Eur. Phys. J. Plus (2025) 140:301
https://doi.org/10.1140/epjp/s13360-025-06235-x

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Quantum expansion of geodesic congruences in Schwarzschild anti-De Sitter spacetime

Osmar E. López-Pérez[a], **Miguel Socolovsky**[b]

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Cd. Universitaria, 04510 Mexico, Mexico

Received: 7 March 2025 / Accepted: 16 March 2025

Abstract We evaluate the Raychaudhuri equation for radial ingoing and outgoing null and timelike geodesic congruences in specific families of metrics. Subsequently, we compute the Feynman propagator specifically for the Schwarzschild anti-De Sitter spacetime. Here, we find that the classical expansion scalars diverge at the singularity. Interestingly, while the propagators for null geodesics remain finite ingoing to the singularity, those for timelike geodesics exhibit divergence.

1 Introduction

It is well known that the Raychaudhuri equation is an identity in Riemannian geometry and therefore lacks independent dynamical content. Only when Einstein's field equations are introduced does the Raychaudhuri equation gain significance in describing the dynamics of spacetime [1]. In this framework, it plays an important role in the proof of classical singularities within general relativity due to the focusing theorem, as demonstrated by Penrose and Hawking [2, 3]. Consequently, studying the Raychaudhuri equation in quantum frameworks provides promising avenues for understanding the nature of singularities and exploring potential resolutions at the quantum level.

Das [4] proposed a quantum version of the Raychaudhuri equation by utilizing Bohmian trajectories, illustrating that geodesic focusing can be prevented when a quantum potential is considered. The broader applications of this quantum formulation to cosmological and black hole spacetimes have been discussed in [5, 6].

Meanwhile, Alsaleh et al. [7] proposed an approach that frames a geodesic congruence as a dynamical system, using ρ as the dynamical variable, where ρ represents the square root of the determinant of the metric induced on a hypersurface. They reformulated the Raychaudhuri equation in terms of ρ and constructed a Lagrangian such that the Raychaudhuri equation arises naturally as the Euler-Lagrange equation. This framework enables a straightforward quantization: they introduced the canonical conjugate momentum associated with ρ, defined a classical Hamiltonian, and promoted both ρ and its conjugate momentum to operators to satisfy the canonical quantization rule $[\hat{\rho}, \hat{\pi}] = i\hbar$. This approach allows them to derive the Hamiltonian operator for quantum geodesic congruences directly.

Gupta et al. [8] later identified an error in one of the equations in the Alsaleh's work, affecting the quantization process and limiting its validity to $(2 + 1)$ dimensions. In their work establish a "correct" quantum version of the Raychaudhuri equation based on Alsaleh et al.'s approach, and concluded that a general application is not feasible. As they noted, "The basic reason behind this is that the Raychaudhuri equation is actually an identity in Riemannian geometry, and naturally is not derived as the equation of motion for a geodesic congruence from a variational principle."

Another approach to quantize geodesic congruences was developed by Socolovsky [9]. His method involves a change in variables that reduces the Raychaudhuri equation to a one-dimensional harmonic oscillator equation in $F(\lambda)$ with a *time*-dependent frequency $\Omega(\lambda)$ [10]. Since Ω is generally non periodic, the equation is not a standard Hill equation but instead belongs to a class of "Hill-type" Eqs. [11]. After defining a λ-dependent Lagrangian leading to the oscillator equation, a Feynman path integral approach [12, 13] is used to construct a propagator $K(F'', \lambda''; F', \lambda')$, which describes the evolution from initial conditions (F', λ') to final conditions (F'', λ'') of the affine parameter and the expansion function F. This propagator, representing a functional integration over all fluctuations in F along its classical trajectory, provides a quantum description of the congruence flow. Examples of finite propagators have been demonstrated for null geodesics reaching a classical singularity. [9, 14]. In this paper, we focus on the Socolovsky's method to explore some space times with classical singularities and test if the propagators associated with radial geodesics ingoing to the singularity are finite or not.

[a] e-mail: osmar.lopez@correo.nucleares.unam.mx (corresponding author)
[b] e-mail: socolovs@nucleares.unam.mx

Published online: 14 April 2025

The paper is organized as follows: in Sect. 2, we review the key ideas presented in [9]. In Sect. 3, we introduce the Raychaudhuri equation for a general spacetime and provide some examples. Section 4 focuses on the Schwarzschild anti-de Sitter spacetime, showing that while propagators for null geodesics remain finite, those for timelike geodesics diverge. Finally, Section 5 presents our conclusions and discussions.

2 Propagators for geodesic congruences

Let $v = (v^\mu)$, $v^\mu = \frac{dx^\mu}{d\lambda}$, be the vector field tangent to an affinely parametrized timelike (T.L.) or null (N) geodesic congruence (λ is the affine parameter) in a 4- dimensional spacetime with local coordinates x^μ, $\mu = 0, 1, 2, 3$, metric $g_{\mu\nu}$, Levi-Civita connection, Ricci tensor $R_{\mu\nu}$, and covariant derivative $D = (D_\mu)$. v obeys equation

$$v \cdot D(v^\mu) = v^\nu D_\nu(v^\mu) = v^\nu(\partial_\nu v^\mu + \Gamma^\mu_{\nu\rho} v^\rho) = 0 \tag{1}$$

with normalization $v^2 = g_{\mu\nu} v^\mu v^\nu = +1$ (0) in the T.L. (N) case (we use the signature $(+, -, -, -)$). The *expansion* of the congruence, that is, the fractional rate of change in the cross-sectional volume (area) to the congruence in the T.L. (N) case, is the scalar

$$\Theta = D \cdot v = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} v^\mu), \tag{2}$$

where $g = det(g_{\mu\nu})$. Through pure geometrical identities, Θ can be shown to obey the Raychaudhuri equation (a Riccati equation)

$$\frac{d\Theta}{d\lambda} = -\frac{1}{n}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} v^\mu v^\nu \tag{3}$$

where $n = 3$ (2) in the T.L. (N) case; $\sigma_{\mu\nu}$ (shear, which measures the change in shape of the congruence without modification of its volume in the T.L. case, or of its area in the N case) and $\omega_{\mu\nu}$ (rotation) are, respectively, the traceless symmetric and antisymmetric parts of the tensor

$$B_{\mu\nu} = D_\nu v_\mu, \tag{4}$$

so that $\Theta = B^\mu_{;\mu}$. One has the decomposition

$$B_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{n}\Theta h_{\mu\nu}, \tag{5}$$

where $h_{\mu\nu}$ is the transverse metric (part of $g_{\mu\nu}$ orthogonal to v) given by $g_{\mu\nu} - v_\mu v_\nu$ in the T.L. case and $g_{\mu\nu} - (v_\mu n_\nu + v_\nu n_\mu)$ in the N case (n^μ is a null vector satisfying $v \cdot n = +1$).

(3) is a purely geometrical equation; its physical meaning only comes after relating the Ricci tensor to the energy-momentum tensor $T_{\mu\nu}$ through the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{6}$$

where $R = R^\mu_\mu$ and Λ is the cosmological constant; finally, all the terms in (3) depend on λ through the x^μ's. In the vacuum, $T_{\mu\nu} = \Lambda = 0$, implying $R_{\mu\nu} = 0$.

(Units: In the geometrical system, $G = c = 1$, so if $[\lambda] = [L]$, then $[\Theta] = [L]^{-1}$, $[\sigma]^2 = [\omega]^2 = [R_{\mu\nu}] = [L]^{-2}$, and $[v] = [L]^0$.)

2.1 Frequency dependent harmonic oscillator

In terms of the function $F(\lambda)$ defined by [10]

$$\Theta(\lambda) = \frac{1}{F^n(\lambda)} \frac{d(F^n(\lambda))}{d\lambda} = n\frac{\dot{F}(\lambda)}{F(\lambda)}, \tag{7}$$

the Raychoudhuri equation (3) becomes

$$\ddot{F}(\lambda) + (\Omega(\lambda))^2 F(\lambda) = 0 \tag{8}$$

with

$$\Omega^2 = \frac{1}{n}(\sigma^2 - \omega^2 + R_{\mu\nu} v^\mu v^\nu), \tag{9}$$

which is nothing but the equation of a classical 1-dimensional harmonic oscillator with λ ("time")-dependent frequency Ω. After Hill [11], (7) is known as a "Hill-type" equation. If at $\lambda = \lambda_0$ the congruence converges to a point i.e., $\Theta(\lambda)$ has a caustic: $\Theta(\lambda) \to -\infty$ as $\lambda \to \lambda_0$, then λ_0 must be a zero of $F(\lambda)$ if $\dot{F}(\lambda_0)$ is finite.

(8) is the Euler-Lagrange equation of the "time"-dependent Lagrangian

$$\mathcal{L}(F, \dot{F}, \lambda) = \frac{1}{2}(\dot{F}^2 - \Omega^2 F^2). \tag{10}$$

For a suitable domain of definition of λ, (8) admits a solution $\bar{F}(\lambda)$ subject to the boundary conditions $F' = \bar{F}(\lambda')$ and $F'' = \bar{F}(\lambda'')$ with, e.g., $\lambda' < \lambda''$.

(Units: $[F]$=$[L]^{1/2}$ since [action]=$[\int d\lambda \mathcal{L}] = [L][\mathcal{L}] = [L]^0$.)

2.2 Path integrals and time-dependent quadratic Lagrangians

It is well known [12, 13] that a Lagrangian of the form

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}((\dot{x}(t))^2 - b(t)(x(t))^2) \tag{11}$$

has associated with it a exactly defined propagator $K(x'', t''; x', t')$ from the quantum state $|x', t' >$ to the quantum state $|x'', t'' >$ given by the path integral

$$\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) e^{i \int_{t'}^{t''} dt \mathcal{L}(x, \dot{x}, t)}, \tag{12}$$

($\hbar = 1$) where, formally,

$$\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) ... = \prod_{t \in (t', t'')} \int_{-\infty}^{+\infty} dx(t) .... \tag{13}$$

The result is

$$K(x'', t''; x', t') = (2\pi i h(t'', t'))^{-1/2} e^{i S[\bar{x}]}, \tag{14}$$

where $\bar{x}(t)$ is the solution of

$$\ddot{x}(t) + b(t) x(t) = 0 \tag{15}$$

with $x(t'') = x''$ and $x(t') = x'$,

$$S[\bar{x}] = \int_{t'}^{t''} dt \mathcal{L}(\bar{x}(t), \dot{\bar{x}}(t), t), \tag{16}$$

and $h(t, t')$ is the solution of

$$\frac{\partial^2 h(t, t')}{\partial t^2} + b(t) h(t, t') = 0 \tag{17}$$

with $h(t', t') = 0$ and $\frac{\partial h(t,t')}{\partial t}|_{t=t'} = 1$.

Since (10) and (11) have the same form, then

$$K(F'', \lambda''; F', \lambda') = (2\pi i h(\lambda'', \lambda'))^{-1/2} e^{i S[\bar{F}]} \tag{18}$$

with

$$S[\bar{F}] = \int_{\lambda'}^{\lambda''} d\lambda \mathcal{L}(\bar{F}(\lambda), \dot{\bar{F}}(\lambda), \lambda) \tag{19}$$

and $h(\lambda, \lambda')$ solution of (17) with t's replaced by λ's, is a Feynman propagator and therefore a quantum object describing the flow of the geodesic congruence from $\lambda = \lambda'$ to $\lambda = \lambda''$. To the pairs (F'', λ'') and (F', λ') might correspond "quantum states" $|F'', \lambda'' >$ and $|F', \lambda' >$.

3 Radial Raychaudhuri equation

Let's consider a space-time with metric

$$g_{\mu\nu} = \text{ diag } (f(r), -\frac{1}{f(r)}, -r^2, -r^2 \sin^2 \theta) \tag{20}$$

with $f(r)$ a smooth function except perhaps on a finite set of points. These function can be dependent on some real constants, for example $f(r, a) = 1 + r^2/a^2$ for adS and $f(r, M,) = 1 - 2M/r$ for Schwarzschild solution.

For this metric we want to obtain the Raychaudhuri Equation (RE) for particular geodesics as they are the radial ones, like

- marginally bound timelike geodesics (see below)
- null geodesics

For this we need the Christoffel symbols. It is well known that for a metric (20) there are nine algebraically independent non-vanishing Christoffel symbols

$$\Gamma^0_{01} = \frac{1}{2}\frac{f'}{f}; \qquad \Gamma^1_{00} = \frac{1}{2}ff';$$
$$\Gamma^1_{11} = -\frac{1}{2}\frac{f'}{f}; \qquad \Gamma^1_{22} = -rf; \qquad \Gamma^1_{33} = -rf\sin^2\theta;$$
$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}; \qquad \Gamma^2_{33} = -\sin\theta\cos\theta; \qquad \Gamma^3_{23} = \cot\theta. \tag{21}$$

where the prime means the derivative respect to the coordinate r. Knowing this we can carry on RE calculations.

3.1 Radial marginally bound timelike geodesics

As we know the geodesic equation can be computed with the help of Euler-Lagrange Equations (ELE). In this case the Lagrangian for the geodesic is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{22}$$

where the dot means the derivative respect to affine parameter λ. In terms of the tangent vector of the geodesic $v^\mu = (v^t, v^r, v^\theta, v^\phi) = \dot{x}^\mu$ the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}v^\mu v^\nu \tag{23}$$

The ELE are

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = 0; \tag{24}$$

for $\mu = 0$ we have

$$\dot{t}f = \text{ cte } = E. \tag{25}$$

The marginally bound geodesic is defined as $E = 1$, thus the temporal component of the tangent vector is $v^t = 1/f$. For radial geodesics $v^\theta = v^\phi = 0$.

On the other hand, for timelike geodesic holds $v^2 = 1$, that is

$$v^2 = g_{\mu\nu}v^\mu v^\nu = (v^t)^2 f - (v^r)^2 f^{-1} = 1 \tag{26}$$

which implies

$$v^r = \pm\sqrt{1-f}. \tag{27}$$

Therefore tangent vectors fields to radial marginally bound timelike geodesics are given by

$$v_\pm = v^\mu_\pm\partial_\mu = \frac{1}{f}\partial_t \pm \sqrt{1-f}\,\partial_r = (1/f, \pm\sqrt{1-f}), \tag{28}$$

and

$$v^\pm_\mu = g_{\mu\nu}v^\nu_\pm = (1, \mp\frac{\sqrt{1-f}}{f}). \tag{29}$$

To calculate all elements of the RE, we need to compute first the tensor $B^\mu_\nu := v^\mu_{;\nu}$ that measures the obstruction for the deviation vector to be parallel transported along geodesics. For this we compute $B_{\mu\nu} := v_{\mu;\nu}$ instead of B^μ_ν, and since there are two tangent vectors, one for the outgoing label by + and the ingoing label by − we will have two tensors $B^\pm_{\mu\nu}$. Then

$$B^\pm_{\mu\nu} := v^\pm_{\mu;\nu} = \partial_\nu v^\pm_\mu - \Gamma^\sigma_{\mu\nu}v^\pm_\sigma. \tag{30}$$

By a straightforward calculation we obtain

$$B^\pm_{\mu\nu} = \sqrt{1-f}\begin{pmatrix} \pm\frac{1}{2}f' & -\frac{1}{2}\frac{f'}{f\sqrt{1-f}} & 0 & 0 \\ -\frac{1}{2}\frac{f'}{f\sqrt{1-f}} & \pm\frac{1}{2}\frac{f'}{(1-f)f^2} & 0 & 0 \\ 0 & 0 & \mp r & 0 \\ 0 & 0 & 0 & \mp r\sin^2\theta \end{pmatrix}. \tag{31}$$

Remembering that this tensor can be decomposed into trace, symmetric-traceless, and antisymmetric parts,

$$B^{\pm}_{\mu\nu} = \frac{1}{3}\Theta_{\pm}h^{\pm}_{\mu\nu} + \sigma^{\pm}_{\mu\nu} + \omega^{\pm}_{\mu\nu} \tag{32}$$

where $\Theta_{\pm} := tr(B^{\pm}) = g^{\mu\nu}B^{\pm}_{\mu\nu}$ is the expansion scalar, $\sigma^{\pm}_{\mu\nu} := B^{\pm}_{(\mu\nu)} - \frac{1}{3}\Theta_{\pm}h^{\pm}_{\mu\nu}$ the shear tensor, $\omega^{\pm}_{\mu\nu} := B^{\pm}_{[\mu\nu]}$ the rotation tensor, and $h^{\pm}_{\mu\nu} := g_{\mu\nu} - v^{\pm}_{\mu}v^{\pm}_{\nu}$ the transverse metric is purely 'spatial,' in the sense that it is orthogonal to $v^{\mu}_{\pm}$.

Since by (31) $B^{\pm}_{\mu\nu} = B^{\pm}_{\nu\mu}$, then $\omega^{\pm}_{\mu\nu} = 0$, and so

$$\begin{aligned}\Theta_{\pm} =& g^{\mu\nu}B^{\pm}_{\mu\nu} \\ =& \pm\frac{\sqrt{1-f}}{r}\left(2 - \frac{1}{2}\frac{rf'}{1-f}\right)\end{aligned} \tag{33}$$

To compute $\sigma^{\pm}_{\mu\nu}$, we need to compute the transverse metric that for a direct substitution we have

$$h^{\pm}_{\mu\nu} = \begin{pmatrix} f-1 & \pm\frac{\sqrt{1-f}}{f} & 0 & 0 \\ \pm\frac{\sqrt{1-f}}{f} & -\frac{1}{f^2} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{34}$$

With this we are ready to calculate the shear tensor that gives us

$$\sigma^{\pm}_{\mu\nu} = \Sigma(r)\begin{pmatrix} \pm\sqrt{1-f} & -\frac{1}{f} & 0 & 0 \\ -\frac{1}{f} & \pm\frac{1}{f^2\sqrt{1-f}} & 0 & 0 \\ 0 & 0 & \mp\frac{r^2}{2\sqrt{1-f}} & 0 \\ 0 & 0 & 0 & \mp\frac{r^2\sin^2\theta}{2\sqrt{1-f}} \end{pmatrix} \tag{35}$$

where

$$\Sigma(r) = \frac{1}{3}\left(f' + 2\frac{1-f}{r}\right). \tag{36}$$

Then

$$\sigma^2_{\pm} = \sigma^{\mu\nu}_{\pm}\sigma^{\pm}_{\mu\nu} = \frac{3}{2}\frac{\Sigma^2}{1-f}. \tag{37}$$

Finally the RE for radial marginally bound timelike geodesics are

$$\frac{d\Theta_{\pm}}{d\lambda} = -\frac{1}{3}\Theta^2_{\pm} - \sigma^2_{\pm} - R_{\mu\nu}v^{\nu}v^{\nu}. \tag{38}$$

As an example of this, let's consider the function $f(r) = 1 - 2M/r$ which is the Schwarzschild case. For this we have

$$\Theta_{\pm} = \pm\frac{3}{2}\sqrt{\frac{2M}{r^3}}, \qquad \Sigma = \frac{2M}{r^2}, \qquad \sigma^2_{\pm} = \frac{3M}{r^3} \tag{39}$$

and then the RE for Schwarzschild space-time is

$$\frac{d\Theta_{\pm}}{d\lambda} = -\frac{9M}{2r^3} \tag{40}$$

as appears in [15] in the section "2.3.7 Another example."

Our goal is the Schwarzschild anti-De Sitter (SaDS) case for which we have $f = 1 - 2M/r + r^2/a^2$ and $R^{\mu\nu} = -\Lambda g_{\mu\nu} = \frac{3}{a^2}g_{\mu\nu}$. In this case, we have

$$\Theta_{\pm} = \pm\sqrt{\frac{2M}{r^3} - \frac{1}{a^3}}\left(2 - \frac{1}{2}\frac{\frac{2M}{r} + \frac{2r^2}{a^2}}{\frac{2M}{r} - \frac{r^2}{a^2}}\right). \tag{41}$$

Notice that

$$\lim_{a\to\infty}\Theta_{\pm} = \pm\frac{3}{2}\sqrt{\frac{2M}{r^3}} \tag{42}$$

that is, the Schwarzschild case is recovered for vanishing cosmological constant. The RE is

$$\frac{d\Theta_{\pm}}{d\lambda} = -\frac{3}{2Mr - \frac{r^4}{a^2}}\left(\frac{3M^2}{r^2} - \frac{r^4}{a^4} - \frac{2Mr}{a^2}\right) - \frac{3}{a^2}. \tag{43}$$

Again, taking the limit $a \to \infty$ we recover the Schwarzschild case (40),.

On the other hand the affine parameter obeys

$$\frac{dr}{d\lambda_{\pm}} = v^r_{\pm} = \pm\sqrt{1-f} = \pm\sqrt{\frac{2M}{r} - \frac{r^2}{a^2}}. \tag{44}$$

Up to an additive constant, one obtains

$$\lambda_{\pm} = \pm\frac{2a}{3}\arcsin\left(\frac{r^{\frac{3}{2}}}{\sqrt{2Ma}}\right). \tag{45}$$

Once again, in the limit $a \to \infty$ at first order we get

$$\lambda_{\pm} \approx \pm\frac{2a}{3}\frac{r^{\frac{3}{2}}}{\sqrt{2Ma}} = \pm\frac{1}{3}\sqrt{\frac{2r^3}{M}} \tag{46}$$

that matches with Eq. (31) of [9] for the Schwarzschild case.

3.2 Null case

The metric for the space-time (20) for radial null geodesics is

$$f dt^2 - f^{-1}dr^2 = 0 \therefore f dt^2 = f^{-1}dr^2 \tag{47}$$

In terms of the tortoise coordinates

$$r^*(r) = \int_0^r \frac{dr'}{f(r')} \tag{48}$$

the metric for radial case takes the form

$$ds^2 = f^2(dt^2 - (dr^*)^2) \tag{49}$$

and then the null geodesic equations can be written in the form

$$\frac{dt}{dr^*} = \pm 1. \tag{50}$$

The solutions in this coordinates are

$$t(r) = \pm r^*(r) + r_0. \tag{51}$$

Strictly speaking one should divide the domain of integration, for the tortoise coordinate, in intervals like $(0, r_1 - \epsilon)$, $(r_1 + \epsilon, r_2 - \epsilon)$, ..., $(r_{n-1} + \epsilon, r_n - \epsilon)$, $(r_n + \epsilon, r)$ where $f(r_i) = 0$ for all $i = 1, ...n.$; then taking the limit $\epsilon \to 0_-$ would differ by and irrelevant constant. In terms of the Eddington-Finkelstein coordinates

$$u := t - r^* \tag{52}$$

and

$$v := t + r^* \tag{53}$$

a null geodesic has the form $u(r) = u_0$ and $v(r) = v_0$ for some constants u_0 and v_0. The 1-forms associate to outgoing ($u = cte$) and ingoing ($v = cte$) geodesics are

$$k^{out}_{\mu} = \partial_{\mu} u = (\partial_t u, \partial u) = (1, -f^{-1}) \tag{54}$$

and

$$k^{in}_{\mu} = \partial_{\mu} v = (\partial_t v, \partial v) = (1, +f^{-1}) \tag{55}$$

, respectively. The corresponding tangent vectors are

$$k^{\mu}_{out} = g^{\mu\nu}k^{out}_{\nu} = (f^{-1}, +1) \tag{56}$$

and

$$k^{\mu}_{in} = g^{\mu\nu}k^{in}_{\nu} = (f^{-1}, -1). \tag{57}$$

Let's label *out* ones by + and the *in* ones by −, then

$$k^{\mu}_{\pm} = (f^{-1}, \pm 1), \qquad k_{\mu}^{\pm} = (1, \mp f^{-1}) \tag{58}$$

To obtain the RE for this case, we need a auxiliary null vector $n^{\mu}_{\pm}$ such that $n^{\mu}_{\pm}k_{\mu}^{\pm} = 1$. For simplicity let's choose for a radial vector, that is, $n^{\nu}_{\pm} = (n^{t}_{\pm}, n^{r}_{\pm}, 0, 0)$, then we get

$$n^{\mu}_{\pm} = \frac{1}{2}(1, \mp f), \qquad n_{\mu}^{\pm} = \frac{1}{2}(f, \pm 1). \tag{59}$$

The transverse metric defined as $h^{\pm}_{\mu\nu} = g_{\mu\nu} - (n^{\pm}_{\mu}k^{\pm}_{\nu} + n^{\pm}_{\nu}k^{\pm}_{\mu})$ is

$$h^{\pm}_{\mu\nu} = \text{ diag }(0, 0, -r^2, -r^2\sin^2\theta). \tag{60}$$

On the other hand the $B^{\pm}_{\mu\nu}$ tensor for this case is

$$B^{\pm}_{\mu\nu} = \begin{pmatrix} \pm\frac{1}{2}f' & -\frac{1}{2}\frac{f'}{f} & 0 & 0 \\ -\frac{1}{2}\frac{f'}{f} & \pm\frac{1}{2}\frac{f'}{f^2} & 0 & 0 \\ 0 & 0 & \mp r & 0 \\ 0 & 0 & 0 & \mp r\sin^2\theta \end{pmatrix}. \tag{61}$$

whose transverse part, defined by $(B^{\perp}_{\pm})_{\mu\nu} = B^{\pm}_{\mu\nu} - B^{\pm}_{\mu\alpha}n^{\alpha}_{\pm}k^{\pm}_{\mu} - k^{\pm}_{\mu}n^{\alpha}_{\pm}B^{\pm}_{\alpha\nu} + k^{\pm}_{\mu}k^{\pm}_{\nu}B^{\pm}_{\alpha\beta}n^{\alpha}_{\pm}n^{\beta}_{\pm}$, is

$$(B^{\perp}_{\pm})_{\mu\nu} = \text{ diag }(0, 0, \mp r, \mp r\sin^2\theta). \tag{62}$$

By a direct calculations we get $\sigma^{\pm}_{\mu\nu} = 0 = \omega^{\pm}_{\mu\nu}$ and

$$\Theta_{\pm} = tr(B^{\perp}) = \pm\frac{2}{r}. \tag{63}$$

Then, the RE for the space-time (20) for null radial geodesics is

$$\frac{d\Theta_{\pm}}{d\lambda} = -\frac{2}{r^2} - R_{\mu\nu}k^{\mu}k^{\nu}. \tag{64}$$

For the SaDS case we get $R_{\mu\nu}k^{\mu}k^{\nu} = -\Lambda g_{\mu\nu}k^{\mu}k^{\nu} = 0$ then the RE is

$$\frac{d\Theta_{\pm}}{d\lambda} = -\frac{2}{r^2} \tag{65}$$

with affine parameters $\lambda = \mp r$.

4 Propagators for radial timelike geodesic congruences

Let's beging with the pure Schwarzschild case. In [9] it was studied that for conjugate points in the massive case, the propagators (18) are finite. However, the case in which the geodesic congruence reaches $r \to 0+$ was not studied. Only the null case was done for a light beam to go from $r = 2M$ to the singularity $r \to 0_+$. The solution for $h(t, t')$ (17) in the massive case is

$$h(t, t') = 3\left(t'^{1/3}t^{2/3} - t'^{2/3}t^{1/3}\right). \tag{66}$$

(This expresion corrects the wrong result (46) in [9], where h is denoted by f and $\lambda'' = t$, $\lambda' = t'$.)

With this, the quantum propagator for the massive geodesic congruences in Schwarzschild is

$$K_{\pm}(F''_{\pm}, \lambda''_{\pm}; F'_{\pm}, \lambda'_{\pm}) = \frac{e^{iS[\bar{F}_{\pm}]}}{\sqrt{2\pi i h(\lambda''_{\pm}, \lambda'_{\pm})}}. \tag{67}$$

$S[\bar{F}_{\pm}]$ does not change and is as in equations (44, 45) in [9].

Let us then analyze this propagator K when the particle starts from the horizon $r = 2M$ and goes to the singularity $r \to 0_+$. From the expression for the expansion in Eq. (39) for the ingoing case we have that when $r \to 0_+$ then $\theta_- \to -\infty$ and by the definition of F in (7), $F''_-(\lambda_-) \to 0$. In this case the parameter $\lambda_{\pm}$ takes the form as in (46), so $\lambda'_-(r = 2M) = -4/3M$ and $\lambda''_-(r = \epsilon) = -1/3\sqrt{2\epsilon^3/M}$, $\epsilon > 0$, so that the limit $\epsilon \to 0$ is then taken. We see that in (67) regardless of the numerator, the convergence or not of $|K_{\pm}(F''_{\pm}, \lambda''_{\pm}; F'_{\pm}, \lambda'_{\pm})|$ is dictated by the denominator and this depends on $h(\lambda''_{\pm}, \lambda'_{\pm})$ (66). Let's see how this one behaves under the parameters $\lambda'_-(r = R) = -1/3\sqrt{2R^3/M}$ and $\lambda''_-(r = \epsilon) = -1/3\sqrt{2\epsilon^3/M}$.

$$h(\lambda''_-(\epsilon), \lambda'_-(R)) = \sqrt{\frac{2}{M}}\left(R\epsilon^{1/2} - R^{1/2}\epsilon\right), \tag{68}$$

For finite R and $\epsilon \to 0$, we have that $h = 0$ so the quantum propagator for the geodesic congruences of massive particles in the Schwarzschild case diverges, contrary to what happens in the null case.

In the Schwarzschild anti-De Sitter case, $f = 1 - 2M/r + r^2/a^2$, and $R^{\mu\nu} = -\Lambda g_{\mu\nu} = \frac{3}{a^2} g_{\mu\nu}$. For the marginally bounded time like geodesics, we have Eq. (45); inverting it and using the change of variable (7) we have the differential Eq. (8) with frequency

$$\Omega^2(\lambda_\pm) = \frac{1}{a^2}\left(1 + \frac{2}{\sin^2(3\frac{\lambda_\pm}{a})}\right) \tag{69}$$

which has period $T = a\pi/3$ so it is a Hill equation [11].

We are interested in the case near the singularity $r = 0$ that is $\lambda_\pm \to 0$, and since physically $a \gg 1$ ($\Lambda = -\frac{3}{a^2} \sim -10^{-52} m^{-2}$) then $|\lambda_\pm/a| \ll 1$ and one has the approximation

$$\Omega^2(\lambda_\pm) \approx \frac{1}{a^2} + \frac{2}{9}\frac{1}{\lambda_\pm^2}. \tag{70}$$

Solving the differential equation by the series method, we have that

$$F(\lambda_\pm) = c_1 F_{2/3}(\lambda_\pm) + c_2 F_{1/3}(\lambda_\pm) \tag{71}$$

with

$$F_{2/3}(\lambda_\pm) = \lambda_\pm^{2/3}\Big(1 + 9\Big[-\frac{1}{42}\left(\frac{\lambda_\pm}{a}\right)^2 + \frac{1}{728}\left(\frac{\lambda_\pm}{a}\right)^4 - \frac{1}{27664}\left(\frac{\lambda_\pm}{a}\right)^6 + \ldots\Big]\Big), \tag{72}$$

$$F_{1/3}(\lambda_\pm) = \lambda_\pm^{1/3}\Big(1 + \frac{9}{10}\Big[-\frac{1}{3}\left(\frac{\lambda_\pm}{a}\right)^2 + \frac{1}{44}\left(\frac{\lambda_\pm}{a}\right)^4 - \frac{1}{1496}\left(\frac{\lambda_\pm}{a}\right)^6 + \ldots\Big]\Big). \tag{73}$$

In terms of Bessel functions of the first kind [16] the solution (71) is

$$F(\lambda_\pm) = \sqrt{\lambda_\pm}\big(c_1 J_{1/6}(\lambda_\pm/a) + c_2 J_{-1/6}(\lambda_\pm/a)\big). \tag{74}$$

$h(t, t')$ follows a similar differential equation (compare (8) and (17)) so

$$h(\lambda''_\pm, \lambda'_\pm) = \sqrt{\lambda''_\pm}\big(c_1(\lambda'_\pm) J_{1/6}(\lambda''_\pm/a) + c_2(\lambda'_\pm) J_{-1/6}(\lambda''_\pm/a)\big). \tag{75}$$

By imposing the boundary conditions $h(\lambda'_\pm, \lambda'_\pm) = 0$ and $\frac{\partial h(\lambda''_\pm, \lambda'_\pm)}{\partial \lambda''_\pm}|_{\lambda''_\pm = \lambda'_\pm} = 1$ the constants $c_1(\lambda'_\pm)$ and $c_2(\lambda'_\pm)$ can be found.

Again the convergence of this propagator is dictated by the value of $h(\lambda''_-, \lambda'_-)$ with $\lambda''_- \to 0$ and λ'_- finite. Writing $h(\lambda''_-, \lambda'_-)$ as in (71) we notice that for $\lambda''_- \to 0$, $h \to 0$ so also in SaDS the propagator diverges for massive particles. So the cosmological constant does not contribute for the converging of the propagator.

5 Conclusion

We found the Raychaudhuri equation for specific cases of geodesic congruences: radial marginally bound timelike geodesics (38) and radial null geodesics (64) in the static space time (20). Then we focus in the SaDS case and showed that to a time like geodesic congruence can be assigned a Feynman propagator describing its flow which near the singularity diverges. It is interesting that for the null case in SaDS the propagators, when going from the event horizon to the singularity converge because (65) is the same as that appearing in [9]. Coincidentally, in [14] the propagators of incoming and outgoing null equatorial principal geodesic congruences in the Kerr metric where the expanding scalars diverge at the ring singularity; however, the propagators remain finite. It would be investigated if there is any example for massive particles near a singularity for which the propagators remain finite.

Acknowledgements O. E. L. P. thanks SECIHTI for a doctoral fellowship.

Data availability statement No Data associated in the manuscript.

Declarations

Conflict of interest The authors declare no Conflict of interest.

References

1. S. Kar, S. Sengupta, The Raychaudhuri equations: a brief review. Pramana - J. Phys. **69**, 49–76 (2007). https://doi.org/10.1007/s12043-007-0110-9
2. S.W. Hawking, Singularities in the universe. Phys. Rev. Lett. **17**, 444–445 (1966). https://doi.org/10.1103/PhysRevLett.17.444
3. R. Penrose, Gravitational collapse and space-time singularities. Phys. Rev. Lett. **14**, 57–59 (1965). https://doi.org/10.1103/PhysRevLett.14.57
4. S. Das, Quantum Raychaudhuri equation. Phys. Rev. D **89**, 084068 (2014). https://doi.org/10.1103/PhysRevD.89.084068
5. A. Farag Ali, S. Das, Cosmology from quantum potential. Phys. Lett. B **741**, 276–279 (2015). https://doi.org/10.1016/j.physletb.2014.12.057
6. A.F. Ali, M.M. Khalil, Black hole with quantum potential. Nucl. Phys. B **909**, 173–185 (2016). https://doi.org/10.1016/j.nuclphysb.2016.05.005
7. S. Alsaleh, L. Alasfar, M. Faizal, A.F. Ali, Quantum no-singularity theorem from geometric flows. Int. J. Modern Phys. A **33**(10), 1850052 (2018). https://doi.org/10.1142/S0217751X18500525
8. S. Gupta Choudhury, A. Dasgupta, N. Banerjee, The Raychaudhuri equation for a quantized timelike geodesic congruence. Eur. Phys. J. C (2021). https://doi.org/10.1140/epjc/s10052-021-09714-4
9. M. Socolovsky, Quantum propagators for geodesic congruences. Theor. Phys. **6**, 9–17 (2021). https://doi.org/10.22606/tp.2021.62001
10. F.J. Tipler, Energy conditions and spacetime singularities. Phys. Rev. D **17**, 2521–2528 (1978). https://doi.org/10.1103/PhysRevD.17.2521
11. G. Teschl, *Ordinary Differential Equations and Dynamical Systems* (American Mathematical Society, Rhode Island, 2012)
12. R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 2012)
13. L.S. Schulman, *Techniques and Applications of Path Integration* (Dover Publications, New York, 2012)
14. J. Trujillo, M. Socolovsky, Principal equatorial null geodesic congruences in the kerr metric, and their quantum propagators. J. High Energy Phys. Gravit. Cosmol. **10**, 906–917 (2024). https://doi.org/10.4236/jhepgc.2024.103055
15. E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, 2004)
16. D.G. Zill, *A First Course in Differential Equations with Modeling Applications* (Cengage Learning, Boston, 2012)