + \Omega_H\varphi^a$, where t^a is the stationary Killing field, φ^a is the axial Killing field and Ω_H is the angular velocity of the horizon. In Ref. 16, it is concluded that $\partial_a J^a[\xi] = 0$ holds *when* the equations of motion are satisfied. This statement is misleading, however, since a full power of Noether's second theorem was not employed to derive $\partial_a J^a[\xi] = 0$ in Ref. 16. As we have frequently mentioned, $\partial_a J^a[\xi] = 0$ can be derived from Noether's second theorem for an arbitrary ξ^b *without* using equations of motion or g_{ab} and matters.

2.3.3. *Asymptotically flat space–time*: *ADM energy*

An asymptotically flat space–time is defined as a space–time whose metric satisfies the vacuum Einstein equation without cosmological constant at $x^2 \to +\infty$ (large space-like separation). In this case, the conserved energy is defined in Cartesian coordinate as[2]

$$E_{\text{ADM}} := \frac{1}{4\kappa}\int_{+\infty} [d^{d-2}x]_{0i}\,(\partial_j h_{ij} - \partial_i h_{jj}), \quad h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}, \tag{48}$$

which is called as the ADM energy (or mass), where i, j run from 1 to $d-1$, $\eta_{\mu\nu}$ is the flat Minkowski metric, and $\int_{+\infty}$ means that the integral is evaluated at $x^2 \to +\infty$.

The ADM energy can be written in a covariant manner as[17]

$$E_{\rm ADM} = \frac{1}{4\kappa}\int_{+\infty} [d^{d-2}x]_{ab}\,\sqrt{-g}\nabla^{[a}\eta^{b]} = \frac{1}{2}Q_{\rm Komar}[\eta], \tag{49}$$

where η^a is an asymptotic time-like Killing vector and satisfies $\nabla_a\eta_b + \nabla_b\eta_a = 0$ at $x^2 \to +\infty$. Since there are many asymptotic Killing vectors, we identify a vector η with another η' if there exists a vector $v_a = \eta_a - \eta'_a$ which vanishes at $x^2 \to +\infty$. Clearly, $Q_{\rm Komar}[\eta] = Q_{\rm Komar}[\eta']$. Under this identification, a collection of all independent asymptotic Killing vectors η generate the isometry of the Minkowski space–time, so that a number of independent vectors are $d(d+1)/2$ (translation and Lorentz transformation). Thus, the ADM energy is regarded as a conserved energy associated with the asymptotic time translation η in the asymptotically flat space–time. Since the ADM energy is (a half of) the Komar integral, we can write

$$E_{\rm ADM} = \frac{1}{4\kappa}\int_{\Sigma_\infty} [d^{d-1}x]_a\,\sqrt{-g}\nabla_b[\nabla^{[a}\eta^{b]}], \tag{50}$$

where Σ_∞ is a space-like surface whose boundary is given by $x^2 \to +\infty$.

2.3.4. *Asymptotically dS/AdS space–time*

As in the case of the asymptotically flat space–time, we define the asymptotically dS or AdS space–time as the space–time whose metric satisfies the vacuum Einstein equation with cosmological constant, $G_{ab} + \Lambda g_{ab} = 0$ at $x^2 \to \infty$. We then regard the isometry of the dS/AdS space–time as a (representative of) asymptotic Killing vectors of this space–time. The isometry of the dS is $\mathrm{SO}(1,d)$, while that of the AdS is $\mathrm{SO}(2,d)$. Since it is possible to make the metric g_{ab} static, the Killing vector η for the time translation always exists. Thus, the energy in these asymptotic space–times is defined using the asymptotic Killing vector η as $E^{\rm as}_{\rm dS/AdS} = Q_{\rm Komar}[\eta]$.

2.4. *Cautions on charges from Noether's second theorem*

As we have already mentioned frequently, Noether's second theorem tells that currents associated with local symmetries are always conserved *without* using equations of motion of dynamical variable. Thus conserved currents and conserved charges do not reflect dynamical properties of the system. Rather they are consequences of constraints (22) for Einstein gravity among the quantities E_G^{ab} and E_ϕ, each of which would vanish at on-shell. Therefore, it does not seem reasonable to define energy in general relativity by either pseudo-tensor or Komar integral including the ADM energy or asymptotic charges. Indeed Noether calls the conservation law from her second theorem *improper*, referring statements by Hilbert and Klein.[12]

In addition, both pseudo-tensor and Komar integral are easily modified by an arbitrary total divergence term, which can be added without changing equations of motion, so that they are not unique. Furthermore, the pseudo-tensor depends on the choice of the coordinate as it is not covariant under general coordinate transformation. The Komar integral, on the other hand, is conserved for an arbitrary vector ξ^a, so that it may depend on a choice of ξ^a.

One may argue to define a physical Noether charge by regarding the local transformation restricted to constant parameters as the "global" transformation. However, this does not work except Quantum Electrodynamics (QED), since the conservation of Noether's charge associated with the "global" transformation is still a part of constraints implied by the local transformation. QED is somewhat special, since the charge can be defined from the matter current, which is U(1) gauge invariant.

In the following section, we introduce our proposal for a proper and covariant definition of charges in general relativity, which are conserved only after equations of motion for gravity and matters are satisfied. We consider several examples in order to compare our definition with those from Noether's second theorem.

3. Our Physical Definition Versus Noether's Second Theorem in General Relativity

In this section, we first explain our recent proposal for the covariant definition of the energy and its generalization in general relativity.[10,11] We then compare our definition with those derived from Noether's second theorem in the previous section for various examples with explicit calculations.

3.1. *Our proposal for conserved non-Noether charge*

We first summarize our proposal to define a conserved charge in general relativity.[10,11] We start with the Einstein equation given by

$$G_{ab} + \Lambda g_{ab} = 2\kappa T_{ab}, \tag{51}$$

where the EMT T_{ab} should be covariantly conserved, $\nabla_a T^a{}_b = 0$, as a consequence of equations of motion for matters, since the left-hand side identically vanishes after applying ∇_a due to the Bianchi identity.

Since $\nabla_a T^a{}_b(x) = 0$ does not give a conserved charge, however, we introduce a special vector ζ^a which satisfies

$$T^a{}_b(x)\nabla_a \zeta^b(x) = 0. \tag{52}$$

We then define a new conserved current density $\sqrt{-g}T^a{}_b(x)\zeta^b(x)$, which indeed satisfies the standard conservation law as

$$\partial_a\big(\sqrt{-g}T^a{}_b(x)\zeta^b(x)\big) = \sqrt{-g}\nabla_a(T^a{}_b(x)\zeta^b(x)) = \sqrt{-g}T^a{}_b(x)\nabla_a\zeta^b(x) = 0. \tag{53}$$

We thus call (52) the conservation condition. In Refs. 10 and 11, we have shown an existence of ζ^a and discussed how to construct it.

A new conserved charge is easily constructed as

$$Q[\zeta] = \int_{\Sigma} [d^{d-1}x]_a \sqrt{-g}\, T^a{}_b(x)\, \zeta^b(x), \tag{54}$$

for a space-like surface Σ, which is manifestly covariant under general coordinate transformations. With a similar argument as discussed for $A^a{}_b$ around (38), it is easy to show that $Q[\zeta]$ is conserved (i.e. it does not depend on a choice of the space-like surface Σ).

Using the conserved charge $Q[\zeta]$, we define the energy and its generalization in general relativity. There are three distinct cases for a choice of ζ, which will be explained with explicit examples in the following sections. We will also make comparisons with other definitions of the energy from Noether's second theorem in the previous section.

3.2. *Energy conservation by symmetry*

If the metric, which is a solution to the Einstein equation (51), is invariant under the time translation, then the (time-like) Killing vector ξ^a, defined by $\nabla_a \xi_b + \nabla_b \xi_a = 0$, exists. Since $T_{ab} = T_{ba}$, it is easy to see that $\zeta^a = \xi^a$ satisfies (52). If the metric does not contain a time coordinate x^0, the Killing vector is given by $\xi^a = -\delta^a_0$ in such a coordinate. Thus, the conserved energy is defined by[10]

$$E := Q(\zeta^a = -\delta^a_0) = -\int_{\Sigma} [d^{d-1}x]_a \sqrt{-g}\, T^a{}_0 = -\int_{\Sigma_0} [d^{d-1}x]_0 \sqrt{-g}\, T^0{}_0, \tag{55}$$

and the conservation is a consequence of the global time translational invariance of the on-shell metric, the solution to the Einstein equation, but is not a consequence implied by the local symmetry of the theory assumed in Noether's second theorem.[c] In the second equality, we present an expression for a constant x^0 space-like surface Σ_0, where $[d^{d-1}x^0]_0 := dx^1 dx^2 \cdots dx^{d-1}$.

3.2.1. *Vacuum energy*

As a warmup, we consider a vacuum described by

$$ds^2 = -f(r)(dx^0)^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{d-2}, \quad f(r) = 1 - \frac{2\Lambda r^2}{(d-2)(d-1)}. \tag{56}$$

As already mentioned, the time-like Killing vector is given by $\xi^a = -\delta^a_0$, though it becomes space-like beyond the cosmological horizon $r > r_H = \sqrt{\frac{(d-2)(d-1)}{2\Lambda}}$ for the positive cosmological constant Λ (dS space–time). By definition, the energy of the

[c] While (55) is not a Noether charge in the general relativity where g_{ab} is dynamical, this energy may be regarded as a conserve charge of Noether's first theorem associated with the isometry for a *fixed background* metric.

vacuum is zero for our definition, $E^{\rm vac}_{\rm our} = 0$, while energies from Noether's second theorem become

$$E^{\rm vac}_{\rm pseudo} = E^{\rm vac}_{\rm Komar} = -\frac{2\Lambda\Omega_{d-2}}{(d-2)\kappa}\int r^{d-2}dr, \quad \Omega_{d-2} := \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{57}$$

Thus, we have

$$E^{\rm vac}_{\rm our} = E^{\rm vac}_{\rm pseudo} = E^{\rm vac}_{\rm Komar} = E^{\rm vac}_{\rm ADM} = 0 \tag{58}$$

for a flat space–time, while

$$\begin{aligned} E^{\rm vac}_{\rm our} = 0, \quad E^{\rm vac}_{\rm pseudo} &= E^{\rm vac}_{\rm Komar} = E^{\rm vac}_{\rm dS/AdS} \\ &= -\frac{2\Lambda\Omega_{d-2}}{(d-2)\kappa}\int r^{d-2}dr \to -\Lambda\times\infty \end{aligned} \tag{59}$$

for nonzero cosmological constant, where the divergence comes from the divergent r integral.

3.2.2. *Schwarzschild black hole*

As a nontrivial example, we consider the Schwarzschild black hole in d dimensions, whose metric is given by

$$ds^2 = -(1+u)d\tau^2 - 2ud\tau dr + (1-u)dr^2 + r^2 d\Omega^2_{d-2} \tag{60}$$

in the Eddington–Finkelstein coordinates, where

$$u := \delta u - \frac{2\Lambda r^2}{(d-2)(d-1)}, \quad \delta u := -\left(\frac{r_g}{r}\right)^{d-3}, \quad r_g^{d-3} := 2GM\theta(r), \tag{61}$$

r_g is the black hole horizon for a case with $\Lambda = 0$, and M is the mass of the black hole. Here, we introduce the step function $\theta(r)$ with $\theta(0) = 0$ to properly treat the singularity at $r = 0$ in the distributional sense. Note that we can replace $\theta(r)$ with other regularizations without changing discussions below.[18]

The constant τ surface is normal to

$$n_a = -(1-u)^{-1/2}\delta^\tau_a, \quad n_a n^a = -1, \tag{62}$$

thus the constant τ surface is always space-like even inside the horizon except in the large r region that $1-u<0$ for the negative Λ (AdS space–time). We illustrate the constant τ surface in the Kruskal–Szekeres like coordinates for $d = 4$ and $\Lambda = 0$ in Fig. 1, where the metric becomes

$$ds^2 = -\frac{4r_g^3 e^{-r/r_g}}{r}(dT^2 - dX^2) + r^2 d\Omega^2, \tag{63}$$

$$\begin{aligned} X &= e^{\frac{r}{2r_g}}\left[\sinh\left(\frac{\tau}{2r_g}\right) + e^{-\frac{\tau}{2r_g}}\frac{r}{2r_g}\right], \\ T &= e^{\frac{r}{2r_g}}\left[\cosh\left(\frac{\tau}{2r_g}\right) - e^{-\frac{\tau}{2r_g}}\frac{r}{2r_g}\right] \end{aligned} \tag{64}$$

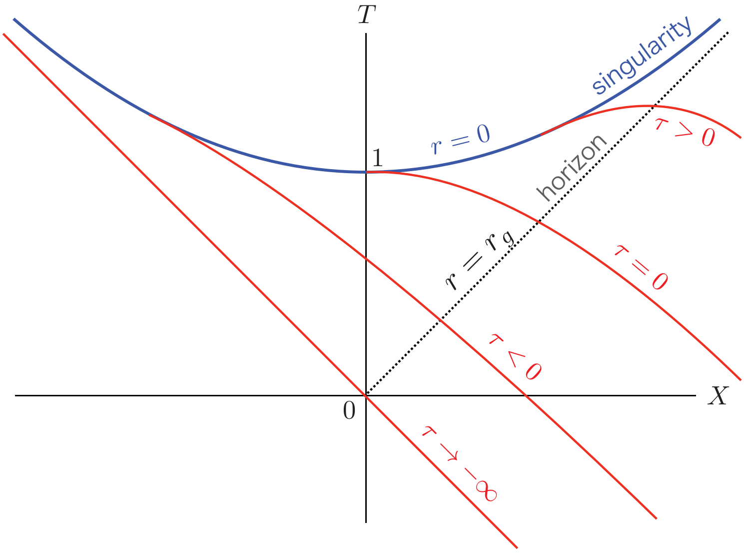

Fig. 1. (Color online) The Schwarzschild black hole in the Kruskal–Szekeres like coordinate. A blue curve defined by $T = \sqrt{1+X^2}$ represents a black hole singularity at $r = 0$ in the Eddington–Finkelstein coordinates, while the dotted black line given by $T = X$ is the horizon as $r = r_g$. Red curves are constant τ surface in the Eddington–Finkelstein coordinates at $\tau \to -\infty$, $\tau < 0$, $\tau = 0$ and $\tau > 0$, respectively. The physical region exists above the surface at $\tau \to -\infty$ and below the singularity surface at $r = 0$.

and

$$\left.\frac{dT}{dX}\right|_\tau = \frac{2r_g \cosh\left(\frac{\tau}{2r_g}\right) - e^{-\frac{\tau}{2r_g}}(r+2r_g)}{2r_g \sinh\left(\frac{\tau}{2r_g}\right) + e^{-\frac{\tau}{2r_g}}(r+2r_g)} \tag{65}$$

for a fixed τ. Toward the singularity ($r \to 0$), the coordinates behave as

$$X \to \sinh\left(\frac{\tau}{2r_g}\right), \quad T \to \cosh\left(\frac{\tau}{2r_g}\right), \quad \left.\frac{dT}{dX}\right|_\tau \to \tanh\left(\frac{\tau}{2r_g}\right), \tag{66}$$

while at horizon ($r = r_g$), they become

$$X = T = \frac{\sqrt{e}}{2} e^{\frac{\tau}{2r_g}}, \quad \left.\frac{dT}{dX}\right|_\tau = \frac{2\sinh\left(\frac{\tau}{2r_g}\right) - e^{-\frac{\tau}{2r_g}}}{2\cosh\left(\frac{\tau}{2r_g}\right) + e^{-\frac{\tau}{2r_g}}}, \tag{67}$$

and at $r \to \infty$, they approach

$$X \to \frac{r}{2r_g} e^{\frac{(r-\tau)}{2r_g}}, \quad T \to -\frac{r}{2r_g} e^{\frac{(r-\tau)}{2r_g}}, \quad \left.\frac{dT}{dX}\right|_\tau \to -1. \tag{68}$$

The relevant component of the EMT is given by[10]

$$T^\tau{}_\tau = \frac{d-2}{4\kappa} \frac{\partial_r (r^{d-3}\delta u)}{r^{d-2}} = -\frac{(d-2)M}{8\pi} \frac{\delta(r)}{r^{d-2}}, \tag{69}$$

whose second expression agrees with the expression for the EMT by other regularizations in the distributional approach.[18] Contrary to the general argument,[19] the EMT is well defined in the distributional sense, since it does not contain ill-defined products of two distributions. The energy is evaluated by the integral of this EMT over the $(d-1)$-dimensional constant τ surface (red curves in Fig. 1 for $d=4$) with the Killing vector[d] $\xi^a = -\delta^a_\tau$ as

$$E^{\rm BH}_{\rm our} = \int d^{d-1}x\sqrt{-g}T^\tau{}_a(-\delta^a_\tau) = \frac{(d-2)M}{8\pi}\Omega_{d-2}\int dr\partial_r\theta(r)$$
$$= \frac{(d-2)\Omega_{d-2}}{8\pi}M[\theta(\infty)-\theta(0)] = \frac{(d-2)\Omega_{d-2}}{8\pi}M, \qquad (70)$$

which exactly gives a mass of the black hole at $d=4$. While we here simply integrate $\partial_r\theta(r)$ over r, a direct use of $\delta(r)$ leads to the same result, showing a correctness of the distributional approach as well as a famous relation $\partial_r\theta(r) = \delta(r)$.

We now consider the black hole energies from Noether's second theorem. Since we take the constant $\xi^a = -\delta^a_\tau$ in the case of the Schwarzschild black hole, the energy from the pseudo-tensor agrees with the Komar energy. In addition, the result by the "volume" integral with the delta function agrees with the one by the "surface integral" without requiring a specific asymptotic behavior. Explicitly

$$E^{\rm BH}_{\rm pseudo} = E^{\rm BH}_{\rm Komar}$$
$$= \frac{1}{\kappa}\int d\Omega_{d-2}\int dr\, r^{d-2}R^\tau{}_\tau\xi^\tau = \frac{1}{2\kappa}\int d\Omega_{d-2}\, r^{d-2}\nabla^{[\tau}\xi^{r]}$$
$$= \Omega_{d-2}\left[\frac{(d-3)M}{4\pi} - \frac{2\Lambda r^{d-1}}{\kappa(d-2)(d-1)}\right]$$
$$= \frac{(d-3)\Omega_{d-2}}{4\pi}M + E^{\rm vac}_{\rm Komar}. \qquad (71)$$

Thus, $E^{\rm BH}_{\rm pseudo/dS/AdS}$ diverges for $\Lambda \neq 0$, while we can define the finite energy by subtracting the "vacuum" contribution as

$$\Delta E^{\rm BH}_{\rm 2nd} := E^{\rm BH}_{\rm 2nd} - E^{\rm vac}_{\rm 2nd} = \frac{(d-3)\Omega_{d-2}}{4\pi}M, \qquad (72)$$

where the word "2nd" represents the pseudo-tensor energy and the Komar energy including the ADM energy and the asymptotically dS/AdS energy.

If we compare (70) with (72), we have

$$\frac{\Delta E^{\rm BH}_{\rm 2nd}}{E^{\rm BH}_{\rm our}} = \frac{2(d-3)}{d-2}, \qquad (73)$$

which becomes unity only at $d=4$. Thus, the covariant definition of the black hole energy in our proposal is in general different from "energies" defined from Noether's

[d] Although constant τ surfaces are space like, the Killing vector ξ^μ is time-like outside the horizon $(r > r_g)$ but space-like inside the horizon $(r < r_g)$ for $\Lambda = 0$. In the case of nonzero cosmological constant, the situation is similar but more complicated.

second theorem, even after subtractions of the divergent vacuum contribution necessary for $\Lambda \neq 0$, though the difference appears only in the normalization. A more distinct difference between the two definitions appears in the case of energies for a compact star.[10]

3.3. *Energy conservation without symmetry*

We next consider a case without Killing vector for the time translation. Even in such a case where $\xi^a = -\delta^a_0$ is not a Killing vector anymore, the energy defined by (55) is time independent if the EMT and the metric satisfy

$$T^a{}_b \nabla_a \xi^b = -T^a{}_b \Gamma^b_{a0} = 0. \tag{74}$$

In this case, the energy E is conserved but the conservation is NOT even a consequence of the global time translational invariance.

3.3.1. *Gravitational collapse*

Let us consider a simple model of gravitational collapses for thick light shells,[20] whose metric in the Eddington–Finkelstein coordinate is given by

$$g_{ab}dx^a dx^b = -(1+u)d\tau^2 - 2u d\tau dr + (1-u)dr^2 + r^2 d\Omega^2_{d-2}, \tag{75}$$

where $x^0 = \tau$, and

$$u(r,\tau) := -\frac{m(r,\tau)}{r^{d-3}} - \frac{2\Lambda r^2}{(d-2)(d-1)}, \tag{76}$$

$$m(r,\tau) := \begin{cases} 2GM\theta(r), & \tau + r > \Delta, \quad \text{I}, \\ 2GM\theta(r) F\left(\dfrac{\tau + r}{\Delta}\right), & 0 \le \tau + r \ge \Delta, \quad \text{II}, \\ 0, & \tau + r < \Delta, \quad \text{III}, \end{cases} \tag{77}$$

where a monotonically increasing function $F(x)$ satisfies $F(0) = 0$ and $F(1) = 1$. The vector $\xi^a = -\delta^a_0$ is NOT a Killing vector due to an existence of the light shell region (II), while it becomes the Killing vector in Schwarzschild (I) and Minkowski (III) regions. See Fig. 2 (left), where solid lines represent infalling lights which reach the origin at $\tau = 0, \tau_0, \Delta$.

Since the metric in (75) gives[11]

$$\Gamma^0_{00} = \frac{1+u}{2} u_\tau - \frac{u}{2} u_r = -\Gamma^r_{r0},$$

$$\Gamma^0_{r0} = \frac{u}{2} u_\tau + \frac{1-u}{2} u_r, \quad \Gamma^r_{00} = -\frac{2+u}{2} u_\tau + \frac{1+u}{2} u_r,$$

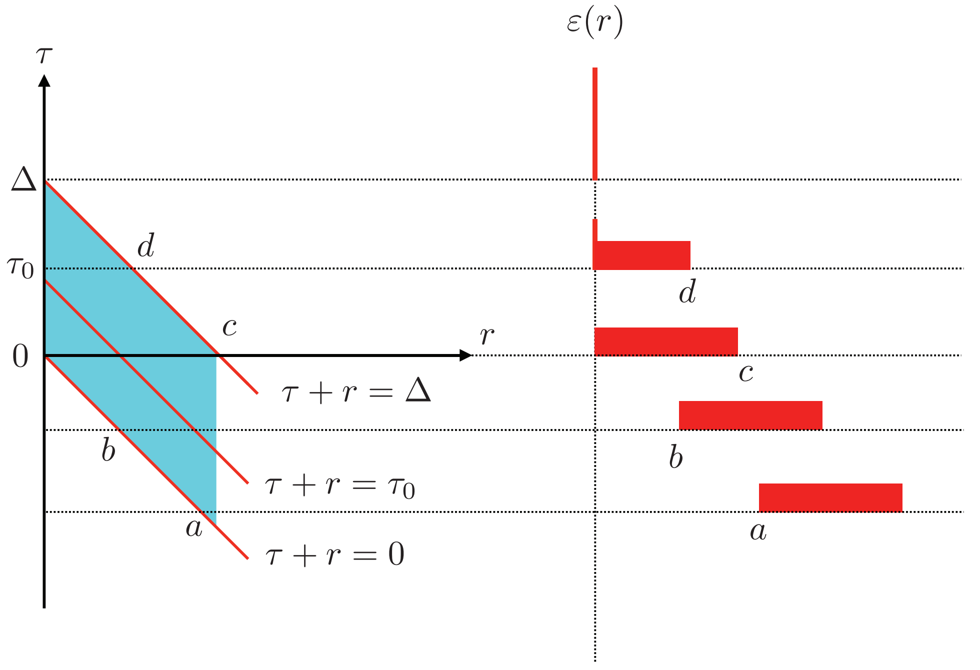

Fig. 2. (Left) Gravitational collapse of thick light shells in the Eddington–Finkelstein coordinate. Solids lines represent infalling lights which reach the origin at $\tau = 0, \tau_0, \Delta$. (Right) The local energy density $\varepsilon(r)$ as a function of r at various τ. Here, we consider $F(x) = x$ case as a simplest example, and $\varepsilon(r)$ has δ function contribution at $r = 0$, represented by a thick vertical line.

and

$$\begin{aligned} T^0{}_0 &= \frac{(d-2)}{4\kappa}\frac{(r^{d-3}\delta u)_r}{r^{d-2}}, \quad T^r{}_r = \frac{(d-2)}{4\kappa}\left[\frac{(r^{d-3}\delta u)_r}{r^{d-2}} - \frac{2(\delta u)_\tau}{r}\right], \\ T^0{}_r &= \frac{(d-2)}{4\kappa}\frac{(\delta u)_\tau}{r} = -T^r{}_0, \quad \delta u := -\frac{m(r,\tau)}{r^{d-3}}, \end{aligned} \tag{78}$$

the condition (74) is satisfied for $\xi^a = -\delta^a_0$ as

$$T^a{}_b\Gamma^b_{a0} = (T^0{}_0 - T^r{}_r)\Gamma^0_{00} + T^0{}_r(\Gamma^r_{00} - \Gamma^0_{r0}) = 0. \tag{79}$$

In this system, the energy (55) is calculated as

$$E(\tau) = -\int d^{d-1}x\sqrt{-g}T^0{}_0 = \frac{(d-2)\Omega_{d-2}}{16\pi G}\int_0^\infty dr\,[m(r,\tau)]_r. \tag{80}$$

For $\tau < 0$ (before the collapse without a black hole), (80) is evaluated as

$$E(\tau) = \frac{(d-2)M\Omega_{d-2}}{16\pi G}\int_{-\tau}^{\Delta-\tau} dr\,\partial_r(\theta F) = \frac{(d-2)M\Omega_{d-2}}{8\pi} := E_{\text{tot}}. \tag{81}$$

For $0 \le \tau \le \Delta$ (during the collapse with a growing black hole), we obtain

$$E(\tau) = E_{\text{tot}}\int_0^{\Delta-\tau} dr\,\partial_r(\theta F) = E_{\text{tot}}\left[1 - \theta(0)F\left(\frac{\tau}{\Delta}\right)\right] = E_{\text{tot}}, \tag{82}$$

which can be evaluated differently using $\partial_r(\theta F) = \delta(r)F + \partial_r F$ as

$$E(\tau) = E_{\rm tot}\left[F\left(\frac{\tau}{\Delta}\right) + \left\{F(1) - F\left(\frac{\tau}{\Delta}\right)\right\}\right] = E_{\rm tot}, \tag{83}$$

where the first term represents a mass of a growing black hole while the second one is an energy of remaining light shells.

Finally for $\tau > \Delta$ (after the collapse with the final black hole), we evaluate the total energy as

$$E(\tau) = E_{\rm tot}\int_0^\infty dr\delta(r) = E_{\rm tot}, \tag{84}$$

which agrees with the mass of the final black hole.

The total energy is conserved as $E(\tau) = E_{\rm tot}$, and we plot typical distributions of the local energy density in Fig. 2 (right).

Other examples have also been discussed in Ref. 11, and gravitational collapses for more general EMTs have been investigated recently in Ref. 21.

3.3.2. *Comparison with energies in Noether's second theorem*

Since $\xi^a = -\delta^a_0$ is constant, the energy from the pseudo-tensor and the Komar energy agree. We thus obtain

$$\begin{aligned} E_{\rm pseudo} &= E_{\rm Komar} \\ &= \frac{\Omega_{d-2}}{2\kappa}\int dr\,\partial_r[r^{d-2}(u_r - u_\tau)] = \frac{\Omega_{d-2}}{2\kappa} r^{d-2}(u_r - u_\tau)\Big|_{r_0}^{r_1}, \end{aligned} \tag{85}$$

where $r_1 = \Delta - \tau$ and $r_0 = -\tau$ for $\tau < 0$, $r_1 = \Delta - \tau$ and $r_0 = 0$ for $0 < \tau < \Delta$, and $u_\tau = 0$ with $r_1 = \infty$ and $r_0 = 0$ for $\tau > \Delta$. We thus obtain

$$E_{\rm 2nd} := E_{\rm pseudo} = E_{\rm Komar} = \frac{(d-3)\Omega_{d-2}}{4\pi}M + E^{\rm vac}_{\rm Komar}, \tag{86}$$

which again gives

$$\frac{E_{\rm 2nd} - E^{\rm vac}_{\rm 2nd}}{E_{\rm our}} = \frac{2(d-3)}{d-4}. \tag{87}$$

3.4. *Conserved charge in the absence of energy conservation*

We finally consider the most general cases, where the Killing vector for time translation is absent and (74) for the constant vector $\xi^a = -\delta^a_0$ is not satisfied. To define a conserved charge, which is a generalization of the energy, we must solve (52) for $\zeta^a(x) = \beta(x)n^a(x)$ with $n^a(x) = \frac{dx^a(\eta)}{d\eta}$ where η is a parameter to characterize the time evolution of space-like surfaces Σ_η. (If we choose η to be the global time x^0, we have $\zeta^a(x) = \beta(x)\delta^a_0$.) As discussed in Ref. 11, a solution to (52) always exists[e]

[e]The existence of such a vector field for a spherically symmetric gravitational system, known as the Kodama vector, was pointed out in Ref. 26.

and is unique once an initial condition for $\beta(x)$ is given at some $\eta = \eta_0$. Thus, using this ζ^a, we can always define a conserved charge (54), which is a generalization of the energy in general relativity.

3.4.1. *Expanding universe*

As an example, we consider a model of homogeneous and isotropic expanding universe in Einstein gravity with a cosmological constant Λ, described by the d-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) metric,[22–25]

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j, \tag{88}$$

where $a(x^0)$ is the scale factor dependent only on time x^0, and the $(d-1)$-dimensional Riemann tensor and the Ricci tensor for $\tilde{g}_{ij}$ becomes

$$\tilde{R}_{ik}{}^{jl} = k\delta^j_{[i}\delta^l_{k]}, \quad \tilde{R}_i{}^j = k(d-2)\delta^i_j, \tag{89}$$

with $k >= 1$ (sphere), 0 (flat space), -1 (hyperbolic space).

The EMT is given by the perfect fluid as

$$T^0{}_0 = -\rho(x^0), \quad T^i{}_j = P(x^0)\delta^i_j, \quad T^0{}_j = T^i{}_0 = 0, \tag{90}$$

where $\nabla_a T^a{}_b = 0$ implies

$$\dot{\rho} + (d-1)(\rho + P)\frac{\dot{a}}{a} = 0, \quad \dot{\rho} := \partial_0\rho,\ \dot{a} := \partial_0 a, \tag{91}$$

while the Einstein equation leads to

$$\begin{aligned} 8\pi G\rho &= \frac{(d-1)(d-2)}{2}\frac{(k+\dot{a}^2)}{a^2} - \Lambda, \\ 8\pi GP &= (2-d)\left[\frac{\ddot{a}}{a} + \frac{(d-3)}{2}\frac{(k+\dot{a}^2)}{a^2}\right] + \Lambda. \end{aligned} \tag{92}$$

In this case, the energy is given by

$$E(x^0) := -\int d^{d-1}x\,\sqrt{-g}T^0{}_0 = V_{d-1}a^{d-1}\rho, \quad V_{d-1} := \int d^{d-1}x\,\sqrt{\tilde{g}}, \tag{93}$$

which is NOT conserved unless $P = 0$, since

$$\frac{\dot{E}}{E} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} \neq 0. \tag{94}$$

To define a conserved charge as a generalization of energy, we take $\zeta^a = -\beta(x^0)\delta^a_0$ to satisfy (52), which leads to[11]

$$-T^0{}_0\dot{\beta} - T^i{}_j\Gamma^j_{i0}\beta = \rho\dot{\beta} - (d-1)P\frac{\dot{a}}{a}\beta = 0, \tag{95}$$

where we use $\Gamma^j_{i0} = \dfrac{\dot{a}}{a}\delta^j_i$. An existence of the second term violates the condition (74) for the energy conservation.

A new conserved charge is thus given by

$$S(x^0) := \int d^{d-1}x\,\sqrt{-g}(-T^0{}_0)\beta = V_{d-1}a^{d-1}\rho\beta, \tag{96}$$

which is manifestly conserved as

$$\frac{\dot{S}}{S} = \frac{\dot{E}}{E} + \frac{\dot{\beta}}{\beta} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} + (d-1)\frac{P}{\rho}\frac{\dot{a}}{a} = 0. \tag{97}$$

The energy nonconservation is compensated by the second term.

What is this conserved charge S? If we define densities $e(x^0) := E(x)/V_{d-1} = \rho(x)v(x^0)$ and $s(x^0) := S(x)/V_{d-1} = e(x^0)\beta(x^0)$, where $v(x^0) := a(x^0)^{d-1}$ is a local volume element at time x^0, we obtain

$$\frac{ds}{dx^0} = \frac{de}{dx^0}\beta + e\frac{d\beta}{dx^0} = \left(\frac{de}{dx^0} + P\frac{dv}{dx^0}\right)\beta, \tag{98}$$

where we use (95). This relation is very similar to the first law of thermodynamics as

$$T ds = de + P dv, \tag{99}$$

if we identify $\beta = \frac{1}{T}$ as an inverse temperature. We thus interpret S as the total entropy of the universe, which is conserved in the FLRW universe.[11,f] In addition, $\beta(x^0)$ is regarded as the time-dependent inverse temperature of the universe. It is easy to see that the temperature decreases as the universe expands, since

$$\frac{\dot{\beta}}{\beta} = (d-1)\frac{P\dot{a}}{\rho a} > 0. \tag{100}$$

Even in more general cases, the entropy S so defined is conserved in general relativity.[11]

Although we assume the Einstein equation (51) for analyses in this section, our definition of the conserved charge (54) works for an arbitrary theory of general relativity whose equation of motion is given by $\tilde{G}_{ab} = 2\kappa T_{ab}$ instead of (51), where $\tilde{G}_{ab}$ is an arbitrary second rank symmetric tensor composed of the metric g_{ab} which satisfies $\nabla_a \tilde{G}^a{}_b = 0$.

3.4.2. *Conserved charge from the second theorem*

Let us consider the conserved charge from Noether's second theorem for the FLRW universe. In the case of the pseudo-tensor, we have

$$A^0{}_0 = \frac{\sqrt{-g}}{2\kappa}[2R^0{}_0 + g^{0c}\Gamma^d_{d0,c} - g^{cd}\Gamma^0_{cd,0}] = 0, \tag{101}$$

where we use

$$R^0{}_0 = (d-1)\frac{\ddot{a}}{a}, \quad \Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij}. \tag{102}$$

Thus $E^{\rm FLRW}_{\rm pseudo} = 0$, which is conserved but physically trivial.

[f] Without a mixing between time and space components for the metric and the EMT, the entropy density s is also conserved.[11]

The conserved current density for the Komar energy is given by

$$J^a[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_b[\nabla^{[a}\xi^{b]}], \tag{103}$$

where we take a nonconstant $\xi^a = \gamma(x^0, r)\delta^a_0$. Here, the $(d-1)$-dimensional metric is parametrized as

$$\tilde{g}_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 h_{kl}dx^k dx^l \tag{104}$$

with the $(d-2)$-dimensional metric h_{kl} for a unit sphere. Since $r = 0$ is not a special point in the $(d-1)$-dimensional space, $\gamma(x^0, r = 0)$ must be finite. Nonzero components of the current density with this choice of ξ^a become

$$\begin{aligned} J^0(x) &= -\frac{a^{d-3}\sqrt{h}}{2\kappa}\partial_r\left(r^{d-2}\sqrt{1-kr^2}\partial_r\gamma\right), \\ J^r(x) &= \frac{r^{d-2}\sqrt{1-kr^2}\sqrt{h}}{2\kappa}\partial_0(a^{d-3}\partial_r\gamma), \end{aligned} \tag{105}$$

where h is the determinant of h_{kl}. For the conservation of the Komar energy, the boundary contribution at $r \to r_\infty$, where $r_\infty = \infty$ for $k \le 0$ or $r_\infty^2 = 1/k$ for $k > 0$, given by

$$\begin{aligned} &\lim_{r\to r_\infty}\int_{x_i^0}^{x_f^0} dx^0 \int d^{d-2}x\, J^r(x) \\ &= \lim_{r\to r_\infty}\frac{\Omega_{d-2}}{2\kappa}a^{d-3}(x^0)r^{d-2}\sqrt{1-kr^2}\partial_r\gamma(x^0, r)\Big|_{x^0=x_i^0}^{x^0=x_f^0} \end{aligned} \tag{106}$$

must vanish,[g] where $\Omega_{d-2} := \int d^{d-2}x\sqrt{h}$ is the volume of the $(d-2)$-dimensional unit sphere. Thus, $\gamma(x^0, r)$ must satisfy

$$\lim_{r\to r_\infty} r^{d-2}\sqrt{1-kr^2}\partial_r\gamma(x^0, r) = 0. \tag{107}$$

Under this condition, the Komar energy is evaluated as

$$\begin{aligned} E^{\mathrm{FLRW}}_{\mathrm{Komar}} &= \int d^{d-1}J^0(x) \\ &= -\frac{\Omega_{d-2}}{2\kappa}a^{d-3}(x^0)r^{d-2}\sqrt{1-kr^2}\partial_r\gamma(x^0, r)|_{r=0}^{r=r_\infty} = 0. \end{aligned} \tag{108}$$

Thus, the Komar energy is conserved but physically trivial, as in the case of the pseudo-tensor.

[g] If the space is a $(d-1)$-sphere ($k > 0$), there should be no need for spatial boundary condition. Using the spherical coordinate and polar angle θ to set r as $\sqrt{k}r = \sin\theta$, the boundary condition (107) reads $\lim_{\theta\to\pi}(\sin\theta)^{d-2}\partial_\theta\gamma = 0$. It is obvious that this equation is trivially satisfied.

3.5. *Initial condition of $\zeta^a(x) = \beta(x)n^a(x)$*

As mentioned before, (52) has a unique solution if the initial condition for $\beta(x)$ is given. *A priori*, there is no principle for a choice of the initial $\beta(x)$. Since $\beta(x)$ physically represents a local inverse temperature, we have to determine a local temperature distribution of matters from the matter EMT $T^a{}_b(x)$ at some x^0 in order to fix the initial value of $\beta(x)$. In the case of the FLRW universe, since matters are uniformly distributed, it is natural to take the initial $\beta(x)$ to be uniform as well. For general cases, however, it has not been known to define the local temperature from matter distributions. We leave this important problem to future investigations.

4. Conclusion and Discussion

In this paper, we have shown that the pseudo-tensor as well as the Komar integral types of the energy including their quasi-local expressions are inappropriate to give the physically meaningful definition of the energy in general relativity. This is because their conservation derived from Noether's second theorem is merely an identity representing a constraint by the local invariance rather than a consequence of the dynamics. Noether's second theorem covers almost all existing definitions of the energy in general relativity including the Abbott–Deser definition[27] in addition to others mentioned in the main text.

In contrast, our proposal utilizes equations of motion to derive the conservation of the energy/entropy without using Noether's theorem. Thus, more than 100 years after Einstein's proposal, our definition finally provides a proper and covariant definition of the energy whose generalization as the entropy is always conserved in general relativity.

The form of the conserved entropy in general relativity depends explicitly on the on-shell g_{ab}, the solution to the Einstein equation, through $\zeta^a(x) = \beta(x)n^a(x)$ in (52), where $\beta(x)$ is determined *after* the Einstein equation is solved. Thus, we cannot predict how the space–time evolves in time using the conservation law of the entropy, unlike the standard conservation law of the energy in the flat space–time, which often gives manifest constraints to dynamics of the system.

As evident from the form of the conserved current, $J^a(x) := T^a{}_b(x)n^b(x)\beta(x)$, the energy/entropy in general relativity is carried only by the matter EMT. This means that gravitational fields including (Ricci flat) gravitational waves cannot carry the energy/entropy in general relativity. Even though one may invent another definition of a conserved energy for gravitational fields, it is still true that there exists the conserved energy/entropy carried only by matters in general relativity. Thus, it is interesting to reanalyze the binary star merger in terms of the conserved entropy, since it has been interpreted that the energy loss through the emission of gravitational waves from rotating binary stars causes their merger. Last but not least, a fact that gravitational fields carry no energy/entropy give a very strong constraint to a theory of quantum gravity if it indeed exists. For example, although a graviton, a quanta of the quantized gravity, carries the energy/entropy, a quantum average of an energy/entropy exchange between matter and gravity field must vanishes in the classical limit ($\hbar \to 0$).

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Appendix A. Noether's Second Theorem

For the sake of readers, we give a derivation of Noether's second theorem.[12] In the case of general relativity, see also an appendix of Ref. 13, which however seems to be not recognized well in the community.

A.1. *Invariant variational theory*

Let us consider an integral of Lagrangian L over an arbitrary d-dimensional region Ω given by

$$S_\Omega = \int_\Omega d^d x \, L(\varphi_n, \varphi_{n,\mu}, \varphi_{n,\mu\nu}), \tag{A.1}$$

where $\varphi_{n,\mu} := \partial_\mu \varphi_n$, $\varphi_{n,\mu\nu} := \partial_\nu \partial_\mu \varphi_n$ and $n = 1, 2, \ldots, N$ labels N different fields. Unlike the standard Lagrangian which, contains at most the first derivatives of φ, the above L also contains the second derivatives of φ_n, which are necessary for Einstein's general relativity. Our discussion below can be extended to a more general L including derivatives of φ_n higher than the second, though the formula becomes more complicated.

A variation of S_Ω is evaluated as

$$\begin{aligned} \delta_v S &= \int_\Omega d^d x \, \left[\frac{\partial L}{\partial \varphi_n} \delta_v \varphi_n + \frac{\partial L}{\partial \varphi_{n,\mu}} \partial_\mu \delta_v \varphi_n + \frac{\partial L}{\partial \varphi_{n,\mu\nu}} \partial_\nu \partial_\mu \delta_v \varphi_n \right] \\ &= \int_\Omega d^d x \, \{ [L]^n \delta_v \varphi_n + \partial_\mu \Theta^\mu (\delta_v \varphi_n) \}, \end{aligned} \tag{A.2}$$

where

$$[L]^n := \frac{\partial L}{\partial \varphi_n} - \partial_\mu \frac{\partial L}{\partial \varphi_{n,\mu}} + \partial_\nu \partial_\mu \frac{\partial L}{\partial \varphi_{n,\mu\nu}}, \tag{A.3}$$

$$\Theta^\mu(\delta_v \varphi_n) = \left(\frac{\partial L}{\partial \varphi_{n,\mu}} - \partial_\nu \frac{\partial L}{\partial \varphi_{n,\mu\nu}} \right) \delta_v \varphi_n + \frac{\partial L}{\partial \varphi_{n,\mu\nu}} \partial_\nu \delta_v \varphi_n. \tag{A.4}$$

If we take an arbitrary variation of φ_n such that $\delta_v \varphi_n = \partial_\mu \delta_v \varphi_n = \partial_\mu \partial_\nu \delta_v \varphi_n = 0$ on the boundary of Ω, the total divergent term, $\partial_\mu \Theta^\mu$, vanishes. Thus, $\delta_v S_\Omega = 0$

for an arbitrary variation of φ_n under this constraint implies $[L]^n = 0$, which gives equations of motion for φ_n.

In addition, we assume that S_Ω is invariant under the following transformation:

$$x^\mu \to (x')^\mu = f^\mu(x), \quad \varphi_n(x) \to \varphi'_n(x') = F_n(\varphi, x), \tag{A.5}$$

whose infinitesimal version is given by

$$(x')^\mu = x^\mu + \delta x^\mu, \quad \varphi'_n(x') = \varphi_n(x) + \delta\varphi_n(x). \tag{A.6}$$

Note that δ can be global as well as local transformations, but is different from the variation δ_v to derive equations of motion for φ_n. Since

$$\frac{\partial(x')^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\partial\delta x^\mu}{\partial x^\nu} \Rightarrow \frac{\partial x^\nu}{\partial(x')^\mu} = \delta^\nu_\mu - \frac{\partial\delta x^\nu}{\partial x^\mu}, \tag{A.7}$$

we obtain

$$\delta(\partial_\mu F(x)) := \frac{\partial F'(x')}{\partial(x')^\mu} - \frac{\partial F(x)}{\partial x^\mu} = \partial_\mu\delta F(x) - \partial_\nu F(x)\partial_\mu\delta x^\nu, \tag{A.8}$$

where $\delta F(x) := F'(x') - F(x)$. This shows that δ does not commute with the derivative ∂_μ due to the second term. We thus introduce another variation $\bar\delta F(x) := F'(x) - F(x)$, which commutes with derivatives as

$$\bar\delta(\partial_\mu F) = \partial_\mu\bar\delta F, \quad \delta F = \bar\delta F + \partial_\mu F\delta x^\mu. \tag{A.9}$$

Then the variation of S_Ω under δ is evaluated as

$$\begin{aligned}
\delta S_\Omega &= \int_\Omega d^dx \left[\frac{\partial L}{\partial\varphi_n}\delta\varphi_n + \frac{\partial L}{\partial\varphi_{n,\mu}}\delta(\varphi_{n,\mu}) + \frac{\partial L}{\partial\varphi_{n,\mu\nu}}\delta(\varphi_{n,\mu\nu}) + L\partial_\mu\delta x^\mu\right] \\
&= \int_\Omega d^dx \, [[L]^n\bar\delta\varphi_n + \partial_\mu\{\Theta^\mu(\bar\delta\varphi_n) + L\delta x^\mu\}] \\
&= \int_\Omega d^dx \, [[L]^n(\delta\varphi_n - \varphi_{n,\mu}\delta x^\mu) \\
&\quad + \partial_\mu t\{\Theta^\mu(\delta\varphi_n) - E^\mu{}_\nu\delta x^\nu - G^{\mu\alpha}{}_\nu\partial_\alpha\delta x^\nu\}] = 0,
\end{aligned} \tag{A.10}$$

where we use $d^dx' = (1 + \partial_\mu\delta x^\mu)d^dx$ and

$$\partial_\alpha L(\varphi_n, \varphi_{n,\mu}, \varphi_{n,\mu\nu}) = \frac{\partial L}{\partial\varphi_n}\partial_\alpha\varphi_n + \frac{\partial L}{\partial\varphi_{n,\mu}}\partial_\alpha\varphi_{n,\mu} + \frac{\partial L}{\partial\varphi_{n,\mu\nu}}\partial_\alpha\varphi_{n,\mu\nu}, \tag{A.11}$$

and we define

$$E^\mu{}_\nu := \frac{\partial L}{\partial\varphi_{n,\mu}}\varphi_{n,\nu} - \partial_\alpha\frac{\partial L}{\partial\varphi_{n,\mu\alpha}}\varphi_{n.\nu} + \frac{\partial L}{\partial\varphi_{n,\mu\alpha}}\varphi_{n,\nu\alpha} - \delta^\mu_\nu L, \tag{A.12}$$

$$G^{\mu\alpha}{}_\nu := \frac{\partial L}{\partial\varphi_{n,\mu\alpha}}\varphi_{n,\nu}. \tag{A.13}$$

A.2. *Noether's first theorem*

Before considering Noether's second theorem, we derive the well-known Noether's first theorem from the invariant variational theory. If we take

$$\delta x^\mu = \epsilon^r f_r^\mu(x), \quad \delta\varphi_n = \epsilon^r F_{r,n}(x,\varphi), \tag{A.14}$$

where ϵ^r $(r = 1, 2, \ldots, R)$ are arbitrary constant parameters while $f_r^\mu(x)$ and $F_{r,n}(x,\varphi)$ are given functions of arguments. Then (A.10) becomes

$$\delta S_\Omega = \epsilon^r \int_\Omega d^d x \, \{[L]^n X_{n,r} + \partial_\mu J^\mu{}_r\} = 0, \tag{A.15}$$

where

$$\begin{aligned} X_{n,r} &:= F_{r,n} - \varphi_{n,\mu} f_r^\mu, \\ J^\mu{}_r &:= \left(\frac{\partial L}{\partial\varphi_{n,\mu}} - \partial_\alpha \frac{\partial L}{\partial\varphi_{n,\mu\alpha}}\right) F_{r,n} \\ &\qquad - E^\mu{}_\nu f_r^\nu + \frac{\partial L}{\partial\varphi_{n,\mu\nu}} \partial_\nu F_{r.n} - G^{\mu\alpha}{}_\nu \partial_\alpha f_r^\nu, \end{aligned} \tag{A.16}$$

and summations over repeated indices including n are understood.

Since we can take Ω arbitrarily small, we obtain

$$[L]^n X_{n,r} + \partial_\mu J^\mu{}_r = 0. \tag{A.17}$$

Thus, if equation of motions are satisfied as $[L]^n = 0$ for $^\forall n$, there appear R conserved currents $J^\mu{}_r$ such that $\partial_\mu J^\mu{}_r = 0$, as a consequence of the global symmetry generated by parameters ϵ^r. This is the famous Noether's first theorem.

A.3. *Noether's second theorem*

Let us consider the local transformation generated by $\xi^r(x)$ as

$$\delta x^\mu = \xi^r f_r^\mu(x), \quad \delta\varphi_n = \xi^r F_{r,n}(x,\varphi) + \xi^r_{,\mu} F_r{}^\mu{}_{,n}(x,\varphi), \tag{A.18}$$

where $r = 1, 2, \ldots, R$ labels R different generators, and we denote $\xi^r_{,\mu} := \partial_\mu \xi^r$, $\xi^r_{,\mu\nu} := \partial_\nu\partial_\mu\xi^r$ and so on, as before. Then Eq. (A.10) becomes

$$\begin{aligned} \int_\Omega d^d x [\xi^r \{ &[L]^n (F_{r,n} - \varphi_{n,\mu} f_r^\mu) - \partial_\mu([L]^n F_r{}^\mu{}_{,n})\} \\ &+ \partial_\mu(A^\mu{}_r \xi^r + B^{\mu,\nu}{}_r \xi^r_{,\nu} + C^{\mu,\nu\alpha}{}_r \xi^r_{,\nu\alpha})] \\ = 0, \end{aligned} \tag{A.19}$$

where

$$\begin{aligned} A^\mu{}_r &:= \left(\frac{\partial L}{\partial\varphi_{n,\mu}} - \partial_\alpha \frac{\partial L}{\partial\varphi_{n,\mu\alpha}}\right) F_{r,n} \\ &\qquad - E^\mu{}_\nu f_r^\nu + [L]^n F_r{}^\mu{}_{,n} + \frac{\partial L}{\partial\varphi_{n,\mu\nu}} \partial_\nu F_{r.n} - G^{\mu\alpha}{}_\nu \partial_\alpha f_r^\nu, \end{aligned}$$

$$\begin{aligned} B^{\mu,\nu}{}_r &:= \left(\frac{\partial L}{\partial \varphi_{n,\mu}} - \partial_\alpha \frac{\partial L}{\partial \varphi_{n,\mu\alpha}}\right) F_r{}^\nu{}_{,n} \\ &\quad + \frac{\partial L}{\partial \varphi_{n,\mu\nu}} F_{r.n} + \frac{\partial L}{\partial \varphi_{n,\mu\alpha}} \partial_\alpha F_r{}^\nu{}_n - G^{\mu\nu}{}_\alpha f_r^\alpha, \\ C^{\mu,\nu\alpha}{}_r &:= \frac{\partial L}{\partial \varphi_{n,\mu\nu}} F_r{}^\alpha{}_n = C^{\nu,\mu\alpha}{}_r, \end{aligned} \tag{A.20}$$

and summations over repeated indices including n are also understood.

As before we can take Ω arbitrarily small. In addition, as opposed to the case of the global symmetry, we can also take $\xi^r = \xi^r_{,\mu} = \xi^r_{,\mu\nu} = 0$ on $\partial\Omega$ (the boundary of Ω). This choice leads to

$$[L]^n (F_{r,n} - \varphi_{n,\mu} f_r^\mu) - \partial_\mu ([L]^n F_r{}^\mu{}_{,n}) = 0, \tag{A.21}$$

which can give R constraints on N equation of motions. Putting this back into (A.19) with an arbitrary Ω and ξ, we obtain

$$\partial_\mu (A^\mu{}_r \xi^r + B^{\mu,\nu}{}_r \xi^r_{,\nu} + C^{\mu,\nu\alpha}{}_r \xi^r_{,\nu\alpha}) = 0, \tag{A.22}$$

which reduces to

$$\begin{aligned} &\partial_\mu (A^\mu{}_r) \xi^r + (A^\nu{}_r + \partial_\mu B^{\mu,\nu}{}_r) \xi^r_{,\nu} + \frac{1}{2} (B^{\mu,\nu}{}_r + B^{\nu,\mu}{}_r + 2\partial_\alpha C^{\alpha,\mu\nu}{}_r) \xi^r_{,\nu\mu} \\ &\quad + \frac{1}{3} (C^{\mu,\nu\alpha}{}_r + C^{\nu,\alpha\mu}{}_r + C^{\alpha,\mu\nu}{}_r) \xi^r_{,\nu\alpha\mu} = 0. \end{aligned} \tag{A.23}$$

Since ξ^r, $\xi^r_{,\nu}$, $\xi^r_{,\mu\nu}$ and $\xi^r_{,\mu\nu\alpha}$ in (A.23) are all arbitrary, we can conclude

$$\begin{aligned} \partial_\mu A^\mu{}_r &= 0, \\ A^\nu{}_r + \partial_\mu B^{\mu,\nu}{}_r &= 0, \\ B^{\mu,\nu}{}_r + B^{\nu,\mu}{}_r + 2\partial_\alpha C^{\alpha,\mu\nu}{}_r &= 0, \\ C^{\mu,\nu\alpha}{}_r + C^{\nu,\alpha\mu}{}_r + C^{\alpha,\mu\nu}{}_r &= 0, \end{aligned} \tag{A.24}$$

as constraints for *off-shell* φ_n. Thus, the constraints are expressed by the form of conservation as

$$\partial_\mu J^\mu{}_r = 0, \quad r = 1, 2, \ldots, R, \tag{A.25}$$

where

$$J^\mu{}_r := A^\mu{}_r = -\partial_\nu \tilde{B}^{\nu,\mu}{}_r, \quad \tilde{B}^{\nu,\mu}{}_r := \frac{1}{2} B^{[\nu,\mu]}{}_r - \frac{1}{3} \partial_\alpha C^{[\nu,\mu]\alpha}{}_r. \tag{A.26}$$

These constraints that $\partial_\mu J^\mu{}_r = 0$, however, are not invariant under (A.18) due to a presence of uncontracted index r.

Equation (A.22) is also regarded as a conservation equation that

$$\partial_\mu J^\mu[\xi] = 0, \tag{A.27}$$

where $J^\mu[\xi]$ is defined as

$$J^\mu[\xi] = A^\mu{}_r \xi^r + B^{\mu,\nu}{}_r \xi^r_{,\nu} + C^{\mu,\nu\alpha}{}_r \xi^r_{,\nu\alpha}. \tag{A.28}$$

This conservation equation is manifestly invariant under (A.18), since uncontracted indices are absent. Using (A.24) one can further rewrite $J^\mu(x)$ as

$$\begin{aligned} J^\mu[\xi] &= -(\partial_\nu B^{\nu,\mu}{}_r)\xi^r + B^{\mu,\nu}{}_r \xi^r{}_{,\nu} + C^{\mu,\nu\alpha}{}_r \xi^r{}_{,\nu\alpha} \\ &= -\partial_\nu (B^{\nu,\mu}{}_r \xi^r) + B^{\{\mu,\nu\}}{}_r \xi^r{}_{,\nu} + C^{\mu,\nu\alpha}{}_r \xi^r{}_{,\nu\alpha} \\ &= -\partial_\nu (B^{\nu,\mu}{}_r \xi^r + 2C^{\nu,\mu\alpha}{}_r \xi^r{}_{,\alpha}) + (2C^{\alpha,\mu\nu}{}_r + C^{\mu,\nu\alpha}{}_r)\xi^r{}_{,\nu\alpha} \\ &= -\partial_\nu (B^{\nu,\mu}{}_r \xi^r + 2C^{\nu,\mu\alpha}{}_r \xi^r{}_{,\alpha}). \end{aligned} \tag{A.29}$$

Thus, the current $J^\mu[\xi]$ turns out to be a total divergence.

Let us remind readers that equations of motion are not employed to derive the conservation equations in Noether's second theorem. Even if we restrict $\xi^\mu_r(x)$ to a constant as $\xi^\mu_r(x) = \epsilon^\mu_r$, *off-shell* conservation equations still hold, so that conservations cannot be regarded as the dynamical ones in the standard Noether's first theorem. Noether herself (as a word by Hilbert and Klein) called such conservations *improper*[12] and distinguished them from *proper* conservations in the first theorem.

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