Singular vectors for the W_N algebras and the BRST cohomology for relaxed highest-weight $\mathcal{L}_k(\mathfrak{sl}(2))$ modules

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Declaration

This is to declare that

- The thesis comprises only my original work towards the degree of Doctor of Philosophy except where indicated in the preface
- Due acknowledgement has been made in the text to all other material used
- The thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices

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Abstract

This thesis presents the computation of singular vectors of the W_n algebras and the BRST cohomology of modules of the simple vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(2))$ associated to the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ in category $\mathcal{R}^\sigma$, which is the category of relaxed highest-weight modules, their spectral flows and non-split extensions.

We will first recall some general theory on vertex operator algebras. We will then introduce the module categories that are relevant for conformal field theory. They are the category $\mathcal{O}$ of highest-weight modules and $\mathcal{R}^\sigma$, where $\mathcal{R}^\sigma$ contains $\mathcal{O}$ as well as the relaxed highest-weight modules with the relaxed spectral flow and non-split extensions. We will then introduce the W_n algebras as well as the simple vertex operator algebra. Properties of the Heisenberg algebra, the bosonic and the fermionic ghosts will be discussed as they are required in the free field realisations of W_n and $\mathcal{L}_k(\mathfrak{sl}(2))$ as well as the construction of the BRST complex.

We will then compute explicitly the singular vectors of W_n algebras in their Fock representations. In particular, singular vectors can be realised as the image of screening operators of the W_n algebras. One can then realise screening operators in terms of Jack functions when acting on a highest-weight state, thereby obtaining explicit formulae of the singular vectors in terms of symmetric functions.

We will then discuss the BRST construction and the BRST cohomology for modules in category $\mathcal{O}$. Lastly we compute the BRST cohomology for $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in category $\mathcal{R}^\sigma$. In particular, we compute the BRST cohomology for the highest-weight modules with positive spectral flow for all degrees and the BRST cohomology for the highest-weight modules with negative spectral flow for one degree. We also compute the BRST cohomology for relaxed highest-weight $\mathcal{L}_k(\mathfrak{sl}(2))$ modules.

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Chapter 0

Introduction

0.1 Background

The ingredients of a two-dimensional conformal field theory include a vertex operator algebra and a module category of the vertex operator algebra that satisfies a number of constraints. The constraints are that the module category has to be closed under conjugation, fusion and that there exists a partition function is invariant under the action of $SL(2, \mathbb{Z})$ (modular invariance).

A large class of vertex operator algebras are the simple affine vertex operators algebras $\mathcal{L}_k(\mathfrak{g})$ constructed from affine Kac-Moody algebras at non-critical levels k. These include the Wess-Zumino-Witten models [77–79] corresponding to k being integral. Another class of vertex operator algebras called the W-algebras are constructed from the affine vertex operator algebras via quantum hamiltonian reduction [34, 35, 39] which is the zeroth cohomology of the so-called BRST complex. W-algebras are parametrised by a finite-dimensional simple Lie algebra $\mathfrak{g}$, a level k and a nilpotent element $f \in \mathfrak{g}$ and we shall denote them by $\mathsf{W}_k(\mathfrak{g}, f)$. Examples of W-algebras include the Virasoro algebra $\mathsf{W}_k(\mathfrak{sl}(2), f)$ [81] and the W_3-algebras, $\mathsf{W}_k(\mathfrak{sl}(3), f)$ [33] where f corresponds to a principal nilpotent element of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$.

Given a conformal field theory, one question to ask is whether it is rational, that is, whether the (appropriate) module category of the corresponding vertex operator algebra is completely reducible (this will also imply that the module category has finitely many irreducible modules [21]). The Wess-Zumino-Witten models, the Virasoro minimal models $\mathsf{M}(p, q) = \mathsf{W}_k(\mathfrak{sl}(2), f)$, $f \neq 0$ [74] are rational, where k, p, q are related by $k + 2 = \frac{p}{q}$ for $p, q \geq 2$, where p, q are coprime. More generally, $\mathsf{W}_k(\mathfrak{g}, f_\theta)$ is rational [4] where f_θ is a principal nilpotent element of $\mathfrak{g}$ with k defined in the paper.

The approach used to prove rationality of Virasoro minimal models $\mathsf{M}(p,q)$ in [74] involves deriving a projection formula of singular vectors in the Virasoro algebra Vir. Singular vectors are in general very hard to compute explicitly. It was shown in [57] that explicit formulae for singular vectors in Fock representations of Vir can be expressed in terms of the Jack symmetric functions. This approach of representing singular vectors using symmetric functions was used in [61] to prove the rationality of $\mathsf{M}(p,q)$ and classify its modules, thereby recovering the result in [74]. The same approach has been used to determine the spectrum of the simple affine vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(2))$ [62].

The approach used in [57] for computing singular vectors has been generalised to the W_n-algebra [7], where $\mathsf{W}_n = \mathsf{W}_k(\mathfrak{sl}(n), f_\theta)$, f_θ being a principal nilpotent element. The first result of this thesis is to extend the results in [57, 7] to give explicit formulae of singular vectors of the W_n-algebra in its Fock representations. This work was published in [60].

The approach used to prove that the simple quotient of $\mathsf{W}_k(\mathfrak{g}, f)$, f principal nilpotent is rational uses BRST cohomology [4]. Specifically, it was shown that all irreducible modules of $\mathsf{W}_k(\mathfrak{g}, f)$ in category $\mathcal{O}$ can be obtained from simple modules in category $\mathcal{O}$ of the simple vertex operator algebra $\mathcal{L}_k(\mathfrak{g})$ via BRST cohomology. Moreover, the so-obtained $\mathsf{W}_k(\mathfrak{g}, f)$ irreducible modules satisfy the constraints of a conformal field theory.

However, for fractional admissible level k, the modules in category $\mathcal{O}$ of $\mathcal{L}_k(\mathfrak{g})$ do not satsify any of the constraints of a conformal field theory. Therefore a larger category than $\mathcal{O}$ is needed. It is conjectured that the right category to consider is the relaxed category with spectral flow $\mathcal{R}^\sigma$. This category includes $\mathcal{O}$ as well as relaxed highest-weight modules and their twisted versions under the spectral flow automorphisms. We also remark that $\mathcal{R}^\sigma$ is not semisimple and thus the theory is a logarithmic conformal field theory. A logarithmic conformal field theory is one where the module category (of the chiral vertex operator algebra) is not semisimple[64, 24]. Similarly, there are W-algebras that are believed to be logarithmic. For example, let $f = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ be a minimal nilpotent element of $\mathfrak{sl}(3)$. Then $\mathsf{W}_k(\mathfrak{sl}(3), f)$ is the Bershadsky-Polyakov algebra. For $k = -3 + \frac{p}{2}$ where p is an odd integrer greater than 3, the simple quotient of this algebra has been proved to be rational [3]. However, for other levels it is believed that the algebra is logarithmic. Therefore, relaxed modules of the Bershadsky-Polyakov algebra must be considered.

It is natural then to ask whether the BRST cohomology takes relaxed modules in category $\mathcal{R}^\sigma$ of $\mathcal{L}_k(\mathfrak{g})$ to relaxed modules in $\mathcal{R}^\sigma$ of simple quotient of the Bershadsky-

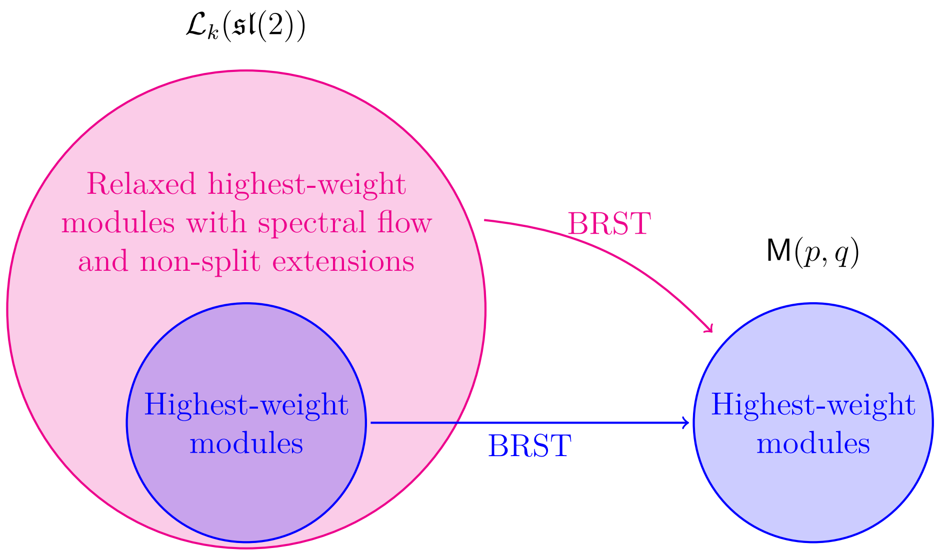


Fig. 1 In the second part of this thesis we will compute the BRST cohomology of some irreducible relaxed highest-weight modules of the simple affine vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(2))$. It turns out that their cohomologies are either zero or a simple highest weight module of the Virasoro minimal model $\mathsf{M}(p,q)$.

Polyakov algebra. Of course we would also like to ask the same for all other W-algebras. The simplest case of this question is the case of $\mathcal{L}_k(\mathfrak{sl}(2))$ which is what the second part of the thesis is about. In particular, we will compute the BRST cohomology of some relaxed modules in $\mathcal{R}^\sigma$ of $\mathcal{L}_k(\mathfrak{sl}(2))$, as shown in Figure 1.

In the future we would like to extend the question of finding the BRST cohomology of relaxed modules to higher ranks, see Figure 2 for example.

0.2 Outline

This thesis is organised as follows:

In Chapter 1 we start with a brief summary of the theory of vertex algebras. We also introduce the vertex algebras and their module categories that are relevant to us.

In Chapter 2, based on the work of [7], we apply the machinery of symmetric functions to compute singuar vectors of the W_N algebra in the Fock representations of W_N. This work was published in [60].

In Chapter 3 we give a detailed proof of certain fundamental results about BRST cohomology, outlined in [16], that takes irreducible highest-weight modules of the

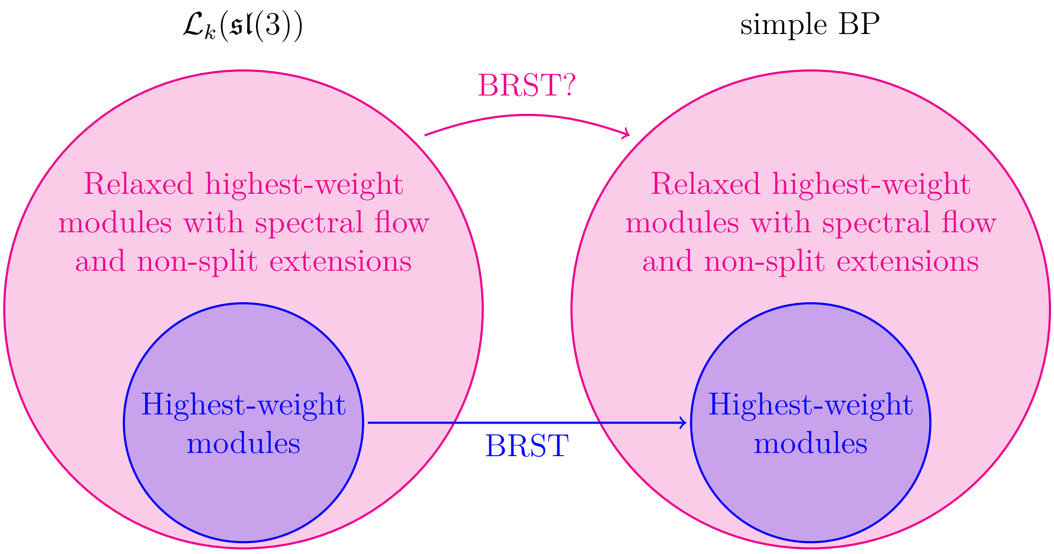


Fig. 2 For most admissible levels k, the simple Bershadsky-Polyakov algebra coresponds to a logarithmic conformal field theory. Moreover, we need to consider relaxed highest-weight modules. An obvious future research direction is to investigate whether the BRST cohomology takes irreducible relaxed modules of the simple affine $\mathfrak{sl}(3)$ vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(3))$ to irreducible relaxed modules of the simple Bershadsky-Polyakov vertex operator algebra.

simple affine vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(2))$ at admissible level k to the irreducible highest-weight modules of the Virasoro minimal model $\mathsf{M}(p,q)$ where $k = -2 + \frac{p}{q}$, $p, q \geq 2$, $(p,q) = 1$. The module category that we work with in this chapter is the category $\mathcal{O}$.

In Chapter 4 we attempt to generalise [16] where we consider the BRST cohomology of $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in the category $\mathcal{R}^\sigma$. This non-semisimple category contains the relaxed $\mathcal{L}_k(\mathfrak{sl}(2))$ modules, the twisted relaxed modules under spectral flow as well as non-split extensions of such modules. We present partial answers for this problem.

Chapter 1

Vertex Algebras

1.1 Vertex Algebras

In this section we recall some basic definitions and properties of vertex operator algebras.

1.1.1 Definitions

Let V be a vector space over $\mathbb{C}$. We assume that V is graded

$$V = \bigoplus_{n \in \mathbb{Z}} V_n. \tag{1.1}$$

Given a countable set $\{A_n \in \operatorname{End} V \mid n \in \mathbb{Z}\}$ of homogenous linear operators A_n of degree $-n$ with respect to the grading on V, we define the formal power series

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \tag{1.2}$$

which is called a field if for any $v \in V$ we have $A_n v = 0$ for n large enough.

Definition 1.1.1. *Two fields $A(z), B(w)$ are local if there exists $N \in \mathbb{N}$ such that*

$$(z - w)^N [A(z), B(w)] = 0 \tag{1.3}$$

Definition 1.1.2. *A vertex algebra is a collection of data*

- *A vector space V*
- *A vacuum vector $|0\rangle \in V$*

- *A linear operator $T : V \to V$*
- *A linear operation*

$$Y(\cdot, z) : V \longrightarrow \operatorname{End} V[\![z^{\pm}]\!] \tag{1.4}$$

satisfying

- $Y(|0\rangle, z) = \mathrm{id}$
- *For any $A \in V$ we have $Y(A, z)|0\rangle \in V[\![z]\!]$ and $\lim_{z\to 0} Y(A, z)|0\rangle = A$*
- $[T, A(z)] = \partial A(z)$ *and* $T|0\rangle = 0$
- *All fields $Y(A, z)$ are local to each other.*

If two fields $A(z), B(w)$ are local, then we can write the product $A(z)B(w)$ as a power series in $z - w$, assuming $|z| > |w|$

$$A(z)B(w) = \sum_{n\geq 0} \frac{Y(A_n \cdot B, w)}{(z-w)^{n+1}} + {:}A(z)B(w){:} \tag{1.5}$$

which is called the operator product expansion of $A(z)$, $B(w)$.

Definition 1.1.3. *The regular terms ${:}A(z)B(w){:}$ in the series expansion in Equation (1.5) is called the normally ordered product of $A(z), B(w)$. Explicitly,*

$${:}A(z)B(w){:} = \sum_{n\leq -1} A_n z^{-n-1} B(w) + B(w) \sum_{n\geq 0} A_n z^{-n-1} \tag{1.6}$$

From now on we will only include the singular terms when we write the operator product expansions of $A(z), B(w)$. That is we write

$$A(z)B(w) \sim \sum_{n\geq 0} \frac{Y(A_n \cdot B, w)}{(z-w)^{n+1}} \tag{1.7}$$

Since $A_n \cdot B = 0$ for n large enough we see that the series in Equation (1.5) always has finite-order pole at $z = w$.

Definition 1.1.4. *A vertex operator algebra is a vertex algebra $(V, |0\rangle, T, Y)$ with the existence of a conformal vector ω. The field corresponding to ω is an energy* *momentum tensor*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{1.8}$$

such that the modes L_n satisfy the commutation relations of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m-1)m(m+1)\delta_{m+n,0} \tag{1.9}$$

where c is central. Furthermore, the translation operator T coincides with L_{-1}. The action of L_0 acts diagonalisably on the vector space of the vertex operator algebra and the grading of the vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is the grading given by the L_0-eigenvalues. Any elements in V_n are said to have conformal weight n.

For any element $A \in V$ with conformal weight h_A, we will rewrite the field corresponding to A as

$$Y(A, z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-h_A} \tag{1.10}$$

Since the conformal vector uniquely determined the translation operator T which is L_{-1}, the data for a vertex operator algebra is $(V, |0\rangle, \omega, Y)$.
From now on we will only consider vertex operator algebras. Given one, we can define subalgebras, ideals and quotients. In particular,

Definition 1.1.5. *A vertex operator algebra ideal $I \subseteq V$ is a T-invariant subspace that satisfies $Y(A, z)B \in I((z))$ for all $A \in V, B \in I$.*

A special property that a vertex operator algebra has is skew-symmetry

Proposition 1.1.6. *[38, Proposition 3.2.5] The identity*

$$Y(A, z)B = e^{zT} Y(B, -z)A \tag{1.11}$$

holds in $V((z))$.

The point of Proposition 1.1.6 is that if $Y(A, z)B \in I((z))$, then $Y(B, z)A \in I((z))$. Therefore for vertex algebras, any left-sided ideal is automatically a two-sided ideal and thus the quotient V/I has a natural vertex algebra structure. In particular, if V has the structure of a vertex operator algebra, so does V/I.
Now that we have defined vertex operator algebras, we will turn to how we can construct them. The theorem below allows us to construct a vertex operator algebra from a set of generators and relations.

Theorem 1.1.7. *[38, Theorem 2.3.11] Suppose that V is a vector space, $|0\rangle$ a non-zero vector, and T an endomorphism of V. Let S be a countable ordered set and $\{a^\alpha\}_{\alpha\in S}$ a collection of vectors in V. Suppose we are also given fields*

$$a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_n z^{-n-h_a} \tag{1.12}$$

such that the following conditions hold

- *For all α, $a^\alpha(z)|0\rangle = a^\alpha + z(\dots)$*
- *$T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all α.*
- *All fields $a^\alpha(z)$ are mutually local*
- *V is spanned by the vectors*

$$a^{\alpha_1}_{j_1}\cdots a^{\alpha_m}_{j_m}|0\rangle \tag{1.13}$$

Then the assignment

$$Y\Big(a^{\alpha_1}_{j_1}\cdots a^{\alpha_m}_{j_m}|0\rangle, z\Big) = \frac{1}{(-j_1-1)!\cdots(-j_m-1)!}{:}\partial_z^{-j_1-1}a^{\alpha_1}(z)\cdots\partial_z^{-j_m-1}a^{\alpha_m}(z){:} \tag{1.14}$$

defines a vertex algebra structure on V. Moreover, if V is a $\mathbb{Z}$-graded vector space, $|0\rangle$ has degree 0, the vectors a^α are homogeneous, T has degree 1, and the fields $a^\alpha(z)$ have degree $\deg a^\alpha$, then V is a $\mathbb{Z}$-graded vertex algebra. Moreover, this is the unique vertex algebra structure on V satisfying the above conditions such that $Y(a^\alpha, z) = a^\alpha(z)$.

Therefore throughout this thesis, we will simply define various vertex operator algebras by stating the underlying vector space, the generating fields, the energy-momentum tensor $T(z)$ as well as the operator product expansion between these fields.
Theorem 1.1.7 ensures that it will be a well-defined vertex operator algebra.
Lastly we will state a general result on vertex operator algebras that will be important in Chapter 3.

Lemma 1.1.8. *[38, Corollary 3.3.8] For any $A \in V$, the mode $A_{-h_A+1} = \int_0 Y(A, z)\,\mathrm{d}z$ of $Y(A, z)$ satisfies*

$$[A_{-h_A+1}, Y(B, w)] = Y(A_{-h_A+1}\cdot B, w) \tag{1.15}$$

1.1.2 Modules of vertex operator algebras

Definition 1.1.9. *Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A vector space M is called a V-module if it is equipped with an operation $Y_M : V \longrightarrow \operatorname{End} M[\![z^{\pm}]\!]$ which assigns to each $A \in V$ a field*

$$Y_M(A, z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \tag{1.16}$$

on M subject to the following axioms:

- $Y_M(|0\rangle, z) = \mathrm{id}_M$
- *for all $A, B \in V, C \in M$ the three expressions*

$$Y_M(A, z)Y_M(B, w)C \in M((z))((w)) \tag{1.17}$$

$$Y_M(B, w)Y_M(A, z)C \in M((w))((z)) \tag{1.18}$$

$$Y_M(Y(A, z-w)B, w)C \in M((w))((z-w)) \tag{1.19}$$

are the same expansions, in their respective domains, of the same element of

$$M[\![z, w]\!]\left[z^{-1}, w^{-1}, (z-w)^{-1}\right] \tag{1.20}$$

To construct modules of a vertex operator algebra, we want to relate them to modules of Lie algebras. For any vertex algebra V, we can attached to it an associative algebra $\mathrm{U}(V)$, see [38, Definition 4.3.1]. In particular, if the modes of all fields $A(z)$ of the generators are elements of some Lie algebra $\mathfrak{g}$, then $\mathrm{U}(V)$ is a completion of $\mathrm{U}(\mathfrak{g})/I$, where $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra and I is the ideal generated by the expressions $(S - s1)$ for any central element $S \in \mathfrak{g}$ and for some fixed $s \in \mathbb{C}$. We call a $\mathrm{U}(V)$ module smooth if for all $v \in M$ and $A \in V$, there is an N such that $A_n \cdot v = 0$ for all $n \geq N$. Then we have

Theorem 1.1.10. *[38, Theorem 5.1.6] There is an equivalence between the category of V-modules and the category of smooth $\mathrm{U}(V)$-modules.*

For our applications, $\mathrm{U}(V)$-modules are all $\mathfrak{g}$ modules so it's enough to consider modules of $\mathfrak{g}$. The Lie algebras $\mathfrak{g}$ that we consider have triangular decompositions

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{1.21}$$

as well as a grading $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ (which will be made explicit when necessary) where $\mathfrak{g}_0$ contains $\mathfrak{h}$. The relaxed decomposition is defined as

$$\mathfrak{g} = \mathfrak{n}_< \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_> \tag{1.22}$$

where $\mathfrak{n}_< = \oplus_{n<0}\mathfrak{g}_n, \mathfrak{n}_> = \oplus_{n>0}\mathfrak{g}_n$. We note that $\mathfrak{g}_0 \oplus \mathfrak{n}_>$ contains the borel subalgebra $\mathfrak{h} \oplus \mathfrak{g}_+$. Given a module $\mathcal{M}$ of $\mathfrak{g}$, for any $\lambda \in \mathfrak{h}^*$ let

$$\mathcal{M}_\lambda = \{v \in \mathcal{M} \mid \forall h \in \mathfrak{h}, hv = \lambda(h)v\} \tag{1.23}$$

Ultimately, we are interested in vertex algebras and their modules from the conformal field theory point of view. Recall that a conformal field theory includes a vertex operator algebra V and a module category of V that satisfies certain constraints:

- closed under conjugation
- closed under fusion
- there exists a partition function that is modular invariant

The category that we need to consider in order to satisfy these assumptions depends on the vertex operator algebra V. As V-modules are modules of the Lie algebra $\mathrm{U}(V)$ which are modules of $\mathfrak{g}$, we will introduce some Lie algebra module category that (conjecturally) satisfies the axioms. The first such category is the category $\mathcal{O}$.

Definition 1.1.11. *[58] The category $\mathcal{O}$ contains objects $\mathcal{M}$ that satisfy*

- *$\mathcal{M}$ is finitely generated*
- *$\mathcal{M} = \bigoplus_{\lambda\in\mathfrak{h}^*} \mathcal{M}_\lambda$, where each $\mathcal{M}_\lambda$ is finite-dimensional*
- *$\mathcal{M}$ is locally $\mathfrak{n}_+$-finite, for each $v \in \mathcal{M}$ the $\mathfrak{n}_+$-module generated by v is finite dimensional*

and the morphisms are $\mathfrak{g}$-module homomorphisms.

For example all modules of the Virasoro minimal models $\mathsf{M}(p, q)$ [74] and the W_N algebras [4], which we will define in the next section, belong to category $\mathcal{O}$.

Definition 1.1.12. *[62] The relaxed category $\mathcal{R}$ contains objects $\mathcal{M}$ that satisfy*

- *$\mathcal{M}$ is finitely generated*

- $\mathcal{M} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{M}_\lambda$, *where each* $\mathcal{M}_\lambda$ *is finite-dimensional*
- $\mathcal{M}$ *is locally* $\mathfrak{n}_>$*-finite, for each* $v \in \mathcal{M}$ *the* $\mathfrak{n}_>$*-module generated by* v *is finite dimensional.*

Given a module $\mathcal{M}$ of the associative algebra $U(V)$ associated to a vertex (operator) algebra V, suppose that $\sigma : U(V) \to U(V)$ is an automorphism of the Lie algebra $U(V)$.

Definition 1.1.13. *The twisted module* $\sigma(\mathcal{M})$ *is defined, for any element* $v \in \mathcal{M}$ *and for any modes* $A_n \in U(V)$*, by*

$$A_n \cdot \sigma(v) = \sigma(\sigma^{-1}(A_n \cdot v)), \qquad \sigma(v) \in \sigma(M). \tag{1.24}$$

Equivalently, $\sigma(\mathcal{M})$ *corresponds to the representation* $\sigma^{-1} \circ \rho : U(V) \to \operatorname{End} \mathcal{M}$ *where* $\rho : U(V) \to \operatorname{End} \mathcal{M}$ *is the representation map of* $\mathcal{M}$*.*

Suppose that $\mathcal{C}$ is a module category of V and let σ be an automorphism of $\mathfrak{g}$. We define $\sigma(\mathcal{C})$ as the set containing the twisted modules $\sigma(M)$ for each $M \in \mathcal{C}$.

Definition 1.1.14. *The relaxed category with spectral flow* $\mathcal{R}^\sigma$ *is the full subcategory of smooth weight modules of* $\mathfrak{g}$ *generated by objects* M*,* $\sigma(M)$*,* $M \in \mathcal{R}$ *as well as all non-split extensions between these objects.*

1.2 The Heisenberg vertex algebra

The Heisenberg Lie algebra $\mathcal{H}$ is the vector space

$$\mathcal{H} = \mathbb{C}\{a_n, 1 \mid n \in \mathbb{Z}\} \tag{1.25}$$

The algebra structure of $\mathcal{H}$ is defined by the Lie bracket

$$[a_m, a_n] = m\delta_{m+n,0}1 \tag{1.26}$$

with 1 central. The Heisenberg algebra has the triangular decomposition

$$\mathcal{H} = \mathcal{H}_- \bigoplus \mathcal{H}_0 \bigoplus \mathcal{H}_+ \tag{1.27}$$

where

$$\mathcal{H}_+ = \mathbb{C}\{a_n \mid n \in \mathbb{Z}_{\geq 1}\} \tag{1.28}$$
$$\mathcal{H}_0 = \mathbb{C}\{a_0, 1\} \tag{1.29}$$
$$\mathcal{H}_- = \mathbb{C}\{a_n \mid n \in \mathbb{Z}_{\leq -1}\} \tag{1.30}$$

With this triangular decomposition we can define Verma modules, commonly called Fock spaces. These modules are constructed from the one dimensional module generated by $|\lambda\rangle, \lambda \in \mathbb{C}$

$$\mathcal{F}_\lambda = \mathrm{U}(\mathcal{H}_-) \bigotimes_{\mathrm{U}(\mathcal{H}_0 \oplus \mathcal{H}_+)} \mathbb{C}|\lambda\rangle \tag{1.31}$$

so that

$$a_0|\lambda\rangle = \lambda|\lambda\rangle \tag{1.32}$$
$$a_n|\lambda\rangle = 0, \qquad n \geq 1 \tag{1.33}$$

As a vector space, the Fock spaces are spanned by the vectors and the highest-weight vector of $\mathcal{F}_\lambda$ is thus $|\lambda\rangle$.

1.2.1 The Heisenberg vertex operator algebra

The module $\mathcal{F}_0$ has the structure of a vertex algebra H with the data $(\mathcal{F}_0, |0\rangle, T, Y)$ where the translation operator is defined as

$$T = \frac{1}{2} \sum_{n \in \mathbb{Z}} : a_n a_{-n-1} : \tag{1.34}$$

H is generated by a single field of conformal weight 1

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \tag{1.35}$$

where the field $a(z)$ satisfies the operator product expansion

$$a(z)a(w) \sim \frac{1}{(z-w)^2} \tag{1.36}$$

with the state-field correspondence defined as

$$Y(a_{-n_k}\cdots a_{-n_1}|0\rangle, z) = \frac{1}{(n_k-1)!\cdots(n_1-1)!}{:}\partial^{n_k-1}a(z)\cdots\partial^{n_1-1}a(z){:} \tag{1.37}$$

where $n_k \geq \cdots n_1 \geq 1$. There is a one parameter family of conformal structures on H, each member of which makes H a vertex operator algebra,

$$T(z) = \frac{1}{2}{:}a(z)a(z){:} + \frac{\alpha_0}{2}\partial a(z), \qquad \alpha_0 \in \mathbb{C} \tag{1.38}$$

where $c = 1 - 3\alpha_0^2$.

1.2.2 Modules of the Heisenberg vertex operator algebra

For all $\lambda \in \mathbb{C}$, the Fock spaces $\mathcal{F}_\lambda$ of the Heisenberg Lie algebra $\mathcal{H}$ are modules for the Heisenberg vertex algebra H. Moreover, as modules of H, these modules are simple. To conclude this section we introduce the spectral flow automorphism of the Heisenberg algebra. For $s \in \mathbb{C}$, let $\sigma^s_{\mathcal{H}}$ be the map

$$\sigma^s_{\mathcal{H}}(a_n) = a_n - s\delta_{n,0}\mathbb{1}, \qquad \sigma^s_{\mathcal{H}}(\mathbb{1}) = \mathbb{1}. \tag{1.39}$$

It is routine to check that $\sigma^s_{\mathcal{H}}$ is an automorphism. Since $\sigma^s_{\mathcal{H}}$ leaves every element but a_0 in $\mathcal{H}$ invariant, we see that $\sigma^s_{\mathcal{H}}(|\lambda\rangle)$ is a highest-weight vector and thus $\sigma^s_{\mathcal{H}}(\mathcal{F}_\lambda)$ is a highest weight module. The action of a_0 acting on the highest weight vector of $\sigma^s_{\mathcal{H}}(\mathcal{F}_\lambda)$ is

$$a_0\sigma^s_{\mathcal{H}}(|\lambda\rangle) = \sigma^s_{\mathcal{H}}\Big(\sigma^{-s}_{\mathcal{H}}(a_0)|\lambda\rangle\Big) = \sigma^s_{\mathcal{H}}((a_0+s)|\lambda\rangle) = (\lambda+s)\sigma^s_{\mathcal{H}}(|\lambda\rangle) \tag{1.40}$$

Thus we see that

$$\sigma^s_{\mathcal{H}}(\mathcal{F}_\lambda) = \mathcal{F}_{\lambda+s} \tag{1.41}$$

That is, twisting $\mathcal{F}_\lambda$ with $\sigma^s_{\mathcal{H}}$ changes it to another Fock space with a different highest weight.

1.2.3 The rank r Heisenberg algebra

In this subsection we use different notations to the ones used for defining $\mathcal{H}$, so we fix $r \geq 2$. The rank r Heisenberg algebra is constructed from an r-dimensional complex vector space $\mathfrak{h}_r$ together with a non-degenerate symmetric bilinear form $(-,-)$. For the application to follow, we pick a basis $\{a^1,\dots,a^r\}$ of $\mathfrak{h}_r$ such that the Gram matrix of $(-,-)$ is the Cartan matrix of $\mathfrak{sl}(r+1)$:

$$\left(a^i,a^j\right) = 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}, \quad i,j = 1,\dots,r. \tag{1.42}$$

Since $(-,-)$ is non-degenerate, it defines a vector space isomorphism $\iota : \mathfrak{h}_r \to \mathfrak{h}_r^*$ by $a \mapsto (a,-)$. The induced non-degenerate symmetric bilinear form will also be denoted by $(-,-)$. We denote the images of the basis vectors a^i by $\alpha^i = \iota(a^i)$ and the elements of the basis of $\mathfrak{h}_r^*$ dual to $\{a^i\}$ by ω_i. Thus, $\omega_i(a^j) = \delta_i^j$. The α^i and ω_i may therefore be identified as simple roots and fundamental weights, respectively, of $\mathfrak{sl}(r+1)$. In this picture, the basis vectors $a^i \in \mathfrak{h}_r$ are the simple coroots of $\mathfrak{sl}(r+1)$. To any vector $a \in \mathfrak{h}_r$, one assigns a field $a(z)$ whose defining operator product expansions are

$$a(z)b(w) \sim \frac{(a,b)\mathbb{1}}{(z-w)^2}, \quad a,b \in \mathfrak{h}_r. \tag{1.43}$$

These fields admit Fourier expansions of the form

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h}_r, \tag{1.44}$$

whose modes satisfy the following commutation relations:

$$\left[a_m, b_n\right] = m(a,b)\delta_{m,-n}\mathbb{1}. \tag{1.45}$$

The Heisenberg Lie algebra $\mathcal{H}_r$ is the infinite-dimensional Lie algebra spanned by the central element $\mathbb{1}$ and the generators a_m, for all $a \in \mathfrak{h}_r$ and $m \in \mathbb{Z}$. We have chosen to denote the central element by $\mathbb{1}$, since we assume that it will act as the identity on any $\mathcal{H}_r$-module. A basis of $\mathcal{H}_r$ is then given by $\mathbb{1}$ and the a^i_m, with $i = 1,\dots,r$ and $m \in \mathbb{Z}$. The Heisenberg Lie algebra admits a triangular decomposition

$$\mathcal{H}_r = (\mathcal{H}_r)_- \oplus (\mathcal{H}_r)_0 \oplus (\mathcal{H}_r)_+, \quad (\mathcal{H}_r)_0 = \bigoplus_{i=1}^{r} \mathbb{C}a^i_0 \oplus \mathbb{C}\mathbb{1}, \quad (\mathcal{H}_r)_\pm = \bigoplus_{i=1}^{r}\bigoplus_{m\geq 1} \mathbb{C}a^i_{\pm m}. \tag{1.46}$$

Verma modules over $\mathcal{H}_r$ are commonly referred to as Fock spaces. These are induced from the one-dimensional modules $\mathbb{C}|\zeta\rangle$, $\zeta \in \mathfrak{h}_r^*$, over $(\mathcal{H}_r)_{\geq 0} = (\mathcal{H})_0 \oplus (\mathcal{H})_+$ that are defined by

$$\mathsf{1}|\zeta\rangle = |\zeta\rangle, \qquad a_n|\zeta\rangle = \delta_{n,0}\zeta(a)|\zeta\rangle, \quad a \in \mathfrak{h}_r, \quad n \geq 0. \tag{1.47}$$

The Fock spaces

$$\mathcal{F}_\zeta = \mathrm{U}(\mathcal{H}_r) \otimes_{\mathrm{U}\left((\mathcal{H}_r)_{\geq 0}\right)} \mathbb{C}|\zeta\rangle \tag{1.48}$$

are well known to be simple $\mathcal{H}_r$-modules, for all $\zeta \in \mathfrak{h}_r^*$.
As a module over itself, the Heisenberg vertex algebra H_r is identified with the Fock space $\mathcal{F}_0$ and the state-field correspondence is given by

$$|0\rangle \longleftrightarrow \mathsf{1}, \quad b^1_{-n_1-1}\cdots b^k_{-n_k-1}|0\rangle \longleftrightarrow :\frac{\partial^{n_1}}{n_1!}b^1(z)\cdots\frac{\partial^{n_k}}{n_k!}b^k(z):, \tag{1.49}$$

where $b^1, \ldots, b^k \in \mathfrak{h}_r$ and normal ordering is defined in the usual way.
The Heisenberg vertex algebra H_r can be endowed with the structure of a vertex operator algebra by choosing an energy-momentum tensor. This choice is not unique. For the purposes of this note, we shall restrict our attention to the following one-parameter family of energy-momentum tensors:

$$T(z) = \sum_{i=1}^r \left[\frac{1}{2}{:}a^i(z)a^{*i}(z){:} + \alpha_0\partial a^{*i}(z)\right], \quad \alpha_0 \in \mathbb{C}. \tag{1.50}$$

Here, the $a^{*i} \in \mathfrak{h}_r$ are dual to the coroots a^i in the sense that $\iota(a^{*i}) = \omega_i$. We note that while the quadratic summand in the above energy-momentum tensor is basis independent, the linear summand is not. The central charge corresponding to this choice of energy-momentum tensor depends on the parameter α_0:

$$c = r - r(r+1)(r+2)\alpha_0^2. \tag{1.51}$$

By definition, the coefficients of the Fourier expansion of the energy-momentum tensor satisfy the commutation relations of the Virasoro algebra as defined in Equation (1.60). Thus, formula (1.50) realises the Virasoro generators L_n, $n \in \mathbb{Z}$, as infinite sums of products of Heisenberg generators:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad L_n = \sum_{i=1}^r \left[\frac{1}{2}\sum_{m\in\mathbb{Z}} {:}a^i_m a^{*i}_{n-m}{:} - \alpha_0(n+1)a^{*i}_n\right]. \tag{1.52}$$

This identification yields an action of the Virasoro algebra on the Fock spaces $\mathcal{F}_\zeta$, $\zeta \in \mathfrak{h}_r^*$. In this way, any highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$ is also a Virasoro highest-weight vector:

$$L_n|\zeta\rangle = h_\zeta \delta_{n,0}|\zeta\rangle, \quad n \geq 0, \quad h_\zeta = \frac{1}{2}(\zeta, \zeta - 2\alpha_0 \varrho). \tag{1.53}$$

Here, $\varrho = \sum_i \omega_i$ is the Weyl vector of $\mathfrak{sl}(r+1)$. We note that while the Fock spaces are simple as Heisenberg modules, they need not be as Virasoro modules.
The primary fields of the free boson theory are called vertex operators (not to be confused with elements of the Heisenberg vertex operator algebra). To define them, we first need to extend $\mathcal{H}_r$ by $\mathbb{C}[\mathfrak{h}_r^*]$, the group algebra of $\mathfrak{h}_r^*$, treating $\mathfrak{h}_r^*$ as an abelian group under vector addition and $\mathbb{C}[\mathfrak{h}_r^*]$ as an abelian Lie algebra. We denote the group algebra basis element corresponding to $\eta \in \mathfrak{h}_r^*$ by e^η and define the commutation relations between the generators a_m and e^η by

$$[a_m, \mathsf{e}^\eta] = \delta_{m,0}\eta(a)\mathsf{e}^\eta, \quad a \in \mathfrak{h}_r, \quad \eta \in \mathfrak{h}_r^*, \quad m \in \mathbb{Z}. \tag{1.54}$$

It is easy to check that this extension of $\mathcal{H}_r$ by $\mathbb{C}[\mathfrak{h}_r^*]$ is a semidirect sum of Lie algebras.
A standard computation now shows that e^η maps the highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$ to a highest-weight vector of a_0-eigenvalue $\zeta(a) + \eta(a) = (\zeta + \eta)(a)$. Following usual practice, we shall identify $\mathsf{e}^\eta|\zeta\rangle$ with $|\zeta + \eta\rangle$. The vertex operator corresponding to $|\zeta\rangle = \mathsf{e}^\zeta|0\rangle$ is

$$\mathrm{V}_\zeta(z) = \mathsf{e}^\zeta z^{a_0} \prod_{m \geq 1} \exp\Big(\frac{a_{-m}}{m} z^m\Big) \exp\Big(-\frac{a_m}{m} z^{-m}\Big), \quad \zeta = \iota(a) \in \mathfrak{h}_r^*. \tag{1.55}$$

These primary fields therefore define linear maps between Fock spaces:

$$\mathrm{V}_\zeta(z) \colon \mathcal{F}_\eta \to z^{(\zeta,\eta)} \mathcal{F}_{\zeta+\eta}[\![z, z^{-1}]\!]. \tag{1.56}$$

It is easy to check from the H_r-primary operator product expansion

$$a(z)\mathrm{V}_\zeta(w) \sim \frac{\zeta(a)\mathrm{V}_\zeta(w)}{z - w} \tag{1.57}$$

that $a(z)$ and $\mathrm{V}_\zeta(w)$ are mutually local for all $a \in \mathfrak{h}_r$ and $\zeta \in \mathfrak{h}_r^*$. The same is therefore true for an arbitrary field of H_r and any vertex operator, by Dong's lemma [50].

Finally, suppose that $\zeta_i = \iota(a^i) \in \mathfrak{h}_r^*$, for $i = 1, \ldots, k$. Then, a standard computation allows one to write the composition of the k vertex operators $\mathrm{V}_{\zeta_i}(z_i)$ as

$$\mathrm{V}_{\zeta_1}(z_1) \cdots \mathrm{V}_{\zeta_k}(z_k) = \prod_{i=1}^{k} \mathsf{e}^{\zeta_i} \cdot \prod_{1 \le i < j \le k} (z_i - z_j)^{(\zeta_i, \zeta_j)} \cdot \prod_{i=1}^{k} z_i^{a_0^i} \tag{1.58}$$

$$\cdot \prod_{m \ge 1} \exp\left(\frac{1}{m} \sum_{i=1}^{k} a_{-m}^i z_i^m\right) \exp\left(-\frac{1}{m} \sum_{i=1}^{k} a_m^i z_i^{-m}\right). \tag{1.59}$$

This explicit formula will be used repeatedly in Chapter 2.

1.3 The Virasoro vertex operator algebra

We define the Virasoro algebra as the vector space

$$V = \mathbb{C}\{L_n \mid n \in \mathbb{Z}\} \oplus \mathbb{C}C \tag{1.60}$$

with the Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m+1)m(m-1)\delta_{m+n,0}, \tag{1.61}$$

$$[L_m, C] = 0 \tag{1.62}$$

for all $m, n \in \mathbb{Z}$. The Virasoro algebra has the triangular decomposition

$$V = V_- \oplus V_0 \oplus V_+ \tag{1.63}$$

where

$$V_+ = \mathbb{C}\{L_n \mid n \in \mathbb{Z}_{\ge 1}\} \tag{1.64}$$

$$V_0 = \mathbb{C}\{L_0, C\} \tag{1.65}$$

$$V_- = \mathbb{C}\{L_n \mid n \in \mathbb{Z}_{\le -1}\} \tag{1.66}$$

With this triangular decomposition we define Verma modules which are constructed from a one dimensional representation $\mathbb{C}|\Delta\rangle$ of the Virasoro subalgebra $V_+ \bigoplus V_0$, where the module structure is defined as

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \tag{1.67}$$
$$C|\Delta\rangle = c|\Delta\rangle, \tag{1.68}$$
$$L_n|\Delta\rangle = 0, \qquad n \geq 1 \tag{1.69}$$

for some $c \in \mathbb{C}$ called the central charge. We then define a Verma module as the induced module

$$\mathcal{V}_\Delta = \mathrm{U}(V_-) \bigotimes_{\mathrm{U}(V_0 \oplus V_+)} \mathbb{C}|\Delta\rangle. \tag{1.70}$$

Every Verma module $\mathcal{V}_\Delta$ contains a unique maximal submodule I. We denote the irreducible quotient $\mathcal{V}_\Delta/I$ by $\mathcal{L}_\Delta$.

1.3.1 Virasoro Vertex Operator Algebra

We can put a vertex operator algebra structure on $\frac{\mathcal{V}_0}{\langle L_{-1}|0\rangle\rangle}$ with the data $\left(\frac{\mathcal{V}_0}{\langle L_{-1}|0\rangle\rangle}, |0\rangle, L_{-2}|0\rangle, Y\right)$. The Virasoro vertex algebra Vir is generated by a single field

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{1.71}$$

that satisfies the operator product expansion

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w} \tag{1.72}$$

The structure of the Virasoro Verma module $\mathcal{V}_0$ at different values of the central charge c is shown in Figure 1.1. From this figure we see that $\mathcal{V}_0$ always has a singular vector $L_{-1}|0\rangle$, which is an eigenvector v of L_0 such that $L_n v = 0$ for all $n \geq 1$. We define the universal Virasoro vertex operator algebra Vir as the data $\left(\frac{\mathcal{V}_0}{\langle L_{-1}|0\rangle\rangle}, |0\rangle, L_{-2}|0\rangle, Y\right)$ where the state-field correspondence $Y : \frac{\mathcal{V}_0}{\langle L_{-1}|0\rangle\rangle} \longrightarrow \mathrm{End}\, \frac{\mathcal{V}_0}{\langle L_{-1}|0\rangle\rangle}[\![z, z^{-1}]\!]$ is defined by

$$Y(L_{-n_k}\cdots L_{-n_1}|0\rangle) = :\frac{1}{(n_1-2)!}\cdots\frac{1}{(n_k-2)!}\partial^{n_1-2}T(z)\cdots\partial^{n_k-2}T(z): \tag{1.73}$$

For generic central charges, the maximal submodule is generated by $L_{-1}|0\rangle$ and the vacuum module of Vir is irreducible, implying that Vir is a simple vertex operator

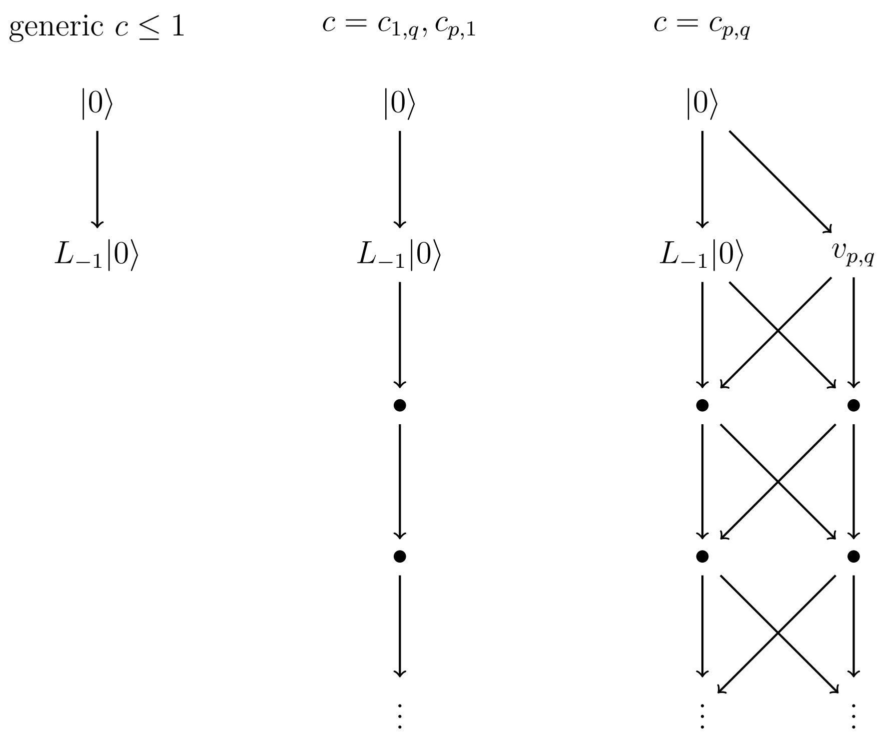


Fig. 1.1 The structure of Virasoro Verma modules of highest-weight 0 at different central charges $c \leq 1$ where $c_{p,q}$ is defined in Equation (1.74). Bullets are additional singular vectors. Arrows means one can apply Virasoro modes to go from one singular vector to another.

algebra. For $p, q \in \mathbb{Z}_{\geq 2}, (p, q) = 1$, let

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \tag{1.74}$$

Then in this case the maximal submodule is generated by two singular vectors $L_{-1}|0\rangle$ and $v_{p,q}$. We remark that singular vectors such as $v_{p,q}$ are very hard to compute in general. Several works [7, 57, 61, 62] have computed explicit formulae of singular vectors for different vertex operators algebras including the next chapter of this thesis, where we compute singular vectors of the W_n algebra.

1.3.2 Minimal models

As we have seen in the previous section, the Virasoro vertex operator algebra is reducible when the central charge is that of Equation (1.74). Quotienting out the maximal ideal we obtain the simple Virasoro vertex operator algebra, denoted by

$\mathsf{M}(p,q)$. These VOAs are called the Virasoro minimal models which was introduced in [14] and were proved to be rational [74]. The irreducible modules $\mathcal{L}_{r,s}^{\mathsf{M}(p,q)}$ of $\mathsf{M}(p,q)$ are all irreducible highest-weight Virasoro modules. They are parametrised by $1 \le r \le p-1, 1 \le s \le q-1$ and the conformal weights of the corresponding highest-weight states are given by

$$h_{r,s} = \frac{(qr-ps)^2 - (p-q)^2}{4pq} \tag{1.75}$$

We remark that $h_{r,s} = h_{p-r,q-s}$.

1.4 Free field realisation of the Virasoro vertex operator algebra

The Virasoro vertex operator algebra Vir of arbitrary central charge can be realised as a vertex operator subalgebra of the Heisenberg algebra. We remark that this is not obvious for the minimal model central charges $c_{p,q}$ as defined in Equation (1.74) as the extra singular vector could be mapped to zero. The explicit embedding is given by

$$T(z) = \frac{1}{2}{:}a(z)a(z){:} + \frac{\alpha_V}{2}\partial a(z) \tag{1.76}$$

where

$$c = 1 - 3\alpha_V^2 \tag{1.77}$$

Fock spaces are now highest-weight modules of the Virasoro algebra with highest weight

$$L_0|\lambda\rangle = \frac{1}{2}\lambda(\lambda - \alpha_V) \tag{1.78}$$

and we refer to Fock spaces as Feigin-Fuchs modules when considered as Virasoro modules. Although Fock spaces are simple as modules over the Heisenberg algebra, they need not be over the Virasoro algebra. The structures of Feigin-Fuchs modules, as shown in Figure 1.2, were determined in [36] . To describe this, let α_+, α_- be the roots of the polynomial $\frac{1}{2}\lambda(\lambda - \alpha_V) = 1$, so that

$$\alpha_+ + \alpha_- = \alpha_V, \qquad \alpha_+\alpha_- = -2. \tag{1.79}$$

Then we have

Theorem 1.4.1. *[36] For $r, s \in \mathbb{Z}$, let*

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_- + \frac{1-s}{2}\alpha_+ \tag{1.80}$$

Then

- *For $\alpha_+^2 \in \mathbb{C}^*$ (or equivalently for $\alpha_-^2 \in \mathbb{C}^*$), the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if $\lambda = \alpha_{r,s}$ for some $r, s \in \mathbb{Z}, rs > 0$*
- *If α_+^2 is non-rational (or equivalently if α_-^2 is non-rational), then the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if and only if $\lambda = \alpha_{r,s}$ for some $r, s \in \mathbb{Z}, rs > 0$*
- *If α_+^2 is positive rational (or equivalently if α_-^2 is positive rational), then the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if and only if $\lambda = \alpha_{r,s}$ for some $r, s \in \mathbb{Z}$*

1.4.1 Vertex operators

We will now construct vertex operators $V_\lambda(z)$ which can be thought of as fields corresponding the highest weight state $|\lambda\rangle$ in the Fock space $\mathcal{F}_\lambda$. We remark that this is a special case of Equation (1.55). We first extend the Heisenberg Lie algebra $\mathcal{H}$ with an element $\widehat{a}$. It has commutation relations

$$[\widehat{a}, a_n] = \delta_{n,0}\mathbf{1} \tag{1.81}$$

for $n \in \mathbb{Z}$. We then define

Definition 1.4.2. *For $\lambda \in \mathbb{C}$, we define a Virasoro vertex operator* $\mathrm{V}_\lambda(z)$

$$\mathrm{V}_\lambda(z) = e^{\lambda\widehat{a}} z^{\lambda a_0} \prod_{m\geq 1} \exp\Big(\lambda \frac{a_{-m}}{m} z^m\Big) \prod_{m\geq 1} \exp\Big(-\lambda \frac{a_m}{m} z^{-m}\Big) \tag{1.82}$$

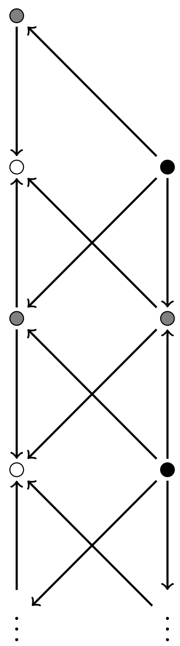


Fig. 1.2 Structure of Feigin-Fuchs modules in Theorem 1.4.1. White dots denote the singular vectors, while the gray and black dots denote the sub-singular vectors. The white dots (singular vectors) generate the maximal semisimple submodule. After quotienting this submodule the grey dots (sub-singular vectors) generate the maximal semisimple submodule in the quotient. Further quotienting the submodule results in a semisimple module, generated by the black dots (sub-singular vectors).

The action of vertex operators on the highest-weight vector of a Fock module $\mathcal{F}_\mu$ is

$$V_\lambda(z)|\mu\rangle = z^{\lambda\mu}\exp\left(\lambda\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right)|\lambda+\mu\rangle \tag{1.83}$$

Therefore vertex operators are maps

$$V_\lambda(z) : \mathcal{F}_\mu \to z^{\lambda\mu}\mathcal{F}_{\lambda+\mu}[\![z,z^{-1}]\!] \tag{1.84}$$

Vertex operators associated to certain values of λ are important because they can be used to construct Virasoro algebra homomorphisms. To begin, from Equation (1.76) we have a free field realisation of the Virasoro vertex operator algebra Vir inside the Heisenberg vertex operator algebra H, so that the Heisenberg vertex operator algebra has an energy momentum tensor $T(z)$ with central charge as in Equation (1.77). The operator product expansion of a vertex operator with $T(z)$ is

$$T(z)V_\lambda(w) = \frac{\frac{1}{2}\lambda(\lambda-\alpha_V)V_\lambda(w)}{(z-w)^2} + \frac{\partial V_\lambda(w)}{z-w} \tag{1.85}$$

Therefore, vertex operators are fields with conformal weight $\frac{1}{2}\lambda(\lambda-\alpha_V)$. Now consider α_+,α_- as defined in Equation (1.79). Then the vertex operators $\mathrm{V}_{\alpha_\pm}(z)$ are fields of conformal weight 1 and their operator product expansions with $T(z)$ can be written as a total derivative

$$T(z)\mathrm{V}_{\alpha_\pm}(w) \sim \frac{\mathrm{V}_{\alpha_\pm}(w)}{(z-w)^2} + \frac{\partial\mathrm{V}_{\alpha_\pm}(w)}{z-w} = \partial_w\left(\frac{\mathrm{V}_{\alpha_\pm}(w)}{(z-w)^2}\right) \tag{1.86}$$

We call $\mathrm{V}_{\alpha_\pm}(z)$ the screening fields. Therefore, we define

Definition 1.4.3. *The Virasoro screening operator is defined to be the residue*

$$\mathcal{S}_V = \oint_0 V_{\alpha_-}(z)\,\mathrm{d}z \tag{1.87}$$

Here, the residue is indicated using a simple anticlockwise contour that encircles 0 once (we absorb the usual factor of $2\pi\mathfrak{i}$ into the definition of the contour integral). Notice that the same definition can be made for α_+, though we will not require it in this thesis. It can be checked that $\mathcal{S}_V$ commutes with the energy momentum tensor,

$$[T(z),\mathcal{S}_V] = \int_0 \partial_w\left(\frac{\mathrm{V}_{\alpha_\pm}(w)}{(z-w)^2}\right)\mathrm{d}w = 0. \tag{1.88}$$

The action of $\mathcal{S}_V$ on a Fock space $\mathcal{F}_\mu$ is

$$\oint_0 V_{\alpha_-}(z)\,\mathrm{d}z|\mu\rangle = \oint_0 z^{\alpha-\mu}\exp\left(\alpha_-\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right)\mathrm{d}z|\mu+\alpha_-\rangle \tag{1.89}$$

Therefore $\mathcal{S}_V$ is a well-defined map whenever the contour in Equation (1.89) closes, that is, when $\alpha_-\mu\in\mathbb{Z}$. For such μ, $\mathcal{S}_V$ defines a Virasoro module homomorphism

$$\mathcal{S}_V : \mathcal{F}_\mu \mapsto \mathcal{F}_{\mu+\alpha_-} \tag{1.90}$$

An important feature of S_V is that the image $S_V|\mu\rangle\in\mathcal{F}_{\mu+\alpha_-}$ will be a singular vector, due to Equation (1.88). With a single screening operator the target Fock space at which singular vectors are constructed are rather limited. It turns out that we can compose multiple screening fields so that we can construct singular vectors in other Fock spaces. To do this we first compose multiple screening fields together. The composition $\mathrm{V}_{\alpha_-}(z_1)\cdots\mathrm{V}_{\alpha_-}(z_n)$, has the following form

$$\mathrm{V}_{\alpha_-}(z_1)\cdots\mathrm{V}_{\alpha_-}(z_r) = e^{\hat{a}\sum_{i=1}^r\alpha_-}\prod_{1\le i<j\le r}(z_i-z_j)^{\alpha_-^2}\prod_{i=1}^r z_i^{\alpha-a_0} \tag{1.91}$$

$$\prod_{m\ge1}\exp\left(\frac{a_{-m}}{m}\sum_{i=1}^r\alpha_-z_i^m\right)\exp\left(-\frac{a_m}{m}\sum_{i=1}^r\alpha_-z_i^m\right) \tag{1.92}$$

We now want to determine the Fock spaces $\mathcal{F}_\mu$ such that we can integrate $\mathrm{V}_{\alpha_-}(z_1)\cdots\mathrm{V}_{\alpha_-}(z_r)|\mu\rangle$. To see this, notice that up to a phase factor, we have

$$\mathrm{V}_{\alpha_-}(z_1)\cdots\mathrm{V}_{\alpha_-}(z_r)|\mu\rangle = \prod_{1\le i\ne j\le r}(z_i-z_j)^{\frac{\alpha_-^2}{2}}\prod_{i=1}^r z_i^{\alpha-\mu} \tag{1.93}$$

$$\prod_{m\ge1}\exp\left(\frac{a_{-m}}{m}\sum_{i=1}^r\alpha_-z_i^m\right)\exp\left(-\frac{a_m}{m}\sum_{i=1}^r\alpha_-z_i^m\right)|\mu+r\alpha_-\rangle \tag{1.94}$$

$$= \prod_{1\le i\ne j\le r}\left(1-\frac{z_j}{z_i}\right)^{\frac{\alpha_-^2}{2}}\prod_{i=1}^r z_i^{\alpha_-\mu+(r-1)\frac{\alpha_-^2}{2}} \tag{1.95}$$

$$\prod_{m\ge1}\exp\left(\frac{a_{-m}}{m}\sum_{i=1}^r\alpha_-z_i^m\right)|\mu+r\alpha_-\rangle \tag{1.96}$$

Therefore we see that we require $\alpha_-\mu + (r-1)\frac{\alpha_-^2}{2}$ to be an integer. Concretely, let $s \in \mathbb{Z}$, then we require μ, recalling that $\alpha_+\alpha_- = -2$, to be

$$\alpha_-\mu + (r-1)\frac{\alpha_-^2}{2} = s - 1 \tag{1.97}$$

$$\mu = \frac{1-r}{2}\alpha_- + \frac{1-s}{2}\alpha_+ \tag{1.98}$$

We therefore let $\alpha_{r,s}$ be as defined as in Equation (1.80). We remark that for any $k \in \mathbb{Z}$,

$$\alpha_{r+kp,s+kq} = \alpha_{r,s} \tag{1.99}$$

Then we have

Theorem 1.4.4. *[67] If $d(d+1)\frac{\alpha_-^2}{2} \notin \mathbb{Z}$ and $d(r-d)\frac{\alpha_-^2}{2} \notin \mathbb{Z}$, for all integers d satisfying $1 \le d \le r-1$, then for each Heisenberg weight $\alpha_{r,s}$, $s \in \mathbb{Z}$, there exists a cycle $\Gamma(r)$ such that*

$$[S_V]^r = \int_{\Gamma(r)} \mathrm{V}_{\alpha_-}(z_1)\cdots \mathrm{V}_{\alpha_-}(z_r)\, dz_1 \cdots dz_r \tag{1.100}$$

defines a non-trivial homomorphism

$$[S_V]^r : \mathcal{F}_{r,s} \longrightarrow \mathcal{F}_{-r,s} \tag{1.101}$$

1.4.2 Felder complexes

In this section we discuss Felder complexes [37] which are in essence complexes of Fock spaces with the screening operators as the differential maps. The main result is that one can realise the irreducible modules of the Virasoro minimal models as the zeroth cohomology of certain Felder complexes. Concretely, recall that the irreducible modules of the Virasoro minimal modules can be displayed in the form of a Kac table. For any central charge with

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} = 1 - 3\alpha_V^2 \tag{1.102}$$

we see that

$$\alpha_V = \frac{\sqrt{2}(p-q)}{pq} \tag{1.103}$$

Let

$$\alpha_+ = \sqrt{\frac{2p}{q}}, \qquad \alpha_- = -\sqrt{\frac{2q}{p}} \tag{1.104}$$

so that $\alpha = \alpha_+ + \alpha_- = \frac{\sqrt{2}(p-q)}{pq}$. Now we define

Definition 1.4.5. *For $r, s \in \mathbb{Z}$, let $\mathcal{F}_{r,s}$ be the Fock space with highest-weight state $|\alpha_{r,s}\rangle$, where*

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_- + \frac{1-s}{2}\alpha_+ \tag{1.105}$$

The L_0 eigenvalue of the highest-weight state $|\alpha_{r,s}\rangle$ of $\mathcal{F}_{r,s}$ is

$$L_0|\alpha_{r,s}\rangle = \frac{(qr-ps)^2 - (p-q)^2}{4pq}|\alpha_{r,s}\rangle. \tag{1.106}$$

We then construct a complex which is referred to in the literature as a Felder complex.

Theorem 1.4.6. *[37] For $1 \le r \le p-1, 1 \le s \le q-1$, let $C = (C^n, d^n)$ be a complex such that, for $n = 2k, 2k+1$,*

$$C^{2k} = \mathcal{F}_{-2kp+r,s}, \qquad d^{2k} = [S_V]^r \tag{1.107}$$

$$C^{2k+1} = \mathcal{F}_{-2kp-r,s}, \qquad d^{2k+1} = [S_V]^{p-r} \tag{1.108}$$

Then the cohomology of C is

$$H^n(C) = \delta_{n,0}\mathcal{L}_{r,s}^{\mathsf{M}(p,q)} \tag{1.109}$$

In order words, a Felder complex is exact except at $\mathcal{F}_{r,s}$, at which the cohomology is isomorphic to an irreducible module of the Virasoro minimal model $\mathcal{L}_{r,s}$. The Feigin-Fuchs modules appearing in a Felder complex, as dipicted in Figure 1.3 contain singular, subsingular and subsubsingular vectors. After quotienting the Feigin-Fuchs modules by its maximal semisimple submodule which are generated by the singular vectors, the subsingular vectors become singular. Further quotienting the maximal submodule in the quotient results in the subsubsingular vectors being singular. Diagrammatically one can "read" off the cohomology of this complex by examining the singular vectors, subsingular vectors and the subsubsingular vectors structure of the Feigin-Fuchs module.

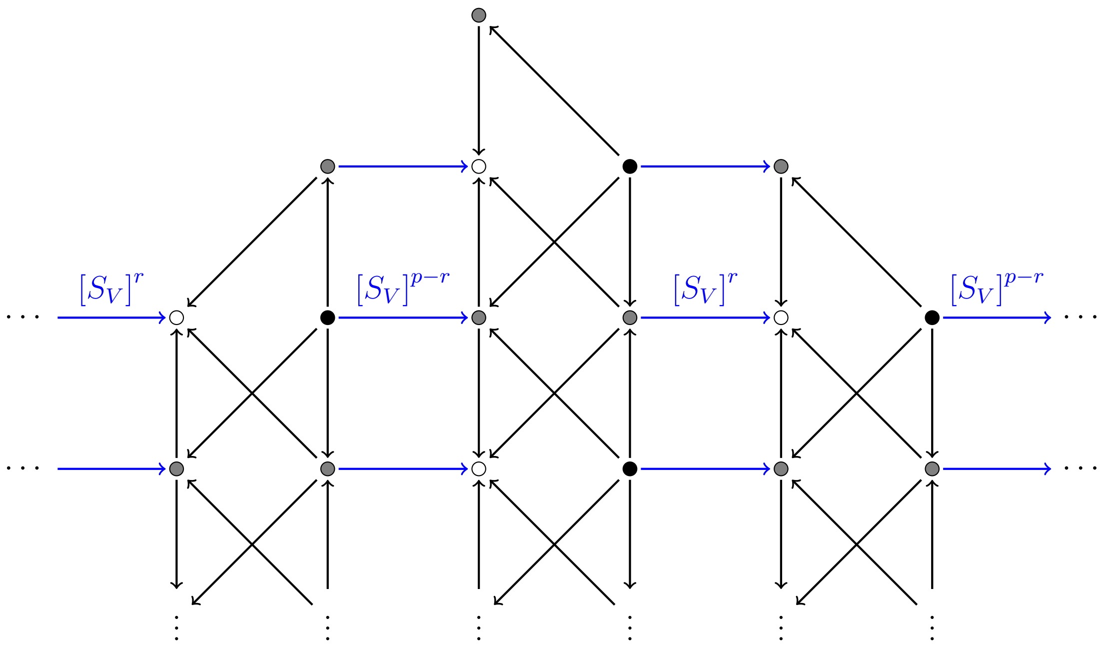


Fig. 1.3 Diagrammatic representations of the Felder complexes. $\mathcal{F}_{r,2q-s}, \mathcal{F}_{r,s}, \mathcal{F}_{r,-s}$, from left to right

1.5 The $\mathsf{V}_k(\mathfrak{sl}(2))$ Vertex Operator Algebra

Recall that $\mathfrak{sl}(2)$ has the commutation relations

$$[h,e] = 2e, \qquad [e,f] = h, \qquad [h,f] = -2f \tag{1.110}$$

We then construct the affine $\mathfrak{sl}(2)$ algebra $\widehat{\mathfrak{sl}}(2)_k$ defined as the vector space

$$\widehat{\mathfrak{sl}}(2)_k = \mathfrak{sl}(2) \otimes \mathbb{C}\left[z, z^{-1}\right] \oplus \mathbb{C}k \tag{1.111}$$

Suppose that J^a is a basis of $\mathfrak{sl}(2)$. Let $J^a_m = J^a \otimes z^m$ for $m \in \mathbb{Z}$ and define the Lie brackets of $\widehat{\mathfrak{sl}}(2)_k$ by

$$[J^a_m, J^b_n] = [J^a, J^b]_{m+n} + m\kappa(J^a, J^b)\delta_{m+n}k \tag{1.112}$$

$$[J^a_m, k] = 0, \qquad \text{for all } J^a \tag{1.113}$$

where $\kappa(\cdot,\cdot)$ is the normalised Killing form of $\mathfrak{sl}(2)$ such that $\kappa(h,h)=2$. Thus we see that $\widehat{\mathfrak{sl}}(2)_k$ has the following commutation relations:

$$[h_m, e_n] = 2e_{m+n}, \qquad [h_m, h_n] = 2m\delta_{m+n,0}k, \qquad [e_m, e_n] = 0 \tag{1.114}$$

$$[h_m, f_n] = -2f_{m+n}, \qquad [e_m, f_n] = h_{m+n} + m\delta_{m+n,0}k, \qquad [f_m, f_n] = 0 \tag{1.115}$$

The algebra $\widehat{\mathfrak{sl}}(2)_k$ has a triangular decomposition $\widehat{\mathfrak{sl}}(2)_k = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ given by

$$\mathfrak{g}_+ = \{e_n, h_{n+1}, f_{n+1} \mid n \geq 0\}, \tag{1.116}$$

$$\mathfrak{h} = \{h_0, k\}, \tag{1.117}$$

$$\mathfrak{g}_- = \{e_{-n-1}, h_{-n-1}, f_{-n} \mid n \geq 0\}. \tag{1.118}$$

Now consider a one dimensional representation $\mathbb{C}|\lambda\rangle$, $\lambda \in \mathbb{C}$ of $\mathfrak{h} \oplus \mathfrak{g}_+$ where $h_0|\lambda\rangle = \lambda|\lambda\rangle$, $\mathfrak{g}_+$ act trivially on $\mathbb{C}|\lambda\rangle$ and k acts as multiplication by the scalar $k \in \mathbb{C}$. The Verma module V_0 of $\widehat{\mathfrak{sl}}(2)_k$ induced from $\mathbb{C}|0\rangle$ contains a singular vector $f_0|0\rangle$. The quotient $\frac{\mathcal{V}_0}{\langle f_0|0\rangle\rangle}$ has the structure of a vertex algebra $\mathsf{V}_k(\mathfrak{sl}(2))$ which we call the universal affine $\mathfrak{sl}(2)$ vertex algebra at level k, $k \neq -2$. It is generated by the fields $e(z), h(z), f(z)$

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1} \tag{1.119}$$

$$h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1} \tag{1.120}$$

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1} \tag{1.121}$$

with the following operator product expansions:

$$h(z)e(w) \sim \frac{2e(w)}{z-w}, \qquad h(z)h(w) \sim \frac{2k}{(z-w)^2}, \qquad e(z)e(w) \sim 0 \tag{1.122}$$

$$h(z)f(w) \sim \frac{-2f(w)}{z-w}, \qquad e(z)f(w) \sim \frac{k}{(z-w)^2} + \frac{h(w)}{z-w}, \qquad f(z)f(w) \sim 0 \tag{1.123}$$

The conformal structure of $\mathsf{V}_k(\mathfrak{sl}(2))$ is given by the Sugawara energy momentum tensor

$$T(z) = \frac{1}{4t}(:h(z)h(z): + 2:e(z)f(z): + 2:f(z)e(z):) \tag{1.124}$$

where $t = k + 2$ and the central charge is

$$c = \frac{3k}{k+2} \tag{1.125}$$

Let $p, q \in \mathbb{Z}_{\geq 2}$ with $(p, q) = 1$. We define k to be an admissible level if

$$t = k + 2 = \frac{p}{q} \tag{1.126}$$

Note that we exclude the case when $q = 1$. The vertex operator algebra $\mathsf{V}_k(\mathfrak{sl}(2))$ at admissible level k is reducible [45] and we denote its simple quotient by $\mathcal{L}_k(\mathfrak{sl}(2))$. Ultimately we are interested in the representation theory of $\mathcal{L}_k(\mathfrak{sl}(2))$ where the modules of $\mathcal{L}_k(\mathfrak{sl}(2))$ are weight modules of $\widehat{\mathfrak{sl}}(2)_k$, in which h_0 act diagonalisably. Therefore we will first describe some of the weight modules of $\widehat{\mathfrak{sl}}(2)_k$.

1.5.1 Weight modules of $\widehat{\mathfrak{sl}}(2)$

Similar to the fact that highest-weight modules of the Virasoro algebra are modules of Vir, highest weight modules of $\widehat{\mathfrak{sl}}(2)_k$ are modules of $\mathsf{V}_k(\mathfrak{sl}(2))$. However in contrast to Vir, the category of highest-weight representations of $\mathcal{L}_k(\mathfrak{sl}(2))$ is not closed under conjugation as required for a conformal field theory. We therefore need to extend our module category to the relaxed category $\mathcal{R}^\sigma$. This category $\mathcal{R}^\sigma$ contains the relaxed highest-weight modules (which we will define later in the section), the twisted relaxed highest-weight modules under the spectral flow automorphisms defined in Equations (1.138) and (1.139) as well as non-split extensions of such modules. We remark that the subcategory of $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in $\mathcal{R}^\sigma$ is not a semi-simple category. Every relaxed highest-weight module in the relaxed category can be induced from a weight module of $\mathfrak{sl}(2)$, which is either a highest weight, a lowest weight or a dense module. We will therefore first describe these weight modules of $\mathfrak{sl}(2)$. Firstly we denote the conjugation automorphism of $\mathfrak{sl}(2)$ by $\overline{\omega}$, where

$$\overline{\omega}(h) = -h, \overline{\omega}(e) = f, \overline{\omega}(f) = e. \tag{1.127}$$

- For $\lambda \in \mathbb{N}$ the highest-weight module $\overline{\mathcal{L}}_\lambda$ of dimension $\lambda + 1$ has a basis of vectors v_μ where $\mu \in \{-\lambda, -\lambda + 2, \ldots, \lambda - 2, \lambda\}$. Each v_μ is an eigenvector of h_0 with $hv_\mu = \mu v_\mu$. These modules are self-conjugate.

- For $\lambda \in \mathbb{C}\backslash\mathbb{N}$ the infinite dimensional highest-weight module $\overline{\mathcal{L}}_\lambda$ has a basis of vectors v_μ where $\mu \in \{\lambda, \lambda-2, \lambda-4, \ldots\}$. Each v_μ is an eigenvector of h_0 with $hv_\mu = \mu v_\mu$.
- For $\lambda \in \mathbb{C}\backslash\mathbb{N}$ the infinite dimensional lowest-weight module $\overline{\omega}\Big(\overline{\mathcal{L}}_\lambda\Big)$ has a basis of vectors v_μ where $\mu \in \{-\lambda, -\lambda+2, -\lambda+4, \ldots\}$. Each v_μ is an eigenvector of h_0 with $hv_\mu = \mu v_\mu$.
- For $\lambda \in \mathbb{C}/2\mathbb{Z}, \Delta \in \mathbb{C}$ the dense module $\overline{\mathcal{R}}_{\lambda,\Delta}$[55] with basis v_μ, $\mu \in \lambda + 2\mathbb{Z}$, is neither highest nor lowest-weight. The action of $\mathfrak{sl}(2)$ on $\overline{\mathcal{R}}_{\lambda,\Delta}$ is given by

$$fv_\mu = v_{\mu-2} \tag{1.128}$$
$$hv_\mu = \mu v_\mu \tag{1.129}$$
$$ev_\mu = \frac{1}{4}(4t\Delta - \mu(\mu+2))v_{\mu+2} \tag{1.130}$$

  It turns out that $\overline{\mathcal{R}}_{\lambda,\Delta}$ is irreducible precisely when the set $\lambda + 2\mathbb{Z}$ does not contain any roots of the equation $4t\Delta - \mu(\mu+2) = 0$.

The simple weight modules of $\mathfrak{sl}(2)$ are then exhausted by the following:

- The $\lambda + 1$ dimensional modules $\overline{\mathcal{L}}_\lambda$, $\lambda \in \mathbb{N}$
- The highest-weight modules $\overline{\mathcal{L}}_\lambda$, $[\lambda] \in \mathbb{C}\backslash\mathbb{N}$
- The lowest-weight modules $\overline{\omega}\Big(\overline{\mathcal{L}}_\lambda\Big)$, $[\lambda] \in \mathbb{C}\backslash\mathbb{N}$
- The dense modules $\overline{\mathcal{R}}_{\lambda,\Delta}$, $[\lambda] \in \mathbb{C}/2\mathbb{Z}, \Delta \in \mathbb{C}$ with $4t\Delta \neq \mu(\mu+2)$ for any $\mu \in \lambda + 2\mathbb{Z}$.

Now, let $\mathfrak{g} = \widehat{\mathfrak{sl}}(2)_k$ and

$$\mathfrak{g}_{>0} = \text{span}\{e_n, h_n, f_n \mid n > 0\} \tag{1.131}$$
$$\mathfrak{g}_0 = \text{span}\{e_0, h_0, f_0, k\} \tag{1.132}$$
$$\mathfrak{g}_{<0} = \text{span}\{e_n, h_n, f_n \mid n < 0\} \tag{1.133}$$

Relaxed highest-weight modules of $\widehat{\mathfrak{sl}}(2)_k$ are constructed by taking a weight module $\overline{\mathcal{M}}$ of $\mathfrak{sl}(2)$ as a module of $\mathfrak{g}_0$ with k acting on $\overline{\mathcal{M}}$ as a constant $k \in \mathbb{C}$. We then extend $\overline{\mathcal{M}}$ to a module over $\mathfrak{g}_0 \oplus \mathfrak{g}_{>0}$ by demanding that all elements in $\overline{\mathcal{M}}$ are annihilated by $\mathfrak{g}_{>0}$. One can then induce this module to $\widehat{\mathfrak{sl}}(2)_k$ modules in the usual way, $\mathcal{M} = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{g}_{\geq 0})} \overline{\mathcal{M}}$. Highest-weight modules, a weight module which is generated by a single weight vector v that is annhilated by $\mathfrak{g}_+$ of $\widehat{\mathfrak{sl}}(2)_k$, are therefore special cases of relaxed highest-weight modules. Now, let ω be the conjugation automorphism of $\widehat{\mathfrak{sl}}(2)_k$, sending

$$\omega(e_n) = f_n, \qquad \omega(h_n) = -h_n \tag{1.134}$$

$$\omega(f_n) = e_n, \qquad \omega(k) = k \tag{1.135}$$

For $\lambda \in \mathbb{C}$, let $\mathcal{V}_\lambda$ be the $\widehat{\mathfrak{sl}}(2)$ Verma module of highest weight λ. The simple relaxed highest-weight modules are therefore obtained as follows:

- Inducing $\overline{\mathcal{L}}_\lambda$ with $\lambda \in \mathbb{N}$. This induced module is highest-weight and is isomorphic to $\mathcal{V}_\lambda/\mathcal{V}_{-\lambda-2}$, where $v_{-\lambda-2} = f_0^{\lambda+1} v_\lambda$. We denote the simple quotient by $\mathcal{L}_\lambda$.

- Inducing $\overline{\mathcal{L}}_\lambda$, the infinite dimensional highest weight module, $\lambda \in \mathbb{C}\backslash\mathbb{N}$. This results in the Verma module $\mathcal{V}_\lambda$. We again denote its simple quotient by $\mathcal{L}_\lambda$.

- Inducing $\overline{\omega}\big(\overline{\mathcal{L}}_\lambda\big)$ with $\lambda \in \mathbb{C}\backslash\mathbb{N}$. This results in the conjugate Verma module $\omega(\mathcal{V}_\lambda)$ which is neither highest nor lowest-weight. We denote the simple quotient by $\omega(\mathcal{L}_\lambda)$.

- Inducing the irreducible dense module $\overline{\mathcal{R}}_{\lambda,\Delta}$, with $4t\Delta \neq \mu(\mu+2)$ for all $\mu \in \lambda + 2\mathbb{Z}$, results in a relaxed Verma module that we denote by $\mathcal{R}_{\lambda,\Delta}$. We denote its simple quotient by $\mathcal{E}_{\lambda,\Delta}$.

For $r = 1, ..., p-1$ and $s = 1, ..., q$, we let

$$\lambda_{r,s} = r - 1 - (s-1)\frac{p}{q} \tag{1.136}$$

$$\Delta_{r,s} = \frac{\left(qr - p(s-1)\right)^2 - q^2}{4pq} \tag{1.137}$$

Then the highest-weight states of the modules $\mathcal{L}_{\lambda_{r,s}}$ have conformal weight $\Delta_{r,s}$. From now on we denote $\mathcal{L}_{r,s} = \mathcal{L}_{\lambda_{r,s}}$. The irreducible relaxed highest-weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$ were classified in [1]:

Theorem 1.5.1. *The irreducible relaxed highest-weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$ are precisely $\mathcal{L}_{r,s}$, $\omega(\mathcal{L}_{r,s})$ for $r = 1, ..., p-1, s = 1, ..., q$ and $\mathcal{E}_{\lambda,\Delta_{r,s}}$ for $\lambda \in \mathbb{R}\backslash 2\mathbb{Z}, \lambda \neq \lambda_{r,s}, \lambda_{p-r,q+2-s} \mod 2$ for $r = 1, ..., p-1, s = 2, ..., q$.*

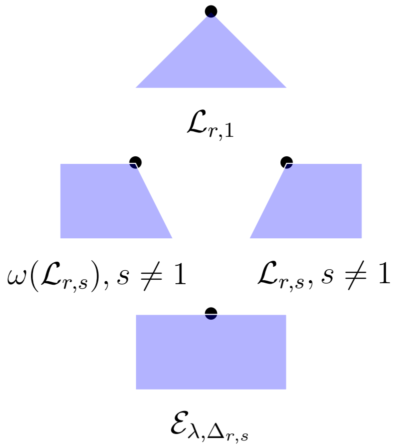


Fig. 1.4 The relaxed highest-weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$. The h_0-eigenvalues increase from right to left and the L_0-eigenvalues increase from top to bottom.

Figure 1.4 shows the structure of the relaxed highest-weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$, where the conformal weight increases from top to bottom and the $\mathfrak{sl}(2)$-weight, the h_0 eigenvalue, increases from right to left. In addition to these relaxed modules, we need to consider twisted relaxed modules under the spectral flow automorphism.

1.5.2 Spectral flow automorphisms of $\widehat{\mathfrak{sl}}(2)$

For each $l \in \mathbb{Z}$, we define the spectral flow automorphism $\sigma^l_{\mathfrak{sl}(2)}$ on $\widehat{\mathfrak{sl}}(2)$ by

$$\sigma^l_{\mathfrak{sl}(2)}(e_n) = e_{n-l}, \qquad \sigma^l_{\mathfrak{sl}(2)}(h_n) = h_n - \delta_{n,0} lk, \tag{1.138}$$

$$\sigma^l_{\mathfrak{sl}(2)}(f_n) = f_{n+l}, \qquad \sigma^l_{\mathfrak{sl}(2)}(k) = k \tag{1.139}$$

Figure 1.5 depicts the structure of the twisted relaxed highest-weight modules under the spectral flow automorphism. We remark that the spectral flow automorphism changes the conformal weights of states,

$$\sigma^l_{\mathfrak{sl}(2)}(L_0) = L_0 - \frac{1}{2} l h_0 + \frac{1}{4} l^2 k \tag{1.140}$$

and in particular the twisted relaxed highest-weight modules in general are no longer conformally bounded below, as opposed to the relaxed highest-weight modules. Furthermore, by noting that $\omega(\mathcal{L}_{r,1}) = \mathcal{L}_{r,1}$ for $r = 1, ..., p-1$, the following modules

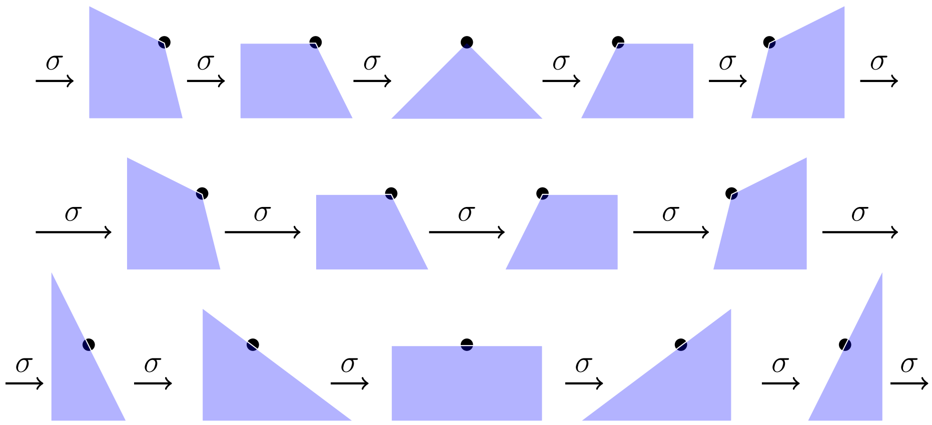


Fig. 1.5 The simple modules of $\mathcal{L}_k(\mathfrak{sl}(2))$. σ denotes the spectral flow automorphism. Again, the h_0-eigenvalues increase from right to left and the L_0-eigenvalues increase from top to bottom.

are related by spectral flow,

$$\sigma^{-1}_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) = \omega(\mathcal{L}_{p-r,q+1-s}) \tag{1.141}$$

$$\sigma_{\mathfrak{sl}(2)}(\mathcal{L}_{r,1}) = \mathcal{L}_{p-r,q} \tag{1.142}$$

for $r = 1, ..., p-1$, $s = 1, ..., q$. Figure 1.5 indicates precisely the simple objects in the relaxed category $\mathcal{R}^\sigma$, however as we remarked earlier $\mathcal{R}^\sigma$ is not a semi-simple category. We will end this section with the introduction of some of the reducible but indecomposable modules in $\mathcal{R}^\sigma$. These are characterised by the non-split exact sequences

$$0 \longrightarrow \mathcal{L}_{r,s} \longrightarrow \mathcal{E}^+_{r,s} \longrightarrow \omega(\mathcal{L}_{r,s}) \longrightarrow 0 \tag{1.143}$$

$$0 \longrightarrow \omega(\mathcal{L}_{r,s}) \longrightarrow \mathcal{E}^-_{r,s} \longrightarrow \mathcal{L}_{r,s} \longrightarrow 0 \tag{1.144}$$

for $r = 1, \cdots, p-1$ and $s = 2, \cdots, q$. We must also include $\sigma^l_{\mathfrak{sl}(2)}\left(\mathcal{E}^\pm_{r,s}\right)$ for $l \in \mathbb{Z}$ in $\mathcal{R}^\sigma_{\mathfrak{sl}(2)}$. Potentially there could be other indecomposable modules that are relevant, see [22, 23, 59, 42].

1.6 The bosonic ghost algebra

The bosonic ghost algebra G is a vertex operator algebra generated by two fields $\beta(z)$ and $\gamma(z)$ with the defining operator product expansions

$$\beta(z)\beta(w) \sim 0, \qquad \beta(z)\gamma(w) \sim \frac{1}{z-w}, \qquad \gamma(z)\gamma(w) \sim 0. \tag{1.145}$$

It has a one parameter family of conformal structures given by

$$T(z) = -(1-\lambda){:}\partial\beta(z)\gamma(z){:} + \lambda{:}\beta(z)\partial\gamma(z){:} \tag{1.146}$$

at which the central charge is

$$c = 12\lambda^2 - 12\lambda + 2 \tag{1.147}$$

For our application, we will only focus on the conformal structures at $c = 2$, where $\lambda = 1$ for the free-field realisation of $\mathsf{V}_k(\mathfrak{sl}(2))$ and $\lambda = 0$ for constructing the BRST complex. Thus, depending on λ, the mode expansions of the fields $\beta(z), \gamma(z)$ are respectively

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-\lambda}, \tag{1.148}$$

$$\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-(1-\lambda)} \tag{1.149}$$

The structure of the operator expansions imply that the modes of the fields are elements of the bosonic ghost Lie algebra $\mathcal{G}$. It has a basis $\{\beta_n, \gamma_n \mid n \in \mathbb{Z}\}$ satisfying the following commutation relations:

$$[\beta_m, \beta_n] = 0, \qquad [\beta_m, \gamma_n] = \delta_{m+n,0}\mathbf{1}, \qquad [\gamma_m, \gamma_n] = 0. \tag{1.150}$$

$\mathcal{G}$ has a triangular decomposition $\mathcal{G} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$

$$\mathfrak{g}_+ = \text{span}\{\beta_n, \gamma_{n+1} \mid n \geq 0\}, \tag{1.151}$$

$$\mathfrak{h} = \text{span}\{\mathbf{1}\}, \tag{1.152}$$

$$\mathfrak{g}_- = \text{span}\{\beta_{-n-1}, \gamma_{-n} \mid n \geq 0\} \tag{1.153}$$

Consider a one dimensional representation $\mathbb{C}|0\rangle$ of $\mathfrak{g}_+ \oplus \mathfrak{h}$ where $\mathfrak{g}_+$ acts trivially on $\mathbb{C}|0\rangle$ and $\{\mathsf{1}\}$ acts as a scalar 1. The Verma module induced from $\mathbb{C}|0\rangle$ is the vacuum module of the bosonic ghost algebra.
We define the ghost degree of any state in a $\mathcal{G}$ module as its eigenvalue of J_0^{G}, where

$$J^{\mathsf{G}}(z) = :\beta(z)\gamma(z): \tag{1.154}$$

and in particular,

$$J^{\mathsf{G}}(z)\beta(z) \sim -\frac{1}{z-w}, \qquad J^{\mathsf{G}}(z)\gamma(z) \sim \frac{1}{z-w} \tag{1.155}$$

1.6.1 Representation Theory of the bosonic ghost algebra

Since the Cartan subalgebra of $\mathcal{G}$ is spanned by $\mathsf{1}$ which is assumed to act as the identity, the highest weight module of $\mathcal{G}$ is unique: it is the vacuum module of G. The representations of G include modules that are not highest-weight. We therefore need to extend our category from the highest weight category to the relaxed category. To introduce the modules in this category, let

$$\mathfrak{g}_{>0} = \text{span}\{\beta_n, \gamma_n \mid n \geq 1\} \tag{1.156}$$

$$\mathfrak{g}_0 = \text{span}\{\beta_0, \gamma_0, \mathsf{1}\} \tag{1.157}$$

$$\mathfrak{g}_{<0} = \text{span}\{\beta_{-n}, \gamma_{-n} \mid n \geq 1\} \tag{1.158}$$

Relaxed modules are constructed by inducing modules over $\mathfrak{g}_0$. To describe the relaxed modules, we first introduce the conjugation automorphism τ of $\mathcal{G}$, by

$$\tau(\beta_n) = -\gamma_n \qquad \tau(\gamma_n) = \beta_n \qquad \tau(\mathsf{1}) = \mathsf{1} \tag{1.159}$$

The simple weight modules of $\mathfrak{g}_0$ are the following

- The Verma module $\overline{\mathcal{V}_0}$ generated by a vector $|0\rangle$ which is annihilated by β_0 and generated freely by γ_0. A basis of this module is $\{\gamma_0^n|0\rangle\}$ where $n \in \mathbb{N}$.
- The module $\tau\left(\overline{\mathcal{V}_0}\right)$ which can be obtained by applying conjugation to $\overline{\mathcal{V}_0}$. It is generated by a vector $\tau(|0\rangle)$ which is annihilated by γ_0 and generated freely by β_0. A basis of this module is $\{\beta_0^n\tau(|0\rangle)\}$, $n \in \mathbb{N}$.

- A one parameter family of modules $\overline{\mathcal{W}_{[\lambda]}}$ parameterised by $[\lambda] \in \mathbb{C}/\mathbb{Z}$, $[\lambda] \neq [0]$, these modules have a basis $\{u_\mu\}$ for $\mu \in \lambda + \mathbb{Z}$. Each u_μ is an eigenvalue of J_0 satisfying $J_0^{\mathsf{G}} u_\mu = \mu u_\mu$.

We can then induce these modules of $\mathfrak{g}_0$ to obtain modules of G. In particular, by inducing $\overline{\mathcal{V}_0}$ we obtain the vacuum module $\mathcal{V}_0$ of G, while inducing $\tau\left(\overline{\mathcal{V}_0}\right)$ gives us the twisted vacuum module $\tau(\mathcal{V}_0)$ under the conjugation automorphism. Finally, inducing the $\overline{\mathcal{W}_\lambda}, \lambda \neq [0]$ results in the new relaxed highest-weight modules $\mathcal{W}_\lambda, \lambda \neq [0]$. The modules $\mathcal{W}_\lambda, \lambda \neq [0]$, are simple. We also need to consider the modules $\mathcal{W}_0^\pm$ which can be characterised by the following exact sequences

$$0 \longrightarrow \mathcal{V}_0 \longrightarrow \mathcal{W}_0^+ \longrightarrow \tau(\mathcal{V}_0) \longrightarrow 0, \qquad 0 \longrightarrow \tau(\mathcal{V}_0) \longrightarrow \mathcal{W}_0^- \longrightarrow \mathcal{V}_0 \longrightarrow 0 \quad (1.160)$$

In addition to these modules, the module category of G also contains modules obtained by twisting the above modules with the spectral flow automorphism σ^l of $\mathcal{G}$, for $l \in \mathbb{Z}$, defined by

$$\sigma^l_{\mathcal{G}}(\beta_n) = \beta_{n-l}, \qquad \sigma^l_{\mathcal{G}}(\gamma_n) = \gamma_{n+l}, \qquad \sigma^l_{\mathcal{G}}(\mathbf{1}) = \mathbf{1} \quad (1.161)$$

These modules are objects in the module category $\mathcal{R}^\sigma_{\mathsf{G}}$. Twisting the module $\mathcal{V}_0$ with spectral flow, as defined in Equation (1.24), will change both the J_0^{G} and L_0 eigenvalues of the states in the module. In particular,

$$\tau(J_n^{\mathsf{G}}) = -J_n - 2\lambda\delta_{n,0}\mathbf{1}, \quad \sigma^l_{\mathcal{G}}\left(J_n^{\mathsf{G}}\right) = J_n^{\mathsf{G}} - l\delta_{n,0}\mathbf{1}, \quad (1.162)$$

$$\tau(L_n^\lambda) = L_n^\lambda + 2\lambda n J_n^{\mathsf{G}}, \qquad \sigma^l_{\mathcal{G}}\left(L_n^\lambda\right) = L_n^\lambda + (1-2\lambda) l J_n^{\mathsf{G}} - \frac{l(l+1-2\lambda)}{2}\delta_{n,0}\mathbf{1} \quad (1.163)$$

For example, Figure 1.6 depicts the new J_0^{G} and L_0 eigenvalues of states in $\sigma^l_{\mathcal{G}}(\mathcal{V}_0)$ under the spectral flow automorphisms.
For the rest of the thesis, we will denote the vacuum module of G by G itself.

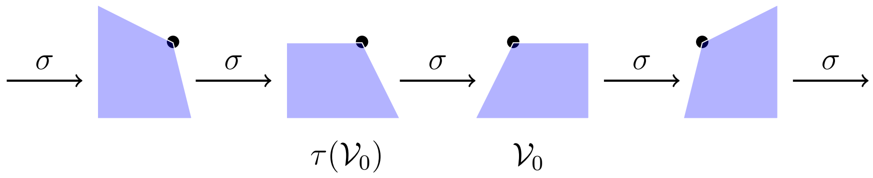


Fig. 1.6 The conformal weights of states in $\mathcal{V}_0$ under spectral flow. The ghost degree increases from left to right and the conformal weight increases from top to bottom.

1.7 Free field realisation of $\mathsf{V}_k(\mathfrak{sl}(2))$

Let $k+2 = \frac{p}{q}$ and let $\alpha_+ = \sqrt{2t} = \sqrt{\frac{2p}{q}}$ and $\alpha_- = -\sqrt{\frac{2q}{p}}$. We can realise $\mathsf{V}_k(\mathfrak{sl}(2))$ as a vertex subalgebra of $\mathsf{H} \otimes \mathsf{G}$. Concretely, there is an (non-trivial) injective map [71]

$$e(z) = \beta(z) \tag{1.164}$$

$$h(z) = -2{:}\beta(z)\gamma(z){:} + \alpha_+ a(z) \tag{1.165}$$

$$f(z) = -{:}\beta(z)\gamma(z)\gamma(z){:} + \alpha_+{:}a(z)\gamma(z){:} + \left(\frac{\alpha_+^2}{2} - 2\right)\partial\gamma(z) \tag{1.166}$$

Under this map, the energy momentum tensor becomes

$$T(z) = \frac{1}{2}{:}a(z)a(z){:} - \frac{1}{\alpha_+}\partial a(z) - {:}\beta(z)\partial\gamma(z){:} \tag{1.167}$$

$$c = 1 - \frac{12}{\alpha_+^2} + 2 = 3 - \frac{6}{t} \tag{1.168}$$

Given this free field realisation the tensor product of any module of the Heisenberg algebra and the bosonic ghost algebra can be restricted to modules of $\mathsf{V}_k(\mathfrak{sl}(2))$. Such modules are called Wakimoto modules. In summary, we have the following list of Wakimoto modules

- The highest weight Wakimoto module $\mathcal{F}_\lambda \otimes \mathsf{G}$
- The conjugate highest weight Wakimoto $\mathcal{F}_\lambda \otimes \tau(\mathsf{G})$
- The relaxed highest weight Wakimoto module $\mathcal{F}_\lambda \otimes \mathcal{W}_\mu$

The conformal weight and h_0 eigenvalue of the highest weight vector of a Wakimoto module $\mathcal{F}_\lambda \otimes \mathsf{G}$ can be computed from the free field realisation of $h(z)$ and $T(z)$

$$h_\lambda = \alpha_+ \lambda, \qquad \Delta_\lambda = \frac{1}{2}\lambda\left(\lambda + \frac{2}{\alpha_+}\right) \tag{1.169}$$

Similarly, the h_0 and L_0 weights of states with lowest conformal weights of $\mathcal{F}_\lambda \otimes \mathcal{W}_\mu$ are

$$h_{\lambda;\mu} = \alpha_+ \lambda + 2\mu, \qquad \Delta_\lambda = \frac{1}{2}\lambda\left(\lambda + \frac{2}{\alpha_+}\right) \tag{1.170}$$

where $\mu \in \lambda + 2\mathbb{Z}$.

1.7.1 Vertex operators

Similar to the Virasoro case, we can construct homomorphisms between $\widehat{\mathfrak{sl}}(2)_k$ modules, thereby constructing singular vectors. Firstly, if we write $e(z), h(z), f(z)$ in terms of their free field realisations as in Equations (1.164) to (1.166), then they have the following operator product expansions with the Heisenberg vertex operators $\mathrm{V}_p(z)$ from Definition 1.4.2,

$$e(z)\mathrm{V}_p(w) \sim 0, \qquad h(z)\mathrm{V}_p(w) \sim \frac{\alpha p \mathrm{V}_p(w)}{z-w}, \qquad f(z)\mathrm{V}_p(w) \sim \frac{\alpha p \mathrm{V}_p(w)\gamma(w)}{z-w}. \tag{1.171}$$

If we define

$$D(z) = :\beta(z)\mathrm{V}_{\alpha_-}(z): \tag{1.172}$$

then $D(z)$ has operator product expansions

$$e(z)D(w) \sim 0, \qquad h(z)D(w) \sim 0, \qquad f(z)D(w) \sim -t\partial_w \frac{\mathrm{V}_{\alpha_-}(z)}{z-w} \tag{1.173}$$

Therefore the zero mode of $D(z)$

$$S_{\mathfrak{sl}(2)} = \oint_0 D(z)\,\mathrm{d}z \tag{1.174}$$

is a $\widehat{\mathfrak{sl}}(2)_k$ module homomorphism whenever the action of $D(z)$ on a highest weight module is well-defined, similar to Equation (1.89). We can compose $\mathfrak{sl}(2)$ screening operators to obtain more module homomorphisms of $\widehat{\mathfrak{sl}}(2)$.

Theorem 1.7.1. *[62] Let $r \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}, t \in \mathbb{C}^*$ and suppose that $d(d+1)/t \notin \mathbb{Z}$ and $d(r-d)/t \notin \mathbb{Z}$, for all integers d satisfying $1 \leq d \leq r-1$, then for each Heisenberg weight $\alpha_{r,s}$, $s \in \mathbb{Z}$, there exists a cycle $\Gamma(r)$ such that*

$$\left[S_{\mathfrak{sl}(2)}\right]^r = \int_{\Gamma(r)} D(z_1)\cdots D(z_r)dz_1\cdots dz_r \tag{1.175}$$

defines a non-trivial homomorphism

$$\left[S_{\mathfrak{sl}(2)}\right]^r : \mathcal{F}_{r,s} \otimes \mathsf{G} \longrightarrow \mathcal{F}_{-r,s} \otimes \mathsf{G} \tag{1.176}$$

1.7.2 Bernard-Felder complexes

Similar to the Virasoro case, any highest-weight module of $\mathcal{L}_k(\mathfrak{sl}(2))$ can be realised as a sub-quotient of a Wakimoto module. Concretely, we can construct a complex [15] of Wakimoto modules such that the cohomology is non-trivial except at one degree, which is a highest-weight module of $\mathcal{L}_k(\mathfrak{sl}(2))$. We first start with the free field realisation of $\mathsf{V}_k(\mathfrak{sl}(2))$ at admissible level, so that $\alpha_+^2 = 2t = \frac{2p}{q} \implies \alpha_+ = \sqrt{\frac{2p}{q}}$. We consider $\mathcal{F}_{r,s} \otimes \mathsf{G}$ where the Fock space was defined in Definition 1.4.5, then the highest-weight states of such Wakimoto modules have h_0 and L_0 eigenvalues

$$h_0|\alpha_{r,s}\rangle \otimes |0\rangle_{\mathsf{G}} = r-1-(s-1)\frac{p}{q} \tag{1.177}$$

$$L_0|\alpha_{r,s}\rangle \otimes |0\rangle_{\mathsf{G}} = \frac{[qr-p(s-1)]^2-q^2}{4pq} \tag{1.178}$$

and we have

Theorem 1.7.2. *[15] Let $k+2=\frac{p}{q}$. For $1 \leq r \leq p-1, 1 \leq s \leq q-1$, let $C = (C^n, d^n)$ be a complex, for $n = 2k, 2k+1$, such that*

$$C^{2k} = \mathcal{F}_{-2kp+r,s} \otimes \mathsf{G}, \qquad d^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{1.179}$$

$$C^{2k+1} = \mathcal{F}_{-2kp-r,s} \otimes \mathsf{G}, \qquad d^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{1.180}$$

Then the cohomology of C is

$$H^n(C) = \delta_{n,0}\mathcal{L}_{r,s}. \tag{1.181}$$

Similar to Theorem 1.4.6, the Bernard-Felder complexes are exact except at zero degree, at which it is a highest-weight module of $\mathcal{L}_k(\mathfrak{sl}(2))$.

1.8 The fermionic ghost algebra B

In this section we will introduce the fermionic ghost algebra which will be used later on when we define the BRST complex. The fermionic ghost algebra is a vertex operator super-algebra generated by two fields $b(z)$ and $c(z)$ that satisfy the following operator product expansions

$$b(z)b(w) \sim 0, \qquad b(z)c(w) \sim \frac{1}{(z-w)^2}, \qquad c(z)c(w) \sim 0 \tag{1.182}$$

This algebra admits a one parameter family of conformal structures,

$$T^{\mathsf{B}}(z) = (1-\lambda){:}\partial b(z)c(z){:} - \lambda{:}b(z)\partial c(z){:} \tag{1.183}$$

at which the central charge is

$$c = -12\lambda^2 + 12\lambda - 2 \tag{1.184}$$

Under this energy momentum tensor, the conformal weights of $b(z)$ and $c(z)$ are λ and $1-\lambda$ respectively. For our purpose, which is constructing the BRST complex, we will specialise to $\lambda = 0$ and therefore consider the fermionic algebra at $c = -2$ only. That is, we have the energy momentum tensor

$$T^{\mathsf{B}}(z) = {:}\partial b(z)c(z){:} \tag{1.185}$$

where the central charge is $c = -2$ with $b(z)$ and $c(z)$ having conformal weights 0 and 1 respectively. Thus the fields have the form

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n}, \qquad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-1} \tag{1.186}$$

The structure of the operator product expansion implies that the modes of the fields satisfy the antibracket relations of the fermionic Lie super-algebra $\mathcal{B}$

$$\{b_m, b_n\} = 0 \qquad \{b_m, c_n\} = \delta_{m+n,0}\mathsf{1} \qquad \{c_m, c_n\} = 0 \tag{1.187}$$

and $\mathsf{1}$ is central. In addition to the energy momentum tensor, we introduce the ghost field

$$J^{\mathsf{B}}(z) = -{:}b(z)c(z){:} = {:}c(z)b(z){:} \tag{1.188}$$

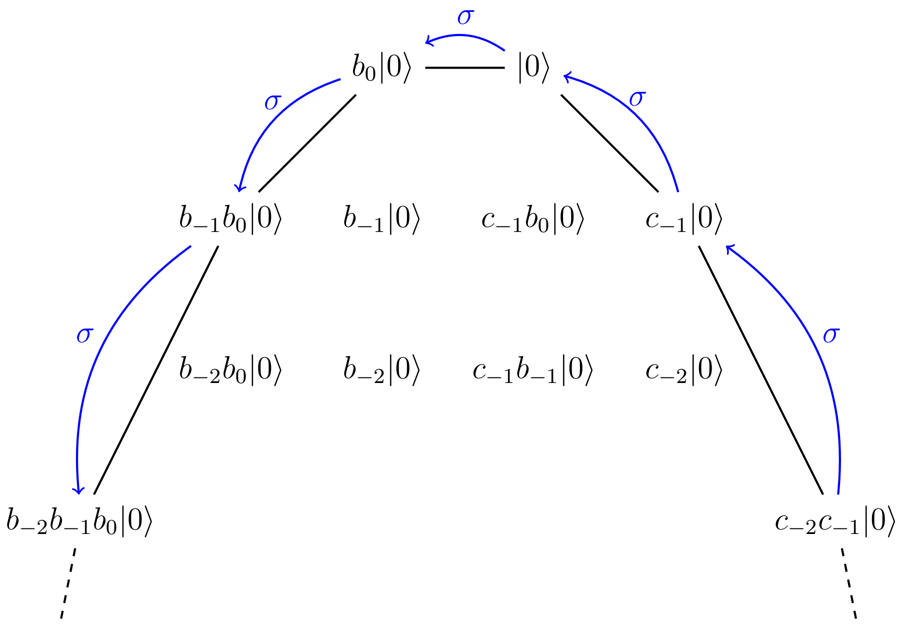


Fig. 1.7 The vacuum module of the fermionic ghost vertex operator super-algebra B. The ghost degree increases from left to right while the conformal weight increases from top to bottom. Blue arrows denote the images of states under the spectral flow automorphism.

so that the operator product expansion of $J^{\mathsf{B}}(z)$ with the generating fields are

$$J^{\mathsf{B}}(z)b(w) \sim -\frac{1}{z-w}, \qquad J^{\mathsf{B}}(z)c(w) \sim \frac{1}{z-w} \tag{1.189}$$

The vacuum module B of the fermionic ghost algebra at $c = -2$ is a highest weight module of $\mathcal{B}$ generated by $|0\rangle$ where

$$b_n|0\rangle = 0, n \geq 1 \tag{1.190}$$

$$c_n|0\rangle = 0, n \geq 0 \tag{1.191}$$

We define the ghost degree on any state in the vacuum module as simply the eigenvalue of J_0^{B}, the zero mode of $J^{\mathsf{B}}(z)$. This ghost number will play an important role when we construct modules of the Virasoro algebra by realising them as the cohomology of a certain complex called the BRST complex. In summary, we have introduced the conformal weight and the ghost degree on the fermionic ghost vacuum, see Figure 1.7.

The fermionic ghost algebra $\mathcal{B}$ also admits spectral flow algebra automorphisms defined as

$$\sigma^l_{\mathcal{B}}(b_n) = b_{n-l}, \qquad \sigma^l_{\mathcal{B}}(c_n) = c_{n+l}, \qquad \sigma^l_{\mathcal{B}}(1) = 1 \tag{1.192}$$

for all $l \in \mathbb{Z}$. This implies that

$$\sigma^l_{\mathcal{B}}\left(J^{\mathsf{B}}_0\right) = J^{\mathsf{B}}_0 + l \tag{1.193}$$

$$\sigma^l_{\mathcal{B}}\left(L^{\mathsf{B}}_0\right) = L^{\mathsf{B}}_0 + lJ^{\mathsf{B}}_0 + \frac{l(l+1)}{2} \tag{1.194}$$

Lemma 1.8.1. *The twisted vacuum module $\sigma^l_{\mathcal{B}}(\mathsf{B})$ under spectral flow is isomorphic to the vacuum module B itself for all l as a $\mathcal{B}$ module.*

Proof. From the action of $\mathcal{B}$ on the twisted module $\sigma^l_{\mathcal{B}}(\mathsf{B})$ defined in Equation (1.24), we see that for each $v \in \mathsf{B}$, there are only finitely many positive modes $b_n, c_n, n \geq 0$ that do not annihilate v. Moreover, the nilpotency of the modes b_n, c_n implies that the conformal weights of the states in $\sigma^l_{\mathcal{B}}(\mathsf{B})$ are bounded below. The nilpotency of the modes b_n, c_n also implies that the space of states with the highest conformal weight is finite dimensional and thus there exists a highest-weight vector. But the only highest-weight module is the vacuum module B as the Cartan subalgebra of $\mathcal{B}$ is just $\{1\}$ and so we must have $\sigma^l_{\mathcal{B}}(\mathsf{B}) \equiv \mathsf{B}$ □

We remark that the states in the vacuum module are permuted around under the spectral flow action, as shown in Figure 1.7.

1.9 The W_n Algebras

1.9.1 The W_3 Algebra

We will first restrict ourselves to the rank 2 Heisenberg vertex algebra H_2 and, in the vein of [33], define a family of subalgebras called the W_3 vertex operator algebras, or W_3 algebras for short. These algebras are parametrised by $\alpha_0 \in \mathbb{C}$ and are generated, in the sense of Theorem 1.1.7, by the energy-momentum tensor defined in Equation (1.50) and an additional primary field $W(z)$ of conformal weight 3.

In the basis $\{a^1, a^2\}$ defined in Section 1.2.3, for which the Gram matrix of the inner product $(-,-)$ is the Cartan matrix of $\mathfrak{sl}(3)$, the energy-momentum tensor $T(z)$ is

$$T(z) = \frac{1}{3}{:}a^1(z)a^1(z){:} + \frac{1}{3}{:}a^1(z)a^2(z){:} + \frac{1}{3}{:}a^2(z)a^2(z){:} + \alpha_0\partial_z a^1(z) + \alpha_0\partial_z a^2(z) \tag{1.195a}$$

and the central charge is $c = 2 - 24\alpha_0^2$. The conformal primary of weight 3 is then

$$\begin{aligned} W(z) = \frac{\sqrt{\beta}}{18\sqrt{3}}\Big[& 2{:}\big(a^2(z) - a^1(z)\big)\big(a^1(z) + 2a^2(z)\big)\big(2a^1(z) + a^2(z)\big){:} \\ & +9\alpha_0\Big({:}\partial a^2(z)\big(a^1(z) + 2a^2(z)\big){:} - {:}\partial a^1(z)\big(2a^1(z) + a^2(z)\big){:}\Big) && (1.195\text{b}) \\ & + 9\alpha_0^2\Big(\partial^2 a^2(z) - \partial^2 a^1(z)\Big)\Big], && (1.195\text{c}) \end{aligned}$$

where

$$\beta = \frac{16}{22 + 5c} = \frac{2}{4 - 15\alpha_0^2} \tag{1.196}$$

in the conventional normalisation, appropriate for $c \neq -\frac{22}{5}\left(\alpha_0 \neq \pm\frac{2}{\sqrt{15}}\right)$. A somewhat involved computation now determines the operator product expansion of $W(z)$ with itself to be

$$\begin{aligned} W(z)W(w) \sim & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} && (1.197) \\ & + \frac{\frac{3}{10}\partial^2 T(w) + 2\beta\Lambda(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w) + \beta\partial\Lambda(w)}{z-w}, && (1.198) \end{aligned}$$

where $\Lambda(z) = {:}T(z)T(z){:} - \frac{3}{10}\partial^2 T(z)$. This, along with the primary nature of $W(z)$, implies the commutation relations

$$\begin{aligned} \big[L_m, W_n\big] &= (2m - n)W_{m+n}, && (1.199) \\ \big[W_m, W_n\big] &= (m - n)\Big[\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2)\Big]L_{m+n} \\ &\quad + \beta(m - n)\Lambda_{m+n} + \frac{c}{360}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}, && (1.200) \end{aligned}$$

where $W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-3}$.

Since Fock spaces are modules over the Heisenberg vertex operator algebra H_2 and we have defined the generators of the W_3 algebra as fields of H_2, each Fock space is a W_3-module, by restriction. In particular, the highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$, $\zeta \in \mathfrak{h}_2^*$, is also a highest-weight vector for W_3:

$$L_n|\zeta\rangle = \delta_{n,0}h_\zeta|\zeta\rangle, \quad W_n|\zeta\rangle = \delta_{n,0}w_\zeta|\zeta\rangle, \quad n \geq 0. \tag{1.201}$$

Here, h_ζ was given in Equation (1.53) and the W_0-eigenvalue is given by

$$w_\zeta = \sqrt{3\beta}(\zeta, \omega_2 - \omega_1)\Big((\zeta, \omega_1) - \alpha_0\Big)\Big((\zeta, \omega_2) - \alpha_0\Big). \tag{1.202}$$

Our main reason for introducing vertex operators in Equation (1.55) is to construct linear maps between Fock spaces that commute with the action of an appropriate subalgebra of the Heisenberg vertex algebra, similar to the case in Section 1.4. Here, we wish to construct maps that commute with W_3, that is, W_3-module homomorphisms. Such module homomorphisms are called screening operators and they are constructed from screening fields, these being vertex operators whose operator product expansions with the fields of W_3 are total derivatives. For this, it clearly suffices to find fields whose operator product expansions with the generating fields $T(z)$ and $W(z)$ are total derivatives.

As the vertex operator $\mathrm{V}_\zeta(w)$ is a conformal primary of weight h_ζ, its operator product expansion with $T(z)$ will be a total derivative if and only if $h_\zeta = 1$. Unsurprisingly, the analogous computation for $W(z)$ is more involved (we used Thielemans' OPEdefs package for Mathematica), noting that a necessary condition for the operator product expansion $W(z)\mathrm{V}_\zeta(w)$ to be a total derivative is that the coefficient of $(z-w)^{-1}$ in this expansion is a total derivative. Analysing this explicitly, for general $\zeta \in \mathfrak{h}_2^*$, and recalling that $h_\zeta = 1$, we conclude that this coefficient will be a total derivative if $\zeta_1 = 0$ or $\zeta_2 = 0$ or if $\zeta_1 = \zeta_2$ and $\alpha_0 = 0$ (where the ζ_i denote Dynkin labels: $\zeta = \sum_i \zeta_i \omega_i$). As we are only interested in screening operators that exist for all values of α_0, it follows that there are exactly four possible weights ζ that can be used to construct screening operators: $\zeta = \alpha_\pm \alpha^1, \alpha_\pm \alpha^2$. Here, similar to Equation (1.79), we define

$$\alpha_+ = \frac{1}{2}\Big(\alpha_0 + \sqrt{\alpha_0^2 + 4}\Big), \quad \alpha_- = \frac{1}{2}\Big(\alpha_0 - \sqrt{\alpha_0^2 + 4}\Big) \tag{1.203}$$

to be the solutions of the quadratic equations $h_{\zeta_i \alpha^i} = 1$, for $i = 1, 2$. It only remains to confirm that the full operator product expansion $W(z)\mathrm{V}_\zeta(w)$, when ζ is one of the above weights, is indeed a total derivative:

$$W(z)\mathrm{V}_{\alpha_\pm\alpha^i}(w) \sim -(-1)^i\sqrt{\frac{6}{4-15\alpha_0^2}}\partial_w\left(\frac{\alpha_0\mathrm{V}_{\alpha_\pm\alpha^i}(w)}{2(z-w)^2} - \frac{:a^{*1}(w)\mathrm{V}_{\alpha_\pm\alpha^i}(w):}{z-w}\right), \quad i=1,2. \tag{1.204}$$

Having identified screening fields for W_3, we construct screening operators by taking residues:

$$\mathcal{S}_{\pm i} = \oint_0 \mathrm{V}_{\alpha_\pm\alpha^i}(w)\ \mathrm{d}w. \tag{1.205}$$

Taking the residue of a screening field $\mathrm{V}_{\alpha_\pm\alpha^i}(z)$ is of course only well-defined when it is acting on a H_2-module for which the exponents of z in the Fourier expansion of $\mathrm{V}_{\alpha_\pm\alpha^i}(z)$ are all integers. In case the H_2-module is the Fock space $\mathcal{F}_\eta$, this is satisfied if and only if $\alpha_\pm(\alpha^i,\eta)\in\mathbb{Z}$. These screening operators define W_3-module homomorphisms since

$$\left[T(z),\mathcal{S}_{\pm i}\right] = -\oint_z T(z)\mathrm{V}_{\alpha_\pm\alpha^i}(w)\ \mathrm{d}w = 0, \quad \left[W(z),\mathcal{S}_{\pm i}\right] = -\oint_z W(z)\mathrm{V}_{\alpha_\pm\alpha^i}(w)\ \mathrm{d}w = 0. \tag{1.206}$$

These identities follow from the mutual locality of Heisenberg fields and vertex operators, see Equation (1.57) as well as the fact that the operator product expansions are total derivatives.
Fortunately, similar to the construction of screening operators in Section 1.4, one can also construct screening operators by integrating compositions Equation (1.58) of multiple screening fields. In particular, composing r_2 copies of $\mathrm{V}_{\alpha_\pm\alpha^2}(w)$ with r_1 copies of $\mathrm{V}_{\alpha_\pm\alpha^1}(z)$ and then acting on $\mathcal{F}_\eta$ gives

$$\mathrm{V}_{\alpha_\pm\alpha^1}(z_1)\cdots\mathrm{V}_{\alpha_\pm\alpha^1}(z_{r_1})\mathrm{V}_{\alpha_\pm\alpha^2}(w_1)\cdots\mathrm{V}_{\alpha_\pm\alpha^2}(w_{r_2})\Big|_{\mathcal{F}_\eta} \tag{1.207}$$

$$\begin{aligned} &= \prod_{1\le i<j\le r_1}(z_i-z_j)^{2\alpha_\pm^2}\cdot\prod_{1\le i<j\le r_2}(w_i-w_j)^{2\alpha_\pm^2}\cdot\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}(z_i-w_j)^{-\alpha\pm^2}\\ &\quad\cdot\prod_{i=1}^{r_1}z_i^{\alpha_\pm(\alpha^1,\eta)}\cdot\prod_{j=1}^{r_2}w_j^{\alpha_\pm(\alpha^2,\eta)}\cdot\mathsf{e}^{r_1\alpha_\pm\alpha^1+r_2\alpha_\pm\alpha^2}\\ &\qquad\cdot\prod_{m\ge1}\exp\left[\alpha_\pm\left(a^1_{-m}\sum_{i=1}^{r_1}\frac{z_i^m}{m}+a^2_{-m}\sum_{i=1}^{r_2}\frac{w_i^m}{m}\right)\right]\exp\left[-\alpha_\pm\left(a^1_{m}\sum_{i=1}^{r_1}\frac{z_i^{-m}}{m}+a^2_{m}\sum_{i=1}^{r_2}\frac{w_i^{-m}}{m}\right)\right]. \end{aligned} \tag{1.208}$$

Up to a complex phase, which we suppress, the first five multivalued factors in this expression can be rewritten in the form

$$\prod_{1\le i\ne j\le r_1}\left(1-\frac{z_i}{z_j}\right)^{\alpha_\pm^2}\cdot\prod_{1\le i\ne j\le r_2}\left(1-\frac{w_i}{w_j}\right)^{\alpha_\pm^2}\cdot\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-\frac{w_j}{z_i}\right)^{-\alpha_\pm^2}\\ \cdot\prod_{i=1}^{r_1} z_i^{\alpha_\pm\left(\alpha^1,\eta\right)+\alpha_\pm^2(r_1-r_2-1)}\cdot\prod_{j=1}^{r_2} w_j^{\alpha_\pm\left(\alpha^2,\eta\right)+\alpha_\pm^2(r_2-1)},\quad (1.209)$$

thereby isolating the non-integer exponents of the z_i and w_j in the last two factors. Finding closed (multivariable) contours over which multivalued functions such as Equation (1.209) can be integrated (to obtain W_3-module homomorphisms) is a highly non-trivial problem. Fortunately, as shown in Theorem 1.4.4, Tsuchiya and Kanie solved this problem for the rank 1 Heisenberg vertex algebra [67] by constructing cycles with non-trivial homology classes over which screening operators can be integrated. These cycles, which we shall denote by $\Gamma(m;t)$ for $m\in\mathbb{Z}_{\ge 0}$ and $t\in\mathbb{C}\setminus\mathbb{Q}_{\le 0}$,[1] allow one to integrate expressions of the form

$$\int_{\Gamma(m;t)}\prod_{1\le i\ne j\le m}\left(1-\frac{z_i}{z_j}\right)^{1/t}\cdot f(z)\,\mathrm{d}z_1\cdots\mathrm{d}z_m,\qquad (1.210)$$

where $f(z)$ is a Laurent polynomial in $z_1,\dots,z_m$ which is invariant with respect to permuting the indices of its variables. We shall not describe the construction of these cycles in any detail. It will, however, be convenient to normalise them by requiring that

$$\int_{\Gamma(m;t)}\prod_{1\le i\ne j\le m}\left(1-\frac{z_i}{z_j}\right)^{1/t}\frac{\mathrm{d}z_1\cdots\mathrm{d}z_m}{z_1\cdots z_m}=1.\qquad (1.211)$$

The cycles $\Gamma(m;t)$ can be used to construct screening operators from the compositions Equation (1.207) whenever the exponents of the z_i and w_j are integers. If this is the case, then the screening operators are defined as

$$\mathcal{S}_\pm^{[r_1,r_2]}=\int_{\Gamma(r_1;1/\alpha_\pm^2)}\int_{\Gamma(r_2;1/\alpha_\pm^2)}\mathrm{V}_{\alpha_\pm\alpha^1}(z_1)\cdots\mathrm{V}_{\alpha_\pm\alpha^1}(z_{r_1})\mathrm{V}_{\alpha_\pm\alpha^2}(w_1)\cdots\mathrm{V}_{\alpha_\pm\alpha^2}(w_{r_2})\\ \mathrm{d}z_1\cdots\mathrm{d}z_{r_1}\,\mathrm{d}w_1\cdots\mathrm{d}w_{r_2}.\qquad (1.212)$$

[1] The range of the parameter t could in principle be extended to $\mathbb{C}\setminus\{0\}$. However, to avoid singularities in certain coefficients, this would require one to use a different normalisation of the Jack symmetric function basis presented in Appendix C. Moreover, some linear independence arguments would become more complicated. For simplicity, we therefore avoid non-positive rational values of the parameter t.

By construction, these screening operators are W_3-module homomorphisms when acting on appropriate Fock spaces.
We parametrise the Fock space weights for which the screening operators Equation (1.212) are defined as follows:

$$\zeta_{u_1,v_1;u_2,v_2} = \Big((1-u_1)\alpha_+ + (1-v_1)\alpha_-\Big)\omega_1 + \Big((1-u_2)\alpha_+ + (1-v_2)\alpha_-\Big)\omega_2, \quad u_1,u_2,v_1,v_2 \in \mathbb{Z}. \tag{1.213}$$

Considering the exponents of the last two factors of Equation (1.209), we conclude that the screening operators define W_3-module homomorphisms between the following Fock spaces:

$$\begin{aligned} \mathcal{S}_+^{[r_1,r_2]} &: \mathcal{F}_{\zeta_{r_1-r_2,s_1;r_2,s_2}} \to \mathcal{F}_{\zeta_{-r_1,s_1;r_1-r_2,s_2}}, \quad r_1,r_2 \in \mathbb{Z}_{\geq 0}, \quad s_1,s_2 \in \mathbb{Z}, \\ \mathcal{S}_-^{[s_1,s_2]} &: \mathcal{F}_{\zeta_{r_1,s_1-s_2;r_2,s_2}} \to \mathcal{F}_{\zeta_{r_1,-s_1;r_2,s_1-s_2}}, \quad r_1,r_2 \in \mathbb{Z}, \qquad s_1,s_2 \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{1.214}$$

Evaluating the action of these screening operators initially appears rather daunting. However, we know from Equation (1.207) that compositions of screening fields factorise into a product of a multivalued function and certain power series in the z_i and w_j that are symmetric with respect to permuting the z_i among themselves and, separately, the w_j among themselves. The theory of symmetric functions provides the tools that allow us to evaluate the action of these screening operators on certain Fock spaces. We refer to Appendix C for a brief review of the theory of symmetric functions.

1.9.2 The W_n Algebra

Continuing the pattern of ranks 1 and 2, the rank $n-1$ Heisenberg vertex operator algebra, with choice of energy-momentum tensor Equation (1.50), has $2(n-1)$ screening operators, $\mathrm{V}_\zeta(w)$ for $\zeta = \alpha_\pm \alpha^i$, where $\alpha_\pm$ was defined in Equation (1.203) and the α^i are the simple roots of $\mathfrak{sl}(n)$.
The W_n vertex operator algebra W_n is usually described as being generated by the Virasoro field $T(z)$ and $n-2$ Virasoro primary fields $W^3(z), \dots, W^n(z)$ of conformal weights $3, \dots, n$, respectively. Unfortunately, explicit formulae for these primaries, for example in terms of Heisenberg fields, rapidly increase in complexity as n increases and there are no known closed formulae for general n. Fortunately our computations do not require explicit expressions for the $W^k(z)$, only the fact that they commute with the screening operators.
We therefore turn to the definition of the W_n vertex operator algebra [51] in terms of a generating function called the quantum Miura transform. This constructs a different

set of generators of W_n that are not conformal primaries in general, but which are easily verified to commute with screening operators. We denote these new generating fields by $U^2(z) = T(z)$, $U^3(z), \dots, U^n(z)$ and their generating function by

$$R_n(z) = -\sum_{k=0}^{n} U_k(z)(\alpha_0\partial)^{n-k} = {:}(\alpha_0\partial_z - \epsilon^1(z))\cdots(\alpha_0\partial_z - \epsilon^n(z)){:}, \qquad (1.215)$$

where the ϵ^i are the weights of the defining representation of $\mathfrak{sl}(n)$ so that $\epsilon^1 + \cdots + \epsilon^n = 0$ and $\alpha^i = \epsilon^i - \epsilon^{i+1}$, for $i = 1, \dots, n-1$.
With the W_n algebra now defined explicitly as the algebra generated by the $U^i(z)$, $i = 2, \dots n$, we construct screening fields in a manner similar to W_3. As mentioned above, the vertex operators $\mathrm{V}_\zeta(w)$ with Heisenberg weights $\zeta = \alpha_\pm\alpha_1, \dots, \alpha_\pm\alpha_{n-1}$ are screening fields, because their operator product expansions with $R_N(z)$ are total derivatives:

$$R_n(z)\mathrm{V}_{\alpha_\pm\alpha_i}(w) \sim \partial_w\left(\frac{{:}R_n^i(w)\mathrm{V}_{\alpha_\pm\alpha_i}(w){:}}{z-w}\right). \qquad (1.216)$$

Here, $R_n^i(z)$ is defined as the product in Equation (1.215), but without the factors involving ϵ^i and ϵ^{i+1}.
As in the rank 2 case, the residues of the screening fields, when defined, commute with the W_n algebra, because their operator product expansions with the generating U^i fields are total derivatives, and therefore define module homomorphisms. Also as in the rank 2 case, one can compose screening fields and integrate them over suitable contours to construct yet more module homomorphisms. Note that it is sufficient to only compose screening fields whose weights are all rescalings of simple $\mathfrak{sl}(n)$ roots by either α_+ or α_-. This is because the two screening operators corresponding to the residues of $\mathrm{V}_{\alpha_+\alpha^i}(w)$ and $\mathrm{V}_{\alpha_-\alpha^j}(w)$ commute and can thus be considered independently. We shall therefore only present calculations involving the $\mathrm{V}_{\alpha_+\alpha^i}(w)$; those involving the $\mathrm{V}_{\alpha_-\alpha^j}(w)$ work in exactly the same way.

We therefore compose r_1 copies of $\mathrm{V}_{\alpha_+\alpha^1}(z^1)$ with r_2 copies of $\mathrm{V}_{\alpha_+\alpha^2}(z^2)$ and so on, evaluating this composition on a Fock space of weight η, to obtain

$$\begin{aligned}
&\prod_{i=1}^{r_1} \mathrm{V}_{\alpha_+\alpha^1}\Big(z_i^1\Big) \cdots \prod_{i=1}^{r_{n-1}} \mathrm{V}_{\alpha_+\alpha^{n-1}}\Big(z_i^{n-1}\Big)\Big|_{\mathcal{F}_\eta} \\
&\quad= \prod_{k=1}^{n-1} \prod_{1\le i<j\le r_k} (z_i^k - z_j^k)^{2\alpha_+^2} \cdot \prod_{k=1}^{n-2}\prod_{i=1}^{r_k}\prod_{j=1}^{r_{k+1}} (z_i^k - z_j^{k+1})^{-\alpha_+^2} \cdot \prod_{k=1}^{n-1}\prod_{i=1}^{r_k} (z_i^k)^{\alpha_+\left(\alpha^k,\eta\right)} \\
&\qquad\cdot \prod_{k=1}^{n-1} \mathsf{e}^{r_k\alpha_+\alpha^k} \cdot \prod_{k=1}^{n-1}\prod_{m\ge1} \exp\Bigg(\frac{\alpha_+ a_{-m}^k}{m}\sum_{i=1}^{r_k}(z_i^k)^m\Bigg) \exp\Bigg(-\frac{\alpha_+ a_m^k}{m}\sum_{i=1}^{r_k}(z_i^k)^{-m}\Bigg) \\
&\quad= \prod_{k=1}^{n-1} \prod_{1\le i\neq j\le r_k} \Bigg(1-\frac{z_i^k}{z_j^k}\Bigg)^{\alpha_+^2} \cdot \prod_{k=2}^{n-1}\prod_{i=1}^{r_k}\prod_{j=1}^{r_{k+1}} \Bigg(1-\frac{z_j^k}{z_i^{k-1}}\Bigg)^{-\alpha_+^2} \cdot \prod_{k=1}^{n-1}\prod_{i=1}^{r_k} (z_i^k)^{\alpha_+^2(r_k-r_{k+1}-1)+\alpha_+\left(\alpha^k,\eta\right)} \\
&\qquad\cdot \prod_{k=1}^{n-1} \mathsf{e}^{r_k\alpha_+\alpha^k} \cdot \prod_{k=1}^{n-1}\prod_{m\ge1} \exp\Bigg(\frac{\alpha_+ a_{-m}^k}{m}\sum_{i=1}^{r_k}(z_i^k)^m\Bigg) \exp\Bigg(-\frac{\alpha_+ a_m^k}{m}\sum_{i=1}^{r_k}(z_i^k)^{-m}\Bigg),
\end{aligned} \tag{1.217}$$

where we define $r_n = 0$. In analogy to the reasoning presented for the W_3 algebra in Section 1.9.1, one can construct a W_n-module homomorphism by choosing an appropriate contour. Integrating over the contours of Tsuchiya and Kanie [67] is well defined whenever

$$\alpha_+^2(r_k - r_{k+1} - 1) + \alpha_+\Big(\alpha^k, \eta\Big) \in \mathbb{Z}, \text{ for all } k = 1, \dots, n-1. \tag{1.218}$$

To parametrise the weights satisfying these constraints, we define

$$\zeta_{\mathbf{u},\mathbf{v}} = \sum_{i=1}^{n-1}((1-u_i)\alpha_+ + (1-v_i)\alpha_-)\omega_i, \quad \mathbf{u} = (u_1,\dots,u_{n-1}),\ \mathbf{v} = (v_1,\dots,v_{n-1}) \in \mathbb{Z}^{n-1}, \tag{1.219}$$

and define screening operators

$$\mathcal{S}_+^{[\mathbf{r}]} = \int_{\Gamma(r_1;1/\alpha_+^2)} \cdots \int_{\Gamma(r_{n-1};1/\alpha_+^2)} \prod_{i=1}^{r_1} \mathrm{V}_{\alpha_+\alpha^1}\Big(z_i^1\Big) \cdots \prod_{i=1}^{r_{n-1}} \mathrm{V}_{\alpha_+\alpha^{n-1}}\Big(z_i^{n-1}\Big) \cdot \prod_{k=1}^{n-1}\prod_{i=1}^{r_k} \mathrm{d}z_i^k, \tag{1.220}$$

where $\mathbf{r} \in \mathbb{Z}_{\ge0}^{n-1}$. These, in turn, induce W_n-module homomorphisms

$$\mathcal{S}_+^{[\mathbf{r}]} \colon \mathcal{F}_{\eta_{\mathbf{r},\mathbf{s}}^+} \to \mathcal{F}_{\theta_{\mathbf{r},\mathbf{s}}^+}, \quad \mathbf{r} \in \mathbb{Z}_{\ge0}^{n-1},\ \mathbf{s} \in \mathbb{Z}^{n-1}, \tag{1.221}$$

where $\eta^+_{\mathbf{r},\mathbf{s}} = \zeta_{(r_1-r_2,\ldots,r_{n-2}-r_{n-1},r_{n-1}),\mathbf{s}}$ and $\theta^+_{\mathbf{r},\mathbf{s}} = \zeta_{(-r_1,r_1-r_2,\ldots,r_{n-2}-r_{n-1}),\mathbf{s}}$. Similar screening operators $\mathcal{S}^{[\mathbf{s}]}_-$ are obtained by swapping the roles of α_+ and α_-, as well as $\mathbf{r}$ and $\mathbf{s}$, in this development.

Chapter 2

W_n singular vectors

In the case of the universal Virasoro vertex operator algebra, the singular vector generating the maximal ideal of the vertex operator algebra is important because it can be used to determine the irreducible representations of the Virasoro minimal models [74]. In [61], symmetric functions are used to compute the corresponding singular vectors, thereby obtaining the same results in [74]. This approach of using symmetric functions to compute singular vectors was used for other vertex operator algebras and classification of the spectrum (in suitable module categories) of the corresponding minimal models [17, 62]. In this chapter we generalise the result in [61] and obtain explicit formulae for singular vectors of the W_N algebra in certain Fock representations. This work was published in [60].

2.1 W_3 singular vectors

We now turn to the computation of singular vectors in Fock spaces, the idea being to realise them as images of highest-weight vectors under a W_3-module homomorphism (screening operator). For definiteness, we shall choose the screening operator $\mathcal{S}_+^{[r_1,r_2]}$ from Equation (1.212) that was constructed from r_1 copies of $\mathrm{V}_{\alpha_+\alpha_1}$ and r_2 copies of $\mathrm{V}_{\alpha_+\alpha_2}$. The computation for $\mathcal{S}_-^{[r_1,r_2]}$ is exactly the same and will be omitted. $\mathcal{S}_+^{[r_1,r_2]}$ has a well defined action on the Fock space $\mathcal{F}_\eta$ of H_2, where $\eta = \zeta_{r_1-r_2,s_1;r_2,s_2}$, sending it into $\mathcal{F}_\theta$, where $\theta = \zeta_{-r_1,s_1;r_1-r_2,s_2}$, see Equation (1.214).
We can now explicitly evaluate the action of the screening operator $\mathcal{S}_+^{[r_1,r_2]}$ on the highest-weight vector $|\eta\rangle \in \mathcal{F}_\eta$. Using Equation (1.207) and Equation (1.209), this

action is

$$\begin{aligned}
\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle &= \int_\Delta \mathrm{V}_{\alpha_+\alpha^1}\Big(z_1^1\Big)\cdots \mathrm{V}_{\alpha_+\alpha^1}\Big(z_{r_1}^1\Big)\mathrm{V}_{\alpha_+\alpha^2}\Big(z_1^2\Big)\cdots \mathrm{V}_{\alpha_+\alpha^2}\Big(z_{r_2}^2\Big)|\eta\rangle \prod_{k=1}^{2}\prod_{i=1}^{r_k} \mathrm{d}z_i^k \\
&= \int_\Delta \prod_{k=1}^{2}\prod_{1\le i\neq j\le r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2}\cdot \prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-\frac{z_j^2}{z_i^1}\right)^{-\alpha_+^2}\cdot \prod_{k=1}^{2}\prod_{i=1}^{r_k}\Big(z_i^k\Big)^{\alpha_+\left(\alpha^k,\eta\right)+\alpha_+^2(r_k-1)+1} \\
&\quad \cdot \prod_{i=1}^{r_1}\Big(z_i^1\Big)^{-\alpha_+^2 r_2}\cdot \prod_{k=1}^{2}\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k}{m}\sum_{i=1}^{r_k}\Big(z_i^k\Big)^m\right)\cdot|\theta\rangle \prod_{k=1}^{2}\prod_{i=1}^{r_k}\frac{\mathrm{d}z_i^k}{z_i^k} \\
&= \int_\Delta \prod_{k=1}^{2}\prod_{1\le i\neq j\le r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2}\cdot \prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-\frac{z_j^2}{z_i^1}\right)^{-\alpha_+^2} \\
&\quad \cdot \prod_{k=1}^{2}\prod_{i=1}^{r_k}\Big(z_i^k\Big)^{s_k}\cdot \prod_{k=1}^{2}\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k \mathsf{p}_m\Big(z^k\Big)}{m}\right)\cdot|\theta\rangle \prod_{k=1}^{2}\prod_{i=1}^{r_k}\frac{\mathrm{d}z_i^k}{z_i^k}.
\end{aligned} \tag{2.1}$$

Here, the integrals are over the product cycle $\Delta = \Gamma(r_1;\alpha_+^{-2})\times\Gamma(r_2;\alpha_+^{-2})$, see Section 1.9.1.
To proceed, we note that the tensor product $\Lambda\otimes_{\mathbb{C}}\Lambda$ is isomorphic to $\mathrm{U}\Big((\mathcal{H}_2)_-\Big) = \mathbb{C}[a_{-m}^k \mid k=1,2,\ m\in\mathbb{Z}_{>0}]$ as an algebra, by Equation (C.9). Concretely, let y_i^1 and y_i^2 denote the variables for the two factors of $\Lambda\otimes_{\mathbb{C}}\Lambda$ and consider the isomorphism

$$\rho_+\colon \Lambda\otimes_{\mathbb{C}}\Lambda \longrightarrow \mathrm{U}\Big((\mathcal{H}_2)_-\Big),\qquad \mathsf{p}_m(y^k)\longmapsto \frac{1}{\alpha_+}a_{-m}^k,\qquad k=1,2,\ m\in\mathbb{Z}_{>0}. \tag{2.2}$$

where $\mathsf{p}_m(y^k)$ is a power sum defined in Equation (C.1). Then, we may write

$$\begin{aligned}
\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k\mathsf{p}_m\Big(z^k\Big)}{m}\right) &= \rho_+\left(\prod_{m\ge 1}\exp\left(\alpha_+^2\frac{\mathsf{p}_m(y^k)\mathsf{p}_m(z^k)}{m}\right)\right) \\
&= \rho_+\left(\prod_{i\ge 1}\prod_{j=1}^{r_k}(1-y_i^k z_j^k)^{-\alpha_+^2}\right),
\end{aligned} \tag{2.3}$$

recognising the Cauchy kernel Equation (C.19) with parameter $t=\alpha_+^{-2}$.
For $k=1$, we expand this Cauchy kernel in terms of Jack polynomials $\mathsf{P}_\lambda^t\Big(y^1\Big)$ and their duals $\mathsf{Q}_\lambda^t\Big(z^1\Big)$ as in Equation (C.19):

$$\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^1\mathsf{p}_m\Big(z^1\Big)}{m}\right) = \rho_+\left(\sum_\lambda \mathsf{P}_\lambda^t\Big(y^1\Big)\mathsf{Q}_\lambda^t\Big(z^1\Big)\right). \tag{2.4}$$

For $k=2$, we first combine the Cauchy kernel with that appearing in the second factor of the integrand of Equation (2.1):

$$\rho_+\left(\prod_{i\geq 1}\prod_{j=1}^{r_2}(1-y_i^2z_j^2)^{-\alpha_+^2}\right)\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\Big(1-(z_i^1)^{-1}z_j^2\Big)^{-\alpha_+^2}=\rho_+\left(\sum_\mu \mathsf{P}_\mu^t\Big(y^2\cup(z^1)^{-1}\Big)\mathsf{Q}_\mu^t\Big(z^2\Big)\right). \tag{2.5}$$

Here, we have noted that the product is a Cauchy kernel in the alphabets $\{y_i^2\}\cup\{(z_\ell^1)^{-1}\}$ and $\{z_j^2\}$. This may be further simplified using skew-Jacks as in Equation (C.22):

$$\mathsf{P}_\mu^t\Big(y^2\cup(z^1)^{-1}\Big)=\sum_\nu \overline{\mathsf{P}_\nu^t\Big(z^1\Big)}\mathsf{P}_{\mu/\nu}^t\Big(y^2\Big). \tag{2.6}$$

where $\overline{\mathsf{P}_\nu^t\Big(z^1\Big)}=\mathsf{P}_\nu^t\Big(\overline{z^1}\Big)$. We recall that the skew-Jack $\mathsf{P}_{\mu/\nu}^t$ is 0 unless $\nu\subseteq\mu$. By considering from Equation (C.24) the definition of $G_n^t(x)$, we also have the integrating kernels

$$\prod_{k=1}^{2}\prod_{1\leq i\neq j\leq r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2}=G_{r_1}^t(z^1)G_{r_2}^t(z^2) \tag{2.7}$$

of the symmetric polynomial inner product Equation (C.23). Finally, the product $\prod_{k=1}^2\prod_{i=1}^{r_k}\Big(z_i^k\Big)^{s_k}$ is a product of rectangular Jack polynomials. However, here we have to be careful with the signs of the s_k. Indeed, Equation (1.53) and Equation (1.213) show that the conformal weights of the highest-weight vectors $|\eta\rangle$ and $|\theta\rangle$ differ by

$$h_\eta-h_\theta=-r_1s_1-r_2s_2. \tag{2.8}$$

This must be non-negative if the screening operator $\mathcal{S}_+^{[r_1,r_2]}$ is to map $|\eta\rangle$ to a singular descendant of $|\theta\rangle$. We shall therefore assume from here on that $s_1,s_2\in\mathbb{Z}_{\leq 0}$. Thus,

$$\prod_{k=1}^{2}\prod_{i=1}^{r_k}\Big(z_i^k\Big)^{s_k}=\overline{\mathsf{P}_{[-s_1^{r_1}]}^t\Big(z^1\Big)}\,\overline{\mathsf{P}_{[-s_2^{r_2}]}^t\Big(z^2\Big)}, \tag{2.9}$$

using Equation (C.15).

Putting all this back in Equation (2.1), the integrand factorises and we get

$$\begin{aligned}\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle=\sum_{\lambda,\mu,\nu}\rho_+\Big(\mathsf{P}_\lambda^t\Big(y^1\Big)\Big)\int_{\Gamma(r_1;t)}G_{r_1}^t(z^1)\overline{\mathsf{P}_{[-s_1^{r_1}]}^t\Big(z^1\Big)}\,\overline{\mathsf{P}_\nu^t\Big(z^1\Big)}\mathsf{Q}_\lambda^t\Big(z^1\Big)\prod_{i=1}^{r_1}\frac{\mathrm{d}z_i^1}{z_i^1}\\ \cdot\rho_+\Big(\mathsf{P}_{\mu/\nu}^t\Big(y^2\Big)\Big)\int_{\Gamma(r_2;t)}G_{r_2}^t(z^2)\overline{\mathsf{P}_{[-s_2^{r_2}]}^t\Big(z^2\Big)}\mathsf{Q}_\mu^t\Big(z^2\Big)\prod_{i=1}^{r_2}\frac{\mathrm{d}z_i^2}{z_i^2}\cdot|\theta\rangle\end{aligned}$$

using Equation (C.23),

$$= \sum_{\lambda,\mu,\nu} \left\langle \mathsf{Q}^t_\lambda, \mathsf{P}^t_{\nu+[-s_1^{r_1}]} \right\rangle^t_{r_1} \left\langle \mathsf{Q}^t_\mu, \mathsf{P}^t_{[-s_2^{r_2}]} \right\rangle^t_{r_2} \rho_+ \Big(\mathsf{P}^t_\lambda \big(y^1\big) \mathsf{P}^t_{\mu/\nu} \big(y^2\big) \Big) |\theta\rangle$$

by Equations (C.16) and (C.25),

$$= \sum_{\substack{\nu\subseteq[-s_2^{r_2}] \\ \ell(\nu)\leq r_1}} b^t_{\nu+[-s_1^{r_1}]}(r_1) b^t_{[-s_2^{r_2}]}(r_2) \rho_+ \Big(\mathsf{P}^t_{\nu+[-s_1^{r_1}]} \big(y^1\big) \mathsf{P}^t_{[-s_2^{r_2}]/\nu} \big(y^2\big) \Big) |\theta\rangle, \tag{2.10}$$

where $l(\nu)$ is the length of ν. As $b^t_{[-s_2^{r_2}]}(r_2)$ is independent of ν (and non-zero), it may be absorbed into the normalisation of the singular vector. Our final result for the singular vector is therefore

$$\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle = \sum_{\substack{\nu\subseteq[-s_2^{r_2}] \\ \ell(\nu)\leq r_1}} b^t_{\nu+[-s_1^{r_1}]}(r_1) \rho_+ \Big(\mathsf{P}^t_{\nu+[-s_1^{r_1}]} \big(y^1\big) \mathsf{P}^t_{[-s_2^{r_2}]/\nu} \big(y^2\big) \Big) |\theta\rangle. \tag{2.11}$$

This form is now easily implemented in computer algebra packages.
The right hand side of Equation (2.11) is easily seen to be manifestly non-zero by noting that the total degree, with respect to the a^2_{-m}, of the summand corresponding to the empty partition $\nu = [\,]$ is maximal and that all other summands have strictly lesser degrees. Since $b^t_{[-s_1^{r_1}]}(r_1)$, $\mathsf{P}^t_{[-s_1^{r_1}]}\big(y^1\big)$ and $\mathsf{P}^t_{[-s_2^{r_2}]}\big(y^2\big)$ are all non-zero, this summand is therefore linearly independent of all others. The conclusion is that Equation (2.11) defines a singular vector for every $r_1, r_2 \in \mathbb{Z}_{\geq 0}$, $s_1, s_2 \in \mathbb{Z}_{\leq 0}$ and $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$.

2.1.1 Examples

We now illustrate the W_3 singular vector formula Equation (2.11) with three examples.

Example 1

For our first example, we compute a singular vector for the case when $t = \frac{4}{5}$, so that

$$\alpha_+ = \frac{\sqrt{5}}{2}, \quad \alpha_- = -\frac{2}{\sqrt{5}}, \quad \alpha_0 = \frac{1}{2\sqrt{5}}, \quad c = \frac{4}{5}. \tag{2.12}$$

This central charge corresponds to that of the 3-state Potts model, described by the W_3 minimal model $\mathsf{W}_3(4,5)$ (the parameters here are the numerator and denominator of t in reduced form). Take $r_1 = r_2 = -s_1 = -s_2 = 1$ for simplicity. Then, the map $\mathcal{S}_+^{[1,1]}$ sends $\mathcal{F}_\eta$, where $\eta = \zeta_{0,-1;1,-1}$, into $\mathcal{F}_\theta$, where $\theta = \zeta_{-1,-1;0,-1}$. We note that

$$h_\eta = \frac{13}{6}, \quad h_\theta = \frac{1}{6}, \quad w_\eta = \frac{187}{9\sqrt{390}}, \quad w_\theta = -\frac{7}{9\sqrt{390}}, \tag{2.13}$$

by Equation (1.53) and Equation (1.202). The conformal weight h_θ is not one of those associated with the 3-state Potts model. Nevertheless, the Fock space $\mathcal{F}_\theta$ has a singular vector at grade 2 in accordance with Equation (2.8). Equation (2.11) writes it in the form

$$\mathcal{S}_+^{[1,1]}|\eta\rangle = \sum_{\nu\subseteq[1]} b^{4/5}_{\nu+[1]}(1)\rho_+\Big(\mathsf{P}^{4/5}_{\nu+[1]}\big(y^1\big)\mathsf{P}^{4/5}_{[1]/\nu}\big(y^2\big)\Big)|\theta\rangle. \tag{2.14}$$

There are only two partitions ν to consider. Using Equation (C.25), Equation (2.2) and SageMath to write Jacks and skew-Jacks in terms of power sums, we have

$$\begin{aligned}
\nu = [0]:\quad b^{4/5}_{[0]+[1]}(1) = \frac{5}{4}, \quad & \rho_+\Big(\mathsf{P}^{4/5}_{[0]+[1]}\big(y^1\big)\Big) = \rho_+\Big(\mathsf{p}_{[1]}\big(y^1\big)\Big) = \frac{2}{\sqrt{5}}a^1_{-1}, \\
& \rho_+\Big(\mathsf{P}^{4/5}_{[1]/[0]}\big(y^2\big)\Big) = \rho_+\Big(\mathsf{p}_{[1]}\big(y^2\big)\Big) = \frac{2}{\sqrt{5}}a^2_{-1}. \\
\nu = [1]:\quad b^{4/5}_{[1]+[1]}(1) = \frac{5}{4}\frac{9}{8}, \quad & \rho_+\Big(\mathsf{P}^{4/5}_{[1]+[1]}\big(y^1\big)\Big) = \rho_+\Big(\frac{5}{9}\mathsf{p}_{[1,1]}\big(y^1\big) + \frac{4}{9}\mathsf{p}_{[2]}\big(y^1\big)\Big) \\
& \qquad = \frac{4}{5}\frac{5}{9}a^1_{-1}a^1_{-1} + \frac{2}{\sqrt{5}}\frac{4}{9}a^1_{-2}, \\
& \rho_+\Big(\mathsf{P}^{4/5}_{[1]/[1]}\big(y^1\big)\Big) = \rho_+\Big(\mathsf{p}_{[0]}\big(y^1\big)\Big) = 1.
\end{aligned} \tag{2.15}$$

The singular vector is therefore explicitly identified as

$$\mathcal{S}_+^{[1,1]}|\eta\rangle = \left(a^1_{-1}a^2_{-1} + \frac{5}{8}a^1_{-1}a^1_{-1} + \frac{\sqrt{5}}{4}a^1_{-2}\right)|\theta\rangle. \tag{2.16}$$

Consider the W_3 Verma module $\mathcal{V}_\vartheta$ whose highest-weight vector $|\vartheta\rangle$ has L_0- and W_0-eigenvalue h_θ and w_θ, as given in Equation (2.13). By direct calculation, $\mathcal{V}_\vartheta$ has a singular vector $|\chi\rangle$, unique up to normalisation, at grade 2:

$$|\chi\rangle = \left(\frac{390}{119}W_{-1}W_{-1} - \frac{\sqrt{390}}{17}W_{-2} + \frac{10\sqrt{390}}{119}L_{-1}W_{-1} + L_{-1}L_{-1}\right)|\vartheta\rangle. \tag{2.17}$$

The free field realisation $f\colon \mathsf{W}_3 \hookrightarrow \mathsf{H}_2$ defined by Equation (1.195) induces a W_3-module homomorphism

$$f_\vartheta\colon \mathcal{V}_\vartheta \longrightarrow \mathcal{F}_\theta, \quad f_\vartheta(U|\vartheta\rangle) = f(U)|\theta\rangle. \tag{2.18}$$

Here, U is an arbitrary element of the W_3 mode algebra, this being the (unital) associative algebra generated by the L_m and W_n subject to Equation (1.199). Explicit calculation now verifies that the image of the singular vector $|\chi\rangle$ under f_ϑ is, of course, that constructed in Equation (2.16):

$$f_\vartheta(|\chi\rangle) = \left(\frac{5}{4}a^1_{-1}a^2_{-1} + \frac{25}{32}a^1_{-1}a^1_{-1} + \frac{5\sqrt{5}}{16}a^1_{-2}\right)|\theta\rangle = \frac{5}{4}\mathcal{S}^{[1,1]}_{+}|\eta\rangle. \tag{2.19}$$

Example 2

For our second example, we compute a grade three singular vector for general central charges. Let $r_1 = 2$ and $r_2 = -s_1 = -s_2 = 1$, so that $\eta = \zeta_{1,-1;1,-1}$ and $\theta = \zeta_{-2,-1;1,-1}$. In order to evaluate the singular vector formula Equation (2.11), we need to compute

$$\begin{gathered} b^t_{[1,1]}(2) = \frac{2}{t+1}\frac{1}{t}, \quad b^t_{[2,1]}(2) = \frac{2}{t+1}\frac{1}{t}\frac{t+2}{2t+1}, \\ \mathsf{P}^t_{[1,1]} = \frac{1}{2}\mathsf{p}_{[1,1]} - \frac{1}{2}\mathsf{p}_{[2]}, \quad \mathsf{P}^t_{[1]/[0]} = \mathsf{p}_{[1]}, \\ \mathsf{P}^t_{[2,1]} = \frac{1}{t+2}\mathsf{p}_{[1,1,1]} + \frac{t-1}{t+2}\mathsf{p}_{[2,1]} - \frac{t}{t+2}\mathsf{p}_{[3]}, \quad \mathsf{P}^t_{[1]/[1]} = 1, \end{gathered} \tag{2.20}$$

again using Equation (C.25), Equation (2.2) and SageMath. The singular vector is thus

$$\begin{aligned} \mathcal{S}^{[2,1]}_{+}|\eta\rangle &= \Big[b^t_{[1,1]}(2)\rho_+\Big(\mathsf{P}^t_{[1,1]}\big(y^1\big)\mathsf{P}^t_{[1]/[0]}\big(y^2\big)\Big) + b^t_{[2,1]}(2)\rho_+\Big(\mathsf{P}^t_{[2,1]}\big(y^1\big)\mathsf{P}^t_{[1]/[1]}\big(y^2\big)\Big)\Big]|\theta\rangle \\ &= \left[\frac{1}{t(t+1)}\left(\frac{1}{\alpha_+^2}a^1_{-1}a^1_{-1} - \frac{1}{\alpha_+}a^1_{-2}\right)\frac{1}{\alpha_+}a^2_{-1}\right. \\ &\qquad \left. + \frac{2}{t(t+1)(2t+1)}\left(\frac{1}{\alpha_+^3}a^1_{-1}a^1_{-1}a^1_{-1} + \frac{t-1}{\alpha_+^2}a^1_{-2}a^1_{-1} - \frac{t}{\alpha_+}a^1_{-3}\right)\right]|\theta\rangle \\ &= \left[\frac{2/\alpha_+}{(t+1)(2t+1)}a^1_{-1}a^1_{-1}a^1_{-1} + \frac{1/\alpha_+}{t+1}a^1_{-1}a^1_{-1}a^2_{-1}\right. \\ &\qquad \left. + \frac{2(t-1)}{(t+1)(2t+1)}a^1_{-2}a^1_{-1} - \frac{1}{t+1}a^1_{-2}a^2_{-1} - \frac{1/\alpha_+}{(t+1)(2t+1)}a^1_{-3}\right]|\theta\rangle. \end{aligned} \tag{2.21}$$

We note that the result is manifestly well defined and non-zero for all α_+ such that $t = \alpha_+^{-2} \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, as expected. This region includes all central charges less than 98.

Example 3

Our final example concerns singular vectors for quite arbitrary central charges (including all $c < 98$). This time, we fix $\theta = 0$ and use Equation (2.11) to construct singular vectors in the Fock space $\mathcal{F}_0$.
First, we note that $\theta = \zeta_{-r_1,s_1;r_1-r_2,s_2} = 0$ may be solved for r_1 and s_1:

$$r_1 = -1 + (-s_1 + 1)t, \quad r_2 = -2 + (-s_1 - s_2 + 2)t. \tag{2.22}$$

Since $r_1, r_2, s_1, s_2 \in \mathbb{Z}$, we will only find singular vectors when $t \in \mathbb{Q}_{>0}$. Writing $t = \frac{u}{v}$, where u and v are coprime integers, it follows that

$$r_1 = mu - 1, \quad -s_1 = mv - 1, \quad r_2 = nu - 2, \quad -s_2 = (n-m)v - 1, \tag{2.23}$$

for some $m, n \in \mathbb{Z}$. Given that $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ and $s_1, s_2 \in \mathbb{Z}_{<0}$, we conclude that m, n and $n - m$ must be positive integers. We thereby obtain, for each fixed $t \in \mathbb{Q}_{>0}$, an infinite sequence of singular vectors, generically indexed by integers $n > m > 0$, of the form $\mathcal{S}_+^{[mu-1,nu-2]} |\zeta_{(m-n)u+1,-mv+1;nu-2,-(n-m)v+1}\rangle$. Among these, the singular vector of lowest grade corresponds, assuming that $u > 1$, to $(m, n) = (1, 2)$. Moreover, the grade of $\mathcal{S}_+^{[u-1,2(u-1)]} |\zeta_{-(u-1),-(v-1);2(u-1),-(v-1)}\rangle$ is $3(u-1)(v-1)$, by Equation (2.8). It is not clear if these singular vectors of the Fock space $\mathcal{F}_0$ correspond, in the sense of Example 1 to singular vectors in the W_3 vacuum Verma module $\mathcal{V}_0$ or not. However, there are five other Fock spaces $\mathcal{F}_\zeta$ whose highest-weight vectors $|\zeta\rangle$ have $h_\zeta = w_\zeta = 0$. This follows from the easily verified fact that both h_ζ and w_ζ are left invariant by the following shifted action of the Weyl group S_3:

$$\sigma \cdot \zeta = \sigma(\zeta - \alpha_0 \varrho) + \alpha_0 \varrho, \quad \sigma \in \mathsf{S}_3. \tag{2.24}$$

Each of these five other Fock spaces has an infinite sequence of singular vectors given by Equation (2.11) and it is interesting to ask whether these also correspond to singualr vectors in $\mathcal{V}_0$ or not. We shall not investigate this question here. We only note the following observation: $\mathcal{F}_{2\alpha_0\varrho}$ has such a singular vector at grade 3 and it corresponds to just one of the two linearly independent grade 3 singular vectors of $\mathcal{V}_0$. Which one is obtained depends on the branch of the square root of β chosen in Equation (1.195).

We conclude by remarking that the question of whether the Fock space singualr vectors constructed here exhaust the singular vectors of $\mathcal{V}_0$ is much easier to answer. They do not. We cannot obtain the two linearly independent singular vectors at grade 1 using Equation (2.11) (for $c \neq 2$; when $c = 2$, W_0 acts non-diagonalisably). Nor can we obtain, when $t = \frac{u}{v} \in \mathbb{Q}_{>0}$, the grade $(u-2)(v-2)$ singular vector whose image is non-zero in the universal W_3 vacuum module. This singular vector can be constructed formally using screening operators, but we do not know how to actually evaluate the integral in this case. What is needed is a certain $\mathfrak{sl}(3)$ analogue of the theory of Jack functions, something which does not appear to have yet been developed (see [65, 75, 76] for work in this direction).

2.2 W_n singular vectors

In this section we generalise the results of Section 2.1.1 and derive explicit formulae for W_n singular vectors in Fock spaces. We remark that this technique does not work for all Fock spaces, only some of them. Recall the conventions and notations from Section 1.9. If we apply the screening operator $\mathcal{S}_+^{[\mathbf{r}]}$ to the H_{n-1} highest-weight vector $|\eta^+_{\mathbf{r},\mathbf{s}}\rangle$, we get

$$\begin{aligned}
\mathcal{S}_+^{[\mathbf{r}]}|\eta^+_{\mathbf{r},\mathbf{s}}\rangle &= \int_{\Gamma(r_1;1/\alpha_+^2)} \cdots \int_{\Gamma(r_{n-1};1/\alpha_+^2)} \prod_{i=1}^{r_1} \mathrm{V}_{\alpha_+\alpha^1}\left(z_i^1\right) \cdots \prod_{i=1}^{r_{n-1}} \mathrm{V}_{\alpha_+\alpha^{n-1}}\left(z_i^{n-1}\right) \cdot |\eta^+_{\mathbf{r},\mathbf{s}}\rangle \prod_{k=1}^{n-1}\prod_{i=1}^{r_k} \mathrm{d}z_i^k \\
&= \int_{\Gamma(r_1;1/\alpha_+^2)} \cdots \int_{\Gamma(r_{n-1};1/\alpha_+^2)} \prod_{k=1}^{n-1} \prod_{1\le i\neq j\le r_k} \left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} \cdot \prod_{k=2}^{n-1}\prod_{i=1}^{r_{k-1}}\prod_{j=1}^{r_k}\left(1-\frac{z_j^k}{z_i^{k-1}}\right)^{-\alpha_+^2} \\
&\quad \cdot \prod_{k=1}^{n-1}\prod_{i=1}^{r_k}\left(z_i^k\right)^{\alpha_+\left(\alpha^k,\eta^+_{\mathbf{r},\mathbf{s}}\right)+\alpha_+^2(r_k-1)+1} \cdot \prod_{k=2}^{n-1}\prod_{i=1}^{r_k}\left(z_i^1\right)^{-\alpha_+^2 r_k} \\
&\quad \cdot \prod_{k=1}^{n-1}\prod_{m\ge1} \exp\left(\frac{\alpha_+ a^k_{-m}}{m}\sum_{i=1}^{r_k}\left(z_i^k\right)^m\right) \cdot |\theta^+_{\mathbf{r},\mathbf{s}}\rangle \prod_{k=1}^{n-1}\prod_{i=1}^{r_k}\frac{\mathrm{d}z_i^k}{z_i^k} \\
&= \int_{\Gamma(r_1;1/\alpha_+^2)} \cdots \int_{\Gamma(r_{n-1};1/\alpha_+^2)} \prod_{k=1}^{n-1} \prod_{1\le i\neq j\le r_k} \left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} \cdot \prod_{k=1}^{n-2}\prod_{i=1}^{r_{k-1}}\prod_{j=1}^{r_k}\left(1-\frac{z_j^k}{z_i^{k-1}}\right)^{-\alpha_+^2} \\
&\quad \cdot \prod_{k=1}^{n-1}\prod_{i=1}^{r_k}\left(z_i^k\right)^{s_k} \cdot \prod_{k=1}^{n-1}\prod_{m\ge1} \exp\left(\frac{\alpha_+ a^k_{-m}\mathsf{p}_m\left(z^k\right)}{m}\right) \cdot |\theta^+_{\mathbf{r},\mathbf{s}}\rangle \prod_{k=1}^{n-1}\prod_{i=1}^{r_k}\frac{\mathrm{d}z_i^k}{z_i^k}. \qquad (2.25)
\end{aligned}$$

As in the W_3 case, we can evaluate these integral formulae for singular vectors in terms of symmetric functions. Recall that the tensor product of $N-1$ copies of the ring of symmetric functions $\Lambda^{\otimes n-1}$ is isomorphic to

$$\mathrm{U}\Big((\mathcal{H}_{n-1})_-\Big) = \mathbb{C}[a^k_{-m} \mid k = 1, \dots, n-1,\ m \in \mathbb{Z}_{>0}] \tag{2.26}$$

as an algebra, by Equation (C.9). Distinguishing the alphabets of the tensor factors by superscripts, so that the alphabet in the i-th tensor factor is denoted by y^i, we define the following algebra isomorphism generalising that of Equation (2.2):

$$\rho_+ \colon \Lambda^{\otimes(n-1)} \to \mathrm{U}\Big((\mathcal{H}_{n-1})_-\Big), \quad \mathsf{p}_n\Big(y^k\Big) \mapsto \frac{1}{\alpha_+} a^k_{-n}. \tag{2.27}$$

This isomorphism allows us to write

$$\begin{aligned}\prod_{m\geq 1} \exp\left(\frac{\alpha_+ a^k_{-m} \mathsf{p}_m\Big(z^k\Big)}{m}\right) &= \rho_+\left(\prod_{m\geq 1} \exp\left(\alpha_+^2 \frac{\mathsf{p}_m(y^k)\mathsf{p}_m(z^k)}{m}\right)\right) \\ &= \rho_+\left(\prod_{i\geq 1}\prod_{j=1}^{r_k} (1 - y^k_i z^k_j)^{-\alpha_+^2}\right),\end{aligned} \tag{2.28}$$

for $k = 1, \dots, n-1$. We now identify, with $t = \alpha_+^{-2}$,

$$\prod_{k=1}^{n-1} \prod_{1\leq i\neq j\leq r_k} \left(1 - \frac{z^k_i}{z^k_j}\right)^{\alpha_+^2} = \prod_{k=1}^{n-1} G^t_{r_k}(z^k) \tag{2.29}$$

as the product of the integrating kernels for the variables z_k. For $k = 1$, as in Equation (2.4), we write

$$\prod_{m\geq 1} \exp\left(\frac{\alpha_+ a^1_{-m} \mathsf{p}_m\Big(z^1\Big)}{m}\right) = \rho_+\left(\sum_{\mu_1} \mathsf{P}^t_{\mu_1}\Big(y^1\Big)\mathsf{Q}^t_{\mu_1}\Big(z^1\Big)\right). \tag{2.30}$$

For $k = 2, \dots, n-1$, similar to Equation (2.5), we have instead

$$\begin{aligned}\prod_{m\geq 1} \exp\left(\frac{\alpha_+ a^k_{-m} \mathsf{p}_m\Big(z^k\Big)}{m}\right) \cdot \prod_{i=1}^{r_{k-1}} \prod_{j=1}^{r_k} \left(1 - \frac{z^k_j}{z^{k-1}_i}\right)^{-\alpha_+^2} &= \rho_+\left(\sum_{\mu_k} \mathsf{P}^t_{\mu_k}\Big(y^k \cup (z^{k-1})^{-1}\Big)\mathsf{Q}^t_{\mu_k}\Big(z^k\Big)\right) \\ &= \rho_+\left(\sum_{\mu_k,\nu_k} \overline{\mathsf{P}^t_{\nu_k}\Big(z^{k-1}\Big)}\mathsf{P}^t_{\mu_k/\nu_k}\Big(y^k\Big)\mathsf{Q}^t_{\mu_k}\Big(z^k\Big)\right).\end{aligned} \tag{2.31}$$

Putting everything together, we have

$$\begin{aligned}
\mathcal{S}_+^{[\mathbf{r}]}|\eta_{\mathbf{r},\mathbf{s}}^+\rangle = &\sum_{\substack{\mu_1,\mu_2,\dots,\mu_{n-1}\\ \nu_2,\dots,\nu_{n-1}}} \rho_+\Big(\mathsf{P}^t_{\mu_1}\big(y^1\big)\Big) \int_{\Gamma(r_1;t)} G^t_{r_1}(z^1)\overline{\mathsf{P}^t_{[-s_1^{r_1}]}\big(z^1\big)}\,\overline{\mathsf{P}^t_{\nu_2}\big(z^1\big)}\mathsf{Q}^t_{\mu_1}\big(z^1\big)\prod_{i=1}^{r_1}\frac{\mathrm{d}z_i^1}{z_i^1}\\
&\cdot\rho_+\Big(\mathsf{P}^t_{\mu_2/\nu_2}\big(y^2\big)\Big)\int_{\Gamma(r_2;t)} G^t_{r_2}(z^2)\overline{\mathsf{P}^t_{[-s_2^{r_2}]}\big(z^2\big)}\,\overline{\mathsf{P}^t_{\nu_3}\big(z^2\big)}\mathsf{Q}^t_{\mu_2}\big(z^2\big)\prod_{i=1}^{r_2}\frac{\mathrm{d}z_i^2}{z_i^2}\\
&\qquad\vdots\\
&\cdot\rho_+\Big(\mathsf{P}^t_{\mu_{n-2}/\nu_{n-2}}\big(y^{n-2}\big)\Big)\\
&\cdot\int_{\Gamma(r_{n-2};t)} G^t_{r_{n-2}}(z^{n-2})\overline{\mathsf{P}^t_{[-s_{n-2}^{r_{n-2}}]}\big(z^{n-2}\big)}\,\overline{\mathsf{P}^t_{\nu_{n-1}}\big(z^{n-2}\big)}\mathsf{Q}^t_{\mu_{n-2}}\big(z^{n-2}\big)\prod_{i=1}^{r_{n-2}}\frac{\mathrm{d}z_i^{n-2}}{z_i^{n-2}}\\
&\cdot\rho_+\Big(\mathsf{P}^t_{\mu_{n-1}/\nu_{N-1}}\big(y^{n-1}\big)\Big)\\
&\cdot\int_{\Gamma(r_{n-1};t)} G^t_{r_{n-1}}(z^{n-1})\overline{\mathsf{P}^t_{[-s_{n-1}^{r_{n-1}}]}\big(z^{n-1}\big)}\mathsf{Q}^t_{\mu_{n-1}}\big(z^{n-1}\big)\prod_{i=1}^{r_{n-1}}\frac{\mathrm{d}z_i^{n-1}}{z_i^{n-1}}\cdot|\theta^+_{\mathbf{r},\mathbf{s}}\rangle\\
=&\sum_{\substack{\mu_1,\mu_2,\dots,\mu_{n-1}\\ \nu_2,\dots,\nu_{n-1}}}\prod_{k=1}^{n-2}\Big\langle\mathsf{Q}^t_{\mu_k},\mathsf{P}^t_{\nu_{k+1}+[-s_k^{r_k}]}\Big\rangle^t_{r_k}\cdot\Big\langle\mathsf{Q}^t_{\mu_{n-1}},\mathsf{P}^t_{[-s_{n-1}^{r_{n-1}}]}\Big\rangle^t_{r_{n-1}}\\
&\quad\cdot\rho_+\Big(\mathsf{P}^t_{\mu_1}\big(y^1\big)\Big)\prod_{k=2}^{n-1}\rho_+\Big(\mathsf{P}^t_{\mu_k/\nu_k}\big(y^k\big)\Big)\cdot|\theta^+_{\mathbf{r},\mathbf{s}}\rangle\\
=&\sum_{\nu_2,\dots,\nu_{n-1}}\left(\prod_{k=1}^{n-2} b^t_{\nu_{k+1}+[-s_k^{r_k}]}(r_k)\cdot b^t_{[-s_{n-1}^{r_{n-1}}]}(r_{n-1})\right)\\
&\cdot\rho_+\Big(\mathsf{P}^t_{\nu_2+[-s_1^{r_1}]}\big(y^1\big)\Big)\prod_{k=2}^{n-2}\rho_+\Big(\mathsf{P}^t_{(\nu_{k+1}+[-s_k^{r_k}])/\nu_k}\big(y^k\big)\Big)\cdot\rho_+\Big(\mathsf{P}^t_{[-s_{n-1}^{r_{n-1}}]/\nu_{n-1}}\big(y^{N-1}\big)\Big)|\theta^+_{\mathbf{r},\mathbf{s}}\rangle.
\end{aligned}\tag{2.32}$$

As before, the factor $b^t_{[-s_{n-1}^{r_{n-1}}]}(r_{n-1})$ does not depend on the summation indices $\nu_2,\dots,\nu_{n-1}$, appears in every summand, and is non-zero, so it can be absorbed into the normalisation of the singular vector. Moreover, the skew-Jack polynomials vanish unless the summation indices $\nu_2,\dots,\nu_{n-1}$ satisfy the relations

$$\nu_k\subseteq\nu_{k+1}+[-s_k^{r_k}],\quad k=2,\dots,n-2,\quad \nu_{n-1}\subseteq[-s_{n-1}^{r_{n-1}}].\tag{2.33}$$

Thus, the singular vector $\mathcal{S}_+^{[\mathbf{r}]}|\eta^+_{\mathbf{r},\mathbf{s}}\rangle \in \mathcal{F}_{\theta^+_{\mathbf{r},\mathbf{s}}}$ is proportional to

$$\sum_{\nu_2,\dots,\nu_{n-1}} \left(\prod_{k=1}^{n-2} b^t_{\nu_{k+1}+[-s_k^{r_k}]}(r_k)\right)\rho_+\Big(\mathsf{P}^t_{\nu_2+[-s_1^{r_1}]}\big(y^1\big)\Big) \\ \cdot \prod_{k=2}^{n-2} \rho_+\Big(\mathsf{P}^t_{(\nu_{k+1}+[-s_k^{r_k}])/\nu_k}\big(y^k\big)\Big) \cdot \rho_+\Big(\mathsf{P}^t_{[-s_{n-1}^{r_{n-1}}]/\nu_{n-1}}\big(y^{n-1}\big)\Big)|\theta^+_{\mathbf{r},\mathbf{s}}\rangle. \quad (2.34)$$

This is our final formula for W_n singular vectors generalising the $n = 3$ case given in Equation (2.11). As before, considering the summand with $\nu_2 = \cdots = \nu_{n-1} = [\,]$ shows that the right hand side is non-zero for every $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{n-1}$, $\mathbf{s} \in \mathbb{Z}_{\leq 0}^{n-1}$ and $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$. This singular vector formula also has the nice property of being comparatively easy to evaluate using computer algebra packages such as SageMath.

Chapter 3

BRST cohomology for $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in category $\mathcal{O}$

In this chapter we will present the proof of the BRST cohmology of $\mathcal{L}_k(\mathfrak{sl}(2))$ highest weight modules presented in [16].

3.1 The BRST complex

Throughout this chapter we fix an admissible level k and set $k+2 = \frac{p}{q} = t$ for coprime integers $p, q \geq 2$. Recall that the energy momentum tensor of the simple affine $\mathfrak{sl}(2)$ vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(2))$ at level k and the fermionic ghost vertex operator super-algebra B are

$$T^{\mathcal{L}_k(\mathfrak{sl}(2))}(z) = \frac{1}{4t}(:h(z)h(z): + 2:e(z)f(z): + 2:f(z)e(z):) \tag{3.1}$$

$$T^{\mathsf{B}}(z) = :\partial b(z)c(z): \tag{3.2}$$

where

$$c_{\mathcal{L}_k(\mathfrak{sl}(2))} = \frac{3k}{k+2}, \qquad c_{\mathsf{B}} = -2 \tag{3.3}$$

The conformal weights of $e(z), h(z), f(z)$ are all 1 and the conformal weights of $b(z)$ and $c(z)$ are 0 and 1 respectively. We introduce the BRST field [10–12, 69] which first appeared in the context of gauge theory

$$Q(z) = :e(z)c(z): + c(z) \tag{3.4}$$

Clearly, from the operator product expansions in Equations (1.122) and (1.182), and noting that for the cross term we have $e(z)c(w) \sim 0$,

$$Q(z)Q(w) \sim 0 \tag{3.5}$$

and we define the BRST operator d_{BRST} as the zero mode of $Q(z)$

$$d_{\text{BRST}} = \int_0 Q(z)\mathrm{d}z \tag{3.6}$$

From Equation (3.5) it is clear that $d_{\text{BRST}}^2 = \frac{1}{2}\{d_{\text{BRST}}, d_{\text{BRST}}\} = 0$. Now, $\mathcal{L}_k(\mathfrak{sl}(2)) \otimes \mathsf{B}$ has the structure of a vertex operator super algebra with respect to the energy momentum tensor $T^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}$. The fermionic ghost vertex operator algebra can be graded by the ghost field $-{:}b(z)c(z){:}$ so that $b(z)$ and $c(z)$ have ghost degrees -1 and 1 respectively. Let B_n be the subspace of B consisting of states with ghost degree n, then for any highest weight module $\mathcal{M}$ of $\mathcal{L}_k(\mathfrak{sl}(2))$, d_{BRST} is a degree one operator acting on $\mathcal{M} \otimes \mathsf{B}$. We define the BRST complex, associated to $\mathcal{M}$, by

$$\cdots \xrightarrow{d_{\text{BRST}}} \mathcal{M} \otimes \mathsf{B}_{n-1} \xrightarrow{d_{\text{BRST}}} \mathcal{M} \otimes \mathsf{B}_n \xrightarrow{d_{\text{BRST}}} \mathcal{M} \otimes \mathsf{B}_{n+1} \xrightarrow{d_{\text{BRST}}} \cdots \tag{3.7}$$

It turns out that when $\mathcal{M} = \mathcal{L}_k(\mathfrak{sl}(2))$, the zeroth cohomology group has the structure of a vertex operator algebra, see Lemma 3.1.1 below. However the conformal structure does not come from $T^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}$. This is because, with respect to the energy momentum tensor above, the operator d_{BRST} is not a homogeneous operator of conformal weight zero. That is,

$$\left[T^{\mathcal{L}_k(\mathfrak{sl}(2))}(z) + T^{\mathsf{B}}(z), d_{\text{BRST}}\right] \neq 0. \tag{3.8}$$

This is due to the fact that both $e(z)$ and $c(z)$ have conformal weight one. To fix this, we introduce a new energy momentum tensor of $\mathcal{L}_k(\mathfrak{sl}(2))$

$$\overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))}(z) = T^{\mathcal{L}_k(\mathfrak{sl}(2))}(z) + \frac{1}{2}\partial h(z) \tag{3.9}$$

under which the conformal weights of $e(z), h(z), f(z)$ are $0, 1, 2$ respectively. One can show that with respect to this new energy momentum tensor, d_{BRST} commutes with this energy momentum tensor,

$$\left[\overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))}(z) + T^{\mathsf{B}}(z), d_{\text{BRST}}\right] = 0 \tag{3.10}$$

We then have

Lemma 3.1.1. *The cohomology* $H^0(\mathcal{L}_k(\mathfrak{sl}(2)) \otimes \mathsf{B}, d_{\text{BRST}})$ *has the structure of a vertex operator algebra, where the energy momentum tensor is given by the modified energy momentum tensor* $\overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}$.

Proof. Let $T = \overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}$ and $d = d_{\text{BRST}}$. Since $\left[d, \overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}\right] = 0$, it is easy to check that both $\ker d$ and $\operatorname{im} d$ are invariant under the action of T. To show that $\ker d$ is a vertex subalgebra, let $A, B \in \ker d$. Now from Lemma 1.1.8 we have

$$[d, Y(A,z)] = Y(dA, z) = 0 \tag{3.11}$$

and therefore

$$dY(A,z)B = Y(A,z)dB = 0 \tag{3.12}$$

so $Y(A,z)B \in \ker d[\![z^{\pm}]\!]$. To show that $\operatorname{im} d$ is an ideal of $\ker d$, suppose that $B \in \ker d$ and $A \in V$. Now,

$$Y(dA,z)B = [d, Y(A,z)]B = dY(A,z)B \tag{3.13}$$

implying that $Y(dA,z)B \in \operatorname{im} d[\![z^{\pm}]\!]$. Therefore the quotient $\ker d/\operatorname{im} d$ is a well-defined vertex operator algebra, where the conformal structure is given by T. □

For admissible level $k+2 = \frac{p}{q}$ with coprime integers $p, q \geq 2$, the irreducible highest weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$, $\mathcal{L}_{r,s}^{\mathcal{L}_k(\mathfrak{sl}(2))}$ are parameterised by $1 \leq r \leq p-1, 1 \leq s \leq q$ and the irreducible highest-weight modules $\mathcal{L}_{r,s}^{\mathsf{M}(p,q)}$ of the Virasoro minimal model $\mathsf{M}(p,q)$ are parameterized by $1 \leq r \leq p-1, 1 \leq s \leq q-1$. Now,

Theorem 3.1.2. *[2, 16] For* $1 \leq r \leq p-1, 1 \leq s \leq q-1$ *we have*

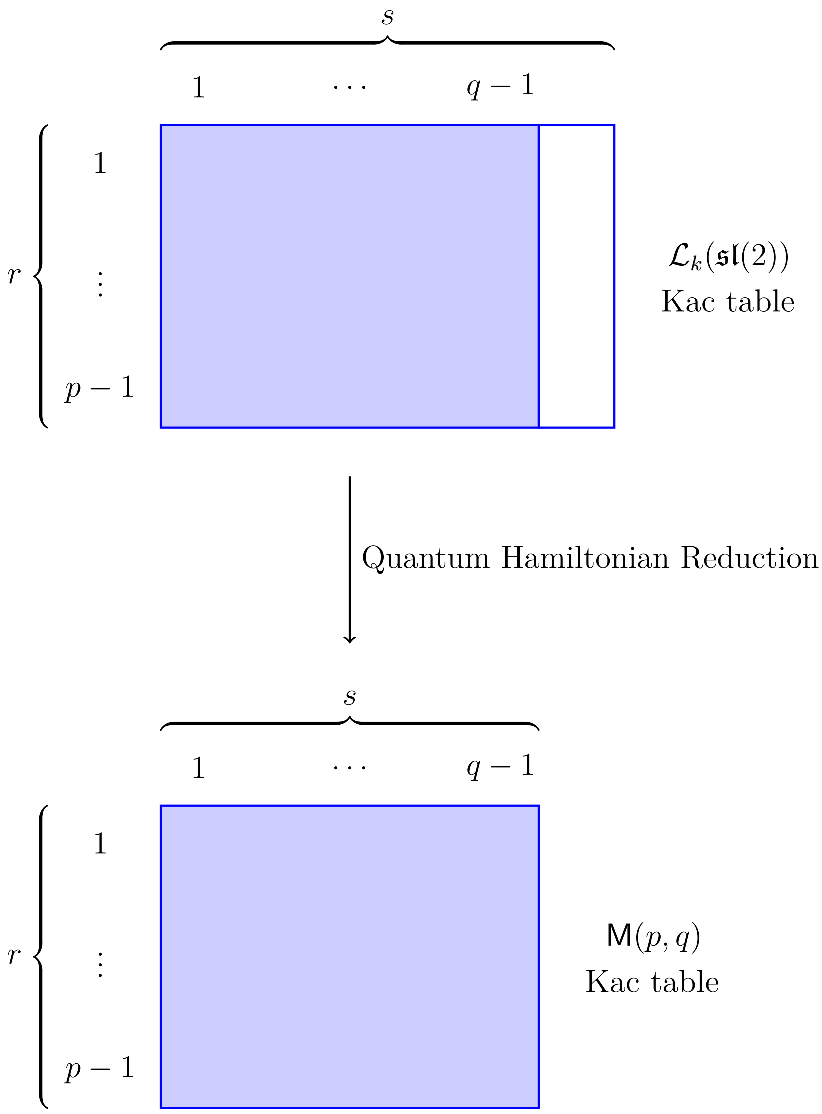
$$H^n(\mathcal{L}_{r,s} \otimes \mathsf{B}, d_{\text{BRST}}) = \delta_{n,0} \mathcal{L}_{r,s}^{\mathsf{M}(p,q)} \tag{3.14}$$

and

$$H^n(\mathcal{L}_{r,q} \otimes \mathsf{B}, d_{\text{BRST}}) = 0 \tag{3.15}$$

for all n.

In summary, the BRST complex is non-trivial at degree zero for all s except at the column $s = q$,



In this chapter we will present a full proof of Theorem 3.1.2 for $s = 1, ..., q-1$ following the outline presented in [16], with some modifications which will allow us to generalise more easily to the case of relaxed highest-weight modules. The case $s = q$ will be dealt with in the next chapter, along with our results for the relaxed highest-weight modules. The proof utilises the free-field realisations of $\mathsf{V}_k(\mathfrak{sl}(2))$ and Vir in Sections 1.4 and 1.7 so we will first set up the parameters needed for this. Firstly, let $T^{FF} = T^{\mathsf{H}} + T^{\mathsf{G}}$ be the energy momentum tensor of $\mathsf{H} \otimes \mathsf{G}$ and let $\alpha^2 = 2t$. Then we have

$$T^{FF} = \frac{1}{2}{:}a(z)a(z){:} - \frac{1}{\alpha}\partial a(z) + {:}\beta(z)\partial\gamma(z){:} \tag{3.16}$$

In particular, $\beta(z)$ and $\gamma(z)$ now have conformal weights 1 and 0 respectively. We can realise $\mathsf{V}_k(\mathfrak{sl}(2))$ as a subalgebra of $\mathsf{H} \otimes \mathsf{G}$ where the explicit embedding of $\mathsf{V}_k(\mathfrak{sl}(2))$ into $\mathsf{H} \otimes \mathsf{G}$ was given in Equations (1.164) to (1.166). Recall that our modified energy momentum tensor for the BRST reduction takes the form $\overline{T}^{\mathcal{L}_k(\mathfrak{sl}(2))} + T^{\mathsf{B}}$. Now we want to represent this modified energy momentum tensor using $a(z), \beta(z), \gamma(z)$ using the free-field realisation defined in Equations (1.164) to (1.166),

$$\overline{T}^{FF} = \frac{1}{2}{:}a(z)a(z){:} + \left(\alpha_+ - \frac{2}{\alpha_+}\right)\partial a(z) - {:}\partial\beta(z)\gamma(z){:} \tag{3.17}$$

In particular, $\beta(z), \gamma(z)$ now has conformal weights 0 and 1 respectively. From now on we will exclusively be working with $\overline{T}^{FF}$. Also, since $k + 2 = \frac{p}{q}$, we have
$\alpha_+ = \sqrt{2t} = \sqrt{\frac{2p}{q}}$ and

$$\overline{T}^{FF} = \frac{1}{2}{:}a(z)a(z){:} + (\alpha_+ + \alpha_-)\partial a(z) - {:}\partial\beta(z)\gamma(z){:} \tag{3.18}$$

Thus we see that the Heisenberg part of $\overline{T}^{FF}$ is precisely the free-field realisation of the Virasoro algebra at minimal model central charge in Equation (1.74). Already this suggests a relationship between $\mathcal{L}_k(\mathfrak{sl}(2))$ and $\mathsf{M}(p,q)$ with $k + 2 = \frac{p}{q}$. As the free-field realisation of $e(z)$ is $\beta(z)$, the BRST operator now takes the form

$$d_{\mathrm{BRST}} = \int_0 {:}\beta(z)c(z){:} + c(z)\mathrm{d}z \tag{3.19}$$

We have the following commutation relations

$$[d_{\mathrm{BRST}}, \beta_n] = 0 \qquad [d_{\mathrm{BRST}}, \gamma_n] = c_n \tag{3.20}$$

$$\{d_{\mathrm{BRST}}, b_n\} = \beta_n + \delta_{n,0}1 \qquad \{d_{\mathrm{BRST}}, c_n\} = 0 \tag{3.21}$$

Now that we have introduced all the parameters for the proof, we will move on and show that the Virasoro and $\mathsf{V}_k(\mathfrak{sl}(2))$ screening operators are BRST exact.

Lemma 3.1.3. *Theorems 1.4.4 and 1.7.1. For $n \geq 1$, let $\alpha_- = -\sqrt{\frac{2q}{p}}$ so that $\left[S_{\mathfrak{sl}(2)}\right]^n$, $[S_V]^n$ are screening operators for the Virasoro and $\mathsf{V}_k(\mathfrak{sl}(2))$ vertex operators algebras respectively. Then there existsa field $\Psi^n(z_1, ..., z_i)$ such that*

$$\{d, \Psi^n\} = \left[S_{\mathfrak{sl}(2)}\right]^n + (-1)^{n+1}[S_V]^n \tag{3.22}$$

Proof. The proof is done by induction. In the case when $n = 1$, we have

$$\{d, b(z)\mathrm{V}_{\alpha_-}(z)\} = \beta(z)\mathrm{V}_{\alpha_-}(z) + \mathrm{V}_{\alpha_-}(z) \tag{3.23}$$

We now assume that there exists $\psi^n(z_1, ..., z_n)$ such that

$$\{d, \psi^n(z_1, ..., z_n)\} = \prod_{i=1}^{n} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) + (-1)^{n+1} \prod_{i=1}^{n} \mathrm{V}_{\alpha_-}(z_i) \tag{3.24}$$

Then we see that

$$\left\{d, \beta(z_1)\cdots\beta(z_n)b(z_{n+1}) \prod_{i=1}^{n+1} \mathrm{V}_{\alpha_-}(z_n) - \psi^n(z_1, ..., z_n)\mathrm{V}_{\alpha_-}(z_{n+1})\right\} \tag{3.25}$$

$$= \prod_{i=1}^{n+1} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) + \prod_{i=1}^{n} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) \cdot \mathrm{V}_{\alpha_-}(z_{n+1}) \tag{3.26}$$

$$- \prod_{i=1}^{n} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) \cdot \mathrm{V}_{\alpha_-}(z_{n+1}) - (-1)^{n+1} \prod_{i=1}^{n+1} \mathrm{V}_{\alpha_-}(z_i) \tag{3.27}$$

$$= \prod_{i=1}^{n+1} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) + (-1)^{n+2} \prod_{i=1}^{n+1} \mathrm{V}_{\alpha_-}(z_i) \tag{3.28}$$

It is obvious to see that $\psi^n(z_1, ..., z_n)$ has $\prod_{i=1}^{n} \mathrm{V}_{\alpha_-}(z_i)$ as a factor and therefore we can integrate

$$\left\{d, \int_\Gamma \psi^n\right\} = \int_\Gamma \prod_{i=1}^{n} \beta(z_i)\mathrm{V}_{\alpha_-}(z_i) + (-1)^{n+1} \int_\Gamma \prod_{i=1}^{n} \mathrm{V}_{\alpha_-}(z_i) \tag{3.29}$$

such that the operator $\int_\Gamma \psi^n$ will be a well-defined map whenever the screening operators are well-defined. Taking $\Psi^n = \int_\Gamma \psi^n$ we have

$$\{d, \Psi^n\} = \left[S_{\mathfrak{sl}(2)}\right]^n + (-1)^{n+1}[S_V]^n \tag{3.30}$$

for all $n \geq 1$ and we are done. □

In other words, the two operators $[S_V]^n$ and $\left[S_{\mathfrak{sl}(2)}\right]^n$ are equivalent when acting on a BRST-cohomology.

3.2 BRST cohomology of $\mathcal{L}_k(\mathfrak{sl}(2))$ highest-weight modules

We will first outline the strategy of proving Theorem 3.1.2. First we take an irreducible highest-weight module of $\mathcal{L}_k(\mathfrak{sl}(2))$ corresponding to $r = 1, ..., p-1$, $s = 1, ..., q-1$ and form the BRST complex

$$\begin{array}{c}
\vdots \\
\Big\uparrow d_{\mathrm{BRST}} \\
\mathcal{L}_{r,s} \otimes \mathsf{B}_1 \\
\Big\uparrow d_{\mathrm{BRST}} \\
\mathcal{L}_{r,s} \otimes \mathsf{B}_0 \\
\Big\uparrow d_{\mathrm{BRST}} \\
\mathcal{L}_{r,s} \otimes \mathsf{B}_{-1} \\
\Big\uparrow d_{\mathrm{BRST}} \\
\vdots
\end{array}$$

We then realise $\mathcal{L}_{r,s}$ as the degree zero cohomology of the Bernard-Felder complex as per Theorem 1.7.2,

$$\cdots \mathcal{F}_{2p-r,s} \otimes \mathsf{G} \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} \mathcal{F}_{r,s} \otimes \mathsf{G} \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} \mathcal{F}_{-r,s} \otimes \mathsf{G} \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} \cdots \tag{3.31}$$

thereby turning the BRST complex into a double complex (the fact that the differentials commute is obvious since they only contain β_n modes)

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \\
\cdots & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{2p-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_1 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \mathcal{F}_{r,s} \otimes \mathsf{G} \otimes \mathsf{B}_1 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_1 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \cdots \\
 & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \\
\cdots & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{2p-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_0 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \mathcal{F}_{r,s} \otimes \mathsf{G} \otimes \mathsf{B}_0 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_0 & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \cdots \\
 & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \\
\cdots & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{2p-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_{-1} & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \mathcal{F}_{r,s} \otimes \mathsf{G} \otimes \mathsf{B}_{-1} & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{r}} & \mathcal{F}_{-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_{-1} & \xrightarrow{\left[S_{\mathfrak{sl}(2)}\right]^{p-r}} & \cdots \\
 & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \Big\uparrow d_{\mathrm{BRST}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

It turns out that taking $\left[S_{\mathfrak{sl}(2)}\right]^n$, $n = r, p-r$ and d_{BRST} cohomologies commute by Proposition 3.4.2. This means that we can determine the BRST cohomology of $\mathcal{L}_{r,s}$ by first determining the BRST cohomology of the corresponding Wakimoto modules instead. Proposition 3.3.1 tells us that the BRST cohomology of a Wakimoto module is

$$H^n(\mathcal{F}_\lambda \otimes \mathsf{G} \otimes \mathsf{B}, d_{\text{BRST}}) = \delta_{n,0}\mathcal{F}_\lambda \tag{3.32}$$

Therefore if we first take d_{BRST} cohomology, all rows will be zero except at the zeroth row, at which it will consist of a sequence of Fock spaces. Moreover, since the Virasoro and the $\mathfrak{sl}(2)$ screening operators are BRST-exact from Lemma 3.1.3, we see that up to a sign, $\left[S_{\mathfrak{sl}(2)}\right]^n$, $[S_V]^n$ are two equivalent maps when acting on a BRST cohomology, that is,

$$\left[S_{\mathfrak{sl}(2)}\right]^n[v] = [S_V]^n[v] \tag{3.33}$$

where $[v]$ is some non-trivial cohomology class. Therefore, we can replace $\left[S_{\mathfrak{sl}(2)}\right]^n$ by $[S_V]^n$ after taking the BRST cohomology of the Wakimoto modules. It turns out this procedure results in a Felder complex at the zeroth row, recall Theorem 1.4.6,

$$\cdots \xrightarrow{[S_V]^r} \mathcal{F}_{2p-r,s} \xrightarrow{[S_V]^{p-r}} \mathcal{F}_{r,s} \xrightarrow{[S_V]^r} \mathcal{F}_{-r,s} \xrightarrow{[S_V]^{p-r}} \cdots$$

The Felder complex is exact except at degree zero, at which we get the irreducible $\mathsf{M}(p,q)$ highest weight module $\mathcal{L}^{\mathsf{M}(p,q)}_{r,s}$.

Proof of Theorem 3.1.2. Consider the double complex $D = \left(D^{i,j}, d_1^i, d_2^j\right)$, for $i = 2k, 2k+1$, where

$$D^{2k,j} = \mathcal{F}_{-2kp+r,s} \otimes \mathsf{G} \otimes \mathsf{B}_j, \qquad d_1^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{3.34}$$

$$D^{2k+1,j} = \mathcal{F}_{-2kp-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_j, \qquad d_1^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{3.35}$$

$$d_2^j = d_{\text{BRST}} \tag{3.36}$$

Then by Theorem 1.7.2,

$$H^n(\mathcal{L}_{r,s}, d_{\text{BRST}}) = H^n\left(H^0(D, d_1), d_2\right) \tag{3.37}$$

from Proposition 3.4.2,

$$= H^0(H^n(D, d_2), d_1) \tag{3.38}$$

by Lemma 3.1.3 and Proposition 3.3.1,

$$= H^0(\delta_{n,0}C, d) \tag{3.39}$$

where $C = (C^i, d^i)$ is the Felder complex from Theorem 1.4.6, this implies

$$= \delta_{n,0}\mathcal{L}^{\mathsf{M}(p,q)}_{r,s} \tag{3.40}$$

□

Therefore to complete the proof of Theorem 3.1.2 we need to show Proposition 3.3.1 and Proposition 3.4.2 which we will do next.

3.3 BRST Cohomology of Wakimoto modules

In this section we will show the following proposition, this is a result in [16], but here we will give a different proof.

Proposition 3.3.1.

$$H^n(\mathcal{F}_\lambda \otimes \mathsf{G} \otimes \mathsf{B}, d_{\text{BRST}}) = \delta_{n,0}\mathcal{F}_\lambda \tag{3.41}$$

Proof. Firstly notice d_{BRST} contains no elements of $\mathcal{H}$ and so $\mathcal{F}_\lambda$ factors through the cohomology

$$H^n(\mathcal{F}_\lambda \otimes \mathsf{G} \otimes \mathsf{B}, d_{\text{BRST}}) = \mathcal{F}_\lambda \otimes H^n(\mathsf{G} \otimes \mathsf{B}, d_{\text{BRST}}) \tag{3.42}$$

That is, it is enough to determine the cohomology of the complex $C = (\mathsf{G} \otimes \mathsf{B}, d_{\text{BRST}})$. For simplicity let $|0\rangle = |0\rangle_{\mathsf{G}} \otimes |0\rangle_{\mathsf{B}}$ and $d = d_{\text{BRST}}$. We can decompose this complex as a tensor product of two complexes

$$C = \bigotimes_{i\geq 0} \text{span}\{\beta^{p_i}_{-i} b^{r_i}_{-i}|0\rangle \mid p_i \geq 0, r_i \in \{0,1\}\} \tag{3.43}$$

$$\bigotimes_{i\geq 1} \text{span}\{\gamma^{q_i}_{-i} c^{s_i}_{-i}|0\rangle \mid q_i \geq 0, s_i \in \{0,1\}\} \tag{3.44}$$

since the vacuum vector lies in the kernel of d and both tensor factors are invariant subspaces under d by Equations (3.20) and (3.21).
Now consider one of the factor $I^B = \bigotimes_{i\geq 0} \mathrm{span}\{\beta_{-i}^{p_i} b_{-i}^{r_i}|0\rangle \mid p_i \geq 0, r_i \in \{0,1\}\}$. The commutation relations of d with the modes β_i, b_i shows that the differential d leaves invariant the space of states generated by the same mode indices. That is, we can further decompose I^B as an infinite tensor product of complexes $I^B = \bigotimes_{i\geq 0} I_i^B$, where

$$I_i^B = \mathrm{span}\{\beta_{-i}^{p_i} b_{-i}^{r_i}|0\rangle \mid p_i \geq 0, r_i \in \{0,1\}\} \tag{3.45}$$

We can then directly compute the cohomology of each tensor factor I_i^B

$$0 \xrightarrow{d_{-2}} \mathrm{span}\{\beta_{-i}^{p_i} b_{-i}|0\rangle\} \xrightarrow{d_{-1}} \mathrm{span}\{\beta_{-i}^{p_i}|0\rangle\} \xrightarrow{d_0} 0 \tag{3.46}$$

where

$$\ker d_{-1} = 0, \qquad \ker d_0 = \mathrm{span}\{\beta_{-i}^{p_i}|0\rangle \mid p_m \geq 0\} \tag{3.47}$$

$$\mathrm{im}\, d_{-2} = 0, \qquad \mathrm{im}\, d_{-1} \cong \mathrm{span}\{\beta_{-i}^{p_i}|0\rangle \mid p_m \geq 1\} \tag{3.48}$$

Hence, we see that

$$\mathrm{H}^n(I_i^B) = \delta_{n,0}\mathbb{C}|0\rangle, \qquad i \geq 0 \tag{3.49}$$

By Lemma A.2.9 we see that

$$\mathrm{H}^n(I^B) = \delta_{n,0}\mathbb{C}|0\rangle \tag{3.50}$$

We will now compute the cohomology of
$I^C = \mathrm{span}\{\gamma_{-i}^{q_i} c_{-i}^{s_i}|0\rangle \mid i \geq 1, q_i \geq 1, s_i \in \{0,1\}\}$. Again, d is invariant on subspaces with the same mode indices. Therefore similar to I^B, we can decompose I^C as
$I^C = \bigotimes_{i\geq 1} I_i^C$, where

$$I_i^C = \mathrm{span}\{\gamma_{-i}^{q_i} c_{-i}^{s_i}|0\rangle \mid q_i \geq 0, s_i \in \{0,1\}\} \tag{3.51}$$

Again we will now compute the cohomology for each tensor factor I_n^C. The complex is

$$0 \xrightarrow{d_{-1}} \mathrm{span}\{\gamma_{-i}^{q_i}|0\rangle\} \xrightarrow{d_0} \mathrm{span}\{\gamma_{-i}^{q_i} c_{-i}|0\rangle\} \xrightarrow{d_1} 0 \tag{3.52}$$

where

$$\ker d_0 = \mathbb{C}|0\rangle, \qquad \ker d_1 = \operatorname{span}\{\gamma_{-i}^{q_i} c_{-i}|0\rangle \mid q_i \geq 0\} \tag{3.53}$$
$$\operatorname{im} d_{-1} = 0, \qquad \operatorname{im} d_0 = \operatorname{span}\{-l_i \gamma_{-i}^{q_i - 1} c_{-i}|0\rangle \mid q_i \geq 1\} \tag{3.54}$$

We therefore conclude that

$$H^n(I_i^C) = \delta_{n,0}\mathbb{C}|0\rangle, \qquad i \geq 1 \tag{3.55}$$

Applying Lemma A.2.9 again we see that

$$H^n(I^C) = \delta_{n,0}\mathbb{C}|0\rangle \tag{3.56}$$

Thus, applying Kunneth's Theorem we finally arrive at

$$H^n(\mathcal{F}_\lambda \otimes \mathsf{G} \otimes \mathsf{B}, d_{\mathrm{BRST}}) = \delta_{n,0}\mathcal{F}_\lambda \tag{3.57}$$

as required. □

3.4 Commutivity of the double complex

In this section we will show that taking cohomologies of the double complex in the proof of Theorem 3.1.2 commute. Recall that the double complex in the proof of Theorem 3.1.2 is $D = \left(D^{i,j}, d_1^i, d_2^j\right)$, for $i = 2k, 2k+1$, where

$$D^{2k,j} = \mathcal{F}_{-2kp+r,s} \otimes \mathsf{G} \otimes \mathsf{B}_j, \qquad d_1^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{3.58}$$
$$D^{2k+1,j} = \mathcal{F}_{-2kp-r,s} \otimes \mathsf{G} \otimes \mathsf{B}_j, \qquad d_1^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{3.59}$$
$$d_2^j = d_{\mathrm{BRST}} \tag{3.60}$$

Firstly, let $T = \overline{T}^{FF} + T^{\mathsf{B}} = T^{\mathsf{H}} + T^{\mathsf{G}} + T^{\mathsf{B}}$ be the modified energy momentum tensor of the vertex operator algebra $\mathsf{H} \otimes \mathsf{G} \otimes \mathsf{B}$, with $L_0 = L_0^{\mathsf{H}} + L_0^{\mathsf{G}} + L_0^{\mathsf{B}}$ being its zero mode. We note the the minimal conformal weight of any Fock spaces appearing in the double complex D is the highest-weight state of $\mathcal{F}_{r,s}$, say Δ. That is, $L_0^{\mathsf{H}}|\alpha_{r,s}\rangle = \Delta|\alpha_{r,s}\rangle$. Then each $D^{i,j}$ in the double complex D can be written as a direct sum of eigenspaces of L_0, that is $D^{i,j} = \bigoplus_{k=0}^{\infty} D^{i,j}_{\Delta+k}$. We note that some of the $D^{i,j}_{\Delta+k}$'s could be zero. Furthermore, since $\left[L_0, \left[S_{\mathfrak{sl}(2)}\right]^n\right] = [L_0, d_{\mathrm{BRST}}] = 0$, we can decompose D as a direct sum of subcomplexes $D = \bigoplus_{k=0}^{\infty} A_k$, where $A_k = \left(A_k^{i,j}, d_1^i, d_2^j\right)$, $A_k^{i,j} = D_{\Delta+k}^{i,j}$. In other words, each $A_k^{i,j}$ in the double complex A_k is just the L_0-eigenspace $D_{\Delta+k}^{i,j}$ of weight $\Delta + k$ in the original double complex $D^{i,j}$. We then have

Remark 3.4.1. *For each $k \in \mathbb{N}$, the double complex $A_k = \left(A_k^{i,j}, d_1^i, d_2^j\right)$ is bounded above and below with respect to the ghost degree. Concretely, $A_k^{i,j} = 0$ for $j > k$ or $j < -k - 1$.*

Proof. We prove this by considering the original double complex $D = (D^{i,j}, d_1^i, d_2^j)$. The minimal conformal weight of any state with ghost degree j is $\Delta + \frac{j(j+1)}{2}$, which is any state containing the modes $c_{-j} \cdots c_{-1}$. Similarly, the minimal conformal weight of any state with ghost degree $-j-1$ is $\Delta + \frac{j(j+1)}{2}$, which is any state containing the modes $b_{-j} \cdots b_0$. Therefore, $A_k^{i,j} = 0$ if $k < j$ and if $k > -j - 1$. □

We then arrive at

Proposition 3.4.2. *For the double complex $D = \left(D^{i,j}, d_1^i, d_2^j\right)$, taking cohomologies commute. That is,*

$$H^j\Big(H^i(D, d_1), d_2\Big) = H^i\Big(H^j(D, d_2), d_1\Big) \tag{3.61}$$

Proof. As we saw before, we first write D as a direct sum of subcomplexes

$$D = \bigoplus_{k=0}^{\infty} A_k \tag{3.62}$$

Then each A_k is a bounded double complex from Remark 3.4.1. From Theorem 1.7.2 and Proposition 3.3.1, since the Bernard-Felder complex as well as the BRST cohomology of Wakimoto module are exact except at one degree, we see that the spectral sequence associated to the total complex of D_k degenerate at most at page two. Therefore by Theorem A.1.9 we see that for each k,

$$H^j\Big(H^i(A_k, d_1), d_2\Big) = H^i\Big(H^j(A_k, d_2), d_1\Big) \tag{3.63}$$

Now since the cohomology functor is additive we see that

$$H^j\Big(H^i(D, d_1), d_2\Big) = H^j\left(H^i\left(\bigoplus_{k=0}^{\infty} A_k, d_1\right), d_2\right) \tag{3.64}$$

$$= \bigoplus_{k=0}^{\infty} H^j\Big(H^i(A_k, d_1), d_2\Big) \tag{3.65}$$

$$= \bigoplus_{k=0}^{\infty} H^i\Big(H^j(A_k, d_2), d_1\Big) \tag{3.66}$$

$$= H^i\Big(H^j(D, d_2), d_1\Big) \tag{3.67}$$

□

Chapter 4

BRST cohomology for $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in category $\mathcal{R}^\sigma$

The chapter contains our results on the BRST cohomology for the relaxed highest-weight modules as well as the twisted ones under spectral flow. We will first discuss Bernard-Felder complexes in category $\mathcal{R}^\sigma$ in Section 4.1. Section 4.2 contains our main results for this chapter, which are Theorem 4.2.1 and Propositions 4.2.2 and 4.2.3. In Section 4.3 we compute the BRST cohomology of a spectrally flowed Wakimoto module. In Section 4.4 we will discuss the commutativity of the double complexes that we obtain for the BRST cohomology for the (spectrally flowed) relaxed highest-weight modules.

4.1 Bernard-Felder complexes in category $\mathcal{R}^\sigma$

In this section we take $\lambda = 1$ for the energy momentum tensor of the bosonic ghost as shown in Equation (1.146). The proof of quantum hamiltonian reduction[16] presented in Theorem 3.1.2 relies on the construction of a double complex involving the Bernard-Felder complex of Theorem 1.7.2 for the irreducible highest weight modules of $\mathcal{L}_k(\mathfrak{sl}(2))$. In order to follow the same strategy for the case of relaxed modules and their spectral flows we need to construct analogs of the Bernard-Felder complexes for the irreducible modules in category $\mathcal{R}^\sigma$. We will therefore devote this section to proving Propositions 4.1.3 and 4.1.10 which do exactly that.
To begin, we will first determine how to apply spectral flow to a Wakimoto module. As Wakimoto modules are modules of $\mathcal{H} \otimes \mathcal{G}$ we want to determine an automorphism of $\mathcal{H} \otimes \mathcal{G}$ that is compatible with the spectral flow automorphism of $\widehat{\mathfrak{sl}}(2)$. First, the free-field realisation of $\mathsf{V}_k(\mathfrak{sl}(2))$ into $\mathsf{H} \otimes \mathsf{G}$, given by Equations (1.164) and (1.165),

induces a map,

$$\iota : \mathsf{U}(\mathsf{V}_k(\mathfrak{sl}(2))) \longrightarrow \mathsf{U}(\mathsf{H} \otimes \mathsf{G}) \tag{4.1}$$

Recall from Equations (1.39), (1.138), (1.139) and (1.161) that the spectral flow automorphisms of $\widehat{\mathfrak{sl}}(2)$, $\mathcal{H}$ and $\mathcal{G}$, for $l \in \mathbb{Z}, s \in \mathbb{R}, t \in \mathbb{Z}$ are given by

$$\sigma^l_{\mathfrak{sl}(2)}(e_n) = e_{n-l}, \qquad \sigma^l_{\mathfrak{sl}(2)}(h_n) = h_n - l\delta_{n,0}k, \qquad \sigma^l_{\mathfrak{sl}(2)}(f_n) = f_{n+l}, \tag{4.2}$$

$$\sigma^s_{\mathcal{H}}(a_n) = a_n - s\delta_{n,0} \qquad \sigma^t_{\mathcal{G}}(\beta_n) = \beta_{n-t} \qquad \sigma^t_{\mathcal{G}}(\gamma_n) = \gamma_{n+t} \tag{4.3}$$

Basically, we want to find s, t such that $\sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}$ is compatible with $\sigma^l_{\mathfrak{sl}(2)}$. Concretely, we have

Lemma 4.1.1. *Let $k + 2 = \frac{p}{q}$ be admissible, $\alpha_+ = \sqrt{\frac{2p}{q}}$ and*

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}} = \sigma_{\mathcal{H}}^{\frac{\alpha_+ l}{2}} \otimes \sigma^l_{\mathcal{G}} \tag{4.4}$$

Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathsf{U}(\mathsf{V}_k(\mathfrak{sl}(2))) & \xrightarrow{\ \iota\ } & \mathsf{U}(\mathsf{H} \otimes \mathsf{G}) \\
\Big\downarrow \sigma^l_{\mathfrak{sl}(2)} & & \Big\downarrow \sigma^l_{\mathcal{H}\otimes\mathcal{G}} \\
\mathsf{U}(\mathsf{V}_k(\mathfrak{sl}(2))) & \xrightarrow{\ \iota\ } & \mathsf{U}(\mathsf{H} \otimes \mathsf{G})
\end{array}$$

Proof. We need to find s, t as functions of l such that the following simultaneous equations hold:

$$\iota\Big(\sigma^l_{\mathfrak{sl}(2)}(e_n)\Big) = \Big[\sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}\Big](\iota(e_n)) \tag{4.5}$$

$$\iota\Big(\sigma^l_{\mathfrak{sl}(2)}(h_n)\Big) = \Big[\sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}\Big](\iota(h_n)) \tag{4.6}$$

$$\iota\Big(\sigma^l_{\mathfrak{sl}(2)}(f_n)\Big) = \Big[\sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}\Big](\iota(f_n)) \tag{4.7}$$

We remark that this is an over-determined system of equations. Now, from Equation (4.5) we have

$$\iota\Big(\sigma^l_{\mathfrak{sl}(2)}(e_n)\Big) = \iota(e_{n-l}) = \beta_{n-l} \tag{4.8}$$

$$\Big[\sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}\Big](\iota(e_n)) = \sigma^s_{\mathcal{H}} \otimes \sigma^t_{\mathcal{G}}(\beta_n) = \beta_{n-t} \tag{4.9}$$

This forces $t = l$. Moving to Equation (4.6),

$$\iota\Big(\sigma^l_{\mathfrak{sl}(2)}(h_n)\Big) = \iota(h_n - l\delta_{n,0}k) = -2{:}\beta\gamma{:}_n + \alpha_+ a_n - l\delta_{n,0}k \tag{4.10}$$

$$\Big[\sigma^s_{\mathcal{H}} \otimes \sigma^l_{\mathcal{G}}\Big](\iota(h_n)) = \Big[\sigma^s_{\mathcal{H}} \otimes \sigma^l_{\mathcal{G}}\Big](-2{:}\beta\gamma{:}_n + \alpha_+ a_n) \tag{4.11}$$

$$= -2({:}\beta\gamma{:}_n - l\delta_{n,0}) + \alpha_+(a_n - s\delta_{n,0}) \tag{4.12}$$

We see that

$$-lk = 2l - \alpha_+ s \implies s = \frac{(k+2)l}{\alpha_+} \tag{4.13}$$

and we conclude that

$$\iota \circ \sigma^l_{\mathfrak{sl}(2)} = \left[\sigma_{\mathcal{H}}^{\frac{(k+2)l}{\alpha_+}} \otimes \sigma^l_{\mathcal{G}}\right] \circ \iota = \left[\sigma_{\mathcal{H}}^{\frac{\alpha_+ l}{2}} \otimes \sigma^l_{\mathcal{G}}\right] \circ \iota \tag{4.14}$$

by noting that $k + 2 = \frac{\alpha_+^2}{2}$. For a consistency check we now want to see if Equation (4.7) holds with $s = \frac{\alpha_+ l}{2}$ and $t = l$ So let $\sigma^l_{\mathcal{H}\otimes\mathcal{G}} = \sigma_{\mathcal{H}}^{\frac{\alpha_+ l}{2}} \otimes \sigma^l_{\mathcal{G}}$ and recall that

$$\iota(f_n) = -{:}\beta(z)\gamma(z)\gamma(z){:}_n + \alpha_+{:}a(z)\gamma(z){:}_n + \left(\frac{\alpha_+^2}{2} - 2\right)\partial\gamma_n. \tag{4.15}$$

First we remark that since $[\gamma_m, \gamma_n] = 0$ for all $m, n \in \mathbb{Z}$, we have

$$\sigma^l_{\mathcal{G}}({:}\gamma(z)\gamma(z){:}_n) = {:}\gamma(z)\gamma(z){:}_{n+2l}, \tag{4.16}$$

Then

$${:}\beta(z)\gamma(z)\gamma(z){:}_n = \sum_{m\le -1} \beta_m{:}\gamma(z)\gamma(z){:}_{n-m} + \sum_{m\ge 0}{:}\gamma(z)\gamma(z){:}_{n-m}\beta_m \tag{4.17}$$

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}({:}\beta(z)\gamma(z)\gamma(z){:}_n) = \sum_{m\le -1} \beta_{m-l}{:}\gamma(z)\gamma(z){:}_{n-m+2l} + \sum_{m\ge 0}{:}\gamma(z)\gamma(z){:}_{n-m+2l}\beta_{m-l} \tag{4.18}$$

$$= \sum_{m\le -l-1} \beta_m{:}\gamma(z)\gamma(z){:}_{n-m+l} + \sum_{m\ge -l}{:}\gamma(z)\gamma(z){:}_{n-m+l}\beta_m \tag{4.19}$$

$$= \sum_{m\le -l-1} \beta_m{:}\gamma(z)\gamma(z){:}_{n-m+l} + \sum_{m\ge 0}{:}\gamma(z)\gamma(z){:}_{n-m+l}\beta_m \tag{4.20}$$

$$+ {:}\gamma(z)\gamma(z){:}_{n+l+l}\beta_{-l} + \cdots + {:}\gamma(z)\gamma(z){:}_{n+1+l}\beta_{-1} \tag{4.21}$$

Now, $:\gamma(z)\gamma(z):_{n+k+l}\beta_{-k} = \beta_{-k}:\gamma(z)\gamma(z):_{n+k+l} - 2\gamma_{n+l}$ so

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(:\beta(z)\gamma(z)\gamma(z):_n) = :\beta(z)\gamma(z)\gamma(z):_{n+l} - 2l\gamma_{n+l} \tag{4.22}$$

Next, we have

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(:a(z)\gamma(z):_n) = \sigma^l\left(\sum_{m\leq -1} a_m\gamma_{n-m} + \sum_{m\geq 0}\gamma_{n-m}a_m\right) \tag{4.23}$$

$$= \sum_{m\leq -1} a_m\gamma_{n-m+l} + \sum_{m\geq 0}\gamma_{n-m+l}a_m - \frac{\alpha_+}{2}l\gamma_{n+l} \tag{4.24}$$

$$= :a(z)\gamma(z):_{n+l} - \frac{\alpha_+}{2}l\gamma_{n+l} \tag{4.25}$$

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\partial\gamma(z)_n) = -n\gamma_{n+l} \tag{4.26}$$

Therefore, putting everything together,

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\iota(f_n)) = -:\beta(z)\gamma(z)\gamma(z):_{n+l} + 2l\gamma_{n+l} + \alpha_+(:a(z)\gamma(z):_{n+l}) - \frac{\alpha_+^2}{2}l\gamma_{n+l} \tag{4.27}$$

$$+\left(\frac{\alpha_+^2}{2} - 2\right)(-n)\gamma_{n+l} \tag{4.28}$$

$$= -:\beta(z)\gamma(z)\gamma(z):_{n+l} + \alpha(:a(z)\gamma(z):_{n+l}) \tag{4.29}$$

$$-\left(\frac{\alpha_+^2}{2} - 2\right)(n+l)\gamma_{n+l} \tag{4.30}$$

and

$$\iota(\sigma^l_{\mathfrak{sl}(2)}(f_n)) = -:\beta(z)\gamma(z)\gamma(z):_{n+l} + \alpha_+(:a(z)\gamma(z):_{n+l}) \tag{4.31}$$

$$-\left(\frac{\alpha_+^2}{2} - 2\right)(n+l)\gamma_{n+l} \tag{4.32}$$

so we see that Equation (4.7) does indeed hold and we are done. □

We will now call $\sigma^l_{\mathcal{H}\otimes\mathcal{G}}$, the automorphism of $\mathcal{H}\otimes\mathcal{G}$ that is compatible with the normal $\widehat{\mathfrak{sl}}(2)$ spectral flow $\sigma^l_{\mathfrak{sl}(2)}$, the free-field spectral flow. Recall from Theorem 1.7.2 that the Bernard-Felder complex $C = (C_n, d_n)$ for an irreducible highest-weight module $\mathcal{L}_{r,s}$ for $1 \leq r \leq p-1, 1 \leq s \leq q-1$ is a complex of Wakimoto modules such that for $n = 2k, 2k+1$,

$$C_{2k} = \mathcal{F}_{-2kp+r,s} \otimes \mathsf{G}, \qquad d_{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{4.33}$$

$$C_{2k+1} = \mathcal{F}_{-2kp-r,s} \otimes \mathsf{G}, \qquad d_{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.34}$$

where cohomology of C is

$$H^n(C) = \delta_{n,0}\mathcal{L}_{r,s} \tag{4.35}$$

Since spectral flow defines an exact functor between the relaxed category $\mathcal{R}^\sigma$ to itself, it commutes with cohomology functors in the relaxed category by Lemma A.2.5. Therefore, spectrally flowing the above Bernard-Felder complex using the free field spectral flow in Lemma 4.1.1, we see that the complex $C = (C_n, d_n)$, for $n = 2k, 2k+1$, where

$$C_{2k} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp+r,s} \otimes \mathsf{G}), \qquad d_{2k} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}\left(\left[S_{\mathfrak{sl}(2)}\right]^r\right) \tag{4.36}$$

$$C_{2k+1} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp-r,s} \otimes \mathsf{G}), \qquad d_{2k+1} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}\left(\left[S_{\mathfrak{sl}(2)}\right]^{p-r}\right) \tag{4.37}$$

has cohomology

$$H^n(C) = \delta_{n,0}\sigma^l_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) \tag{4.38}$$

With some simplifications we see that the above complex becomes

$$C_{2k} = \mathcal{F}_{-2kp+r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}), \qquad d_{2k} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}\left(\left[S_{\mathfrak{sl}(2)}\right]^r\right) \tag{4.39}$$

$$C_{2k+1} = \mathcal{F}_{-2kp-r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}), \qquad d_{2k+1} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}\left(\left[S_{\mathfrak{sl}(2)}\right]^{p-r}\right) \tag{4.40}$$

Therefore it remains to determine the image of the $\mathfrak{sl}(2)$ screening operators under the free field spectral flow.

Lemma 4.1.2. *The free-field spectral flow $\sigma^l_{\mathcal{H}\otimes\mathcal{G}}$ preserves the $\mathfrak{sl}_2$ screening operator $\left[S_{\mathfrak{sl}(2)}\right]^n$ homomorphism. That is,*

$$\sigma^l_{\mathcal{H}\otimes\mathcal{G}}\left(\left[S_{\mathfrak{sl}(2)}\right]^n\right) = \left[S_{\mathfrak{sl}(2)}\right]^n \tag{4.41}$$

for all $l \in \mathbb{Z}, n \geq 1$, where the maps are defined as in Theorem 1.7.2.

Proof. Recall from Equation (1.172) the screening field is

$$\beta(z)V_{\alpha_-}(z) = \beta(z)e^{\alpha_-\widehat{a}}z^{\alpha_- a_0}\prod_{m\geq 1}\exp\left(\alpha_-\frac{a_{-m}}{m}z^m\right)\prod_{m\geq 1}\exp\left(-\alpha_-\frac{a_m}{m}z^{-m}\right) \tag{4.42}$$

We let

$$\Omega(z) = e^{\alpha_-\widehat{a}}\prod_{m\geq 1}\exp\left(\alpha_-\frac{a_{-m}}{m}z^m\right)\prod_{m\geq 1}\exp\left(-\alpha_-\frac{a_m}{m}z^{-m}\right) \tag{4.43}$$

so that we can write

$$V_{\alpha_-}(z) = \Omega(z)z^{\alpha_- a_0} \tag{4.44}$$

We first remark that

$$[\sigma^l(a_0), e^{\alpha_-\widehat{a}}] = \left[a_0 - \frac{\alpha_+ l}{2}, e^{\alpha_-\widehat{a}}\right] = [a_0, e^{\alpha_-\widehat{a}}] \tag{4.45}$$

so the Heisenberg automorphism does not affect $e^{\alpha_-\widehat{a}}$. Now notice that

$$\sigma^l_{\mathcal{G}}(\beta(z)) = \sum_{n\in\mathbb{Z}}\beta_{n-l}z^{-n} = z^{-l}\sum_{n\in\mathbb{Z}}\beta_{n-l}z^{-n+l} = z^{-l}\beta(z) \tag{4.46}$$

$$\sigma_{\mathcal{H}}^{\frac{\alpha_+ l}{2}}\Big(V_{\alpha_-}(z)\Big) = \sigma^l(z^{\alpha_- a_0}\Omega(z)) = z^{\alpha_-\left(a_0-\left(-\frac{1}{\alpha_-}\right)l\right)}\Omega(z) = z^l V_{\alpha_-}(z) \tag{4.47}$$

Therefore

$$\begin{aligned}\sigma^l_{\mathcal{H}\otimes\mathcal{G}}\Big(\beta(z)V_{\alpha_-}(z)\Big) &= \sigma^l_{\mathcal{G}}(\beta(z))\sigma_{\mathcal{H}}^{\frac{\alpha_+ l}{2}}\Big(V_{\alpha_-}(z)\Big) &(4.48)\\ &= z^{-l}\beta(z)z^l V_{\alpha_-}(z) &(4.49)\\ &= \beta(z)V_{\alpha_-}(z) &(4.50)\end{aligned}$$

Now,

$$\begin{aligned}\sigma^l_{\mathcal{H}\otimes\mathcal{G}}\Big(\Big[S_{\mathfrak{sl}(2)}\Big]^n\Big) &= \int_{\Gamma(n)}\prod_{i=1}^n \sigma^l_{\mathcal{H}\otimes\mathcal{G}}\Big(\beta(z_i)\mathrm{V}_{\alpha_-}(z_i)\Big)\mathrm{d}z_1\cdots\mathrm{d}z_r &(4.51)\\ &= \int_{\Gamma(n)}\prod_{i=1}^n \beta(z_i)\mathrm{V}_{\alpha_-}(z_i)\mathrm{d}z_1\cdots\mathrm{d}z_r &(4.52)\\ &= \Big[S_{\mathfrak{sl}(2)}\Big]^n &(4.53)\end{aligned}$$

□

We have therefore obtained the Bernard-Felder complexes for the twisted irreducible highest weight modules under spectral flow. To state our result more precisely, Theorem 1.7.2 and Lemma 4.1.2 imply that

Proposition 4.1.3. *Fix an admissible level $k+2=\frac{p}{q}$ and let $r=1,...,p-1, s=1,...,q-1$. Let $C=(C^n,d^n)$ be a complex such that, for $n=2k,2k+1$,*

$$C^{2k} = \mathcal{F}_{-2kp+r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}), \qquad d^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{4.54}$$
$$C^{2k+1} = \mathcal{F}_{-2kp-r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}), \qquad d^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.55}$$

Then the cohomology of C is

$$H^n(C) = \delta_{n,0}\sigma^l_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) \tag{4.56}$$

To summarise Proposition 4.1.3, since spectral flow $\sigma^l_{\mathfrak{sl}(2)}$ is an algebra automorphism and thus lifts to an exact functor from $\mathcal{R}^\sigma$ to itself, it commutes with the cohomology functors of any complexes. All we needed to determine was the image of the morphisms $\left[S^-_{\mathfrak{sl}(2)}\right]^n$ under spectral flow.

We now want to construct Bernard-Felder complexes for the irreducible modules $\mathcal{E}_{\lambda,\Delta_{r,s}}$ where $r=1,...,p-1, s=2,...q$. We want to follow a similar strategy to Proposition 4.1.3, but where we shall realise each $\mathcal{E}_{\lambda,\Delta_{r,s}}$ as the image of an exact functor applied to $\omega(\mathcal{L}_{r,s})$, the conjugate highest-weight module. This functor is called twisted localisation and was introduced in [54].

Firstly, let $\mathrm{U}\big(\widehat{\mathfrak{sl}}(2)\big)_e$ be the localisation of $\mathrm{U}\big(\widehat{\mathfrak{sl}}(2)\big)$ with respect to the set $\{e_0^n \mid n \geq 0\}$ as defined in Corollary B.2.2. Then for each $\mu \in \mathbb{C}$ we define a functor

$$\mathscr{E}^\mu : \mathrm{U}\big(\widehat{\mathfrak{sl}}(2)\big)\textbf{-Mod} \longrightarrow \mathrm{U}\big(\widehat{\mathfrak{sl}}(2)\big)\textbf{-Mod} \tag{4.57}$$
$$\mathcal{M} \longmapsto \mathrm{Res}^{\mathrm{U}(\widehat{\mathfrak{sl}}(2))_e}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2))} \circ (\Omega^\mu_e)^* \circ \mathrm{Ind}^{\mathrm{U}(\widehat{\mathfrak{sl}}(2))_e}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2))}(\mathcal{M}), \tag{4.58}$$

where

$$\mathrm{Ind}^{\mathrm{U}(\widehat{\mathfrak{sl}}(2))_e}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2))}(\mathcal{M}) = \mathrm{U}\big(\widehat{\mathfrak{sl}}(2)\big)_e \otimes_{\mathrm{U}(\widehat{\mathfrak{sl}}(2))} \mathcal{M}, \tag{4.59}$$
$$\Omega^\mu_e(-) = \sum_{i\geq 0} \binom{\mu}{i} \mathrm{ad}(e_0)^i(-)e_0^{-i}, \tag{4.60}$$

$\mathrm{Res}_{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)}^{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)_e}$ is the restriction functor and $(\Omega_e^\mu)^*$ is the twisted action induced by Ω_e^μ. We remark that $\Omega_e^\mu \in \mathrm{Aut}\Big(\mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)_e\Big)$. To see that $\mathscr{E}^\mu$ is exact, notice that it is composed of three functors:

$$\mathrm{Ind}_{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)}^{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)_e}(\mathcal{M}) : \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)\text{-}\mathbf{Mod} \longrightarrow \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)_e\text{-}\mathbf{Mod} \tag{4.61}$$

$$(\Omega_e^\mu)^* : \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)_e\text{-}\mathbf{Mod} \longrightarrow \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)_e\text{-}\mathbf{Mod} \tag{4.62}$$

$$\mathrm{Res}_{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)}^{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)_e} : \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)_e\text{-}\mathbf{Mod} \longrightarrow \mathrm{U}\Big(\widehat{\mathfrak{sl}}(2)\Big)\text{-}\mathbf{Mod} \tag{4.63}$$

Therefore it is enough to show each of these functors is exact. From Lemma B.1.10 we see that $\mathrm{Ind}_{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)}^{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)_e}(\mathcal{M})$ is exact. The proofs for the exactness of $(\Omega_e^\mu)^*, \mathrm{Res}_{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)}^{\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)_e}$ are easy so we omit them. We now want to apply $\mathscr{E}^\mu$ to $\omega(\mathcal{L}_{r,s})$ for $r = 1, ...p-1, s = 2, ..., q$ and determine μ so that $\mathscr{E}^\mu(\omega(\mathcal{L}_{r,s}))$ is precisely $\mathcal{E}_{\lambda,\Delta_{r,s}}$. Recall that $\omega(\mathcal{L}_{r,s})$ is a relaxed highest-weight module generated by a relaxed highest-weight state $|-\lambda_{r,s}\rangle$ that is annililated by f_0, where $-\lambda_{r,s}$ denotes the h_0 eigenvalue. Next recall that L_0 acts on grounds states as $L_0|_{gs} = \frac{1}{4t}(h_0^2 + 2e_0f_0 + 2f_0e_0)$ which coinsides with the quadratic Casimir of the horizontal subalgebra $\mathfrak{sl}(2)$ and so $\mathrm{ad}(e_0)^n(L_0|_{gs}) = 0$ for $n \geq 1$. Thus,

$$\Omega_e^\mu(L_0|_{gs}) = L_0|_{gs} \tag{4.64}$$

We also have

$$\Omega_e^\mu(e_0) = e_0 \tag{4.65}$$

$$\Omega_e^\mu(h_0) = h_0 - 2\mu \tag{4.66}$$

$$\Omega_e^\mu(f_0) = f_0 + \mu h_0 e_0^{-1} - \mu(\mu-1)e_0^{-1} \tag{4.67}$$

We first let

$$\phi(\mathsf{q}) = \prod_{i=1}^{\infty}\Big(1-\mathsf{q}^i\Big) \qquad \eta(\mathsf{q}) = \mathsf{q}^{\frac{1}{24}}\prod_{i=1}^{\infty}\Big(1-\mathsf{q}^i\Big) \tag{4.68}$$

Now, we have

Proposition 4.1.4. *Let $\lambda \in \mathbb{C}$ such that $\lambda \neq \lambda_{r,s}, \lambda_{p-r,q+2-s} \mod 2$ and $r' = r \mod 2$. Then*

$$\operatorname{ch} \mathscr{E}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(\omega(\mathcal{V}_{r',s})) = \frac{z^\lambda \mathsf{q}^{\Delta_{r',s} - \frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}}}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.69}$$

where $\mathcal{V}_{r',s}$ is a $\widehat{\mathfrak{sl}}(2)$ Verma module of highest weight $\lambda_{r',s}$.

Proof. Recall that $\mathscr{E}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)} = \operatorname{Res}_U^{U_e} \circ \left(\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\right)^* \circ \operatorname{Ind}_U^{U_e}$. A PBW basis for the induced module $\operatorname{Ind}_U^{U_e}(\omega(\mathcal{V}_{r',s}))$ is

$$\{e_0^{\alpha_0} J_{-n_1}^{\alpha_1} \cdots J_{-n_k}^{\alpha_k} |-\lambda_{r',s}\rangle \mid J \in \{e, f, h\}, \alpha_0 \in \mathbb{Z}, 1 \leq \alpha_1, ..., \alpha_k, 1 \leq n_1 \leq \cdots \leq n_k\} \tag{4.70}$$

From Remark B.2.3 we see that, for $n \geq 1$,

$$h_0 e^{-n}|-\lambda_{r',s}\rangle = (-\lambda_{r',s} - 2n) e^{-n}|-\lambda_{r',s}\rangle \tag{4.71}$$
$$L_0 e^{-n}|-\lambda_{r',s}\rangle = \Delta_{r',s}|-\lambda_{r',s}\rangle \tag{4.72}$$

Therefore,

$$\operatorname{ch} \operatorname{Ind}_U^{U_e}(\omega(\mathcal{V}_{r',s})) = \frac{z^{-\lambda_{r',s}} \mathsf{q}^{\Delta_{r',s} - \frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}}}{\phi(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.73}$$

The twisted actions $\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}$ of e_0, h_0, f_0, L_0 acting on the relaxed highest weight vector $|-\lambda_{r',s}, \Delta_{r',s}\rangle$, as we saw from Equations (4.65) to (4.67), are

$$\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(e_0)|-\lambda_{r',s}\rangle = e_0|-\lambda_{r',s}\rangle \tag{4.74}$$
$$\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(h_0)|-\lambda_{r',s}\rangle = (\lambda + (\lambda_{r,s} - \lambda_{r',s}))|-\lambda_{r',s}\rangle \tag{4.75}$$
$$\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(f_0)|-\lambda_{r',s}\rangle = -\frac{1}{4}(\lambda_{r,s} + \lambda)(-2\lambda_{r',s} + \lambda_{r,s} + \lambda + 2) e_0^{-1}|-\lambda_{r',s}\rangle \tag{4.76}$$
$$\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(L_0|_{gs})|-\lambda_{r',s}\rangle = \Delta_{r',s}|-\lambda_{r',s}\rangle \tag{4.77}$$

Since $\lambda_{r,s} - \lambda_{r',s} = 0 \mod 2$, we see that

$$\operatorname{ch} \operatorname{Ad}(e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)})^* \circ \operatorname{Ind}_U^{U_e}(\omega(\mathcal{V}_{r',s})) = \frac{z^{\lambda} \mathsf{q}^{\Delta_{r',s} - \frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}}}{\phi(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.78}$$

Notice that the expression $\frac{1}{2}(-\lambda_{r,s}-\lambda)(-2\lambda_{r',s}+\lambda_{r,s}+\lambda+2)$ in Equation (4.76) is only zero if $\lambda = -\lambda_{r,s} = \lambda_{p-r,q+2-s}+2$ or $\lambda = \lambda_{r,s} \mod 2$. Since we demand that $\lambda \neq \lambda_{r,s}, \lambda_{p-r,q+2-s} \mod 2$, we see that $\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(f_0)|-\lambda_{r',s}\rangle$ will never be 0. This implies that the states involving e_0^{-1} in the PBW basis will not be annihilated after applying the restriction functor $\mathrm{Res}_U^{U_e}$. Thus,

$$\mathrm{ch}\,\mathscr{E}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(\omega(\mathcal{V}_{r',s})) = \mathrm{ch}\,\mathrm{Res}_U^{U_e}\circ\left(\Omega_e^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\right)^*\circ\mathrm{Ind}_U^{U_e}(\omega(\mathcal{V}_{r',s})) \tag{4.79}$$

$$= \frac{z^\lambda \mathsf{q}^{\Delta_{r',s}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.80}$$

□

We can now compute

Proposition 4.1.5. *For $1 \le r \le p-1, 2 \le s \le q$,*

$$\mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s})) = \mathcal{E}_{\lambda,\Delta_{r,s}}. \tag{4.81}$$

Proof. We first start with the BGG resolution of $\mathcal{L}_{r,s}$ [53],

$$\cdots \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{V}_{r,s} \longrightarrow \mathcal{L}_{r,s} \longrightarrow 0 \tag{4.82}$$

where for $k \ge 1$,

$$\mathcal{M}_{2k-1} = \mathcal{V}_{2kp-r,s}\bigoplus\mathcal{V}_{-2(k-1)p-r,s} \tag{4.83}$$

$$\mathcal{M}_{2k} = \mathcal{V}_{2kp+r,s}\bigoplus\mathcal{V}_{-2kp+r,s} \tag{4.84}$$

Since both the conjugate automorphism and twisted localisation are exact functors, applying both to Equation (4.82) we get

$$\cdots \longrightarrow \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{M}_2)) \longrightarrow \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{M}_1)) \longrightarrow \cdots \tag{4.85}$$

$$\cdots \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{r,s})) \longrightarrow \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s})) \longrightarrow 0 \tag{4.86}$$

Now, let

$$\mathcal{N}_{2k-1} = \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{2kp-r,s}))\bigoplus\mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}\left(\omega\left(\mathcal{V}_{-2(k-1)p-r,s}\right)\right) \tag{4.87}$$

$$\mathcal{N}_{2k} = \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{2kp+r,s}))\bigoplus\mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{-2kp+r,s})) \tag{4.88}$$

Then the above exact sequence becomes

$$\cdots \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_1 \longrightarrow \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{r,s})) \longrightarrow \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s})) \longrightarrow 0 \tag{4.89}$$

The above exact sequence now allows us to compute the character of $\mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s}))$. From Proposition 4.1.4 we have

$$\operatorname{ch} \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{V}_{r,s})) = \frac{z^\lambda \mathsf{q}^{\frac{(qr-p(s-1))^2-q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}}}{\phi(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.90}$$

and

$$\operatorname{ch} \mathcal{N}_{2k-1} = \frac{z^\lambda \left(\mathsf{q}^{\frac{(q(2kp-r)-p(s-1))^2-q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}} + \mathsf{q}^{\frac{(q(-2(k-1)p-r)-p(s-1))^2-q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}} \right)}{\phi(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.91}$$

$$\operatorname{ch} \mathcal{N}_{2k} = \frac{z^\lambda \left(\mathsf{q}^{\frac{(q(2kp+r)-p(s-1))^2-q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}} + \mathsf{q}^{\frac{(q(-2kp+r)-p(s-1))^2-q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}} \right)}{\phi(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.92}$$

Thus, we have

$$\operatorname{ch} \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s})) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch} \mathcal{N}_k \quad = \tag{4.93}$$

$$= \frac{z^\lambda \mathsf{q}^{-\frac{q^2}{4pq}-\frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24}+\frac{1}{24}}}{\phi(\mathsf{q})^2} \frac{\mathsf{q}^{-\frac{1}{24}}}{\phi(\mathsf{q})} \sum_{k\in\mathbb{Z}} \left(\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2}{4pq}} - \mathsf{q}^{\frac{(qr+p(s-1)-2kpq)^2}{4pq}} \right) \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.94}$$

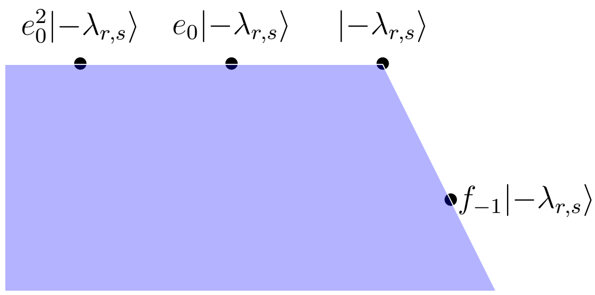
$$= \frac{z^\lambda \chi^{\mathsf{M}(p,q)}_{r,s-1}}{\eta(\mathsf{q})^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{4.95}$$

Equation (4.95) follows from

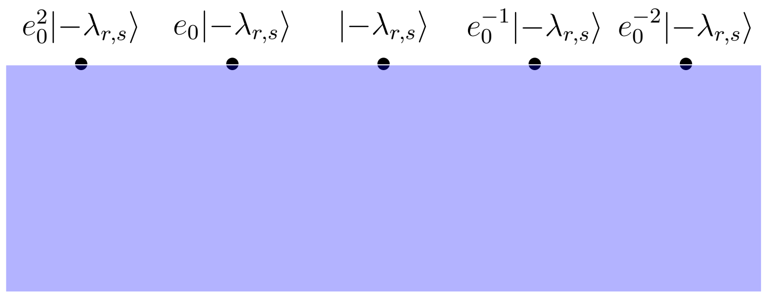
$$-\frac{q^2}{4pq} - \frac{c_{\mathcal{L}_k(\mathfrak{sl}(2))}}{24} + \frac{1}{24} = -\frac{q}{4p} - \frac{1}{24}\left(3 - \frac{6q}{p}\right) + \frac{1}{24} = -\frac{1}{12} \tag{4.96}$$

Therefore we see that the character of $\operatorname{ch} \mathscr{E}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}(\omega(\mathcal{L}_{r,s}))$ is precisely the character of the irreducible module $\mathcal{E}_{\lambda,\Delta_{r,s}}$ [48]. This completes the proof. □

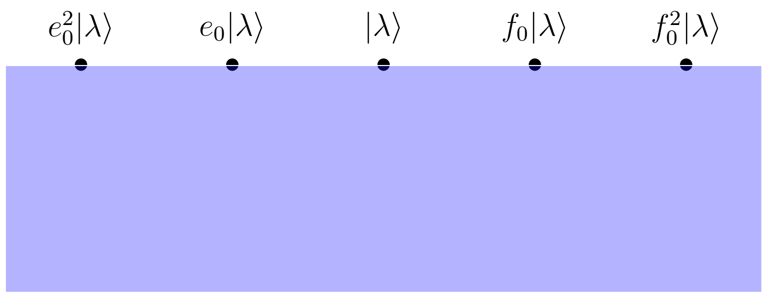
In summary, we start from a conjugate highest-weight module $\omega(\mathcal{L}_{r,s})$, where $|-\lambda_{r,s}\rangle$ is a relaxed highest-weight vector with h_0 and L_0 eigenvalues $-\lambda_{r,s}, \Delta_{r,s}$ respectively,



and we induced $\omega(\mathcal{L}_{r,s})$ from a module over $\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)_k\right)$ to a module over $\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)_k\right)_e$.



We then apply a twist to the module under the automorphism $\Omega_e^{-\lambda_{r,s}-\lambda}$. This will change the h_0 eigenvalue of the highest-weight state and f_0 will act like e_0^{-1} up to a constant. Restricting the module back to a module of $\widehat{\mathfrak{sl}}(2)_k$ we get precisely $\mathcal{E}_{\lambda,\Delta_{r,s}}$. Again, $|\lambda\rangle$ has h_0 and L_0 eigenvalues $\lambda, \Delta_{r,s}$ respectively.



Now we want to apply the twisted localisation to the Bernard-Felder complex associated to $\omega(\mathcal{L}_{r,s})$. In order to do this we need to figure out the twisted localisation for a Wakimoto module that is compatible with the $\widehat{\mathfrak{sl}}(2)_k$ twisted localisation. Since the free field realisation sends e_0 to β_0 from Equation (1.164), we let $\mathrm{U}(\mathcal{H} \otimes \mathcal{G})_\beta$ be the localisation of $\mathrm{U}(\mathcal{H}\otimes\mathcal{G})$ with respect to the set $\{\beta_0^n \mid n\in\mathbb{N}\}$. Then let

$$\mathscr{B}^\mu : \mathrm{U}(\mathcal{H}\otimes\mathcal{G})\text{-}\mathbf{Mod} \longrightarrow \mathrm{U}(\mathcal{H}\otimes\mathcal{G})\text{-}\mathbf{Mod} \tag{4.97}$$
$$\mathcal{M} \mapsto \mathrm{Res}_{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})_\beta} \circ \left(\Omega_\beta^\mu\right)^* \circ \mathrm{Ind}_{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})_\beta}(\mathcal{M}) \tag{4.98}$$

where

$$\Omega_\beta^\mu(-) = \sum_{i\geq 0}\binom{\mu}{i}\mathrm{ad}(\beta_0)^i(-)\beta_0^{-i} \tag{4.99}$$

Let ι be the inclusion map that realises $\mathrm{U}(\mathsf{V}_k(\mathfrak{sl}(2)))$ as a subalgebra of $\mathrm{U}(\mathsf{H}\otimes\mathsf{G})$ through the free-field realisation defined in Equations (1.164) to (1.166). Let $\mathrm{Res}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2))}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})}$ be the restriction functor from a $\mathrm{U}(\mathcal{H}\otimes\mathcal{G})$ module to a $\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)$ module. Specifically, if $\mathcal{M}$ is a $\mathrm{U}(\mathcal{H}\otimes\mathcal{G})$ module then it becomes a $\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)$ module with the action defined as $x\cdot v = \iota(x)\cdot v$ for any $x\in\mathrm{U}\left(\widehat{\mathfrak{sl}}(2)\right)$. We remark that although $\iota(x)$ is in general an infinite sum of elements in $\mathrm{U}(\mathcal{H}\otimes\mathcal{G})$, $x\cdot v$ is always a finite sum of states in $\mathcal{M}$ as $\mathcal{M}$ is smooth. Then we have

Lemma 4.1.6. *The following diagram commutes*

$$\begin{array}{ccc}
\mathrm{U}(\mathcal{H}\otimes\mathcal{G})\text{-}\mathcal{R}^\sigma_{\mathcal{H}\otimes\mathcal{G}} & \xrightarrow{\mathrm{Res}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2)_k)}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})}} & \mathrm{U}\left(\widehat{\mathfrak{sl}}(2)_k\right)\text{-}\mathcal{R}^\sigma_{\widehat{\mathfrak{sl}}(2)_k} \\
\Big\downarrow \mathscr{B}^\mu & & \Big\downarrow \mathscr{E}^\mu \\
\mathrm{U}(\mathcal{H}\otimes\mathcal{G})\text{-}\mathcal{R}^\sigma_{\mathcal{H}\otimes\mathcal{G}} & \xrightarrow{\mathrm{Res}_{\mathrm{U}(\widehat{\mathfrak{sl}}(2)_k)}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})}} & \mathrm{U}\left(\widehat{\mathfrak{sl}}(2)_k\right)\text{-}\mathcal{R}^\sigma_{\widehat{\mathfrak{sl}}(2)_k}
\end{array}$$

The proof of Lemma 4.1.6 is obvious since $\iota(e_0)=\beta_0\in\mathrm{U}(\mathsf{H}\otimes\mathsf{G})$. In other words, $\mathscr{B}^\mu=\mathscr{E}^\mu$ when we regard a $\mathcal{H}\otimes\mathcal{G}$ module as an $\widehat{\mathfrak{sl}}(2)$ module. Now we want to apply $\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}$ to a Bernard-Felder complex. Since $[\beta_0,\beta_n]=0$ for all $n\in\mathbb{Z}$, we have

Lemma 4.1.7. *The $\mathfrak{sl}(2)$ screening operators in Theorem 1.7.2, which we refer to as $d_{\mathfrak{sl}(2)}$, are invariant under the twisted localisation $\mathscr{B}^{\frac{1}{2}(-\lambda_{r,s}-\lambda)}$.*

Recall that $\sigma_{\mathfrak{sl}(2)}^{-1}(\mathcal{L}_{p-r,q+1-s})=\omega(\mathcal{L}_{r,s})$ for $1\leq r\leq p-1, 2\leq s\leq q$ and $\sigma_{\mathcal{H}}^l(\mathcal{F}_{r,s})=\mathcal{F}_{r,s-l}$ by Equation (1.41). By letting $l=-1$ in Proposition 4.1.3, we therefore see that the Bernard-Felder complex $C = \left(\bigoplus_{n\in\mathbb{Z}} C^n, d^n\right)$ with

$$C^{2k} = \mathcal{F}_{-2kp+p-r,q+2-s} \otimes \sigma_{\mathcal{G}}^{-1}(\mathsf{G}), \qquad d^{2k} = \left[S^{-}_{\mathfrak{sl}(2)}\right]^r \tag{4.100}$$

$$C^{2k+1} = \mathcal{F}_{-2kp-(p-r),q+2-s} \otimes \sigma_{\mathcal{G}}^{-1}(\mathsf{G}), \qquad d^{2k+1} = \left[S^{-}_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.101}$$

has cohomology

$$H^n(C) = \delta_{n,0}\sigma_{\mathfrak{sl}(2)}^{-1}(\mathcal{L}_{p-r,q+1-s}), \tag{4.102}$$

$$= \delta_{n,0}\omega(\mathcal{L}_{r,s}) \tag{4.103}$$

Therefore, we now want to localise each Wakimoto module appearing in the Felder complex in Equations (4.100) and (4.101). Since the localisation functor $\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}$ only acts on the bosonic ghost modules, we will first determine $\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right)$. Let J_0, L_0 act as $J_0|_{gs}, L_0|_{gs}$ on ground states, so that $J_0|_{gs} = \beta_0\gamma_0, L_0|_{gs} = 0$. From the definition of localisation we now compute

$$\Omega^{\mu}_{\beta}(\beta_0) = \beta_0 \tag{4.104}$$

$$\Omega^{\mu}_{\beta}(\gamma_0) = \gamma_0 + \mu\beta_0^{-1} \tag{4.105}$$

$$\Omega^{\mu}_{\beta}(J_0|_{gs}) = J_0|_{gs} + \mu \tag{4.106}$$

$$\Omega^{\mu}_{\beta}(L_0|_{gs}) = L_0|_{gs} \tag{4.107}$$

Specifically,

Proposition 4.1.8. *We have*

$$\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right) = \mathcal{W}_{\left[-\frac{\lambda_{r,s}+\lambda}{2}\right]} \tag{4.108}$$

Proof. This proof is very similar to Proposition 4.1.4, firstly we recall that

$$\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)} = \mathrm{Res}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})_\beta}_{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})} \circ \left(\Omega_{\beta}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\right)^* \circ \mathrm{Ind}^{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})_\beta}_{\mathrm{U}(\mathcal{H}\otimes\mathcal{G})} \tag{4.109}$$

A PBW basis for the induced module is

$$\{\gamma^{q_n}_{-n_i}\cdots\gamma^{q_1}_{-n_1}\beta^{p_m}_{-m_i}\cdots\beta^{p_1}_{-m_1}\beta^{p_0}_0|0\rangle \mid i\geq 1, p_0\in\mathbb{Z}, p_i, q_i\geq 0, 1\leq m_1,...,\leq m_i, 1\leq n_1\leq\cdots\leq n_j\} \tag{4.110}$$

From Remark B.2.3 we see that, for $n \geq 1$,

$$J_0\beta_0^{-n}\sigma_{\mathcal{G}}^{-1}(|0\rangle) = n\beta_0^{-n}\sigma_{\mathcal{G}}^{-1}(|0\rangle) \tag{4.111}$$
$$L_0\beta_0^{-n}\sigma_{\mathcal{G}}^{-1}(|0\rangle) = 0 \tag{4.112}$$

Therefore,

$$\operatorname{ch}\operatorname{Ind}_U^{U_\beta}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right) = \frac{\mathsf{q}^{-\frac{c_\mathsf{G}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} w^n \tag{4.113}$$

The twisted actions of $\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}$ of $\beta_0, \gamma_0, J_0, L_0$ acting on the relaxed highest-weight state $\sigma_{\mathcal{G}}^{-1}(|0\rangle)$ are

$$\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(\beta_0)\sigma_{\mathcal{G}}^{-1}(|0\rangle) = \beta_0\sigma_{\mathcal{G}}^{-1}(|0\rangle) \tag{4.114}$$
$$\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(\gamma_0)\sigma_{\mathcal{G}}^{-1}(|0\rangle) = -\frac{1}{2}(\lambda_{r,s}+\lambda)\beta_0^{-1}\sigma_{\mathcal{G}}^{-1}(|0\rangle) \tag{4.115}$$
$$\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(J_0|_{gs})\sigma_{\mathcal{G}}^{-1}(|0\rangle) = \left(-1-\frac{1}{2}(\lambda_{r,s}+\lambda)\right)\sigma_{\mathcal{G}}^{-1}(|0\rangle) \tag{4.116}$$
$$\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(L_0|_{gs})\sigma_{\mathcal{G}}^{-1}(|0\rangle) = 0 \tag{4.117}$$

Therefore,

$$\operatorname{ch}\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right) = \frac{w^{-1-\frac{1}{2}(\lambda_{r,s}+\lambda)}\mathsf{q}^{-\frac{c_\mathsf{G}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} w^n \tag{4.118}$$
$$= \frac{w^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\mathsf{q}^{-\frac{c_\mathsf{G}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} w^n \tag{4.119}$$

As $\lambda \neq -\lambda_{r,s} \mod 2$, we see that the expression in Equation (4.115) will never be zero. Therefore, states involving β_0^{-1} in the PBW basis will never be annihilated after applying the restriction functor $\operatorname{Res}_U^{U_\beta}$. Therefore,

$$\operatorname{ch}\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right) = \operatorname{ch}\operatorname{Res}_U^{U_\beta}\circ\left(\Omega_\beta^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\right)^*\circ\operatorname{Ind}_U^{U_\beta}\left(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\right) \tag{4.120}$$
$$= \frac{w^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\mathsf{q}^{-\frac{c_\mathsf{G}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} w^n \tag{4.121}$$

Since irreducible modules of the bosonic ghost algebra G are completely characterised by their characters, we see that

$$\mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\Big(\sigma_{\mathcal{G}}^{-1}(\mathsf{G})\Big) = \mathcal{W}_{[-\frac{1}{2}(\lambda_{r,s}+\lambda)]} \tag{4.122}$$

□

Thus, applying the localisation functors $\mathscr{E}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}, \mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}$ to the complex in Equations (4.100) and (4.101) we get that the complex $C = (\bigoplus_{n\in\mathbb{Z}} C_n, d_n)$ given by

$$C^{2k} = \mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\Big(\mathcal{F}_{-2kp+p-r,q+2-s} \otimes \sigma_{\mathcal{G}}^{-1}(\mathsf{G})\Big), \qquad d^{2k} = \mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\Big(\Big[S^-_{\mathfrak{sl}(2)}\Big]^r\Big) \tag{4.123}$$

$$C^{2k+1} = \mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\Big(\mathcal{F}_{-2kp+p-r,q+2-s} \otimes \sigma_{\mathcal{G}}^{-1}(\mathsf{G})\Big), \quad d^{2k+1} = \mathscr{B}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}\left(\Big[S^-_{\mathfrak{sl}(2)}\Big]^{p-r}\right) \tag{4.124}$$

has cohomology

$$H^n(C) = \delta_{n,0}\mathscr{E}^{-\frac{1}{2}(\lambda_{r,s}+\lambda)}(\omega(\mathcal{L}_{r,s})) \tag{4.125}$$

Applying Proposition 4.1.5 and Lemma 4.1.7 to the complex above we get

Proposition 4.1.9. *Let* $r = 1, ..., p-1, s = 2, ..., q$ *and* $C = \left(\bigoplus_{n\in\mathbb{Z}} C^n, d^n\right)$ *be the complex with*

$$C^{2k} = \mathcal{F}_{-2kp+p-r,q+2-s} \otimes \mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}, \qquad d^{2k} = \Big[S^-_{\mathfrak{sl}(2)}\Big]^r \tag{4.126}$$

$$C^{2k+1} = \mathcal{F}_{-2kp-p+r,q+2-s} \otimes \mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}, \qquad d^{2k+1} = \Big[S^-_{\mathfrak{sl}(2)}\Big]^{p-r} \tag{4.127}$$

Then the cohomology of C *is*

$$H^n(C) = \delta_{n,0}\mathcal{E}_{\lambda,\Delta_{r,s}} \tag{4.128}$$

We can do a consistency check for Proposition 4.1.9 by computing the Euler characteristic of the complex. Recall from Equations (1.165) and (1.167) we can write

$h_0 = h_0^{\mathsf{H}} + h_0^{\mathsf{G}}, L_0 = L_0^{\mathsf{H}} + L_0^{\mathsf{G}}$ in terms of modes of H, G. We then have

$$h_0^{\mathsf{H}}|\alpha_{-2kp+p-r,q+2-s}\rangle = -2kp - \lambda_{r,s} \tag{4.129}$$

$$h_0^{\mathsf{H}}|\alpha_{-2kp-p+r,q+2-s}\rangle = -2kp - \lambda_{r,s} - 2(p-r) \tag{4.130}$$

$$L_0^{\mathsf{H}}|\alpha_{-2kp+p-r,q+2-s}\rangle = \frac{(qr-p(s-1)+2kpq)^2-q^2}{4pq} \tag{4.131}$$

$$L_0^{\mathsf{H}}|\alpha_{-2kp-p+r,q+2-s}\rangle = \frac{(qr+p(s-1)-2(k+1)pq)^2-q^2}{4pq} \tag{4.132}$$

$$h_0^{\mathsf{G}}\Big|-\frac{1}{2}(\lambda_{r,s}+\lambda)\Big\rangle = (\lambda_{r,s}-\lambda)\Big|-\frac{1}{2}(\lambda_{r,s}+\lambda)\Big\rangle \tag{4.133}$$

$$L_0^{\mathsf{G}}\Big|-\frac{1}{2}(\lambda_{r,s}+\lambda)\Big\rangle = 0 \tag{4.134}$$

Then we have

$$\operatorname{ch}\mathcal{F}_{-2kp+p-r,q+2-s} = \frac{z^{-2kp-\lambda_{r,s}}\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}}}{\phi(\mathsf{q})} \tag{4.135}$$

$$\operatorname{ch}\mathcal{F}_{-2kp-p+r,q+2-s} = \frac{z^{2kp-\lambda_{r,s}-2(p-r)}\mathsf{q}^{\frac{(qr+p(s-1)-2(k+1)pq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}}}{\phi(\mathsf{q})} \tag{4.136}$$

$$\operatorname{ch}\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]} = \frac{z^{\lambda_{r,s}+\lambda}\mathsf{q}^{-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.137}$$

where $c^{\mathsf{H}}, c^{\mathsf{G}}$ are central charges corresponding to $L_0^{\mathsf{H}}, L_0^{\mathsf{G}}$ respectively. Therefore

$$\operatorname{ch}\mathcal{F}_{-2kp+p-r,q+2-s}\otimes\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]} = \frac{z^{\lambda_{r,s}+\lambda-2kp-\lambda_{r,s}}\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^3}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.138}$$

$$= \frac{z^{\lambda}\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^3}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.139}$$

$$\operatorname{ch}\mathcal{F}_{-2kp-p+r,q+2-s}\otimes\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]} = \frac{z^{\lambda_{r,s}+\lambda-2kp-\lambda_{r,s}-2(p-r)}\mathsf{q}^{\frac{(qr+p(s-1)-2(k+1)pq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^3}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.140}$$

$$= \frac{z^{\lambda}\mathsf{q}^{\frac{(qr+p(s-1)-2(k+1)pq)^2-q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^3}\sum_{n\in\mathbb{Z}} z^{2n} \tag{4.141}$$

Computing the Euler characteristic we get

$$\sum_{k\in\mathbb{Z}}\left(\operatorname{ch}\mathcal{F}_{-2kp+p-r,q+2-s}\otimes\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}-\operatorname{ch}\mathcal{F}_{-2kp-p+r,q+2-s}\otimes\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right) \tag{4.142}$$

$$=\frac{z^\lambda\mathsf{q}^{-\frac{q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}}}{\phi(\mathsf{q})^3}\sum_{n\in\mathbb{Z}}z^{2n}\sum_{k\in\mathbb{Z}}\left(\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2}{4pq}}-\mathsf{q}^{\frac{(qr+p(s-1)-2(k+1)pq)^2}{4pq}}\right) \tag{4.143}$$

$$=\frac{z^\lambda\mathsf{q}^{-\frac{q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}+\frac{1}{24}}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}}z^{2n}\frac{\mathsf{q}^{-\frac{1}{24}}}{\phi(\mathsf{q})}\sum_{k\in\mathbb{Z}}\left(\mathsf{q}^{\frac{(qr-p(s-1)+2kpq)^2}{4pq}}-\mathsf{q}^{\frac{(qr+p(s-1)-2(k+1)pq)^2}{4pq}}\right) \tag{4.144}$$

$$=\frac{z^\lambda\mathsf{q}^{-\frac{q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}+\frac{1}{24}}\chi^{\mathsf{Vir}}_{r,s}}{\phi(\mathsf{q})^2}\sum_{n\in\mathbb{Z}}z^{2n} \tag{4.145}$$

$$=\frac{z^\lambda\chi^{\mathsf{M}(p,q)}_{r,s-1}}{\eta(\mathsf{q})^2}\sum_{n\in\mathbb{Z}}z^{2n} \tag{4.146}$$

$$=\operatorname{ch}\mathcal{E}_{\lambda,\Delta_{r,s-1}} \tag{4.147}$$

Equation (4.146) follows from

$$-\frac{q^2}{4pq}-\frac{c^{\mathsf{H}}}{24}-\frac{c^{\mathsf{G}}}{24}+\frac{1}{24}=-\frac{q}{4p}-\frac{1}{24}\left(1-\frac{12}{\alpha_+^2}\right)-\frac{2}{24}+\frac{1}{24} \tag{4.148}$$

$$=-\frac{q}{4p}-\frac{1}{24}\left(1-\frac{6q}{p}\right)-\frac{2}{24}+\frac{1}{24} \tag{4.149}$$

$$=-\frac{1}{12} \tag{4.150}$$

Spectrally flowing the complex in Proposition 4.1.9, noting Equation (1.99), we arrive at

Proposition 4.1.10. *For $r=1,...,p-1, s=2,...,q,$ let $C=(C^n,d^n)$ be a complex such that for $n=2k,2k+1$,*

$$C^{2k}=\mathcal{F}_{-2kp+p-r,q-s+2-l}\otimes\sigma^l_{\mathcal{G}}\left(\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right),\qquad d^{2k}=\left[S^-_{\mathfrak{sl}(2)}\right]^r \tag{4.151}$$

$$C^{2k+1}=\mathcal{F}_{-2kp-(p-r),q-s+2-l}\otimes\sigma^l_{\mathcal{G}}\left(\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right),\qquad d^{2k+1}=\left[S^-_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.152}$$

Then the cohomology of C is

$$H^n(C)=\delta_{n,0}\sigma^l_{\mathfrak{sl}(2)}\left(\mathcal{E}_{\lambda,\Delta_{r,s}}\right) \tag{4.153}$$

We have therefore obtained the Bernard-Felder complexes for all irreducible modules in category $\mathcal{R}^\sigma$.

4.2 BRST cohomology for $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in category $\mathcal{R}^\sigma$

For the rest of the chapter we take $\lambda = 0$ for the energy momentum tensor of the bosonic ghosts given in Equation (1.146). In this section we will state and prove our (partial) results for the BRST cohomology of the irreducible $\mathcal{L}_k(\mathfrak{sl}(2))$ modules in category $\mathcal{R}^\sigma$. The more technical propositions that are used will be proved in later sections.

Theorem 4.2.1. *For $1 \leq r \leq p-1, 1 \leq s \leq q-1$ and $l \geq 1$,*

$$H^n\Big(\sigma^l_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = 0, \qquad \textit{for all } n \tag{4.154}$$

Proof. From Proposition 4.1.3 we see that the double complex that we should consider is (notice the shift in ghost degree) $\sigma^l(D) = \Big(\sigma^l(D)^{i,j}, d_1^i, d_2^j\Big)$, for $i = 2k, 2k+1$, where

$$\sigma^l(D)^{2k,j} = \mathcal{F}_{-2kp+r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}) \otimes \mathsf{B}_{j-l}, \qquad d_1^{2k} = \Big[S_{\mathfrak{sl}(2)}\Big]^r \tag{4.155}$$

$$\sigma^l(D)^{2k+1,j} = \mathcal{F}_{-2kp-r,s-l} \otimes \sigma^l_{\mathcal{G}}(\mathsf{G}) \otimes \mathsf{B}_{j-l}, \qquad d_1^{2k+1} = \Big[S_{\mathfrak{sl}(2)}\Big]^{p-r} \tag{4.156}$$

$$d_2^j = d_{\mathrm{BRST}} \tag{4.157}$$

Propositions 4.3.1 and 4.4.3 tell us that

$$0 = H^i\Big(H^j\Big(\sigma^l(D), d_2\Big), d_1\Big) = H^j\Big(H^i\Big(\sigma^l(D), d_1\Big), d_2\Big) = H^{j-l}\Big(\sigma^l_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) \tag{4.158}$$

for all j and we are done. □

Since $\sigma(\mathcal{L}_{r,1}) = \mathcal{L}_{p-r,q}$, Theorem 4.2.1 explains the result from Theorem 3.1.2, that

$$H^n(\mathcal{L}_{r,q} \otimes \mathsf{B}, d_{\mathrm{BRST}}) = 0, n \in \mathbb{Z} \tag{4.159}$$

see Figure 4.1. Next we will state our result for negative spectral flow

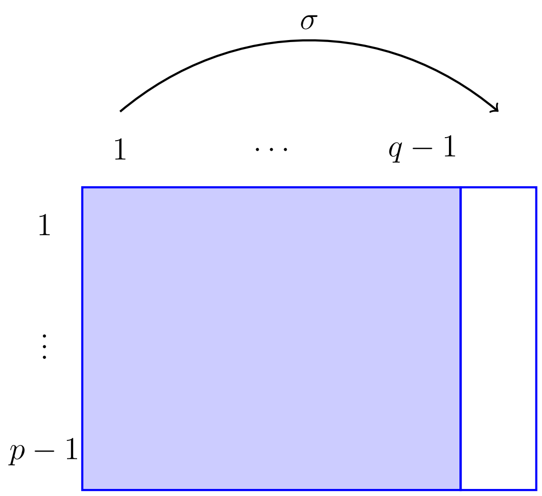


Fig. 4.1 Spectrally flowing modules from column $s = 1$ gives us modules in column $s = q$. In particular, $\sigma_{\mathfrak{sl}(2)}(\mathcal{L}_{r,1}) = \mathcal{L}_{p-r,q}$.

Proposition 4.2.2. *For* $1 \leq r \leq p-1, 1 \leq s \leq q-2$ *and* $1 \leq l \leq q-1-s$*,*

$$H^l\left(\sigma^{-l}_{\mathfrak{sl}(2)}(\mathcal{L}_{r,s}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\right) = \mathcal{L}^{\mathsf{M}(p,q)}_{r,s+l} \tag{4.160}$$

Proof. From Proposition 4.1.3, the double complex that we should consider is (notice the shift in ghost degree) $\sigma^{-l}(D) = \left(\sigma^{-l}(D)^{i,j}, d_1^i, d_2^j\right)$, for $i = 2k, 2k+1$, where

$$\sigma^{-l}(D)^{2k,j} = \mathcal{F}_{-2kp+r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{4.161}$$

$$\sigma^{-l}(D)^{2k+1,j} = \mathcal{F}_{-2kp-r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.162}$$

$$d_2^j = d_{\mathrm{BRST}} \tag{4.163}$$

Proposition 4.3.2 tells us that if we compute the cohomology with respect to d_2, the double complex reduces to a Felder complex concentrated at ghost degree l which is when $j = 0$. Since we can also replace the $\mathfrak{sl}(2)$ screening operators with Virasoro screening operators as they are BRST-exact from Lemma 3.1.3. Therefore we get a complex $C = (C^i, d^i)$, where $i = 2k, 2k+1$, along at $j = 0$,

$$C_{2k} = \mathcal{F}_{-2kp+r,s+l}, \qquad d_{2k} = [S_V]^r \tag{4.164}$$

$$C_{2k+1} = \mathcal{F}_{-2kp-r,s+l}, \qquad d_{2k+1} = [S_V]^{p-r} \tag{4.165}$$

Next from Corollary 4.4.8 we know that

$$H^0\Big(H^0\big(\sigma^{-l}(D), d_1\big), d_2\Big) = H^0\Big(H^0\big(\sigma^{-l}(D), d_2\big), d_1\Big) \tag{4.166}$$

and therefore,

$$\begin{aligned} H^l\Big(\sigma_{\mathfrak{sl}(2)}^{-l}(\mathcal{L}_{r,s}), d_{\mathrm{BRST}}\Big) &= H^0\Big(H^0(D, d_1), d_1\Big) && (4.167)\\ &= H^0\Big(H^0(D, d_2), d_1\Big) && (4.168)\\ &= H^0(C, d) && (4.169)\\ &= \mathcal{L}_{r,s+l}^{\mathsf{M}(p,q)} && (4.170)\end{aligned}$$

□

We remark that we must impose the condition $1 \leq l \leq q-1-s$, otherwise the cohomology functors will not commute for the double complex $\sigma^{-l}(D)$.

Proposition 4.2.3. *For $1 \leq r \leq p-1, 2 \leq s \leq q$, we have*

$$H^n\Big(\sigma_{\mathfrak{sl}(2)}\Big(\mathcal{E}_{\lambda,\Delta_{r,s}}\Big), d_{\mathrm{BRST}}\Big) = \delta_{n,0} \bigoplus_{k=0}^{\infty} \mathcal{L}_{r,s-1}^{\mathsf{M}(p,q)}. \tag{4.171}$$

That is, the zeroth cohomology is an infinite direct sum of copies of $\mathcal{L}_{r,s-1}^{\mathsf{M}(p,q)}$.

Proof. From Proposition 4.1.10, the double complex that we should consider is $\sigma(D) = \Big(\sigma(D)^{i,j}, d_1^i, d_2^j\Big)$, for $i = 2k, 2k+1$, where

$$\sigma(D)^{2k,j} = \mathcal{F}_{-2kp+p-r,q-s+1} \otimes \sigma_{\mathcal{G}}\left(\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right) \otimes \mathsf{B}_j, \qquad d_1^{2k} = \Big[S_{\mathfrak{sl}(2)}\Big]^r \tag{4.172}$$

$$\sigma(D)^{2k+1,j} = \mathcal{F}_{-2ip+p-r,q-s+1} \otimes \sigma_{\mathcal{G}}\left(\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right) \otimes \mathsf{B}_j, \quad d_1^{2k+1} = \Big[S_{\mathfrak{sl}(2)}\Big]^{p-r} \tag{4.173}$$

$$d_2^j = d_{\mathrm{BRST}} \tag{4.174}$$

Note that $\sigma_{\mathcal{G}}\left(\mathcal{W}_{\left[\frac{\lambda_{r,s}+\lambda}{2}\right]}\right)$ is conformally bounded and so the double complex satisfies Remark 3.4.1. Thus the cohomology functors d_1, d_2 commute by Proposition 3.4.2. Now, Proposition 4.3.3 tells us that if we compute the cohomology with respect to d_2, we get a complex $C = (C^i, d^i)$, at ghost degree $j = 0$,

$$C^{2k} = \bigoplus_{q_0=0}^{\infty} \mathcal{F}_{-2ip+p-r,q-s+1} \otimes \gamma_0^{q_0}|\psi\rangle, \qquad d^{2k} = [S_V]^r \tag{4.175}$$

$$C^{2k+1} = \bigoplus_{q_0=0}^{\infty} \mathcal{F}_{-2ip+p-r,q-s+1} \otimes \gamma_0^{q_0}|\psi\rangle, \qquad d^{2k+1} = [S_V]^{p-r} \tag{4.176}$$

Therefore,

$$\begin{aligned} H^n\Big(\sigma_{\mathfrak{sl}(2)}(\mathcal{E}_{r,s}), d_{\text{BRST}}\Big) &= H^n\Big(H^0(\sigma(D), d_1), d_2\Big) &(4.177)\\ &= H^0(H^n(\sigma(D), d_2), d_1) &(4.178)\\ &= H^0(\delta_{n,0}C, d) &(4.179)\\ &= \delta_{n,0} \bigoplus_{k=0}^{\infty} \mathcal{L}^{\mathsf{M}(p,q)}_{p-r,q-s+1} &(4.180)\\ &= \delta_{n,0} \bigoplus_{k=0}^{\infty} \mathcal{L}^{\mathsf{M}(p,q)}_{r,s-1} &(4.181) \end{aligned}$$

□

4.3 BRST cohomology for $\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathcal{G})$

In this section we fix $\lambda \in \mathbb{C}$. Here we will discuss the BRST cohomology for spectrally flowed Wakimoto modules, where we will prove Propositions 4.3.1 and 4.3.2. We first remark that the main reason we have these two different results is because $\beta_0\sigma^l_{\mathcal{G}}(|0\rangle_{\mathsf{G}}) = 0$ for $l \geq 1$ whereas $\beta_0\sigma^{-l}_{\mathcal{G}}(|0\rangle_{\mathsf{G}}) \neq 0$ for $l \geq 1$.

4.3.1 Case for $\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G})$ with $l \geq 1$

Proposition 4.3.1. *For $l \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{C}$, we have*

$$H^n\Big(\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}, d_{\text{BRST}}\Big) = 0, \qquad \textit{for all } n \tag{4.182}$$

Proof. This proof is similar to the proof for Proposition 3.3.1. We first let

$$|\psi\rangle = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}(|\lambda\rangle \otimes |0\rangle_{\mathsf{G}}) \otimes |0\rangle_{\mathsf{B}} \tag{4.183}$$

Then we can decompose the complex as a tensor product of complexes

$$C = \bigotimes_{m\geq l} \text{span}\Big\{\beta^{p_m}_{-m} b^{r_m}_{-m}|\psi\rangle \mid p_m \geq 0, r_m \in \{0,1\}\Big\} \tag{4.184}$$

$$\bigotimes_{n\geq -l+1 n\neq 0} \text{span}\{\gamma^{q_n}_{-n} c^{s_n}_{-n}|0\rangle \mid q_n \geq 0, s_n \in \{0,1\}\} \tag{4.185}$$

$$\otimes\, \text{span}\{\gamma^{q_0}_0|\psi\rangle \mid q_0 \geq 0\} \otimes \text{span}\{b^{p_0}_0|\psi\rangle \mid p_0 \in \{0,1\}\} \tag{4.186}$$

Now, let

$$C_0 = \bigotimes_{m\geq l} \text{span}\Big\{\beta^{p_m}_{-m} b^{r_m}_{-m}|\psi\rangle \mid p_m \geq 0, r_m \in \{0,1\}\Big\} \tag{4.187}$$

$$\bigotimes_{n\geq -l+1 n\neq 0} \text{span}\{\gamma^{q_n}_{-n} c^{s_n}_{-n}|0\rangle \mid q_n \geq 0, s_n \in \{0,1\}\} \tag{4.188}$$

$$\text{span}\{\gamma^{q_0}_0|\psi\rangle \mid q_0 \geq 0\} \tag{4.189}$$

$$I^B_0 = \text{span}\{b^{p_0}_0|\psi\rangle \mid p_0 \in \{0,1\}\} \tag{4.190}$$

so that $C = C_0 \otimes I^B_0$. Now consider the cohomology of I^B_0

$$0 \xrightarrow{d^{-2}} \text{span}\{b_0|\psi\rangle\} \xrightarrow{d^{-1}} \text{span}\{|\psi\rangle\} \xrightarrow{d^0} 0 \tag{4.191}$$

Then we have

$$\ker d^{-1} = 0, \qquad \ker d^0 = \text{span}\{|\psi\rangle\} \tag{4.192}$$

$$\text{im}\, d^{-2} = 0, \qquad \text{im}\, d^{-1} = \text{span}\{|\psi\rangle\} \tag{4.193}$$

Therefore we see that

$$H^n\Big(I^B_0\Big) = 0 \tag{4.194}$$

for all n. By the Künneth theorem we see that

$$H^n(C, d_{\text{BRST}}) = \bigoplus_{i+j=n} H^i(C_0) \otimes H^j\Big(I^B_0\Big) = 0 \tag{4.195}$$

for all n. □

4.3.2 Case for $\sigma^{-l}_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G})$ with $l \geq 1$

We now move onto the BRST cohmology for negatively spectral flowed Wakimoto modules. The proof for this is a generalisation of the proof for Proposition 3.3.1.

Proposition 4.3.2. *For $l \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{C}$, we have*

$$H^n\Big(\sigma^{-l}_{\mathcal{H}\otimes\mathcal{B}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = \delta_{n,l}\sigma^{-l}_{\mathcal{H}}(\mathcal{F}_\lambda), \qquad \forall l \geq 1 \tag{4.196}$$

Proof. Firstly from Lemma 1.8.1 we see that the complex $\Big(\sigma^{l}_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}_\bullet, d_{\mathrm{BRST}}\Big)$ is equivalent to $\Big(\sigma^{l}_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}}(\mathcal{F}_\lambda \otimes \mathsf{G} \otimes \mathsf{B}_\bullet), d_{\mathrm{BRST}}\Big)$, where $\sigma^{l}_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}} = \sigma^l_{\mathcal{H}} \otimes \sigma^l_{\mathcal{G}} \otimes \sigma^l_{\mathcal{B}}$. As B is irreducible, any vector in the vacuum generates B. Again, let $|0\rangle = |0\rangle_{\mathsf{G}} \otimes |0\rangle_{\mathsf{B}}$ and we define the vector

$$|\psi\rangle = \sigma^{-l}(|0\rangle) = \sigma^{-l}(|0\rangle_{\mathsf{G}}) \otimes c_{-l}\cdots c_{-1}|0\rangle_{\mathsf{B}} \tag{4.197}$$

Notice that $|\psi\rangle$ has ghost degree l but we would like it to have ghost degree zero instead. We therefore introduce a complex C where $C_n = \sigma^{-l}_{FF}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}_{n+l}$. A PBW basis for $|\psi\rangle$ as well as the decomposition for the complex C is

$$C = \mathrm{span}\{\beta^{p_i}_{-i} b^{r_i}_{-i}|\psi\rangle \mid i \geq -l, p_i \geq 0, r_i \in \{0,1\}\} \tag{4.198}$$
$$\bigotimes \mathrm{span}\{\gamma^{q_i}_{-i} c^{s_i}_{-i}|\psi\rangle \mid i \geq l+1, q_i \geq 0, s_i \in \{0,1\}\} \tag{4.199}$$

Again we can decompose $I^B = \bigotimes_{i\geq -l} I^B_i$, $I^C = \bigotimes_{i \geq l+1} I^C_i$, where I^B_i, I^C_i were defined in Equations (3.45) and (3.51). We can then directly compute the cohomology of each tensor factor I^B_i

$$0 \xrightarrow{d_{-2}} \mathrm{span}\{\beta^{p_i}_{-i} b_{-i}|\psi\rangle\} \xrightarrow{d_{-1}} \mathrm{span}\{\beta^{p_i}_{-i}|\psi\rangle\} \xrightarrow{d_0} 0 \tag{4.200}$$

where

$$\ker d_{-1} = 0, \qquad \ker d_0 = \mathrm{span}\{\beta^{p_i}_{-i}|\psi\rangle \mid p_i \geq 0\} \tag{4.201}$$
$$\mathrm{im}\, d_{-2} = 0, \qquad \mathrm{im}\, d_{-1} \cong \mathrm{span}\{\beta^{p_i+1}_{-i}|\psi\rangle \mid p_i \geq 0\} \tag{4.202}$$

Hence, we see that

$$H^n(I^B_i) = \delta_{n,0}\mathbb{C}|\psi\rangle, \qquad i \geq -l \tag{4.203}$$

Using Künneth's Theorem we see that

$$H^n(I^B) = \delta_{n,0}\mathbb{C}|0\rangle \tag{4.204}$$

We will now compute the cohomology for each tensor factor I^C_n. The complex is

$$0 \xrightarrow{d_{-1}} \mathrm{span}\{\gamma^{q_i}_{-i}|\psi\rangle\} \xrightarrow{d_0} \mathrm{span}\{\gamma^{q_i}_{-i}c_{-i}|\psi\rangle\} \xrightarrow{d_1} 0 \tag{4.205}$$

where

$$\ker d_0 = \mathbb{C}|\psi\rangle, \qquad \ker d_1 = \mathrm{span}\{\gamma^{q_i}_{-i}c_{-i}|\psi\rangle \mid q_i \geq 0\} \tag{4.206}$$

$$\mathrm{im}\, d_{-1} = 0, \qquad \mathrm{im}\, d_0 = \mathrm{span}\{(q_i+1)\gamma^{q_i}_{-i}c_{-i}|\psi\rangle \mid q_i \geq 0\} \tag{4.207}$$

We therefore conclude that

$$H^n(I^C_i) = \delta_{n,0}\mathbb{C}|\psi\rangle \tag{4.208}$$

Using Künneth's Theorem again we have

$$H^n(I^C) = \delta_{n,0}\mathbb{C}|\psi\rangle \tag{4.209}$$

Thus, applying Künneth's Theorem one last time we finally arrive at

$$H^n(C) = \delta_{n,0}\sigma^{-l}_{\mathcal{H}}(\mathcal{F}_\lambda) \tag{4.210}$$

Since we had $C^n = \sigma^{-l}_{FF}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}^{n+l}$, we conclude that

$$H^n(\sigma^{-l}_{FF}(\mathcal{F}_\lambda \otimes \mathsf{G}), d_{\mathrm{BRST}}) = \delta_{n,l}\sigma^{-l}_{\mathcal{H}}(\mathcal{F}_\lambda) \tag{4.211}$$

□

Summarising our results, for $l \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathbb{C}$, the BRST cohomology of a spectrally flowed Wakimoto module is given by,

$$H^n\Big(\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = 0, \qquad \text{for all } n, \tag{4.212}$$

$$H^n\Big(\sigma^{-l}_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = \delta_{n,l}\sigma^{-l}_{\mathcal{H}}(\mathcal{F}_\lambda). \tag{4.213}$$

4.3.3 Case for $\sigma_{\mathcal{H}\otimes\mathcal{G}}\big(\mathcal{F}_\lambda \otimes \mathcal{W}_{[\zeta]}\big)$

Proposition 4.3.3. *Let* $\lambda \in \mathbb{C}, [\zeta] \neq [0] \in \mathbb{C}/\mathbb{Z}$*, then*

$$H^n\Big(\sigma_{\mathcal{H}\otimes\mathcal{G}}\big(\mathcal{F}_\lambda \otimes \mathcal{W}_{[\zeta]}\big) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = \delta_{n,0} \bigoplus_{k=0}^{\infty} \sigma_{\mathcal{H}}(\mathcal{F}_\lambda) \tag{4.214}$$

Proof. Let $C = \Big(\sigma_{\mathcal{G}}\big(\mathcal{W}_{[\zeta]}\big) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big)$, and let $|\psi\rangle = \sigma_{\mathcal{G}}(|0\rangle) \otimes |0\rangle$, then C can be decomposed into the tensor product of complexes

$$C = \bigotimes_{m\geq 0} \mathrm{span}\{\beta_{-m}^{p_m} b_{-m}^{r_m} |\psi\rangle \mid p_m \geq 0, r_m \in \{0,1\}\} \tag{4.215}$$

$$\bigotimes_{n\geq 1} \mathrm{span}\{\gamma_{-n}^{q_n} c_{-n}^{s_n} |\psi\rangle \mid q_n \geq 0, s_n \in \{0,1\}\} \tag{4.216}$$

$$\bigotimes \mathrm{span}\{\gamma_0^{q_0} |\psi\rangle \mid q_0 \geq 0\} \tag{4.217}$$

With an analysis similar to Proposition 4.3.2, we see that

$$H^n\Big(\sigma_{\mathcal{H}\otimes\mathcal{G}}\big(\mathcal{F}_\lambda \otimes \mathcal{W}_{[\zeta]}\big) \otimes \mathsf{B}, d_{\mathrm{BRST}}\Big) = \delta_{n,0} \bigoplus_{q_0=0}^{\infty} \sigma_{\mathcal{H}}(\mathcal{F}_\lambda) \otimes \gamma_0^{q_0} |\psi\rangle \tag{4.218}$$

$$= \delta_{n,0} \bigoplus_{k=0}^{\infty} \sigma_{\mathcal{H}}(\mathcal{F}_\lambda) \tag{4.219}$$

□

4.4 Commutativity of the double complexes

4.4.1 Case for positive l, $l \geq 1$

Let $\sigma(D) = \Big(\sigma(D)^{i,j}, d_1^i, d_2^j\Big)$, for $i = 2k, 2k+1$, where

$$\sigma(D)^{2k,j} = \sigma_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp+r,s} \otimes \mathsf{G}) \otimes \mathsf{B}_{j-1}, \qquad d_1^{2k} = \Big[S_{\mathfrak{sl}(2)}\Big]^r \tag{4.220}$$

$$\sigma(D)^{2k+1,j} = \sigma_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp-r,s} \otimes \mathsf{G}) \otimes \mathsf{B}_{j-1}, \qquad d_1^{2k+1} = \Big[S_{\mathfrak{sl}(2)}\Big]^{p-r} \tag{4.221}$$

$$d_2^j = d_{\mathrm{BRST}} \tag{4.222}$$

Then we have

Lemma 4.4.1. $\sigma(D)$ *is a direct sum of bounded complexes*

Proof. Since $\sigma_{\mathcal{G}}(\mathsf{G})$ is conformally bounded, the argument is the same as Remark 3.4.1. □

Next, we have

Proposition 4.4.2. *The cohomology functors d_1, d_2 commute when applied to the double complex $\sigma(D)$, that is,*

$$H^j\Big(H^i(\sigma(D), d_1), d_2\Big) = H^i\Big(H^j(\sigma(D), d_2), d_1\Big) \tag{4.223}$$

Proof. This can be proven with the same argument used in Proposition 3.4.2 along with Lemma 4.4.1. □

Now, let Let $\sigma^l(D) = \Big(\sigma^l(D)^{i,j}, d_1^i, d_2^j\Big)$, for $i = 2k, 2k+1$, where

$$\sigma^l(D)^{2k,j} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp+r,s} \otimes \mathsf{G}) \otimes \mathsf{B}_{j-l}, \qquad d_1^{2k} = \Big[S_{\mathfrak{sl}(2)}\Big]^r \tag{4.224}$$

$$\sigma^l(D)^{2k+1,j} = \sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_{-2kp-r,s} \otimes \mathsf{G}) \otimes \mathsf{B}_{j-l}, \qquad d_1^{2k+1} = \Big[S_{\mathfrak{sl}(2)}\Big]^{p-r} \tag{4.225}$$

$$d_2^j = d_{\text{BRST}} \tag{4.226}$$

We now have

Proposition 4.4.3. *For $l \geq 2$, the two double complexes $\sigma(D), \sigma^l(D)$ are isomorphic as double complexes of vector spaces.*

Proof. Firstly for any i, j we denote the highest-weight vector in the Fock space appearing in $\sigma(D)^{i,j}, \sigma^l(D)^{i,j}$ to be $\sigma(|\alpha_i\rangle), \sigma^l(|\alpha_i\rangle)$ respectively. We then redefine the reference vectors in $\sigma_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}$, $\sigma^l_{\mathcal{H}\otimes\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathsf{G}) \otimes \mathsf{B}$ for $\lambda \in \mathbb{C}$ as $c_0\sigma_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}}(|\lambda\rangle \otimes |0\rangle \otimes |0\rangle), c_0\sigma^l_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}}(|\lambda\rangle \otimes |0\rangle \otimes |0\rangle)$ respectively. Now, for each $i, j \in \mathbb{Z}$, consider the map

$\phi^l : Xc_0\sigma_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}}(|\alpha_i\rangle \otimes |0\rangle \otimes |0\rangle) \mapsto \tau^l(X)c_0\sigma^l_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}}(|\alpha_i\rangle \otimes |0\rangle \otimes |0\rangle)$ where

$$\tau^l : \mathrm{U}(\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}) \longrightarrow \mathrm{U}(\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}) \tag{4.227}$$
$$a_n \mapsto a_n \tag{4.228}$$
$$\beta_n \mapsto \beta_{n-l+1} \tag{4.229}$$
$$b_0 \mapsto b_0 \tag{4.230}$$
$$b_n \mapsto b_{n-l+1} \text{ if } n \neq 0 \tag{4.231}$$
$$\gamma_0 \mapsto \gamma_0 \tag{4.232}$$
$$\gamma_{-l+1} \mapsto \gamma_{l-1} \tag{4.233}$$
$$\gamma_n \mapsto \gamma_{n+l-1} \text{ if } n \neq 0, -l+1 \tag{4.234}$$
$$c_{-l+1} \mapsto c_{l-1} \tag{4.235}$$
$$c_n \mapsto c_{n+l-1} \text{ if } n \neq 0, -l+1 \tag{4.236}$$

To show that this is an isomorphism of double complexes, it is enough to show that the following diagrams are commutative

$$\begin{array}{ccc} \sigma(D)^{i,j} & \xrightarrow{\;d_1\;} & \sigma(D)^{i,j} \\ \downarrow{\scriptstyle \phi^l} & & \downarrow{\scriptstyle \phi^l} \\ \sigma^l(D)^{i,j} & \xrightarrow{\;d_1\;} & \sigma^l(D)^{i,j} \end{array}$$

$$\begin{array}{ccc} \sigma(D)^{i,j} & \xrightarrow{\;d_2\;} & \sigma(D)^{i,j} \\ \downarrow{\scriptstyle \phi^l} & & \downarrow{\scriptstyle \phi^l} \\ \sigma^l(D)^{i,j} & \xrightarrow{\;d_2\;} & \sigma^l(D)^{i,j} \end{array}$$

To show the commutativity of the first diagram, notice that we have the following commutative diagram

$$\begin{array}{ccc}
\sigma(D)^{i,j} & \xrightarrow{\quad d_1 \quad} & \sigma(D)^{i,j} \\
\Big\downarrow \phi^l & & \Big\downarrow \phi^l \\
\sigma^l(D)^{i,j} & \xrightarrow{\phi^l \circ d_1 \circ (\phi^l)^{-1}} & \sigma^l(D)^{i,j}
\end{array}$$

Now, since $\phi^l(\beta_n) = \sigma^l(\beta_n)$ for all n and that d_1 only contains β modes and not γ, see Equation (1.172), we see that $\phi^l \circ d_1 \circ (\phi^l)^{-1} = \sigma^l \circ d_1 \circ (\sigma^l)^{-1} = d_1$ and so the first diagram is commutative.
For the second diagram, if we decompose each vertical complexes $\bigoplus_{j\in\mathbb{Z}} \sigma(D), \bigoplus_{j\in\mathbb{Z}} \sigma^l(D)$ for each $i \in \mathbb{Z}$ as in **??**, we get

$$\bigoplus_{j\in\mathbb{Z}} \sigma(D) = \sigma_{\mathcal{H}}(\mathcal{F}_{\alpha_i}) \otimes \bigotimes_{m\geq 1} \text{span}\{\beta_{-m}^{p_m} b_{-m}^{r_m}|\psi\rangle \mid p_m \geq 0, r_m \in \{0,1\}\} \tag{4.237}$$

$$\otimes \bigotimes_{n\geq 1} \text{span}\{\gamma_{-n}^{q_n} c_{-n}^{s_n}|\psi\rangle \mid q_n \geq 0, s_n \in \{0,1\}\} \tag{4.238}$$

$$\otimes\, \text{span}\{b_0^{r_0}|\psi\rangle \mid r_0 \in \{0,1\}\} \otimes \text{span}\{\gamma_0^{q_0}|\psi\rangle \mid q_0 \geq 0\} \tag{4.239}$$

$$\bigoplus_{j\in\mathbb{Z}} \sigma^l(D) = \sigma^l_{\mathcal{H}}(\mathcal{F}_\bullet) \otimes \bigotimes_{m\geq l} \text{span}\{\beta_{-m}^{p_m} b_{-m}^{r_m}|\psi\rangle \mid p_m \geq 0, r_m \in \{0,1\}\} \tag{4.240}$$

$$\otimes \bigotimes_{n\geq -l+1\neq 0} \text{span}\{\gamma_{-n}^{q_n} c_{-n}^{s_n}|\psi\rangle \mid q_n \geq 0, s_n \in \{0,1\}\} \tag{4.241}$$

$$\otimes\, \text{span}\{b_0^{r_0}|\psi\rangle \mid r_0 \in \{0,1\}\} \otimes \text{span}\{\gamma_0^{q_0}|\psi\rangle \mid q_0 \geq 0\} \tag{4.242}$$

Then we see that ϕ^l maps each tensor factor to another. The crucial property is that

$$d_2\phi^l(c_0\sigma(|0\rangle)) = \phi^l d_2(c_0\sigma^l(|0\rangle)) \tag{4.243}$$

It is clear that ϕ^l, d_2 commute for each tensorand and therefore we conclude that these two double complexes are isomorphic as vector spaces. □

We remark that ϕ^l does not preserve the module structures of the double complexes. It is only a vector space isomorphism. The reason for this is that $\sigma(C)$ is an exact complex so to show that $\sigma^l(C)$ is an exact complex is is enough to show that $\sigma^l(C)$ and $\sigma(C)$ are isomorphic double complexes of vector spaces. We finally have

Corollary 4.4.4. *The cohomology functors d_1, d_2 commute when taking cohomologies of the double complex $\sigma^l(D)$.*

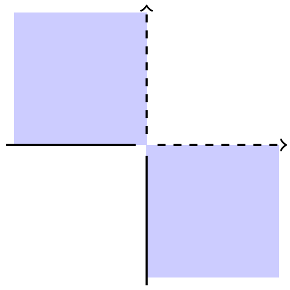


Fig. 4.2 Regions of the double complex in which Lemma 4.4.5 holds

4.4.2 Case for $-l$, $l \geq 1$

We will first give the proof for $l = 0$ (This is the proof presented in Bershadsky and Ooguri's paper)

Lemma 4.4.5. *Let* $\left(\bigoplus_{i,j} D_{i,j}, d_1, d_2\right)$ *be a double complex and* d_1, d_2 *be the horizontal and vertical differentials respectively. Suppose that*

- *the horizontal sequences are exact except at* $D_{0,j}$ *and the vertical sequences are exact except at* $D_{i,0}$
- *the vertical sequences are bounded, that is for a sufficiently large* J *we have* $D_{i,j} = \{0\}$ *and* $D_{i,-j} = \{0\}$ *for any* $j > J$.

Then

$$\ker(d_1 d_2) = \ker(d_1) + \ker(d_2) \tag{4.244}$$

for $D_{i,-j}$ *where* $i \geq 0, j > 0$ *or* $D_{-i,j}$ $i > 0, j \geq 0$.

The double complex $\mathcal{F}_\bullet \otimes \mathsf{G} \otimes \mathsf{B}_\bullet$ satisfies the first assumption due to Propositions 4.1.3 and 4.3.2. The second assumption is satisfied due to the fact that the conformal operator L_0 commutes with both d_1, d_2. Therefore we can restrict the double complex by L_0 eigenspaces. We proceed to the proof.

Proof. We will use mathematical induction for the proof. Suppose that Equation (4.244) holds at $D_{i+1,-(j+1)}$, we show that it also holds for $D_{i,-j}$.
($\supseteq$) If $v \in \ker\left(d_1\big|_{D_{i,-j}}\right)$ or $v \in \ker\left(d_2\big|_{D_{i,-j}}\right)$ then $v \in \ker\left(d_1 d_2\big|_{D_{i,-j}}\right)$ since d_1, d_2 commute so $\ker\left(d_1\big|_{D_{i,-j}}\right) + \ker\left(d_2\big|_{D_{i,-j}}\right) \subseteq \ker\left(d_1 d_2\big|_{D_{i,-j}}\right)$.

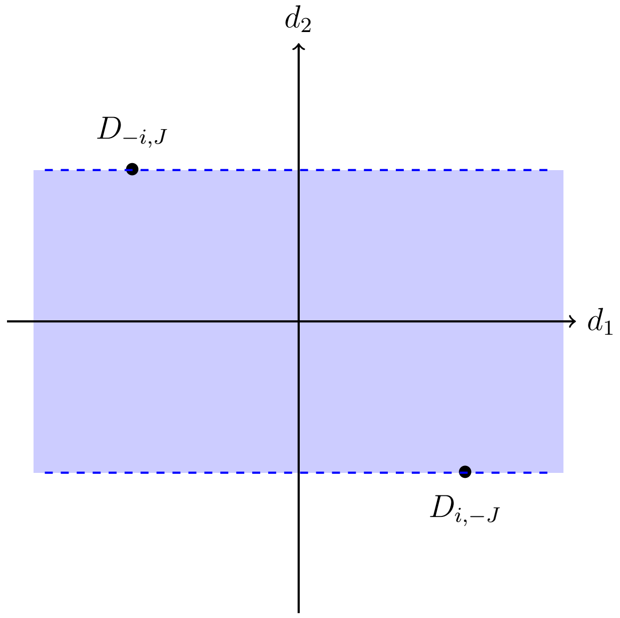


Fig. 4.3 Diagram for Lemma 4.4.5

($\subseteq$) We now want to show $\ker\left(d_1d_2\big|_{D_{i,-j}}\right) \subseteq \ker\left(d_1\big|_{D_{i,-j}}\right) + \ker\left(d_2\big|_{D_{i,-j}}\right)$. For $v_{i,-j} \in \ker\left(d_1d_2\big|_{D_{i,-j}}\right)$, since $d_2d_1v_{i,-j} = 0$, $d_1v_{i,-j}$ belongs to $\ker\left(d_2\big|_{D_{i+1,-j}}\right)$. Because of exactness at $D_{i+1,-j}$ with respect to d_2, there is some $v_{i+1,-j-1} \in D_{i+1,-j-1}$ such that $d_1v_{i,-j} = d_2v_{i+1,-j-1}$. This implies that $0 = d_1^2v_{i,-j} = d_1d_2v_{i+1,-j-1}$ so $v_{i+1,-j-1} \in \ker\left(d_1d_2\big|_{D_{i+1,-j-1}}\right)$. According to our induction assumption, we have $v_{i+1,-j-1} \in \ker\left(d_1\big|_{D_{i+1,-j-1}}\right) + \ker\left(d_2\big|_{D_{i+1,-j-1}}\right)$ and so $v_{i+1,-j-1} = d_1v_{i,-j-1} + d_2v_{i+1,-j-2}$ where $d_1v_{i,-j-1} \in \ker\left(d_1\big|_{D_{i+1,-j-1}}\right), d_2v_{i+1,-j-2} \in \ker\left(d_2\big|_{i+1,-j-1}\right)$ by exactness. Substituting this into $d_1v_{i,-j} = d_2v_{i+1,-j-1}$ we have $d_1v_{i,-j} = d_2d_1v_{i,-j-1} \Rightarrow d_1(v_{i,-j} - d_2v_{i,-j-1}) = 0$ so $v_{i,-j} - d_2v_{i,-j-1} \in \ker\left(d_1\big|_{D_{j,-i}}\right)$ and therefore $v_{j,-i} \in \ker\left(d_1\big|_{D_{j,-i}}\right) + \ker\left(d_2\big|_{D_{j,-i}}\right)$.
By interchanging d_1, d_2 we can prove the case for $D_{-i,j}$ for $i > 0, j \geq 0$.
Since the double complex is bounded, this result can be easily verified for the modules on the boundaries of the double complex as it is bounded, see Figure 4.3.
For the top edge, we have the sequence

$$\cdots \xrightarrow{d_2} D_{-i,J} \xrightarrow{d_2} 0 \tag{4.245}$$

Obviously $D_{-i,J} = \ker\Big(d_2\Big|_{D_{-i,J}}\Big)$. Thus we obtain

$$\ker\Big(d_1 d_2\Big|_{D_{-i,J}}\Big) \subseteq D_{-i,J} = \ker\Big(d_2\Big|_{D_{-i,J}}\Big) \subseteq \ker\Big(d_1\Big|_{D_{-i,J}}\Big) \oplus \ker\Big(d_2\Big|_{D_{-i,J}}\Big) \tag{4.246}$$

For the bottom edge, we have

$$0 \xrightarrow{d_2} D_{i,-J} \xrightarrow{d_2} \cdots , \tag{4.247}$$

and the induction assumption is trivially satisfied for $D_{i+1,-J-1} = 0$. This completes the proof. □

Proposition 4.4.6. *[16] Under the assumptions of Lemma 4.4.5, we have*

$$H^0\Big(H^0(D, d_2), d_1\Big) = \frac{\ker\Big(d_1\Big|_{D_{0,0}}\Big) \bigcap \ker\Big(d_2\Big|_{D_{0,0}}\Big)}{\ker\Big(d_2\Big|_{D_{0,0}}\Big) \bigcap \operatorname{im}\Big(d_1\Big|_{D_{-1,0}}\Big) + \ker\Big(d_1\Big|_{D_{0,0}}\Big) \bigcap \operatorname{im}\Big(d_2\Big|_{D_{0,-1}}\Big)} \tag{4.248}$$

and by symmetry this implies that

$$H^0\Big(H^0(D, d_2), d_1\Big) = H^0\Big(H^0(D, d_1), d_2\Big) \tag{4.249}$$

Proof. Firstly consider the complex D, showing degrees $-1, 0, 1$ respectively

$$\cdots \xrightarrow{d_1} \frac{\ker d_2|_{D_{-1,0}}}{\operatorname{im} d_2|_{D_{-1,-1}}} \xrightarrow{d_1} \frac{\ker d_2|_{D_{0,0}}}{\operatorname{im} d_2|_{D_{0,-1}}} \xrightarrow{d_1} \frac{\ker d_2|_{D_{1,0}}}{\operatorname{im} d_2|_{D_{1,-1}}} \xrightarrow{d_1} \cdots \tag{4.250}$$

Since the induced map d_1 sends $[v] \mapsto [d_1 v]$, we see that

$$H^0\Big(H^0(D, d_2), d_1\Big) = \frac{\{w_{0,0} + d_2(D_{0,-1}) \mid d_2(v_{0,0}) = 0, d_1(v_{0,0}) \in d_2(D_{1,-1})\}}{\{d_1(v_{-1,0}) + d_2(D_{0,-1}) \mid d_2(v_{-1,0}) = 0\}} \tag{4.251}$$

From the first assumption we have $d_1 v_{0,0} = d_2 v_{1,-1}$ where $v_{1,-1} \in D_{1,-1}$. Since $d_1 d_2 v_{1,-1} = 0$ implies $v_{1,-1} = d_1 v_{0,-1} + d_2 v_{1,-2}$. Therefore $d_1 v_{0,0} = d_2(d_1 v_{0,-1} + d_2 v_{1,-2}) = d_2 d_1 v_{0,-1}$ so $d_1(v_{0,0} - d_2 v_{0,-1}) = 0$. Because we are considering the coset $w_{0,0} + d_2(D_{0,-1})$, without loss of generality we can choose a representative $v_{0,0}$ such that $d_1 v_{0,0} = 0$. Let

$$M = \ker\Big(d_1\big|_{D_{0,0}}\Big) \bigcap \ker\Big(d_2\big|_{D_{0,0}}\Big) \tag{4.252}$$

$$I = d_1\Big(\ker\Big(d_2\big|_{D_{-1,0}}\Big)\Big) \tag{4.253}$$

$$J = \operatorname{im}\Big(d_2\big|_{D_{0,-1}}\Big) \tag{4.254}$$

Then we see that

$$H^0\Big(H^0(D, d_2), d_1\Big) = \frac{M + J}{I + J}, \tag{4.255}$$

$$= \frac{M + I + J}{I + J}, \qquad \text{since } I \subseteq M \tag{4.256}$$

$$= \frac{M}{M \cap (I + J)}, \qquad \text{by second isomorphism theorem} \tag{4.257}$$

$$= \frac{M}{I + M \cap J} \tag{4.258}$$

$$= \frac{\ker\Big(d_1\big|_{D_{0,0}}\Big) \bigcap \ker\Big(d_2\big|_{D_{0,0}}\Big)}{d_1\Big(\ker\Big(d_2\big|_{D_{-1,0}}\Big)\Big) + \ker\Big(d_1\big|_{D_{0,0}}\Big) \bigcap \operatorname{im}\Big(d_2\big|_{D_{0,-1}}\Big)}. \tag{4.259}$$

Next we want to show that

$$d_1\Big(\ker\Big(d_2\big|_{D_{-1,0}}\Big)\Big) = \ker\Big(d_2\big|_{D_{0,0}}\Big) \bigcap \operatorname{im}\Big(d_1\big|_{D_{-1,0}}\Big) \tag{4.260}$$

($\subseteq$) is obvious since $d_1\Big(\ker\Big(d_2\big|_{D_{-1,0}}\Big)\Big) \subseteq \operatorname{im}\Big(d_1\big|_{D_{-1,0}}\Big)$ and if $d_1(v_{-1,0}) \in d_1\Big(\ker\Big(d_2\big|_{D_{-1,0}}\Big)\Big)$ then $d_2 d_1(v_{-1,0}) = d_1 d_2(v_{-1,0}) = 0$ so $d_1(v_{-1,0}) \in \ker\Big(d_2\big|_{D_{0,0}}\Big)$

($\supseteq$) For the other direction, let $d_1(v_{-1,0})$ be an element in $\ker\Big(d_2\big|_{D_{0,0}}\Big) \bigcap \operatorname{im}\Big(d_1\big|_{D_{-1,0}}\Big)$. Then $d_2 d_1(v_{-1,0}) = 0$ so by Lemma 4.4.5 we can write $v_{-1,0} = w_1 + w_2$ where $w_1 \in \ker\Big(d_1\big|_{D_{-1,0}}\Big)$, $w_2 \in \ker\Big(d_2\big|_{D_{-1,0}}\Big)$ so

$d_1 v_{-1,0} = d_1(w_1 + w_2) = d_1(w_2) \in d_1\left(\ker\left(d_2\big|_{D_{-1,0}}\right)\right)$. Thus we see that

$$H^0\left(H^0(D, d_2), d_1\right) = \frac{\ker\left(d_1\big|_{D_{0,0}}\right) \bigcap \ker\left(d_2\big|_{D_{0,0}}\right)}{\ker\left(d_2\big|_{D_{0,0}}\right) \bigcap \operatorname{im}\left(d_1\big|_{D_{-1,0}}\right) + \ker\left(d_1\big|_{D_{0,0}}\right) \bigcap \operatorname{im}\left(d_2\big|_{D_{0,-1}}\right)} \tag{4.261}$$

and we are done. □

Now, for $l \geq 1$, let $\sigma^{-l}(D) = \left(\sigma^{-l}(D)^{i,j}, d_1^i, d_2^j\right)$, for $i = 2k, 2k+1$, where

$$\sigma^{-l}(D)^{2k,j} = \mathcal{F}_{-2kp+r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{4.262}$$

$$\sigma^{-l}(D)^{2k+1,j} = \mathcal{F}_{-2kp-r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.263}$$

$$d_2^j = d_{\text{BRST}} \tag{4.264}$$

as in the proof of Proposition 4.2.2. Then we see that the double complex $\sigma^{-l}(D)$ satisfies the assumptions of Lemma 4.4.5 and therefore Proposition 4.4.6 holds for this double complex. We will now show that this also satisfies Lemma 4.4.5. More precisely, we have

Proposition 4.4.7. *The double complex $\sigma^{-l}(D)$ satisfies*

$$\ker\left(d_{\text{BRST}} d_{\mathfrak{sl}(2)}\right) = \ker d_{\text{BRST}} + \ker d_{\mathfrak{sl}(2)} \tag{4.265}$$

for $\sigma^{-l}(D)^{i,-j}$ where $i \geq 0, j > 0$ or $\sigma^{-l}(D)^{-i,j}$ $i > 0, j \geq 0$.

Proof. Let D be the double complex defined in the proof of Theorem 3.1.2 and let

$$\sigma^{-l} = \sigma^{-l}_{\mathcal{H}\otimes\mathcal{G}\otimes\mathcal{B}} \tag{4.266}$$

We first want to show the following equalities for all $i, j \in \mathbb{Z}$,

- $\ker\left(d_1\big|_{\sigma^{-l}(D)^{i,j}}\right) = \sigma^{-l}\left(\ker\left(d_1\big|_{D^{i,j}}\right)\right)$
- $\ker\left(d_2\big|_{\sigma^{-l}(D)^{i,j}}\right) = \sigma^{-l}\left(\ker\left(d_2\big|_{D^{i,j}}\right)\right)$

For the first equality, since d_1 is invariant under the free-field spectral flow by Lemma 4.1.2, we see that for any $i, j \in \mathbb{Z}$ and $v \in D^{i,j}$, $\sigma^{-l}(d_1 v) = d_1 \sigma^{-l}(v)$. Thus

$v \in \ker\left(d_1\big|_{D^{i,j}}\right)$ if and only if $\sigma^{-l}(v) \in \ker\left(d_1\big|_{\sigma^{-l}(D)^{i,j}}\right)$ and so we get

$$\ker\left(d_1\big|_{\sigma^{-l}(D)^{i,j}}\right) = \sigma^{-l}\left(\ker\left(d_1\big|_{D^{i,j}}\right)\right) \tag{4.267}$$

For the second equality, by comparing the proofs of Proposition 3.3.1 and Proposition 4.3.2, in particular Equations (3.47) and (3.53), Equations (4.201) and (4.206), we see that $v \in \ker\left(d_2\big|_{D^{i,j}}\right)$ if and only if $\sigma^{-l}(v) \in \ker\left(d_2\big|_{\sigma^{-l}(D)^{i,j}}\right)$. In order words, we have

$$\ker\left(d_2\big|_{\sigma^{-l}(D)^{i,j}}\right) = \sigma^{-l}\left(\ker\left(d_2\big|_{D^{i,j}}\right)\right) \tag{4.268}$$

Now suppose that $v \in \ker\left(d_1 d_2\big|_{D^{i,j}}\right)$. Then

$$\begin{aligned}
v \in \ker\left(d_1 d_2\big|_{D^{i,j}}\right) &\iff d_1 d_2 v = 0, && (4.269)\\
&\iff d_2 d_1 v = 0, \quad \text{since the differentials commute}, && (4.270)\\
&\iff \sigma^{-l}(d_1 v) \in \sigma^{-l}\left(\ker\left(d_2\big|_{D^{i+1,j}}\right)\right) && (4.271)\\
&\iff \sigma^{-l}(d_1 v) \in \ker\left(d_2\big|_{\sigma^{-l}(D)^{i+1,j}}\right) && (4.272)\\
&\iff d_2\sigma^{-l}(d_1 v) = 0 && (4.273)\\
&\iff d_2 d_1 \sigma^{-l}(v) = 0, \quad \text{since } \sigma^{-l}(d_1) = d_1, && (4.274)\\
&\iff \sigma^{-l}(v) \in \ker\left(d_1 d_2\big|_{\sigma^{-l}(D)^{i,j}}\right) && (4.275)
\end{aligned}$$

and we see that

$$\ker d_1 d_2\Big|_{\sigma^{-l}(D)^{i,j}} = \sigma^{-l}\left(\ker d_1 d_2\Big|_{D^{i,j}}\right) \tag{4.276}$$

Finally we have for $\sigma^{-l}(D)^{i,j}$ where $i \geq 0, j < 0$ or $i < 0, j \geq 0$,

$$\begin{aligned}
\ker\left(d_1 d_2\big|_{\sigma^{-l}(D)^{i,j}}\right) &= \sigma^{-l}\left(\ker\left(d_1 d_2\big|_{D^{i,j}}\right)\right) && (4.277)\\
&= \sigma^{-l}\left(\ker\left(d_1\big|_{D^{i,j}}\right) + \ker\left(d_2\big|_{D^{i,j}}\right)\right) && (4.278)\\
&= \sigma^{-l}\left(\ker\left(d_1\big|_{D^{i,j}}\right)\right) + \sigma^{-l}\left(\ker\left(d_2\big|_{D^{i,j}}\right)\right) && (4.279)\\
&= \ker\left(d_1\big|_{\sigma^{-l}(D)^{i,j}}\right) + \ker\left(d_2\big|_{\sigma^{-l}(D)^{i,j}}\right) && (4.280)
\end{aligned}$$

□

Corollary 4.4.8. *Let $l \geq 1$, let $\sigma^{-l}(D) = \left(\sigma^{-l}(D)^{i,j}, d_1^i, d_2^j\right)$, for $i = 2k, 2k+1$, where*

$$\sigma^{-l}(D)^{2k,j} = \mathcal{F}_{-2kp+r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k} = \left[S_{\mathfrak{sl}(2)}\right]^r \tag{4.281}$$

$$\sigma^{-l}(D)^{2k+1,j} = \mathcal{F}_{-2kp-r,s+l} \otimes \sigma_{\mathcal{G}}^{-l}(\mathsf{G}) \otimes \mathsf{B}_{j+l}, \qquad d_1^{2k+1} = \left[S_{\mathfrak{sl}(2)}\right]^{p-r} \tag{4.282}$$

$$d_2^j = d_{\mathrm{BRST}} \tag{4.283}$$

Then we have

$$H^0\Big(H^0\Big(\sigma^{-l}(D), d_1\Big), d_2\Big) = H^0\Big(H^0\Big(\sigma^{-l}(D), d_2\Big), d_1\Big) \tag{4.284}$$

Proof. This follows immediately from Propositions 4.4.6 and 4.4.7 □

Chapter 5

Conclusion

In the first part of this thesis we have computed singular vectors of W_n-algebras in terms of Jack symmetric functions in certain Fock representations. We saw that this was possible only if the monodromy in the integral inner product for Jack functions is trivial, which in turns restricts the possibility of Fock spaces that allow such computation. An obvious future research direction is to compute singular vectors in the other Fock spaces of the W_n-algebras. We believe that in the case of W_3-algebras, the integral of composition of screening operators gives us a complex $\mathfrak{sl}(3)$ Selberg integral. Therefore the problem of computing singular vectors of the W_3-algebras in other Fock spaces is in some way equivalent to deriving a closed formula for the complex $\mathfrak{sl}(3)$ Selberg integral. Some work has been done for the real $\mathfrak{sl}(3)$ Selberg integral, see [65, 76]. In particular, an explicit formula for the real $\mathfrak{sl}(3)$ Selberg integral was derived in terms of Gamma functions.
In the second part of this thesis we computed the BRST cohomology of some simple modules of $\mathcal{L}_k(\mathfrak{sl}(2))$. We found that the BRST cohomology of a positively spectrally flowed irreducible highest-weight module is trivial. On the other hand, the BRST cohomology of a negatively spectrally flowed irreducible highest-weight module is non-exact. In particular, the cohomology is an irreducible highest-weight module of a Virasoro minimal model at a degree other than zero and we expect that all other degrees are zero. We also computed the BRST cohomology of a simple relaxed highest-weight module $\sigma_{\mathfrak{sl}(2)}(\mathcal{E}_{r,s})$ and found that its cohomology is exact except at degree zero, where it is a direct sum of a countably-infinite number of irreducible highest-weight modules of a Virasoro minimal model. In general, the main difficulty in computing the BRST cohomology of modules in $\mathcal{R}^\sigma$ is that these modules are not conformally bounded in general, as opposed to highest-weight modules in $\mathcal{O}$.

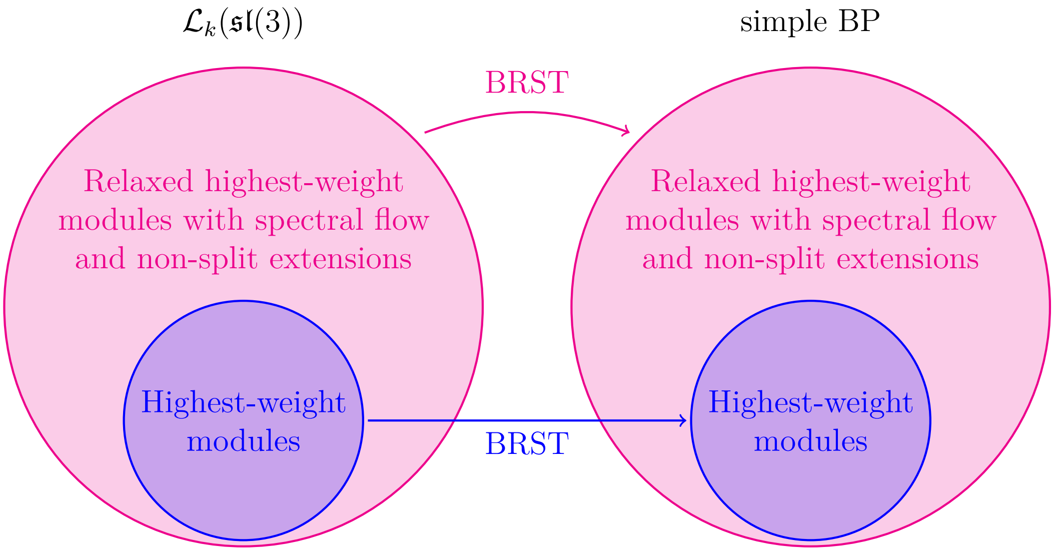


Fig. 5.1

An obvious future research direction is to compute the BRST cohomology of the irreducible relaxed highest-weight modules of the simple affine $\mathfrak{sl}(3)$ vertex operator algebra $\mathcal{L}_k(\mathfrak{sl}(3))$. We believe that the BRST cohomology of some of these irreducible relaxed highest-weight modules should be the relaxed highest-weight modules of the simple Bershadsky-Polyakov vertex operator algebra, see Figure 5.1.

Appendix A

Homological algebra

The claim we want to establish in this appendix is that infinite tensor product of cochain complexes commutes with cohomology. Let R be a ring and we refer left R-modules as simply R-modules. Here we consider cochain complexes over R-modules.

A.1 Cochain complexes

Definition A.1.1. *A cochain complex $C = (C^n, d^n)$ is a sequence of R-modules $C^n, n \in \mathbb{Z}$ with R-module homomorphisms d^n*

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots \tag{A.1}$$

such that $d^{n+1} \cdot d^n = 0$.

Definition A.1.2. *A cochain map f between two cochain complexes $C_1 = (C_1^n, d_1^n), C_2 = (C_2^n, d_2^n)$ is a sequence of R-module homomorphisms $f^n : C_1^n \longrightarrow C_2^n$ such that $f^{n+1} \circ d_1^n = d_2^n \circ f^n$ for all $n \in \mathbb{Z}$. In particular, the following diagram commutes*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_1^{n-2}} & C_1^{n-1} & \xrightarrow{d_1^{n-1}} & C_1^n & \xrightarrow{d_1^n} & C_1^{n+1} & \xrightarrow{d_1^{n+1}} & \cdots \\
 & & \downarrow f^{n-1} & & \downarrow f^n & & \downarrow f^{n+1} & & \\
\cdots & \xrightarrow{d_2^{n-2}} & C_2^{n-1} & \xrightarrow{d_2^{n-1}} & C_2^n & \xrightarrow{d_2^n} & C_2^{n+1} & \xrightarrow{d_2^{n+1}} & \cdots
\end{array}$$

Definition A.1.3. *The tensor product* $C_1 \otimes C_2 = (C^n, d^n)$ *of two cochain complexes* $(C_1^n, d_1^n), (C_2^n, d_2^n)$ *is defined as*

$$(C_1 \otimes C_2)^n = \bigoplus_{i+j=n} \left(C_1^i \otimes C_2^j \right) \tag{A.2}$$

with differential defined as

$$d^n(x_1 \otimes x_2) = \sum_{i+j=n} d_1^i(x_1) \otimes x_2 + (-1)^j x_1 \otimes d_2^j(x_2) \tag{A.3}$$

for $x_1 \in C_1^i, x_2 \in C_2^j$.

We have a useful theorem that relates the cohomology of the tensor product of two cochain complexes with the cohomologies of it's tensorands

Theorem A.1.4 (Künneth formula)**.**

$$H^n(C) = \bigoplus_{i+j=n} \left(H^i(C_1) \otimes H^j(C_2) \right) \tag{A.4}$$

A.1.1 Double complexes

Definition A.1.5. *A double complex* $\left(D^{i,j}, d_1^i, d_2^j \right)$ *is a collection of R-modules* $\{D_{i,j} \mid i, j \in \mathbb{Z}\}$ *together with differentials* d_1^i, d_2^j

$$d_1^i : D^{i,j} \longrightarrow D^{i+1,j} \tag{A.5}$$

$$d_2^j : D^{i,j} \longrightarrow D^{i,j+1} \tag{A.6}$$

such that $d_1^{i+1} \circ d_1^i = d_2^{j+1} \circ d_2^j = 0$ *and* $d_1^i d_2^j + d_2^j d_1^i = 0$.

A cochain map $f : \left(D^{i,j}, d_1^i, d_2^j \right) \longrightarrow \left(\Delta^{i,j}, \delta_1^i, \delta_2^j \right)$ between two double complexes is a collection of maps $f : D^{i,j} \longrightarrow \Delta^{i,j}$ such that the following diagrams commute

$$\begin{array}{ccc} D^{i,j} & \xrightarrow{d_1^i} & D^{i+1,j} \\ \big\downarrow f & & \big\downarrow f \\ \Delta^{i,j} & \xrightarrow{\delta_1^i} & \Delta^{i+1,j} \end{array} \qquad \begin{array}{ccc} D^{i,j} & \xrightarrow{d_2^j} & D^{i,j+1} \\ \big\downarrow f & & \big\downarrow f \\ \Delta^{i,j} & \xrightarrow{\delta_2^j} & \Delta^{i,j+1} \end{array}$$

A.1.2 Spectral sequences

Definition A.1.6. *A filtered complex* (C^n, d^n, F) *is a cochain complex of R-modules with a filtration at each degree, that is, for each* $n \in \mathbb{Z}$ *there are submodules* $F^p C_n \in C_n$ *such that*

$$\cdots \subseteq F^{p+1}C^n \subseteq F^p C^n \subseteq F^{p-1}C^n \subseteq \cdots \tag{A.7}$$

with the property that

$$d^n F^p C^n \subseteq F^p C^{n+1} \tag{A.8}$$

for all $n \in \mathbb{Z}$.

The filtration F on the cochain complex induces a filtration on the cohomology,

$$F^p H^n(C) = \iota(H^n(F^p C)) \tag{A.9}$$

where the map ι is defined as

$$\iota : H^n(F^p C) \hookrightarrow H^n(C) \tag{A.10}$$

$$x + \operatorname{im} d^{n-1}(F^p C^{n-1}) \mapsto x + \operatorname{im} d^{n-1}(C^{n-1}) \tag{A.11}$$

Definition A.1.7. *A spectral sequence is a sequence of bigraded objects* $(E_r^{p,q}, d_r^{p,q}), p, q \in \mathbb{Z}, r \geq 0$, *each at page* r, *where*

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{A.12}$$

are differential maps of degree $(r, -r+1)$ *and for all* p, q, r,

$$E_r^{p,q} = H^{p,q}(E_r, d_r) \tag{A.13}$$

A spectral sequence is said to degenerate (or collapse) at page r if

$$E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q} \tag{A.14}$$

for all $p, q \in \mathbb{Z}$. We have

Theorem A.1.8. *[56] Each filtered complex* (C, d, F) *determines a spectral sequence,* $\{E^{*,*}, d_r\}, r = 1, 2, \dots$ *with* d_r *of bidegree* $(r, 1-r)$ *and*

$$E_1^{p,q} = H^{p+q}\left(\frac{F^p C}{F^{p+1} C}\right) \tag{A.15}$$

Suppose further that the filtration is bounded, that is for each n *there exists* $s(n), t(n)$ *such that*

$$0 \subseteq F^{s(n)} C^n \subseteq F^{s-1} C^n \subseteq \cdots \subseteq F^{t+1} C_n \subseteq F^{t(n)} C^n = C^n \tag{A.16}$$

Then the spectral sequence converges to $H(C, d)$*, that is,*

$$E_\infty^{p,q} \equiv \frac{F^p H^{p+q}(C, d)}{F^{p+1} H^{p+q}(C, d)} \tag{A.17}$$

Given a double complex $(D^{i,j}, d_1^i, d_2^j)$, often we want to know whether taking cohomologies commutes, that is whether

$$H^j(H^i(C, d_1), d_2) = H^i(H^j(C, d_2), d_1) \tag{A.18}$$

To answer this question we will first introduce the total complex $(Tot(D)^n, d^n)$ of a double complex, defined as

$$Tot(D)^n = \bigoplus_{i+j=n} D^{i,j}, \qquad d^n \sum_{\substack{i,j \in \mathbb{Z} \\ i+j=n}} x_{i,j} = \sum_{\substack{i,j \in \mathbb{Z} \\ i+j=n}} d_1^i x_{i,j} + d_2^j x_{i,j} \tag{A.19}$$

The total complex has two natural filtrations, they are

$$_I F^p Tot(D)^n = \bigoplus_{i \geq p} D^{i,n-i} \tag{A.20}$$

$$_{II} F^p Tot(D)^n = \bigoplus_{i \geq p} D^{n-i,i} \tag{A.21}$$

Theorem A.1.9. *[56] Given a double complex* $\left(D^{i,j}, d_1^i, d_2^j\right)$*, each filtration defined in Equations (A.20) and (A.21) on the double complex give rise to a spectral sequence with second pages equal to*

$$_I E_2^{p,q} = H^p(H^q(D, d_2), d_1) \tag{A.22}$$

$$_{II} E_2^{p,q} = H^q(H^p(D, d_1), d_2) \tag{A.23}$$

Furthermore, if the double complex is bounded, that is there is an s, t such that $D^{i,j} = 0$ for $j > t$ and $j < s$ (or similarly for $i > t$ and $i < s$). Then both spectral sequences converge to the filtered quotients of the total complex

$$H^n(Tot(D), d_1 + d_2) \tag{A.24}$$

In particular, if both spectral sequences degenerate on the second page, then taking cohomologies commute,

$$H^i\Big(H^j(D, d_2), d_1\Big) = H^j\Big(H^i(D, d_1), d_2\Big) \tag{A.25}$$

A.2 The category of directed systems $\mathbf{DS}_I(R\text{-}\mathbf{Mod})$

Let $\langle I, \leq \rangle$ be the set of natural numbers $\mathbb{N}$ equipped with the natural ordering. In general $\langle I, \leq \rangle$ can be any directed set. Let $\{A_i \mid i \in I\}$ be a sequence of R-modules and $f_{ij} : A_i \longrightarrow A_j$ be a homomorphism for all $i \leq j$ such that

- f_{ii} is the identity on A_i
- $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$

Then the pair $\langle A_i, f_{ij} \rangle$ is called a directed system over I. We then define $\mathbf{DS}_I(R\text{-}\mathbf{Mod})$ to be the category of directed systems whose objects are directed systems of R-modules $\langle A_i, f_{ij} \rangle$ over I. The morphisms between two objects $\langle A_i, f_{ij} \rangle, \langle B_i, g_{ij} \rangle$ in this category are collection of R-homomorphisms $\{\phi_i \mid i \in I\}$ such that the following diagram commutes

$$\begin{array}{ccc} A_i & \xrightarrow{\phi_i} & B_i \\ \Big\downarrow f_{ij} & & \Big\downarrow g_{ij} \\ A_j & \xrightarrow{\phi_j} & B_j \end{array}$$

whenever $i \leq j$. To see that this defines a proper morphism on this category, notice that the identity morphism is just $\{1_R \mid i \in I\}$ so it remains to show the composition property. We define composition of two morphisms $\{\phi_i \mid i \in I\}, \{\psi_i \mid i \in I\}$ to be $\{\psi_i \circ \phi_i \mid i \in I\}$. Suppose that we have three directed systems $A = \langle A_i, f_{ij} \rangle, B = \langle B_i, g_{ij} \rangle, C = \langle C_i, h_{ij} \rangle$ with morphisms $A \xrightarrow{\{\phi_i\}} B \xrightarrow{\{\psi_i\}} C$ then it suffices to show the following diagram commutes

$$\begin{array}{ccc} A_i & \xrightarrow{\psi_i \circ \phi_i} & C_i \\ \Big\downarrow f_{ij} & & \Big\downarrow h_{ij} \\ A_j & \xrightarrow{\psi_j \circ \phi_j} & C_j \end{array}$$

Now

$$\psi_j \circ \phi_j \circ f_{ij} = \psi_j \circ g_{ij} \circ \phi_i \tag{A.26}$$

$$= h_{ij} \circ \psi_i \circ \phi_i \tag{A.27}$$

We conclude that $\mathbf{DS}_I(R\text{-}\mathbf{Mod})$ is a category. We then have [63]

Lemma A.2.1. *Let $\langle A_i, f_{ij} \rangle$ be a direct system of R-modules over a directed set I, and let $\iota_i : A_i \to \bigoplus A_i$ be the ith injection. Let S be the submodule generated by $\iota_j \circ f_{ij}(x_i) = \iota_i(x_i)$ for $x_i \in A_i$. Then*

(i) Each element of $\varinjlim A_i$ has a representative of the form $\iota_i(x_i) + S$ for some i.

(ii) $\iota_i(x_i) + S = 0$ if and only if $f_{ij}(x_i) = 0$ for some $j \geq i$.

Proof. (i) An arbitrary element in $\varinjlim A_i$ is of the form

$$x = \sum_i \iota_i x_i + S \tag{A.28}$$

Since I is a directed set there exists a $j \geq i$ for all i appearing in the sum. Now let $y_i = f_{ij} x_i \in A_j$ and

$$y = \sum_i y_i \in A_j \tag{A.29}$$

Then we see that

$$\sum_i (\iota_i(x_i) - \iota_j(y)) = \sum_i (\iota_i(x_i) - \iota_j(f_{ij}(x_i))) \in S \tag{A.30}$$

Thus $x + S = y + S$ which is what we wanted

(ii) if $f_{ij}(x_i) = 0$ for some $j \geq i$ then

$$\iota_i(x_i) + S = \iota_i(x_i) + (\iota_j(f_{ij}(x_i)) - \iota_i(x_i)) + S = S \tag{A.31}$$

Conversely, if $\iota_i(x_i) + S = 0$ then we have an expression

$$\iota_i(x_i) = \sum_j r_j(\iota_k(f_{jk}(x_j)) - \iota_j(x_j)) \in S \tag{A.32}$$

for some $r_j \in R$. We define

$$r(j,k,x_j) = \iota_k(f_{jk}(x_j)) - \iota_j(x_j) \tag{A.33}$$

Clearly $r_j r(j,k,x_j) = r(j,k,r_j x_j)$ so we can assume

$$\iota_i(x_i) = \sum_j (\iota_k(f_{jk}(x_j)) - \iota_j(x_j)) \in S \tag{A.34}$$

Now, choose m such that $m \geq j,k$ for all j,k in the above expression. Then

$$\begin{aligned} \iota_m(f_{im}(x_i)) &= \iota_m(f_{im}(x_i)) - \iota_i(x_i) + \iota_i(x_i) &\text{(A.35)}\\ &= r(i,m,x_i) + \sum_j r(j,k,x_j) &\text{(A.36)} \end{aligned}$$

Now, recall that $r(j,k,x_j) = \iota_k(f_{jk}(x_j)) - \iota_j x_j$ and that
$\iota_m f_{jm}(x_j) = \iota_m f_{km} f_{jk}(x_j)$ so we can write

$$\begin{aligned} r(j,k,x_j) &= \iota_m f_{jm}(x_j) - \iota_j(x_j) + \iota_m f_{km}(-f_{jk}(x_j)) - \iota_k(-f_{jk}(x_j)) &\text{(A.37)}\\ &= r(j,m,x_j) + r(k,m,-f_{jk}(x_j)) &\text{(A.38)} \end{aligned}$$

so that $\iota_m f_{im}(x_i) = \sum_l r(l,m,x_l)$. Now it is easy to check that $r(l,m,x_l) + r(l,m,x'_l) = r(l,m,x_i + x'_i)$ so we can assume that each l appearing in the summand $\sum_l r(l,m,x_l)$ are all different. Therefore we have

$$\begin{aligned} \iota_m f_{im}(x_i) &= \sum_l r(l,m,x_l) &\text{(A.39)}\\ &= \sum_l (\iota_m f_{lm}(x_l) - \iota_l(x_l)) = \iota_m\left(\sum_l f_{lm}(x_l)\right) - \sum_l \iota_l(x_l). &\text{(A.40)} \end{aligned}$$

Now, since $\iota_m f_{im}(x_i) \in A_m$, we see that $\iota_l(x_l) = 0$ if $l \neq m$. Since ι_l is injective for all l, we must have $x_l = 0$. Therefore the above expression reduces to

$$\begin{aligned} \iota_m f_{im}(x_i) &= \iota_m f_{mm}(x_m) - \iota_m(x_m) &\text{(A.41)}\\ &= 0 &\text{(A.42)} \end{aligned}$$

since f_{mm} is the identity map. This means that $f_{im}(x_i) = 0$ since ι_m is injective. □

A.2.1 The direct limit functor $\varinjlim$

Definition A.2.2. *For each i, let ι_i be the inclusion of A_i into the direct sum $\bigoplus_i A_i$. We define the direct limit $\varinjlim A_i$ of a direct system $\langle A_i, f_{ij} \rangle$ by*

$$\varinjlim A_i = \bigoplus A_i \Big/ S \tag{A.43}$$

where $\iota_i : A_i \to \bigoplus_i A_i$ is the embedding map and S is the submodule generated by $\iota_j \circ f_{ij}(x_i) = \iota_i(x_i)$ for $x_i \in A_i$

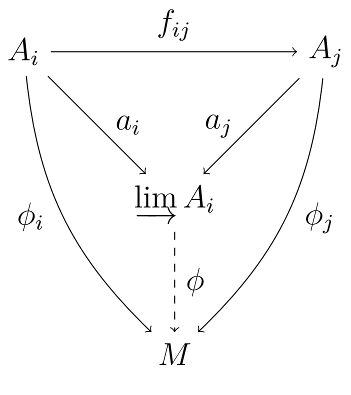
We now let

$$a_i : A_i \longrightarrow \varinjlim A_i \tag{A.44}$$

$$x_i \mapsto \iota(x_i) + S \tag{A.45}$$

then we claim the direct limit satisfies the following universal property

Lemma A.2.3. *Suppose that we have maps $\phi_i : A_i \longrightarrow M$. Then there exists a map ϕ such that the following diagram commutes*



Proof. let

$$\phi : \varinjlim A_i \longrightarrow M \tag{A.46}$$

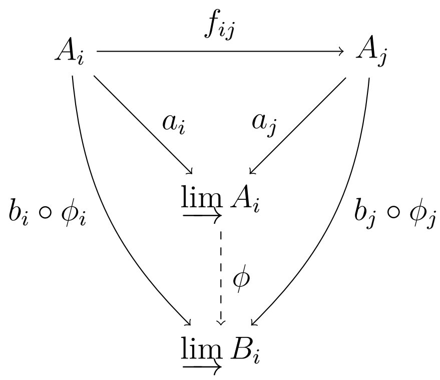
$$a_i(x_i) + S \mapsto \phi_i(x_i) \tag{A.47}$$

We only need to show that this is map is well defined. Suppose that $a_j f_{ij}(x_i)$ is in the same coset, and therefore $\phi(a_j f_{ij}(x_i) + S) = \phi_j(f_{ij}(x_i)) = \phi_i(x_i)$ □

We now want to show that the direct limit is actually a functor from the category of directed systems of R-modules over I to R-modules

$$\varinjlim : \mathbf{DS}_I(R\text{-}\mathbf{Mod}) \longrightarrow R\text{-}\mathbf{Mod} \tag{A.48}$$

It remains to show the image of morphisms under this functor. That is, given a morphism between two directed systems $\{\phi_i\} : \langle A_i, f_{ij}\rangle \longrightarrow \langle B_i, g_{ij}\rangle$ we would like to define a morphism $\phi : \varinjlim A_i \longrightarrow \varinjlim B_i$. To do this we employ the universal property of direct limits



which guaranteeds the existence of a map $\phi : \varinjlim A_i \longrightarrow \varinjlim B_i$. We therefore let $\varinjlim \phi_i = \phi$.

We now show that the direct limit functor is exact

Proposition A.2.4. *Let $A = \langle A_i, f_{ij}\rangle, B = \langle B_i, g_{ij}\rangle, C = \langle C_i, h_{ij}\rangle \in \mathbf{DS}_I(R\text{-}\mathbf{Mod})$ and suppose that we have a short exact sequence*

$$0 \longrightarrow A \xrightarrow{\{\phi_i\}} B \xrightarrow{\{\psi_i\}} C \longrightarrow 0 \tag{A.49}$$

then the sequence

$$0 \longrightarrow \varinjlim A_i \xrightarrow{\varinjlim \phi_i} \varinjlim B_i \xrightarrow{\varinjlim \psi_i} \varinjlim C_i \longrightarrow 0 \tag{A.50}$$

is exact.

Proof. Let $\phi = \varinjlim \phi_i$ and suppose that $\phi(x) = 0$ for some $x \in \varinjlim A_i$. We can then write $x = \iota_i x_i + S$ and therefore

$$\phi(\iota_i^A x_i + S_A) = \iota_i^B \phi_i(x_i) + S_B = 0 \tag{A.51}$$

by the definition of ϕ. Therefore there exists a j such that $g_{ij}\phi_i(x_i) = 0$ and since ϕ is a morphism we see that

$$g_{ij}\phi_i(x_i) = \phi_j f_{ij}(x_i) = 0 \tag{A.52}$$

since ϕ_i is injective we have $f_{ij}(x_i) = 0$ and so $x = \iota_i^A + S = 0$, therefore ϕ is injective. Now, suppose that $z \in \varinjlim C_i$ so we can write $z = \iota_i^C(z_i) + S_C$. Therefore there exists $y_i \in B_i$ such that $\psi_i(y_i) = z_i$ and so

$$z = \iota_i^C \psi_i(y_i) + S_C \tag{A.53}$$

but by the definition of ψ we see that

$$\psi(\iota_i^B(y_i) + S_B) = \iota_i^C \psi_i(y_i) + S_C = z \tag{A.54}$$

and so we see that ψ is surjective. □

One of the nice things about an exact functor is that it commutes with cohomology functors.

Lemma A.2.5. *Suppose that we have a cochain complex of R-**Mod***

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial_{n-1}} C^n \xrightarrow{\partial_n} C^{n+1} \longrightarrow \cdots \tag{A.55}$$

and an exact functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. That is for every exact sequence in $\mathcal{C}$

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{A.56}$$

we have an exact sequence

$$0 \longrightarrow \mathcal{F}M \longrightarrow \mathcal{F}N \longrightarrow \mathcal{F}P \longrightarrow 0 \tag{A.57}$$

in $\mathcal{D}$. Then, $\mathcal{F}$ commutes with cohomology, that is

$$\mathcal{F}H^n(C) = H^n(\mathcal{F}C) \tag{A.58}$$

Proof. By the definition of a functor we see that

$$\operatorname{im}(\mathcal{F}\partial_{n-1}) = \mathcal{F}(\partial_{n-1})\mathcal{F}(C^{n-1}) = \mathcal{F}(\partial_{n-1}C^{n-1}) = \mathcal{F}(\operatorname{im}(\partial_{n-1})) \tag{A.59}$$

We now let $B_n = \operatorname{im} \partial_{n-1}$ and $Z_n = \ker \partial_n$, so that $B_n \subseteq Z_n \subseteq C^n$. Then by applying $\mathcal{F}$ to the exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\iota} C^n \xrightarrow{\partial_n} B_{n+1} \longrightarrow 0 \tag{A.60}$$

we get the exact sequence

$$0 \longrightarrow \mathcal{F}(Z_n) \xrightarrow{\mathcal{F}(\iota)} \mathcal{F}(C^n) \xrightarrow{\mathcal{F}(\partial_n)} \mathcal{F}(B_{n+1}) \longrightarrow 0 \tag{A.61}$$

We therefore see that

$$\ker(\mathcal{F}(\partial_n)) = \mathcal{F}(Z_n) = \mathcal{F}(\ker \partial_n) \tag{A.62}$$

Now, applying $\mathcal{F}$ to the exact sequence

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H^n(C) \longrightarrow 0 \tag{A.63}$$

we get the exact sequence

$$0 \longrightarrow \mathcal{F}(B_n) \longrightarrow \mathcal{F}(Z_n) \longrightarrow \mathcal{F}(H^n(C)) \longrightarrow 0 \tag{A.64}$$

Therefore we see that

$$H^n(\mathcal{F}C) = \frac{\ker \mathcal{F}(\partial_n)}{\operatorname{im} \mathcal{F}(\partial_{n-1})} = \frac{\mathcal{F}(\ker(\partial_n))}{\mathcal{F}(\operatorname{im}(\partial_{n-1}))} = \mathcal{F}H^n(C) \tag{A.65}$$

□

We will now work in the category of cochain complexes of R-**Mod**,which we call ch(R-**Mod**),

Definition A.2.6. *A directed system in* $\mathbf{DS}_I(\text{ch}(R\text{-}\mathbf{Mod}))$ *is a sequence of cochain complexes* $\langle (C_i, d_i), f_{i,j} \rangle$ *where each* (C_i, d_i) *is a complex*

$$\cdots \xrightarrow{d_i^{n-2}} C_i^{n-1} \xrightarrow{d_i^{n-1}} C_i^n \xrightarrow{d_i^n} C_i^{n+1} \xrightarrow{d_i^{n+1}} \cdots \tag{A.66}$$

such that $d_i^{n+1} \circ d_i^n = 0$. *Each* $f_{ij} = \{f_{ij}^n\}$ *is a cochain map* $f_{ij}^n : C_i^n \longrightarrow C_j^n$. *Moreover, the cohomology of each complex forms a directed system*

$$H^n(C) = \left\langle \frac{\ker d_i^n}{\operatorname{im} d_i^{n-1}}, f_{ij}^n \right\rangle \tag{A.67}$$

To see that the maps

$$f_{ij}^n : H^n(C_i, d_i) \longrightarrow H^n(C_j, d_j) \tag{A.68}$$

is well-defined, observe that

$$f_{ij}^n(x + d_i^{n-1}a) = f_{ij}^n x + f_{ij}^n d_i^{n-1} a \tag{A.69}$$

$$= f_{ij}^n x + d_j^{n-1} f_{ij}^{n-1} a \tag{A.70}$$

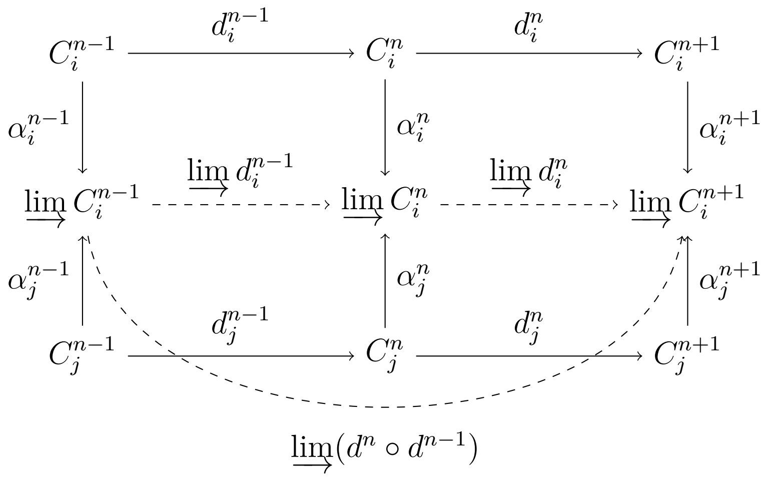
$$\tag{A.71}$$

so f_{ij}^n's are well-defined.

Definition A.2.7. *We can also define $\varinjlim C$ as the cochain complex of R-modules*

$$\cdots \xrightarrow{\varinjlim d_i^{n-2}} \varinjlim C_i^{n-1} \xrightarrow{\varinjlim d_i^{n-1}} \varinjlim C_i^n \xrightarrow{\varinjlim d_i^n} \varinjlim C_i^{n+1} \xrightarrow{\varinjlim d_i^{n+1}} \cdots \tag{A.72}$$

Clearly $\varinjlim d^n \circ \varinjlim d^{n-1} = \varinjlim (d^n \circ d^{n-1}) = 0$ by the commutative diagram below



and therefore this complex is well-defined.

Proposition A.2.4 and Lemma A.2.5 then imply the following

Lemma A.2.8. *Let C be a cochain complex in $\mathbf{DS}_I(R\text{-}\mathbf{Mod})$. Since exact functors preserve cohomology, we see that the direct limit commutes with taking cohomology, that is*

$$H^n\left(\varinjlim C\right) = \varinjlim H^n(C) \tag{A.73}$$

A.2.2 Infinite tensor product of cochain complexes

This section discusses the cohomology of infinite tensor products of cochain complexes in detail. While we can apply the Künneth Theorem for the cohomology of a finite tensor product of complexes, the Theorem may not hold for infinite tensor products. First we will define an infinite tensor product of complexes. Suppose that we have a countable set of complexes $\{C_k \mid k \geq 1\}$. $D_i = \bigotimes_{k=1}^{i} C_k$, the tensor product of the first k complexes. Let $f_{i,j} : \bigotimes D_i \longrightarrow D_j$ be the inclusion map onto the first i tensorand of complexes. Then $\langle D_i, f_{i,j}\rangle$ is a directed system in the category of cochain complexes. By construction, the direct limit $\varinjlim D_i$ is

$$\varinjlim D_i = \bigotimes_{k\geq 1} C_k \tag{A.74}$$

and that

$$\varinjlim H^n(D_i) = \bigotimes_{k\geq 1} H^n(C_k) \tag{A.75}$$

Using the Künneth formula, we see that

$$H^n(D_i) = \bigoplus_{n_1+\cdots+n_i=n} H^{n_1}(C_{k_1}) \otimes \cdots \otimes H^{n_i}(C_{k_i}) \tag{A.76}$$

Since direct limit commutes with cohomology functors by Lemma A.2.8, we finally arrive at

Lemma A.2.9.

$$H^n\left(\bigotimes_{k\geq 1} C_k\right) = \bigotimes_{k\geq 1} H^n(C_k) \tag{A.77}$$

Appendix B

Localisation

B.1 Definitions

Let R be a non-commutative domain and let S be a multiplicatively closed set, $0 \notin S$ satisfying the (left) Ore condition

$$\forall r \in R,\ s \in S,\ Sr \cap Rs \neq \varnothing \tag{B.1}$$

Definition B.1.1. *Define an equivalence relation $\sim$ on $S \times R$ by $(s_1, r_1) \sim (s_2, r_2)$ if there exists $a_1, a_2 \in R$ such that*

$$a_1 s_1 = a_2 s_2 \in S \tag{B.2}$$

$$a_1 r_1 = a_2 r_2 \in R \tag{B.3}$$

and we write $S^{-1}R = S \times R \big/ \sim$.

Given two equivalence classes $(s_1, r_1), (s_2, r_2)$, the Ore condition in Equation (B.1) says that there are $a \in R, x \in S$ such that $xs_1 = as_2$. Since $(s_1, r_1) \sim (xs_1, xr_1)$ and $(s_2, r_2) \sim (as_2, ar_2)$, it follows that we can choose representatives (s_1, r_1) and (s_2, r_2) of any two classes so that $s_1 = s_2$. This allows us to define addition on $S^{-1}R$.

Definition B.1.2. *Given two representatives $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, the Ore condition in Equation (B.1) says that there are $a \in R, x \in S$ such that $xs_1 = as_2$. Define addition on $S^{-1}R$ by*

$$(s_1, r_1) + (s_2, r_2) = (t, xr_1 + ar_2), \qquad t = xs_1 = as_2 \tag{B.4}$$

Definition B.1.3. *Given two representatives* $(s_1, r_1), (s_2, r_2)$*, by Equation* (B.1) *we have* $Rs_2 \cap Sr_1 \neq \varnothing$ *which implies that* $as_2 = xr_1$*. Define multiplication on* $S^{-1}R$ *by*

$$(s_1, r_1) \times (s_2, r_2) = (xs_1, ar_2). \tag{B.5}$$

We then have

Definition B.1.4. *The localisation of* R *is the ring* $(S^{-1}R, +, \times)$*. Lemmas B.1.5 to B.1.7 shows that* $(S^{-1}R, +, \times)$ *is well-defined.*

We will now proceed to proving Lemmas B.1.5 to B.1.7.

Lemma B.1.5. *The equivalence relation* $\sim$ *in Definition B.1.1 is well-defined.*

Proof. The relation is obviously reflexive and symmetric and we only show transitivity. Suppose that $(s_1, r_1) \sim (s_2, r_2)$ and $(s_2, r_2) \sim (s_3, r_3)$. Then there exists $a_1, a_2, b_2, b_3 \in R$ such that

$$a_1 r_1 = a_2 r_2 \qquad b_2 r_2 = b_3 r_3 \tag{B.6}$$

$$a_1 s_1 = a_2 s_2 \qquad b_2 s_2 = b_3 s_3 \tag{B.7}$$

By Ore's condition we have $S(a_2 s_2) \cap R(b_2 s_2) \neq \varnothing$ so there exists $c \in R, x \in S$ such that $xa_2 s_2 = cb_2 s_2$. As R is a domain and $0 \notin S$ this implies that $xa_2 = cb_2$. Then we have

$$xa_1 r_1 = xa_2 r_2 = cb_2 r_2 = cb_3 r_3 \tag{B.8}$$

$$xa_1 s_1 = xa_2 s_2 = cb_2 s_2 = cb_3 s_3 \tag{B.9}$$

so we see that $(s_1, r_1) \sim (s_3, r_3)$ and we are done. □

Lemma B.1.6. *Addition appearing in Definition B.1.2 is well-defined.*

Proof. We need to show that the operation is independent on the choice of a, x, r_1, s_1 and r_2, s_2. Suppose that there are $a' \in R, x' \in S$ such that $x's_1 = a's_2$. Since $S(xs_1) \cap R(x's_1) \neq \varnothing$, we have $yxs_1 = bx's_1$ for some $b \in R, y \in S$. Since $xs_1 = as_2, x's_1 = a's_2$ we also have $yas_2 = ba's_2$. This implies that $yx = bx', ya = ba'$. Thus

$$(xs_1, xr_1 + ar_2) \sim (yxs_1, yxr_1 + yar_2)) = (bx's_1, bx'r_1 + ba'r_2) \sim (x's_1, x'r_1 + a'r_2) \tag{B.10}$$

and so the sum is independent of the choice and a or x.
To see that it does not depend on (s_1, r_1), we will first show that if we replace (s_1, r_1) with (bs_1, br_1) such that $bs_1 \in S$, then multiplcation is still well-defined. To see this, as $S(bs_1) \cap Rs_2 \neq \varnothing$ we have $x'bs_1 = a's_2$. So we have

$$(s_1, r_1) + (s_2, r_2) = (t, xr_1 + ar_2), \qquad t = xs_1 = as_2 \tag{B.11}$$

$$(bs_1, br_1) + (s_2, r_2) = (t', x'br_1 + a'r_2), \qquad t' = x'bs_1 = a's_2 \tag{B.12}$$

Now, $S(xs_1) \cap R(x'bs_1) \neq \varnothing$ so $yxs_1 = cx'bs_1 \implies yx = cx'b$. Now, $yas_2 = yxs_1 = cx'bs_1 = ca's_2$ so $ya = ca'$. Thus we see that $(yt, yxr_1 + yar_2) = (ct', cx'br_1 + ca'r_2)$, implying that

$$(t, xr_1 + ar_2) = (yt, yxr_1 + yar_2) = (ct', cx'br_1 + ca'r_2) = (t', x'br_1 + a'r_2) \tag{B.13}$$

So we see that multiplication is well-defined after replacing (s_1, r_1) with (bs_1, br_1). Now, in general suppose that $(s_1, r_1) \sim (s'_1, r'_1)$. By definition there exists b, b' such that $br_1 = b'r'_1, bs_1 = b's'_1$ so $(bs_1, br_1) = (b's'_1, b'r'_1)$. The above argument shows that multiplcation is well-defined if we replace (s_1, r_1) with (bs_1, br_1) and (s'_1, r'_1) with $(b's'_1, b'r'_1)$. But $(bs_1, br_1) = (b's'_1, b'r'_1)$ so we see that multiplication is still well-defined after placing (s_1, r_1) with (s'_1, r'_1). The case for (s_2, r_2) is exactly the same so we are done. □

Lemma B.1.7. *Multiplication appearing in Definition B.1.3 is well-defined.*

Proof. Suppose that $a's_2 = x'r_1$, so that $(s_1, r_1) \times (s_2, r_2) = (x's_1, a'r_2)$. Since $R(xs_1) \cap S(x's_1) \neq \varnothing$ we have $yxs_1 = y'x's_1 \implies yx = y'x'$. Therefore

$$(xs_1, ar_2) \sim (yxs_1, yar_2) = (y'x's_1, y'a'r_2) \sim (x's_1, a'r_2) \tag{B.14}$$

Now we want to check multiplication is independent of (s_1, r_1). As we have mentioned before we can just consider elements of the form (bs_1, br_1) such that $bs_1 \in S$. Now, originally we have $as_2 = xr_1$. As $Rs \cap Sb \neq \varnothing$ we have $cx = yb$. Therefore $cas_2 = cxr_1 = ybr_1$. So we have $(bs_1, br_1) \times (s_2, r_2) = (ybs_1, car_2)$. But then $(xs_1, ar_2) \sim (cxs_1, car_2) \sim (ybs_1, car_2)$.
Lastly we need to check multiplication is independent of (s_2, r_2). Again we can just check for (bs_2, br_2). From Ore's condition we have $a'bs_2 = x'r_1$ so we have $(s_1, r_1) \times (s_2, r_2) = (x's_1, a'br_2)$. Again we can assume $x' = x$ so $a'bs_2 = x'r_1 = xr_1 = as_2 \implies a'b = a$ and so $(x's_1, a'br_2) \sim (xs_1, ar_2)$. □

Now that we have shown $S^{-1}R$ is a well-defined ring, let ϕ be the map $\phi : R \longrightarrow S^{-1}R$ where $\phi(r) = (1, r)$. Then $\ker \phi = \{r \in R \mid \exists s \in S, sr = 0\}$. Since R is a domain we see that ϕ is injective and therefore we can realise R as a subring of $S^{-1}R$. Furthermore, any element $(s, r) \in S^{-1}R$ can be written as a product $(s, 1)(1, r)$. Noting this, we have

Definition B.1.8. *Suppose that M is an R-module. The localisation $R_S \otimes_R M$ of M with respect to S is the tensor product $S^{-1}R \otimes_R M$.*

Elements in $S^{-1}R \otimes_R M$ can be written in the form

$$(s, r) \otimes m = (s, 1)(1, r) \otimes m = (s, 1) \otimes rm = (s, 1) \otimes m', \qquad m' \in M. \tag{B.15}$$

We will write the element $(s, 1) \otimes m'$ as (s, m'). It is easy to check, using the definition in Definition B.1.1 that if $s \in S, m \in M$, $(s_1, m_1) \sim (s_2, m_2)$ if and only if there exists $a_1, a_2 \in R$ such that

$$a_1 s_1 = a_2 s_2 \in S \tag{B.16}$$

$$a_1 m_1 = a_2 m_2 \in M \tag{B.17}$$

Lemma B.1.9. *A homomorphism $\phi : U \longrightarrow V$ between R-modules can be lifted to a homomorphism $\overline{\phi} : R_S \otimes_R U \longrightarrow R_S \otimes_R V$ between R_S-modules by $\overline{\phi}((s, u)) = (s, \phi(u))$.*

Proof. We want to show that $\overline{phi}$ is indeed a module homomorphism. Suppose that $(s_1, u_1), (s_2, u_2) \in R_S \otimes_R U$. Then as per Definition B.1.2,

$$\overline{\phi}((s_1, u_1) + (s_2, u_2)) = \overline{\phi}(t, xu_1 + au_2) \tag{B.18}$$

$$= (t, \phi(xu_1 + au_2)) \tag{B.19}$$

$$= (t, x\phi(u_1)) + (t, a\phi(u_2)) \tag{B.20}$$

$$= (s_1, \phi(u_1)) + (s_2, \phi(u_2)) \tag{B.21}$$

Now, suppose that $(s_1, r) \in S^{-1}R, (s_2, u) \in S^{-1}U$. Then

$$(s_1, r)\overline{\phi}((s_2, u)) = (s_1, r)(s_2, \phi(u)) \tag{B.22}$$

$$= (s_1, r)(s_2, 1) \otimes \phi(u) \tag{B.23}$$

as per Definition B.1.3, we have $(s_1, r)(s_2, 1) = (x, s_1 a)$ for some $a \in R, x \in S$,

$$
\begin{aligned}
&= (x, s_1 a) \otimes \phi(u) && \text{(B.24)}\\
&= (x, \phi(s_1 a u)) && \text{(B.25)}\\
&= \overline{\phi}((x, s_1 a u)) && \text{(B.26)}\\
&= \overline{\phi}((s_1, r)(s_2, u)) && \text{(B.27)}
\end{aligned}
$$

and we are done. □

Lemma B.1.10. *The functor*

$$
\begin{aligned}
\mathrm{Ind}_S^{S^{-1}R}(-) : R\text{-}\mathbf{Mod} &\longrightarrow S^{-1}R\text{-}\mathbf{Mod} && \text{(B.28)}\\
M &\mapsto S^{-1}R \otimes_R M && \text{(B.29)}
\end{aligned}
$$

is exact.

Proof. Given a exact sequence of R-modules

$$0 \longrightarrow U \xrightarrow{\phi} V \longrightarrow W \longrightarrow 0, \tag{B.30}$$

we want to show that

$$0 \longrightarrow S^{-1}R \otimes_R U \xrightarrow{\overline{\phi}} S^{-1}R \otimes_R V \longrightarrow S^{-1}R \otimes_R W \longrightarrow 0 \tag{B.31}$$

is exact. Since the tensor product functor is always right-exact, it suffices to show left-exactness. Suppose that $\overline{\phi}((s, u)) = 0$ hence that $(s, \phi(u)) = (1, 0)$. This implies that there exists $a \in S$ such that $a\phi(u) = 0 \implies \phi(au) = 0 \implies au = 0$ since ϕ is injective. Now $(s, u) \sim (as, au) \sim (as, 0) \sim (1, 0)$ so $\overline{\phi}$ is injective. □

B.2 The localisation of universal enveloping algebras

Lemma B.2.1. *[54, Lemma 4.2] Let R be an associative algebra and let S be a multiplicatively closed subset generated by locally* ad*-nilpotent elements of R. Then S satisfies the Ore condition.*

In particular, we have

Corollary B.2.2. *Let* $\mathfrak{g}$ *be a Lie algebra and let* $e \in \mathfrak{g}$ *be a locally* ad*-nilpotent element of* $\mathfrak{g}$*. Then* R_S *is a well-defined ring where*

$$R = \mathrm{U}(\mathfrak{g}), \qquad S = \{e^n \mid n \in \mathbb{N}\}. \tag{B.32}$$

Lastly, with the above notations, we have

Remark B.2.3. *Suppose that* $X \in R$ *and* $[X, e] = ce \in R$ *for some* $c \in \mathbb{C}$*. Then* $[X, e^{-1}] = -ce^{-1}$ *in* R_S*.*

Appendix C

The ring of symmetric functions

The purpose of this section is to review various results from the theory of symmetric functions that will be used to evaluate the action of screening operators on certain Fock spaces. The standard reference for symmetric functions and their myriad properties is Macdonald's book [52] to which we refer the reader for more details. Let Λ_n denote the ring of symmetric polynomials in the n variables $z_1, \ldots, z_n$. This is the subring of $\mathbb{C}[z_1, ..., z_n]$ that consists of the polynomials that are invariant with respect to permuting the indices of the z_i. It admits numerous interesting generators such as the power sums

$$\mathsf{p}_k = \sum_{i=1}^{n} z_i^k, \quad k \geq 1. \tag{C.1}$$

For $1 \leq k \leq n$, the p_k are algebraically independent and freely generate Λ_n, that is,

$$\Lambda_n = \mathbb{C}[\mathsf{p}_1, \cdots, \mathsf{p}_n]. \tag{C.2}$$

We can therefore use partitions $\lambda = [\lambda_1, \lambda_2, \ldots]$, whose parts λ_i are bounded by n, to define

$$\mathsf{p}_\lambda = \mathsf{p}_{\lambda_1} \cdots \mathsf{p}_{\lambda_k}. \tag{C.3}$$

These power sums, labelled by partitions whose parts do not exceed n, thus form a basis of Λ_n:

$$\Lambda_n = \bigoplus_{\lambda, \lambda_1 \leq n} \mathbb{C}\mathsf{p}_\lambda. \tag{C.4}$$

Another family of symmetric polynomials is given by the monomial symmetric polynomials

$$\mathsf{m}_\lambda = \sum_\sigma z_1^{\lambda_{\sigma(1)}} \cdots z_n^{\lambda_{\sigma(n)}}, \tag{C.5}$$

where σ runs over all distinct permutations of the partition λ. In this case, λ is not constrained by a bound on its individual parts, but by their number $\ell(\lambda)$ (the length of λ) which is at most n. Note that each monomial summand of m_λ has coefficient 1. For example,

$$\mathsf{m}_{[2,2]}\Big(z_1, z_2\Big) = z_1^2 z_2^2, \quad \mathsf{m}_{[2,2]}\Big(z_1, z_2, z_3\Big) = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2. \tag{C.6}$$

The monomial symmetric polynomials also form a basis of Λ_n:

$$\Lambda_n = \bigoplus_{\lambda, \ell(\lambda) \leq n} \mathbb{C}\mathsf{m}_\lambda. \tag{C.7}$$

The respective restrictions on parts and lengths of partitions in the definitions of these symmetric polynomials can be avoided by taking a formal limit to infinitely many variables. The resulting ring Λ is called the ring of symmetric functions and, unsurprisingly, its elements are called symmetric functions. The ring Λ_n of symmetric polynomials in n variables can then be easily recovered from Λ by setting all but the first n variables to 0. This amounts to a projection

$$\pi_n : \Lambda \to \Lambda_n, \quad f(x_1, x_2, \dots) \mapsto f(x_1, \dots, x_n, 0, 0, \dots). \tag{C.8}$$

In Λ, the power sums p_k are algebraically independent for all $k \geq 1$ and they freely generate Λ, that is,

$$\Lambda = \mathbb{C}[\mathsf{p}_1, \mathsf{p}_2, \dots]. \tag{C.9}$$

Similarly, the restrictions on the sizes of the parts and the lengths of the partitions labelling power sums and monomial symmetric functions, respectively, no longer apply. Both classes of symmetric functions give bases of Λ:

$$\Lambda = \bigoplus_\lambda \mathbb{C}\mathsf{p}_\lambda = \bigoplus_\lambda \mathbb{C}\mathsf{m}_\lambda. \tag{C.10}$$

We note that $\pi_n(\mathsf{m}_\lambda) = 0$ if and only if $\ell(\lambda) > n$, but that no such truncations exist for the power sums p_k: their images under π_n are all non-zero.

C.1 The Jack functions

There exists another family of bases of Λ and Λ_n labelled by partitions, called the Jack symmetric functions and Jack symmetric polynomials (or just Jack functions or polynomials for short), respectively. These are defined using the dominance partial ordering of partitions: if λ and μ are both partitions of the same non-negative integer, then we write $\lambda \geq \mu$ (and say that λ dominates μ) if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \tag{C.11}$$

for all $i \geq 1$.
For each $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ (the non-positive rationals are excluded to avoid certain normalisation problems), the Jack functions P^t_λ are uniquely defined by the following two properties:

1. For any partition λ, P^t_λ admits an upper triangular decomposition of the form

$$\mathsf{P}^t_\lambda = \mathsf{m}_\lambda + \sum_{\lambda > \mu} v_{\lambda,\mu}(t)\mathsf{m}_\mu, \qquad v_{\lambda,\mu}(t) \in \mathbb{C}. \tag{C.12}$$

2. The Jack functions form an orthogonal basis of Λ with respect to the inner product defined by

$$\langle \mathsf{p}_\lambda, \mathsf{p}_\mu \rangle^t = t^{\ell(\lambda)} \delta_{\lambda\mu} \prod_{i \geq 1} i^{m_i} m_i!, \tag{C.13}$$

   where m_i denotes of number of parts of λ equal to i.

For each $n \geq 1$, the Jack polynomials in Λ_n may be defined as the images of the corresponding Jack functions in Λ under the projection π_n. As with monomial symmetric polynomials, we have $\pi_n(\mathsf{P}^t_\lambda) = 0$ if and only if $\ell(\lambda) > n$. For $\ell(\lambda) \leq n$, the Jack polynomials

$$\mathsf{P}^t_\lambda\Big(z_1, \ldots, z_n\Big) = \pi_n(\mathsf{P}^t_\lambda) \tag{C.14}$$

are linearly independent and form a basis of Λ_n. For the application to follow, we mention the following important examples in Λ_n called the *rectangular* Jack polynomials. In these, the partition has the form $\lambda = [m^n]$ in which all n parts are equal to m. Rectangular Jack polynomials have a very simple form:

$$\mathsf{P}^t_{[m^n]}\big(z_1,\ldots,z_n\big) = \mathsf{m}_{[m^n]}\big(z_1,\ldots,z_n\big) = \prod_{i=1}^{n} z_i^m. \tag{C.15}$$

This follows because all partitions of mn that are strictly dominated by $[m^n]$ have length greater than n. They also have extremely simple products with other Jacks. For $\ell(\lambda) \le n$, denote by $\lambda + [m^n]$ the partition with parts $\lambda_i + m$. Then,

$$\mathsf{P}^t_{[m^n]}\big(z_1,\ldots,z_n\big)\mathsf{P}^t_{\lambda}\big(z_1,\ldots,z_n\big) = \mathsf{P}^t_{\lambda+[m^n]}\big(z_1,\ldots,z_n\big). \tag{C.16}$$

We emphasise that rectangular Jack polynomials are independent of the parameter t.

C.2 Inner product for Jack functions

The Jack functions and polynomials satisfy many properties that shall be essential for what follows. We list some of them here for convenience.

1. We denote by Q^t_λ the elements of the basis dual to the P^t_λ with respect to the inner product Equation (C.13). Since the Jack functions form an orthogonal basis, P^t_λ is proportional to Q^t_λ:

$$\mathsf{Q}^t_\lambda = b^t_\lambda \mathsf{P}^t_\lambda, \quad b^t_\lambda = \frac{1}{\langle \mathsf{P}^t_\lambda, \mathsf{P}^t_\lambda\rangle^t}. \tag{C.17}$$

   The proportionality constant b^t_λ is given explicitly by

$$b^t_\lambda = \prod_{s\in\lambda} \frac{a(s)t + l(s) + 1}{(a(s)+1)t + l(s)}, \tag{C.18}$$

   where $a(s)$ and $l(s)$ denote the arm and leg lengths, respectively, of the box s in the Young diagram of λ.

2. The Jack functions and their duals admit a kind of generating function called the *Cauchy kernel*:

$$\prod_{i,j}(1 - y_i z_j)^{-1/t} = \prod_{m\ge 1} \exp\left(\frac{1}{t}\frac{\mathsf{p}_m\big(y\big)\mathsf{p}_m\big(z\big)}{m}\right) = \sum_\lambda \mathsf{P}^t_\lambda\big(y\big)\mathsf{Q}^t_\lambda\big(z\big). \tag{C.19}$$

   In this identity, the two alphabets $\{y_i\}$ and $\{z_j\}$ may be finite or infinite. The sum is across all partitions λ.

3. Given partitions λ and μ, the skew Jack functions $\mathsf{P}^t_{\lambda/\mu}$ and $\mathsf{Q}^t_{\lambda/\mu}$ are defined to be the unique symmetric functions satisfying

$$\left\langle \mathsf{P}^t_{\lambda/\mu}, \mathsf{Q}^t_{\nu} \right\rangle^t = \left\langle \mathsf{P}^t_{\lambda}, \mathsf{Q}^t_{\mu}\mathsf{Q}^t_{\nu} \right\rangle^t \qquad \text{and} \qquad \left\langle \mathsf{Q}^t_{\lambda/\mu}, \mathsf{P}^t_{\nu} \right\rangle^t = \left\langle \mathsf{Q}^t_{\lambda}, \mathsf{P}^t_{\mu}\mathsf{P}^t_{\nu} \right\rangle^t \tag{C.20}$$

for all partitions ν. Let us write $\mu \subseteq \lambda$ if the Young diagram of μ is contained in that of λ. Then, $\mathsf{P}^t_{\lambda/\mu} = \mathsf{Q}^t_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$. Finally, the ordinary and dual skew Jack functions are proportional:

$$\mathsf{Q}^t_{\lambda/\mu} = \frac{b^t_\lambda}{b^t_\mu} \mathsf{P}^t_{\lambda/\mu}. \tag{C.21}$$

4. Consider an alphabet $z = (z_1, z_2, \dots)$, partitioned into two subsets $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Any symmetric function in z may obviously be decomposed into symmetric functions in x and y. For Jack functions, this decomposition is

$$\begin{aligned} \mathsf{P}^t_\lambda\big(z\big) &= \mathsf{P}^t_\lambda\big(x \cup y\big) = \sum_\nu \mathsf{P}^t_\nu\big(x\big)\mathsf{P}^t_{\lambda/\nu}\big(y\big), \\ \mathsf{Q}^t_\lambda\big(z\big) &= \mathsf{Q}^t_\lambda\big(x \cup y\big) = \sum_\nu \mathsf{Q}^t_\nu\big(x\big)\mathsf{Q}^t_{\lambda/\nu}\big(y\big). \end{aligned} \tag{C.22}$$

Both sums may clearly be restricted to partitions satisfying $\nu \subseteq \lambda$.

5. The Jack polynomials $\mathsf{P}^t_\lambda\big(z_1, \dots, z_n\big)$ are orthogonal with respect to the inner product

$$\langle f, g \rangle^t_n = \int_{\Gamma(n;t)} G^t_n(x) f(x) \overline{g(x)} \, \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n}, \tag{C.23}$$

where $\Gamma(n;t)$ is the cycle normalised in Equation (1.211), $\overline{g(x_1, x_2, \dots)} = g(x_1^{-1}, x_2^{-1}, \dots)$ and

$$G^t_n(x) = \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{1/t} \tag{C.24}$$

is called the integrating kernel. With respect to this integral inner product, the Jack polynomials satisfy

$$\begin{aligned} \Big\langle \mathsf{P}^t_\lambda\Big(x\Big), \mathsf{Q}^t_\mu\Big(x\Big)\Big\rangle^t_n &= \delta_{\lambda,\mu} b^t_\lambda(n), \\ b^t_\lambda(n) &= \prod_{s\in\lambda} \frac{n + a'(s)t - l'(s)}{n + (a'(s)+1)t - l'(s) - 1}, \end{aligned} \tag{C.25}$$

where $a'(s)$ and $l'(s)$ denote the arm and leg colengths, respectively, of the box s in the Young diagram of λ.

References

[1] D Adamović and A Milas. Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$. *Math. Res. Lett.*, 2:563–575, 1995. arXiv:hep-th/9509025.

[2] T Arakawa. Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction. *Int. Math. Res. Not.*, 2004:729–767, 2004.

[3] T Arakawa. Rationality of Bershadsky-Polyakov Vertex Algebras. *Comm. Math. Phys.*, 323:627–633, 2013.

[4] T Arakawa. Rationality of W-algebras: principal nilpotent cases. *Ann. Math.*, 182:565–604, 2015.

[5] T Arakawa, T Creutzig, and A Linshaw. W-algebras as coset vertex algebras. arXiv:1801.03822 [math.QA].

[6] T Arakawa and C Jiang. Coset vertex operator algebras and $\mathcal{W}$-algebras of A-type. arXiv:1701.06880 [math.RT], 2018.

[7] H Awata, Y Matsuo, S Odake, and J Shiraishi. Excited states of the Calogero-Sutherland model and singular vectors of the W_n algebra. *Nucl. Phys.*, B449:347–374, 1995. arXiv:hep-th/9503043.

[8] F Bais, P Bouwknegt, K Schoutens, and M Surridge. Coset construction for extended Virasoro algebras. *Nucl. Phys.*, B304:371–391, 1988.

[9] F Bais, P Bouwknegt, M Surridge, and K Schoutens. Extensions of the Virasoro Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir invariants. *Nucl. Phys.*, B304:348–370, 1988.

[10] C Becchi, A Rouet, and R Stora. The abelian Higgs-Kibble model, unitarity of the S-operator. *Phys. Lett.*, B52:344–346, 1974.

[11] C Becchi, A Rouet, and R Stora. Renormalization of the abelian Higgs-Kibble model. *Comm. Math. Phys.*, 42:127–162, 1975.

[12] C Becchi, A Rouet, and R Stora. Renormalization of gauge theories. *Ann. Phys.*, 98:287–321, 1976.

[13] A Belavin. KdV-type equations and W-Algebras. *Adv. Stud. Pure Math.*, pages 117–125, 1989.

[14] A Belavin, M Polyakov, and A Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys.*, B241:333–380, 1984.

[15] D Bernard and G Felder. Fock representations and BRST cohomology in SL(2) current algebra. *Comm. Math. Phys.*, 127:145–168, 1990.

[16] M Bershadsky and H Ooguri. Hidden $sl(n)$ symmetry in Conformal Field Theories. *Comm. Math. Phys.*, pages 49–83, 1989.

[17] O Blondeau-Fournier, P Mathieu, D Ridout, and S Wood. The super-Virasoro singular vectors and Jack superpolynomials relationship revisited. *Nucl. Phys.*, B913:34–63, 2016. arXiv:1605.08621 [math-ph].

[18] O Blondeau-Fournier, P Mathieu, D Ridout, and S Wood. Superconformal minimal models and admissible Jack polynomials. *Adv. Math.*, 314:71–123, 2017. arXiv:1606.04187 [hep-th].

[19] P Bouwknegt, J MacCarthy, and K Pilch. *The $\mathcal{W}_3$ algebra. Modules, semi-infinite cohomology and BV algebras.* Lecture Notes in Physics. Springer, Heidelberg, 1996.

[20] P Bouwknegt and K Schoutens. W-symmetry in conformal field theory. *Phys. Rept.*, 223:183–276, 1993. arXiv:hep-th/9210010.

[21] G Mason C Dong, H Li. Twisted representations of vertex operator algebras. *Math. Ann.*, 310:571–600, 1998.

[22] T Creutzig, S Kanade, T Liu, and D Ridout. Cosets, Characters and Fusion for Admissible-Level $osp(1|2)$ Minimal Models. *Nucl. Phys.*, B938:22–55, 2018.

[23] T Creutzig, T Liu, D Ridout, and S Wood. Unitary and Non-Unitary $N = 2$ Minimal Models. *J. High Energy Phys.*, 2019.

[24] T Creutzig and D Ridout. Modular Data and Verlinde Formulae for Fractional Level WZW Models II. *Nucl. Phys.*, B875:423–458, 2013.

[25] J de Boer and T Tjin. The Relation Between Quantum W-Algebras and Lie Algebras. *Comm. Math. Phys.*, 160:317–332, 1994. arXiv:hep-th/9302006.

[26] P Desrosiers, L Lapointe, and P Mathieu. Supersymmetric Calogero-Moser-Sutherland models and Jack superpolynomials. *Nucl. Phys.*, B606:547–582, 2001. arXiv:hep-th/0103178.

[27] P Desrosiers, L Lapointe, and P Mathieu. Superconformal field theory and Jack superpolynomials. *J. High Energy Phys.*, 1209:037, 2012. arXiv:1205.0784 [hep-th].

[28] The Sage developers. The Sage Mathematics Software System (version 8.0), 2017. `http://www.sagemath.org`.

[29] A Díaz and J Figueroa-O'Farrill. A new explicit construction of W_3 from the affine algebra $A_2^{(1)}$. *Nucl. Phys.*, B349:237–252, 1991.

[30] V Dotsenko and V Fateev. Conformal algebra and multipoint correlation functions in 2D statistical models. *Nucl. Phys.*, B240:312–348, 1984.

[31] L Faddeev and V Popov. Feynman diagrams for the Yang-Mills field. *Phys. Lett.*, B25:29–30, 1967.

[32] V Fateev and A Zamolodchikov. Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems. *Soviet Phys. JETP*, 62:215–225, 1985.

[33] V Fateev and A Zamolodchikov. Conformal Quantum Field Theory Models in Two-Dimensions Having $Z(3)$ Symmetry. *Nucl. Phys.*, B280:644–660, 1987.

[34] B Feigin and E Frenkel. Quantization of the Drinfeld-Sokolov reduction. *Phys. Lett.*, B246:75–81, 1990.

[35] B Feigin and E Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras. *Int. J. Mod. Phys.*, A7:197–215, 1992.

[36] B Feigin and D Fuchs. Representations of the Virasoro algebra. *Adv. Stud. Contemp. Math*, 7:465–554, 1990.

[37] G Felder. BRST approach to minimal models. *Nucl. Phys.*, B317:215–236, 1989.

[38] E Frenkel and D Ben-Zvi. *Vertex Algebras and Algebraic Curves: Second Edition.* Mathematical surveys and monographs. American Mathematical Society, 2004.

[39] E Frenkel, V Kac, and M Wakimoto. Characters and fusion rules for W-algebras via quantized Drinfeld-Sokolov reduction. *Comm. Math. Phys.*, 147:295–328, 1992.

[40] I Frenkel and Y Zhu. Vertex Operator Algebras Associated to Representations of Affine and Virasoro Algebras. *Duke Math. J.*, 66:123–168, 1992.

[41] D Friedan, E Martinec, and S Shenker. Conformal Invariance, Supersymmetry and String Theory. *Nucl. Phys.*, B271:93–165, 1986.

[42] M Gaberdiel. Fusion rules and logarithmic representations of a WZW model at fractional level. B618:407–436, 2001.

[43] M Gaberdiel and R Gopakumar. Minimal Model Holography. *J. Phys.*, A46:214002, 2013. arXiv:1207.6697 [hep-th].

[44] P Goddard, A Kent, and D Olive. Virasoro algebras and coset space models. *Nucl. Phys.*, B152:88–92, 1985.

[45] M Gorelik and V Kac. On simplicity of vacuum modules. *Adv. Math.*, 211:621–677, 2007.

[46] H Jack. A Class of Symmetric Polynomials with a Parameter. *Proc. Roy. Soc. Edinburgh*, A69:1–18, 1970–71.

[47] M Kato and Y Yamada. Missing link between Virasoro and $\widehat{sl(2)}$ Kac-Moody algebras. *Progr. Theoret. Phys. Suppl.*, 110:291–302, 1992.

[48] K Kawasetsu and D Ridout. Relaxed Highest-Weight Modules I: Rank 1 Cases. *Comm. Math. Phys.*, 368:627–663, 2019.

[49] T Kugo and I Ojima. Manifestly Covariant Canonical Formulation of the Yang-Mills Field Theories. I. *Progr. Theoret. Phys. Suppl.*, 60:1869–1889, 1978.

[50] H Li. Local system of vertex operators, vertex superalgebras and modules. *J. Pure Appl. Algebra*, 109:143–195, 1996.

[51] A Lukyanov. Quantization of the Gel'fand-Dikii brackets. *Funct. Anal. Appl.*, 22:255–262, 1988.

[52] I Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 2nd edition, 1995.

[53] F Malikov. Verma modules over Kac-Moody algebras of rank 2. *Leningrad Math. J*, 2(2):269–286, 1991.

[54] O Mathieu. Classification of irreducible weight modules. *Annales de l'Institut Fourier*, 50(2):537–592, 2000.

[55] V Mazorchuk. *Lectures on* $\mathfrak{sl}_2(\mathbb{C})$ *modules*. Imperial College Press, 2009.

[56] J McCleary. *User's guide to Spectral Sequences*. Cambridge University Press, 2001.

[57] K Mimachi and Y Yamada. Singular vectors of the Virasoro algebra in terms of Jack symmetric polynomials. *Comm. Math. Phys.*, 174:447–455, 1995.

[58] R.V. Moody and A Pianzola. *Lie Algebras with Triangular Decompositions.* Wiley Interscience, New York Toronto, 1995.

[59] D Ridout. Fusion in Fractional Level sl(2)-Theories with $k = -1/2$. *Nucl. Phys.*, B848:216–250, 2011.

[60] D Ridout, S Siu, and S Wood. Singular vectors for the W_N algebras. *J. Math. Phys.*, 59:031701, 2018.

[61] D Ridout and S Wood. From Jack polynomials to minimal model spectra. *J. Phys.*, A48:045210, 2015. arXiv:1409.4847 [hep-th].

[62] D Ridout and S Wood. Relaxed singular vectors, Jack symmetric functions and fractional level $\widehat{sl}(2)$ models. *Nucl. Phys.*, B894:621–664, 2015. arXiv:1501.07318 [hep-th].

[63] J Rotman. *An Introduction to Homological Algebra.* Springer, 2009.

[64] D Ridout T Creutzig. Modular Data and Verlinde Formulae for Fractional Level WZW Models I. *Nucl. Phys.*, B865:83–114, 2012.

[65] V Tarasov and A Varchenko. Selberg type integrals associated with $\mathfrak{sl}_3$. *Lett. Math. Phys.*, 65:173–185, 2003. arXiv:math/0302148 [math.QA].

[66] K Thielemans. A MATHEMATICA Package for Computing Operator Product Expansions. *Int. J. Mod. Phys.*, C2:787–798, 1991.

[67] A Tsuchiya and Y Kanie. Fock space representations of Virasoro algebra and intertwining operators. *Publ. Res. Inst. Math. Sci.*, 22:259–327, 1986.

[68] A Tsuchiya and S Wood. On the Extended W-Algebra of Type sl_2 at Positive Rational Level. *Int. Math. Res. Not.*, 2015:5357–5435, 2015. arXiv:1302.6435 [math.QA].

[69] I V Tyutin. Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism. 1975. arXiv:hep-th/0812.0580.

[70] M Vasiliev. Higher-Spin Gauge Theories in Four, Three and Two Dimensions. *Int. J. Mod. Phys.*, D05:763–797, 1996. arXiv:hep-th/9611024.

[71] M Wakimoto. Fock representation of the algebra $A_1^{(1)}$. *Comm. Math. Phys.*, 104:605–609, 1986.

[72] M Wakimoto and H Yamada. Irreducible decompositions of Fock representations of the Virasoro algebra. *Lett. Math. Phys.*, 7:513–516, 1983.

[73] M Wakimoto and Y Yamada. The Fock representations of the Virasoro algebra and the Hirota equations of the modified KP hierarchies. *Hiroshima Math. J.*, 16:427–441, 1986.

[74] W Wang. Rationality of Virasoro vertex operator algebras. *Int. Math. Res. Not.*, 1993:197–211.

[75] S Warnaar. A Selberg integral for the Lie algebra A_n. *Acta Math.*, 203:269–304, 2009. arXiv:0708.1193 [math.CA].

[76] S Warnaar. The $\mathfrak{sl}_3$ Selberg integral. *Adv. Math.*, 224:499–524, 2010. arXiv:0901.4176.

[77] J Wess and B Zumino. Consequences of anomalous ward identities. *Phys. Lett.*, 37:95–97, 1971.

[78] E Witten. Global aspects of current algebra. *Nucl. Phys.*, B223:422–432, 1983.

[79] E Witten. Nonabelian bosonization in two dimensions. *Comm. Math. Phys.*, 92(4):455–472, 1984.

[80] S Yanagida. Singular vectors of $N = 1$ super Virasoro algebra via Uglov symmetric functions. arXiv:1508.06036 [math.QA].

[81] A Zamolodchikov. Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory. *Theoret. and Math. Phys.*, 65:1205–1213, 1985.