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Aspects of Derivative Screening

in

Extended Brans-Dicke Theories

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Aspekte von /emphderivative screening in erweiterten Brans-Dicke-Theorien:

Wir betrachten das Problem der Kosmologischen Konstante im Zusammenhang mit dem ähnlichen Hierarchie-Problem des Higgs-Boson. Wir beschreiben Modelle, die ein *running* der Energiedichte des Vakuums mit der kosmischen Expansion vorhersagen. Diese Modelle sind mit der Brans-Dicke-Theorie (BD-Theorie) verwandt, die faszinierende theoretische und empirische Konsequenzen hat; insbesondere erklärt die (BD-Theorie) Abweichungen zwischen Messungen kosmologischer Parametern, die in [1, 2, 3, 4, 5, 6, 7] diskutiert werden. In BD-ΛCDM kann die effektive gravitative Kopplung $4 - 9\%$ größere Werte annehmen als die Gravitationskonstante G_N. Um mit den getesteten Vorhersagen Allgemeiner Relativitätstheorie kompatibel zu sein, müssen BD-Effekte auf lokalen Skalen abgeschirmt sein. Wir untersuchen erweiterte BD-Theorien, die Abweichungen von G auf kosmologischen Skalen erlauben und lokal mit G_N kompatibel sind. Wir zeigen, dass Abschirmungsmechanismen nicht die von BD-ΛCDM vorhergesagten signifikanten Abweichungen von G erklären können. Abschließend untersuchen wir für diese Erweiterungen die Stabilität klassischer Theorien höherer Ableitungsordnungen unter Quanten-Korrekturen, was in einem allgemeineren Kontext bereits in [8, 9] getan wurde.
Diese Masterarbeit hat eine Publikation hervorgebracht:
"Difficulties in reconciling non-negligible differences between the local and cosmological values of the gravitational coupling in extended Brans-Dicke theories",
Adrià Gómez-Valent and Prajwal Hassan Puttasiddappa,
JCAP09(2021)040; arXiv:2105.14819 [astro-ph.CO]

Aspects of Derivative Screening in Extended Brans-Dicke Theories:

We discuss Cosmological Constant (CC) problem in conjunction with a similar problem of Higgs fine-tuning. We move on to describe models where we expect the running of vacuum energy density with the cosmic expansion. These models are related to Brans-Dicke (BD) theory which has alluring theoretical and observational consequences; in particular, BD theory with a constant vacuum energy density (BD-ΛCDM), alleviates cosmological tensions. These are discussed in previous works like, [1, 2, 3, 4, 5, 6, 7]. In BD-ΛCDM, the effective gravitational couplings at cosmological scales can take values $4-9\%$ greater than Newton constant G_N. The BD effects have to be screened in the local scales where we have tight constraints for theories that deviate from GR. In this context, we explore extended BD theories that can allow variations in G at cosmological scales and reconcile G_N at local scales. We show that the screening mechanisms cannot explain large deviations of G as predicted by BD-ΛCDM [10]. In the last part, for our particular extensions, we study the stability of classical higher-order derivative theories under quantum corrections, which has been done in a very general context in [8, 9].
This MSc thesis work has given rise to a published paper:
"Difficulties in reconciling non-negligible differences between the local and cosmological values of the gravitational coupling in extended Brans-Dicke theories",
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Contents

Part I

Fine Tuning Problem

1 Cosmological Constant

In this chapter we shall discuss aspects of Cosmological Constant and its association to vacuum energy density. We introduce the CC problem and the fine tuning problem at the classical level. For this chapter we import knowledge from [1, 2, 11] along with other references cited in the text.

The discovery of accelerated expansion of the Universe with the observations of supernova and the cosmic microwave background (CMB) was a major breakthrough in modern cosmology research. It predicted diffusive Dark Energy (DE) which is evenly spread across the cosmos and is responsible for the acceleration of the Universe. The other important discovery of the last century is the presence of 'Dark Matter' (DM) which dominates the fraction of matter content of the Universe. From observations and numerical simulations of the dynamics of galaxies, it was predicted that DM exists around the galaxies and clusters. As the name indicates, DM particles are 'dark' as they have not been found interacting with rest of the accessible Universe through any of the know forces. The amount and nature of DM plays an important role in cosmology, particularly in the formation of structures. Non-relativistic (cold), massive particles would explain structures of the Universe and is well supported by Cosmic Microwave Background (CMB) spectrum. Relativistic DM models are not able to explain the small structures of the Universe. Thus, the kinematic characteristics of DM particles are constrained by the structure formation in the Universe which in turn narrows down the expected DM mass range for ongoing experiments. Even though there are some interesting anomalies in our detection, there has not been a direct detection of DM particles. In this thesis we will not discuss DM sector, rather we will focus on yet another mysterious component of the Universe, Dark Energy (DE).

Cosmological constant (CC) would be the simplest candidate for DE. As the name indicates, its energy density remains constant in time. The bare CC, represented by Λ_B which we add to the Einstein Hilbert Action, together with Cold Dark Matter (CDM) makes the ΛCDM model of cosmology. This model has proved its consistency through a wide range of observations. CC is usually associated with some form of vacuum energy. A small energy scale of CC would be sufficient to account for the cosmic acceleration. But, if one associates DE with vacuum energy from the quantum field theory (QFT), as we will see, the energy scales are at least 10^{55} bigger than the former leading to the famous CC problem. We shall discuss this along with the required fine tuning of bare CC term to obtain the observed dark energy, in the next section. We can show from simple thermodynamics that CC is associated with vacuum energy and it scales with volume. In de Sitter universe (just vacuum, no matter), as there is no exchange of energy, we set $dQ = 0$. Then from the first law of thermodynamics, the variation of internal energy is proportional to the internal pressure. Thus the universe with only vacuum energy density ρ^{Λ}_{vac} (which is a constant, indicated by Λ) $dU = d(\rho^{\Lambda}_{vac}V)$, we can write,

$$
\begin{aligned}
& dU = dQ|_{(=0 \text{ for CC})} - p^{\Lambda}_{vac}dV \\
\implies & \rho^{\Lambda}_{vac}dV = -p^{\Lambda}_{vac}dV.
\end{aligned}
$$

Constant vacuum energy density behaves as the CC with constant equation of state (EoS) $w^{\Lambda}_{vac} = p^{\Lambda}_{vac}/\rho^{\Lambda}_{vac} = -1$. Violation of strong energy condition is implied by $w^{\Lambda}_{vac} < -1/3$ which explains that it is responsible for acceleration of the Universe. From these arguments, we can put the CC term as a matter component in addition to the Einstein-Hilbert (EH) action. But historically, CC term has more drama attached to it.

Soon after Albert Einstein presented the theory of General Relativity (GR) in 1915, along with weak field solutions (like the Schwarzschild solution), people tried to apply this theory to the Universe. It was then understood that the Universe was made up of matter which would eventually collapse due to attractive gravitational force. Einstein introduced cosmological constant Λ to his equation in 1917, believing in the static Universe. This term will evidently work as repulsive force as we just saw. In the same year Willem de Sitter found solutions for Einstein equation with CC (and no matter) and showed that we get accelerating Universe with only vacuum energy which was supposed to make Universe static. Moreover, Georges Lemaître found that Einstein's static solutions suffered from stability problems - a small perturbation of matter would result in collapse or forever expansion of the Universe[1]. Vesto Slipher's observations of galaxies in 1910-20's and Edwin Hubble's (also significant contributions from astronomer Henrietta S. Leavitt and predictions by Lemaître) redshift measurements in 1929 indicated an expansion of the Universe favouring the idea of de Sitter. So in 1931, Einstein renounced the idea of CC and set $\Lambda = 0$. He also went ahead to abandon the idea of static Universe and proposed the Friedmann-Einstein model of a universe with positive curvature and the Einstein-de Sitter model of a flat universe with only matter. In the later half of the 20th century, measurements of CMB indicated a flat Universe. Also, redshift measurements by SNIa, not only re-invoked Λ but was measured to be greater than zero with very high accuracy. It dominates the current expanding Universe with its energy density approximately given by $\rho^{\Lambda}_{O} \sim 10^{-29}\text{g/cm}^3 = 10^{-47}\text{GeV}^4$.

Einstein equations in vacuum are derived from Einstein-Hilbert action. We introduce a CC term Λ_B in the action. The action and resulting field equations are

[1] Lemaître went further to propose the first cosmological model.

* I have tried to give a very short summary of the historical events without any references. For a more detailed historical view see [2] and references therein

$$S_{EH} = \int d^4x \frac{1}{16\pi G_N} \sqrt{-g}(R - 2\Lambda_B) \tag{1.1}$$

$$G_{\mu\nu} + \Lambda_B g_{\mu\nu} = 0. \tag{1.2}$$

$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor given by trace-reversed Ricci tensor $R_{\mu\nu}$. It is the geometrical part of the equation quantifying the curvature of pseudo-Riemannian spacetime manifold. The 'beauty' of Einstein equation is in the realization that curvature of spacetime manifold is proportional to energy momentum of the matter fields present[2]. In (1.2), curvature is sourced by the bare CC term Λ_B, denoting constant vacuum energy density as we mentioned earlier. The name "bare" carries the same meaning as in particle physics indicating that, parameters appearing in the Lagrangian are not exactly the quantities we measure. In this case Λ_B is not actually observed in the cosmos and it should get some corrections, say, from the matter fields present in the Universe. When we take this in account, we have an effective Λ_E which is of small value as we have measured it from observations. As we shall see, the corrections from zero point energy of mater fields lead to "fine-tuning problem".

From Bianchi identities, we have vanishing covariant derivative of the Einstein tensor $\nabla^\mu G_{\mu\nu} = 0$. This is purely due to algebraic and differential symmetries associated with Riemann Curvature tensor encoded in Bianchi identities. Considering metric compatibility $\nabla^\mu g_{\mu\nu} = 0$,

$$\nabla^\mu [G_{\mu\nu} + \Lambda_B g_{\mu\nu}] = 0 \implies \nabla^\mu \Lambda_B = 0 \implies \Lambda_B \equiv \text{ constant.} \tag{1.3}$$

The CC term in (1.2) could be transferred to the *r.h.s* by defining the energy momentum tensor of the CC as,

$$T^{\Lambda_B}_{\mu\nu} = -\rho^{\Lambda}_B g_{\mu\nu} = \frac{\Lambda_B}{8\pi G_N} g_{\mu\nu} \tag{1.4}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T^{\Lambda_B}_{\mu\nu}. \tag{1.5}$$

Again applying covariant derivative on both sides of equation (1.5), the Bianchi identities imply conservation of energy momentum tensor $\nabla^\mu T^{\Lambda_B}_{\mu\nu} = 0$. With constant gravitational coupling G_N, covariantly conserved energy momentum tensor

[2]The Reimann curvature tensor $R_{\mu\nu\alpha\beta}$ is a mathematical structure which has the tracefull part, the Ricci tensor (its trace is the Ricci scalar $g^{\mu\nu} R_{\mu\nu} = R$), which is entirely determined by the energy density $T_{\mu\nu}$. Riemann curvature tensor also includes traceless Weyl Tensor $C_{\mu\nu\alpha\beta}$ ((B.5)) with same symmetries as the former and is invariant under conformal transformations.

implies that we can have a "cosmological *constant*" (as described in (1.3)) incorporated in the Einstein equation. Since the Einstein equation has this beauty of equating geometrical quantity of *l.h.s* to the matter component in *r.h.s*, transfer of CC term is not just a mathematical step and needs more attention. By applying divergence operator on (1.5), the *l.h.s* vanishes due to Bianchi identities. However, on the *r.h.s* we have the following two possibilities.

1. From purely theoretical point of view, a generic energy momentum tensor should be conserved locally,

   $$\partial^\mu T_{\mu\nu} = 0 \implies T_{\mu\nu} \propto \eta_{\mu\nu}$$
   $$\text{but one could still keep } \nabla^\mu T_{\mu\nu} \neq 0 \implies T_{\mu\nu} \propto g_{\mu\nu}.$$

   This is counter-intuitive as the latter expression is just a 'covariantized' form of the former and we can always have a transformation from local Minkowski (flat) spacetime to a curved spacetime. But one has to be careful here as general covariance principle also allows any combinations of invariant quantities possible. That is,

   $$T^{\Lambda_B}_{\mu\nu} = a_0 g_{\mu\nu} + a_1 R_{\mu\nu} + (...)$$
   with a_i's composed of constants like $R, R^2, R_{\mu\nu}R^{\mu\nu}$,etc .

   Hence, one could in principle have modified conservation laws like in Rastall gravity [12],

   $$\nabla^\mu T_{\mu\nu} = 0 + \alpha \nabla^\nu R \quad \text{or} \quad = \beta R^{\mu\nu} \nabla \mu R$$
   where α, β have to be fixed from observations.

   Clearly, only a_0 is a constant which we equate with ρ_B^Λ. For $i > 0$, a_i's involve higher derivatives and should not be our immediate concern.

2. We could also think of gravitational coupling as a variable $G(t)$ instead of a Newtons constant G_N and retain covariant conservation of stress-energy tensor. This case indicates violation of strong equivalence principle and such a theory could have interesting impacts on cosmology which will be discussed in the next chapter.

Hence, by either having a time dependent gravitational coupling $G(t)$ or covariantly non-conserved energy momentum tensor[3] a "time dependent cosmological constant" $\Lambda(t)$ would still be compatible in the Einstein equation, fulfilling Bianchi identities.

[3] if there were matter fields, this possibility could be thought of as the vacuum exchanging energy with matter fields

Now let us consider matter action in terms of a real scalar field ϕ with some potential $V(\phi)$. In the classical treatment, its energy-momentum tensor is now given by $T^{\phi}_{\mu\nu}$,

$$S_\phi[\phi, g_{\mu\nu}] = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] \tag{1.6}$$

$$T^{\phi}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{1.7}$$

The combined action and the corresponding field equations respectively read,

$$\begin{aligned} S &= S_{EH}[g_{\mu\nu}] + S_\phi[\phi, g_{\mu\nu}] \\ &= \int d^4x \frac{1}{16\pi G_N}\sqrt{-g}\left[R - 2\Lambda_B - \frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)\right] \end{aligned} \tag{1.8}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_B g_{\mu\nu} = 8\pi G_N T^{\phi}_{\mu\nu} \tag{1.9}$$

In general, all matter fields contribute to *r.h.s* of the above field equation. Our focus has been one a single field ϕ which can be associated with the Higgs field[4]. All massive gauge bosons, quarks and charged leptons get masses through their coupling to Higgs field. The well defined renormalizable Higgs potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \quad (\lambda > 0) \tag{1.10}$$

has the property to trigger the phenomenon of Spontaneous Symmetry Breaking (SSB). In fact, SSB is the only known way to generate masses in a gauge invariant way.

- For $m^2 > 0$ the potential has a minimum at $\langle\phi\rangle = 0$, there is no symmetry breaking. Then the ground state of (1.7) is given by,

  $$\langle 0|T^{\phi}_{\mu\nu}|0\rangle \equiv \langle T^{\phi}_{\mu\nu}\rangle = \langle V(\phi)\rangle g_{\mu\nu} = 0\ . \tag{1.11}$$

  The ground state is nothing but the vacuum state of scalar field and hence there is no kinetic energy contribution. Hence the ground state is just the vacuum expectation value of scalar field potential.

- For $m^2 < 0$, SSB is triggered and potential has a minimum at $v/\sqrt{2} = \langle\phi\rangle = \sqrt{\frac{-6m^2}{\lambda}}$. The ground state will have contribution to vacuum energy at classical level given by,

  $$\langle 0|T^{\phi}_{\mu\nu}|0\rangle \equiv \langle T^{\phi}_{\mu\nu}\rangle = \langle V(\phi)\rangle g_{\mu\nu} = (-\frac{m^2}{2\lambda})g_{\mu\nu} = -\rho^{\phi vac}_{clas}\, g_{\mu\nu}\ . \tag{1.12}$$

[4]This is a simplification for further discussions. However, one must have introduced Higgs as a complex doublet of scalar fields in SM. Nevertheless, this simplification will not affect the nature of the problem

We can calculate this quantity once we rewrite the vacuum energy density in terms of the two free parameters of the potential, the vacuum expectation value v and mass m as,

$$\rho_{clas}^{\phi vac} \equiv \langle V(\phi) \rangle = -\frac{3m^4}{2\lambda} = \frac{m^2 v^2}{4} \approx 2.4 \times 10^8 \text{GeV} \ . \tag{1.13}$$

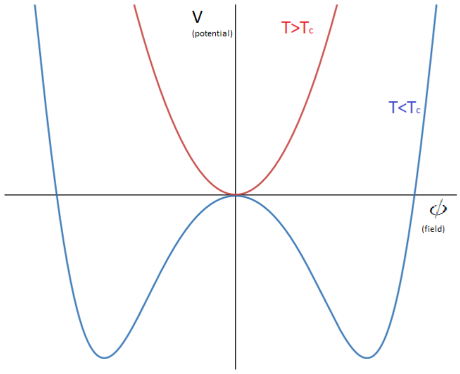


Figure 1.1: Here we show diagrammatic representation of SSB which is often found in literature (for example [13]). The symmetric potential in red and the potential after symmetry breaking is blue curve is the broken potential. A simple analogy of Paramagnet to Ferromagnet phase-transition can be made where we have (rotational) symmetric potential for high temperatures beyond a critical temperature T_c below which we have broken phase.

In last equation we have plugged in the values of $v = 246\text{GeV}$ and $m = 125.35\text{GeV}$. This energy density is *induced* at classical level by the phenomena of electroweak phase transition (SSB) of Higgs potential. This gives an additional term in (1.9).

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_B g_{\mu\nu} = 8\pi G_N (T_{\mu\nu}^{\phi} + \langle T_{\mu\nu} \rangle) \tag{1.14}$$

The first term on the *r.h.s* is for ordinary matter which can be radiation, dust etc. and the second part is the *induced* part from classical vacuum. This can be seen as an addition to the bare CC term. We define $\frac{\rho_{clas}^{-\phi vac}}{8\pi G_N} = \Lambda_{clas}^{\phi vac}$, then effective CC, Λ_E is given by,

$$\Lambda_E = \Lambda_B + \Lambda_{clas}^{\phi vac} . \tag{1.15}$$

Since this is the quantity which we measure, we can accordingly replace Λ_B in the field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_E g_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{\phi} \ . \tag{1.16}$$

The subscript *clas* reminds us that we are in the classical limit. By comparing the value obtained in (1.13) with the observed vacuum energy density ρ_O^Λ,

$$\left(\frac{\rho_{clas}^{\phi vac}}{\rho_O^\Lambda}\right) \approx 10^{55} \ . \tag{1.17}$$

This discrepancy in our observed value when compared with the theoretical predictions is the famous *cosmological constant problem.* The CC problem can be recognized in (1.15) as the observational value is low in spite of huge contribution from $\Lambda_{clas}^{\phi vac}$. We have only calculated the classical contribution. But the latter has contributions from the quantum zero point energy which we will discuss in the next sections. The latter appears to somehow cancel out such that we observe a small CC. And also, the bare CC term Λ_B which we put in the EH Lagrangian has to be finely tuned up to at least 55 decimal places. This is the *fine tuning problem*[5].

[5]The nature of the problems discussed here are not specific to electroweak phase transition. One would come across similar problems if QCD phase transition is considered for example.

2 Higgs mass fine-tuning and hierarchy problem

In this chapter we describe some standard regularization procedures and renormalization schemes. Our focus will be on ϕ^4 scalar field theory. We discuss divergences in the loop integrals, the ways in which we can parameterize and later isolate these divergences. The procedures of regularization will introduce a new parameter, an energy scale μ. As a consequence of this scale dependence of coupling constants we obtain the flow equations or the beta functions. We also introduce fixed points and conclude this chapter by listing some important problems of the Higgs. These problems by themselves are very interesting and have cosmological consequences. The structure of these problems are also closely related to the CC problem which was introduced in the previous chapter. For this chapter, we derive insights from various sources like [13, 14, 15, 16, 17].

2.1 Regularization and Renormalization

Lets us consider a simple Lagrangian with quartic potential just like we had it in (1.10) and systematically introduce regularization and renormalization by actually calculating important divergent integrals at 1-loop level. The bare quantities denoted with subscript B do not include quantum corrections and are not the quantities we measure in real world. They just appear in the Lagrangian and receive radiative corrections and has to be renormalized[1].

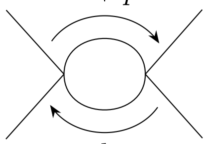
$$\mathcal{L}_B = \frac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{2}m_B^2 \phi_B^2 - \frac{1}{4!}\lambda_B \phi^4 \ . \tag{2.1}$$

We can calculate the quantum corrections for m_B and λ_B from 1-loop diagrams. For a basic 2-particle scattering amplitude at 1-loop level,

$$i\mathcal{M}(p_1, p_2, p_1', p_2') = \Gamma^{(4)} = -i\lambda_B + [iV(s) + iV(t) + iV(u)] \ , \tag{2.2}$$

[1]Here we shall limit ourselves to one loop level to obtain some intuition which will help us in later chapters. You can see some of the loop diagrams in, (7.9) and (7.6) which are all divergent. There are also loop diagrams which contributes to radiative corrections of tree level 2-particle scattering.

where the first term is from the tree level contribution and each loop contribution (s, t, u are Mandelstam variables) is proportional to $(i\lambda)^2$ because of two external leg vertices for each diagram. We indicate the mass of the virtual particle by m_L which could be mass of any particle, for example quark, massive boson, etc., in that particular loop. For the s-channel diagram, representing, internal loop momentum with k and external legs as p,



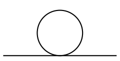
$$
\begin{aligned}
V(p^2) &= \frac{(-i\lambda_B)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_L^2} \frac{i}{(p+k)^2 - m_L^2} \\
&= \frac{(-i\lambda_B)^2}{2} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 + x(1-x)p^2 - m_L^2)^2} \; \text{(where, } l^2 = k + xp\text{)}.
\end{aligned} \tag{2.3}
$$

One way to evaluate such integrals is to rotate the integration contour for l^0 from real to imaginary axis $l^0 = il_E^0$. Now the integral is on Euclidean space. This is Wick rotation and such a rotation is possible as the integrands are analytic in first and third quadrant. We have shown the mathematical steps in Appendix (C). The integral reduces to

$$
V(p^2) = \frac{i\lambda_B^2}{32\pi^2} \int_0^1 dx \left[\frac{x(1-x)p^2 + m_L^2}{l_E^2 + x(1-x)p^2 + m_L^2} + \log\Big(l_E^2 + x(1-x)p^2 + m_L^2\Big) \right]_0^\infty . \tag{2.4}
$$

In the limit $|l_E| \to \infty$, the integrand is logarithmically divergent - quantum corrections to 2-particle scattering amplitude becomes very sensitive to quantum fluctuations in high energy scales.

We come across divergences of 2-point function (7.9) as well. For example at 1-loop level, we have the following diagram,

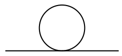


$$
\begin{aligned}
&\equiv \frac{i\lambda_B}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_L^2} \\
&= \frac{i\lambda_B}{2} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 - m_L^2} = \frac{i\lambda_B}{32\pi^2} \int d^2k_E \frac{k_E^2}{k_E^2 - m_L^2} \\
&= \frac{i\lambda_B}{32\pi^2} \left[k_E^2 - m_L^2 \ln\left(1 + \frac{k_E^2}{m_L^2}\right) \right]_0^\infty .
\end{aligned} \tag{2.5}
$$

This time, the divergence is worse than the former method because, along with subdominant logarithmic divergence, we also have a dominant quadratic divergence as $|k_E| \to \infty$. This could be due to our carelessness while writing down integral. The integral is over all possible momenta of the internal loops and we are using the bare quantities in our calculations. We need to be smart in our ignorance. We shall try to parametrize the high energy (UV) behaviour of the theory so that theory is well defined in the scales at which we are interested in. This procedure of parametrizing our ignorance is *Regularization* and can be achieved in a numerous of ways.

2.1.1 Regularization

Regularization and a systematic Renormalization scheme will help us make sense out of our calculations which are yielding infinities. One way to regularize a theory is to parametrize the sensitivity to short distances scales by introducing a cutoff parameter. With such a cutoff, say Λ_{cut}, we have now modified the theory which ran to all momentum scales to a theory which is well defined till Λ_{cut}. In our new theory, the UV divergences are replaced by - sensitivity to cutoff scale such that for fixed couplings, in the limit of cutoff $\Lambda_{cut} \to \infty$ physical quantities diverge instead of divergence due to $l_E, k_E \to \infty$. We can check this by setting limit of integration in (2.5) as Λ_{cut} instead of integrating to infinity.



$$\equiv \frac{i\lambda_B}{32\pi^2}\left[\Lambda_{cut}^2 - m_L^2 \ln\left(1 + \frac{\Lambda_{cut}^2}{m_L^2}\right)\right] . \tag{2.6}$$

Now divergence for $|k_E| \to \infty$ disappeared. Instead, physical divergences are now only in the continuum limit $\Lambda_{cut} \to \infty$. As long as the cutoff is large compared to physical scales of interest, we have a well defined behaviour of the theory because we have simply removed the UV part. This is *cutoff regularization.* There are other ways in which we could have introduced the cut off parameter for example- Lattice regularization, Pauli-Villars regularization etc. Most of these regularization schemes are only convenient to specific purposes. For example the above equation is still divergent in the limit $\Lambda_{cut} \to \infty$. The divergence in other schemes, for example, dimensional regularization, may not be as severe as in cutoff scheme (see (2.10)). Some of these regularization schemes should be done at each loop order of the perturbation theory (example, dimensional regularization). The cutoff regularization has the problem of unitarity. Lorentz symmetry and gauge invariance may not be manifestly preserved in some of the schemes. We have to make some compromises when it comes to associating meaning to these schemes. Together with a renormalization scheme, these procedures are of high practical importance.

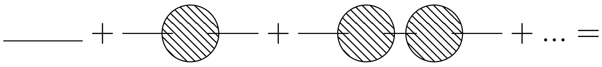
One of the use full regularization scheme is *Dimensional Regularization*, which is to parametrize the sensitivity to UV scale by expressing our theory in a lower dimension. We shall apply this scheme to both 1-loop divergent terms of 4-point function (2.4) and 2-point function (2.5) . Let us first generalize (2.3) to d dimensions in Euclidean space, and in order to keep λ_B dimensionless, we add a parameter μ. As we saw from previous examples of regularization, all *regularization schemes introduce one more parameter to the naive continuum theory.*

$$V(p^2) = \frac{i(\mu^\epsilon \lambda_B)^2}{2} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + x(1-x)p^2 - m_L^2)^2} \tag{2.7}$$

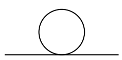
$\epsilon = 4 - d$ is the small deviation of the spacetime dimension from 4. We again refer to Appendix (C) for detailed calculation of the loop integrals.

$$V(p^2) = \frac{i\lambda^2(\mu)}{32\pi^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{m_L^2 - x(1-x)p^2}\right)\right] \tag{2.8}$$

The integral has an extra parameter of mass dimension 1, μ which was not present in our original Lagrangian. The logarithmic divergence for higher momenta in (2.4) has been removed. We can choose μ at will as this is not associated with momentum scale. We can also do the same calculation for the 2-point function. Let us generalize it to sum of all 1PI diagrams[2] at one loop level $\Sigma(k^2)$. Such a generalization would be helpful because we can then express exact propagator $\Delta(p^2)$ as a geometric series in $\Sigma(k^2)$ (represented by the blob).

$\Delta(p^2) =$  $= \frac{1}{p^2 + m_B^2 - \Sigma(k^2)}$ (2.9)

If the above is actually an exact propagator, it should have a pole for $-p^2 = m^2$. Where, the mass parameter is the physical mass instead of bare mass m_B. It has to be renormalized to remove this ambiguity. Lets first regularize and keep the renormalization for the next section. For the first blob, in simple case of 1 loop approximation, there is only one loop diagram contribution it. But at 2 loop level we have 2 more diagrams (the last two diagrams in (7.9)). So, we shall apply dimensional regularization to the 1-loop diagram (2.5),



$$\begin{aligned} \equiv \Sigma(k^2) &= \frac{-i\mu^\epsilon \lambda_B}{2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m_L^2} = \frac{-i\lambda_B \mu^\epsilon \pi^{d/2}}{2(2\pi)^d} \frac{\Gamma(1 - d/2)}{(m_L^2)^{(1-d/2)}} \\ &= \frac{i\lambda(\mu)}{32\pi^2} m_L^2 \left(\frac{2}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m_L^2} + \mathcal{O}(\epsilon)\right). \end{aligned} \tag{2.10}$$

We have denoted $\lambda_B(\mu^2)^{2-d/2}$ as $\lambda(\mu)$ - see Appendix (C) such that in 4 dimensions, $\lambda(\mu) \to \lambda_B$. The regularization has removed the high momentum divergence with the help of a new parameter μ. In comparison to cutoff regularization, we have got better results, as $\Lambda_{cut} \to \infty$ is just a pole as $d \to 4$. Moreover, the new

[2]PI stands for 'particle irreducible'. In 1PI diagrams, we cannot cut any internal propagator such that we end up with two disconnected diagrams. In a QFT, of all the possible Feynman diagrams, only 1PI diagrams contribute for physical observables.

parameter, μ is not a momentum cutoff parameter so we need not take $\mu \to \infty$)to have a continuum theory. Unlike cutoff regularization, dimensional regularization is perturbative regularization which allows momentum integrals at all scales for individual loops. For a physical application, we would want to provide a finite integral measure directly in the path integral. In a physical application, the regulated theory indicates that the couplings will scale with the new parameter Λ_{cut} or μ. This implies that, *a change in cutoff can be compensated by changing the bare couplings so that all physical quantities remain invariant.* Moreover, all the quantities we used in the initial Lagrangian are bare quantities. Now we implement *renormalization schemes* in order to remove the divergent parts from them. Thus, after renormalization we should be left with the same number of parameters we started with.

2.1.2 Renormalization

In (2.9), all the divergences are encapsulated in $\Sigma(k^2)$ which upon dimensional regularization (2.10), still has pole as $\epsilon \to 0$. Dimensional regularization is usually used along with minimal subtraction renormalization scheme or rather *modified minimal subtraction* $\overline{MS}$ scheme. The former is to choose the counter terms so as to remove the pole $\epsilon \to 0$. In the latter scheme, we also remove Euler-Mascheroni constant and $\log 4\pi$.

The removed divergent parts are given by δm and $\delta\lambda$

$$\begin{aligned} \delta m &= -\frac{i\lambda(\mu)}{32\pi^2} m_L^2 \left(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi\right) \\ \delta\lambda &= \frac{3\lambda^2(\mu)}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi\right) \end{aligned} \tag{2.11}$$

For Δ to be exact propagator, it should have a pole $-p^2 = m^2$ where m is the physical mass. Hence we can say that, the physical renormalized mass is the difference,

$$\begin{aligned} m^2 &= m_B^2 - [\Sigma - \delta m] \\ &= m_B^2 - \frac{\lambda(\mu) m_L^2}{32\pi^2} \ln \frac{\mu^2}{m_L^2} \\ \lambda &= \lambda_B - \frac{3\lambda^2(\mu)}{32\pi^2} \log \frac{\mu^2}{m_L^2}. \end{aligned} \tag{2.12}$$

So far we have been successful in isolating all the divergences in the physical couplings using methods of regularization; now we have to get rid of them. The way to get rid of divergence is by implementing *wave function renormalization* where we replace the bare fields ϕ_B with renormalized fields ϕ which are defined by rescaling the former as shown below. The idea is to absorb the cutoff dependent divergences through rescaling factor $\mathcal{Z}$. The Lagrangian with rescaled fields is now,

$$
\begin{aligned}
\phi_B &\longrightarrow \phi = \frac{\phi_B}{\sqrt{\mathcal{Z}}}, \quad \text{(rescaling)} \\
L_B &\longrightarrow L = \frac{1}{2}\mathcal{Z}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_B^2\mathcal{Z}\phi^2 - \frac{\lambda_B}{4!}\mathcal{Z}^2\phi^4
\end{aligned}
\tag{2.13}
$$

We also write expressions for rescaling the bare couplings so that we get a Lagrangian with renormalized couplings (m, λ) which we actually measure in the experiments.

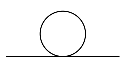
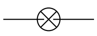
$$
\begin{aligned}
m_B^2 &= \mathcal{Z}^{-1}(m^2 + \delta m^2), \\
\lambda_B &= \mathcal{Z}^{-2}(\delta\lambda + \lambda) \\
\text{and, } \mathcal{Z} - 1 &= \delta\mathcal{Z}
\end{aligned}
\tag{2.14}
$$

The quantum effects are encoded in $\delta\mathcal{Z}, \delta m, \delta\lambda$ which represent our freedom to adjust the couplings in original action. They form the *counter* Lagrangian δL which absorb the infinities from the loop calculation we formed earlier. The new Lagrangian L is given by,

$$
\begin{aligned}
L_B \to L &= L + \delta L \\
L &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^2 \\
&+ \frac{1}{2}\delta\mathcal{Z}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\lambda}{4}\delta\lambda\phi^2
\end{aligned}
\tag{2.15}
$$

We have to note that, the Feynman diagrams would involve renormalized mass and coupling constant. As we are adding extra counter terms to the Lagrangian, these result in interactions which correspond to new vertices. The exact propagator in (2.24) would have additional counter terms.

$$\Delta^{-1} = p^2 + m_B^2 - \Pi(p^2)$$

$\Pi(p^2) \equiv$  $+$  $+ \ldots$ (2.16)

$$= \Sigma(k^2) + i(p^2\delta\mathcal{Z} - \delta m^2)$$

In principle, we can expand Σ to any orders in loop. At 1-loop level, the sum of all 1PI diagrams $\Sigma(k^2)$ (see (2.5)), receives a counter term equivalent to a propagator $(p^2\delta\mathcal{Z} - \delta m)$ which is represented with crossed vertex which reminds us that they are corrections at perturbation level. We consider counter terms to 1-loop order, which includes only a tree level propagator for 1 loop diagram shown in (2.16). We can get higher order terms by expanding δm as a power series in λ. Nevertheless we have to fix $\delta\mathcal{Z}$, δm and $\delta\lambda$ using some *renormalization conditions* such that the divergences are cancelled.

1.
- The renormalized exact propagator should represent the classical propagator with m being actual physical mass. As a classical propagator it (i) should have a pole at physical mass $-p^2 = m^2$ and (ii) residue of this pole should be i.
- We expand $\Pi(p^2)$ in external momenta p^2 around arbitrary value of m^2 upto second order (derivative with respect to p^2 is indicated by $'$)

  $$\begin{aligned}\Pi(p^2) &= \Sigma(k^2) + i(p^2\delta\mathcal{Z} - \delta m^2) \\ &= \Pi(m^2) + (p^2 - m^2)\Pi'(m^2) + \tilde{\Pi}(p^2) + i(p^2\delta\mathcal{Z} - \delta m^2)\end{aligned}$$

  The first two terms are respectively quadratic and logarithmic divergent whereas the third term is not divergent. Since we want exact propagator to have pole at physical mass,(i) we would want $\Pi(m^2) = m_B^2 - m^2$. Then the exact propagator is just a classical propagator. It would have been convenient to have started with physical mass in the initial Lagrangian. Then, on the shell $-p^2 = m^2$, the quantum correction cancel to give, $\Pi(m^2) = 0$. This would satisfy the first condition of m^2 being the pole.

  (ii) The second condition can be met by setting $\Pi'(m^2) = 0$, (this is true because the integral Σ is independent of p) which leads to $\delta\mathcal{Z} = 0$ and thus the residue is i. From (2.16) we set the complete divergent integral Σ in the above equation to the remaining parameter, δm which has the opposite sign. In the previous section we regularized this integral.
- With cutoff regularization

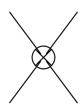
  $$\delta m^2 = \frac{i\lambda}{32\pi^2}\left[\Lambda_{cut}^2 - m_L^2 \ln\left(1 + \frac{\Lambda_{cut}^2}{m_L^2}\right)\right]. \tag{2.17}$$

  Now, the 1-loop quantum correction Σ is divergent, even in the continuum limit $\Lambda_{cut} \to \infty$. After all this calculations, we see that, we need not integrate over all the energy scales [3].
- And in case of dimensional regularization, the 1-loop integral in (2.10) implies,

  $$\delta m^2 = \frac{i\lambda(\mu)}{32\pi^2} m_L^2 \left(\frac{2}{\epsilon} - \gamma_E + \ln\frac{\mu^2}{m_L^2} + \mathcal{O}(\epsilon)\right) \tag{2.18}$$

2. The 4-point function (scattering amplitude) including the counter terms is written as follows (also Feynman diagram of counter term vertex).

$$\Gamma^{(4)} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)] - i\delta\lambda$$



[3] We can simply put the cutoff scale right in the partition function (7.1).

The renormalization condition then would be to set λ to be scattering amplitude at zero momentum : $s = 4m^2$ (we ignore t and u-channel contribution). Which implies from (2.8),

$$\begin{aligned}\delta\lambda &= -\lambda^2[V(4m^2)] \\ &= \frac{\lambda^2(\mu)}{2}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 dx\left[\frac{1}{[m_L^2 - x(1-x)4m^2]^{2-d/2}}\right] \\ &= \frac{3\lambda^2(\mu)}{32\pi^2}\int_0^1 dx\left[\frac{2}{\epsilon} - \gamma_E + \log\left(\frac{4\pi}{m_L^2 - x(1-x)4m^2}\right)\right]\end{aligned} \tag{2.19}$$

This is *on-shell renormalization*, in which we fixed the counter terms to get the actual physical mass and scattering amplitude.

2.2 Running of couplings and Fixed Points

Even upon removal of divergent parts, the resulting mass and coupling is still not sufficiently 'practical' couplings as it still depends on the renormalization scale. We would also need initial values of renormalized couplings to find the physical couplings at any given energy scale. The physical quantities like, scattering matrix, or exact propagator in (2.9) has to be independent of these conditions, especially independent of renormalization scale μ. For example, lets us consider a theory for massless scalar field with only quartic self coupling. Physical coupling λ to be independent of μ we should have,

$$\begin{aligned} 0 &= \mu\frac{\partial}{\partial\mu}\lambda \\ &= \mu\frac{\partial}{\partial\mu}\left[\lambda_B - \frac{3\lambda^2(\mu)}{32\pi^2}\log\mu^2\right] \\ \implies \beta(\lambda) &= \mu\frac{\partial\lambda}{\partial\mu} = \frac{3\lambda^2(\mu)}{16\pi^2} + \mathcal{O}(\lambda^2).\end{aligned} \tag{2.20}$$

This is an example of *Beta Function.* If we define $\beta = \mu\frac{\partial\lambda}{\partial\mu}$ and also include bare mass, then we can rewrite the above equation as[4],

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \mu\frac{\partial m_B}{\partial\mu}\frac{\partial}{\partial m_B}\right)\lambda = 0, \tag{2.21}$$

which is a simple version of *Callan-Symanzik Equation.* For cutoff regularization, we have a similar expression

$$\frac{d}{d\Lambda_{cut}}\lambda = \left(\frac{\partial}{\partial\Lambda_{cut}} + \frac{dm_B^2}{d\Lambda_{cut}}\frac{\partial}{\partial m_B^2} + \frac{d\lambda_B}{d\Lambda_{cut}}\frac{\partial}{\partial\lambda_B}\right)\lambda = 0. \tag{2.22}$$

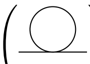
[4]In d= 4, λ would be independent (because the quartic coupling is marginally irrelevant in 4 dimensions) of μ.

Instead of λ one could have any other physical quantity like, the exact propagator for example,

$$\frac{d}{d\Lambda_{cut}}(\Delta(p^2))^{-1} = \left(\frac{\partial}{\partial\Lambda_{cut}} + \frac{dm_B^2}{d\Lambda_{cut}}\frac{\partial}{\partial m_B^2} + \frac{d\lambda_B}{d\Lambda_{cut}}\frac{\partial}{\partial\lambda_B}\right)\Delta^{-1} = 0, \tag{2.23}$$

Δ, as we saw earlier is a geometric series in $\Sigma(k^2)$. At 1-loop level,

$$\Delta^{-1} = p^2 - m_B^2 - \left[\left(\quad\right) + \mathcal{O}(\text{2-loop}) + ...\right] \tag{2.24}$$



Now using cutoff regularized 2-point function of (2.6) in the above equation at 1-loop level, and then using (2.23), we have,

$$\Lambda_{cut}\frac{dm^2}{d\Lambda_{cut}} = -\frac{\lambda_B}{16\pi^2}(\Lambda_{cut}^2 - m_L^2). \tag{2.25}$$

With the above examples we want to emphasis that beta function gives us the flow of couplings or rather the theories themselves *w.r.t* renormalization scale μ. For scalar field, we see that the beta function is positive. That is the coupling increases with energy scale. This is similar to QED. Solution to the beta function (2.20) is,

$$-\frac{1}{\lambda}\bigg|_{\lambda_0}^{\lambda_{UV}} = \frac{3}{16\pi^2}\ln\frac{\mu_{UV}}{\mu_0} \implies \lambda_{UV} \approx \frac{\lambda_0}{1 - \frac{3}{16\pi^2}\lambda_0 \ln\frac{\mu_{UV}}{\mu_0}}. \tag{2.26}$$

In practical applications, μ_0 would determine the energy scale of the experiment performed Λ_{exp}, and λ_0 would be the value of experimentally obtained coupling constant. This would involve known interactions from which we can precisely compute λ_0. $\mu_{UV} \equiv \Lambda_{cut}$ determines the cut-off energy scale where we believe the considered quantum field theory still works. Coupling at high energies then seem to be increasing with Λ_{cut} as shown below in Figure(2.1). From the solution above, we see that there is a pole when $\mu_{UV} = \mu_0 \exp\left(\frac{16\pi^2}{3}\right)$. This is Landau pole which determines the scale up to which our perturbative approach dealt in this chapter is valid. There may be new physics at very high energies which are yet to be discovered. This is very important for the Hierarchy problem described in the next section. The QED beta function is very similar to our analysis of scalar field model. For QCD, we have a negative beta function which also has a Landau pole indicating the breakdown of perturbation theory. Here we expect non-perturbative physics for example hardronization.

For a generic coupling g, the solution to beta function $\beta(g) = \mu\partial g/\partial\mu$ is,

$$\int_{g_0}^{g_{UV}} \frac{dg}{\beta(g)} = \ln\left(\frac{\mu_{UV}}{\mu_0}\right) \text{ diverges for } \mu_{UV} \to \infty \text{ if } \begin{cases}\beta(g^*) = 0 \implies g \to g^* \\ g \to \infty\end{cases}. \tag{2.27}$$

For $\mu_{UV} \to 0$ then we have IR flow of couplings and for $\mu \to \infty$ we have UV flow. The divergent UV flow can be result of either $g \to \infty$ or $\beta(g^*) \to 0$ in the limit where g goes to some g^*. The points where couplings change sign, that is, those points where evolution curves meet $x - axis$ in the above plot we have $\beta(0) = 0$ which are called the *fixed points*. See Figure(2.1) and the description below[5].

1. $\beta > 0$ (the violet curve): Beta function always remain positive. This is the case in QED where the coupling increases with energy until it reaches Landau pole beyond which we cannot use the perturbative approach to track its growth. This is also the case for scalar field (ϕ^4) model described in this chapter. Equation (2.20) gives the beta function for such a theory where the coupling increases with energy. As discussed in the next section, we can still ask if we have non trivial zero of the beta function. It turns out that ϕ^4 theory has trivial zero which is, beta function vanishes for vanishing coupling.

2. $\beta < 0$ (the green curve): This is the case where the coupling vanishes for very high energies. QCD is one example of this scenario. The flow however goes to trivial zero which we know as *asyptotic freedom* where coupling goes to zero at large energies.

3. There are also non-trivial fixed points indicated by red and blue dots. The red curve represents a the beta function which is positive till it reaches the red point. The flow then reverses the direction, that is beta function is negative for larger couplings. The fixed point is attained at high energies from the both the directions and hence this is a UV fixed point. The blue point is an example of IR fixed point where the couplings flow towards this fixed point.

Beta functions can tell us about the flow of couplings and moreover, they are scheme independent. We shall introduce them again in the context of Wilsonian Renormalization Group flow and Callan-Symanzik equation.

2.3 Problems of Triviality, Hierarchy and Fine Tuning

Equation (2.26) is seemingly simple yet encapsulates some of the fundamental problems. One of those is realised when we want to extend the above solution to all energy scales $\mu_{UV} \to \infty$, then we would want $\mu_0 \to 0$ to avoid divergence of the coupling at higher energies. But then, the theory should be independent of scales and μ_0 should remain zero at all scales which implies we simply have non-interacting, trivial theory at all scales. The question one could ask is, if this is a valid behaviour at high energy scales. However, we have to notice that this *triviality problem* has

[5] The colour indication is in accordance with Figure(2.1)

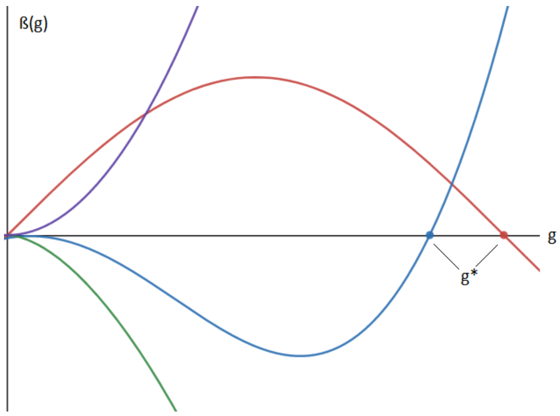


Figure 2.1: Examples of beta functions and fixed points is shown in this β vs g plot. This plot is inspired by [18]. All the curves have a trivial fixed point where the coupling vanishes. There are also IR and UV fixed points $g^* \neq 0$ indicated by blue and red points respectively.

appeared due to presence of Landau pole. If one can still continue the theory with non-perturbative approach beyond Λ_{cut} then we have no such issues.

The global symmetry underlying Standard Model(SM) of particles helped us discover many particles in the previous century (see for example [19], for historical introduction). By gauging the global symmetries we demand the Lagrangian to be invariant under local diffeomorphism, which leads to Gauge Theory. The SM is a gauge theory with gauge group $G_{SM} \equiv SU(3)_C \times SU(2)_W \times U(1)_Y$, where the subscripts indicate the respective charges- color, weak isospin and weak hypercharge. The Higgs field which is usually represented by a complex (doublet) scalar field, through electro-weak symmetry breaking $SU(2)_W \times U(1)_Y \to U(1)_{EM}$ gains mass but also gives mass to weak W and Z bosons. The vacuum solution breaks the symmetry but the symmetry of the Lagrangian is still maintained. The Higgs particle is the excitation around the vacuum expectation value. The mass term of Higgs is valid under gauge symmetry but fermions cannot have bare mass terms because, such terms will violate gauge symmetry [6]. So, SSB is the only standard mechanism we have at our disposal to give masses to elementary particles while preserving the gauge symmetry. Fermions, on the other hand, through Yukawa couplings to Higgs, also get their mass in SM.

SSB was triggered for $m^2 < 0$ implying a imaginary mass. This already hinders our need for fundamental understanding. On the other hand, apart from the classical potential, there are also quantum corrections in QFT for the scalar field. We saw these corrections for mass and quartic coupling at 1-loop level in the previous section. It was shown in [20] that we could have SSB even with $m = 0$ through radiative corrections. One also has to remember that the Higgs vacuum expectation value is gauge dependent, that is it can be zero or non-zero depending on the choice of gauge

[6]It mixes the left and right handed particles. In SM left and right handed particles have different gauge quantum numbers and interact differently

[21, 22]. In simple words, the Higgs need not be simple doublet field whose vacuum expectation value is within our reach in the particle physics experiments. It could have been a much higher dimensional field depending on the choice of gauge and its representation with gauge dependent vacuum.

In our calculations of radiative corrections, we have noted that problems with diverging integrals appear. For example, in calculations to compute corrections to Higgs mass using cutoff regularization (2.6), the loop integration over all possible momenta of virtual particle was cutoff with limit Λ_{cut}. This limit could be proxy for some unknown energy scale for example the scale of Grand Unified Theory Λ_{GUT}, Planck scale Λ_P etc. The corrections scale with the cutoff, and for $\Lambda_{cut} \sim \Lambda_P$, there is an *unnatural* separation between Higgs mass and Λ_P. From definitions of exact propagator,

$$m = m_B - m_{correction} \quad \text{where, } m_{correction} = [\Sigma - \delta m]. \tag{2.28}$$

Due to quadratic divergence, the quantum corrections $m_{corrections}$ to Higgs mass is greater than Higgs mass itself. For example[7], the *top* quark, with mass $m_t \sim 172\text{GeV} > m \sim 125\text{GeV}$ (m is just the Higgs mass) leads to a very large correction. The separation between high energy cutoff scale and electro-weak scale is the *Hierarchy Problem*. This can be compared to problems encountered with CC in (1.15). There appears to be a significant dependence on the *new* physics scale which is endowed by the mass of the particle in the loop. This implies we have an *unnatural*, *fine tuning* in the initial conditions to have a small value of observed Higgs mass.

Interestingly, one can also think of this problem being only manifested in cutoff regularization. This suspicion is valid because the results of quadratic divergence upon applying cutoff regularization, disappears in dimensional regularization (2.10). The problem in cutoff regularization appears when we believe a fundamental theory at high energy scales exist (UV-complete theory) and embed Higgs as a low energy effective theory. However, any scheme will introduce a new parameter which will eventually lead to logarithmic divergences at 1-loop level.

There are various proposals which could be possible solutions to the problems mentioned above. It has to be noted that this Hierarchy problem is generic to any scalar field, hence the dark energy models which use scalar fields to mimic the dynamics of a cosmological constant (for example, Quintessence) are not addressed to solve the Hierarchy problem but rather to explain the observed acceleration of the Universe alone. However in a broader perspective, one can attempt to address both issues using new models which will be discussed in the further sections. Theories like Supersymmetry try to address this problem by posing a bigger symmetry where the scalar field mass is protected similar to the chiral symmetry which protects fermion mass [23]. There are also proposals of Technicolor[24], Relaxation [25], extra dimensions and Composite Higgs [26].

[7] The loop could be simply by a top quark in which case, the mass of top quark determines the Higgs mass; but it is actually the Higgs which gives the mass to top quark through Yukawa coupling.

Since $\lambda \sim m^2/v^2$, it is relatively hard to obtain precise values of λ in comparison to m and we need to aim for very precise value of the latter in order to get good estimates for λ. The major dependence is on the top quark mass. However, the potential is assumed to be stable as long as scale μ dependent $\lambda(\mu)$ is positive. This is necessary to make distinction between 'true' vacuum and 'false' vacuum. If the vacuum we have found is not 'true' and there exists much deeper minima, then there could be tunneling towards that minima to the deeper minima. This implies decay of electro-weak vacuum and this, apparently has cosmological consequences [27, 28]. Precision calculations of λ indicate that the data favours *metastable* potential [29, 30]. For overview of the problem and its consequence see [31] and references therein. We can of course introduce new physics in the intermediate scales ($\sim \mathcal{O}(10^{11})$GeV - this is approximately the scale at which the potential becomes unstable) and hence could possibly make potential stable [28].

3 Vacuum Energy

After looking at the quantum corrections to bare couplings of Higgs through loop diagrams, we shall do a similar exercise to extend our notion of CC problem. Vacuum energy in flat spacetime is not of any concern in particle physics experiments because the observable quantities are dependent on differences in energy and not the absolute energy itself. Hence our precise calculations of particle physics which are in flat spacetime, are not affected by the presence of vacuum energy. However, GR encapsulates the complete energy density present in the (local) spacetime. The first section of this chapter deals with vacuum energy in flat spacetime which supplements our discussions on curved spacetime in the following section. Our objective will be to represent energy density in terms of propagators. We make use of [32] along with [1] which gives an extensive review on the topic. For the second section we also refer to standard texts and papers like [33, 34, 35, 36, 37, 38].

3.1 Zero Point Energy: Flat Spacetime

Along with the classical contribution, CC receives contribution from quantum zero point fluctuations. The latter will still remain even if we could somehow vanish the classical contributions of the last section. Thus the second terms on the *r.h.s* in (1.14) and (1.15) involves the quantum contributions. Hence it becomes important to discuss Zero Point Energy (ZPE), the quantum ground state of energy momentum tensor. We shall do this first, for a free theory and then for interactions in the next section. Our approach in this section will be semi-classical, which means we shall not discuss quantizing the gravity part of the Einstein equation and rather focus on ZPE.

3.1.1 ZPE: Free Theory

For free theory, $\lambda = 0$ in our potential of the scalar field (1.10). Now with this potential $V(\phi) = m^2\phi^2/2$ we can derive the field equations by taking the variations of (1.6) *w.r.t* ϕ. We arrive at the Klein-Gordon equation

$$\nabla^2\phi - m^2\phi = 0. \tag{3.1}$$

This is similar to differential equation for a simple (linear) harmonic oscillator. Note that $\nabla^2 = \nabla_\nu\nabla^\nu = g_{\mu\nu}\nabla^\mu\nabla^\nu$, with metric signature $(-,+,+,+)$ and on flat spacetime $g \to \eta$ and $\nabla \equiv \partial$. To solve the above equation, we go to the Fourier space where, (the bold letters indicate 3-vectors)

$$\phi(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{k}, t) \text{ and } \quad -\ddot{\phi} + (\mathbf{k}^2 + m^2)\phi = 0 \tag{3.2}$$

Promoting the scalar fields as operators, it can be expressed in terms of annihilation and creation operators $a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}}$ which are quantum operators and follow the commutation rules [13]

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E}} \left(a_{\mathbf{k}} e^{-iEt+i\mathbf{k}.\mathbf{x}} + a^{\dagger}_{\mathbf{k}} e^{iEt-i\mathbf{k}.\mathbf{x}} \right); \; \left[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k'}} \right] = \delta^3(k - k'). \tag{3.3}$$

The Lorentz invariant normalization factor $\sqrt{2E}$ (with factor of 2 for convenience) has been used like in[1][13] but the only difference is that we have $(2\pi)^{3/2}$ instead of its square. The time component of the 4-momentum $k^{\mu} \equiv (E, \mathbf{k})$, represents the frequency or energy of the harmonic oscillator ($|\mathbf{k}| = k$ is the magnitude of 3-momentum vector)

$$\omega(k) \equiv E = \sqrt{k^2 + m^2} \tag{3.4}$$

The fields are quantized which is evident from commutation relation of ladder operators as this is equivalent to canonical commutation relation $[\phi(t, \mathbf{x}), \pi(t, \mathbf{x'})] = i\delta^3(\mathbf{x} - \mathbf{x'}))$ where $\pi(t, \mathbf{x'})$ is the conjugate field operator. With the above quantum definition of the field $\phi(\mathbf{x}, t)$, we now turn to calculate vacuum state of the energy momentum tensor. In general, with four velocities u_{μ} normalized as $g_{\mu\nu}u^{\mu}u^{\mu} = u_{\mu}u^{\mu} = -1$, the energy momentum is given by,

$$\langle T_{\mu\nu} \rangle = \langle \rho u_{\mu} u_{\nu} + (g_{\mu\nu} + u_{\mu}u_{\nu})p \rangle. \tag{3.5}$$

From this expression we can extract mean energy density $\langle \rho \rangle$ and pressure $\langle p \rangle$ by projecting energy momentum tensor along $u^{\mu}u^{\nu}$ and $(g^{\mu\nu} + u^{\mu}u^{\nu})$ respectively. We have computed the energy momentum tensor for a scalar field in (1.7),

$$\langle T_{\mu\nu} \rangle = \begin{cases} \langle T_{00} \rangle & = \langle 0 | \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\eta^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}m^2\phi^2 \right] |0\rangle \\ \langle T_{ij} \rangle & = \langle 0 | \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\eta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 \right] |0\rangle . \end{cases} \tag{3.6}$$

[1]

$$\int \frac{d^4k}{(2\pi)^{3/2}} \delta(p^2 - m^2)\theta(E) = \int \frac{d^4k}{(2\pi)^{3/2}} \left(\frac{1}{E}\delta(E - \sqrt{\mathbf{p} + m}) + (\frac{1}{E}\delta(E + \sqrt{\mathbf{p} + m}) \right) \theta(E)$$

(for positive energy the second term vanishes)

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2E}$$

is a Lorentz invariant integral.

We point out that for slowly varying fields (negligible kinetic energy), the scalar field energy density behaves like that of a cosmological constant with $\langle T_{00}\rangle = -\langle T_{ij}\rangle$. With (3.3), we can calculate mean values of each of the quantities in the above expression

$$\langle\dot{\phi}^2\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(k)}E^2(k)\ ;\ \langle\eta^{ij}\partial_i\phi\partial_j\phi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{2E(k)}\ ;\ \langle\phi^2\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(k)}.$$

In the inertial reference frame of the observer, four velocities are given by $u^\mu = (1,\mathbf{0})$ such that energy momentum tensor is diagonal. With above integral quantities plugged into (3.6) we can have mean energy and pressure density in this frame as

$$\langle\rho\rangle = \langle[u^\mu u^\nu T_{\mu\nu}]\rangle = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}E(k) = \frac{1}{(2\pi)^3}\frac{1}{2}\int d^3k\sqrt{k^2+m^2} \tag{3.7}$$

$$\langle p\rangle = \frac{1}{3}\langle[g^{\mu\nu} + u^\mu u^\mu T_{\mu\nu}]\rangle = \frac{1}{6}\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{E(k)} = \frac{1}{6}\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{\sqrt{k^2+m^2}}. \tag{3.8}$$

In out preferred frame, there are no off-diagonal terms. The EoS is not similar to that of CC ($w^\Lambda_{vac} = -1$). In fact, the average quantities are expressed in terms of integrals whose solution might result in expected EoS of -1. But, both of the integrals are indefinite and diverge for large momentum. Usually such integrals are ignored in the absence of gravity because relative energies are observable and not absolute energy. For instance, starting with the Lagrangian of the scalar field

$$L = -\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2,$$

and its canonical momentum $\pi(x) = \frac{\partial L}{\partial\dot{\phi}} = \dot{\phi}$, we can construct the Hamiltonian $H = \int d^3\mathbf{x}[\pi(x)\dot{\phi} - L]$. That is,

$$\begin{aligned} H &= \int d^3\mathbf{x}[-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2] = \int\frac{d^3k}{(2\pi)^3} \\ &= \int\frac{d^3k}{(2\pi)^3}E_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}\right]\right). \end{aligned} \tag{3.9}$$

Here we have denoted energy associated with particular state as $E_{\mathbf{k}}$. From commutation relations of annihilation and creation operators in (3.3), the second term which is usually associated with energy of the scalar particle in its ground state yields $\delta(0)$ which diverges to infinity. We can reformulate it using the relation $\delta(0) = V/(2\pi)^3$ where V is the space volume which runs to infinity, we can write energy density as,

$$\rho = \int\frac{d^3k}{(2\pi)^3}\frac{\omega_{\mathbf{k}}}{2} \tag{3.10}$$

which is simply the replica of (3.7). The ground state energy given by $\omega_{\mathbf{k/2}}$ appears in all quantized oscillators. Sum of all the ground state energies of these oscillators constitutes Zero Point Energy (ZPE) $(\hbar/2)\sum_{\mathbf{k}}\omega_{\mathbf{k}}$. For historical facts on ZPE refer to [1, 2]. Since the sum is over all possible $\mathbf{k}$ it gives infinities. While doing standard quantum mechanics or QFT, we escape these quantities by claiming that only relative energies are measurable not the absolute quantities. Whereas, in the gravitational context we place all these energy densities to the *r.h.s* of the Einstein Equations. Since all forms of energy contributes for curvature of the spacetime, say, the Universe with only ZPE will not yield flat Minkowski spacetime as a solution. This implies that space time is unavoidably curved! We will leave the consequences of treating this problem in curved spacetime in the subsequent sections and now focus on making sense of the divergent integrals.

As we saw in the previous chapter, we employed some renormalization schemes to make sense out of divergent integrals. One of them is through cut-off regulator which introduces a high energy scale as a cut off such that we have a well defined effective theory for scales lower than it. This made sense in our previous chapter because we need not have the information of Higgs physics to make soaps and plastics. Because while dealing with the energy scales of chemistry, involving atomic or molecular interactions, the Higgs influence is simply not relevant. But vacuum energy is not soap! - it involves contribution from all the particles. Even Planck's scale as cutoff will not solve problem as CC problem remains at all scales. Our expression for Λ_B in (1.15) has become worse than the previous chapter with ZPE contribution

$$\Lambda_E = \Lambda_B + \frac{8\pi G_N}{2(2\pi)^3} \int d^3k \sqrt{k^2 + m^2}. \tag{3.11}$$

As discussed earlier, imposing 3-momentum cut-off will break Lorentz symmetry. The regularization schemes should maintain the symmetry of the original theory (in SM the gauge symmetries has to preserved). So we use dimensional regularization instead. We start with redefining all the quantities in $d-1$ dimensions. Then the field in (3.3) would be,

$$\phi(\mathbf{x}, t) = \int \frac{d^{(d-1)}k}{(2\pi)^{(d-1)/2}} \frac{1}{\sqrt{2E}} \left(a_{\mathbf{k}} e^{-iEt + i\mathbf{k}.\mathbf{x}} + a^\dagger_{\mathbf{k}} e^{iEt - i\mathbf{k}.\mathbf{x}} \right). \tag{3.12}$$

In $d-1$ dimensions, the energy and pressure density integrals are

$$\begin{aligned} \langle \rho \rangle &= \frac{\mu^\epsilon}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} E = \frac{\mu^\epsilon m^d}{2(4\pi)^{d-1}} \frac{\Gamma(-d/2)}{\Gamma(-1/2)} \\ \langle p \rangle &= \frac{\mu^\epsilon}{2(d-1)} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{k^2}{E(k)} = \frac{\mu^\epsilon m^d}{4(4\pi)^{d-1}} \frac{\Gamma(-d/2)}{\Gamma(1/2)}. \end{aligned} \tag{3.13}$$

In the first equality, the integrals look like Euclidean integrals in $d-1$ dimensions carried out for loop integrals in the previous section. We have introduced a new energy scale μ for dimensional justification and $\epsilon = 4 - d$ in the second equality and have followed the same procedure which are explicitly done in the Appendix(C.4). As we expected, the regularized integrals of energy and pressure densities gives the correct EoS (since $\Gamma(-1/2) = -2\Gamma(1/2)$),

$$w^\Lambda = \frac{\langle p \rangle}{\langle \rho \rangle} = -1.$$

Finally we find the approximate behaviour around the isolated poles of Gamma function by expanding around $\epsilon = 0$

$$\langle\rho\rangle = \frac{1}{2}\beta_\Lambda^{(1)}\left[\gamma_E - \frac{2}{\epsilon} - \frac{3}{2} - \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right], \quad \text{where, } \beta_\Lambda^{(1)} = \frac{m^4}{32\pi^2} \tag{3.14}$$

is the one loop beta function. For $\overline{MS}$ renormalization, we can choose the counter term to be $\langle\delta\rho\rangle = \frac{m^4}{64\pi^2}\left(\gamma_E - \frac{2}{\epsilon} + \ln 4\pi\right)$. Now the physical (renormalized up to 1-loop) vacuum energy density is given by,

$$\langle\rho\rangle_{vac}^{(1)} = \langle\rho\rangle_B + \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right). \tag{3.15}$$

This equation can be compared with (3.11). From (3.13), the average energy momentum is given by,

$$\langle T\rangle = -\langle\rho\rangle + (d-1)\langle p\rangle = \frac{\mu^\epsilon m^2}{2}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{E(k)}.$$

The above integral in QFT is known under the name of Feynman propagator D_F which is evaluated at the same point. Feynman propagator in (3.16) itself is represented by a line segment indicating propagation of a scalar field from point x_1 to x_2. But in our case of zero point energy, the integral is evaluated at the same point, $x_1 = x_2$. It is represented by a vacuum to vacuum diagram also called as bubble diagram as shown below.

$$D_F(x_1 - x_2) = i\int\frac{d^4k}{(2\pi)^4}\frac{\exp\{ik_\mu(x_1^\mu - x_2^\mu)\}}{k^2+m^2} = \underline{\qquad\quad} \tag{3.16}$$

$$\text{in our case, } D_F(x_1 - x_2) = i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} = \bigcirc. \tag{3.17}$$

We can solve time integral of (3.17) to realize $\langle T\rangle = m^2 D_F$ and hence, $\langle\rho\rangle = -m^2 D_F/4$. Now, we rewrite the effective cosmological constant and restate the CC problem from (3.11) as,

$$\Lambda_E = \Lambda_B - 8\pi G_N\frac{m^2}{4}D_F. \tag{3.18}$$

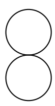
In our calculations so far, there are at least three assumptions which will have a significant effect on estimation of ZPE. The same non zero vacuum energy density appears in Quantum mechanics as the ground state energy. But it is simply neglected due to the fact that in laboratories we only measure relative energies and not absolute energy. In general theory of gravity, all forms of energy gravitate. ZPE cannot be neglected if we switch on the gravitational interactions. The above formalism is also not a good estimate of actual vacuum energy density because it is non zero for massive particle which contradicts our conclusion. Photons for example are safe because they are not massive and interaction terms would not contribute to energy density. To have a democratic treatment it is better to express in terms of (high energy) cutoff scale rather than mass of the particle, in which case, the orders of magnitude might change. Also, the above expression is only first order estimation of ZPE. If we go beyond free theory and include interactions we get corrections to our leading order estimation.

3.1.2 ZPE: With Interactions

Lets quickly calculate ZPE with corrections from interactions and proceed with ZPE in curved spacetime in the next section. We again make use of [13, 32]. Feynman propagator introduced in the previous section is actually a part of a much useful quantities called *correlation function.* Feynman propagator are called as *two point correlation function* or *two point Green's function* and expressed as $D_F = \langle\Omega| T\phi(x_1)\phi(x_2) |\Omega\rangle$. This simply means, it is amplitude for propgation of a particle (excitation of a field) between x_2 and x_1 and by definition, it is only a function of the difference $x_2 - x_1$. $\langle\Omega|$ denotes ground state in full interaction theory. It is standard to use $\langle 0|$ to denote ground state of free theory which is different from $\langle\Omega|$. T is time ordering symbol. One can of course think of three point or n point correlation function. In fact, the Feynman propagator for vacuum to vacuum bubble is actually one point correlation function.

So we consider the full potential in (1.10) with $\lambda \neq 0$. Then we expect all sorts of loop diagrams contributing to the free theory correlation function. They can be divided into connected and disconnected diagrams. The simplest disconnected diagram at two loop is given below in which the two loops intersect at a vertex given by $(i\lambda)\int d^4x$. This is still a vacuum to vacuum diagram and hence adds to ZPE. The corresponding energy density is given by,

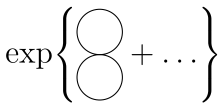
$$\frac{\lambda}{4!}\langle\phi^4\rangle = \frac{3\lambda}{4!}\langle\phi^2\rangle^2 = \frac{\lambda}{8}D_F^2(0),$$



where, factor 3 accounts for possible combinations. Along with free theory ZPE contribution given in (3.18), we have the above term appearing in interaction theory. Hence the total vacuum energy density is given by,

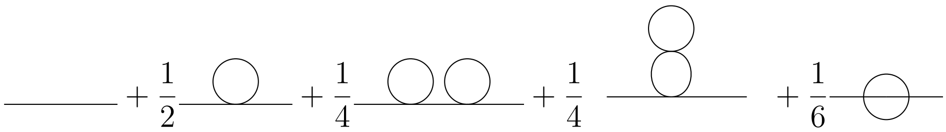
$$\langle\rho\rangle = -\frac{m^2}{4}D_F(0) + \frac{\lambda}{8}D_F^2(0) \tag{3.19}$$

As said earlier, vacuum bubbles are not relevant for calculating physical quantities like cross section *etc.* Only connected diagrams given by correlation function $\langle\Omega| T\phi(x_1)\phi(x_2) |\Omega\rangle$ are relevant. All the disconnected diagrams gets canceled upon using an identity which suggests exponentiation of all disconnected diagrams (it is explained and proved in Chapter 4 of [13]). For example, two loop disconnected diagram is as shown below



Hence, $\langle\Omega| T\phi(x_1)\phi(x_2) |\Omega\rangle$is just sum of all connected diagrams. Also, because of causality conditions, the commutator $[\phi(x_1)\phi(x_2)]$ is a well defined quantity rather than simply $\phi(x_1)\phi(x_2)$. We still need to renormalize mass which is similar to our calculation for Higgs mass in the previous section (see (2.10)). Up to two loops, the two point connected correlation function is given below.

$$\langle\Omega| T[\phi(x_1)\phi(x_2)] |\Omega\rangle =$$



(3.20)

A detailed derivation of the same but using functional methods is given in Part III. At one loop, we can use the integral in (2.10) and write the relation between renormalized mass m_{renorm}, bare mass m and the loop integral which simply reduces to $-\lambda D_F/2$ at leading order of the coupling λ, as, (similar to (2.28))

$$m^2_{\text{renorm}} = m^2 - \frac{\lambda}{2} D_F(0).$$

We can use this expression to express (3.19) in terms of renormalized mass as

$$\langle\rho\rangle = -\frac{m^2_{\text{renorm}}}{4} D_F(0). \tag{3.21}$$

Before going further, we remind ourselves that, all these calculations are in flat spacetime. The dependence of ϕ is only in the loop diagrams of the perturbation theory. Otherwise, ZPE is only depending on the couplings. To summarize, we can define a effective potential,

$$V_E(\phi) = V_{clas} + \hbar V_1 + \hbar^2 V_2 + \dots,$$

which reduces to classical contribution derived in (1.13) if $\hbar = 0$. For $\hbar \neq 0$ the quantum contributions are switched on and at every loop order the potential gets $\hbar$ contribution. At every loop order, again we have disconnected bubble diagram contribution with no external legs V_D^i's where i is a natural number indicating the loop order. Along with this pat, every loop order also has loop diagrams with external ϕ legs (as shown in the diagrams above) and hence this is field dependent. That is, if we set $\phi = 0$, then the external legs vanish. Here, the couplings (like mass) has to be renormalized. These contributions from connected diagrams are denoted by V_C^i's

$$\begin{aligned} V_E(\phi) &= V_{clas} + \hbar(V_D^1 + V_C^1(\phi)) + \hbar^2(V_D^2 + V_C^2(\phi)) + ... \\ \rho_E^{\phi vac} &= \rho_B^{\phi vac} - \rho_{clas}^{\phi vac} + \hbar(V_D^1 + V_C^1(\phi)) + \hbar^2(V_D^2 + V_C^2(\phi)) + ... \\ 10^{-47}\mathrm{GeV}^4 &= \rho_B^{\phi vac} - 10^8\mathrm{GeV}^4 + \hbar(V_D^1 + V_C^1(\phi)) + \hbar^2(V_D^2 + V_C^2(\phi)) + ... \; . \end{aligned} \tag{3.22}$$

In the second line we associate energy densities to average values of these potentials and redefine *fine tuning* problem. Evidently this expression is complicated compared to our classical estimation in (1.15). The bare vacuum energy density (like the bare mass term appearing in Lagrangian) has to be tuned very precisely from all the classical and quantum contributions at all loops to obtain a very small, yet non-zero CC. This problem is a very annoying for any physics student. Supersymmetry among others is one way to get rid of loop corrections. We expect special symmetries to be present in nature to cancel out all the loop order corrections. But since such special symmetries are broken at very high energies itself, one would however end up with classical contribution to fine tuning. Another interesting model is to have dynamical symmetry breaking where we introduce another variable say, Φ such that mass term in ϕ^4 theory (1.10) is now $m(\Phi)$. Now, symmetry breaking could have happened for any value of Φ, but accidentally, the phase transition happened at the particular value observed today. But we are still far from having a fundamental theory which can explain the fine tuning of vacuum energy. Nevertheless, studying ZPE in curved spacetime is very necessary at least for the following reason. If at all there is vacuum energy $\Lambda \neq 0$ then, it should gravitate then, a trivial Minkowski metric is not a solution for Einstein equation in vacuum. This poses a question on all our calculations on flat spacetime and motivates to study ZPE in curved spacetime.

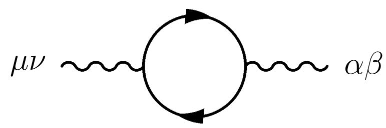
3.2 ZPE: Curved Spacetime

In this section we shall briefly discuss ZPE in curved spacetime in a semi-classical approach without dwelling into the rigorous calculations[2]. This is important because, the non zero ZPE density should result in some curvature of spacetime which is a non-trivial solution of Einstein equation. Hence, the trivial Minkowski metric cannot be a solution of Einstein equations in vacuum. On the flat background we could use standard techniques to subtract the divergent parts of the diverging integrals and hence obtain a renormalized ZPE. Here in renormalization of ZPE in curved spacetime the counter terms are external gravitational field dependent. We can try to visualize it through Feynman diagrams where, like flat spacetime, we have matter external legs attached to matter loops (see (3.20)). But now, we also have graviton legs attached to the matter loops (see (3.23)). In principle one also has loops of

[2] We would like to reproduce results of [1], for which we follow [33] but also refer to standard texts like [34, 35], or the modern book [36] and notes [38, 37].

gravitons themselves, but then we come across the problems of renormalization of gravity.

Suppose we start with EH action then one can consider the perturbations $h_{\mu\nu}$ around Minkowski metric $\eta_{\mu\nu}$ as the graviton field. Then, we can write propagators for this field similar to scalar fields. The interaction diagrams will introduce terms which are quadrtic or higher orders in $h_{\mu\nu}$. One has to introduce counter terms in our original classical EH action. EH action is surprisingly very simple but still elegant even though there is no reason for its simplicity. One can include complicated higher derivative terms to the Lagrangian. It was shown in [39] that, pure gravity is renormalizable at one loop level because in from Einstein equations $R, R_{\mu\nu}$ are zero. This also implies higher orders like $R^2, R_{\mu\nu}R^{\mu\nu}$ appearing in the loop orders are also zero[3]. But in reality we do have matter, in which case it is not renormalizable. For example, consider one loop matter correction and start with EH action which only includes Ricci scalar R. For divergent one loop vacuum polarization diagrams as shown below, we can use dimensional regularization and counter term renormalization to obtain finite results in 4 dimensions. In QED the counter term for vacuum polarization is of type $\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu}$ where Z_3 is the wavefunction renormalization parameter. For gravity, we do not have such terms in the original EH Lagrangian. But if one thinks of adding Higher Derivative (HD) terms like $R_{\mu\nu}R^{\mu\nu}, R^2$, one would have add more terms to counter the divergence caused by these terms. Hence we say gravity is perturbatively not renormalizable. The non-renormaliziblity has to do with high energy or short distance behaviour of the theory and does not lead to any new predictions; in fact, quantum effects of gravity does not play any significant role in late time Universe.



We concern ourselves with effects of spacetime curvature in our calculation of ZPE for which we will follow semi-classical approach where the background classical gravitational field interacts with quantum matter fields (hence, we need not alter the classical potential contribution to ZPE calculated in (1.13)). For such a theory, we have an action as shown below[4]. This is different from the action for a scalar field in flat spacetime with two additional features. The first evident difference is, the non-minimal coupling of the scalar field and Ricci Curvature, with ξ being dimensionless coupling. In the absence of this coupling, we have problems of renormalization of counter terms[5]. The second difference is the integral measure $\int d^4x\sqrt{-g}$ which is

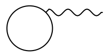
[3]This is true in four dimensions as we have Chern-Gauss-Bonnet Theorem [40]. We discuss the details later.

[4]for a scalar field ϕ, $\partial_\mu\phi \equiv \nabla_\mu\phi$.

[5]See importantace of this conformal coupling in the context of Inflation [41, 42], and actual

the volume element of generic $d-$dimensional curved spacetime with line element $ds^2 = g^{\mu\nu}(x)dx^\mu dx^\nu$ and $g = \det(g^{\mu\nu})$. From analogy with bubble diagrams in interaction theory, the integral measure corresponds to a vertex (see Feynman rules in [13]). In curved spacetime we can think of diagrams as shown below with external legs of graviton represented by a wiggly line.

$$S[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\xi\phi^2 R - V(\phi)\right] \; ; \tag{3.23}$$



$$(\Box - m^2 - \xi R)\phi = 0. \tag{3.24}$$

The Klein-Gordon equation for scalar field in curved space time in (3.24) is derived by differentiating the action (3.23) *w.r.t* scalar field. The solution solving this wave equation can be expanded in terms of annihilation and creation operators as we did in (3.3). Following [34], it is given by,

$$\phi(x) = \sum_{\mathbf{k}}\left(A_{\mathbf{k}}f_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}f^*_{\mathbf{k}}(x)\right),$$

where, $A^\dagger_{\mathbf{k}}, A_{\mathbf{k}}$ are creation and annihilation operators and, $f_{\mathbf{k}}, f^*_{\mathbf{k}}$ being mode functions and its conjugate. The solutions are not unique and hence the vacuum in curved spacetime in not uniquely determined. For example, there are ambiguities in defining a solution of a kind $f \sim e^{-k_0 t + \mathbf{k}\cdot\mathbf{x}}$ near singularities because, the time-like geodesics are incomplete (see Appendix(F.1)). Nevertheless, we can have these solution indexed in such a way that it includes an index which specifies the Cauchy surface for which they forms a complete orthonormal set [34].

Particle creation and annihilation in curved spacetime is also not trivial because we have creation and annihilation induced by gravitational field. Similarly the two point function which we can derive directly from (3.24) is not only having self interactions but also interacts with the gravitational field. It is given by, Feynman propagator in $d-$dimensional curved background $D_F(x_1, x_2) \equiv -i\langle 0|T\phi(x_1)\phi(x_2)|\langle\rangle 0)$, where we define $|0\rangle$ such that, $A_{\mathbf{k}}|0\rangle = 0$ then it obeys,

$$(\Box - m^2 - \xi R(x_1))D_F(x_1, x_2) = -\delta^d(x_1, x_2), \tag{3.25}$$

where, $\delta^d(x_1, x_2) = |g(x_1)|^{-1/2}\delta(x_1 - x_2)$ is the $d-$dimensional Dirac delta function (see Appendix(F.2)), $\Box = \nabla_\mu\nabla^\mu$ and the minus sign is for sign convention. We are interested in short distance aspects of these two point functions, because this would give us a good description of ZPE contribution to vacuum energy density of QFT (which we calculated in previous section but) in a realistic curved spacetime background. We know from our experience on flat spacetime that, we will encounter divergent integrals once we start calculating expectation value of field and energy momentum tensor associated with it. Previously, standard renormalization tools like cut-off and dimensional regularization were used. We can make use of physically more meaningful *Adiabatic Regularization* which is well suited for our present context

derivation of its value from equivalence principle [43] and also used extensively in literature [44]. In previous section, we had full (bare) energy momentum tensor which we renormalized order by order. We also noted that ZPE in flat spacetime is physically irrelevant. Here, we could have started with an assumption that metric has very slow dependence of time, then as a consequence, creation and annihilation operators operations correspond to physical particles and the vacuum state is also time independent. This is the idea of adiabatic limit, and the order of adiabatic expansion hence implies the order of time derivative of metric (for example in Friedmann - Lemaître - Robertson - Walker metric it is the number of time derivatives of the scale factor). There is also an assumption of certain symmetry like homogeneity of spacetime in order to have physical meaning for quantized fields. The adiabatic expansion is similar WKB approximation through which we obtain finite expectation values of energy momentum tensor[44]. Same results are obtained if one uses other techniques [45]. Since we can express the vacuum energy density using propagators, instead of introducing new regularization method, let us rather try to express the propagator in curved spacetime and try to isolate the divergences. Here, we want to expand the metric up to 4th order in adiabatic expansion, that is, expansion of a generic metric tensor in the neighbourhood a point where we can approximate the spacetime to flat. We make use of Riemann Normal Coordinates from Appendix(F.3) which reduces our computation efforts in writing down the Taylor expansion of metric tensor near the origin. Consider expansion of a function W around the origin point Q, with Reimann coordinates y^μ, we have at any point x in the vicinity of Q,

$$W(x) = W\Big|_Q + (\partial_\mu W)\Big|_Q \, y^\mu + \frac{1}{2}\,(\partial_{\mu\nu} W)\Big|_Q \, y^\mu y^\nu + \dots \,.$$

The definition of Riemann coordinates makes it easy to compute the coefficients $\partial_{\mu\dots} W$.

We want to have such expansion series for our propagator in (3.25). Let us take the operator (inverse propagator) $H = (\Box + m^2 - \xi R)\Big|_P$ at some point P with coordinates $x \equiv x^\mu$ in the near neighbourhood of origin Q with (primed) coordinates $x' \equiv x'^\mu$. We include the expansion of d'Alembert operator $\Box = \nabla^\nu\nabla_\nu = g^{\mu\nu}\nabla_\mu\nabla_\nu)$ in Riemann coordinates and keep only first order in curvature, that is, we keep $-\xi R$ term as it is. Then we have,

$$\begin{aligned} \Box\Big|_{P\equiv x} &= \eta_{\mu\nu}\Big|_Q \,\partial_\mu\partial_\nu - \frac{2}{3}\,R^\alpha_\beta\Big|_Q \,y^\beta\,\partial_\alpha + \frac{1}{3}\,R^{\mu\ \nu}_{\alpha\beta}\Big|_Q \,y^\alpha y^\beta\,\partial_\mu\partial_\nu + \dots \\ H &= \eta_{\mu\nu}\,\partial_\mu\partial_\nu - \frac{2}{3}\,R^\alpha_\beta\,y^\beta\,\partial_\alpha + \frac{1}{3}\,R^{\mu\ \nu}_{\alpha\beta}\,y^\alpha y^\beta\,\partial_\mu\partial_\nu - m^2 + \xi R + \dots \end{aligned} \tag{3.26}$$

Notice that all the coefficients are curvatures evaluated at Q, from here on we shall not indicate in explicitly. Before proceeding further, lets define covariantized propagator as,

$$G(x_1, x_2) = |g(x_1)|^{1/2} D_F(x_1, x_2).$$

The solution for G from equation (3.25) is obtained as usual by expressing it as a Fourier expansion. We generalize to d-dimensional Euclidean space (to avoid $+i\epsilon$ in denominators), and with the inner product is given by, $k \cdot x = \eta^{\mu\nu} k_\mu x_\nu$,

$$G(y) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} G(k).$$

An iterative procedure is used in [33], $G(k) = G_0(k) + G_1(k) + G_2(k) + ...$, where the each term had geometrical coefficient and the subscripts indicated the number of derivatives of the metric it has. $G_0(k) = (k^2 + m^2)^{-1}$ because, we have $\eta^{\mu\nu}(\partial_\mu \partial_\nu - m^2)e^{iky} = -(k^2 + m^2)e^{iky}$ and for next term $G_1(k)$ the derivatives vanish at origin. However, curvature terms appear in coefficients of $G_2(k)$. From (3.26),

$$-R^\alpha_\beta \int \frac{d^d k}{(2\pi)^d} y^\beta \partial_\alpha e^{iky} \frac{1}{k^2 + m^2} = R^\alpha_\beta \int \frac{d^d k}{(2\pi)^d} e^{iky} \left[\frac{\delta^\beta_\alpha}{k^2 + m^2} - \frac{2k_\alpha k^\beta}{k^2 + m^2} \right].$$

After a similar step for the next term, we have, local expansion for the propagator in curved spacetime,

$$G(y)\Big|_P = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \left[\frac{1}{k^2 + m^2} + \frac{1}{3} \frac{(1 - 3\xi)R}{(k^2 + m^2)^2} - \frac{2}{3} \frac{R^{\mu\nu} k_\mu k_\nu}{(k^2 + m^2)^3} + ... \right]. \quad (3.27)$$

One can see that, in this expression can be continued to have higher orders in curvature which will be compensated by higher negative powers of (k^2+m^2) in the denominator. Such infinite terms would correspond to infinite number of Feynman diagrams for every Feynman diagram of flat spacetime. Surprisingly from power counting, all these diagrams have lesser superficial degree of divergence than the diagrams of flat spacetime. The generic expansion up to adiabatic forth order in coefficients is given in Appendix(F.3). There, we also write the propagator in DeWitt-Schwinger (DS) representation and list the coefficients in coincidence limit. These are the standard results listed from [34] (also see [35]) and for detailed derivation one can refer the original papers and books[33, 44] and [38]. The divergence is clearly due to terms expressing gravitational field rather than dynamics of scalar field. Renormalization involves steps similar to our previous sections, where we add a counter term and try to absorb the infinities by redefining the bare coupling coefficients.

One thing we are sure from the above illustration is that, renormalizability of scalar field in curved spacetime demands classical action to have higher derivative (HD) terms. Even though we are discussing the scalar field in curved spacetime the divergent quantities are all geometrical. As possible counter terms in the action, we can consider three possible scalars a) Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ b) $R_{\mu\nu}R^{\mu\nu}$ c) R^2. Their combination would be a general Lagrangian containing HD terms. The theorem which relates all three of these scalars is given by Chern-Gauss-Bonnet Theorem which implies

$$(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) = \mathcal{T},$$

where $\mathcal{T}$ is a topological invariant quantity [39]. We make use of this expression to express Kretschmann scalar in terms of other two scalars, hence it would not appear in our Lagrangian. Then the full Lagrangian (HD terms and Ricci Scalar) for gravitational sector is like $L_{HD+EH} = \gamma_1^B R_{\mu\nu}R^{\mu\nu} + \gamma_2^B R^2 + \gamma_3^B R$ where, without being rigorous, we have absorbed signs and constant coefficients into γ_i^B's; superscript B indicating that these are bare couplings. This is a Lagrangian of what is known as Quadratic Gravity. Since Weyl tensor is just the Riemann tensor with all the traces removed, one can express the obtained full gravitational Lagrangian in terms of Weyl tensor $C_{\mu\nu\alpha\beta}$ (B.5), but we shall restrain being rigorous at this stage[6] and make use of standard results of [1] and references therein. By our qualitative arguments and renaming the bare coefficients as α_i^B's, we can write the following full gravitational action (with HD, Ricci scalar). We also add bare cosmological constant Λ_B such that it quantifies the vacuum energy density in semiclassical Einstein equations, that is, it contributes to expectation value, $\langle T_{\mu\nu}\rangle^B \equiv \rho_\Lambda^B g_{\mu\nu}$. Now the anticipated action looks like,

$$S_{\text{EH}} + S_{\text{HD}} = -\int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N^B}R - \rho_\Lambda^B\right) + \int d^4x\sqrt{-g}\left(\alpha_1^B C^2 + \alpha_2^B R^2\right). \tag{3.28}$$

One can find equations of motion as shown below (3.29). The bare constants are α_1^B, α_2^B and bare CC, Λ_B is associated with vacuum energy density $\langle T_{\mu\nu}\rangle^B$. The tensors $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(2)}$ are quadratic in Riemann Tensor which are listed below [37] (also see [46]).

$$G_{\mu\nu} + \alpha_1^B H_{\mu\nu}^{(1)} + \alpha_2^B H_{\mu\nu}^{(2)} = -8\pi G_N^B \langle T_{\mu\nu}\rangle^B \tag{3.29}$$

$$\begin{aligned} H_{\mu\nu}^{(1)} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\sqrt{-g}R^2\right] \\ &= 2\nabla_\nu\nabla_\mu R - 2g_{\mu\nu}\nabla_\rho\nabla^\rho R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} H_{\mu\nu}^{(2)} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}\right] = 2\nabla_\alpha\nabla_\nu R_\mu^\alpha - \nabla_\rho\nabla^\rho R_{\mu\nu} \\ &\quad -\frac{1}{2}g_{\mu\nu}\nabla_\rho\nabla^\rho R - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2R_\mu^\rho R_{\rho\nu}. \end{aligned} \tag{3.31}$$

As we saw earlier, there are divergent quantities appearing in $\langle T_{\mu\nu}\rangle^B$. Now that we have added counter terms, all the divergences can be absorbed by redefining the bare

[6]In fact, particularly in 4 dimensions, we will also have a total derivative term and a Gauss-Bonnet term which can be omitted safely as they are surface terms and negligible.

couplings $G_N^B, \alpha_1^B, \alpha_2^B$ into effective 1-loop couplings $G_N, \alpha_1^{(1)} \alpha_2^{(1)}, \langle T_{\mu\nu} \rangle^{(1)}$ or $(\rho_{vac}^{(1)})$ at 1-loop level as given below (from [1]). In spite being renormalized quantities, these cannot be immediately associated with measured values (for example G_N from Cavendish experiments) simply because these are results for 1-loop effective theory. At 2-loop level for example, we will have to again redefine the now obtained 1-loop finite parameters and treat them as if they were bare couplings.

$$\alpha_1^{(1)} = \alpha_1(\mu) - \frac{1}{2(4\pi)^2} \frac{1}{120} \left(\ln \frac{m^2}{\mu^2} + \text{finite const} \right) \tag{3.32}$$

$$\alpha_2^{(1)} = \alpha_2(\mu) - \frac{1}{4(4\pi)^2} \left(\frac{1}{6} - \xi \right)^2 \left(\ln \frac{m^2}{\mu^2} + \text{finite const} \right) \tag{3.33}$$

$$\frac{1}{16\pi G_N^{(1)}} = \frac{1}{16\pi G(\mu)} + \frac{m^2}{2(4\pi)^2} \left(\frac{1}{6} - \xi \right) \left(\ln \frac{m^2}{\mu^2} + \text{finite const} \right) \tag{3.34}$$

$$\rho_{vac}^{(1)} = \rho_{vac}(\mu) + \frac{m^4 1}{4(4\pi)^2} \left(\ln \frac{m^2}{\mu^2} + \text{finite const} \right) . \tag{3.35}$$

Here we could also add $\hbar$ to the second term in the *r.h.s* of the above equations (as we did in (3.22)) to symbolize that it is 1-loop order. Regularization has introduced scale (μ) dependence and ρ_{vac} is energy scale dependent vacuum energy density. Upon removing the finite terms through some renormalization scheme we have corresponding running of these quantities which can be expressed using beta functions (at 1-loop),

$$\beta_1 = \mu \frac{d\alpha_1(\mu)}{d\mu} = \frac{1}{120(4\pi)^2}, \qquad \beta_2 = \mu \frac{d\alpha_2(\mu)}{d\mu} = \frac{1}{2(4\pi)^2} \left(\frac{1}{6} - \xi \right)^2 , \tag{3.36}$$

$$\beta_{1/G} = \mu \frac{d}{d\mu} \frac{1}{16\pi G(\mu)} = \frac{m^2}{(4\pi)^2} \left(\frac{1}{6} - \xi \right), \quad \beta_\Lambda = \mu \frac{d\rho_{vac}(\mu)}{d\mu} = \frac{m^4}{2(4\pi)^2} . \tag{3.37}$$

The beta function indicating the quantum corrections for vacuum energy density has dependence on mass in quartic order. This implies the quantum corrections is quickly growing with particle mass. This result of curved spacetime is no different from that of flat spacetime[7] and the problem still remains. One can reach out to supersymmetry, and supergravity in vain as they are not solving CC problem either [47]. Theories of quantum gravity particularly in the lines of string theory also cannot solve CC problem as it relies on supersymmetry to manifest itself in real world.

Though we have been able to do some calculations, the procedure followed here actually fails to describe the very dynamic nature of the ZPE. Neither the beta function (3.37) nor (3.35) is not giving us the real value of vacuum energy density. We still need some initial input to select one of many flows. Also, as argued in [1], the propagator expansion in curved spacetime, for example in (3.27) tells us that we have at the first order a propagator in Minkowski metric. Since ZPE does not exist in flat

[7] We can compare with (3.15) and (3.14).

spacetime, our expansion of metric (in (F.3)) would miss the dynamical correction to vacuum energy density. The vacuum energy does not exist on background on which we are perturbatively expanding. This is seen in the first term of the propagator in curved spacetime background which corresponds to a vacuum to vacuum bubble diagram of Minkowski metric that actually does not exist. The dynamics of vacuum energy in curved spacetime is determined by bubble diagrams with legs. Thus the aim of having beta function or the running of coupling *w.r.t* change in curvature is not yet reached. In fact, we do not have a method to derive them without doing metric perturbations. In the following chapter we shall start with Running Vacuum Model which gives a phenomenological description of running of vacuum energy.

Part II

Extended Brans-Dicke Theories and Screening Problems

4 Motivations for BD-ΛCDM

The results from previous chapter showed that vacuum energy density in flat and curved spacetime give the same result. This is because perturbative expansion performed does not not make sense as it still includes in the first term, the irrelevant flat spacetime diagram. Here we discuss an alternative phenomenological approach to describe the running of vacuum energy density in the cosmological context. For this we make use of [1]. For further details we also recommend [48, 49]. In the second part, we discuss scalar tensor theory along with ghost instabilities. We have made use of [50] extensively and cite other important references in the text. Building on this we will make connections to BD-ΛCDM cosmology in the next chapter.

4.1 Running Vacuum Model

Here we shall give an overview of Running Vacuum Model and how it can lead us to Brans-Dicke Theory. As evident from the previous chapter the running description will not provide us the value of vacuum energy density but rather describes its flow in curved spacetime. Usually this flow is *w.r.t* some new parameter like energy scale μ introduced by a regularization procedure. We would like to look at evolution of vacuum energy density in cosmological context hence we associate this free parameter to the curvature of spacetime. In Friedmann –Lemaître –Robertson –Walker (FLRW) universe which adheres the cosmological principle of isotropic and homogeneous universe, we have curvature given by,

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right].$$

Here dots ”” represent the differentiation *w.r.t* coordinate time, a is the scale factor and κ is the spatial curvature parameter. Defining Hubble parameter $H = \frac{\dot{a}}{a}$, we can write the curvature as,

$$R = 12H^2 + 6\dot{H}.$$

For further discussions we note that the FLRW metric for flat space $\kappa = 0$ in cartesian coordinates is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. From the Einstein equations we can obtain the Friedmann equations and pressure equations from which we have,

$$\begin{aligned} 3H^2 &= 8\pi G\rho \\ 3H^2 + 2\dot{H} &= -8\pi G p. \end{aligned} \tag{4.1}$$

The general covariance also allows higher order terms like $R^2, R_{\mu\nu}R^{\mu\nu}$ as mentioned before. This is gives the effective action with ladder of operators[1]. From analogy to electromagnetism and also from scalar potential in (1.10) we can say that requirement of covariance would imply having even powers in our action. Now we have $R \sim H^2$ but also $R \sim \dot{H}$. For the sake of maintaining covariance we can demand the powers of H to be even, but powers of $\dot{H}$ can be any (for example, terms like $H^2\dot{H}$ would be valid in our effective action). However, order of $\ddot{H}$ is counted as 3. We follow the notions of [1] (we recommend the same for detailed arguments on odd and even powers), where all the terms like $H^4, \dot{H}^2, H^2\dot{H}$ are denoted simply by H^4. In the same spirit, we note the curvature in FLRW universe to be H^2. Now we want to associate the curvature to the scale parameter and investigate the flow of vacuum energy density $\rho_{vac}(H)$ in the curved spacetime. We realize that the scale parameter appears logarithmically in the beta functions hence we can take $\mu \equiv H$. This makes sense because in natural units both have same dimensions.

Given the problems with beta functions of vacuum energy, let us consider flow of gravitational coupling. From (3.34) we have,

$$\frac{1}{G(H)} = \frac{1}{G_N} + \frac{m^2}{2\pi}\left(\frac{1}{6} + \xi\right)\ln\frac{H^2}{H_0^2} \implies G(H) = \frac{G_N}{1 + C_1\ln(H^2/H_0^2)}, \tag{4.2}$$

where H_0 is the current Hubble constant and $G_N \equiv G(H_0)$ is the current gravitational coupling that we know as Newton's constant. The constant $C_1 = (1/2\pi)(1/6 - \xi)m^2/M_P^2$ includes all the constants. The varying G is related to the vacuum energy as we argued in the first chapter (see around (1.4)). The relation is from Bianchi identities and assumption of conserved matter energy density ρ_m. In FLRW universe the conservation equation is $\dot{\rho}_m = -3H(\rho_m + p_m)$ and using this, the total energy momentum (along with Gravitational coupling G) conservation equation $\nabla^\mu[G(T_{\mu\nu} + g^{\mu\nu}\rho_{vac})] = 0$ is,

$$\begin{aligned} \frac{d}{dt}[G(\rho_m + \rho_{vac})] + 3GH(\rho_m + p_m) =& (\rho_m + \rho_{vac})\dot{G} + G\dot{\rho}_\Lambda = 0 \\ &\implies \left[(\rho_m + \rho_{vac})\frac{dG}{dH} + G\frac{d\rho_{vac}}{dH}\right]\dot{H} = 0. \end{aligned}$$

Since $\dot{H} \neq 0$, we can solve the expression inside the brackets. We now make use of Friedmann equations and from (4.2) the solution to the above equation is,

$$\rho_{vac}(H) = C_0 + \frac{3C_1}{8\pi}M_P^2 H^2.$$

[1]We describe effective theory in Part III, also see Appendix(D.1)

Here, C_0 is a constant and hence has no dependence on H. If we take variation $w.r.t$ $\ln H^2$, we obtain the anticipated beta function for ρ_{vac}. To make sense of the above equation we normalize it by taking the current value of vacuum energy density as $\rho^0_{vac} = C_0 + \frac{3C_1}{8\pi} M_P^2 H_0^2$,

$$\rho_{vac} = \rho^0_{vac} + \frac{3C_1}{8\pi} M_P^2 (H^2 - H_0^2). \tag{4.3}$$

We expect C_0 to have mass dimension 4. Appearance of M_P in the second term is hence justified because now, $M_P^2 H^2$ has mass dimension 4 as well. The second term can be seen as the first correction to the vacuum energy density. This part corresponds to the mildly ($\sim H^2$) evolving vacuum energy whereas the first term is a non-zero constant $\rho^0_{vac} \neq 0$. Now the free parameter C_1 can be constrained from observations. In principle, one could have a series expansion in the even powers of H accompanied by a constant coefficient which has to be constrained through phenomenological analysis. However, these are correction terms and are expected to be negligible, at least at present time. One can consider inflationary scenarios using higher order Ricci scalar, for example Starobinsky's inflation [51] where terms of order H^4 (and higher) are essential. Early universe models in running vacuum scenario are investigated for example in [48, 49]. In the late universe, that is from radiation domainant era till today, we need not take H^4 (or higher) into account. In the very late universe we can also take $\rho_m \to 0$ leaving the dominant ρ_{vac} in Friedmann equation. In this limit, taking G, ρ_{vac} to be varying or constants, possible models were considered in [1, 2]. We could also consider models including $\dot{H}$ in (4.3) which can be seen as vacuum energy realized from Ricci scalar [5],

$$\rho_{vac} = \rho^0_{vac} + \frac{3C_1}{8\pi} M_P^2 (H^2 + \frac{1}{2}\dot{H}) = \rho^0_{vac} + \frac{C_1}{32\pi} M_P^2 \,(R) \equiv \rho_{vac}(R).$$

In radiation dominated era we have $R/H^2 \ll 1$ and thus the correction term to the constant vacuum energy density is negligible. The Ricci scalar in terms of trace of energy momentum tensor can be expressed as, $R = 8\pi G_N(\rho_m + 4\rho_{vac})$. Hence the radiation dominated era is in accordance with standard cosmology and not altered by running vacuum energy density. We choose a particular case where we do include interactions between matter sector and vacuum energy and assume conservation of matter but keep the time time dependent gravitational coupling $G(H)$. This model is called *type II (R)RVM* [5]. The acronym roughly stands for (Ricci) Running Vacuum Model. This model resembles to the famous Brans-Dicke theory. We shall discuss this connection in the next chapter.

4.2 Scalar-Tensor Theory

For a 4-dimensional metric theory, Lovelock's theorem states that [52, 53], Einstein equations are the only possible equations of motion which are of second order in derivatives. Of course, the equations of motion can include a cosmological constant without any problem. So these extensions of GR should violate the Lovelock's theorem.

One can think of higher dimensional gravity theories in the lines of Kaluza-Klein Theory [54, 55], which is a 5-dimensional classical theory describing a geometrical unification of gravitation and electromagnetism. It involves a 5-dimensional metric with 15 independent components - 10 of which can be related to a 4-dimensional metric. The remaining 5 independent components can be represented by an electromagnetic potential A_μ which is a 4-vector and a scalar field ϕ. Large extra dimensions are strictly constrained from particle physics experiments, however we could have very small extra dimensions. The proposal is to have a loop along the 5th dimension so that we have a *compact* space along the extra dimension. Theories like String theory make use of this dimensional reduction in a complicated way. We can roughly see that higher dimensional theories can be reduced to 4-dimensional effective theories which involve new degrees of freedom, like a scalar field. String theory also motivates braneworld models[2] for dark energy where a lower dimensional brane is embedded in a higher dimensional spacetime with very large (or infinite) extra dimensions. A famous model is the DGP model [57] where a 3-brane is embedded in 5-dimensional Minkowski space with an infinite size extra dimension. 4-dimensional gravity emerges in small distances and 5-dimensional gravity plays the role at large distances. These theories have significant phenomenology as they can explain late time acceleration. However, all these theories can be recast into to an effective scalar-tensor theory.

Theories involving higher derivative terms in the Lagrangian for example, R^2, Gauss-Bonnet model, etc. or generally $f(R)$ theories, also add a scalar degree of freedom to the gravitational (metric) tensor field. So modification of Einstein's GR is equivalent to adding new degrees of freedom. Most of the modified gravity theories at least effectively behaves as scalar tensor theories[50].

Horndeski Theory provides us a framework for the most general scalar-tensor theory whose equations of motion are second order in derivatives. This is very important because in general the Lagrangian can have second or higher order derivatives which corresponds to higher derivatives in the equations of motion. This would lead to ghost instabilities. Let us first describe these instabilities and the associated Ostragradsky Theorem. To introduce these aspects we follow [50]. Consider a Lagrangian which has a second order derivative of a scalar field $\phi(t)$ which is only a function of time,

$$L = \frac{A}{2}\ddot{\phi}^2 - V(\phi) \xrightarrow{\text{variations } w.r.t\ \phi} \frac{\delta L}{\delta \phi} = A\ddddot{\phi} - \frac{dV}{d\phi}. \tag{4.4}$$

$A \neq 0$ is a constant and $V(\phi)$ is some potential for the field. We can make use of an auxiliary field ψ to rewrite the Lagrangian as, $L = -A\dot{\psi}\dot{\phi} - \frac{A}{2}\psi^2 - V(\phi) + A\frac{d}{dt}(\psi\dot{\phi})$.

[2]Also see the books [11, 56] for a review on braneworld models and Scalar-Tensor Theory respectively.

If we set the auxiliary field as $\psi = \ddot{\phi}$ then we recover the original Lagrangian in (4.4). We can further redefine the fields as $\chi = (\phi + \psi)/\sqrt{2}$ and $\varphi = (\phi - \psi)/\sqrt{2}$, then we can write the Lagrangian as,

$$L = -\frac{A}{2}\dot{\chi}^2 + \frac{a}{2}\dot{\varphi}^2 - V(\varphi, \chi).$$

The Lagrangian now has two dynamical degrees of freedom with opposite signs on the kinetic terms. The one with negative sign is indicating the ghost-like instabilities[3]. However, we have to construct the kinetic matrix (which is just the collecting the coefficients of the higher order kinetic terms in a matrix) and check if it is degenerate (determinant is zero). If the matrix is degenerate then there is a possibility to absorb higher derivatives by combining equations of motions. Hence the ghost instabilities originate in theories with non-degenerate Lagrangian. The theorem is known under the name of Ostragradsky ghost instabilities. If we consider a scalar field ϕ which is invariant under shift symmetry, $\phi \to \phi + a$, where the field is shifted along a constant a, then we can add higher derivative terms which are all invariant under such transformations. To be more precise, we consider transformations which are analogous to Galilei transformations ($\dot{x} \to \dot{x} + v$), where the field is not only shifted by a constant but also has a linear term attached to it. That is, with a constant vector b_μ (shift in gradient) and a coordinate vector x^μ, we have Galilean shift symmetry,

$$\phi \to \phi + a + b_\mu x^\mu \,, \qquad \text{and} \qquad \partial_\mu \phi \to \partial_\mu \phi + b_\mu. \tag{4.5}$$

Then we can have derivative couplings in the Lagrangian which is invariant under shift symmetry up to a total derivative term [4].

$$L = f(\phi,\, \partial\phi,\, (\partial\partial\phi), ...)$$

However the Ostragradsky theorem puts restriction on terms which are allowed in the Lagrangian. Even though we could have higher order equations of motion without ghost instabilities in general, here we shall restrict ourselves to Lagrangians which will yield equations of motion at the most up to second order. In flat spacetime, the canonical kinetic term is given by $X = -\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The most general Lagrangian in four dimensions whose equations of motions finally reduce to second order in

[3] The energy of the ghost field is not bounded from below which leads to fundamental problems of vacuum stability etc. We make use of ghosts in cosmological scenarios as dark energy candidates (maybe under the name of 'phantom') as it can have EoS $w < -1$. For examples of such candidates see section(8.1) and the cited references.

[4] See [58] as well to relaize importance of this symmetry in theories which have derivative couplings.

derivatives is given by,

$$
\begin{aligned}
S &= \int d^4x \sum_{i=1}^{5} L_i \\
L_1 &= c_1\phi \\
L_2 &= -\frac{1}{2}(\partial\phi)^2 = c_2 X \\
L_3 &= -c_3 X\Box\phi \\
L_4 &= c_4 X[\Box\phi^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] \\
L_5 &= -\frac{c_5}{3}X[\Box\phi^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi]
\end{aligned}
\tag{4.6}
$$

Here, c_i's ($i = 1, 2, 3, 4, 5$) are constants and $\Box \equiv \eta^{\mu\nu}\partial_\nu\partial_\mu = \partial^\mu\partial_\mu$. L_1 simply corresponds to the tadpole, which we shall ignore as we are interested in equations of motion, but it is still valid to consider it in the Lagrangian[5]. To realize such a theory in curved spacetime, we covariantize (4.6). Since there are second derivatives in the action and covariant derivatives do not commute they rather represent the spacetime curvature,

$$
[\nabla_\mu, \nabla_\nu](\nabla^\alpha\phi) = \nabla_\mu\nabla_\nu(\nabla^\alpha\phi) - \nabla_\nu\nabla_\mu(\nabla^\alpha\phi) = R^\alpha_{\beta\mu\nu}(\nabla^\beta\phi) \neq 0\,,
$$

we will have curvature terms in equations of motion for ϕ. Since we have much higher derivatives, we also get derivatives of curvature (Ricci) tensor, for example, terms like $\nabla^\alpha(R_{\mu\nu}...)$. Such terms might introduce ghosts. So we add curvature dependent counter terms, by hand, in order to get rid of these terms [50, 59]. The new covaraint action is called the Covariant Galileon or Horndeski action,

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g} \sum_{i=2}^{5} L_i \\
L_2 &= G_2(X,\phi) = K(X,\phi) \\
L_3 &= -G_3(X,\phi)\Box\phi \\
L_4 &= G_{4X}(X,\phi)[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi] + G_4(X,\phi)R \\
L_5 &= -\frac{G_{5X}(X,\phi)}{6}[\Box\phi^3 - 3\Box\phi(\nabla_\mu\nabla\nu\phi)^2 \\
&\quad + 2(\nabla_\mu\nabla_\nu\phi)^3] + G_5(X,\phi)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\,.
\end{aligned}
\tag{4.7}
$$

Here, the canonical kinetic term is $X = -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and $\Box \equiv g^{\mu\nu}\nabla_\nu\nabla_\mu = \nabla^\mu\nabla_\mu$. $G_{\mu\nu}$ is the Einstein tensor and R is the Ricci scalar. Gi's are free functions, of ϕ and X as indicated. We use subscript X to denote derivative *w.r.t* X; for example, $\partial G_i/\partial X \equiv G_{iX}$. We shall later make specific choices of these terms for our purposes.

[5]The extra terms under shift symmetry, $b_\mu x^\mu + c \sim \partial_\alpha(b_\mu x^\mu x^\alpha + cx^\alpha)$ indicates that we can write it as a total derivative, which can correspond to a conserved charge and is allowed like in any other theory.

It is interesting to note that detection of gravitational waves and electromagnetic waves from Neutron star merger event GW170817 [60] has reduced some terms in this Lagrangian. For example we can eliminate G_5 along with G_{5X} completely. G_4 has to only be a function of ϕ, hence $G_{4X} = 0$. See [61, 62, 63, 64] and also a review [65].

Scalar-Tensor Theory is also known as Horndeski theory. Horndeski first proposed the most general scalar-tensor theory with second order equations of motion [66]. It was rediscovered a decade ago under an interesting name, 'Fab Four' [67]. In the upcoming sections we use specific sub-classes of theories from Horndeski theory. Particularly, we are interested in Brans-Dicke Theory in the next chapter.

5 BD-ΛCDM Cosmology

In this chapter we first introduce Brans-Dicke Model, with a constant vacuum energy density, or simply BD-ΛCDM and list its basic equations. The original motivation for the theory is briefly discussed in Appendix(E). *We then explore its connections to Running Vacuum Models which were discussed in the previous chapter. This was originally shown in [3, 4] and also in [5, 6]. We shall also see this model in the context of scalar-tensor theory, which allows us to think of its generalizations and possible extensions. We also give a note on H_0 and σ_8 tensions in general and in the context of BD-ΛCDM [6, 7].*

5.1 Introduction

Brans-Dicke (BD) theory [68, 69, 70] describes the evolution of gravitational coupling G by assigning a time dependent scalar BD field $\psi(t)$ which is non-minimally coupled to the curvature scalar. At present $t = t_0$, the value of the BD field reduces to the Newton's constant and depicts the effective gravitational coupling for any other t.

$$\psi(t) \equiv 1/G(t) \quad \text{at present, } \psi(t_0) \equiv \psi_0 \equiv 1/G_N \equiv M_P^2$$

The scalar field also has a kinetic term which comes with a dimensionless factor, the Brans-Dicke parameter $\omega_{\rm BD}$ in the action. In the limit of very large values of this parameter $\omega_{\rm BD} \to \infty$ BD theory reduces to GR[1]. We include a constant energy density for the vacuum energy density ρ_{vac} which is now indicated by $\rho_{vac} \equiv \rho_\Lambda$. This model is called by the name BD-ΛCDM [3, 4, 5, 6] and the action is given by,

$$S_{\rm BD\Lambda} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(\psi R - \frac{\omega_{\rm BD}}{\psi}\nabla^\mu\psi\nabla_\mu\psi\right) - \rho_\Lambda\right] + \int d^4x\sqrt{-g}L_m(\Phi_i, g_{\mu\nu}). \tag{5.1}$$

Since in this theory the scalar field is a dynamical variable, we get its equations of motion by taking the variations of the above action *w.r.t* ψ. We also have modified Einstein equations upon taking variations *w.r.t* $g^{\mu\nu}$.

[1] In Appendix(E) we discuss how BD was motivated and its dissimilarity from GR.

$$\psi G_{\mu\nu} + g_{\mu\nu}\left[\Box\psi + \frac{\omega_{\rm BD}}{2\psi}(\nabla\psi)^2\right] - \nabla_\mu\nabla_\nu\psi - \frac{\omega_{\rm BD}}{\psi}\nabla_\mu\psi\nabla_\nu\psi = 8\pi(T_{\mu\nu} - g_{\mu\nu}\rho_\Lambda) \tag{5.2}$$

$$\psi R + 2\omega_{\rm BD}\Box\psi - \frac{\omega_{\rm BD}}{\psi}\nabla_\mu\nabla^\nu\psi = 0\,. \tag{5.3}$$

We again note that $\Box \equiv \nabla_\mu\nabla^\mu$, Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}$ where R is the Ricci scalar. There are two free new parameters in our formulation of BD-ΛCDM cosmology. Along with time dependent BD field ψ, we also have a constant parameter $\omega_{\rm BD}$ which sometimes used in terms of its inverse $\epsilon_{\rm BD} \equiv 1/\omega_{\rm BD}$. For large values of $\omega_{\rm BD}$ or as $\epsilon_{\rm BD} \to 0$, then from the scalar field equations we see that we go back to having GR.

Ricci scalar appearing in (5.3) can be removed by obtaining the trace from gravitational field equation (5.2). Then we have modified Klein-Gordon equation which on FLRW background is given by,

$$(2\omega_{\rm BD} + 3)\Box\psi = 8\pi(T - 4\rho_\Lambda) \implies \ddot{\psi} + 3H\dot{\psi} = \frac{8\pi}{2\omega_{\rm BD} + 3}(T - 4\rho_\Lambda) \tag{5.4}$$

Here, dots represent derivative *w.r.t* to cosmic time such that we have Hubble rate $H = \dot{a}/a$. We are working with those models where we have varying gravitational coupling G but have conservation in the energy momentum sector. Full energy momentum tensor can be,

$$\bar{T}_{\mu\nu} = T_{\mu\nu} - \rho_\Lambda g^{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}.$$

Here, ρ includes all the relativistic and non-relativistic matter part along with radiation, dark matter and also cosmological constant ρ_Λ. p corresponds to total pressure in the same sense. There is no coupling between ψ and matter sector. However, we have for constant vacuum energy density an EoS of -1 implying $\rho_\Lambda = -p_\Lambda$. For FLRW metric we can also rewrite the gravitational field equations in terms of Friedmann and pressure equations. We make use this to write two independent equations for ρ and p similar to (4.1).

$$\begin{aligned} 3H^2 + 3H\frac{\dot{\psi}}{\psi} - \frac{\omega_{\rm BD}}{2}\left(\frac{\dot{\psi}}{\psi}\right)^2 &= \frac{8\pi}{\psi}\rho, \\ 2\dot{H} + 3H^2 + \frac{\ddot{\psi}}{\psi} + 2H\frac{\dot{\psi}}{\psi} + \frac{\omega_{\rm BD}}{2}\left(\frac{\dot{\psi}}{\psi}\right)^2 &= -\frac{8\pi}{\psi}p. \end{aligned} \tag{5.5}$$

The Friedmann(-like) equations in (5.2) as usual talk about evolution and geometry[2] of the universe function of fluid density. But in this case we also have evolution

[2]We point out that, we have taken flat FLRW metric with $\kappa = 0$.

of gravitational coupling. We do not yet have analytical solutions for equations (5.4). However we can make realistic ansatz and proceed to have some solutions. We now proceed with this idea and look at the connection between BD-ΛCDM and Running Vaccum Model.

5.1.1 Connections to Running Vacuum Models

We know that for a particular fluid in standard FLRW cosmology, the solution to conservation equations yields a power law in scale factor. This explains how the energy density of a particular fluid varies as a function of scale factor. So we expect power law solutions for the BD field,

$$\psi = \psi_0 \, a^{-\epsilon}, \quad \text{with } |\epsilon| \ll 1 \,. \tag{5.6}$$

We take this as an ansatz. Here, we are assuming a mild evolution of BD field using a small parameter ϵ such that we do not have any significant departure from the well tested GR based ΛCDM. or simply GR-ΛCDM. The sign of ϵ determines if the gravitational coupling was strong ($\epsilon < 0$) or weak ($\epsilon > 0$) in the cosmic history. The scale factor is normalized using the current value $a_0 = 1$. Now we can rewrite the Klein-Gordon and Friedmann equations (5.4), (5.5) by using the time derivatives of the BD field. With,

$$\frac{\dot{\psi}}{\psi} = -\epsilon H, \quad \frac{\ddot{\psi}}{\psi} = -\epsilon \dot{H} + \epsilon^2 H^2,$$

and realizing the fact that radiation has no trace and matter has no pressure, we have,

$$\epsilon \dot{H} + (3\epsilon - \epsilon^2) H^2 = -\frac{8\pi}{3 + 2\omega_{\rm BD}} \frac{\rho_m - 4\rho_\Lambda}{\psi} \tag{5.7}$$

$$3H^2 - 3H^2 \left(\epsilon + \frac{\epsilon^2}{6} \omega_{\rm BD} \right) = \frac{8\pi}{\psi} (\rho_m + \rho_r + \rho_\Lambda) \tag{5.8}$$

$$H^2 \left(3 - 2\epsilon + \epsilon^2 + \frac{\epsilon^2}{2} \omega_{\rm BD} \right) - \dot{H}(2 - \epsilon) = -\frac{8\pi}{\psi} \left(\frac{\rho_r}{3} - \rho_\Lambda \right) . \tag{5.9}$$

In the Friedmann field equation (5.8), let us ignore the radiation energy density and reorganize the terms to write it in the standard form

$$H^2 = \frac{8\pi}{3\psi} \frac{1}{1 - C} (\rho_m + \rho_\Lambda), \quad \text{where } C = \epsilon \left(1 + \frac{\epsilon}{6} \omega_{\rm BD} \right) . \tag{5.10}$$

We can also bring back Newtons constant as well because $1/\psi \equiv G(a) = G_N a^\epsilon$. Also we expand[3] the fraction $1/(1 - C)$ to linear order and write,

$$H^2 = \frac{8\pi G_N}{3} \frac{a^\epsilon}{1 - C} \left[\rho_m^0 a^{-3+\epsilon} + \rho_\Lambda a^\epsilon + C a^\epsilon (\rho_m^0 a^{-3} + \rho_\Lambda) \right] . \tag{5.11}$$

[3] We realize that C need not be a small quantity. The term $\epsilon^2 \omega_{\rm BD}$ can also be of order unity. We can still make linear approximations for very small scale factors.

We see that the last term can be rewritten by making use of (5.10). From (5.10), we have $3(1-C)H^2/8\pi G_N = (\rho_m^0 a^{-3} + \rho_\Lambda)$. We make an approximation, $\rho_\Lambda a^\epsilon \sim \rho_\Lambda(1+\epsilon \ln a) \sim \rho_\Lambda$. This approximation is justified from our arguments in the previous section where we argued that evolution of ρ_Λ is very mild after radiation era. Since we are concerned with matter and present epoch this approximation is valid. Again taking up to linear terms in C,

$$\begin{aligned} H^2 &= \frac{8\pi G_N}{3}\left[\rho_m^0 a^{-3+\epsilon} + \rho_{DE}(H)\right], \quad \text{with} \\ \rho_{DE}(H) &= \rho_\Lambda + \frac{3C}{8\pi G_N} H^2 = \rho_\Lambda + \frac{3C}{8\pi} M_P^2 H^2. \end{aligned} \tag{5.12}$$

This equation is very similar to the one obtained from the previous chapter, for example, (4.3). It is also called the Dynamical Dark Energy [4, 2, 1] because here the dynamics of BD field is responsible for dynamics of dark energy (DE). We see that C is entirely depended on the value of ϵ. We obtain CC of standard cosmology model ΛCDM, if $|\epsilon| \to 0$. In fact, as shown in [4] ϵ is related to BD parameter $\omega_{\rm BD}$ in the same way as $\epsilon_{\rm BD}$ hence, $\epsilon \equiv \epsilon_{\rm BD}$.

5.1.2 Connections to Scalar-Tensor Theories

In equation (4.7) we wrote the action for a general scalar-tensor theory. Along with higher derivative terms in the Lagrangian, we had coefficients, G_i's ($i = 2, 3, 4, 5$) and counter terms G_{4X}, G_{5X} which are all arbitrary functions of ψ and kinetic energy $X = -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. This action consists of most of the modified gravity. For example, the first term $L_2 = K(\phi, X)$ includes all the models which make use of non-canonical Kinetic terms. For some examples see section(8.1) and cited references. They present interesting phenomenology and have been used to explain the accelerated expansion of the Universe.

By fixing the relevant coefficients, we can realize different models for gravity. To obtain GR, we put all of them to zero except, having a constant G_4 which reassembles gravitational coupling constant. Brans-Dicke model can also be obtained by putting $G_2 = X, G_4 = \phi$ and keeping all other coefficients zero.

In this chapter we are focusing on Brans-Dicke theory along with a constant vacuum energy. However, in the next chapter we have extended interests along with which we extend our original model by adding nonlinear terms. In this context, we not only exploit K-essence models, particularly with $G_2 = X^2$ but also make use of cubic Galileons[71, 59] with $G_3 = X$. Galileons has connections to DGP model[57] and Massive gravity theories[72]. We also make use of L_4 in the last chapter. We list some important models in the Table(5.1).

Constraints on BD model parameters, the effective gravitational coupling and $\omega_{\rm BD}$ are obtained using both local and cosmological data. Locally we have Cavendish-like table top experiments and satellites like Cassini[73] which give constraints on the value of gravitational coupling within solar system. Cosmic microwave background also provides constraints on the BD parameters at cosmological scales. The

Models\Coefficients	G_2	G_3	G_4	G_5
GR	0	0	constant$(1/G_N)$	0
BD	X	0	ϕ	0
k-essence	$K(\phi, X)$	0		0
Galileons	X	X	0	0
GGC	$a_1X + a_2X^2$	$f(\phi)X$	0	0
Kinetic Gravity Braiding	X^2	$f(\phi)X$	0	0
Kinetic Coupling	0	0	0	constant(ξ)

Table 5.1: We schematically show different scalar tensor models realized from of the Horndeski action (4.7). Here, GGC refers to Galileon-Ghost Condensate.

interesting part is that, in many models extracted from Horndeski theory reduce to BD cosmology at cosmological scales [74]. Hence BD-ΛCDM can be used as the boundary condition while studying cosmology of scalar-tensor theories.

5.2 H_0 and σ_8 tension in the light of BD-ΛCDM

The standard cosmology model which we refer to as GR-ΛCDM, is based on five things; (i) the force of gravity is explained by GR, (ii) presence of a cosmological constant which is responsible for accelerated universe, (iii) presence of cold DM, (iv) cosmological principle implying homogeneity and isotropy at large scales of the universe and an (v) inflationary epoch preceding the radiation era. Even though GR-ΛCDM has been very successful in matching the observational data. In the recent years factors like new observation methods, new telescopes and observational facilities, new tools to analyse large data, have resulted in significant improvements size and precision of the data. In this new era of precision cosmology, we have encountered cosmological tensions which has led many theorists to make progress by constructing models to explain these tensions. On the other hand, the observational cosmologists have been attempting not only very precise measurements and to achieve better control of systematic errors, but are also using new techniques to unravel the reality.

H_0 and σ_8 tensions are often mentioned in the literature. The Hubble constant H_0 is a very important parameter in cosmology as it is the expansion rate of current universe. The local (or late universe) measurements of H_0 are obtained using cosmic distance ladder, cosmic chronometers and using time-delay measurements of quasars when they pass through strong lenses. We also obtain constraints on H_0 using early universe probes like CMB and Baryon Acoustic Oscillations (BAO). The tension arises due to the fact that value of H_0 measured using late universe probes does not match with that of early universe probes (also see interesting results obtained by making use of Tip of Red Giant Branch [75]). The measurements from CMB are obtained assuming GR-ΛCDM. Using CMB temperature anisotropy, polarization and lensing data from Planck Collaboration [76], H_0 is measured to be 67.36 ± 0.54km/s/Mpc. The distance ladder measurements obtained from SH0ES Collaboration [77] gives $H_0 = 73.2 \pm 1.3$km/s/Mpc which implies we have 4.1σ tension between these two values. If we include time delay measuremnts from H0LICOW Collaboration which uses 6 gravitationally lensed quasars we have $H_0 = 73.3 \pm 1.7$km/s/Mpc, where the tension is much more significant with 4.8σ. Measuring the redshift of a source continuously for a long time would be an obvious model independent approach to measure the expansion rate. Recently there has been interesting proposals for model-independent approaches for example method of Clustering of Standard Candles [78] can be used to reconstruct not only H_0 but also other cosmological parameters.

The σ_8 is a measure of the amplitude of the linear power spectra. This quantifies the amplitudes of the initial density perturbations of the early universe. The CMB would have imprints of the primordial density perturbations on it. However, we usually consider the scales of length $8h^{-1}$Mpc to infer this parameter which explains the number 8 in its notation. We need to note that σ_8 does not belong to the 'basic' set of parameters[4] but instead it is a derived parameter. This is because σ_8 depends on the Hubble constant H_0, more specifically on h which is a dimensionless number quantifying the expansion rate such that we have $H_0 = h100$ km/s/Mpc. h belongs to the basic set of parameters (see for example [80] for detailed discussion on h). The tension in σ_8 comes from measuring the amount of large-scale structures (LSS) in the universe from early universe probe, the CMB (Planck) and late universe probes like weak lensing (for example, BOSS, Kilo Degree Survey- KiDS, 2dFLenS) and galaxy surveys (for example, Dark Energy Survey- DES). A combination of KiDS, BOSS, and 2dFLenS data shows $\sigma_8 = 0.760^{+0.021}_{-0.023}$ which is 2.2σ tension with Planck. A related parameter $S_8 = \sigma_8(\Omega_m^0/0.3)^{1/2}$ is 3.1σ tension tension with Planck measurements. σ_8 depends on h, and power spectra also have dependence on h. However, σ_8 does not show the impact of h on amplitude of power spectra because for different values of h there is a change in reference scale $R = 8h^{-1}$Mpc. This ambiguity has been pointed out recently in [81], where they propose to use σ_{12} such that for $h \sim 0.67$, we have $8h^{-1}$Mpc $\sim$ 12Mpc. Notice that with such formalism one can abandon units like h^{-1}Mpc and now the new parameter σ_{12} is the measure of density fluctuations in a 12 Mpc sphere.

These tensions might be an artifact caused by systematic error in the data. If they are not, this might be hinting at new physics. Any model which explains or eases the tension, should alleviate both H_0 and σ_8 tensions at the same time. Also, if a model loosens one of the tension it should do so without worsening the other. This is a *golden rule* mentioned in [7].

Apart from its alluring theoretical connections to running vacuum models, BD-ΛCDM and related Type-II (R)RVM has very interesting inputs to observational cosmology. In fact, in [82], the former has been categorized under 'Promising Models' as the tensions loosen up to $\leq 3\sigma$, whereas, the latter has been put under 'Good Models' as it alleviates the tension to $\leq 2\sigma$. In a recent analysis of BD-ΛCDM using

[4]See [79] for the distinction between basic and derived parameter sets.

data from Planck 2018, supernova Type Ia from DES and Pantheon compilation, cosmic chronometers, BAO, LSS, the prior on H_0 from SH0ES and the strong-lensing data from the H0LICOW collaboration, it was found that $H_0 = 71.30^{+0.80}_{-0.84}$ km/s/Mpc and $\sigma_8 = 0.789 \pm 0.013$. BD-ΛCDM has two new variables, the field ψ and $\omega_{\rm BD}$. We can express dynamics of BD field in terms of a dimensionless field $\varphi \equiv G_N\psi$, that is, effective gravitational coupling, $G(\varphi) = G_N/\varphi$. In the case of large $\omega_{\rm BD}$, we have $\epsilon_{\rm BD} = 0$ and assume constant $\varphi < 1$ then we have larger values for gravitational coupling. In this case, the Friedmann equation in (5.5) now reduces to,

$$H^2 = \frac{8\pi G_N}{3\varphi}\rho(a) \xrightarrow{\text{for } a=1,} H_0^2 = \frac{8\pi G_N}{3\varphi}\rho^0,$$

where ρ^0 is the total energy density at present time. For $\varphi < 1$, $G(\varphi)$ increases, and so does H_0 and this can help in lowering the tension. We can also take non-zero (negative) values for $\epsilon_{\rm BD}$ to show that the σ_8 tensions are reduced as well. For a detailed study see [7]. The exact numerical evolution of φ from radiation dominated era to present epoch is also shown in Figure 4 of [7] which is obtained by performing a numerical fit to the observational data included in their analysis. The field φ mildly evolves staying within the interval $0.918 \lesssim \varphi \lesssim 0.932$ from radiation dominated era till now.

6 Screening and Difficulties

We have seen interesting theoretical and observational consequences of BD-ΛCDM. The exact numerical evolution of φ suggests that, gravitational coupling at large scales is 4-9% larger than the locally measured Newton's constant. The variations in G is constrained from the Solar System and Cavendish-like experiments which prompts G_N as a valid constant in the local scales. In order to have BD-like behaviour at the cosmological scale, but still recovering GR locally, we have to provide a screening mechanism. We are interested in extended Brans-Dicke theory where we add higher derivative terms of the Horndeski Lagrangian to the original BD Lagrangian (with a cosmological constant). In this context, we are looking at higher order derivative screening mechanisms, like Vainshtein and K-mouflage screening. We realize that these mechanisms can only explain very tiny departures of effective value of G from the locally measured coupling. We also propose as a possible remedy; suppose BD parameter ω_{BD} takes unrealistic value close to $-3/2$, then we obtain the aforementioned behaviour of scalar field. But such attempts not only indicate significant departures from GR at some intermediate scales but also suggests ω_{BD} is fine tuned. This chapter reflects the results and arguments of [10] quite closely.

6.1 Field Equations of Extended Brans-Dicke Theory

We have BD-ΛCDM action given by, (5.1). The numerical fit to observational data gives a mild evolution of effective gravitational coupling in the cosmic history. The model requires cosmological gravitational coupling G to be $4-9\%$ larger than the one measured on earth G_N. In the local scales we have constraints from the Solar System and Cavendish-like experiments on gravitational coupling. So we have to restore GR at the local scales while allowing variations at cosmological scales[1]. Here we study 'Extended' Brans-Dicke Theory denoted by 'eBD' in the subscript. We include higher derivative terms from Horndeski action in order to allow for screening mechanisms. Here, we consider the following extension,

$$S_{\rm eBD} = S_{\rm BD\Lambda} + \int d^4x \frac{\sqrt{-g}}{16\pi} \left(f \,\Box\psi + \theta \,\nabla^\nu\psi\nabla_\nu\psi\right) \nabla^\mu\psi\nabla_\mu\psi \,. \tag{6.1}$$

[1] See (6.6) for a comparison of local and cosmological values of BD variables

We have a cubic and a quartic self-interaction terms with constant couplings f and θ with mass dimensions to the power -6 and -8 in natural units, respectively. The couplings expected to be negligible so that they do not harm the cosmological behaviour of BD-ΛCDM. However, they dominate the scalar field dynamics in the local scales resulting in screening mechanisms. As already pointed out in section 4.2, these terms of the Horndeski Lagrangian do not alter the speed of propagation of gravitational waves and are not affected by binary neutron star merger event GW170817 [60]. We would like to do our calculations in Jordan frame where we have scalar field coupled to the gravitational sector (see Appendix(E)).

We now replace the dimensionful BD field ψ with a dimensionless scalar field φ so that we retrieve the Newton's gravitational constant $G_N = 1/M_P^2$ in the equations.

$$\varphi \equiv G_N \psi \,, \text{ the effective gravitational coupling } \; G(\varphi) = \frac{G_N}{\varphi} \,. \tag{6.2}$$

The modified Einstein equation and Klein-Gordon equations are obtained by taking the variation of the action (6.1) *w.r.t* to the metric and the scalar field φ respectively. With notations $(\nabla\varphi)^2 \equiv \nabla_\mu\varphi\nabla^\mu\varphi$, $\Box \equiv \nabla_\mu\nabla^\mu$, we write the field equations,

$$\begin{aligned} &\varphi G_{\mu\nu} + g_{\mu\nu}\left[\Box\varphi + \frac{\omega_{\rm BD}}{2\varphi}(\nabla\varphi)^2\right] - \nabla_\mu\nabla_\nu\varphi - \frac{\omega_{\rm BD}}{\varphi}\nabla_\mu\varphi\nabla_\nu\varphi \\ &\quad + \frac{g_{\mu\nu}}{2}\frac{f}{G_N^2}\nabla^\alpha\varphi\nabla_\alpha(\nabla\varphi)^2 + \frac{f}{G_N^2}\Box\varphi\nabla_\mu\varphi\nabla_\nu\varphi - \frac{f}{G_N^2}\nabla_\mu(\nabla\varphi)^2\nabla_\nu\varphi \\ &\quad + \frac{2\theta}{G_N^3}(\nabla\varphi)^2\left[\nabla_\mu\varphi\nabla_\nu\varphi - \frac{g_{\mu\nu}}{4}(\nabla\varphi)^2\right] = 8\pi G_N(T_{\mu\nu} - \rho_\Lambda g_{\mu\nu}) \,, \end{aligned} \tag{6.3}$$

$$\begin{aligned} 0 = {} & R - \frac{\omega_{\rm BD}}{\varphi^2}(\nabla\varphi)^2 + \frac{2\omega_{\rm BD}}{\varphi}\Box\varphi \\ & + \frac{f}{G_N^2}\Box(\nabla\varphi)^2 - \frac{2f}{G_N^2}\nabla_\mu\left(\Box\varphi\nabla^\mu\varphi\right) - \frac{4\theta}{G_N^3}\nabla_\nu\left[\nabla^\nu\varphi(\nabla\varphi)^2\right] \,, \end{aligned} \tag{6.4}$$

$G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = -(2/\sqrt{-g})\delta S_m/\delta g^{\mu\nu}$ is the total energy-momentum tensor which includes relativistic and non-relativistic matter fields. Its trace is $T = g^{\mu\nu}T_{\mu\nu}$ which is just the matter energy density ρ_m. As expected in our scenario, the scalar field equation has curvature dependence. This can be removed by taking the trace of (6.3) and then substituting it for Ricci scalar. Detailed derivation is given in Appendix(B.2). Now we have,

$$\begin{aligned} (3 + 2\omega_{\rm BD})\Box\varphi = {} & 8\pi G_N(T - 4\rho_\Lambda) + \frac{4\theta\varphi}{G_N^3}\nabla_\nu\left[\nabla^\nu\varphi(\nabla\varphi)^2\right] \\ & + \frac{2f\varphi}{G_N^2}\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)\left(\nabla^\mu\nabla^\nu\varphi + \frac{1}{\varphi}\nabla^\mu\varphi\nabla^\nu\varphi\right)\right] \\ & + \frac{2f\varphi}{G_N^2}\left[-\frac{1}{2\varphi}\Box\varphi(\nabla\varphi)^2 - R_{\mu\nu}\nabla^\mu\varphi\nabla^\nu\varphi\right] . \end{aligned} \tag{6.5}$$

We expect the non-linear terms to dominate in the local scales and remain negligible in cosmological scales. The local and cosmological constraints on BD variables -the scalar field φ and $\epsilon_{\rm BD}$, are given below in (6.6). They are indicated by superscripts (l) and (c) respectively. By local constraints we mean the constraints put in the Solar System from Cassini mission [73]. The cosmological values are obtained from [7]. The screening mechanisms due to higher derivative terms are commonly called Vainshtein mechanism [83]. However we preserve this name for the particular case of screening mechanism due to non-linear cubic term. We refer the screening mechanism from k-essence term as K-mouflage screening[84]. In the following sections we shall discuss these mechanisms in extended Brancs-Dicke Theory context.

$$\varphi \equiv \begin{cases} \text{local } \varphi^{(l)} \to 1 \\ \text{cosmological } \varphi^{(c)} \to \sim 0.9 \end{cases} \quad \epsilon_{\rm BD} \equiv \begin{cases} \text{local } \epsilon_{\rm BD}^{(l)} \sim \mathcal{O}(10^{-5}) \\ \text{cosmological } \epsilon_{\rm BD}^{(c)} \sim \mathcal{O}(10^{-3}) \end{cases} \tag{6.6}$$

6.2 Pure Brans-Dicke Solution

The objective is to obtain scalar field profile around a spherically symmetric (static) mass, which is placed in empty space. We perform the all the calculation in weak-field limit. This approximation is similar to FLRW universe in time scales much smaller than inverse of Hubble function (expansion timescale) H^{-1}. This this limit, we can neglect the background matter energy density and cosmological constant. We expand the metric around the background Minkowski metric $\eta_{\mu\nu}$ and the scalar field around the cosmological background value $\varphi^{(c)}$. We denote the fluctuation of the field as $\delta\varphi \equiv \phi$.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \varphi = \varphi^{(c)} + \delta\varphi \equiv \varphi^{(c)} + \phi.$$

For any point far away from the source, the weak field limit applies. To be more precise, we can say for a point, which is at distance $\mathcal{R}$ from a source, the weak field limit holds if $\mathcal{R} >> r_{Sch}$ where r_{Sch} is the Schwarzschild radius. In this limit by definition, the fluctuations are very small, that is $|\eta_{\mu\nu}| \gg |h_{\mu\nu}|$ and $|\varphi^{(c)}| \gg |\phi|$. For pure BD theory, we have $f = 0, \theta = 0$ in (6.5) and (6.3). The trace of energy momentum tensor is just the matter density $T = -\rho$. In the weak-field limit, considering only leading order of the perturbed scalar and metric fields, their respective field equations are,

$$\partial^i \partial_i \phi = \frac{-8\pi G_N \rho}{3 + 2\omega_{\rm BD}}\,, \tag{6.7}$$

$$\varphi^{(c)} G_{\mu\nu}(h) + \eta_{\mu\nu}\partial^\alpha\partial_\alpha\phi - \partial_\mu\partial_\nu\phi = 8\pi G_N T_{\mu\nu}\,, \tag{6.8}$$

$G_{\mu\nu}(h)$ is the perturbed Einstein tensor given in Appendix(B.1). These are coupled equations. In [85], they have suggested a way to make (6.8) ϕ independent, which is by introducing $h_{\mu\nu} = H_{\mu\nu} - \eta_{\mu\nu}\frac{\phi}{\varphi^{(c)}}$. Then we can write (6.8) as,

$$G_{\mu\nu}(H) = \frac{8\pi G_N}{\varphi^{(c)}} T_{\mu\nu} \, . \tag{6.9}$$

The $0i$ components vanish. Then we have $-\nabla^2 H_{00} = \frac{8\pi G_N}{\varphi^{(c)}}\rho \implies H_{00} = \frac{2G_N M}{\varphi^{(c)} r}$. From scalar field equation we obtain,

$$\phi(r) = \frac{2G_N M}{(3+2\omega_{\rm BD})r} \, . \tag{6.10}$$

This is the scalar field profile. When we include non-linear terms, we still expect the field equations reduce to that of pure Brans-Dicke for large distances.

6.3 Vainshtein Mechanism

First we consider only the cubic interaction term so we put $\theta = 0$ in (6.5). We perform the perturbations in weak field limit as we did in the previous section for (6.5) but now we keep terms up to second order,

$$\begin{aligned}&(\eta^{\mu\nu} + h^{\mu\nu})\partial_\mu\partial_\nu\phi + \eta^{\mu\nu}\Gamma^\kappa_{\mu\nu}(h)\partial_\kappa\phi \\ &\qquad = \frac{-8\pi G_N \rho}{3+2\omega_{\rm BD}} + \frac{2f\varphi^{(c)}}{G_N^2(3+2\omega_{\rm BD})}\left[(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi)^2 - (\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi)\right] \, .\end{aligned} \tag{6.11}$$

From the perturbed quantities given in Appendix(B.1) we see that Christoffel symbols are already in first order in h. In static limit we have,

$$\partial^i\partial_i\phi = \frac{-8\pi G_N \rho}{3+2\omega_{\rm BD}} + \frac{2f\varphi^{(c)}}{G_N^2(3+2\omega_{\rm BD})}\left[(\partial^i\partial_i\phi)^2 - (\partial_i\partial_j\phi)(\partial^i\partial^j\phi)\right] \, . \tag{6.12}$$

Here we recognize the Laplace operator $\partial^i\partial_i = \nabla^2$. We make use of spherical coordinates in which case, the Laplace operator is give by,

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) = \frac{d^2}{dr^2} + \frac{2}{r}\left(\frac{d}{dr}\right)^2 \, .$$

We consider spherical mass distribution with $\rho = \rho(r)$ and rewrite the above equation as,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{-8\pi G_N \rho(r)}{3+2\omega_{\rm BD}} + \frac{4f\varphi^{(c)}}{G_N^2(3+2\omega_{\rm BD})}\left[\frac{1}{r^2}\left(\frac{d\phi}{dr}\right)^2 + \frac{2}{r}\frac{d\phi}{dr}\frac{d^2\phi}{dr^2}\right] \, . \tag{6.13}$$

Multiplying r^2 and integrating once,

$$r^2\frac{d\phi}{dr} = \frac{-2G_N M(r)}{3+2\omega_{\rm BD}} + \frac{4f\varphi^{(c)}}{G_N^2(3+2\omega_{\rm BD})} r\left(\frac{d\phi}{dr}\right)^2 \, . \tag{6.14}$$

For very large distances $r > \mathcal{R}$ we can use a constant mass M instead of distribution $M(r)$. The above looks like a quadratic equation and has two solutions, one of which does not reduce to Brans-Dicke solution at large distances. Since we want BD solution at large distances, we consider the other solution. We rearrange the constants to define two characteristic lengths r_c and r_V as given below.

$$\frac{d\phi}{dr} = \frac{rG_N^2(3+2\omega_{\mathrm{BD}})}{8f\varphi^{(c)}}\left[1-\sqrt{\frac{32Mf\varphi^{(c)}}{r^3(3+2\omega_{\mathrm{BD}})^2G_N}}\right] \to \frac{d\phi}{dr} = \frac{r}{r_c^2}\left[1-\sqrt{1+\frac{r_V^3}{r^3}}\right] , \tag{6.15}$$

$$\text{where,} \quad r_c = \left[\frac{8f\varphi^{(c)}}{G_N^2(3+2\omega_{\mathrm{BD}})}\right]^{1/2} \quad ; \quad r_V = \left[\frac{32Mf\varphi^{(c)}}{G_N(3+2\omega_{\mathrm{BD}})^2}\right]^{1/3} , \tag{6.16}$$

We see that, for vanishing coupling, r_V vanishes. In fact both r_c and r_V are functions of f. The former controls the power of the screening mechanism and the latter is known as the Vainshtein radius or Screening radius. If $3+2\omega_{\mathrm{BD}}$ is a negative number then r_c becomes complex. Hence we do not associate it with a physical length scale. But r_V is a physical length scale which also depends on the mass M of the object concerned. Sign of $3+2\omega_{\mathrm{BD}}$ is crucial because if $\varphi^{(c)} < 1$ then we need it to be positive so that we match the cosmological value of scalar field with the local value of $\varphi^{(l)} = 1$. Also, the solution has to reduce to Brans-Dicke solution of previous section for $r \gg r_V$. For small values of the fraction under the square root, we can expand it and integrate it to see that it indeed returns scalar field profile (6.10). A general solution to the above equation is given in terms of hypergeometric function ${}_2F_1$ [85, 86].

$$\phi(r) = \left(\frac{r_V}{r_c}\right)^2 \int_\infty^{r/r_V} dx\, x\,[1-\sqrt{1+x^{-3}}] = \left(\frac{r_V}{r_c}\right)^2 g\left(\frac{r}{r_V}\right) , \tag{6.17}$$

$$\text{with,} \quad g(x) = \frac{x^2}{2}\left[1-{}_2F_1\left(-\frac{1}{2},-\frac{2}{3},\frac{1}{3},-x^{-3}\right)\right] . \tag{6.18}$$

In the Figure(6.1), we plot the function $g\left(\frac{r}{r_V}\right)$; where we can easily see the change of behaviour near screening radius. For $r \gg r_V$ we recover unscreened pure Brans-Dicke behaviour but for $r \leq r_V$ we have screened region where we do not see any variation of g and hence ϕ.

The local value of the scalar field is expected to be $\varphi^{(l)} = 1$. We make use of the fact that the function $g(x) \to 2.103$ when $x \to 0$ and $\phi = \varphi^{(l)} - \varphi^{(c)}$, to find

$$1 = \varphi^{(c)} + 2.103\left(\frac{r_V}{r_c}\right)^2 . \tag{6.19}$$

Making use of the definitions of r_c, r_V we can write this expression in terms of Schwarzschild radii,

$$(1-\varphi^{(c)})\, r_V = 4.206\,\frac{r_{Sch}(M)}{3+2\omega_{\mathrm{BD}}} . \tag{6.20}$$

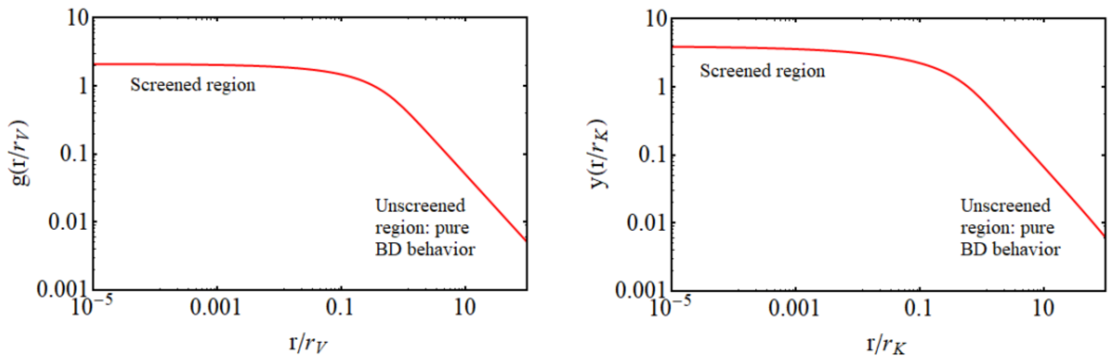


Figure 6.1: *Left PLot:* We plot the function $g(x)$ (6.18) as a function of $\frac{r}{r_V}$. This represents the shape of ϕ (6.17). The screening here refers to Vainshtein screening. *Right PLot:* We plot the function $y(x)$ (6.28) as a function of $\frac{r}{r_K}$. This represents the shape of ϕ (6.27). The screening here refers to K-mouflage screening.

Within screening radius, we have obvious relation $r_V \gg \mathcal{R} \gg r_{Sch}(M)$, which implies,

$$\frac{4.206}{(1-\varphi^{(c)})(3+2\omega_{\rm BD})} \gg 1\,. \tag{6.21}$$

The first term in the denominator is of the order $(1-\varphi^{(c)}) \sim \mathcal{O}(0.1)$ and the second term $(3+2\omega_{\rm BD})^{-1} \sim \epsilon^{(c)}_{\rm BD}/2 \sim \mathcal{O}(10^{-3})$ from (6.6). Clearly the inequality is not satisfied. We need much bigger gradients for φ facilitating the running of the scalar field. In other words, we see that the Vainshtein radius is very small, in fact smaller than Schwarzschild radius. If this is the case, our weak field limit is not sufficient in these limits. If $\omega_{\rm BD}$ takes values closer to $-3/2$ then the equality holds. But neither the cosmological data nor the local measurements suggest such values to $\omega_{\rm BD}$. We shall explore this path later. If we want reasonable values for r_V, that is $r_V \gg \mathcal{R}$, then we need to choose coupling f accordingly. But in this case, we can only explain small differences between cosmological value and the local value. For example, $1-\varphi^{(c)} \lesssim \mathcal{O}(10^{-8})$ for the Milky Way and $1-\varphi^{(c)} \lesssim \mathcal{O}(10^{-12})$ for the Solar System. We show this in Figure(6.2). We encounter same problem with K-mouflage mechanism.

6.4 K-mouflage Mechanism

The procedure for this section is similar to the previous one. In this section we put $f=0$ in (6.5) and obtain scalar field equation in the weak field limit,

$$\partial^i\partial_i\phi = \frac{-8\pi G_N}{3+2\omega_{\rm BD}} + \frac{4\theta\varphi^{(c)}}{G_N^3(3+2\omega_{\rm BD})}\left[(\partial_j\phi)(\partial^j\phi)\partial_i\partial^i\phi + 2(\partial^i\phi)(\partial^j\phi)\partial_i\partial_j\phi\right]. \tag{6.22}$$

For the last term we make use of

$$2(\partial^i\phi)(\partial^j\phi)\partial_i\partial_j\phi = \partial_i[\partial^i\phi(\partial_j\phi)(\partial^j\phi)] - (\partial_j\phi)(\partial^j\phi)\partial_i\partial^i\phi$$

to obtain,

$$\partial^i\partial_i\phi = \frac{-8\pi G_N}{3+2\omega_{\rm BD}} + \frac{4\theta\varphi^{(c)}}{G_N^3(3+2\omega_{\rm BD})}\partial_i[\partial^i\phi(\partial_j\phi)(\partial^j\phi)]\,. \tag{6.23}$$

Now we assume spherical symmetry and make use of divergence theorem to write the following cubic equation.

$$\frac{d\phi}{dr} = \frac{-2G_N M}{(3+2\omega_{\rm BD})r^2} + \frac{4\theta\varphi^{(c)}}{G_N^3(3+2\omega_{\rm BD})}\left(\frac{d\phi}{dr}\right)^3\,. \tag{6.24}$$

Of all the possible solutions, the physical solution has positive discriminant with $\theta(3+2\omega_{\rm BD}) < 0$. For $\theta(3+2\omega_{\rm BD}) > 0$ we do not get BD profile at large distances. The discriminant cannot be zero as well (for nonzero $\omega_{\rm BD}$ and mass M). Using Cardano's formula, we get the solutions similar to Vainshtein screening. Here we define the length scales and solution to cubic equation as,

$$\tilde{r}_c = \frac{4\theta\varphi^{(c)}}{G_N^4 M} \quad ; \quad r_K = \left(\frac{-108\,\theta\,\varphi^{(c)}M^2}{G_N(3+2\omega_{\rm BD})^3}\right)^{1/4}\,. \tag{6.25}$$

$$\frac{d\phi}{dr} = \frac{1}{\tilde{r}_c^{1/3}r^{2/3}}\left[\left(1+\sqrt{1+\left(\frac{r}{r_K}\right)^4}\right)^{1/3} - \left(-1+\sqrt{1+\left(\frac{r}{r_K}\right)^4}\right)^{1/3}\right], \tag{6.26}$$

$\tilde{r}_c, r_K$ play the same role as r_c, r_V. Only r_K is a physical length scale which is always real. A solution to the above equation for $r > \mathcal{R}$ is,

$$\phi(r) = -\left(\frac{r_K}{\tilde{r}_c}\right)^{1/3} y\left(\frac{r}{r_K}\right), \tag{6.27}$$

$$y\,(z) = \int_\infty^z dx\,x^{-2/3}\left[\left(-1+\sqrt{1+x^4}\right)^{1/3} - \left(1+\sqrt{1+x^4}\right)^{1/3}\right]\,. \tag{6.28}$$

We plot the function in Figure(6.1). In this case, if we realize that the function $y(x) \to 3.984$ when $x \to 0$, we obtain a similar expression to (6.20),

$$(1-\varphi^{(c)})\,r_K = 5.976\,\frac{r_{Sch}(M)}{3+2\omega_{\rm BD}}\,, \tag{6.29}$$

Given the same values for $(1-\varphi^{(c)}) \sim \mathcal{O}(0.1)$ and the second term $(3+2\omega_{\rm BD})^{-1} \sim \mathcal{O}(10^{-3})$ from (6.6), we see that we have similar issues like we had in Vainshtein mechanism.

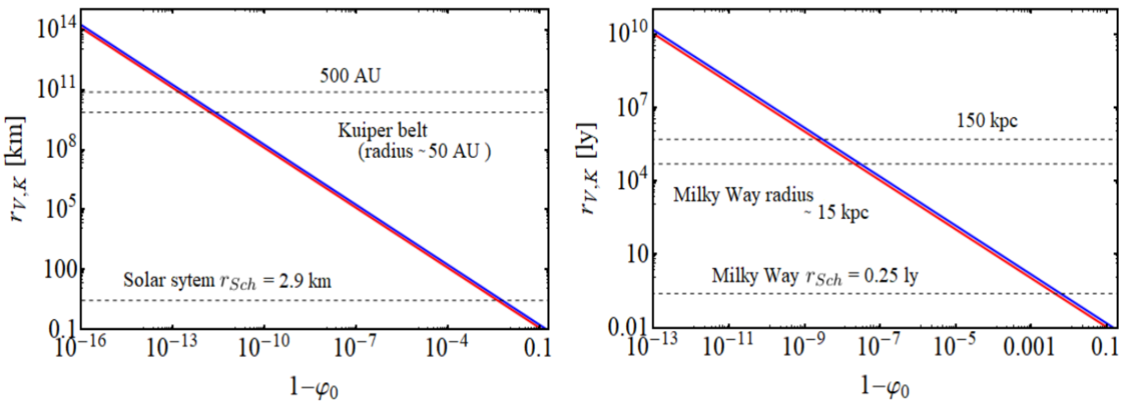


Figure 6.2: Here we plot screening radius as a function of $\phi = (1 - \varphi^{(c)})$. In both the plots red line is the size of Vainshtein radius r_V and blue line is the size of K-mouflage radius r_K to obtain which we use $\epsilon_{\text{BD}} = 2 \cdot 10^{-3}$ in (6.20) and (6.29) respectively. From both the plots we see that we need very small values of $(1 - \varphi^{(c)})$ to obtain reasonable values for screening radius. *Left PLot:* We see the screening radius associated to the Solar System as a function of ϕ. The black dashed lines indicate Schwarzschild radius (2.9 km), radius of Kupier belt ($\sim$ 50AU) and a radius which is ten times radius of Kupier belt. *Right PLot:* We see the screening radius associated to the Milky Way as a function of ϕ. The black dashed lines indicate Schwarzschild radius (0.25ly), physical radius (15kpc), and a radius ten times the physical radius.

6.5 Screening with a varying $\omega_{\rm BD}$ parameter

So far we have considered $\omega_{\rm BD}$ as a constant. But the results of previous sections shows that, very low values of $\epsilon_{\rm BD}$ plays a significant role in reducing the variation of scalar field. Since both Vainshtein and K-mouflage mechanisms have not been able to explain non-negligible differences of scalar field and their presence is the theory is not improving the situation. Instead we expect $\omega_{\rm BD}$ to be taking different values throughout the space such that we have differences of scalar field as expected. We see that to satisfy inequalities in (6.20) and (6.29), $\omega_{\rm BD}$ has to be close to $-3/2$. This would generate large gradients of scalar field. This can supplement the Vainshtein and K-mouflage screening mechanisms by quickly taking the scalar field to nonlinear regime. However $\omega_{\rm BD} \sim -3/2$ is not allowed by any data; neither in cosmological scales nor in local scales. There might still be some regions of space where $\omega_{\rm BD}$ can take such values. Here we consider non-constant $\omega_{\rm BD}$ such that it results in screening-like behaviour where ϕ takes constant values in small scales. So we set nonlinear interactions to zero with $f = 0, \theta = 0$. Now the scalar field equation is,

$$
\begin{aligned}
0 =& R + \frac{1}{\varphi}\left(\omega'_{\rm BD} - \frac{\omega_{\rm BD}}{\varphi}\right)(\nabla\varphi)^2 + (\nabla\varphi)^2 + \frac{2\omega_{\rm BD}}{\varphi}\Box\varphi \\
\implies & (3 + 2\omega_{\rm BD})\Box\varphi = 8\pi G_N(T - 4\rho_\Lambda) - \omega'_{\rm BD}(\nabla\varphi)^2
\end{aligned}
\tag{6.30}
$$

The prime denotes derivatives $w.r.t\ \varphi$. In the second line we have substituted the Ricci scalar from taking the trace of metric field equation. In the weak field limit,

$$
\partial^i\partial_i\phi = \frac{8\pi G_N}{3 + 2\omega_{\rm BD}}T - \frac{\omega'_{\rm BD}}{3 + 2\omega_{\rm BD}}\partial_i\phi\partial^i\phi\,. \tag{6.31}
$$

Again, considering spherical symmetry,

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{-8\pi G_N\rho(r)}{3 + 2\omega_{\rm BD}} - \frac{\omega'_{\rm BD}}{3 + 2\omega_{\rm BD}}\left(\frac{d\phi}{dr}\right)^2\,. \tag{6.32}
$$

Far away from the source, the density distribution simply vanishes. So for $r > \mathcal{R}$,

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\omega'_{\rm BD}}{3 + 2\omega_{\rm BD}}\left(\frac{d\phi}{dr}\right)^2\,, \tag{6.33}
$$

which is easily solvable to obtain,

$$
r^2\frac{d\phi}{dr} = \frac{C}{\sqrt{3 + 2\omega_{\rm BD}}}\,. \tag{6.34}
$$

C is an integration constant which has to be determined. This is set by demanding the field varies like BD field sufficiently far away from the source at some radius r_*.

$$
C = r_*^2\frac{d\varphi}{dr}\bigg|_{r_*}\sqrt{3 + 2\omega^{(c)}_{\rm BD}} = \frac{-r_{Sch}}{\sqrt{3 + 2\omega^{(c)}_{\rm BD}}}\,. \tag{6.35}
$$

$\omega_{\rm BD}$ is a function of scalar field φ; and what we need is $\phi \equiv \delta\varphi$. If we choose $\omega_{\rm BD}(\varphi)$ then we can compute ϕ by performing integration of (6.34). Then we will have a profile for BD parameter $\omega_{\rm BD}(r)$. Instead, we can also select the shape of ϕ, then we compute $\omega_{\rm BD}(r)$ which leads us to $\omega_{\rm BD}(\varphi)$. Here, we guess the following phenomenological expression for ϕ,

$$\phi(r) = \frac{1}{(r+r_S)^{n+1}}\left[\phi_{Scr} r_S^{n+1} + \frac{r_{Sch} r^n}{3+2\omega_{\rm BD}}\right], \tag{6.36}$$

where r_S is the screening radius and n is a positive number. For different values of n we have different slopes which connects large scale pure BD behaviour with screened regions. In the screened regions, ϕ remains constant ϕ_{Scr} non-negligible value as expected. The screening radius can be obtained by specifying the free parameters in the above expression. For $\varphi^{(c)} \sim 0.9, \phi_{Scr} = 0.1$ and $r_{Sch} = 2.9$km and assuming $r_S \sim 50$AU we plot the shapes of $\phi(r), \omega_{\rm BD}(r)$ for different values of n. Even though we now have expected variations of gravitational coupling, from the shapes of the plot we see that there has to be lot of fine tuning to get $\omega_{\rm BD} \sim -3/2$. At this point we should have significant deviations from GR. However, we can still look for other shapes of ϕ but in any case the problems mentioned will still remain.

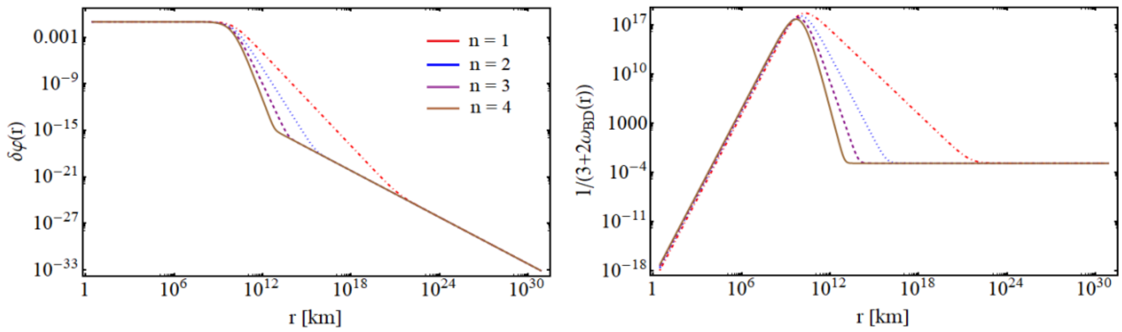


Figure 6.3: *For both the plots we have considered Solar System values -* $r_{Sch} = 2.9km$ *and set* $r_V = 50AU$*,* $\varphi^{(c)} = 0.9$*.* n *is a positive number in* (6.36)*. Left Plot:* $\phi \equiv \delta\varphi$ (6.36) *for various values of* n*. Right Plot:* Plots for associated $\omega_{\rm BD}(r)$ (6.34).

We studied extended BD-ΛCDM where we included cubic and quartic self-interaction terms from the Horndeski family. In the weak field limit we obtained Brans-Dicke scalar field profile assuming Vainshtein and K-mouflage screening mechanisms. We have seen that both of these screening mechanisms are only able to explain very small variations of G which is quantified through the difference $\varphi^{(l)} - \varphi^{(c)} \lesssim \mathcal{O}(10^{-8})$ for the Milky Way and $\varphi^{(l)} - \varphi^{(c)} \lesssim \mathcal{O}(10^{-12})$ for the Solar System. However, BD-ΛCDM allows for $\varphi^{(l)} - \varphi^{(c)} \sim \mathcal{O}(0.1)$. This difference in gravitational couplings allowed by BD-ΛCDM have proven to be very useful to loosen the cosmological tensions.

We can consider next higher order derivatives available from Horndeski Lagrangian. As mentioned in the previous chapter, measurements of speed of gravitational waves have constrained Horndeski Lagrangian such that $G_{4X} = 0$. Even in this case we do not anticipate any significant changes to our results because in the weak field limit, again the small value of $\epsilon_{\rm BD}$ will prevent large deviations. The introduction of a non-constant potential for the scalar field does not solve the problem neither. The very criteria for such a potential regardless of its shape would be to generate large gradients of $\delta\varphi \equiv \phi = \varphi^{(l)} - \varphi^{(c)}$. This is the origin of the aforementioned problems. Hence, we do not expect the latter to be solved with the aid of Chameleon[87] or Symmetron[88] mechanisms which make use of environment dependent potentials. They are of course able to screen Brans-Dicke effects in dense enough environments, but they are not capable of explaining the large differences of the gravitational coupling that are needed by BD-ΛCDM.

Part III

Stability Against Quantum Corrections

7 Functional Methods

We start with introducing important generating functionals like, Schwinger functional and effective action. The latter is a functional of the mean field and its corresponding equations of motion includes all quantum effects in it and hence exact (for vanishing source). We use [89] as our reference. In the second section we revisit renormalization and flow equations but in Wilsonian picture. Appendix(D.1) *also give a note on effective field theory which supplements this section. With the example of gravitational coupling, we insist on computing flow equations for dimensionless couplings as their divergence is related to physical divergence. In this particular example, appearance of non-trivial UV fixed point indicates asymptotic safety scenario. In general non-trivial fixed points are reached in an interacting theory for non vanishing anomalous dimension and coupling. Finding such points plays a major role in the following chapter. Here we make use of [16, 90] as references.*

7.1 Generating Functionals and Correlation Functions

Correlation functions play an important role in QFT. For example, 1-point correlation function is just the expectation value; 2-point correlation function is the Green's function, interpreted as propagators. Moreover, we can treat QFT as a statistical field theory, if we can define a *generating functional* (analogous to partition functions in statistics) such that the correlation functions are its moments. In path integral formalism, a generic generating functional of source function $J(x)$ reads,

$$Z[J] = \int D\chi \exp\left(-S[\chi] + \int J(x)\chi(x)d^4x\right). \tag{7.1}$$

We could in principle have many sources which set the background. It generalizes a constant background to an inhomogeneous (for example, it can be x dependent magnetic field background). $S[\chi]$ is just the classical action functional and the integral measure is given by[1],

$$\int D\chi = \Pi_x \int_{-\infty}^{\infty} d\chi(x). \tag{7.2}$$

Any n-point function is obtained by performing functional differentiation of $Z[J]$ n number of times. So, the only condition on $Z[J]$ is that it has to be differentiable.

$$\langle \chi(x_1)\chi(x_2)...\chi(x_n)\rangle = G^n(x_1, x_2, ..., x_n) = \frac{1}{Z}\frac{\partial^n Z[J]}{\partial J(x_1)\partial J(x_2)...\partial J(x_n)} \tag{7.3}$$

Thus one can see $Z[J]$ as a *generating functional.* In a more traditional way we can also show this by writing $Z[J]$ as an expansion in powers of J for a weak background source.

$$\begin{aligned} \exp\left(\int d^4x J(x)\chi(x)\right) &= 1 + \int d^4x J(x)\chi(x) + \frac{1}{2}\int d^4x d^4y J(x)\chi(x)J(y)\chi(y) + ... \\ \implies Z[J] = Z[0]&\left(1 + \int d^4x J(x)\langle\chi(x)\rangle + \frac{1}{2}\int d^4x d^4y J(x)J(y)\langle\chi(x)\chi(y)\rangle + ...\right). \end{aligned} \tag{7.4}$$

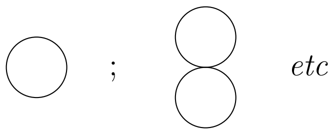
The n-point functions are the time-ordered Green's function written in terms of Heisenberg operators (indicated by T and $\hat{}$ respectively).

$$\langle \chi(x_1)...\chi(x_n)\rangle = \frac{\int D[\chi]e^{-S[\chi]}\chi(x_1)...\chi(x_n)}{\int D[\chi]e^{-S[\chi]}} = \langle 0| T\hat{\chi}(x_1)...\hat{\chi}(x_n)|0\rangle \tag{7.5}$$

If we take 1-point functional for example, the expectation value would be background J dependent. Since we need expectation values independent of a background, we just set $J = 0$ after differentiation. The generating functional generates all possible interactions, even the ones which do not affect the observable quantities. The useful interactions for particle physics are those which affect the cross-section, depicted by connected Feynman diagrams. On the other hand, unconnected diagrams which are also called *vacuum bubbles* or *vacuum to vacuum transitions*, can be factored out and are cancelled by the same factor in the denominator thus they do not play any role in calculating cross-sections[2]. One such example would be the 2 loop diagram in (7.6).

[1] The integral is for all positions, however small you can get! This looks a problem which will be corrected to be written as (7.17) in a regularized sense.

[2] Connected diagrams mean we can go from one end of the diagram to the other end by following the edges in the Feynman diagrams. Disconnected diagrams mean the diagrams are disconnected from external points.

 (7.6)

The mean field $\langle\chi\rangle$ or the 1-point correlation function which we have been repeatedly pointing out has relevance to our discussion on CC because it refers to 1-loop vacuum to vacuum transition (the first diagram in the above equation). Bubble diagrams (vacuum to vacuum) are propagators evaluated at the same point which contribute to CC. To have only connected diagrams, we define a new functional, *Schwinger functional* by taking logarithm of the partition function. In our present discussion, we are considering particle physics perspective, where ground state is non zero.

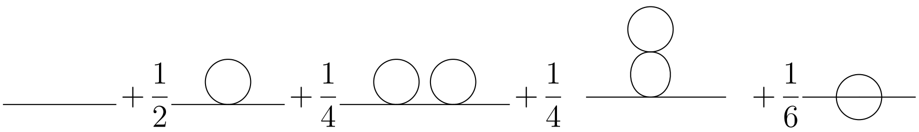
$$W[J] = \ln Z[J] \tag{7.7}$$

$W[J]$ is the Schwinger functional and the correlation functions are again defined by taking functional derivatives. But now, the resulting n-point functions are connected indicated by subscript c.

$$G_c^n(x_1, x_2, ..., x_n) = \left.\frac{\partial^n W[J]}{\partial J(x_1)\partial J(x_2)...\partial J(x_n)}\right|_{J=0}$$

example: 2-point function

$$\begin{aligned} G_c^{(2)}(x_1, x_2) &= \frac{\partial^2 W[J]}{\partial J(x_1)\partial J(x_2)} = \frac{\partial}{\partial J(x_1)}\left(\frac{1}{Z}\frac{\partial Z}{\partial J(x_2)}\right) \\ &= \frac{1}{Z}\frac{\partial^2 Z}{\partial J(x_1)\partial J(x_2)} - \frac{1}{Z^2}\frac{\partial Z}{\partial J(x_1)}\frac{\partial Z}{\partial J(x_2)} \\ &= \langle\chi(x_1)\chi(x_2)\rangle - \langle\chi(x_1)\rangle\langle\chi(x_2)\rangle \end{aligned} \tag{7.8}$$

$G_c^{(2)} =$  (7.9)

The terms representing unconnected diagrams are subtracted from 2-point correlation function in (7.8) to give only connected diagrams (7.9). Schwinger functional still contains extra information. We need 1-particle reducible diagrams - the diagrams that cannot be cut into two parts by cutting one internal line. For example the third diagram in (7.9) with two loops can be cut in the internal line between the loops and hence a reducible diagram.

We define a new generating functional by performing a Legendre transformation of the Schwinger functional. This is usually referred to as effective action which is a functional of mean field $\langle\chi\rangle \equiv \varphi$,

$$\Gamma[\varphi] = W[J] - \int d^4x J(x)\varphi(x)\ ;\ \varphi(x) = \langle\chi(x)\rangle = \frac{\partial W[J]}{\partial J(x)} \tag{7.10}$$

Now that we have an action, we can get the field equation in a given background J. For comparison we also write the classical field equation derived from the classical action $S[\chi]$

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = J(x)\ ;\quad \frac{\delta S[\chi]}{\delta\chi} = 0 \tag{7.11}$$

For $J = 0$ the first equation is the exact field equation. Unlike $S[\chi]$, $\Gamma[\varphi]$, which is a functional of the mean field φ, includes all the quantum effects and thus it is the exact quantum field equation. Now, the second functional derivative gives the inverse propagator $\Gamma^{(2)} = G^{-1}$. Similarly, for higher derivatives we get n point vertices as shown below, which are *one particle irreducible* (1PI). These diagrams cannot be cut on any internal propagator into two disconnected diagrams. This is very useful because the scattering amplitude is nothing but the amputated, connected Green function.

$$\Gamma^n(x_1, x_2, ..., x_n) = \frac{\partial^n\Gamma[\varphi]}{\partial\varphi(x_1)\partial\varphi(x_2)...\partial\varphi(x_n)}. \tag{7.12}$$

We also mention an important identity, the *Background Field Identity*, where we make use of the above definitions of Schwinger functional and source functional to show[3],

$$\begin{aligned} e^{-\Gamma} &= e^{-(-W+\int j\varphi)} = \int \mathcal{D}\phi e^{-S+\int_x(J\chi-J\varphi)} \\ &= \int \mathcal{D}\phi\ \exp[-S + \int_x \frac{\delta\Gamma}{\delta\varphi}(\chi-\varphi)] \\ &= \int \mathcal{D}\phi\ \exp\left(-S[\chi] + \int_x \frac{\delta\Gamma}{\delta\varphi}\phi(x)\right). \end{aligned} \tag{7.13}$$

The macroscopic field χ can be decomposed into a background mean field φ and 'true' quantum fluctuations ϕ around it such that $\langle\phi\rangle = 0$. Then microscopic (quantum fluctuations) field around the background macroscopic field is, $\phi = \chi - \varphi$. To have quantum equations of motion, we just have to take the derivative of the above *w.r.t* φ. In the absence of interactions $\phi = 0$ the effective action reduces to macroscopic action. The quantum equations of motion are called Dyson-Schwinger. In (7.13), the classical action $S[\varphi + \phi]$ is complemented by quantum equation of

[3]The notations of integral is simply $\int_x \equiv \int d^4x$; or for example $\int_{x,y} \equiv \int d^4x \int d^4y$. The number of dimensions on which the integral is performed need not always be 4. Unless mentioned we can take it to be d dimensions in general.

motion already. Hence it is an implicit equation and we make use of the *Saddle Point Approximation* where we expand the classical action to obtain equations of motion up to 1-loop.

$$\begin{aligned}\Gamma[\varphi] &= \underbrace{S[\varphi]}_{\text{classical}} \underbrace{- \ln \int \mathcal{D}\phi \, \exp\left(-S[\varphi+\phi] - S[\varphi] + \int_x \frac{\delta\Gamma}{\delta\varphi}\phi(x)\right)}_{\text{fluctuation}}, \\ S[\varphi+\phi] &= S[\varphi] + \underbrace{\int_x S^{(1)}[\varphi]\phi(x)}_{\text{last term of fluctuation term}} + \frac{1}{2}\int_{x,y} S^{(2)}[\varphi]\phi(x)\phi(y) + \dots,\end{aligned} \tag{7.14}$$

where we neglected higher order terms and $\frac{\delta\Gamma}{\delta\varphi}$ so that we can extract the effective 1-loop contribution to the effective action. One loop effective action is a Gaussian integral given by (in the second line c is a constant),

$$\begin{aligned}\Gamma_{1l} &= -\ln \int \mathcal{D}\phi \exp\left(\frac{1}{2}\int_{x,y} S^{(2)}(x,y)\phi(x)\phi(y)\right) \\ &= -\ln(\det\sqrt{(S^{(2)})}\,.\,c) \\ &= \frac{1}{2}\mathrm{Tr}\;\; \ln S^{(2)}[\varphi].\end{aligned} \tag{7.15}$$

It is evident that $\Gamma^{(2)}$ and $S^{(2)}$ have a difference of 1-loop and using this we can now extrapolate to have 2-loop contributions and so on. We will discuss the renormalization and flow equations in section(9.1). To avoid confusion with effective average action appearing in that section, we shall represent effective action by S_E in the following section. Now lets start with Wilsonian approach to renormalization and the framework of EFT to revisit beta functions and fixed points.

7.2 Wilsonian Renormalization

Quantum fluctuations modify the strength of the couplings. The loop diagrams leads to diverging integrals. We introduced the idea of cutoff regularization to trade physical divergences ($k \to \infty$) against sensitivity to cutoff scale. Our approach so far has been of a reductionist. We can instead explain it from the perspective of Effective Field Theory (EFT). Here, we shall refrain from giving historic and fundamental motivation in the aspects of phase transitions and study of critical exponents but rather focus on modern renormalization techniques including the flow equation in the form of Callan-Symanzik Equation[4]. Most of the insights are from the [90, 16]. However, we have given very short introduction to EFT in Appendix(D.1) mostly based on [91] which complements this section.

[4] In Appendix(D.3) we have revisited fixed points and have given a short note on linearization of beta functions around fixed points.

If a quantum field theory (QFT) is well defined upto a fundamental energy scale Λ but our experiments are restricted to much lower energy scales $\Lambda_{exp} \ll \Lambda$ then we can choose a cutoff in between these scales such that $\Lambda_{cut} < \Lambda$ and divide all the quantum field into high and low energy components $\chi = \chi_L + \chi_H$. Only low energy field components are sufficient to make predictions for experiments as $\Lambda_{exp} \ll \Lambda_{cut}$. We can integrate out the high energy fields as they do not appear in our calculations, for example in the computation of the cross-section of a scattering process. So, with decomposition of fields we can write the relevant part of the action which is only dependent on the low energy fields by integrating out all the high energy fields as shown below.

$$\begin{aligned}
\chi &= \int_{|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\chi \; + \; \int_{\Lambda_{cut}<|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\chi \\
&= \chi_L + \chi_H, \\
e^{-S_E[\chi_L]} &= \int D\chi_H \;\; e^{-S[\chi_L,\chi_H]},
\end{aligned} \tag{7.16}$$

then the partition function (7.1) in d-dimensions can be written as follows

$$Z[J_L] = \int D\chi_L D\phi_H \exp\bigg\{-S[\chi_L,\chi_H] + \int_x J_L[x]\chi_L[x]\bigg\} \tag{7.17}$$

$$\longrightarrow Z_E[J_L] = \int D\chi_L \exp\bigg\{-S_E[\chi_L] + \int_x J_L[x]\chi_L[x]\bigg\}. \tag{7.18}$$

The action $S_E(\chi_L)$ is called Wilsonian Effective Action[5]. We have to note that, S_E depends on the choice of cutoff Λ_{cut} because this determines the energy beyond which fluctuations are integrated out[6]. To define the theory at some other energy scale, say Λ', we can simply formulate the action in a similar fashion by integrating out modes above Λ'. For $\Lambda' < \Lambda_{cut}$, we can consider this as a renormalization group flow and we should be able to derive a flow equation similar to (2.23). Let us take the exact case and put all the local sources to zero $J = 0$. It is apparent that we are allowed to express effective partition function as a functional of couplings $g_i(\Lambda_{cut})$ which depends on cutoff energy scale,

$$Z_E[g_i(\Lambda_{cut})] = \int D\chi_L \exp\{-S_E[\chi_L]\}.$$

Since the integral is only over low energy modes this will result in the exact partition functional with exact couplings g_0. These are the same couplings which we obtained in experiments. In other words we have an EFT which valid up to certain energy scale that is a part of some more comprehensive theory. Suppose we lower the energy

[5]We can see the relation to thermodynamic free energy when written as above. For a review see [92]

[6]Note that, having such a cutoff makes S_E non-local because at the length scales associated with Λ_{cut} the high energy modes are not present as they are integrated out

scale $\Lambda' < \Lambda_{cut}$, knowing that the partition function in the higher energy scale Λ_{cut} ends up giving the 'exact' partition function which has couplings g_0. Then we want the partition function to be independent of cutoff scale (this is similar to reasons mentioned to write (2.23)). Which is why we write,

$$\Lambda_{cut}\frac{dZ_E[g]}{d\Lambda_{cut}} = \left(\Lambda_{cut}\frac{\partial}{\partial\Lambda_{cut}}\bigg|_{g_i} + \Lambda_{cut}\frac{\partial g_i(\Lambda_{cut})}{\partial\Lambda_{cut}}\frac{\partial}{\partial g_i}\bigg|_{\Lambda_{cut}}\right) Z_E(g) = 0, \tag{7.19}$$

or in terms of correlation function[7] [93, 94],

$$\left(\Lambda_{cut}\frac{\partial}{\partial\Lambda_{cut}}\bigg|_{g_i} + \Lambda_{cut}\frac{\partial g_i(\Lambda_{cut})}{\partial\Lambda_{cut}}\frac{\partial}{\partial g_i}\bigg|_{\Lambda_{cut}}\right) G^n(\{x_k\}, \Lambda_{cut}, g_i).$$

We could also write a generic effective action including all possible interactions operator along with a kinetic term. Then accounting for wavefunction renormalization, we can do the rescaling of field using $\mathcal{Z}$ ($\phi = \frac{\phi_B}{\sqrt{\mathcal{Z}}}$). But now, this factor is defined at Λ_{cut}. One has the anomalous dimension defined as,

$$\gamma = -\frac{1}{2}\Lambda_{cut}\frac{\partial \ln \mathcal{Z}_{\Lambda_{cut}}}{\partial\Lambda_{cut}},$$

which also looks like a beta function and hence appears as an additional term in the Callan-Symanzik equation. The factor 1/2 in is due to the particular rescaling mentioned above. While adding this term to (7.19) for a n-point function, the anomalous dimension is multiplied by n because of the wavefunction rescaling. *As one can see, the beta function along with the anomalous dimension depends on all the couplings.* The n-point function (with 1-loop correction) includes a tree level diagram, 1PI loop diagrams. We will also have counter terms for vertex. For a two point function $G_c^{(2)}$,

$$\begin{aligned} G_c^{(2)} &\equiv \underline{\qquad\quad} + \text{loop diagrams} + \text{counter terms} \\ &\equiv \frac{i}{p^2} + \frac{i}{p^2}\left(A\ln\frac{\Lambda_{cut}^2}{m_L^2} + B\right) + \text{counter term.} \end{aligned} \tag{7.20}$$

A and B are some constants in accordance with (2.10). The counter terms are given in (2.16). From Callan-Symanzik equation, we again see that the counter term has to be of the form given in (2.16). So beta function can be thought of as a combination of all the coefficients of divergent logarithms. Even though the beta function and gamma function depend on the coupling which in tern depend on the energy scales, the renormalized connected Green's function do no depend on the cutoff scale.

The use of dimensionless constants is helpful, because their divergence implies divergence of physical quantities. Consider the example of renormalized Newton's

[7] Refer [13] for details.

coupling $G_N(\Lambda_{cut}) = G_N/\mathcal{Z}_G(\Lambda_{cut})$ where $\mathcal{Z}_G$ is the wavefunction renormalization factor of graviton [95]. It is dependent on the cutoff energy Λ_{cut}. Dimensionless gravitational coupling is $\tilde{G}_N = G_N(\Lambda_{cut})\Lambda_{cut}^{d-2} = G_N\Lambda_{cut}^{d-2}/\mathcal{Z}_G$ for which, Callan-Symanzik equation is given by,

$$\begin{aligned}
&\left(\Lambda_{cut}\frac{\partial}{\partial\Lambda_{cut}} + \Lambda_{cut}\beta(g_i)\frac{\partial}{\partial g_i} - \Lambda_{cut}\frac{\partial \ln \mathcal{Z}_G}{\partial\Lambda_{cut}}\right)\tilde{G}_N = 0 \\
\Longrightarrow\ &\Lambda_{cut}\frac{\partial(G_N\Lambda_{cut}^2)}{\partial\Lambda_{cut}} - \frac{1}{2}\Lambda_{cut}\frac{\partial \ln \mathcal{Z}_G}{\partial\Lambda_{cut}}\tilde{G}_N = ((d-2)+\gamma)\tilde{G}_N = 0,
\end{aligned} \tag{7.21}$$

where we included anomalous dimension as well. Now this beta function leads to two types of fixed points. The first type is a Gaussian fixed point where we take anomalous dimension to be zero hence considering the theory to be classical (non-interacting). In this case, $\tilde{G}_N^* = 0$ solves the equation. There is also a non-Gaussian fixed point $\tilde{G}_N^* \neq 0$ when anomalous dimension is $-(d-2)$. This fixed point is reached in an interacting theory. The direction from which this point is reached becomes important and will be discussed in the next section. It is interesting to note that all the quantum aspects are in this second term. On the first term of *r.h.s* of (7.21) we have a running with number of dimensions, and the second term includes contributions from quantum fluctuations. Suppose for $\Lambda_{cut} \to \Lambda$ the dimensionless coupling grows, then the quantum contributions are significant only if they are very large. As we will see, in the case of Newton's gravitational coupling the second term counter-balances the canonical dimension. Now, renormalized gravitational coupling scales as $G_N(\Lambda_{cut}) \sim \tilde{G}_N^*/\Lambda_{cut}^{d-2}$ which becomes small in high energy limits. Similarly, the dimensionless coupling will reach a fixed point $\tilde{G}_N^*$ in this limit. This is parameterized as $\tilde{G}_N = \Lambda_{cut}^2/(M_P^2 + k^2/\tilde{G}_N^*)$ [90]. This trend is shown in Figure(7.1).

For our applications we stick with the beta function definition as rate of flow of renormalization group of couplings constant. With this definition a positive beta function would mean that the renormalized coupling increases with energy scale. With all the couplings rescaled such that they are dimensionless we can express their vanishing variation *w.r.t* Λ_{cut} as beta functions. We denote them with a tilde (for example, $\tilde{g} = g\Lambda_{cut}^{D-d}$ where D is the dimension of the coupling).

$$\begin{aligned}
&\beta_i(\tilde{g}_j) = \Lambda_{cut}\frac{\partial(g_i\Lambda_{cut}^{d_i-d})}{\partial\Lambda_{cut}} = (d_i-d)\tilde{g}_i + \beta_i(g_j) \\
&\qquad\qquad \text{for example, dimensionless gravitational coupling } \tilde{G}_N, \\
&\beta(\tilde{G}_N) = \Lambda_{cut}\frac{\partial(G_N\Lambda_{cut}^2)}{\partial\Lambda_{cut}} = 2\tilde{G}_N + \Lambda_{cut}^2\frac{\partial G_N}{\partial\Lambda_{cut}}.
\end{aligned} \tag{7.22}$$

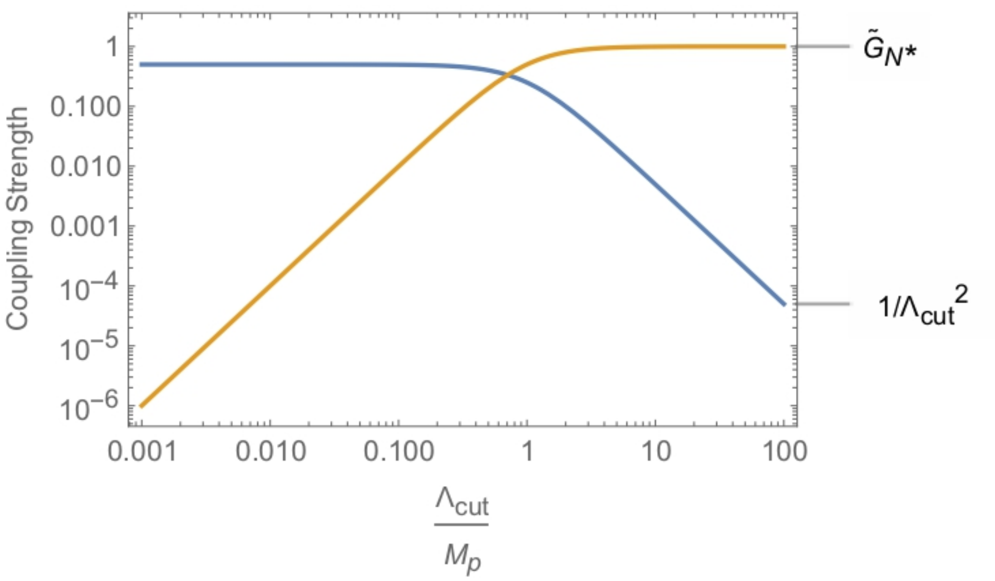


Figure 7.1: Here we show how the coupling strengths of Newton's gravitational constant and dimensionless gravitational constant vary with scales. The energy scales are labeled as Λ_{cut}. They are normalized with Planck's mass M_P. The dimensional coupling G_N remains constant at low energies and decreases like $1/\Lambda_{cut}^2$ at high energies. The dimensionless coupling $\tilde{G}_N$ however reaches a fixed point $\tilde{G}_N^*$ at high energies. We can clearly see the transition when $\Lambda_{cut} \sim M_P$.

8 Validity Regime for Classical Higher Derivative Screening

This chapter derives its motivation from [8]. First we define generic K-mouflage (and Galileons) theories in effective field theory perspective. We define strong coupling regime which is the region of interest in the context of screening mechanisms. The aim is to see stability of classical solutions in these theories. The stability is confirmed if quantum corrections (at 1-loop) do not dominate over classical solution even when higher order operators are dominant. We use [96] for calculations and reproduce results of [8].

8.1 Motivation

So far, we studied classical K-mouflage and Galileon theories as an extension to Brans-Dicke theory or as a part of general Horndeski theory. The main reason for considering such extensions is to obtain a screening mechanism for the scalar field so that we retrieve GR in local scales but still maintain BD behaviour at larger scales. The considered screening mechanisms requires no potential and is completely dependent on the dominant dynamics from the kinetic or higher derivative terms. To this point, we found that higher derivative terms in the extension could not screen the BD effects. The considered two sub-classes of theories can be thought as Effective Field Theories which include a ladder of all possible derivative terms in the Lagrangian. Galileon theories are in fact effective theories of DGP model. In this spirit, we would like to know how these classes of theories behave in the UV regime. But first let us set the simple objective to define a regime of validity of screening solutions. This would be our regime of interest where we have dominant dynamics coming from non-canonical kinetic terms or higher derivative terms. The objective is to check if 1-loop contributions are small compared to the classical theory.

$P(X)$ theories belong to the class of *k-essence models* mentioned in the previous chapter. Unlike quintessence models, here we do not introduce a (slowly varying) potential to realize cosmic acceleration. Instead, the kinetic energy X of a scalar field φ drives the acceleration. Hence these can be considered as modified matter models which have been widely studied and have rich phenomenology. A list some of the examples mentioned in [11].

1. Ghost Condensate Model [58]: $P = -X + X^2/\Lambda_0^4$

Energy momentum tensor is given by, $T_{\mu\nu} = P'\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}P$ and equation of state is given by, $w = p/\rho = (1 - X/\Lambda_0^4)/(1 - 3X/\Lambda_0^4)$. For $-1 < w < -1/3$, range of kinetic energy is the interval $(1/2, 2/3)$. One can slightly modify this model with some extra terms for example,

a) Dilatonic Ghost Condensate [97]: $P = -X + e^{\Lambda_0\varphi/\sqrt{8\pi G_N}} X^2/\Lambda_0^4$

b) Galileon Ghost Condensate [98]: $P = -X + X^2/\Lambda_0^4 + fX\Box\varphi$.

In fact, it was shown that cosmological data favors Galileon ghost condensate model over standard ΛCDM model [98]. The equation of state has an attractor solution unlike models with only cubic galileon term in which behaviour of equation of state is not consistent with data in matter dominated era. Hence, such models are useful to explain the acceleration of the late universe.

2. Dirac-Born-Infeld (DBI) theories [99]: $P = \Lambda_0^4\sqrt{1+X} - \Lambda_0^4$

   These theories arise in Type IIB String theory (due to dynamics of D-branes - hence called D-acceleration) which can also explain the late acceleration of the Universe; as a cosmological constant [100] and general dark energy component [101].

3. We can also consider such theories to drive inflation in the early epochs. These models are called k-inflation [102] which can lead to slow-roll or power law like inflation.

These models introduce new degrees of freedom and are seen as EFTs in the cosmological context. They become interesting in the so called *strong coupling regime*. This is the regime in which a screening mechanism helps these theories escape the local constraints. The screening mechanism relies on high order terms in the Lagrangian which dominate the dynamics. To make way for such dynamics, we should rely on the fact that effects of higher order terms are radiatively stable. But there is no reason for why they should not cause problems with diverging couplings and thus spoil the macroscopic behaviour. When applied as an extension of Brans-Dicke theories we have seen in previous chapter that, higher derivative screening mechanisms were insufficient to reconcile the Newton's gravitational constant in the solar system scales and galaxy scales. Hence, one has to look at high energy aspects of these theories to see if at some scale they spoil the macroscopic behaviour. For this reason, we shall now follow the works of [8] and try to reproduce their results to define the regime of validity for effective K-essence and Galileon theories. All our calculations in this chapter and the next one will be on flat spacetime.

8.2 Validity Regime for Effective K-mouflage and Galileon Theories

We first look at the dynamics of $P(X)$ theories in flat space time. We start with effective action,

$$S_E^K = \int d^4x \Lambda_0^4 P(X), \quad \text{where, } X = -\frac{g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}{2\Lambda_0^4}. \tag{8.1}$$

This includes all possible terms, for example, terms like $(\partial\varphi)^2, (\partial\varphi)^4$ etc. The scalar field φ couples to the rest of the matter through an effective coupling $\beta(\phi)$. We refer to $P(X)$ as K in the following. From dimensional analysis, we see that $P(X)$ is dimensionless and Λ_0 is some constant with dimensions of energy. The classical equation of motion is the modified Klein-Gordon equation[1] with the source given by the trace of energy momentum of matter field T_ψ.

$$\partial_\mu[K'\partial^\mu\varphi] = -\frac{T_\psi}{M_P}\,.$$

We have again considered Jordan frame where the scalar field couples to the metric. Let us consider simple case where *r.h.s* be a point source $T = -M\delta^3(\vec{r})$. Then we have equations of motion,

$$K'\Box\varphi - \frac{2}{\Lambda_0^4}K''\partial^\mu\partial^\nu\varphi\partial_\mu\varphi\partial_\nu\varphi = -\frac{M}{M_P}\delta^3.$$

We can directly see the 'fifth force' if we integrate the equations of motion. The fifth force has to be such that it is very small compared to Gravitational force at local scales but should mimic Newtons's inverse square law at large distances. Such a solution would be a screening solution. The screening mechanism is dependent on the dominance of non-canonical kinetic terms. We can spilt the regimes as *strong coupling regime* where we expect to have screening mechanism and a *weak coupling regime* where the fifth force behaves like Newton's force $\partial^\mu\varphi \sim M/r^2M_P$. To make this distinction, we define Λ_0 appearing in our equations to be the scale at which the screening mechanisms unravel themselves. Later we shall address the question whether we can claim this scale as the cutoff scale of the theory itself. If this is the cutoff scale of the effective field theory, then $P(X)$ is too generic and consists of many operators which have dominant behaviour in strong coupling regime. We shall elaborate on this towards the end.

We expect $|X|$ to be greater than or of order unity $|X| \gg 1$ in the strong coupling regime. In this regime, without loss of generality, $P(X)$ can be given as $P(X) \sim \theta(-X)^N$, where N is a constant (> 1 in the strong coupling regime). This definition of regime is not physically strict, rather, the dynamics of scalar field can change from

[1]Primes denote derivative *w.r.t* X, that is, $dK/dX = K'$, $d^2K/dX^2 = K''$.

one to another without any problems. Then in this limit, the fifth force mediated by φ can be seen upon integrating the above equation of motion once,

$$
\begin{aligned}
K'(X)\partial^\mu\varphi &= \frac{M}{r^2 M_P} \implies -\theta X^{N-1}\sqrt{X\Lambda_0^4} \sim \left(\frac{M}{r^2 M_P}\right) \\
-X &\sim \left(\frac{M}{\theta\Lambda_0^2 M_P r^2}\right)^{2/(2N-1)} \sim (\frac{r_K}{r})^{4/(2N-1)} \\
\text{where } r_K &\sim \sqrt{\frac{M}{\theta\Lambda_0^2 M_P}}.
\end{aligned} \tag{8.2}
$$

So for $r \ll r_K$ we can compare the fifth force with Newton's force,

$$
\frac{F_K}{F_N} \sim \frac{1}{\theta}\left(\frac{r}{r_K}\right)^{\frac{-2}{2N-1}+2} \sim \frac{1}{\theta}\frac{r}{r_K}^{\frac{4(N-1)}{2N-1}}. \tag{8.3}
$$

To check the validity of our EFT (8.1), we check if the loop contributions are less than the classical screening solutions obtained above. To do this, we follow a similar routine as the previous chapter, we split the field into background and fluctuation $\varphi \equiv \bar{\varphi} + \delta\varphi$. The fluctuations will be denoted by $\delta\varphi = \phi$. Now, the kinetic term can be split as $X = \bar{X} + \delta X$ where $\bar{X}$ contains all the derivatives of the background field and,

$$
\delta X = -\frac{1}{\Lambda_0^4}\partial^\mu\bar{\varphi}\partial_\mu\phi - \frac{1}{2\Lambda_0^4}\partial^\mu\phi\partial_\mu\phi.
$$

We can write the Lagrangian $L_K = L_K + \delta L_K$ where δL_K corresponds to δS_E which can be written as a series

$$
\delta S_K = \int \Lambda_0^4 d^4x[K'\delta X + K''(\delta X)^2 + ...].
$$

Note that in the action, the integral over linear terms like $\partial\phi$ are boundary terms which we assume to vanish by the divergence theorem. Collecting terms up to quadratic order we have,

$$
\delta L_K = K'\partial^\mu\phi\partial_\mu\phi + \frac{K''}{2\Lambda_0^4}(\partial^\mu\bar{\varphi}\partial_\mu\phi)^2.
$$

If we separate the background field dependence from ϕ, we can write the fluctuation part of the effective action as[2],

$$
\delta S_K = \int d^4x[\bar{\mathcal{Z}}^K_{\mu\nu}[\bar{\varphi}]\partial^\mu\phi\partial^\nu\phi], \quad \text{with} \quad \bar{\mathcal{Z}}^K_{\mu\nu}[\bar{\varphi}] = -K'\delta_{\mu\nu} + \frac{K''}{2\Lambda_0^4}(\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi})^2. \tag{8.4}
$$

Similarly, for generic $P(B)$ models, with $B \equiv \Box\varphi$, we have $L_B + \delta L_B$, with δ_B taken up to quadratic order, becoming $\delta L_B = B'(\delta B) + B''(\delta B)^2 + \mathcal{O}((\delta B)^3)$. δB is

[2]we have suppressed the subscript E representing effective action

obtained, again, by first splitting the classical field to background and fluctuation part and separating out the background field dependence. So $\delta B = \delta_{\mu\nu}\partial^\mu\partial^\nu\phi$ and $(\delta B)^2 = \partial^\mu\partial^\nu\phi\partial_\mu\partial_\nu\phi$ where we have again ignored the linear terms in ϕ.

$$\delta L_B = (B'\delta_{\mu\nu} + B''\partial_\mu\partial_\nu\phi)\,\partial^\mu\partial^\nu\phi$$

Now as an example, we take the Galileon Lagrangian which is a particular model of generic $P(X,B)$ theories. It is given by[3],

$$L_G = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + c_{1,1}\Box\varphi\partial^\mu\varphi\partial_\mu\varphi \equiv X - c_{1,1}BX.$$

Notice that since B appears in the linear order, we do not have any fluctuation contribution Galileon terms to δS_G. They only appear in the background level and we can easily see that,

$$\delta(X - fBX) = (\partial\phi)^2 + 2c_{1,1}\bar{B}[-\partial^\mu\phi\partial_\mu\bar{\varphi} - (\partial\phi)^2].$$

Taking up to quadratic order with vanishing boundary terms, also dividing each term by relevant powers of Λ_0, we have a similar equation for Galileons [8],

$$\delta S_G = \int d^4x[\bar{\mathcal{Z}}^G_{\mu\nu}[\bar{\varphi}]\partial^\mu\phi\partial^\nu\phi], \text{ with } \bar{\mathcal{Z}}^G_{\mu\nu} = -\delta_{\mu\nu} + 4c_{1,1}\left(\frac{\Box\bar{\varphi}}{\Lambda_0^3}\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu\bar{\varphi}}{\Lambda_0^3}\right). \quad (8.5)$$

We should not recognize $\bar{\mathcal{Z}}$ as the rescaling parameter $\mathcal{Z}$ which appeared while discussing wave function renormalization (2.13). For now looking at the above equation we can relate this to the background effective 'metric'. In fact, we could also write a mass term for the scalar field which depends on the background field,

$$\delta L = \bar{\mathcal{Z}}_{\mu\nu}[\bar{\varphi}]\partial^\mu\phi\partial^\nu\phi - M^2[\bar{\varphi}]\phi^2 + \frac{\phi}{M_p}T. \quad (8.6)$$

The higher derivative screening mechanism is reliant on dominant kinetic terms which we can now recognize as $\bar{\mathcal{Z}}[\bar{\varphi}] \geq \mathcal{O}(1)$. Similarly we have thin shell screening mechanisms like Chameleon[87] where $M[\bar{\varphi}]$ dominates making the range of the force very small in a massive environment. But the associated fifth force behaves like Newtonian gravity at large scales. In analogy with a canonical scalar field, we can associate this to

$$\bar{\mathcal{Z}}_{\mu\nu} \equiv \sqrt{\tilde{g}}\tilde{g}_{\mu\nu}$$

where, $\tilde{g}$ represents the determinant of the effective metric $\tilde{g}_{\mu\nu}$. The covariant derivatives *w.r.t* the effective metric can be denoted by ∇. We can write a covariant the expression which due to divergence theorem reduces to,

$$\delta S_{K,G} = \int d^4x\,\left[\sqrt{\tilde{g}}\,\tilde{g}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right] \xrightarrow[\text{by parts}]{\text{integration}} \delta S_{K,G} = \int d^4x\,\phi\left[\sqrt{\tilde{g}}\,\tilde{g}_{\mu\nu}\nabla^\mu\nabla^\nu\right]\phi. \quad (8.7)$$

[3]The coupling $c_{1,1}$ has different dimensions compared to that of f of extended-BD (see around (6.1)). $c_{1,1}$ has dimensions of -7, where as the latter is of -6 mass dimension. However, we have brought back G_N in terms of M_P by redefining the BD field $\varphi = \psi\sqrt{G_N}$ and approximating the theory on flat spacetime.

The effective action at 1-loop is $S_E \approx S + \Gamma_{1l}$ where the last term is the effective 1-loop action. We can define a regime of validity where the quantum fluctuations, particularly the 1-loop contributions, are negligible compared to classical theory. This tells us up to what scales we can continue with classical equations. We can make use of the Background Field Identity given in (7.13) to obtain (7.15), $\Gamma_{1l}[\varphi] \sim \mathrm{Tr}\ln[\delta^2(\delta S_{K,G})/\delta\phi^2]$. This equation is exact as *l.h.s* is a functional of the macroscopic field φ. To compute this we should have started with a known exact form of the fundamental action; the action in (8.1) is very general and has all possible functions of kinetic term. We shall see more on this in the next section. Since we do not have such a theory yet, we can use techniques like heat kernel method to compute 1-loop contributions [8, 34]. We have obtained the results by splitting the classical field φ as, $\varphi \to \varphi + \epsilon\phi$. Suppose we do similar splitting in the case of Einstein Hilbert action we expect the 1-loop divergences to be of the form given below,

$$\Gamma_{1l} \sim \int d^4x\sqrt{\tilde{g}}\left(\tilde{R}^2 + 2\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}\right).$$

Now we make use of this by obtaining the curvature terms from the effective metric[4]. We see that the effective 1-loop action looks like massless propagator in curved spacetime. We have performed similar calculations in Appendix(F.3) where we computed DeWitt-Schwinger coefficients. This rough estimation works for any $\bar{\mathcal{Z}}$ including Galileons. The definition of the regime of validity for classical theory can be symbolically represented as [8],

$$\begin{aligned} &|L_{\Gamma_{1l}}| \ll |L_E| \\ &\sqrt{\tilde{g}}\left|\tilde{R}^2\right| \ll \Lambda_0^4 P(X) \implies \left[\left(\frac{\partial\tilde{g}}{\tilde{g}}\right)^2 + \frac{\partial^2\tilde{g}}{\tilde{g}}\right]^2 \ll \Lambda_0^4 P(X) \\ &\left|\frac{\partial\bar{\mathcal{Z}}}{\bar{\mathcal{Z}}}\right|^4, \left|\frac{\partial^2\bar{\mathcal{Z}}}{\bar{\mathcal{Z}}}\right|^2 \ll \Lambda_0^4 P(X), \end{aligned} \tag{8.8}$$

where $\bar{\mathcal{Z}}$ can be roughly recognized as velocity of the background field and $\partial\bar{\mathcal{Z}}$ as acceleration (and the derivatives ∂ are assumed to be in Cartesian coordinates). From (8.3) we have the definition of acceleration in the strong coupling regime and, also making use of (8.2), we have validity regime of K-mouflage theories,

$$\left(\frac{r}{r_K}\right)^{\frac{4}{2N-1}}\left(\frac{r}{r_K}\right)^{\frac{4(N-1)}{2N-1}} \ll r\Lambda_0 \implies \left(\frac{r}{r_K}\right)^{\frac{N}{2N-1}} \ll r\Lambda_0. \tag{8.9}$$

This inequality has to be maintained to have a valid classical solution. This also suggests that $N > 1/2$. Within the screening regime, $r < r_K$ and roughly taking $\Lambda_0 \sim 1/r_K$ we see that *l.h.s* the inequality gets smaller for larger values of N.

[4] One should also include power law divergences and a detailed analysis is done in [103, 104] using heat kernel method. We shall not include for time being but will consider in the next section with a different approach.

Similarly, for Galileons we can make use of our classical solution (6.15) in strong coupling regime, $F_G/F_N \sim r_V^3$, to symbolically represent validity regime,

$$(\Lambda_0 r_V)^3 \ll r\Lambda_0. \tag{8.10}$$

So far through symbolic arguments we tried to see, to what extent we could have stable classical solution, which at some scales could be spoiled due quantum fluctuations. This is not the entire picture and our approach has some drawbacks. First, the loop divergences we have included are logarithmically divergent. But power law divergences will play a major role in determining influence of high energy physics at the classical level. Secondly, in our EFT for higher derivatives, it is difficult to choose the high energy cutoff. Let the cutoff of a full theory be Λ, with the EFT cutoff scale Λ_{cut} and strong coupling scale Λ_0. In the limit $\Lambda_{cut} < \Lambda$ the loop divergences of the EFT should be independent of Λ_{cut} which is shown through (7.19) or (2.21). But to extrapolate the theory to be valid up to Λ but still include the same couplings of the low energy EFT (which matches with experiments) requires fine tuning of high energy contributions as shown in first chapter.

1. In EFT picture of higher derivative theories, Λ_0 is the scale where *irrelevant* higher order terms become dominant in the Lagrangian and accordingly have screening mechanisms at this scale. Suppose if we choose $\Lambda_0 \sim \Lambda_{cut}$ then we have to include a ladder of higher order terms in the Lagrangian (for example (D.1)) itself. This is the scale where perturbative unitarity breaks down which implies breakdown of perturbation theory[105]. Unless we have some sort of symmetry, such that all the higher order terms gets summed up to give only classical couplings the effective theories are not isolated from the UV physics. We have DBI theory which is a good example for such a theory[5]. The higher derivative terms make the propagator to diverge very quickly at high energies.

2. We could also have $\Lambda_0 \ll \Lambda_{cut}$ which reflects the hierarchy between strong coupling regime and cutoff scales. In this case, we can continue doing low energy EFT, just like in the Standard Model of particle physics where we do not know the full theory, yet can make low energy predictions. This is true because the Standard Model theory is a renormalizable theory. We would want our screening mechanisms to fall into such simple framework. However, we have non-renormalizability theorem for Galileons [109]. Thus, $\Lambda_0 \ll \Lambda_{cut}$ leads to hierarchy problem and seems *unnatural*.

3. If we try to continue in the path of Callan-Symanzik (7.19) or for high energy modes using Polchinski's equation (D.5), we have the dependence of the effective action on the cutoff scale Λ_{cut} or Λ' respectively which is a unphysical. We have shown Polchinski equation in a simple context in Appendix(D.2).

[5]See [8, 9] and references therein for detailed claims on symmetries, analyticity and also on non-standard UV completions. For perturbation renormalizability at 1-loop, see [106] and related matter loop corrections for Galileons [107], and also see [108]

Introducing more loops and propagators will immediately make it very complicated equation to solve. The functional renormalisation group equation on the other hand handles non-pertubative theories systematically where we only come across 1-loop structure.

4. The other way to bypass this problem is to look for asymptotic safety as discussed in previous section. The way to move forward is to use non-perturbative methods to find non-trivial UV fixed points [9, 110]. We start the next section with some introduction to the former. We introduce the Exact Functional Renormalization Equation or Wetterich equation and use it to look for fixed points.

9 Exact Renormalization Group Approach for Higher Derivative Theories

Here we derive Exact Renormalization Group equation for K-mouflage and Galileon theories. This is the exact flow equation for effective average action and includes all the quantum fluctuations even though it is 1-loop equation. To proceed in this line, we truncate the theory up to dimension 8 operators which is in accordance with our previous work in chapter 6. From beta functions of some important couplings, we see that there is no running of the couplings. Here, we again refer to [8] and also [9]. Our results are in fact particular case of [9] who also take the discussion further to discuss UV completion in standard Wilsonian way and non-standard ways.

9.1 Exact Functional Renormalization Group Equation

The idea of Wilson renormalization was to integrate out the fluctuations shell by shell. We integrated out high energy modes and concentrated on the flow of effective couplings appearing in the effective action. We now make use of the effective action $\Gamma[\varphi]$ given in (7.10) which is a functional of mean field $\langle\chi\rangle \equiv \varphi$. In this chapter we denote effective action using a bar on top to distinguish it from a more useful generating functional, the effective averaged action Γ_k. Effective action now denoted by $\bar{\Gamma}_k$, includes all the quantum effects, the subscript k is the distance/energy scale of interest. The classical action S can be extracted from it for specific scale. Effective average action is also a scale dependent action. It interpolates between classical S and effective action $\bar{\Gamma}_k$, which in principle could be the action of full theory. The scale dependence implies that it includes all the fluctuations appearing at the scale k butit rather acts as the dynamic cutoff which in the limit $k \to 0$ (large distances, hence includes all quantum fluctuations) gives the effective action and for $k \to \infty$ or Λ_0 gives the classical action. Now we have a flow for infinitesimal change in scale $k \longrightarrow k - \delta k$. Hence variation *w.r.t* k is called *flow equation* or *exact functional renormalization group equation* or simply *exact renormalization group equation* (ERGE) or *Wetterich equation* [111, 89]. The running in k can be thought as zooming in or out on a microscope.

The effective average action comes with an infrared (IR) cutoff such that it only accounts for scales which are greater than a certain cutoff p; that is $k \geq p$. Conceptually it is similar to cutoff regularization (discussed in the first chapter) which involves an addition of a large mass to the propagator. For example consider changing the loop integrand $(k^2 + i\epsilon)^{-1} \to (k^2 + i\epsilon)^{-1} - (k^2 - M + i\epsilon)^{-1}$, then for very large M, the propagator does not make a difference for small k, but acts like a cutoff for $k \geq M$. The way it is implemented here is through a regulator in the Schwinger effective generating functional $W_k[J]$.

$$\exp(W_k[J]) = \int \mathcal{D}\chi \exp\left(-S[\chi] + \int_x J(x)\chi(x) - \Delta_k S_k[\chi]\right). \tag{9.1}$$

The IR cutoff part which has been added in the tail of the equation in momentum space reads,

$$\Delta_k S[\chi] = \frac{1}{2}\int_p \chi(-p) R_k(p^2) \chi(p)$$

R_k is called a regulator which more or less serves works like a step function. Some examples would be,

$$\begin{aligned} R_k(p^2) &= \frac{p^2}{e^{p^2/k^2} - 1} \quad \text{exponential cutoff} \\ R_k(p^2) &= (k^2 - p^2)\,\Theta(k^2 - p^2) \quad \text{step function} \\ &= \begin{cases} k^2 \text{ for } p^2 < k^2 \\ 0 \text{ for } p^2 > k^2 \end{cases} \end{aligned} \tag{9.2}$$

As a result of the IR cutoff, we expect a suppression of small momentum modes by R_k which acts like a k-dependent mass term. As an example, we can consider an inverse propagator, now including step function R_k, which vanishes for $p^2 > k^2$ and attains a constant value for $p^2 < k^2$. The regularized propagator looks like,

$$(S + \Delta_k S)^{(2)} = p^2 + R_k(p^2) + \ldots\,. \tag{9.3}$$

The effective action $\overline{\Gamma}_k$ is the Legendre transform of the generating functional (7.10) $\bar{\Gamma}[\varphi] = W_k + \int J\varphi$. The effective average action is expressed as a subtraction of IR piece from the effective action.

$$\begin{aligned} \exp(-\Gamma[\varphi]) &= \int \mathcal{D}\chi \;\exp\left(-S[\varphi + \phi] + \int_x \frac{\delta\Gamma}{\delta\phi}\phi(x) - \Delta_k S\right) \\ \Gamma_k[\varphi] &= \overline{\Gamma}_k - \Delta_k S \end{aligned} \tag{9.4}$$

- For $k \to \infty$ we expect to have classical microscopic action. At this limit terms depending on fluctuations ϕ become less dominant and $S[\varphi + \phi] \to S[\varphi]$. The dominating term is the IR cutoff. The effective action reduces to classical action at the fundamental level.

$$\lim_{k\to\infty} R_k(p) \longrightarrow \infty \implies \lim_{k\to\infty} \Gamma_k[\varphi] \longrightarrow S[\varphi],$$

By setting the cutoff part of the integral to have large values, we keep the propagator convergent; we then just have a Gaussian integral. Symbolically, in loop expansion we would come across integrands like $(k^2 + m^2)^{-n}$ which now becomes $(k^2 - R + m^2)$ and has no divergences for large k due to presence of large R (in our context this is scale dependent). Thus loop expansion has no IR divergences even if $m^2 = 0$.

- For small momentum modes,

$$\lim_{k\to 0} R_k(p) \longrightarrow 0 \implies \lim_{k\to 0} \Gamma_k[\varphi] \longrightarrow \Gamma[\varphi].$$

Here we have not used any unphysical regularization cutoff like we did in the first chapter. However, in this method all the necessary small modes are included. Hence we can conclude that in the theory space, $\Gamma_k[\phi]$ flows between macroscopic (quantum included) effective action $\Gamma[\varphi]$ and microscopic action $S[\varphi]$.

The practical use of this is through the flow equations. Given all the definitions of scale dependent generating functionals, it is not hard to derive the flow equations. We start with (9.1) where the regulator term is the only k-dependent term.

$$\partial_k W_k = \frac{1}{Z_k}\partial_k Z_k = \frac{1}{Z_k}\int D\chi(-\partial_k \Delta S_k)e^{-S[\chi]-\Delta S_k - \int J\chi}$$

The variation of the regulator $\frac{-1}{2}\int_p \chi(-p)(\partial_k R_k(p))\chi(p)$ can be separated from the rest of the functional integral over fields. Also by making use of the definition of n-point functions (7.5) in second line, and using (7.8), (7.9) in the next steps,

$$\begin{aligned}
\partial_k W_k &= \frac{-1}{2Z_k}\int_p (\partial_k R_k(p)) \int D\chi e^{-S[\chi]-\Delta S_k - \int J\chi}\chi(-p)\chi(p) \\
&= -\frac{1}{2}\int_p (\partial_k R_k(p))\langle\chi(-p)\chi(p)\rangle \\
&= -\frac{1}{2}\int_p (\partial_k R_k(p))\left(G_k^{(2)} + \langle\chi(-p)\rangle\langle\chi(p)\rangle\right) \\
&= -\frac{1}{2}\int_p (\partial_k R_k(p))\, G_k^{(2)} - \partial_k \Delta S_k.
\end{aligned} \tag{9.5}$$

This is however the flow for the Schwinger functional. The effective functional is defined as functional of mean field $\varphi \equiv \langle\chi\rangle$ which is independent of k. From its relation to Schwinger functional we have,

$$\partial_k \bar{\Gamma}[\varphi] = -\partial_k W_k[J] - \int_x \left[-\left(\frac{\delta W_k}{\delta J}\partial_k J(p)\right) + (\partial_k J(p))\varphi\right].$$

Notice that, taking the supremum of the second term leads to equations of motion $\frac{\delta W_k}{\delta J} - \varphi = \langle\chi\rangle - \varphi = 0$. Taking another variation *w.r.t* J gives,

$$\frac{\delta\varphi}{\delta J} = G_k^{(2)}(x_1, x_2).$$

From variations of quantum equations of motion (7.11) $\frac{\delta J}{\delta\varphi} = \bar{\Gamma}_k^{(2)} = \frac{\delta^2\bar{\Gamma}}{\delta\varphi(x_1)\delta\varphi(x_2)}$, due to which,

$$\bar{\Gamma}_k^{(2)} G_k^{(2)} = 1 = (\Gamma_k^{(2)} + R_k) G_k^{(2)}.$$

$\bar{\Gamma}_k^{(2)}$ can be thought of as the inverse of 2 point function. In the second equality we have made use of the fact that, two variations of ΔS_k *w.r.t* φ gives the regulator R_k. At some supremum, $J_s up$, the flow of effective averaged action is given by,

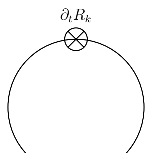
$$\begin{aligned}\partial_k\Gamma_k[\varphi] &= \partial\left(\int J\varphi - W_k - \Delta S_k\right)\\ &= -\partial_k W_k[J] - \partial_k\Delta S_k[\varphi]\end{aligned} \tag{9.6}$$

Now we can use the last result of (9.5) and replace the 2-point function by its inverse (in matrix representation), to have the final flow equation of average effective action in the theory space given by,

$$\partial_k\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\Big[(\Gamma^{(2)} + R_k)^{-1}\partial_k R_k\Big]. \tag{9.7}$$

The integral over momentum modes is written as a trace. We multiply both sides by k then we write the derivatives *w.r.t* k as derivatives *w.r.t* $\ln k$. We replace $\ln k$ with t to write $\partial_t = k\partial_k$ which makes the flow equation dimensionless.

$$\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\Big[(\Gamma_k^{(2)} + R_k)^{-1}\partial_t R_k\Big] =$$



. (9.8)

The most interesting part of the flow equation is that the functionals appearing are functionals of the macroscopic field φ. We have made no assumptions on cutoffs and scales of relevance etc. Hence *the flow equation is exact.* The finiteness of this integral in IR is ensured by R_k and in UV it is controlled implicitly by $\partial_k R_k(p^2)$ because this term is almost zero everywhere except at $p^2 = k^2$. So we are only calculating on shell contributions keeping the UV and IR convergence in the integrals which is comparable to Wilsonian flow equations. However, this flow equation includes all non-perturbative effects. This is a one loop equation and that is all that is needed. All the higher loop effects can be obtained by iteration of the flow equation.

The flow equation is a functional differential equation, except for few cases, it cannot be solved exactly. We can however evaluate the trace on the *r.h.s* by expanding it in terms of a small coupling constant. For practical applications, as an ansatz, we truncate the theory space (which otherwise has many terms) and look at flow in finite dimensional subspace which can potentially include features of the full theory space.

The loop expansion of effective averaged action goes like $\Gamma_k = S + \Gamma_{k,1l} + ...$, and for 1-loop, flow equation is

$$\partial_t \Gamma_{k,1l} = \frac{1}{2} \operatorname{Tr} \left[\partial_t R_k (S^{(2)} + R_k)^{-1} \right] = \frac{1}{2} \partial_t \operatorname{Tr} \ln \Big(S^{(2)} + R_k \Big). \tag{9.9}$$

The solution to this gives back (7.15),

$$\Gamma_{k,1l} = \frac{1}{2} \operatorname{Tr} \ln \Big(S^{(2)} + R_k \Big) + c \implies \Gamma_k \sim S + \frac{1}{2} \operatorname{Tr} \ln \Big(S^{(2)} + R_k \Big).$$

This confirms the consistency with the perturbative approach. Now lets us use these techniques to find (if any) fixed points in higher derivative theories.

9.2 Exact Renormalization Group Equations for Higher Derivative Theories

K-mouflage along with Galileon theories can be expressed using a polynomial $P_k(X, B) = \sum_{n,m=0}^{\infty} c_{n,m}(k) X^n B^m$ where $X = -(\partial\varphi)^2/2\Lambda^2, B = \Box\varphi/\Lambda^3$. For our extended-BD theories discussed in previous part, we can truncate the theory by taking up to dimension 8 operators,

$$P_k(X, B) = c_{0,0} + c_{1,0} X + c_{0,2} B^2 + c_{1,1} X B + +c_{2,0} X^2. \tag{9.10}$$

The operators on *r.h.s* are of dimensions $0, 4, 6, 7, 8$ respectively[1]. The couplings are dimensionful which in terms of dimensionless couplings (with tilde) can be expressed as,

$$\tilde{c}_{n,m}(k) = c_{n,m} k^{4n+3m-4}.$$

The coupling $c_{1,0}$ can be recognized as the rescaling parameter from wavefunction renormalization (2.13), $c_{1,0} \equiv \mathcal{Z}/2$. To obtain fixed points, we have to first derive the running of dimensionless couplings. Let us first focus on $P(X)$ theories $\Gamma_k \sim \int_x \Lambda^4 P_k(X) + \mathcal{O}(\partial^2)$ and use the step function regulator from (9.2),

$$R_k = \mathcal{Z}_k (k^2 + \Box) \Theta(\Box + k^2)$$

We recall that in (8.7), as we had not defined effective average action for higher derivative theories, we could not use Γ_{1l} (7.15). But now we can make use of this

[1]Dimensions are in accordance with (8.1), $[\varphi] \equiv 1, [\partial\varphi] \equiv 2, [X] \equiv 4, [B] \equiv 3$.

through (9.9). In momentum space, taking step function regulator into account, ERGE is,

$$\frac{\partial P_k(X)}{\partial k} = \frac{1}{(2\pi)^4}\int_{p^2<k^2} d^4p \left[\frac{\mathcal{Z}_k k}{\mathcal{Z}_k k^2 - \mathcal{Z}_k p^2 + \bar{\mathcal{Z}}_{\mu\nu}p^\mu p^\nu}\right], \tag{9.11}$$

where we have used (8.7) by taking its variations $w.r.t\ \varphi$ twice $S_k^{(2)} \sim \frac{\delta^2(\delta S_{K,G}[\varphi])}{\delta\phi}$ (see (7.15)). We again point out that $\mathcal{Z}_k$ is the wave function rescaling parameter and $\bar{\mathcal{Z}}$ is the background field dependent factor which behaves like an effective metric. The presence of $\bar{\mathcal{Z}} \sim P' - P''(\partial\bar{\varphi})^2$ from (8.4) implies we have many terms in the denominator if we do not consider any truncation and the equation is not exact. Multiplying k on both sides we have,

$$\frac{\partial P_k}{\partial t} = \frac{1}{(2\pi)^4}\int_{p^2<k^2} d^4p \left[\frac{\mathcal{Z}_k k^2}{\mathcal{Z}_k k^2 + \mathcal{X}_{k,\mu\nu}p^\mu p^\nu}\right] \tag{9.12}$$

where $\mathcal{X}_{k,\mu\nu} = (\bar{\mathcal{Z}}_{\mu\nu} - \mathcal{Z}_k\delta_{\mu\nu})$. For $P(X)$ theories alone, from our truncation (9.10), we have $P_k(X) = c_{0,0} + c_{1,0}X + c_{2,0}X^2$. Now we expand $r.h.s$ up to second order in X and perform the integral in spherical coordinates up to momentum shell k,

$$\begin{aligned}
\frac{\partial P_k}{\partial t} &= \frac{1}{(2\pi)^4}\int_{p^2<k^2} d^4p \left[1 - \frac{\mathcal{X}_{k,\mu\nu}p^\mu p^\nu}{\mathcal{Z}_k k^2} + \left(\frac{\mathcal{X}_{k,\mu\nu}p^\mu p^\nu}{\mathcal{Z}_k k^2}\right)^2 + ...\right] \\
\text{where, } \mathcal{X}_{k,\mu\nu} &= -4c_{2,0}X\delta_{\mu\nu}, \\
\frac{\partial P_k}{\partial t} &= \frac{4\pi}{(2\pi)^4}k^4\left[1 - \frac{2c_{2,0}X}{c_{1,0}}\frac{k^2}{6} + \frac{4c_{2,0}^2X^2}{c_{1,0}^2}\frac{k^4}{8}\right] \\
&= \frac{4\pi}{(2\pi)^4}k^4\left[1 - \frac{1}{3}\frac{\tilde{c}_{2,0}X}{\tilde{c}_{1,0}} + \frac{1}{2}\frac{\tilde{c}_{2,0}^2X^2}{\tilde{c}_{1,0}^2}\right]
\end{aligned} \tag{9.13}$$

Now matching the coefficients from $l.h.s$, we get equations similar to beta functions for couplings $\tilde{c}_{1,0}$ and $\tilde{c}_{2,0}$. Since we are on flat spacetime, we have ignored the vacuum energy. Otherwise we should have included a scale dependent vacuum potential along with the coupling $c_{0,0}$. The beta functions for $c_{1,0}$ is,

$$\beta_{1,0} \approx \frac{\partial(k^4\tilde{c}_{1,0})}{\partial t} \approx -\frac{k^4}{12\pi^3}\frac{\tilde{c}_{2,0}}{\tilde{c}_{1,0}}. \tag{9.14}$$

For fixed points we have $\beta_{i,0} = 0$. For $c_{2,0}$, we have $\beta_{2,0} \approx \frac{k^4}{8\pi^3}\frac{\tilde{c}_{2,0}^2}{\tilde{c}_{1,0}^2}$ where we notice that we have a trivial fixed point for $\tilde{c}_{2,0} = 0$. Also, we see that the beta function vanishes for no values of $\tilde{c}_{1,0}$. However, we know that at least for classical canonical kinetic term $\tilde{c}_{1,0} \neq 0$. The couplings considered in our truncation only have trivial fixed points rather than non-trivial UV fixed points. If it was the latter case one would have to linearize beta functions around fixed points and look at the direction of the flow - that is if the flow is towards IR or UV (Appendix(D.3)). These are know as relevant' and 'irrelevant' flow respectively. UV fixed points however do not indicate that we have reached a fundamental theory. They would imply a very generic prediction for all our experiments irrespective of quantum corrections which is exactly our aim. From these fixed points the couplings flow slightly on a hypersurface when we reduce the scale. If we find irrelevant flows in some couplings, then this implies that there are some quantum fluctuations which dominate at low scales. If we find a non-trivial UV fixed point at all then we can have a scenario like asymptotic safety similar to what we have for gravitational coupling. However, in the considered $P(X)$ truncation we do not find any such fixed points. Hence these theories are valid as effective field theories without any problems from the quantum sector.

Now we shall take into account the complete truncation (9.10) involving Galileons. We start by deriving a very general expression for the inverse propagator as done in [9] and then use it in our context to find if any non-trivial UV fixed points exists. First, let us distinguish between derivatives *w.r.t* X and B by specifying them as subscripts. For example, $P_X \equiv dP/dX; P_B \equiv dP/dB; P_{X,B} \equiv d^2P/dXdB$. We also suppress the subscript k. We can split the fields $\varphi \to \bar{\varphi} + \epsilon\phi$, with for small ϵ. The inverse propagator is obtained by taking functional derivative - this can be expressed in a form where a kernel $\frac{\delta\Gamma}{\delta\varphi}$ is acting on the 'test' function ϕ [112].

$$\delta\Gamma[\varphi] = \int_x \frac{\delta\Gamma[\bar{\varphi}]}{\delta\bar{\varphi}}\phi(x).$$

Let us start with finding the kernel. The variation of effective action twice *w.r.t* background field and collecting terms up to quadratic order, we have,

$$\begin{aligned} \Gamma[\varphi]^{(2)} &= \int_x \frac{\delta^2\Gamma_k}{\delta\varphi(x')\delta\varphi(x)}\varphi(x), \\ &= P_X\delta(\delta X) + P_{XX}(\delta X)^2 + P_{XB}\delta X\delta B + P_{BB}(\delta B)^2. \end{aligned} \tag{9.15}$$

Now we use integration by parts and separate the terms into kernel and test function. Every term in the above equation is represented in a separate line on the *r.h.s.*

$$\begin{aligned} \Gamma[\varphi]^{(2)} &= [\partial_\mu P_X\partial^\mu]\phi(x) + [P_X\partial^\mu\partial_\mu]\phi(x) \\ &- [\partial^\nu(P_{XX}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi})\partial^\mu]\phi(x) - [(P_{XX}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi})\partial^\mu\partial^\nu]\phi(x) \\ &- [\partial^\rho(P_{XB}\partial_\rho\bar{\varphi})\partial^\mu\partial_\mu]\phi(x) - [2\partial_\mu(P_{XB}\partial_\nu\bar{\varphi})\partial^\mu\partial^\nu]\phi(x) - [\Box P_{XB}\partial_\nu\bar{\varphi}]\partial^\mu\phi(x) \\ &- [(\partial_\mu\partial_\nu P_{BB})\partial^\mu\partial^\nu]\phi(x) - [2\partial_\mu P_{BB}\Box\partial^\mu]\phi(x) - [P_{BB}\Box^2]\phi(x)\ . \end{aligned}$$

This is exactly the equation obtained in [9] where the expressions are regrouped to write,

$$\begin{aligned} \bar{\mathcal{Z}}_{\mu\nu}[\bar{\varphi}] &= P_X\delta_{\mu\nu} - (P_{XX}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi}) - \partial^\rho(P_{XB}\partial_\rho\bar{\varphi})\delta_{\mu\nu} \\ &\quad - 2\partial_\mu(P_{XB}\partial_\nu\bar{\varphi}) - (\partial^\rho\partial_\rho P_{BB})\delta_{\mu\nu}, \\ \bar{\mathcal{Z}}_\mu[\bar{\varphi}] &= \partial_\mu P_X - \partial^\rho(P_{XX}\partial_\rho\bar{\varphi}\partial_\mu\bar{\varphi}) - \Box(P_{XB}\partial_\mu\bar{\varphi}), \\ \implies \int_x \left[\frac{\delta\Gamma}{\delta\bar{\varphi}}\right]\phi(x) &= (\bar{\mathcal{Z}}_{\mu\nu}[\bar{\varphi}]\partial^\mu\partial^\nu + \bar{\mathcal{Z}}_\mu[\bar{\varphi}]\partial^\mu + 2\partial_\mu P_{BB}\Box\partial^\mu + P_{BB}\Box^2)\phi\,. \end{aligned} \tag{9.16}$$

Now we take IR regulator of the form [9, 113],

$$R_k = [\mathcal{Z}_1(k)(k^2 + \Box) + \mathcal{Z}_2(k)(k^4 - \Box^2)]\Theta(k^2 + \Box).$$

We then follow steps similar to the previous example of $P(X)$ theories; this included expansion of the fraction on *r.h.s* of (9.12) up to second order with the only change being, we stop at the leading order this time. Hence we only obtain beta functions for some of the constants whereas others vanish at first order. Also, $\mathcal{Z}_1 \equiv c_{1,0}$ and $\mathcal{Z}_2 \equiv c_{0,2}$.

$$k\frac{\partial P}{\partial k} = \int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{Z}_1 k^2 + \mathcal{Z}_2 k^4}{(\mathcal{Z}_1 k^2 + \mathcal{Z}_2 k^4) + [\bar{\mathcal{Z}}_{\mu\nu} + 2P_{BB}p^2\delta_{\mu\nu} - P_{XX}\delta_{\mu\nu} - (\mathcal{Z}_1 - \mathcal{Z}_2 p^2)\delta_{\mu\nu}]p^\mu p^\nu}. \tag{9.17}$$

On performing the integration and comparing with *l.h.s* we get the beta functions of all the couplings in our truncation. Ignoring the vacuum energy $c_{0,0}$, we list the beta functions for dimensionless couplings.

$$\begin{aligned}
\tilde{\beta}_{1,0} &\approx k\frac{\partial \tilde{c}_{1,0}}{\partial k} \approx -\frac{4}{12\pi^3}\frac{\tilde{c}_{2,0}}{\tilde{c}_{1,0}} \\
\tilde{\beta}_{1,1} &\approx k\frac{\partial \tilde{c}_{1,1}}{\partial k} \approx \frac{4\tilde{c}_{1,1}}{24\pi^3} \\
\tilde{\beta}_{0,2} &\approx k\frac{\partial \tilde{c}_{0,2}}{\partial k} \approx \frac{\tilde{c}_{0,2}}{24\pi^3}
\end{aligned} \tag{9.18}$$

Again, we see that we have to have $\tilde{c}_{1,0} \neq 0$ which validates our previous result. Also we see constant beta functions for $\tilde{c}_{1,1}, \tilde{c}_{0,2}$. This might be indicating that higher derivative couplings are not running. From (7.22) this implies that there is no quantum contribution (and without a fixed point, we cannot have asymptotic safety scenario). However, we have to go for next order in our expansion in denominator of (9.17) to make final conclusions. Our results can be related to the one obtained for a much bigger truncation (and background ansatz) in [9]. There, they also proceed with linearizing beta functions near fixed points to obtain the direction of running. It is also shown that $\tilde{c}_{1,0}$ has the marginal direction whereas other couplings are repelled away from UV and run towards their trivial fixed points $\tilde{c}_{n,m} = 0$ in the IR. Here we have not discussed the aspects of UV completion for which we again refer to [9]. To conclude, we have shown that we do not have running of couplings which allows us to use K-mouflage and Galileon theory as effective field theories.

10 Summary and Future Prospects

In this thesis, we studied Higher Order Derivative theories in the context of extended Brans-Dicke theories. We pointed out difficulties in explaining non-negligible differences between effective cosmological gravitational coupling and the locally measured Newton's gravitational constant. We have also check the stability of classical screening solutions against quantum corrections. Using ERG approach, we show the absence of running of couplings.

The first part included an elaborate discussion on Fine Tuning Problem in the context of CC and Higgs. We introduced standard regularization techniques and renormalization schemes. An invitation to beta functions and fixed points was provided. Using these tools we not only commented on Hierarchy and Fine Tuning Problem of Higgs but also used them to compute vacuum energy in flat spacetime. We then proceeded with computation of vacuum energy density in curved spacetime. In the propagator expansion in curved spacetime, the first term clearly corresponds to vacuum to vacuum bubble diagrams of the flat spacetime. Bubble diagrams with graviton hair will only contribute to calculation of vacuum energy density in curved spacetime. The appearance of bubble diagram without hair, explains why the results of vacuum energy density in flat and curved spacetime are the same. There is already a need for a more realistic and practical tool to describe the vacuum energy density in generic, nontrivial, curved background.

Since we do not yet have a theoretical way to express the actual vacuum energy density in curved spacetime, we look at the beta functions for the gravitational coupling. This eventually leads to Running Vacuum Models which presents many interesting upshot. The model describes the running of vacuum energy density as a function of curvature or H^2. However, in such models we have a nonzero constant background value ρ^0_{vac} which corresponds to present value of vacuum energy density. In the second part we show simple connections between these models and Brans-Dicke Model. This leaves us with BD-ΛCDM which is simply Brans-Dicke Model with a constant vacuum energy density. After list some important equations we give a short note on observational consequences of this model, particularly in the context of H_0 and σ_8 tensions. In this model we have a scalar field which quantifies the effective gravitational coupling, has a small evolution curve in the cosmic history. The model predicts that in the cosmological scale, gravitational coupling could be $4 - 9\%$ stronger than what we locally measure. Experiments put tight constraints for a varying gravitational coupling within Solar System. This calls for a screening mechanism, which does not affect the behaviour of theory in the large scale but reduce it to GR in the Solar System scale.

In this spirit, we look at 'extended' Brans-Dicke Theory. The 'extensions' to the original BD-ΛCDM are simply borrowed from the Horndeski Lagrangian which provides very general higher order derivatives of the scalar field in the Lagrangian yet results in equations of motion which are of second order. The nonlinear terms included results in screening the BD effects in the local scales. We use cubic Galileon and quartic K-essence terms leading to Vainshtein and K-mouflage screening mechanisms respectively. However, we see that the screening mechanisms can only explain tiny departure of local gravitational coupling at cosmological scale. This remains a problem for BD-ΛCDM which predicts much bigger deviations in effective gravitational coupling. However, we recognize, very small values of $\epsilon_{\rm BD} \equiv 1/\omega_{\rm BD}$ at cosmological scales (allowed by the cosmological data) as the root cause of the problem as it leads to very mild spatial variation of the scalar field. For $\omega_{\rm BD} \to -3/2$, we can explain the difference between gravitational couplings of two different realms. This limit not only indicates very significant deviations from GR at intermediate scales, also as the deviations happen for a particular (low) field value, one demands explanation for this fine tuning. Moreover neither cosmological data nor Solar system measurements allow for $\omega_{\rm BD} \to -3/2$. "Our analysis suggests that the current value of the cosmological gravitational coupling can only be extremely close to G_N in the context of the scalar-tensor theories of gravity that reduce to BD-ΛCDM at large scales, and this can be used as a boundary condition in any cosmological study dealing with these theories"[10].

In the third part of this thesis, we started the discussion with looking at validity and stability of classical solutions against quantum 1-loop corrections. We simply reproduced the computations of [8] in a detailed manner. We define validity regime where 1-loop radiative effects are less compared to the classical solution. This study in effective field theory perspective is relevant for higher derivative screening mechanisms because, the latter rely on domination of higher dimensional operators within the screening radius. In EFT perspective, such domination indicates, breakdown of perturbation theory. The radiative corrections might possibly spoil the classical screening behaviour. We see that for K-essence and Galileons, the quantum corrections are low even deep inside the screening radius. We also describe briefly why this approach is not the complete picture. We make use of flow equations of effective averaged action which is by definition a functional of macroscopic fields implying it has all the information of the quantum level, to study the stability of higher derivative theories in the UV scales. Using the general flow equation [9], and truncating the theory of our interest (up to dimension 8 operators), we investigate the beta functions. We see that at first order expansion, there is no running of couplings.

Part IV

Appendix

A Lists

A.1 List of Figures

A.2 List of Tables

B Perturbed Quantities and Equations of Motion

B.1 Perturbed and Important Geometrical Quantities

We list the formulas of some geometrical quantities up to first order in the perturbed metric $h_{\mu\nu}$ which includes Christoffel symbols, Ricci tensor, Ricci scalar and Einstein tensor. They are respectively given by,

$$\Gamma^{\alpha}_{\beta\kappa}(h) = \frac{\eta^{\alpha\mu}}{2}\left(h_{\mu\beta,\kappa} + h_{\mu\kappa,\beta} - h_{\beta\kappa,\mu}\right) , \tag{B.1}$$

$$R_{\mu\nu}(h) = \frac{1}{2}\left[\partial^{\alpha}\partial_{\nu}h_{\alpha\mu} + \partial^{\alpha}\partial_{\mu}h_{\alpha\nu} - \partial_{\mu}\partial_{\nu}h - \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h_{\mu\nu}\right] , \tag{B.2}$$

$$R(h) = \partial^{\alpha}\partial^{\mu}h_{\alpha\mu} - \eta^{\alpha\mu}\partial_{\alpha}\partial_{\mu}h\,, \tag{B.3}$$

$$G_{\mu\nu}(h) = \frac{1}{2}\left[\partial^{\alpha}\partial_{\nu}h_{\alpha\mu} + \partial^{\alpha}\partial_{\mu}h_{\alpha\nu} - \partial_{\mu}\partial_{\nu}h - \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h_{\mu\nu} + \eta_{\mu\nu}(\eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h - \partial^{\alpha}\partial^{\beta}h_{\alpha\beta})\right] . \tag{B.4}$$

We also note that unperturbed Weyl tensor (only defined for dimensions $d > 2$) is given by,

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu} + \frac{1}{d-2}\left(g_{\mu\sigma}R_{\rho\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\mu\rho}R_{\sigma\nu} - g_{\nu\sigma}R_{\rho\mu}\right) + \frac{1}{(d-1)(d-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R. \tag{B.5}$$

B.2 Equations of Motion

Here we give detailed derivation of the field equations[114] for chapter(6.1)

B.2.1 Cubic interaction term

Field equation

Action

$$S_{eBD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(\psi R - \frac{\omega}{\psi}(\nabla\psi)^2 + f\Box\psi(\nabla\psi)^2\right) - \rho_\Lambda\right] + S_m \tag{B.6}$$

Differentiate with respect to ψ

$$\delta S = 0 = \delta\psi R - \delta\left(\frac{\omega}{\psi}(\nabla\psi)^2\right) + \text{differentiate self interaction term} + \left(\frac{\delta S_m}{\delta\psi} = 0\right) \tag{B.7}$$

The second term variation is

$$\delta\left(\frac{\omega}{\psi}(\nabla\psi)^2\right) = -\frac{\omega}{\psi^2}\nabla^\mu\psi\nabla_\mu\psi\delta\psi + \left(\frac{\omega}{\psi}\delta(\nabla\psi)^2 = \omega\frac{2}{\psi}\nabla^\mu(\delta\psi)\nabla_\mu\psi = -\frac{2\omega\Box\psi}{\psi}\right) \tag{B.8}$$

The third term in the RHS can be simplified as follows,

$$\begin{aligned}\delta(\nabla\psi)^2 &= \nabla^\mu(\delta\psi)\nabla_\mu\psi + \nabla^\mu\psi\nabla_\mu(\delta\psi)\\ &= 2\nabla^\mu(\delta\psi)\nabla_\mu\psi\\ &= -2\Box\psi\delta\psi\end{aligned} \tag{B.9}$$

In the third equality, we have used the fact that, divergence of a vector vanishes at the boundary; that is,

$$\nabla^\mu(\nabla_\mu\psi\delta\psi) = 0 = (\nabla^\mu\nabla_\mu\psi)\delta\psi + (\nabla^\mu\delta\psi)\nabla_\mu\psi \implies (\nabla^\mu\delta\psi)\nabla_\mu\psi = -(\nabla^\mu\nabla_\mu\psi)\delta\psi.$$

Now, lets differentiate the self interaction term with respect to BD field, considering f to be a constant.

$$\delta(f(\Box\psi)(\nabla\psi)^2) = f\delta(\Box\psi)(\nabla\psi)^2 + f\delta((\nabla\psi)^2)\Box\psi \tag{B.10}$$

We can employ the divergence theorem for both of these terms. Let these terms be X and Y where the later can be written as, $2(\nabla^\mu(\delta\psi)\nabla_\mu\psi)\Box\psi f$ respectively. Now, consider

$$\begin{aligned}\nabla_\mu[\nabla^\mu(\delta\psi)(\nabla\psi)^2 f] &= 0 = X + \nabla_\mu(\nabla\psi)^2\nabla^\mu(\delta\psi)f\\ 2\nabla^\mu[\delta\psi\Box\psi\nabla_\mu\psi f] &= 0 = Y + 2\nabla^\mu(\Box\psi)\nabla_\mu\psi f\delta\psi + 2(\Box\psi)^2 f\delta\psi\end{aligned}$$

Substituting for X and Y in Eq(B.10), the differentiation of self interaction term is,

$$\delta(f(\Box\psi)(\nabla\psi)^2) = -f\nabla_\mu(\nabla\psi)^2\nabla^\mu(\delta\psi) - 2f\nabla^\mu(\Box\psi)\nabla_\mu\psi\delta\psi - 2f(\Box\psi)^2\delta\psi \tag{B.11}$$

As the first contains differentiation of $\delta\psi$ we can use the same divergence theorem to simplify. Considering that term as A, upon divergence theorem, $A = -\Box(\nabla\psi)^2 f\delta\psi$. Hence the above equation can be rewritten as,

$$\delta(f(\Box\psi)(\nabla\psi)^2) = f\Box(\nabla\psi)^2\delta\psi - 2f\nabla^\mu(\Box\psi)\nabla_\mu\psi\delta\psi - 2f(\Box\psi)^2\delta\psi \tag{B.12}$$

Now the including Eq(B.9) and Eq(B.12) in Eq(B.7), it can be written as,

$$\frac{\delta S}{\delta\psi} = 0 = R - \frac{\omega}{\psi^2}\nabla^\mu\psi\nabla_\mu\psi + \frac{2\omega}{\psi}\Box\psi + f\Box(\nabla\psi)^2 - 2f\nabla^\mu(\Box\psi)\nabla_\mu\psi - 2f(\Box\psi)^2 \tag{B.13}$$

Now we can use the dimensionless field $\varphi = G_N\psi$, thus the above equation changes to,

$$0 = R - \frac{\omega}{\varphi^2}\nabla^\mu\varphi\nabla_\mu\varphi + \frac{2\omega}{\varphi}\Box\varphi + \frac{f}{G_N^2}\Box(\nabla\varphi)^2 - \frac{2f}{G_N^2}\nabla^\mu(\Box\varphi)\nabla_\mu\varphi - \frac{2f}{G_N^2}(\Box\varphi)^2 \tag{B.14}$$

Lets further deal with term, $f\Box(\nabla\psi)^2$ explicitly the following,

$$\Box(\nabla\psi)^2 = \nabla^\mu[2\nabla_\nu\psi\nabla_\mu(\nabla^\nu\psi)] = 2[(\nabla^\mu\nabla_\nu\psi)(\nabla_\mu\nabla^\nu\psi) + \nabla_\nu\psi\Box(\nabla^\nu\psi)] \tag{B.15}$$

Now with the definition of Ricci tensor as the commutators acting on a Vector field, $R_{\mu\nu}(\nabla^\mu\psi) = \nabla_\mu\nabla_\nu(\nabla^\mu\psi) - \nabla_\nu\nabla_\mu(\nabla^\mu\psi)$, we can replace the LHS term in the above equation by,$R_{\mu\nu}(\nabla^\mu\psi) + \nabla_\mu(\Box\psi)$.

Metric Field Equation

Upon variation of the action in Eq(B.6) with respect to $g_{\mu\nu}$, the first term will be,

$$\delta(\psi R) = \psi R_{\mu\nu}\delta g^{\mu\nu} + \psi g^{\mu\nu}[\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})] \tag{B.16}$$

In the second term in the above equation, the terms in the square bracket is not the boundary term (as there is an extra field). The actual boundary terms are (which vanish at the boundary),

$$\nabla_\lambda[\psi g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\nu})] = \nabla_\lambda\psi g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\nu}) + \psi g^{\mu\nu}\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu})$$

$$\nabla_\nu[\psi g^{\mu\nu}(\delta\Gamma^\lambda_{\lambda\mu})] = \nabla_\nu\psi g^{\mu\nu}(\delta\Gamma^\lambda_{\lambda\mu}) + \psi g^{\mu\nu}\nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})$$

The last terms here looks like divergence of a vector integrated over curved space time volume - by Gauss Theorem, it vanishes. Now we have write variation of Christoffel symbols in terms of variation of metric.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\nabla_\nu g_{\mu\rho}) + (\nabla_\mu g_{\nu\rho}) - (\nabla_\rho g_{\mu\nu})$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\delta\{g^{\lambda\rho}[(\nabla_\nu g_{\mu\rho}) + (\nabla_\mu g_{\nu\rho}) - (\nabla_\rho g_{\mu\nu})]\} = \frac{1}{2}g^{\lambda\rho}[\nabla_\nu(\delta g_{\mu\rho}) + \nabla_\mu(\delta g_{\nu\rho}) - \nabla_\rho(\delta g_{\mu\nu})]$$
$$\delta\Gamma^{\lambda}_{\lambda\mu} = \frac{1}{2}\delta\{g^{\lambda\rho}[(\nabla_\lambda g_{\mu\rho}) + (\nabla_\mu g_{\lambda\rho}) - (\nabla_\rho g_{\lambda\mu})]\} = \frac{1}{2}g^{\lambda\rho}\nabla_\mu(\delta g_{\rho\lambda}) \tag{B.17}$$

Given that we have to evaluate, $\nabla_\lambda\psi g^{\mu\nu}(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_\nu\psi g^{\mu\nu}(\delta\Gamma^{\lambda}_{\lambda\mu})$. The metric is not raised and we have been varying with respect to $\delta g^{\mu\nu}$. So we need to convert $g_{\mu\nu} \to g^{\mu\nu}$ using, (See Appendix A of [Alejandro Guarnizo et.al 2010])

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu} \tag{B.18}$$

$$-\nabla_\lambda\psi g^{\mu\nu}(\delta\Gamma^{\lambda}_{\mu\nu}) + \nabla_\nu\psi g^{\mu\nu}(\delta\Gamma^{\lambda}_{\lambda\mu}) = \nabla_\lambda\psi[-g^{\mu\nu}(\delta\Gamma^{\lambda}_{\mu\nu}) + g^{\mu\lambda}(\delta\Gamma^{\lambda}_{\lambda\mu}) \tag{B.19}$$
$$= \nabla_\lambda\psi[\nabla_\mu(\delta g^{\mu\lambda}) - g_{\mu\nu}\nabla^\lambda(\delta g^{\mu\nu})] \tag{B.20}$$

We had changed the variable $\lambda \to \nu$ in the above equation. We will change it back, which is now,

$$\nabla_\nu\psi\nabla_\mu\delta g^{\mu\nu} - \nabla_\lambda\psi g_{\mu\nu}\nabla^\lambda\delta g^{\mu\nu}$$

As the terms involve the derivatives of the variation, we can simplify these terms again using the divergence theorem.

$$\nabla_\mu(\nabla_\nu\psi\delta g^{\mu\nu}) = \nabla_\mu\nabla_\nu\psi\delta g^{\mu\nu} + \nabla_\nu\psi\nabla_\mu\delta g^{\mu\nu}$$

$$\nabla^\lambda(\nabla_\lambda\psi g_{\mu\nu}\delta g^{\mu\nu}) = \nabla^\lambda\nabla_\lambda\psi g_{\mu\nu}\delta g^{\mu\nu} + \nabla_\lambda\psi g_{\mu\nu}\nabla^\lambda\delta g^{\mu\nu}$$

LHS vanishes, Hence Eq(B.16) is,

$$\delta(\psi R) = \psi R_{\mu\nu}\delta g^{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\psi\delta g^{\mu\nu} \tag{B.21}$$

Now we can vary the second term in Eq(B.6) which is straight forward

$$\delta(-\frac{\omega}{\psi}(\nabla\psi)^2) = -\frac{\omega}{\psi}(\nabla_\mu\psi\nabla_\nu\psi)\delta g^{\mu\nu} \tag{B.22}$$

Also, variation of S_m gives the Energy momentum tensor $T_{\mu\nu}$ which also added with the constant energy density of vacuum.

Derivative of the interaction term is,

$$\delta(f\Box\psi(\nabla\psi)^2) = f\delta(\Box\psi)(\nabla\psi)^2 + f\Box\psi\delta(\nabla\psi)^2 \tag{B.23}$$

Let us focus on the first term on RHS as we have seen the last term will be $\delta g^{\mu\nu}(f\Box\psi\nabla_\mu\psi\nabla_\nu\psi)$. To see the variation of d'Alembert operator we need variation of Christoffel Symbols given by Eq(B.17),

$$\delta(\Box\psi) = \delta g^{\mu\nu}\nabla_\mu\nabla_\nu\psi - g^{\mu\nu}(\delta\Gamma^\lambda_{\mu\nu})\nabla_\lambda\psi \tag{B.24}$$

$$= \delta g^{\mu\nu}\nabla_\mu\nabla_\nu\psi + \nabla_\lambda\psi\nabla_\mu(\delta g^{\lambda\mu}) - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\psi\nabla^\lambda(\delta g^{\mu\nu}) \tag{B.25}$$

In the second equality we have used Eq(B.18) to raise the indices of metric under variation. We can again use divergence theorem to remove that variation of metric under derivative.

$$\delta(f(\nabla\psi)^2\Box\psi)$$
$$= \delta g^{\mu\nu}\underbrace{[f(\nabla\psi)^2\nabla_\mu\nabla_\nu\psi - \nabla_\mu[f(\nabla\psi)^2\nabla_\nu\psi] + \frac{1}{2}g_{\mu\nu}\nabla^\lambda[f(\nabla\psi)^2\nabla_\lambda\psi]\nabla_\nu\psi]}_{\text{Eq(B.24)}} \tag{B.26}$$
$$+ f\Box\psi\delta(\nabla\psi)^2$$

The first term in the second equation cancels when we see the variation root of determinant of g. We can try to solve the derivatives but it does not matter. Hence,

$$\delta(f\Box\psi(\nabla\psi)^2) = \delta g^{\mu\nu}[-f\psi\nabla_\mu(\nabla\psi)^2\nabla_\nu + \frac{1}{2}g_{\mu\nu}\nabla_\lambda\psi\nabla^\lambda(\nabla\psi)^2 + f\Box\psi\nabla_\mu\psi\nabla_\nu\psi] \tag{B.27}$$

Putting all the bits together Eq(B.21), Eq(B.22) and Eq(B.27) or simply (keeping the total derivatives without simplification in the previous step),

$$\frac{\delta S}{\delta g^{\mu\nu}} =$$
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{\psi}(\nabla_\mu\nabla_\nu\psi - g_{\mu\nu}\Box\psi) - \frac{\omega}{\psi^2}[\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}(\nabla\psi)^2] \tag{B.28}$$
$$+ \frac{1}{\psi}[f(\nabla\psi)^2\nabla_\mu\nabla_\nu\psi - \nabla_\mu[f(\nabla\psi)^2]\nabla_\nu\psi + \frac{1}{2}g_{\mu\nu}\nabla^\lambda[f(\nabla\psi)^2]\nabla_\lambda\psi\nabla_\nu\psi]$$

Making substitutions to the earlier equation,

$$\varphi G_{\mu\nu} + g_{\mu\nu}[\Box\varphi + \frac{\omega}{2\varphi}(\nabla\varphi)^2] - [\nabla_\mu\nabla_\nu\varphi - \frac{\omega}{\varphi}\nabla_\mu\varphi\nabla_\nu\varphi] \tag{B.29}$$

$$+\frac{f}{G_N^2}\Box\varphi\nabla_\mu\varphi\nabla_\nu\varphi + \frac{fg_{\mu\nu}}{2G_N^2}\nabla_\lambda\varphi\nabla^\lambda(\nabla\varphi)^2 - \frac{f}{G_N^2}\nabla_\mu[\nabla\varphi]^2\nabla_\nu\varphi \tag{B.30}$$

$$= 8\pi G_N(T_{\mu\nu} - \rho_\Lambda g_{\mu\nu}) \tag{B.31}$$

B.2.2 Quartic term

Differential w.r.t field

$$\delta[\theta(\nabla^\mu\psi\nabla_\mu\psi)(\nabla^\nu\psi\nabla_\nu\psi)] = 2\theta(\nabla^\mu\psi\nabla_\mu\psi)\underbrace{\delta(\nabla^\mu\psi\nabla_\mu\psi)}_{2\delta(\nabla_\mu\psi)} \tag{B.32}$$

RHS can be simplified using the divergence theorem

$$\nabla_\mu[(\nabla^\mu\psi\nabla_\mu\psi)(\delta\psi)] = 0 = \nabla_\mu(\nabla^\mu\psi\nabla_\mu\psi)\delta\psi + \text{RHS of the above equation} \tag{B.33}$$

Finally we have,

$$\frac{\delta(\nabla\psi)^4}{\delta\psi} = 4\theta\nabla_\mu[\nabla^\mu\psi(\nabla^\mu\psi\nabla_\mu\psi)] \tag{B.34}$$

Differential w.r.t metric

This is straight forward

$$\frac{\delta(g^{\mu\nu}(\nabla_\mu\psi\nabla_\nu\psi)(\nabla\psi)^2)}{\delta g^{\mu\nu}} = \theta(\nabla\psi)^2[\nabla^\mu\psi\nabla^\nu\psi] \tag{B.35}$$

There is also an additional term in the equations of motion as we are also differentiating $\sqrt{-g}$ *w.r.t* the metric.

C Feynman Methods and Loop Integrals

C.1 Feynman Methods

We list some useful methods for loop integration.

$$\frac{1}{A_1A_2} = \int_0^1 \frac{1}{[xA_1 + (1-x)A_2]^2} = \int_0^1 dxdy\ \delta(x+y-1)\frac{1}{[xA_1 + yA_2]^2} \tag{C.1}$$

We can have a general formula for

$$\begin{aligned}\frac{1}{A_1A_2...A_n} &= \int_0^1 dx_1dx_2...dx_n\ \delta(\sum_i x_i - 1)\frac{(n-1)!}{[x_1A_1 + x_2A_2 + ... + x_nA_n]^n}\\ \frac{1}{A_1^{m_1}A_2^{m_2}...A_n^{m_n}} &= \int_0^1 dx_1dx_2...dx_n\ \delta(\sum_i x_i - 1)\frac{\Pi_i x_i^{m_i-1}}{\sum_i [x_iA_i]^{\sum_i m_i}}\frac{\Gamma(m_1 + ... + m_n)}{\Gamma(m_1)...\Gamma(m_n)}\end{aligned} \tag{C.2}$$

C.2 Loop Integrals: 4 dimensions

$$\begin{aligned}V(p^2) &= \frac{(-i\lambda_B)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_L^2}\frac{i}{(p+k)^2 - m_L^2}\\ &= \frac{(-i\lambda_B)^2}{2}\int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2 + x(1-x)p^2 - m_L^2)^2}\ ;\quad (l^2 = k + xp)\end{aligned} \tag{C.3}$$

We start integration over energy component then, clearly the integrand has poles at $l_0 = \pm\sqrt{\vec{l}^2 + x(1-x)p^2 - m_L^2}$ and the contour is along the real axis. If we subtract a small value $i\epsilon$ from the denominator, we see that the poles are located slightly below and above the positive and real real axis. We can calculate the the residue,

$$V(p^2) = \frac{i\lambda_B^2}{2}\int_0^1 dx \int \frac{d^3\vec{l}}{32\pi^3}\frac{1}{(l_E^2 - x(1-x)p^2 + m_L^2)^{3/2}} \tag{C.4}$$

Now performing other three integration will reduce the power of denominator by 1/2 and multiply by an overall factor. In the last integral, we see that there is a logarithmic divergence $\sim \frac{i\lambda_B^2}{32\pi^2}\int_0^1 dx \log(l^2 - x(1-x)p^2 + m_L^2)$.

But the most often used method is Wick rotation where we rotate the integration contour from real axis to imaginary axis by substituting $l^0 = il_E^0$. Now, the integral is reduced to one in Euclidean space where we can use rotational symmetry to integrate over the volume of 3-sphere.

$$\int \frac{d^4 l_E}{(2\pi)^4} \longrightarrow \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty l_E^3 dl_E \quad \text{and} \quad \int d\Omega_4 = 2\pi^2 \tag{C.5}$$

Now we use standard integration procedures by substitution as given below.

$$\int \frac{x^3}{x^2+a^2}dx = \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{x^2+a^2}2xdx \overset{(u=x^2+a^2)}{=} \frac{1}{2}\int \frac{u-a^2}{u^2}du = \frac{1}{2}\left(\frac{1}{u} - a^2 \ln u\right)$$

(C.3) can be solved similarly (we apply the limits in the last step).

$$\begin{aligned} V(p^2) &= \frac{i\lambda_B^2}{2}\int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4}\frac{1}{(l_E^2 - x(1-x)p^2 + m_L^2)^2} \ ; \quad (l^0 = il_E^0) \\ &= \frac{i\lambda_B^2}{32\pi^2}\int_0^1 dx \int dl_E \frac{l_E^3}{(l_E^2 - x(1-x)p^2 + m_L^2)^2} \\ &= \frac{i\lambda_B^2}{32\pi^2}\int_0^1 dx \left[\frac{x(1-x)p^2 + m_L^2}{l_E^2 + x(1-x)p^2 + m_L^2} + \log\Big(l_E^2 + x(1-x)p^2 + m_L^2\Big)\right]_0^\infty . \end{aligned} \tag{C.6}$$

C.3 Loop Integrals: d dimensions

Solving the integral (2.7) which is a d dimensional integral is not very different from our previous calculation in 4-dimensions. By analytic continution to d dimensions, we have,

$$V(p^2) = \frac{(-i\mu^\epsilon \lambda_B)^2}{2}\int_0^1 dx \int \frac{d^d l}{(2\pi)^d}\frac{1}{(l_E^2 + x(1-x)p^2 - m_L^2)^2}. \tag{C.7}$$

The evaluation of the d dimensional volume integral can be simplified using the Gamma functions $\Gamma(n)$. Because,

$$\begin{aligned} (\sqrt{\pi})^d &= \left(\int dx e^{-x^2}\right)^d = \int d^d x \exp\left\{-\sum_{i=1}^d x_i^2\right\} = \int d\Omega_{d-1}\int_0^\infty dx x^{d-1} e^{-x^2} \\ &= \int (d\Omega_{d-1})\left(\frac{1}{2}\Gamma(d/2)\right). \end{aligned} \tag{C.8}$$

We can use this to separate the integral as follows

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+a^2)^2} = \int \frac{d\Omega_{d-1}}{(2\pi)^d}\int_0^\infty \frac{l_E^{d-1}}{(l_E^2+a^2)^2}dl_E \ ; \text{ where } \int d\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

integral over l_E is $\displaystyle\int_0^\infty \frac{l_E^{d-1}}{(l_E^2+a^2)^2}dl_E = \frac{1}{2}\int_0^\infty \frac{(l_E^2)^{d/2-1}}{(l_E^2+a^2)^2}d(l_E^2)$.

Continuing with the integration by substitution,

$$\frac{1}{2}\int_0^\infty \frac{(l_E^2)^{d/2-1}}{(l_E^2+a^2)^2}d(l_E^2) \overset{(u=a^2/l_E^2+a^2)}{=} \frac{1}{2}\left(\frac{1}{a^2}\right)^{2-d/2}\int_0^1 u^{1-d/2}(1-u)^{d/2-1}du. \quad (C.9)$$

Now we use beta function and gamma function relation $\beta(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. The complete integral would be then,

$$\mu^{2\epsilon}\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+a^2)^2} = \frac{(\mu^2)^{2-d/2}}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{\mu^2}{a^2}\right)^{2-d/2}$$

Due to poles of Gamma function the integral has the poles as well. The approximate behaviour for $\epsilon/2 = 2-d/2$, is given in terms of Euler-Mascheroni constant γ_E

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \quad \text{where, } \gamma_E = 0.5772$$

We set $d = 4-\epsilon$ and expand exponential factor $1/a^2$ around $\epsilon = 0$.

$$\left(\frac{1}{a^2}\right)^{\epsilon/2} \overset{(x=1/a^2)}{=} \exp\left\{\frac{\epsilon}{2}\ln x\right\} = 1 + \frac{\epsilon}{2}\ln x + ... \quad (C.10)$$

We have included a factor of 4π which comes from the Gamma function and volume integral and denoted $\lambda_B(\mu^2)^{2-d/2}$ as $\lambda(\mu)$.

$$V(p^2) \overset{d\to 4}{=} \frac{i\lambda^2(\mu)}{32\pi^2}\int_0^1 dx\left[\frac{2}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m_L^2 - x(1-x)p^2}\right)\right] \quad (C.11)$$

C.4 Divergent Integrals: Vacuum Energy

In $d-1$ dimensions, the energy and pressure ($d-1$ appears due to d-dimensional energy momentum tensor) density integrals are

$$\begin{aligned}
\langle\rho\rangle &= \frac{1}{2}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\sqrt{k^2+m^2} = \frac{1}{4}\int\frac{d\Omega_{d-2}}{(2\pi)^{d-1}}\int d(k^2)(k^2)^{d/2-3/2}\sqrt{k^2+m^2} \\
&= \frac{1}{4}\int\frac{d\Omega_{d-2}}{(2\pi)^{d-1}}\int du\frac{-m^2}{u^2}\left[m^{d/2-3/2}(1-u)^{(d/2-3/2)}u^{(d/2-3/2)}\right]\frac{m}{\sqrt{u}}\ ; \quad \left(u = m^2/(k^2+m^2)\right) \\
&= \frac{m^d}{4}\int\frac{d\Omega_{d-2}}{(2\pi)^{d-1}}\int du\ (1-u)^{(d/2-3/2)}u^{(-d/2-1)}, \\
\langle p\rangle &= \frac{1}{2(d-1)}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{k^2}{\sqrt{k^2+m^2}} = \frac{1}{4(d-1)}\int\frac{d\Omega_{d-2}}{(2\pi)^{d-1}}\int d(k^2)\frac{(k^2)^{d/2-1/2}}{\sqrt{k^2+m^2}} \\
&= \frac{m^d}{4(d-1)}\int\frac{d\Omega_{d-2}}{(2\pi)^{d-1}}\int du(1-u)^{(d/2-1/2)}u^{(-d/2+1/2)}.
\end{aligned} \quad (C.12)$$

For dimensional reasons we plugged in a new parameter μ along with $\epsilon/2 = 2-d/2$ in (2.7). Similarly, we make of these parameters here as well. We then have to refer to (C.9) to express the integral in terms of Gamma functions and using volume integral (C.8) for $d-2$ dimensions,

$$\begin{aligned}\langle\rho\rangle &= \frac{\mu^\epsilon m^d}{2(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(-1/2)} \\ \langle p\rangle &= \frac{\mu^\epsilon m^d}{4(4\pi)^{d-1}}\frac{\Gamma(-d/2)}{\Gamma(1/2)}.\end{aligned} \tag{C.13}$$

Then we expand around $\epsilon = 0$ as we did in (C.10) that is, $m^{-\epsilon} \sim (1-\epsilon m)$ and $(4\pi)^{\epsilon/2} \sim 1 + \frac{\epsilon}{2}\ln(4\pi)$,

$$\langle\rho\rangle = \frac{1}{2}\beta_\Lambda^{(1)}\left[\gamma_E - \frac{2}{\epsilon} - \frac{3}{2} - \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] \tag{C.14}$$

where $\beta_\Lambda^{(1)} = \frac{m^4}{32\pi^2}$ is the one loop beta function.

D Effective Field Theory

D.1 EFT: A Short Note

[For detailed discussion see for example, [18, 91]]

We write the effective action as an expansion in terms of local field operators of low energy as,

$$S_E[\chi_L] = \int d^d x L_E \quad \text{where,} \quad L_E = \sum_i c_i F_i[\chi_L, \partial\chi_L]. \tag{D.1}$$

We have written the effective Lagrangian explicitly in terms of coupling constants c_i called Wilson coefficients and functions F_i of low energy fields χ_L and its derivatives $\partial\chi_L$. The Lagrangian in principle could contain infinite terms in the sum (allowed by the symmetries of the theory). With our assumption of a single fundamental scale Λ for the theory, and coupling with mass dimension $[c_i] = \gamma_i$, we can write it in terms of dimensionless couplings denoted by $\tilde{c}_i$ as, $c_i = \frac{\tilde{c}_i}{\Lambda_{cut}^{\gamma_i}}$. The order of these couplings could have a wide range in principle but we assert on *naturalness* claiming that, they are of order 1, $\tilde{c}_i \sim \mathcal{O}(1)$ unless we have specific explanation for their smallness or largeness, for example we encountered such problem in case of the CC and Higgs, where we had unnaturally small value for the observed value. Now the values of couplings determine how much does each term F_i in the Lagrangian, contribute in some measurements (we assume the measurable quantity to be dimensionless).

$$c_i \equiv \tilde{c}_i \left(\frac{\Lambda_{exp}}{\Lambda_{cut}}\right)^{\gamma_i} = \begin{cases} \mathcal{O}(1) & \text{if } \gamma_i = 0 \\ \gg 1 & \text{if } \gamma_i < 0 \\ \ll 1 & \text{if } \gamma_i > 0 \end{cases} \tag{D.2}$$

We can see that, due to largeness of $\Lambda_{cut}(< \Lambda)$, only the first two cases are significant. We can *truncate* the infinite terms in the Lagrangian depending upon precision we need to compare with the experimentally obtained values. Since Lagrangian L_E has mass dimensions of the spacetime d, then each term has dimension $[F_i] = D_i$. With the couplings c_i with dimensions γ_i, we can write, $D_i = d + \gamma_i$. Now, we can classify the terms in the summation (D.1), as given in Table(D.1)

Naive dimensional analysis tells us that only finite number of relevant (and marginal) terms exist in the expansion (D.1). We can have any arbitrary number of irrelevant operators but as we see from (D.2), such couplings comes are suppressed by higher powers of Λ. We have to remember that, we are looking at free theory Lagrangian and the quantum effects can change the scaling of couplings described in (D.2). In

Dimension	Relevance as $\Lambda_{exp} \to 0$	Renormalizability
$D_i < d$ $(\gamma_i < 0)$	grows; relevant operators	super-renormalizable
$D_i = d$ $(\gamma_i = 0)$	constant; marginal operators	renormalizable
$D_i > d(\gamma_i > 0)$	falls; irrelavant operators	non-renormalizable

Table D.1: Relevance of Operators in Effective Lagrangian and Renormalizability

perturbation theory, we can make use of the dimensional regularization in which case, our Λ is just μ. In fact the couplings can be written as $\tilde{c}_i(\mu)$ and γ_i are anomalous dimensions facilitating regularization. The non renormalizable, higher dimensional irrelevant operators contribute for the precision of measurements of a low energy process and tells us about the physics at cutoff scale.

Symmetries are defined while writing the Lagrangian. From our *naturalness* assertion, mass of a scalar field in 4-dimensions, comes with 'relevant' operator $m^2 \sim \Lambda^2$ (natural). This is the case for the Higgs field which was discussed in first part. In order for EFT to work, we claim all the mass terms to be forbidden. This imposed symmetry (scale invariance) can then be broken at low energy through which the particles get masses. This is like chiral symmetry for fermions where their masses vanish imposing this symmetry. Unlike chiral symmetry, Higgs mass is not protected by any symmetry and hence is not consistent with our low energy effective picture of the Standard Model. Supersymmtery would be one of the way to solve this problem. Indeed this problem of *naturalness* is exactly what we came across as Higgs mass fine tuning problem.

EFT is based on the idea that the picture of the Universe depends on the energy scales in which we see it. For example, decay of muons can be approximated by Fermi's Theory of weak interactions where W boson is integrated out. In this picture, we have a four fermion vertex parametrized by Fermi constant, G_F. This 'effective' picture is valid up to 10^{-6} corrections. For precise measurements of muon lifetime lifetime (which is $\mathcal{O}(10^{-5})/\text{GeV}^2$), we will have to include the W boson propagator of Standard Model theory. To summarize, Fermi Theory of weak interactions is the first order approximation of the much bigger theory, for example Standard model which will give corrections of the order, $\mathcal{O}(\frac{p^2}{M_W^2})$. Another example would be dimension 5 operators which are necessary to include the right handed fermions to the SM. The Standard Model of particles is also an effective field theory of a much bigger theory in the UV scale which we don't know yet.

EFT involves following steps, before one proceeds to compute the observable quantities.

- Identify relevant degrees of freedom : this gives us an idea of available number of fields in Lagrangian.

- Identify relevant Symmetries : this determines the kind of operators appearing in Lagrangian. All of those operators has to be included.

$$L_E[\chi] = L_{kinetic}[\phi] + \sum_i c_i F_i[\chi_L, \partial\chi_L] \tag{D.3}$$

Usage of EFT can be two ways, bottom-up approach and top-bottom approach. The latter approach refers to a situation where the high energy theory is known but we employ EFT to understand low energy behaviour of the theory. For example, QCD. The former approach refers to cases like SM where the underlying theory is unknown but we can write a EFT Lagrangian with all possible operators whose couplings are measured in the experiments. The precision of the experiments tell us how many terms are needed to explain the full theory.

We are aware that we can only do experiments up to some energy scales. Most of the theories we have so far which have been tested very well are phenomenological theories and for a pure theorist, there exists a fundamental theory which explains our universe in high energy scales - and the low energy physics can be calculated exactly from this underlying theory (even if the computation is hard). In this context, Standard model is a phenomenological theory where we can test our effective theories of low energy by doing precision experiments.

So far, by setting a high energy cutoff we built a effective theory which captured relevant terms at this energy scale so as to make predictions. Evidently the drawback of this approach is that we would not know if there were relevant interactions in the high energy which we integrated out. So we are stuck with many theories each valid up to certain energy scales. A vast number of QFTs have a same low energy theory, this is called *universality*.

In perturbative theories we encountered Landau pole (in QED for example) where the perturbative approach breaks down. In our EFT approach we can evade such problems because existence of such pole is just an indication that theory is simply incomplete and there exists a much bigger theory at high energy scale where it has to incorporated.

D.2 High Energy Interactions

We can handle interactions by splitting the full action into $S_\Lambda[\chi_L + \chi_H] = S_0[\chi_L] + S_0[\chi_H] + S_\Lambda[\chi_L, \chi_H]$. The pieces $S_0[\chi_L], S_0[\chi_H]$ contain no interaction terms hence are tree level actions. From (7.16), realizing the effective action *l.h.s* also includes tree level action for χ_L which cancels with the first piece of the splitting, we have,

$$\begin{aligned} S_E[\chi_L] &= -\ln\left[\int D\chi_H \exp\{-S_\Lambda[\chi_L + \chi_H]\}\right] \\ \xrightarrow{split} S_E^{int}[\chi_L] &= -\ln\left[\int D\chi_H \exp(-S_0[\chi_H] - S_\Lambda[\chi_L, \chi_H])\right]. \end{aligned} \tag{D.4}$$

Now, the trouble is in including the high energy interactions $S_\Lambda[\chi_L, \chi_H]$ while the other term on the *r.h.s* only contributes to the vacuum energy. Since we shall restrict ourselves to weak field or Minkowski limits in the further calculations, we shall ignore contribution of the latter in this section. The real space propagator for high energy modes is the integral,

$$G_H(x_1, x_2) = \int_{\Lambda_{cut} < |p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot x}\chi}{p^2 + m^2}.$$

This is full integral up to the fundamental energy scale Λ which implies integration to smallest length scales. To have a flow, we lower the scale by $\delta\Lambda$, that is, $\Lambda' = \Lambda - \delta\Lambda$, in the lowest order, the propagator is

$$G_{\Lambda'}(x, y)| = \frac{1}{(2\pi)^d} \frac{{\Lambda'}^{d-1}}{{\Lambda'}^2 + m^2} \int d\Omega^{d-1} e^{i\Lambda' p\cdot(x-y)}.$$

As we can observe, the integral space has reduced. To lowest order in $\delta\Lambda$ we have to consider only up to single χ_H propagator. Treating χ_L as external legs then we have only two possibilities to incorporate χ_H as a single propagator. It has to be either a loop connects the same point which can have many χ_L legs or a propagator connecting two n-point vertex which can each have many χ_L legs[1]. This mechanism is like zoom out mechanism. We start out with high resolution-integrating smallest scales and slowly vary the (first order) the scale to zoom out to finally obtain the full theory in the macroscopic level. This is encoded Polchinski's equation which is infinitesimal limit of $e^{-S_{\Lambda'}}$,

$$-\Lambda' \frac{\partial S_{\Lambda'}}{\partial \Lambda'} = \int d^d x d^d y \left[\frac{\delta S_{\Lambda'}}{\delta \chi_L(x)} G_{\Lambda'}(x, y) \frac{\delta S_{\Lambda'}}{\delta \chi_L(y)} - G_{\Lambda'}(x, y) \frac{\delta^2 S_{\Lambda'}}{\delta \chi_L(x) \delta \chi_L(x)} \right]. \quad \text{(D.5)}$$

In practice, we expand $S_\Lambda[\chi_L, \chi_H]$ in derivatives of the potentials which is called the loacl potential approximation. Even though this equation gives us a hope to attain knowledge of full theory, this is in principle includes many interactions and loops if we consider higher orders of $\delta\Lambda$. This complicates calculations. However, to our rescue, we shall later use much more systematic way of dealing with high energy theory.

D.3 Fixed Points Revisited

The fixed points of beta functions were described in section(2.2). In the similar fashion, we have, $\beta_i(g_j^*) = 0$, where g_j^* denotes the values of the couplings at fixed point. At this point the theory is scale invariant. We can think of these points as UV or IR points in a theory space, and beta function would determine the trajectory on this space. This scale invariance is quantum in nature. The scale invariance of classical action plays no role here as they might be broken due to quantum fluctuations. For example we can relate this with Weinberg-Coleman model[20] which uses radiative correction to attain SSB even when $m_B = 0$ in (1.10). Best

[1] One can hence expect non-locality.

example would be from QCD where, fixed point $g^* = 0$ denotes asymptotic freedom. This point where couplings vanish are called Gaussian Fixed Point in contrary to non-Gaussian Fixed Points which are fixed points in interacting theory. To get to a non-Gaussian fixed point one can see from (7.22) that we have to tune the quantum part (the second term) very carefully to cancel out the first term. Hence it is rare to come across multiple (non-trivial) fixed points.

One looks at the vicinity of fixed points to see the direction of the flow - whether it is from IR to UV or otherwise. At close vicinity of these points $g_j^* + \delta g_j$, we can approximate the trajectory using linear beta functions given by following expansion,

$$\beta_i(g_j) = \beta_i(g_j^*) - \Lambda_{cut} \frac{\partial g_i}{\partial \Lambda_{cut}} \bigg|_{g_j^* + \delta g_j} = B_{ij}(g^*)(g_j - g_j^*) + \mathcal{O}(\delta g)^2.$$

The first term in the expansion is zero by definition. The stability matix $B_{ij} = -\frac{\partial \beta_i(g_j)}{\partial g_j}$ is a constant matrix and in principle, can be infinite dimensional. From the analogy of (7.22), we can expect the eigenvalues of this matrix to be $d_i - d$ in case of a classical theory. If we renormalize the theory, then we have anomalous dimension γ_i for i number of fields. This is in fact the difference, $\Delta_i - d_i$ such that, in classical theory we have $\Delta_i = d_i$. So the eigenvalues can be written in general as, $\Delta_i - d$ where we also include the contribution from rescaling parameter $\mathcal{Z}$. The corresponding eigenvectors be $\vec{\theta}_j$. Consider a direction along one of the eigenvector denoted by subscript l, now the linearized flow equation is simply,

$$\Lambda_{cut} \frac{\partial \vec{\theta}_l}{\partial \Lambda_{cut}} = (\Delta_l - d)\vec{\theta}_l + \mathcal{O}(\theta^2) \implies \vec{\theta}_l(\Lambda_{cut}) = \left(\frac{\Lambda_{cut}}{\Lambda'} \right)^{\Delta_l - d} \theta_l(\Lambda').$$

Now if we have operators with $\Delta_i > d$, the coupling reduces as the scale Λ_{cut} is reduced. Also if $\Delta_i < d$ then those couplings grow as scale is reduced. The lowering of the scale refers to moving towards IR theory. In the former case, the couplings are called as irrelevant as they take us to the critical point even if we included it in the action. In latter case, the couplings are relevant. Starting from a fixed point, if we lower the scale, we evolve along the trajectory, the renormalized trajectory, either eternally or to meet another critical point. There is yet other possibility of having $\Delta_i = d$ in which the couplings are called marginal where the couplings remain unchanged in the flow of scale. The quantum part can contribute a small running which can be irrelevant or relevant. The there cases of relevant, irrelevant and marginal couplings correspond to the respective operators as described in Appendix(D.1).

From the above example of Newton's gravitational coupling, we can see that, the first term in (7.22), dimensionless coupling $\tilde{G}_N \equiv G_N \Lambda_{cut}^2$ grows with the scale. Lets assume, below Planck scale, all the quantum fluctuations (second term) are suppressed. Then we know that dimensional G_N is constant. The Planck scale marks the regime above which the quantum fluctuations balance (G_N decreases with increasing scale) the canonical running in the first term leading to a constant $\tilde{G}_N$ which denotes the fixed point $\tilde{G}_N^*$. This is the region of *asymptotic safety*. In principle this regime is of non-perturbative in nature [90]. Hence, finding a UV fixed point can be seen as one of the first step in asymptotic safety program. However, asymptotic safety makes strange predictions. For example, in low energy we have a theory for gravity in 4 spacetime dimension which reduces to 2 dimensions in high energies.

To summarize, we started out by setting a high energy cutoff and we built a effective theory which captured relevant terms at this energy scale. We looked at flow of couplings which represents the flow of theory (in theory space spanned by all the couplings). We also interpreted trivial Gaussian fixed point where all the couplings vanishes (asymptotically free) and the non-trivial fixed points where couplings runs to a non-vanishing constant (asymptotically safe). Usually these aspects are proposed to describe UV completion through asymptotic safety program. Renormalizability does not imply UV completion because renormalizable theories do not promise of being the fundamental theories. On the other hand, non renormalizable theories can be candidates of UV complete theories [115]. Even though there are problems with number of dimensions of the theory at high energies compared with low energy theory, and possibility of unknown couplings in high energies, we can think of UV completion as a guiding principle. In the following, we shall use the beta functions in the context of general Galileon and K-essence theories (with particular truncation - operators up to dimension 8) to see if we can find (non-trivial) fixed points which are compatible with asymptotic safety program.

E Equivalence Principle and Conformal Frames

We pointed out the 'beauty' of Einstein equation in the first chapter which lies in equating the geometry with the matter content. Brans-Dicke theory was motivated from the difficulties encountered in applying Mach's principle into GR. Mach's arguments can simply stated as, the state of inertia is achieved through interactions with all the matter in the Universe [116]. The book [116] also gives a very simple illustration. Consider a state where we stand still with arms stretched out to our sides. In this state the stars above us does not appear to move. But when we pirouette, our arms go upwards due to centrifugal force. In the rotating state, the stars are rotating as well. Then this would "be a remarkable coincidence if the inertial frame in which (y)our arms hung freely, just happened to be the reference frame in which typical stars are at rest" [116]. GR abounded the Newtonian picture of absolute time and space. Even though we have a dynamical spacetime in GR, Newtonian limits can be obtained by making weak field limit. In other words we can have a (approximate) state of inertia in a very small region of space and time. This is encoded in Equivalence Principle which is incorporated in GR which states that "at every point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation" [116].

Often there is a distinction made between strong and weak Equivalence Principle, which are further branched for example very strong principle and medium strong principle etc. Let us only consider Strong and Weak Equivalence Principle. Strong Equivalence Principle (SEP) roughly translates to having constants of nature to be *universal* constants. Violation of it would imply time varying constants. Weak Equivalence Principle (WEP) is a phenomenological principle and unlike for SEP, we have strong experimental constraints for theories which violate WEP. WEP can be roughly stated as there should no other long range force that couples to matter universally other than gravity. If the force is carried by a scalar field then we expect it to be coupled to complete energy momentum tensor without depending on the composition, just like gravity. A detection of violation can be made, for example, if one observes anomalous acceleration of objects, then one can explain this using introduction of a fifth force. For a nice description see [117] and also [56].

Even though GR has dynamical picture of spacetime, through Equivalence Principle it solves the problem of inertia but still does not explain many other aspects concerning the Mach's principle. In a theory based on Mach's principle, Minkowski flat spacetime should be the only solution to Einstein equations without any matter (and of course also without vacuum energy). This is achieved in GR only when we put boundary conditions on the field equations. For example we could have a condition such that we have flat universe at infinity. Now we have inertial frame which is unrelated to the energy or matter content if the space time. With this condition, we have Minkowski metric as solutions to Einstein equations with no matter. But according to Mach's principle, the entire matter content of the universe should determine the state of inertia [118]. Einstein introduced cosmological constant in order to obtain a closed universe such that he evades the necessity of applying boundary conditions. But de Sitter showed that presence of such a term would imply non-zero matter content. Brans-Dicke theory was an attempt to formulate a gravitational theory which was more satisfactory than GR. BD theory violates only SEP and introduces a varying gravitational coupling in the form a scalar field, which we refer as BD field. There is no violation of well tested WEP because the scalar field is not directly coupled to the matter sector.

Given that the scalar field nonminimally couples to gravitational sector, there is a possibility of transferring it to the matter sector. This is in principle possible due to the fact that GR is not invariant under conformal transformations. We can redefine the conformal factor such that coupling to gravitational sector, we call this Jordan frame, upon conformal transformation will have coupling to matter sector is called Einstein frame. This simply means that we are going from one conformal frame to another frame but these two frames are now very different. One can now ask which of these frames is real. Let us take the example of BD theory[1].

$$L_{\text{BD}} = \sqrt{-g}\left(\phi R - \frac{\omega_{\text{BD}}}{2\phi} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + L_{\text{matter}}\right) \tag{E.1}$$

We denote quantities in Eisntein frame with a *bar* on top. The conformal transformations of metric and root of its determinant is given by,

$$\begin{aligned} g^{\mu\nu} &= \Omega^2 \bar{g}^{\mu\nu} \\ \sqrt{-g} &= \Omega^{-4}\sqrt{-\bar{g}}\,. \end{aligned}$$

Here, Ω is the transformation function which is a arbitrary function of spacetime coordinate. The transformation of Ricci scalar is [119],

$$R = [\overline{R}\Omega^2 - 2(d-1)\Box\ln(\Omega) - \frac{(d-1)(d-2)}{\Omega^2} g^{\mu\nu}\nabla_\mu \ln(\Omega)\nabla_\nu \ln(\Omega)] \tag{E.2}$$

and with [56],

$$f = \ln(\Omega) \implies f_\mu = \frac{\partial_\mu \ln\Omega}{\Omega},$$

the first term in the Lagrangian can be written as,

$$L_1 = \sqrt{-\bar{g}}\Omega^{-2}\phi\left(\bar{R} + 6\bar{\Box}f - 6\bar{g}^{\mu\nu}f_\mu f_\nu\right)\,.$$

[1]Refer to [56] for more details.

We want to remove the nonminimal coupling. So we put $\Omega^{-2}\phi \implies \Omega = \sqrt{\phi}$. The second term is the de Alembert operator in Eisntein frame which is give by,

$$\bar{\Box} f = \frac{1}{\sqrt{-\bar{g}}} \partial_\mu \left(\sqrt{-\bar{g}} \bar{g}^{\mu\nu} \partial_\nu f \right),$$

this vanishes in the action due to divergence theorem. The third term in (E.1) is $f_\mu = \frac{1}{2}\frac{\partial_\mu \phi}{\phi}$. We have now the first term in Einstein frame,

$$L_1 = \sqrt{-\bar{g}} \left(\frac{1}{2}\bar{R} - \frac{3}{4\phi^2} \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right)$$

The derivatives in Jordan frame can be transformed to derivatives in Einstein frame by, $\nabla^\mu \equiv \bar{g}^{\mu\nu}\nabla_\nu$. If we make use of this for second term of (E.1) it is not hard to see that, in Einstein Frame, the Lagrangian should have the form,

$$\bar{L} \sim \sqrt{-\bar{g}} \left(\frac{1}{2}\bar{R} - \frac{1}{2}\bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + L_{\text{matter}}(\bar{g}) \right). \tag{E.3}$$

F Normalization and Riemann Normal Coordinates

F.1 Normalization of States

In (3.3) we defined creation and annihilation operators[1] such they created a particle and hence increased the number of particles in the state by 1 or annihilated a particle in the state. That is, we can have a state with momentum $\mathbf{p}$ created by the operator $a^\dagger_{\mathbf{k}}$ from vacuum $|\mathbf{p}\rangle = \sqrt{2E}a^\dagger_{\mathbf{k}}|0\rangle$. One can define a Hamiltonian of free theory as given in (3.9). Then we have inner product

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2E(2\pi)^2\delta(\mathbf{p}-\mathbf{q}).$$

The field operator in position space was also defined in the same equation. This operator would create a particle at position $\mathbf{x}$. That is, $\phi(\mathbf{x})|0\rangle = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E}}e^{-i\mathbf{k}\cdot\mathbf{x}}|\mathbf{p}\rangle$. We can further define a equivalent of a wavefunction by taking the expectation value of this operator,

$$\langle 0|\phi(\mathbf{x})|\mathbf{p}\rangle = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E}}\left(a_{\mathbf{k}}e^{-iEt+i\mathbf{k}.\mathbf{x}} + a^\dagger_{\mathbf{k}}e^{iEt-i\mathbf{k}.\mathbf{x}}\right)\sqrt{2E}a^\dagger_{\mathbf{k}}\langle 0| = e^{i\mathbf{k}\cdot\mathbf{x}}. \quad \text{(F.1)}$$

Similarly, one can define two point function $\langle 0|\phi(\mathbf{x}_1,\mathbf{x}_2)|0\rangle$ or the time order product of ϕ which is nothing but the propagator defines in (3.16). These definitions are in trivial flat spacetime and without any interactions. For theories with interactions, we defined the vacuum state $\langle\Omega|$ which is different from that of free theory. Let us define the interaction Hamiltonian H_I which can be equivalent to $\int d^3x\frac{\lambda}{4!}\phi^4(\mathbf{x})$ from (1.10) in our scalar theory. As discussed in [13], we can get $|\Omega\rangle$ simply by evolving $|0\rangle$ in time. the two point function in interaction theory taken in the limit of $T\to\infty$,

$$\langle\Omega|T\phi(\mathbf{x}_1,\mathbf{x}_2)|\Omega\rangle = \lim_{T\to\infty}\frac{\langle 0|T\left(\phi(\mathbf{x}_1,\mathbf{x}_2)\exp\left\{-i\int\limits_{-T}^{T}H(t)dt\right\}|0\rangle\right)}{\langle 0|T\left(\exp\left\{-i\int\limits_{-T}^{T}H(t)dt\right\}|0\rangle\right)}$$

For Klein-Gordon equation in curved spacetime (3.24), we continued with similar method and defined creation and annihilation operators $A^\dagger_{\mathbf{k}}, A_{\mathbf{k}}$ and the solution

[1]We have made extensive use of [13] especially for notations and refer the same for a detailed overview.

field operators $f_{\mathbf{k}}, f^*_{\mathbf{k}}$. For example, we can have $f_1 = \phi$ and $f_2 = n^\mu \nabla_\mu \phi$ where n^μ is a timelike unit vector normal to a spacelike hypersurface where f_2 is defined. Here we make use of [37] to see distinction from flat spacetime, first by looking at the inner product $\langle f|_1 \, |f\rangle_2 = i \int (f_2^* \partial_\mu f_1 - f_1 \partial_\mu f_2^*) d\Sigma^\mu$ where $d\Sigma$ is the volume element in the given spacelike hypersurface. The integrals and especially delta functions appearing commutation relations (for example $[A^\dagger_{\mathbf{k}}, A_{\mathbf{k'}}] = \delta(\mathbf{k}, \mathbf{k'})$) are evaluated on this volume element such that (say in position space) $\int \delta(\mathbf{x}, \mathbf{x'}) d\Sigma = 1$. In Minkowski space, solutions of Klein-Gordon equation are like in (F.1), when written explicitly with time coordinate t as $\phi \sim e^{-iE_{\mathbf{k}}t + \mathbf{k}\cdot\mathbf{x}}$. This is Lorentz invariant and hence it is the unique solution of the Klien-Gordon equation. With this we could define a unique vacuum state.

In curved spacetime, we cannot have unique set of solutions $\{f_{\mathbf{k}}\}$ and hence no unique vacuum state [37]. This is due to the fact that in general curved manifold, we can have singularities such that timelike geodesics are incomplete - for example on the event horizon of a Black Hole. One can still have a set of orthonormal solutions $\{f_i, f_i^*\}$, where i includes an index specifying the Cauchy surface[34]. The one point functions or the local expectation values in curved spacetime, need not be associated with a particle; similarly the n-point functions (see section(3.2)). This has very interesting physical consequences like particle creation by gravitational fields and is discussed in [37].

F.2 Dirac Delta Function on d-dimensional Curved Spacetime

In the section(3.2) we introduced d-dimensional Dirac Delta Function on curved spacetime $\delta^d(x_1, x_2) = |-g(x_1)|^{-1/2} \delta(x_1 - x_2)$. The relation between the usual delta function $\delta(x_1 - x_2)$ in flat spacetime to the one in curved spacetime can be derived from their definitions,

$$\int_{-\infty}^{+\infty} f(x_1)\delta(x_1 - x_2) dx = f(x_2) \rightarrow \int_{-\infty}^{+\infty} f(x_1)\delta^d(x_1 - x_2)\sqrt{-g(x_1)} d^d x = f(x_2),$$

$$\implies \delta^d((x_1, x_2)) = \frac{\delta(x_1 - x_2)}{\sqrt{-g(x_1)}} = \frac{\delta(x_1 - x_2)}{\sqrt{-g(x_2)}} = |-g(x_1)|^{-1/4} \delta(x_1 - x_2) |-g(x_2)|^{-1/4}.$$

The normalization (completeness relation) is given by, $\mathbb{1} = \int d^d x \, \langle x| \sqrt{-g(x)} \, |x\rangle$.

F.3 Reimann Normal Coordinates

Let us set a point Q to be the origin and a point P in its near neighbourhood on a 4 dimensional Manifold (in our case, a pseudo-Riemannian manifold). Then there is a unique geodesic joining these two points. Lets define coordinates at origin Q as

$x' = x'^{\mu}$. Let the point $P \equiv x$ have its coordinates x^{μ}, then Riemann coordinates $y^{\mu} = x^{\mu} - x'^{\mu}$ of a point P is given by

$$y^{\mu} = \lambda \xi^{\mu} \quad \text{where, } \xi^{\mu} = \left.\frac{dy^{\mu}}{d\lambda}\right|_{Q \equiv x'},$$

are constant tangents to the geodesic at Q. The basic idea of Riemann coordinates is to use the geodesics passing through a point to define coordinates for nearby points, the condition being, the geodesic do not cross or in other words working in sufficiently small neighbourhood (for further details see [120]). Since, ξ^{μ} is constant tangent vector or independent of λ, geodesic through Q is simply,

$$\left.\frac{d^2 y^{\mu}}{d\lambda^2}\right|_{Q \equiv x'} = 0 \implies \Gamma^{\mu}_{\alpha\beta} \frac{dy^{\alpha}}{d\lambda} \frac{dy^{\beta}}{d\lambda} = \Gamma^{\mu}_{\alpha\beta} \xi^{\alpha} \xi^{\beta} = \Gamma^{\mu}_{\alpha\beta} y^{\alpha} y^{\beta} = 0, \tag{F.2}$$

which is the definition of Riemann Coordinates. Note that the vanishing Christoffel connections are evaluated at point Q. Coordinates to any point x be z^{μ} which is at the vicinity of point $Q \equiv x'$ can be given by a expansion series

$$z^{\mu} = z^{\mu}\big|_{Q \equiv x'} + C^{\mu}_{\alpha} \left.\frac{dy^{\alpha}}{d\lambda}\right|_{Q \equiv x'} + \frac{1}{2!} C^{\mu}_{\alpha\beta} \left.\frac{dy^{\alpha}}{d\lambda}\right|_{Q \equiv x'} \left.\frac{dy^{\beta}}{d\lambda}\right|_{Q \equiv x'} + ...,$$

where $C^{\mu...}_{\alpha...}$ are constant coefficients choosing which has been simplified due to our definition (F.2). To supplement it, we also have, symmetric derivatives of Christoffel symbols at Q also vanish This can be shown by first writing down the expansion for Christoffel symbols as below and using (F.2)

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta}\big|_{Q} + \left(\partial_{\rho} \Gamma^{\mu}_{\alpha\beta}\right)\big|_{Q} y^{\rho} + \frac{1}{2!} \left(\partial_{\rho} \partial_{\sigma} \Gamma^{\mu}_{\alpha\beta}\right)\big|_{Q} y^{\rho} y^{\sigma} + ...$$

That is, we have, $\partial_{(\rho} \Gamma^{\mu}_{\alpha\beta)}\big|_{Q} = 0, \quad \partial_{(\rho_1, \rho_2, ..., \rho_n} \Gamma^{\mu}_{\alpha\beta)}\big|_{Q} = 0$

where we have used compact notations ($\partial_{\rho_1} \partial_{\rho_2} ... \equiv \partial_{\rho_1, \rho 2, ...}$) for multiple derivatives in the latter expression and parenthesis indicate the symmetrization.

Riemann tensor at Q is given by $R^{\mu}_{\nu\alpha\beta} = \partial_{\beta} \Gamma^{\mu}_{\nu\alpha} - \partial_{\alpha} \Gamma^{\mu}_{\nu\beta}$ all evaluated at Q. We can show that,

$$\partial_{(\beta} \Gamma^{\mu}_{\alpha)\nu}\big|_{Q} = -\frac{1}{3} R^{\mu}_{(\alpha\beta)\nu}\big|_{Q} \implies g_{\mu\nu,\alpha\beta} = -\frac{1}{3} \left(R_{\mu\alpha\beta\nu} + R_{\mu\beta\alpha\nu}\right)\big|_{Q}.$$

Metric tensor $g_{\mu\nu}$ in the vicinity of point Q can be approximated by Minkowski metric $\eta_{\mu\nu}$. The Taylor expansion of the metric $g_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu,\alpha\beta}\, y^{\alpha} y^{\beta} + ...$

$$\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu}\big|_{Q} - \frac{1}{3} R_{\mu\alpha\nu\beta}\big|_{Q} y^{\alpha} y^{\beta} + -\frac{1}{6} R_{\mu\alpha\nu\beta;\gamma}\big|_{Q} y^{\alpha} y^{\beta} y^{\gamma} \\ &\quad + \left(-\frac{1}{20} R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45} R_{\alpha\mu\beta\lambda} R^{\lambda}_{\gamma\nu\delta}\right)\big|_{Q} y^{\alpha} y^{\beta} y^{\gamma} y^{\delta} ... \\ g(x) &= 1 - \frac{1}{3} R_{\alpha\beta}\big|_{Q} y^{\alpha} y^{\beta} - \frac{1}{6} R_{\alpha\beta;\gamma}\big|_{Q} y^{\alpha} y^{\beta} y^{\gamma} \\ &\quad + \left(\frac{1}{18} R_{\alpha\beta} R_{\gamma\delta} - \frac{1}{90} R_{\lambda\alpha\beta}{}^{\kappa} R_{\lambda\gamma\delta\kappa} - \frac{1}{20} R_{\alpha\beta;\gamma\delta}\right)\big|_{Q} y^{\alpha} y^{\beta} y^{\gamma} y^{\delta} ... . \end{aligned} \tag{F.3}$$

We use $\eta_{\mu\nu}$ to raise or lower the indices. Inverse of metric tensor can be similarly expanded,

$$g^{\mu\nu}(x) = \eta^{\mu\nu}\Big|_Q + \frac{1}{3}\, R^{\mu\ \nu}_{\ \alpha\ \beta}\Big|_Q\, y^\alpha y^\beta + \dots\ .$$

We have skipped intermediate steps one of which is putting non-linear terms $\partial_\alpha g_{\mu\nu} = 0$ at Q. One can derive second derivatives of metric tensor from the expression for derivatives of Christoffel Symbols. The next order terms are of 3rd order. Covariant propagator in momentum space expanded up to adiabatic fourth order for the coefficients, is given below.

$$\begin{aligned} G(k) \approx & \frac{1}{k^2-m^2} - \left(\frac{1}{6}-\xi\right)\frac{R}{(k^2-m^2)^2} + \frac{i}{2}\left(\frac{1}{6}-\xi\right)R_{;\alpha}\frac{\partial}{\partial k_\alpha}\frac{1}{(k^2-m^2)^2} \\ & - \frac{1}{2}a_{\alpha\beta}\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial k_\beta}\frac{1}{(k^2-m^2)^2} + \frac{\left(\frac{1}{6}-\xi\right)^2 R^2 + 2a^\lambda_\lambda}{(k^2-m^2)^3} \end{aligned} \tag{F.4}$$

where,

$$\begin{aligned} a_{\alpha\beta} = & \frac{1}{2}\left(\xi - \frac{1}{6}\right)R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}(R_{\alpha\beta;\lambda})^{;\lambda} - \frac{1}{30}R^\lambda_\alpha R_{\lambda\beta} \\ & + \frac{1}{60}R^{\kappa\ \lambda}_{\ \alpha\ \beta}R_{\kappa\lambda} + \frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta}. \end{aligned} \tag{F.5}$$

The systematic analysis of divergences and renormalization is done in DeWitt-Schwinger representation of the Feynman propagator. Using this, we can separate the divergent terms and then discuss the renormalization. We list the standard results from [38, 34] (also see [35] and we also refer to original papers like, [33, 44] where one can find detailed derivation of these results). To start with, in integral representation,

$$\frac{1}{k^2-m^2} \equiv -i\int_0^\infty ds e^{is(k^2-m^2)}.$$

Feynman propagator of equation in DeWitt-Schwinger (DS) representation is given by (assuming x is a point in the near vicinity of x'),

$$G_{DS}(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi)^{(d+1)/2}}\int_0^\infty ds(is)^{-(d+1)/2}\exp\left\{-im^2 s + \left(\frac{\sigma(x,x')}{2is}\right)\right\}F(x,x';is),$$

where s is a scalar parameter. The function $\sigma(x,x') = 1/2\ y^\alpha y_\alpha$ is half of the proper distance between x and x'; Δ is Van Vleck determinant and the function $F(x,x';is)$ has asymptotic adiabatic expansion,

$$F(x,x';is) = a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + \dots \approx \sum_{j=0}^\infty a_j(x,x')(is)^j.$$

The divergence can arise for $d \leq 2$ because we have,

$$G^{DS}(x,x') \sim \int_0^\infty ds (is)^{-(d+1)/2} e^{-im^2 s + \left(\frac{\sigma}{2is}\right)} \left[a_0(x,x') + a_1(x,x') is + a_2(x,x')(is)^2\right],$$

where the function σ is the half of proper distance between x and x'.

In 'coincidence' limit, we have,

$$\begin{aligned}
a_0(x) &= 1 \\
a_1(x) &= \left(\frac{1}{6} - \xi\right) R \\
a_2(x) &= \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2.
\end{aligned}$$

G Bibliography

[1] Joan Solà Peracaula. *Cosmological constant and vacuum energy: old and new ideas*. J.Phys.Conf.Ser. 453 (2013) 012015, arXiv:1306.1527.

[2] Adrià Gómez-Valent. *Vacuum energy in Quantum Field Theory and Cosmology*. Ph.D. Thesis, Departament de Física Quàntica i Astrofísica Universitat de Barcelona, arXiv:1710.01978.

[3] Joan Solà Peracaula. *Brans-Dicke gravity: from Higgs physics to (dynamical) dark energy*. Int.J.Mod.Phys. D27 (2018) no.14, 1847029, doi.org/10.1142/S0218271818470296.

[4] Javier de Cruz Perez and Joan Solà Peracaula. *Brans-Dicke cosmology mimicking running vacuum*. Mod. Phys. Lett. A33 (2018) 1850228, doi.org/10.1142/S0217732318502280.

[5] Joan Solà Peracaula, Adria Gomez-Valent, Javier de Cruz Perez, and Cristian Moreno-Pulido. *Running vacuum against the H0 and sigma8 tensions*. EPL 134 (2021) 19001, doi.org/10.1209/0295-5075/134/19001.

[6] Javier de Cruz Perez, Joan Solà Peracaula, Adria Gomez-Valent, and Cristian Moreno-Pulido. *BD-LCDM and Running Vacuum Models: Theoretical background and current observational status*. arXiv:2110.07569.

[7] Javier de Cruz Perez, Joan Solà Peracaula, Adria Gomez-Valent, and Cristian Moreno-Pulido. *Brans-Dicke cosmology with a Λ-term: a possible solution to ΛCDM tensions*. Class. Quantum Grav. 37 (2020) 245003, doi.org/10.1088/1361-6382/abbc43.

[8] Claudia de Rham and Raquel H. Ribeiro. *Riding on irrelevant operators*. JCAP11(2014)016, arXiv:1405.5213.

[9] Antonio Padilla and Ippocratis D. Saltas. *Vainshtein in the UV and a Wilsonian analysis of derivatively coupled scalars*. JCAP06(2018)039, doi.org/10.1088/1475-7516/2018/06/039.

[10] Adrià Gómez-Valent and Prajwal Hassan Puttasiddappa. Difficulties in reconciling non-negligible differences between the local and cosmological values of the gravitational coupling in extended Brans-Dicke theories. *JCAP*, 09:040, 2021.

[11] Luca Amendola and Shinji Tsujikawa. *Dark Energy: Theory and Observations.* Cambridge University Press (2010), ISBN: 9781107453982.

[12] Peter Rastall. *Generalization of the Einstein Theory.* Phys. Rev. D 6, 3357, doi.org/10.1103/PhysRevD.6.3357, 1972.

[13] Michael E Peskin and Daniel V Schroeder. *An Introduction to quantum field theory.* Addison-Wesley, Reading, USA, 1995.

[14] Kher Sham Limy. *Planck Scale Boundary Conditions And The Standard Model.* This diploma thesis has been carried out at the Max Planck Institute for Nuclear Physics under the supervision of Prof. Dr. Manfred Lindner, 2011.

[15] Gustavo Burdman. *Lecture Notes on Quantum Field Theory II.* Department of Mathematical Physics, University of Sao Paulo, 2021.

[16] David Skinner. *Lecture notes: Quantum Field Theory II.* Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

[17] Markus A. Luty. *Lecture Notes on Renormalization.* Physics 851 Notes, Spring 2007.

[18] Iain W Stewart. *Effective Field Theory, EFT Course 8.851, Lecture Notes.* Massachusetts Institute of Technology, (Online course on edX), doi.org/10.1098/rsta.2011.0103.

[19] David J Griffiths. *Introduction to elementary particles; 2nd rev. version.* Physics textbook. Wiley, New York, NY, 2008.

[20] Sidney Coleman and Erick Weinberg. *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking.* Phys. Rev. D 7, 1888 (1973), doi:https://doi.org/10.1103/PhysRevD.7.1888.

[21] Axel Maas. *Brout-Englert-Higgs physics: From foundations to phenomenology.* Progress in Particle and Nuclear Physics Volume 106, May 2019, Pages 132-209, doi.org/10.1016/j.ppnp.2019.02.003.

[22] Shmuel Elitzur. *Impossibility of spontaneously breaking local symmetries.* Phys. Rev. D 12, 3978 (1975), doi.org/10.1103/PhysRevD.12.3978.

[23] Csaba Csáki and Philip Tanedo. *Beyond the Standard Model.* 2013 European School of High-Energy Physics, Paradfurdo, Hungary, 5 - 18 Jun 2013, pp.169-268 (CERN-2015-004), doi.org/10.5170/CERN-2015-004.169.

[24] Leonard Susskind. *The Gauge Hierarchy Problem, Technicolor, Supersymmetry, and all that.* Review Section of Phys.Rept. 104 (1984) 181-193, 10.1016/0370-1573(84)90208-4.

[25] Joerg Jaeckel, Viraf M. Mehta, and Lukas T. Witkowski. *Musings on cosmological relaxation and the hierarchy problem*. Phys. Rev. D 93, 063522 (2016), doi.org/10.1103/PhysRevD.93.063522.

[26] Csaba Csáki, Salvator Lombardo, and Ofri Telem. *TASI Lectures on Non-Supersymmetric BSM Models*. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics : Anticipating the Next Discoveries in Particle Physics (TASI 2016), 501-570.

[27] Tommi Markkanen, Arttu Rajantie, and Stephen Stopyra. *Cosmological Aspects of Higgs Vacuum Metastability*. Front. Astron. Space Sci. 5 (2018) 40, doi.org/10.3389/fspas.2018.00040.

[28] Vincenzo Branchina, Emanuele Messina, and Marc Sher. *The lifetime of the electroweak vacuum and sensitivity to Planck scale physics*. Phys. Rev. D 91, 013003 (2015), doi.org/10.1103/PhysRevD.91.013003.

[29] Dario Buttazzo, Giuseppe Degrassi, Pier Paolo Giardino, Gian F. Giudice, Filippo Sala, Alberto Salvio, and Alessandro Strumia. *Investigating the near-criticality of the Higgs boson*. J. High Energ. Phys. 2013, 89 (2013), doi.org/10.1007/JHEP12(2013)089.

[30] Gino Isidori, Giovanni Ridolfi, and Alessandro Strumia. *On the metastability of the Standard Model vacuum*. Nucl.Phys.B609:387-409,2001, doi.org/10.1016/S0550-3213(01)00302-9.

[31] Masaharu Tanabashi et al. (Particle Data Group). *Status of Higgs Boson Physics in Review of Particle Physics*. Phys. Rev. D 98, 030001 (2018), doi.org/10.1103/PhysRevD.98.030001.

[32] Jerome Martin. *Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask)*. Comptes Rendus Physique 13, 566 (2012), arXiv:1205.3365.

[33] T. S. Bunch and Leonard Parker. *Feynman propagator in curved spacetime: A momentum-space representation*. Phys. Rev. D 20, 2499, 15 November 1979, doi.org/10.1103/PhysRevD.20.2499.

[34] Leonard E Parker and David J. Toms. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge University Press, 2009, ISBN: 9780521877879.

[35] P. C. W. Davies N. D. Birrell. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge University Press, Online publication date: August 2012, Print publication year: 1982, ISBN: 9780511622632.

[36] Iosif L. Buchbinder and Ilya Shapiro. *Introduction to Quantum Field Theory with Applications to Quantum Gravity*. Oxford University Press., Print ISBN-13: 9780198838319.

[37] Lawrence H. Ford. *Quantum Theory in Curved Spacetime*. Lectures given at the IX Jorge Andr´e Swieca Summer School, Campos dos Jord˜ao, SP, Brazil, February 1997, and at Soochow University, Taipei, Taiwan, June 1997, arXiv:gr-qc/9707062.

[38] Aram A. Saharian. *Lecture notes on Quantum Field Theory in Curved Spacetime*. Department of Physics, Yerevan State University.

[39] Gerard 't Hooft and M. J. G. Veltman. One loop divergencies in the theory of gravitation.

[40] John F. Donoghue, Mikhail M. Ivanov, and Andrey Shkerin. *EPFL Lectures on General Relativity as a Quantum Field Theory*. 2017.

[41] Valerio Faraoni. *Conformally Coupled Inflation*. Galaxies 2013, 1, 96-106; ISSN 2075-4434, doi:10.3390/galaxies1020096.

[42] Valerio Faraoni. *A Crucial Ingredient of Inflation*. International Journal of Theoretical Physics, Vol. 40, No. 12, December 2001.

[43] Sebastiano Sonego and Valerio Faraoni. *Coupling to the curvature for a scalar field from the equivalence principle*. Class. Quantum Grav. 10 1185 (1993).

[44] Leonard Parker and S. A. Fulling. *Adiabatic regularization of the energy-momentum tensor of a quantized field in homogeneous spaces*. Phys. Rev. D 9, 341 (1974), doi.org/10.1103/PhysRevD.9.341.

[45] John S Dowker and Raymond Critchley. *Effective Lagrangian and energy-momentum tensor in de Sitter space*. Phys. Rev. D 13, 3224 (1976), doi.org/10.1103/PhysRevD.13.3224.

[46] Stephen L Adler, JudyLieberman, and Yee JackNg. *Regularization of the stress-energy tensor for vector and scalar particles propagating in a general background metric*. Annals of Physics, Volume 106, Issue 2, August 1977, Pages 279-321, doi.org/10.1016/0003-4916(77)90313-X.

[47] Arthur Hebecker. *Lectures on Naturalness, String Landscape and Multiverse*. Part of the Lecture Notes in Physics book series (LNP, volume 979), Springer, https://link.springer.com/book/10.1007/978-3-030-65151-0.

[48] J. A. S. Lima, Spyros Basilakos, and Joan Sola. *Expansion History with Decaying Vacuum: A Complete Cosmological Scenario*. Monthly Notices of the Royal Astronomical Society, Volume 431, Issue 1, 1 May 2013, Pages 923–929,, doi.org/10.1093/mnras/stt220.

[49] E. L. D. Perico, J. A. S. Lima, Spyros Basilakos, and Joan Sola. *Complete Cosmic History with a dynamical Lambda(H) term*. Phys. Rev. D 88, 063531, (2013), doi.org/10.1103/PhysRevD.88.063531.

[50] Tsutomu Kobayashi. *Horndeski theory and beyond: a review*. Rept.Prog.Phys. 82 (2019) no.8, 086901, doi.org/10.1088/1361-6633/ab2429.

[51] Alexei A. Starobinsky. *A new type of isotropic cosmological models without singularity*. Phys.Lett.B 91 (1980) 99-102, doi.org/10.1016/0370-2693(80)90670-X.

[52] David Lovelock. *The Einstein tensor and its generalizations*. J. Math. Phys. 12, 498 (1971), doi.org/10.1063/1.1665613.

[53] David Lovelock. *The four-dimensionality of space and the einstein tensor*. J. Math. Phys. 13, 874 (1972), doi.org/10.1063/1.1666069.

[54] Th. Kaluza. Zum Unitätsproblem der Physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1921:966–972, 1921.

[55] Oskar Klein. *Quantentheorie und fünfdimensionale Relativitätstheorie*. Zeitschrift für Physik volume 37, pages895–906 (1926), doi.org/10.1007/BF01397481.

[56] Y. Fujii and K. Maeda. *The scalar-tensor theory of gravitation*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 7 2007.

[57] Gia Dvali, Gregory Gabadadze, and Massimo Porrati. *4D Gravity on a Brane in 5D Minkowski Space*. Phys.Lett.B485:208-214,2000, doi.org/10.1016/S0370-2693(00)00669-9.

[58] Nima Arkani-Hamed, Hsin-Chia Cheng, Markus A. Luty, and Shinji Mukohyama. *Ghost Condensation and a Consistent Infrared Modification of Gravity*. JHEP 0405 (2004) 074, doi.org/10.1088/1126-6708/2004/05/074.

[59] C. Deffayet, G. Esposito-Farèse, , and A. Vikman. *Covariant Galileon*. Phys. Rev. D 79, 084003 (2009), doi.org/10.1103/PhysRevD.79.084003.

[60] The Virgo Collaboration The LIGO Scientific Collaboration. *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*. Phys. Rev. Lett. 119 161101 (2017), doi.org/10.1103/PhysRevLett.119.161101.

[61] C. D. Kreisch and E. Komatsu. *Cosmological Constraints on Horndeski Gravity in Light of GW170817*. JCAP12(2018)030, doi.org/10.1088/1475-7516/2018/12/030.

[62] Jose María Ezquiaga and Miguel Zumalacárregui. *Dark Energy after GW170817: dead ends and the road ahead*. Phys. Rev. Lett. 119, 251304 (2017), doi.org/10.1103/PhysRevLett.119.251304.

[63] Paolo Creminelli and Filippo Vernizzi. *Dark Energy after GW170817*. Phys. Rev. Lett. 119, 251302 (2017), doi.org/10.1103/PhysRevLett.119.251302.

[64] Alexandru Dima and Filippo Vernizzi. *Vainshtein Screening in Scalar-Tensor Theories before and after GW170817: Constraints on Theories beyond Horndeski*. Phys. Rev. D 97, 101302 (2018), doi.org/10.1103/PhysRevD.97.101302.

[65] Ryotaro Kase and Shinji Tsujikawa. *Dark energy in Horndeski theories after GW170817: A review*. Int. J. Mod. Phys. D28 (2019) no.05, 1942005, doi.org/10.1142/S0218271819420057.

[66] Gregory Walter Horndeski. *Second-order scalar-tensor field equations in a four-dimensional space*. Int. J. Theor. Phys. 10 (1974) 363, doi.org/10.1007/BF01807638.

[67] Christos Charmousis, Edmund J. Copeland, Antonio Padilla, and Paul M. Saffin. *General second order scalar-tensor theory, self tuning, and the Fab Four*. Phys. Rev. Lett. 108, 051101, doi.org/10.1103/PhysRevLett.108.051101, 2012.

[68] C. Brans and R.H. Dicke. *Mach's principle and a relativistic theory of gravitation*. Phys. Rev., 124, 925 (1961), doi.org/10.1103/PhysRev.124.925.

[69] C. Brans. *Mach's principle and a relativistic theory of gravitation. II*. Phys. Rev., 125, 2194 (1962), doi.org/10.1103/PhysRev.125.2194.

[70] R.H. Dicke. *Mach's principle and Invariance under transformation of unit*. Phys. Rev., 125, 2163 (1962), doi.org/10.1103/PhysRev.125.2163.

[71] Alberto Nicolis, Riccardo Rattazzi, and Enrico Trincherini. *The galileon as a local modification of gravity*. Phys.Rev.D79:064036,2009, doi.org/10.1103/PhysRevD.79.064036.

[72] Claudia de Rham and Lavinia Heisenberg. *Cosmology of the Galileon from Massive Gravity*. Phys.Rev.D84:043503,2011, doi.org/10.1103/PhysRevD.84.043503.

[73] B. Bertotti, L. Iess, and P. Tortora. A test of general relativity using radio links with the Cassini spacecraft. *Nature*, 425:374–376, 2003.

[74] C. Skordis A. Avilez. *Cosmological constraints on Brans-Dicke theory*. Phys. Rev. Lett. 113, 011101 (2014), doi.org/10.1103/PhysRevLett.113.011101.

[75] Wendy L. Freedman. *Measurements of the Hubble Constant: Tensions in Perspective*. ApJ 919 16 (2021), doi.org/10.3847/1538-4357/ac0e95.

[76] Planck Collaboration. *Planck 2018 results. VI. Cosmological parameters*. A&A 641, A6 (2020), doi.org/10.1051/0004-6361/201833910.

[77] Adam G. Riess et al. *Cosmic Distances Calibrated to 1 percent Precision with Gaia EDR3 Parallaxes and Hubble Space Telescope Photometry of 75 Milky Way Cepheids Confirm Tension with LambdaCDM*. ApJL 908 L6 (2021), doi.org/10.3847/2041-8213/abdbaf.

[78] Luca Amendola and Miguel Quartin. *Measuring the Hubble function with standard candle clustering*. MNRAS 504 (3), 3884-3889 (2021), doi.org/10.1093/mnras/stab887.

[79] S. Eidelman et al. Review of particle physics. Particle Data Group. *Phys. Lett. B*, 592(1-4):1, 2004.

[80] Darren Croton. *Damn You, Little h! (or, Real-World Applications Of The Hubble Constant Using Observed And Simulated Data)*. Publications of the Astronomical Society of Australia, 30, E052, doi.org/10.1017/pasa.2013.31.

[81] Ariel G. Sanchez. *Arguments against using h^{-1}Mpc units in observational cosmology*. Phys. Rev. D 102, 123511, doi.org/10.1103/PhysRevD.102.123511.

[82] Eleonora Di Valentino, Olga Mena, Supriya Pan, Luca Visinelli, Weiqiang Yang, Alessandro Melchiorri, David F. Mota, Adam G. Riess, and Joseph Silk. *In the Realm of the Hubble tension − a Review of Solutions*. Class. Quantum Grav. 38, 153001 (2021), doi.org/10.1088/1361-6382/ac086d.

[83] A. I. Vainshtein. To the problem of nonvanishing gravitation mass. *Phys. Lett. B*, 39:393–394, 1972.

[84] E. Babichev, C. Deffayet, and R. Ziour. k-Mouflage gravity. *Int. J. Mod. Phys. D*, 18:2147–2154, 2009.

[85] Kazuya Koyama. Testing Brans-Dicke Gravity with Screening by Scalar Gravitational Wave Memory. *Phys. Rev. D*, 102(2):021502, 2020.

[86] Fabian Schmidt, Wayne Hu, and Marcos Lima. Spherical Collapse and the Halo Model in Braneworld Gravity. *Phys. Rev. D*, 81:063005, 2010.

[87] Justin Khoury and Amanda Weltman. Chameleon cosmology. *Phys. Rev. D*, 69:044026, 2004.

[88] Kurt Hinterbichler and Justin Khoury. Symmetron Fields: Screening Long-Range Forces Through Local Symmetry Restoration. *Phys. Rev. Lett.*, 104:231301, 2010.

[89] Stefan Floerchinger and Christof Wetterich. *Lectures on Quantum Field Theory 2*. Summer Semester (2019).

[90] Manuel Reichert. *Lecture notes: Functional Renormalisation Group and Asymptotically Safe Quantum Gravity*. XV Modave Summer School in Mathematical Physics, 8-14 September 2019, Modave, Belgium.

[91] Matthias Neubert. *Effective Field Theory and Heavy Quark Physics*. Lecture notes for Physics in D>=4, Theoretical Advanced Study Institute (TASI) 2004, arXiv:hep-ph/0512222.

[92] Kenneth G.Wilson and John Kogut. *The renormalization group and the expansion*. Physics Reports, Volume 12, Issue 2, August 1974, Pages 75-199, doi.org/10.1016/0370-1573(74)90023-4.

[93] Curtis G. Callan. Jr. *Broken Scale Invariance in Scalar Field Theory*. Phys. Rev. D 2, 1541, 15 October 1970, doi.org/10.1103/PhysRevD.2.1541.

[94] Kurt Symanzik. *Small distance behaviour in field theory and power counting*. Communications in Mathematical Physics volume 18, pages227–246 (1970), doi.org/10.1007/BF01649434.

[95] Daniel F. Litim. *Renormalisation group and the Planck scale*. Phil. Trans. R. Soc. A 369 (2011)2759-2778, doi.org/10.1098/rsta.2011.0103.

[96] Philippe Brax and Patrick Valageas. *Quantum Field Theory of K-mouflage*. Phys. Rev. D 94, 043529 (2016), doi.org/10.1103/PhysRevD.94.043529.

[97] Federico Piazza and Shinji Tsujikawa. *Dilatonic ghost condensate as dark energy*. JCAP 0407 (2004) 004, doi.org/10.1088/1475-7516/2004/07/004.

[98] Simone Peirone, Giampaolo Benevento, Noemi Frusciante, and Shinji Tsujikawa. *Cosmological data favor Galileon ghost condensate over LCDM*. Phys. Rev. D 100, 063540 (2019), doi.org/10.1103/PhysRevD.100.063540.

[99] Eva Silverstein and David Tong. *Scalar speed limits and cosmology: Acceleration from D-cceleration*. Phys. Rev. D 70, 103505, 8 November, 2004, doi.org/10.1103/PhysRevD.70.103505.

[100] Changrim Ahn, Chanju Kim, and Eric V. Linder. *Dark Energy Properties in DBI Theory*. Phys.Rev.D80:123016,2009, doi.org/10.1103/PhysRevD.80.123016.

[101] Changrim Ahn, Chanju Kim, and Eric V. Linder. *Cosmological Constant Behavior in DBI Theory*. Phys. Lett. B684:181-184,2010, doi.org/10.1016/j.physletb.2009.12.069.

[102] C. Armendariz-Picon, T. Damour, and V. Mukhanov. *k-Inflation*. Phys.Lett.B458:209-218,1999, doi.org/10.1016/S0370-2693(99)00603-6.

[103] Anderi Barvinskyand Grigorii A. Vilkovisky. *The generalized Schwinger-Dewitt technique in gauge theories and quantum gravity*. Physics Reports, Volume 119, Issue 1, March 1985, Pages 1-74, doi.org/10.1016/0370-1573(85)90148-6.

[104] Anderi Barvinskyand Grigorii A. Vilkovisky. *Covariant perturbation theory (II). Second order in the curvature. General algorithms*. Nuclear Physics B, Volume 333, Issue 2, 26 March 1990, Pages 471-511, doi.org/10.1016/0550-3213(90)90047-H.

[105] Nemanja Kaloper and Antonio Padilla. *Sequestering the Standard Model Vacuum Energy*. Phys.Rev.Lett. 112 (2014) 091304, arXiv:1309.6562.

[106] Lavinia Heisenberg and Christian F. Steinwachs. One-loop renormalization in Galileon effective field theory. *JCAP*, 01:014, 2020.

[107] Lavinia Heisenberg. Quantum Corrections in Galileons from Matter Loops. *Phys. Rev. D*, 90(6):064005, 2014.

[108] Lavinia Heisenberg and Christian F. Steinwachs. Geometrized quantum Galileons. *JCAP*, 02:031, 2020.

[109] Garrett Goon, Kurt Hinterbichler, Austin Joyce, and Mark Trodden. *Aspects of Galileon Non-Renormalization*. JHEP 11 (2016) 100, doi.org/10.1007/JHEP11(2016)100.

[110] Christian F. Steinwachs. Non-perturbative quantum Galileon in the exact renormalization group. *JCAP*, 04:038, 2021.

[111] Christof Wetterich. *Exact evolution equation for the effective potential*. Physics Letters B, Volume 301, Issue 1, February 1993, Pages 90-94, doi.org/10.1016/0370-2693(93)90726-X.

[112] Eberhard Engel and Reiner M. Dreizler. *Appendix A in Density Functional Theory*. Springer, Part of the Theoretical and Mathematical Physics book series (TMP), 2011.

[113] A. Codello, N. Tetradis, and O. Zanusso. *The renormalization of fluctuating branes, the Galileon and asymptotic safety*. JHEP 1304 (2013) 036, doi.org/10.1007/JHEP04(2013)036.

[114] Israel Quiros, Ricardo García-Salcedo, Tame Gonzalez, F. Antonio Horta-Rangel, and Joel Saavedra. Brans–Dicke Galileon and the variational principle. *Eur. J. Phys.*, 37(5):055605, 2016.

[115] Karen Crowther and Niels Linnemann. *Renormalizability, fundamentality and a final theory: The role of UV-completion in the search for quantum gravity*. The British Journal for the Philosophy of Science, Volume 70, Number 2, June 2019, doi.org/10.1093/bjps/axx0524.

[116] Steven Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley and Sons, New York, 1972.

[117] Sean M. Carroll. *Spacetime and Geometry*. Cambridge University Press, 7 2019.

[118] Jiri Bicak. *Selected solutions of Einstein's field equations: their role in general relativity and astrophysics*. Lect.Notes Phys. 540 (2000) 1-126, arXiv: gr-qc/0004016.

[119] Robert M. Wald. *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984.

[120] Leo Brewin. *Riemann Normal Coordinates, Smooth Lattices and Numerical Relativity*. Class.Quant.Grav. 15 (1998) 3085-3120, doi.org/10.1088/0264-9381/15/10/014, also see his *collection of notes on Riemann normal coordinates*.

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 31.03.2022 ....