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Precision Holography and its Applications to Black Holes

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and its

Applications to Black Holes

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Precision holography and its applications to black holes

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Chapter 1

Holography and the AdS/CFT correspondence

(1.1) Introduction

One of the most exciting discoveries made in the context of String Theory in the last decade are holographic dualitites, which relate gravity theories in $(d+1)$ dimensions to Quantum Field Theories (QFTs) in d dimensions. The most prominent example is the AdS/CFT correspondence [5, 6, 7] in which the gravity theory is given by String Theory in asymptotically Anti de Sitter space and the boundary theory by a QFT whose renormalization group flows to a fixed point in the UV.

The very idea that a gravity theory be related to a QFT in one dimension less has been conjectured earlier in the context of black hole physics. According to the Bekenstein-Hawing formula, the entropy $S = A/4$ (in units $G = c = \hbar = k = 1$) of a black hole scales with the horizon area A. However in a (local) QFT, the entropy of a system, being an extensive quantity, should scale with the volume of the system. This led 't Hooft and Susskind to conjecture that a description of the microscopic degrees of freedom of black holes and hence of gravity are given by a QFT in one dimension less [8]. This idea has been named *holographic principle*, since the lower-dimensional QFT was thought to contain all the physical information of the higher-dimensional gravity theory.

In 1997, Maldacena found the first concrete example of a holographic duality [5]: By considering the low-energy limit of N parallel D3 branes he conjectured that IIB String Theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory (without gravity) in four dimensions, which can be imagined to live on the boundary of AdS_5. Naively one might think this to be unrelated to the holographic principle, since the string theory side is living in a

10-dimensional space. However in the limit of large N and large 't Hooft coupling the string theory in $AdS_5 \times S^5$ is described by its low-energy supergravity approximation, which in turn can be Kaluza-Klein reduced on the S^5 to a gravity theory on AdS_5 coupled to the Kaluza-Klein modes. This five-dimensional gravity theory is then dual to four-dimensional strongly coupled large N Super Yang-Mills theory on the boundary, thus implementing the holographic principle for $d = 4$.

Soon thereafter, more examples which resembled above setup were found. In all of these examples, the string theory side consists of a geometry asymptotic to $AdS_{d+1} \times X^{9-d}$, where X^{9-d} is a $(9-d)$-dimensional compact space. Whereas the $\mathcal{N} = 4$ Super Yang-Mills theory discussed above is conformal and has a vanishing β-function, the d-dimensional QFT dual to asymptotic AdS will in general have a renormalization group flow which however flows to a conformal fixed point in the UV. It is also important to mention that all known holographic dualities in string theory are strong-weak dualities, meaning that the regime where the curvature in the bulk is small enough such that stringy corrections to supergravity calculations can be neglected corresponds to a strongly coupled boundary theory.

A precise formulation of holographic dualities in string theory was proposed in [6, 7]. There it is assumed that the duality between a $(d+1)$-dimensional bulk theory and a d-dimensional QFT is defined by an equality of (Euclidean signature) partition functions,

$$Z_{QFT}[\phi_0] \equiv \langle \exp(-\int d^d x \phi_0 \mathcal{O}_\phi) \rangle_{QFT} = Z_{bulk}[\phi|_{bdry} \sim \phi_0]. \tag{1.1}$$

The partition functions in this equality are functions of a generating source ϕ_0. At the bulk side on the right, this source ϕ_0 has an interpretation as the boundary value of a bulk field ϕ (up to a potential divergent prefactor), whereas in the QFT ϕ_0 couples to a dual operator $\mathcal{O}_\phi$. The right hand side simplifies in the limit that the bulk theory becomes classical. In the examples mentioned in the previous paragraph this limit corresponds to the limit of large N and large 't Hooft coupling, in which the string theory can be approximated by its low-energy supergravity description. The supergravity limit is equivalent to the saddle point approximation of the bulk partition function,

$$Z_{bulk}[\phi_0] = \exp(-I_S(\phi)), \tag{1.2}$$

where $I_S(\phi)$ is the on-shell action of the supergravity theory with boundary condition $\phi|_{bdry} = \phi_0$. In the supergravity limit, one can use the relation (1.1) to calculate (connected) correlation functions of dual operators in the field theory,

$$\langle \mathcal{O}_\phi(x_1)\mathcal{O}_\phi(x_2)\ldots\mathcal{O}_\phi(x_n)\rangle_c = (-1)^n \frac{\delta}{\delta\phi_0(x_1)} \frac{\delta}{\delta\phi_0(x_2)} \cdots \frac{\delta}{\delta\phi_0(x_n)} W_{QFT}[\phi_0]|_{\phi_0=0}, \tag{1.3}$$

where $W_{QFT} \equiv \ln Z_{QFT} = \ln Z_{bulk} = -I_S$ is the generator of connected correlation functions in the QFT. Given this framework, one usually proceeds in two steps: In the first steps one tries to identify the dual QFT to a given bulk theory. This can usually only be achieved if the bulk as well as the boundary theory can be obtained by taking low energy limits of brane configurations, following the example of [5] for parallel D3 branes. In this limit the d-dimensional world volume theory of the brane configuration decouples from the bulk, giving rise to the QFT on the boundary, while on the gravity side one zooms in in the near-horizon region of the back-reacted branes. The latter yields a 10-dimensional geometry which typically can be Kaluza-Klein reduced to a $(d+1)$-dimensional gravity theory. An important check of the duality is a correspondence of global symmetries on both sides. In the case of D3 branes for example, the conformal group in four dimensions $SO(4,2)$ of the Super Yang-Mills theory corresponds to the isometry group of AdS_5, the $SU(4) \simeq SO(6)$ R-symmetry group to the isometry group of S^5, and the Monotonen-Olive duality $SL(2,Z)$ to the S-duality of IIB String theory. In addition, both sides are invariant under 16 Poincaré and 16 conformal supercharges.

Given this duality of theories, the second step is to match the spectrum on both sides, which means to match bulk fluctuations around the background to dual operators on the boundary. A first guideline to achieve this is to map fluctuations to operators transforming in the same representations under the global symmetries. However if symmetries are not restrictive enough, it is necessary to compare dynamic information on both sides by calculating correlation functions. This comparison in turn is complicated by the strong-weak nature of the duality. In general, correlation functions renormalize as the coupling is changed from the regime where the bulk description is valid to the regime in which perturbation theory in the boundary theory can be applied. Only if there are enough supersymmetries to protect the correlation functions through non-renormalization this comparison can be performed and the map between bulk fluctuations and dual operators can be refined by dynamic information.

A comprehensive introduction into the AdS/CFT correspondence is clearly beyond the scope of this thesis, see [9, 10] for further reference. In what follows, we will restrict ourselves to key concepts which will be central to the discussion in later chapters.

(1.2) The holographic dictionary

In the approximation (1.2) the problem of finding the generating functional on the boundary is reduced to solving the classical supergravity equations for the bulk fields ϕ given the Dirichlet boundary data ϕ_0 and evaluating the bulk action on this solution. But one can also use (1.2) together with (1.3) to read off field theory data from a given bulk solution. It turns out that there are two linearly independent solutions in the bulk for each field, the *normalizable* and the *non-normalizable* mode, which are related respectively to the vacuum expectation value (vev) of the dual operator and deformations of the dual field theory by the dual operator.

Let us explore these solutions in the case of a free scalar field in a fixed AdS background described by the action

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2), \tag{1.4}$$

where the AdS background in Poincaré coordinates is given by

$$ds^2 = \frac{dz^2 - dx_0^2 + dx_1^2 + \ldots + dx_d^2}{z^2}. \tag{1.5}$$

The equation of motion is given by

$$(-\nabla_g^2 + m^2)\phi = 0, \tag{1.6}$$

where ∇_g^2 denotes the Laplace operator in AdS. The solution to this equation can be written as

$$\phi(z,x) = a_+\phi^+(z,x) + a_-\phi^-(z,x), \tag{1.7}$$

where $\phi^\pm$ are linearly independent and behave asymptotically as

$$\phi^\pm \sim z^{\alpha_\pm}, \qquad \alpha_\pm = \frac{d}{2} \pm \sqrt{(d/2)^2 + m^2}. \tag{1.8}$$

The solution (1.7) can thus asymptotically be written as

$$\phi(z,x) \sim \phi_0(x) z^{\alpha_-} + \ldots + \phi_n(x) z^{\alpha_+} + \ldots, \tag{1.9}$$

where $\phi_0(x)$ and $\phi_n(x)$ denote the non-normalisable and normalisable mode respectively.[1] Obviously, since $\alpha_- < \alpha_+$ and the boundary is at $z \to 0$, $\phi_0(x)$ plays the role of the boundary source. On the boundary, $\phi_0(x)$ couples in the generating functional to the dual operator $\mathcal{O}_\phi$ via $\langle \exp(\int dx \phi_0 \mathcal{O}_\phi)\rangle$ and a non-trivial $\phi_0(x)$ corresponds to a deformation of the boundary action by precisely this term.

To identify the field theory interpretation of $\phi_n(x)$, it will be helpful to look at how isometries in the bulk map to the boundary. Given the metric (1.5), which diverges at the boundary $z \to 0$, we can define a boundary metric by multiplying the bulk metric with z^2 and restricting it to the boundary,

$$ds_0^2 = (z^2 ds^2)|_{z=0}. \tag{1.10}$$

The transformation $z \to \lambda z$, $x \to \lambda x$ is an isometry of (1.5) which induces a boundary dilatation $ds_0^2 \to \lambda^2 ds_0^2$. This is referred to as the fact that AdS only defines a conformal structure at the boundary. We can now use this result to translate dependencies on the radial coordinate z to the conformal dimension of the coefficient. As $\phi(z,x)$ is invariant under AdS isometries $\phi_0(x)$ must have conformal dimension α_-. The dimension of the dual operator $\mathcal{O}_\phi$, to which ϕ_0 couples via $\langle \exp(\int dx \phi_0 \mathcal{O}_\phi)\rangle$ must be

$$\Delta = d - \alpha_- = \alpha_+. \tag{1.11}$$

Furthermore we see that $\phi_n(x)$ has just the right conformal dimension to be identified with the vev of $\mathcal{O}_\phi$. For general $\phi_0(x)$ however the relation between $\phi_n(x)$ and $\langle \mathcal{O}_\phi \rangle$ turns out to be more complicated and requires properly addressing the subtleties of renormalizing infinities which arise in the bulk. This is done by the framework of holographic renormalization, which we will summarize in the next section.

[1] The case in which the Breitenlohner-Freedman bound [11] $m^2 \geq -(d/2)^2$ is saturated requires special treatment. Furthermore in the case $-(d/2)^2 < m^2 < -(d/2)^2 + 1$ there is a second dual QFT in which α_+ and α_- are interchanged [12].

(1.3) A preview of holographic renormalization

Holographic renormalization [13, 14, 15, 16, 17, 18, 19, 20, 21] starts with the observation that the action I_S in (1.2) evaluated on an asymptotically AdS geometry will be divergent. Even if we truncate the bulk action to pure gravity with cosmological constant there still remains the divergence corresponding to the infinite volume of AdS space. The divergences can be cancelled by adding counter-terms to the regulated on-shell action, which are local in the sources. These counterterms can be made bulk-covariant by expressing them in terms of local functionals of the bulk fields. In essence, holographic renormalization corresponds to the well-known UV renormalization in the QFT.

Let us illustrate the calculation of counterterms for the case of a free scalar in a fixed AdS background, which we already explored in the last section.[2] The first step is to asymptotically expand the field equations to determine the dependence of arbitrary solutions on the boundary conditions. This is done by expanding the fields in the Fefferman-Graham expansion (in the new radial coordinate $\rho = z^2$),

$$\Phi(\rho, x) = \rho^{(d-\Delta)/2}\left[\phi_{(0)}(x) + \rho\phi_{(2)}(x) + \ldots + \rho^{\Delta-d/2}(\phi_{(2\Delta-d)}(x) + \tilde{\phi}_{(2\Delta-d)}(x)\log\rho) + \ldots\right], \tag{1.12}$$

and inserting it in the equation of motion (1.6). At each order in ρ this results in a recursion formula which determines higher order coefficients in terms of lower order coefficients,

$$\phi_{(2n)} = \frac{1}{2n(2\Delta - d - 2n)}\nabla_0^2\phi_{(2n-2)}, \tag{1.13}$$

where $n < \Delta - d/2$ and we have used the notation ∇_0^2 to denote the Laplace operator with respect to the (flat) boundary metric. Iterating (1.13), we can express all $\phi_{(2n)}$ with $n < \Delta - d/2$ as local functionals of ϕ_0. It can also be easily checked that the coefficient of any power of ρ not appearing in (1.12) necessarily vanishes.

The further discussion now depends on whether $\Delta - d/2$ is an integer. If $\Delta - d/2$ is an integer, (1.13) cannot be applied to obtain $\phi_{(2\Delta-d)}$, which means that the latter is not determined by the asymptotic expansion of the equations of motion. Furthermore one has to add a logarithmic term to the expansion (1.12) to satisfy the equation of motion at order $\Delta - d/2$. If $\Delta - d/2$ is not an integer, one can still add an (undetermined) coefficient $\phi_{(2\Delta-d)}$ to the expansion, but the coefficient of the logarithmic term $\tilde{\phi}_{(2\Delta-d)}$ vanishes in this case.

The undetermined coefficient $\phi_{(2\Delta-d)}$ corresponds to the normalizable mode found in (1.7). It is not surprising that it is not determined in terms of $\phi_{(0)}$ since $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$ are just the first coefficients of the two linearly independent solutions in (1.6). In order to fix $\phi_{(2\Delta-d)}$ we have to impose additional boundary conditions, for example that the solution is smooth in the interior.

[2] For illustration purposes we neglect here the backreaction of the scalar on the geometry. This can only be done in special cases like the free scalar, and only if one is interested in a subset of possible correlation functions [18]. In general one should always solve the full set of gravity-scalar equations.

In the next step we would like to isolate the divergences in (1.4). To this aim we insert the expansion (1.12) in (1.4) and introduce a radial cutoff at $\rho = \epsilon$. The part of the on-shell action which diverges as $\epsilon \to 0$ is then given by the boundary action

$$S_{reg} = \int_{\rho=\epsilon} d^d x \left(\epsilon^{-\Delta+d/2} a_{(0)} + \epsilon^{-\Delta+d/2+1} a_{(2)} + \ldots - a_{(2\Delta-d)} \log \epsilon \right). \tag{1.14}$$

Fortunately, the powers of ϵ work out in such a way that all $a_{(n)}$ can be expressed as local functionals of the source $\phi_{(0)}$ and are independent of the normalizable mode $\phi_{(2\Delta-d)}$. This important property allows us to define a counterterm action which is local in $\phi_{(0)}$ simply by

$$S_{ct}[\phi(x,\epsilon)] = -S_{reg}[\phi_{(0)}[\phi(x,\epsilon)]]. \tag{1.15}$$

The fact that we have determined the counterterm action for general boundary coundition $\phi_{(0)}$ means that its form does not depend on a particular solution but applies for extracting data from all solutions of the field equations with the given boundary conditions. However as indicated in (1.15), the counterterm action should be defined in terms of the bulk fields $\phi(x,\rho)$ instead of the sources $\phi_{(0)}$ in order to transform in a well-defined manner under bulk diffeomorphisms. The inverse expression $\phi_{(0)}[\phi(x,\rho)]$ can be obtained by inverting the expansion (1.12).

This allows us to define the renormalized action

$$S_{ren} = \lim_{\epsilon\to 0} \left(S_{on-shell} + S_{ct} \right), \tag{1.16}$$

with which we can compute the exact renormalized 1-point function in the presence of arbitrary source $\phi_{(0)}$,

$$\langle \mathcal{O}_\phi \rangle_s \equiv \frac{\delta S_{ren}}{\delta \phi_{(0)}} = -(2\Delta - d)\phi_{(2\Delta-d)} + C[\phi_{(0)}], \tag{1.17}$$

where $C[\phi_{(0)}]$ is a local functional of $\phi_{(0)}$. Note that (1.16) leaves the freedom of adding additional finite counterterms to (1.16) which corresponds to a change of scheme in the renormalization of the boundary theory. In (1.17) a change of scheme corresponds to a change of $C[\phi_{(0)}]$.

With (1.17), higher point functions can in principle be calculated via

$$\begin{aligned} \langle \mathcal{O}_\phi(x_1)\ldots\mathcal{O}_\phi(x_n)\rangle &= \frac{\delta^n S_{ren}}{\delta\phi_{(0)}(x_1)\delta\phi_{(0)}(x_2)\ldots\delta\phi_{(0)}(x_n)} \\ &= -(2\Delta-d)\frac{\delta^{n-1}\phi_{(2\Delta-d)}(x_1)}{\delta\phi_{(0)}(x_2)\ldots\phi_{(0)}(x_n)} + \text{contact-terms}, . \end{aligned} \tag{1.18}$$

where the scheme-dependent contact-terms in the second line arise from the functional derivatives of $C[\phi_0]$. Note that we have presupposed a functional dependence of the normalizable mode $\phi_{(2\Delta-d)}$ on the source $\phi_{(0)}$. This seems in conflict with the above statement that $\phi_{(2\Delta-d)}$ is an independent mode of the equations of motion. However, after imposing an additional boundary condition in the interior, namely that the solution be smooth, $\phi_{(2\Delta-d)}$ is fixed and its behavior as $\phi_{(0)}$ is changed can be studied. Hence in contrast to the counterterms, which are local functionals of $\phi_{(0)}$, $\phi_{(2\Delta-d)}$ depends on $\phi_{(0)}$ in a very non-local way.

Retrieving the full functional dependence of $\phi_{(2\Delta-d)}$ on $\phi_{(0)}$ would however require solving the non-linear field equations with arbitrary Dirichlet boundary conditions, which is too difficult given current techniques. A viable alternative is to linearize the field equation around a given background to obtain the infinitesimal dependence of $\phi_{(2\Delta-d)}$ on changes of $\phi_{(0)}$, which allows one to calculate two-point functions at the background value of $\phi_{(0)}$. Higher point functions similarly require an expansion to the given order.

For arbitrary bulk fields $\mathcal{F}(x,\rho)$, most of above discussion generalizes in a straightforward way. We again start by asymptotically expanding the fields in a Fefferman-Graham expansion,

$$\mathcal{F}(x,\rho) = \rho^m \left[f_{(0)}(x) + f_{(2)}(x)\rho + \ldots + \rho^n (f_{(2n)}(x) + \tilde{f}_{(2n)}(x)\log\rho) + \ldots \right]. \tag{1.19}$$

One of the fields will be the metric of the asymptotically AdS geometry, which is given by

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j, \\ g_{ij}(x,\rho) &= g_{(0)ij}(x) + g_{(2)ij}(x,\rho)\rho + \ldots \end{aligned} \tag{1.20}$$

Assuming the metric aymptotically behaves as in (1.20) will restrict the boundary behavior of the other fields through the field equations, and with it the leading power m in (1.19). Furthermore the expansion will proceed in integer steps in the power of ρ if only integer powers of ρ arise in the asymptotic field equations. Given m, the power $m+n$ of the undetermined term is also determined by the field equations. Analogously to the example of the scalar a logarithmic term has to be added if the undetermined term arises at an order which is a multiple of the step size of the expansion. Whereas the terms up to the power m are used to define the counterterm action, the undetermined term $f_{(2n)}(x)$ again is related to the 1-point function.

There is one complication however in generalizing the holographic renormalization of the scalar to non-scalar fields: In the example of pure gravity in AdS, in which $m = 0$ and $n = d$, the trace and divergence of $g_{(2n)ij}$ are asymptotically determined as local functions of $g_{(0)ij}$. This fact is due to Ward identities of the dual operator, in this case the conformal and diffeomorphism Ward identity of the dual energy-momentum tensor. Also note that in the general case of multiple fields, the coefficients $\mathcal{F}^i{}_{(2n)}$ in the Fefferman-Graham expansion depend not only on the source $\mathcal{F}^i{}_{(0)}$ but on all sources $\mathcal{F}^j{}_{(0)}$ turned on in the problem at hand.

Although the presented method of holographic renormalization satisfactorily solves the problem of extracting holographic correlation functions given the bulk field equations with specified boundary conditions it includes the somewhat clumsy step of first asymptotically expanding the fields and then inverting the expansion to retrieve the covariant counterterm action. This issue is addressed in the Hamiltonian formulation of holographic renormalization [19, 20]. In this formalism, the asymptotic expansion in terms of a radial variable is replaced by an expansion in terms of the dilatation operator, which is an asymptotic symmetry of asymptotic AdS spaces. As the dilatation operator is formulated as functional derivative on the solution space of the field equations w.r.t. boundary conditions on an arbitrary radial hypersurface near the boundary, the elements in this expansion are covariant from the outset. By Hamilton-Jacobi theory, the holographic 1-point function, obtained by varying the on-shell action w.r.t. the boundary condition on the hypersurface, is related to the radial canonical momentum π. The renormalized 1-point function is then given by the term of weight Δ in the dilatation expansion of the canonical momentum,

$$\langle \mathcal{O}_\phi \rangle = \pi_{\phi(\Delta)}. \tag{1.21}$$

The Hamiltonian formulation allows one to determine the counterterms to the momenta by calculationally efficient recursion relations. Furthermore it is advantageous for proving general statements that are independent of the particular solution at hand, like Ward identities.

(1.4) Chiral primaries and the Kaluza-Klein spectrum

In the last section we have discussed the extraction of field theory data given bulk field equations and geometry in a $(d+1)$-dimensional asymptotically AdS space. However, as mentioned in the introduction, in all known dualities the string theory lives in a 10-dimensional background, as for example $AdS_{d+1} \times X^{9-d}$. The $(d+1)$-dimensional field equations can then be obtained by linearizing and Kaluza-Klein reducing the 10-dimensional field equations around the $AdS_{d+1} \times X^{9-d}$ background. Given the lower-dimensional modes, we would then like to know how they map to dual operators.

As mentioned in the introduction, mapping bulk fields to dual operators for general dualities is far from being trivial. The most powerful tool at hand is to use existing global symmetries like supersymmetry, R-symmetry and conformal symmetry. Of particular importance here is supersymmetry since it is able to protect multiplets from changing their constitution and dimensions by renormalization as the coupling is changed from strong to weak 't Hooft-coupling, or equivalently from the regime with weakly curved bulk description to the perturbative regime in the boundary description. Protected multiplets, which are also called *short multiplets* or *BPS multiplets*, have the property that they span a shorter spin range than general multiplets. Their lowest dimension state, the *chiral primary* state, is not only annihilated by all conformal supercharges, as in the case of a general multiplet, but also by a combination of Poincaré supercharges. We will be mostly interested in 1/2-BPS chiral primaries, which are annihilated by half of possible combinations of Poincarè supercharges.

The theory obtained by Kaluza-Klein reducing 10-dimensional supergravity contains only 1/2-BPS multiplets. This is because by this method only fields with spin ≤ 2 appear which have to fit into multiplets with a spin range ≤ 2. But only 1/2-BPS multiplets fulfill this requirement; 1/4-BPS, 1/8-BPS and long multiplets have a spin range of 3, $7/2$ and 4 respectively. A further important property of 1/2-BPS multiplets is that the conformal dimension of its chiral primary is fixed in terms of its R-charge, as can be shown from the superconformal algebra.

Furthermore, the spectrum of operators in the boundary theory is expected to be dual to single-particle states as well as bound states of multiple particles in the bulk. Multiple particle states could then be constructed out of operator product expansions of their single particle constituents. This behavior can be reproduced if we remind ourselves that all known holographic dualities are dualities of large N theories where N corresponds to the number of indices corresponding to a symmetry, for example a gauge symmetry. As all multiple trace operators with respect to this symmetry can be constructed out of multiplying single trace operators in a operator product expansion, it is natural to identify single trace operator with single particle states and multiple trace operator with bound states of particles. Thus when matching the supergravity spectrum to the spectrum of operators it suffices to consider chiral multiplets of single-trace operators.

In the latter part of the thesis, we will make extensive use of the duality between IIB Supergravity on $AdS_3 \times S^3 \times M_4$, where M_4 can be either T^4 or $K3$, and the dual two-dimensional $\mathcal{N} = (4,4)$ superconformal theory, which is a deformation of the sigma-model on the symmetric orbifold M_4^N/S_N. The volume of the Ricci-flat compact space M_4 on the bulk side is taken to be of the order of the string scale, thus when considering the low energy effective theory we can neglect all but the zero modes. Reducing IIB Supergravity to six dimensions yields $N = 4b$ supergravity coupled to n_t tensor multiplets, where $n_t = 5, 21$ in the case of $M_4 = T^4, K3$ respectively. The radius of the S^3 however is of the same size as that of AdS_3, so we need to retain the whole Kaluza-Klein tower.

(1.5) KALUZA-KLEIN HOLOGRAPHY

After relating the spectrum on both sides, we would like to compute holographic correlation functions as outlined in section 1.3. The subtleties of this process are addressed in the method of Kaluza-Klein holography [22]. At first it might seem that the Kaluza-Klein reduction of 10-dimensional supergravity leads to an infinite number of fields all coupled together, and hence it would be intractable to extract 1-point and higher-point functions. However, if the aim is to extract higher-point functions for vanishing source, we solely need to retain the perturbation expansion in the given number of fields. For example, to extract 3-point functions, we need to keep quadratic terms in the field equations. If the aim is to extract 1-point functions from specific bulk solutions asymptotic to $AdS_{d+1} \times X^{9-d}$ we can make use of the fact that the fall-off of the fields near the boundary is fixed by their mass, or equivalently by the conformal dimension of the dual operator. Only interaction terms involving modes with lower conformal dimension can contribute to the 1-point function of a given operator.

The lower-dimensional field equations are obtained by expanding the 10-dimensional fields perturbatively around a background $\Phi_b(x, y)$ as

$$\begin{aligned} \Phi(x,y) &= \Phi_b(x,y) + \delta\Phi(x,y), \\ \delta\Phi(x,y) &= \sum_I \psi^I(x) Y^I(y), \end{aligned} \tag{1.22}$$

where x is a coordinate in the $(d+1)$ non-compact directions, y is a coordinate in the compact directions and Y^I denotes collectively all harmonics (scalar, vector, tensor and their covariant derivatives) on the compact space.

The expansion (1.22) however is not unique. There will be gauge transformations

$$X^{M'} = X^M - \xi^M(x,y), \tag{1.23}$$

where $X^M = \{x, y\}$ that transform the fluctuations ψ^I to each other or the background solution Φ_b. One possibility to address this ambiguity is to pick a gauge, for example de Donder gauge, in which a subset of modes are set to zero. This has however the disadvantage that a given solution has to be brought to this gauge-fixed form before being able to extract data from it. Alternatively one constructs combinations of modes which transform as scalars, vectors and tensors under these gauge transformations and reduce to single modes in de-Donder gauge. Schematically at second order in the fluctuations these are given by

$$\hat{\psi}^Q = \sum_R a_{QR}\psi^R + \sum_{R,S} a_{QRS}\psi^R\psi^S. \tag{1.24}$$

After the reduction the equations of motion for the gauge-invariant modes will be of the form

$$\mathcal{L}_I\hat{\psi}^I = \mathcal{L}_{IJK}\hat{\psi}^J\hat{\psi}^K + \mathcal{L}_{IJKL}\hat{\psi}^J\hat{\psi}^K\hat{\psi}^L + \ldots, \tag{1.25}$$

where the differential operator $\mathcal{L}_{I_1\ldots I_n}$ contains higher derivatives. These higher derivatives however can be removed by a non-linear shift of the lower-dimensional fields, which is called the Kaluza-Klein map and allows one to integrate the equations of motion to an action,

$$\phi^I = \hat{\psi}^I + \mathcal{K}^I_{JK}\hat{\psi}^J\hat{\psi}^K + \ldots \tag{1.26}$$

Integrating to an action is necessary in order for holographic renormalization discussed in section 1.3 to be applicable.

In addition, there is a subtlety related to extremal correlators which further contributes to the non-linear relation between 1-point functions and Kaluza-Klein modes. Extremal correlators are correlators between operators with conformal dimensions (Δ_i, Δ), s.t $\sum \Delta_i = \Delta$. It has been shown at cubic order [23, 24], that extremal correlators do not arise from bulk couplings, since their existence would cause conformal anomalies known to be zero. Instead they arise from additional boundary terms in the 10-dimensional action. The extremal correlators modify the expression for the 1-point function to

$$\langle \mathcal{O}^I \rangle = \pi^I_{(\Delta)} + \sum_{JK} a^I_{JK}\pi^J_{(\Delta_1)}\pi^K_{(\Delta_2)} + \ldots, \tag{1.27}$$

where the numerical constants a^ϕ_{JK} are related to extremal 3-point functions and the dots denote contributions from extremal higher-point functions.

In total, (1.24), (1.26) and (1.27) all contribute to non-linear terms in the relations between 1-point functions and Kaluza-Klein modes which are schematically given by

$$\langle \mathcal{O}^I_\Delta(\vec{x}) \rangle = [\psi^I(\vec{x})]_\Delta + \sum_{JK} b^I_{JK}[\psi^I(\vec{x})]_{\Delta_1}[\psi^K(\vec{x})]_{\Delta-\Delta_1} + \ldots, \tag{1.28}$$

where $\vec{x}$ is now the d-dimensional boundary coordinate, b^I_{JK} are numerical coefficients and we have used the notation

$$\delta\phi(\rho,\vec{x},y) = \sum_{I,m}[\psi^I(\vec{x})]_{2m}\rho^m\Psi^I(y) \tag{1.29}$$

for the asymptotic coefficients of the Fefferman-Graham radial coordinate ρ.

Chapter 2

The fuzzball proposal for black holes

(2.1) Black hole puzzles

Among the most challenging questions of black hole physics of the last 30 years are the origin of the Bekenstein-Hawking entropy, whether information is lost in black hole evaporation and how singularities are resolved in a full theory of quantum gravity.

According to the no-hair theorem of Einstein-Maxwell gravity, black holes are solely characterized by their mass, charge and angular momentum. Nevertheless to prevent the total entropy in the universe to decrease if matter falls in a black hole, which would be a violation of the second law of thermodynamics, it is necessary to assign black holes an intrinsic entropy. Following a formal analogy between the laws of thermodynamcis and the laws of black hole mechanics, this entropy should be proportional to the horizon area of the black hole, the Bekenstein-Hawking entropy. The discovery of Hawking radiation by semiclassical quantiztion of the black hole geometry then showed that black holes indeed emit black body radiation according to their assigned temperature and furthermore fixed the precise prefactor of their entropy. [25] Since then it has been a longstanding issue of gravitational physics to find the microscopic degrees of freedom corresponding to the Bekenstein-Hawking entropy and to explain why their number grows as $\exp(A/4G)$ with the horizon area A.

Hawking radiation also gave rise to the information paradox. Since the radiation is exactly thermal and the black hole finally evaporates, it seems as if information is lost in this process (see figure 2.1). In quantum-mechanical terms, conservation of information is equivalent to unitarity. If a final state in a quantum-mechanical process arises from unitary evolution,

$$|\psi\rangle_f = e^{-iHt}|\psi\rangle_i, \tag{2.1}$$

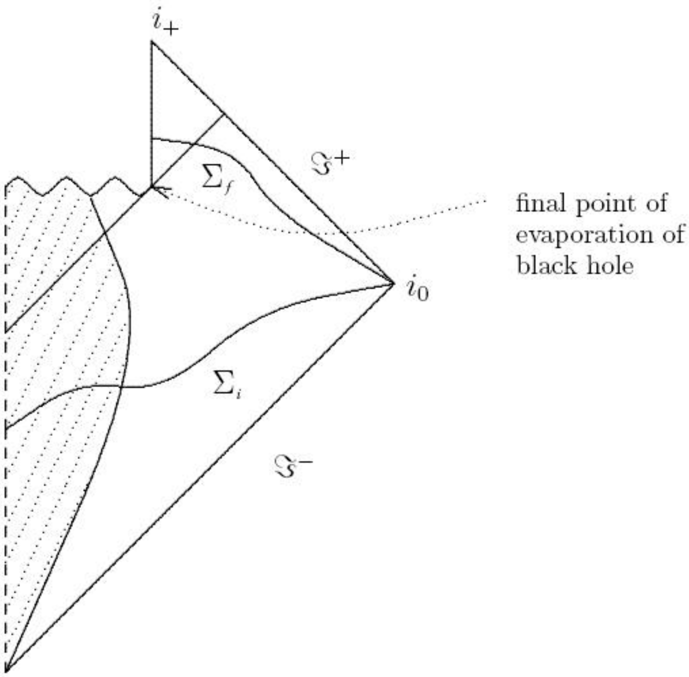

Figure 2.1: Penrose diagram of information loss: The inital data $|\psi\rangle_i$ provided on Σ_i partly falls into the black hole, leaving the data on Σ_f in a mixed state. The evolution from Σ_i to Σ_f is non-unitary. (Figure adapted from [26].)

it means that the initial state can be reconstructed by inverting the evolution,

$$|\psi\rangle_i = e^{iHt}|\psi\rangle_f, \tag{2.2}$$

and hence information has been preserved. However, in Hawking's calculation, entanglement between infalling and outgoing pair quanta at the horizon causes the state at Σ_f to be mixed, since the infalling quanta have been destroyed. If the state at Σ_i is pure, the evolution from Σ_i to Σ_f is necessarily non-unitary.

An alternative is to assume that information leaks out in subtle correlations of the Hawking radiation which are invisible in the semiclassical approximation. In this scenario the Hawking radiation is conceptually not very different from the black body radiation of a piece of burning coal. However, for a macroscopic black hole with mass well above the Planck mass, this requires that information must be non-locally transmitted from the infallen matter near the center of the black hole to the horizon. As a result, it seems that the information paradox requires either giving up unitarity or locality in a full quantum theory of gravity.

Assuming that AdS/CFT is valid gives an implicit solution to the information paradox: Since the evolution in the dual QFT is unitary the evolution in the bulk gravity theory must be as well, so information must be conserved. Unfortunately however, this argument does not reveal *how* information escapes the black hole. It is not known how to calculate in the dual QFT correlations measured by an infalling observer.

The fuzzball proposal [27, 28], arising out of string theory, proposes to resolve the information paradox and to provide a microscopic description of the Bekenstein-Hawking entropy. The basic idea is to replace the black hole by a large number of horizonless solutions which asymptote to the black hole geometry but differ at the horizon scale. These horizonless solutions are thought to correspond to microscopic states in the black hole ensemble, and upon averaging over these geometries, the original black hole with its horizon is retrieved. The fuzzball proposal solves the information paradox because each individual microstate geometry does not possess a horizon which implies that information can escape the black hole after a very long time once one takes into account its pure state. After quantization of the solution phase space, it furthermore allows for a statistical explanation of the Bekenstein-Hawking entropy as the microstates are given by the (quantized) individual geometries.

So far, all candidate solutions which have been found only involve low energy supergravity fields. However there are only a few, atypical states which are believed to be well described by supergravity. Typical microstate geometries are expected to contain regions of string scale curvature in which higher string modes and higher modes arising in the compact space of a ten- or eleven-dimensional fuzzball solution become important.

Although the fuzzball proposal was originally formulated for black holes in asymptotically flat spacetimes, it can be analyzed using AdS/CFT if the near-horizon limit of the black hole has a known holographic dual at the boundary. In fact, as we will elaborate on below, AdS/CFT strongly supports the fuzzball program.

(2.2) BLACK HOLE ENTROPY COUNTING BY STRING THEORY

Already before the proposal of the fuzzball program, string theory has been able to count the entropy of black holes, mostly extremal BPS black holes. [29] As an example we review here the case of the five-dimensional black hole arising from the bound D1-D5-P system with respective charges Q_1, Q_5 and Q_p compactified on $S^1 \times M_4$, where M_4 is either T^4 or $K3$. Both D1 and D5 branes wrap the S^1 with radius $R_z >> \sqrt{\alpha'}$, while the volume of the compact space is taken of the order of the string length, $vol(M_4) \sim \alpha'^2$. The momentum P then denotes the momentum of excitations along the circle. This system preserves 1/8 of the supersymmetry. The solution in the decoupling limit is given by

$$ds^2 = \frac{1}{\sqrt{h_1 h_5}} \left(-(dt^2 - dz^2) + \frac{Q_p}{r^2}(dz - dt)^2 \right) \tag{2.3}$$

$$+\sqrt{h_1 h_5} dx^m dx^m + \sqrt{\frac{h_1}{h_5}} ds^2(M_4),$$

where $h_i = 1 + Q_i/r^2$, x^m denote the coordinates in the transverse direction and $ds^2(M_4)$ denotes the metric on the compact space. The corresponding RR 2-form potential and dilaton is given by

$$e^{-2\Phi} = \frac{h_5}{h_1}, \qquad C_2 = (h_1^{-1} - 1) dt \wedge dz. \tag{2.4}$$

The charges Q_i can be expressed in terms of integral charges N_i via

$$\begin{aligned} Q_1 &= \frac{N_1 g_s \alpha'^3}{V}, \\ Q_5 &= N_5 g_s \alpha', \\ Q_p &= \frac{N_p g_s^2 \alpha'^2}{R_z^2}, \end{aligned} \tag{2.5}$$

with $V = (2\pi)^{-4} vol(M_4)$. The Bekenstein-Hawking entropy of this black hole in the Einstein frame $ds_E^2 = e^{-\Phi/2} ds^2$ can be calculated to be

$$S_{BH} = \frac{A_{10}}{4G_{10}} = \frac{A_5}{4G_5} = 2\pi\sqrt{N_1 N_5 N_p}, \tag{2.6}$$

where (A_{10}, G_{10}) and (A_5, G_5) are the horizon area and gravitational constant in ten and five dimensions respectively. The gravitational constants are given by

$$\begin{aligned} G_{10} &= 8\pi\kappa_{10} = 8\pi^6 g_s^2 \alpha'^4, \\ G_5 &= \frac{G_{10}}{(2\pi)^5 RV}. \end{aligned} \tag{2.7}$$

This supergravity description of the D1-D5-P system is valid if all $Q_i >> \sqrt{\alpha'}$.

The microscopic calculation of the entropy counts the excitations of the low-energy theory of the D-brane system. Since the volume of M_4 is of order of the string scale the system is described by an effective (1+1)-dimensional theory living on the circle. This low-energy theory, which is conjectured to be the a deformation of the $\mathcal{N} = (4,4)$ sigma model on the symmetric orbifold $(M_4)^{N_1 N_5}/S^{N_1 N_5}$, is also the AdS/CFT dual to IIB string theory on $AdS_3 \times S^3 \times M_4$, since the decoupling limit of the D1-D5-P solution above is $BTZ \times S^3 \times M_4$, and BTZ can be obtained by orbifolding AdS_3. The perturbative description of the dual theory is valid for all $Q_i << \alpha'$.

However, unlike in [29] where the microscopic entropy was counted in the perturbative regime of the orbifold CFT and related to the supergravity regime by using non-renormalization theorems, we strictly speaking do not need the details of the CFT here. If we invoke AdS/CFT, the only necessary assumption is that there is a CFT dual to $AdS_3 \times S^3 \times M_4$ in the supergravity regime which is unitary. By either analyzing the asymptotic conformal symmetries [30] or by computing the conformal anomaly [13] one then finds that the central charge of this CFT dual is given by

$$c = \frac{3l}{2G_3} = 6N_1 N_5. \tag{2.8}$$

In the supersymmetric case the momentum P in the bound state correspond to the left-moving excitation level in the dual CFT, while the right-moving excitations are in their ground state. Due to the unitarity of the dual CFT we can calculate the degeneracy at high excitation number N_p with Cardy's formula [31],

$$d(c, N_p) \sim e^{2\pi\sqrt{N_p c/6}}. \tag{2.9}$$

As a result, the microscopic calculation of the entropy yields

$$S_{mic} = \ln d(c, N_p) = 2\pi\sqrt{N_1 N_5 N_p}, \tag{2.10}$$

which precisely agrees with (2.6).

(2.3) THE D1-D5 TOY MODEL

Even though black hole entropy counting, which has been succesfully performed for many more extremal and near-extremal black holes, offers an important glimpse of the microscopic origin of the Bekenstein-Hawking entropy, it only partly addresses the questions raised in the beginning of section (2.1). The counting is performed in the dual QFT and it is a priori not clear how the QFT states are related to gravitational states. Particularly it is not known how global properties of the gravitational description like horizons are encoded in the QFT. As a result, it does not give information on how the entropy is related to the horizon of the black hole and how the information paradox is resolved. In contrast, the fuzzball proposal goes further by suggesting an explicit representation of the microscopic states in terms of gravitational degrees of freedom. As we will see below, these gravitational degrees of freedom are related to the dual microscopic states by AdS/CFT.

The idea of the fuzzball proposal is to replace *naive* black hole solutions like (2.3) by an ensemble of solutions with the same asymptotic charges but a different geometry at the horizon scale. Unfortunately, the fuzzball solutions corresponding to the macroscopic 3-charge D1-D5-P black hole are quite intricated. An interesting toy model, which we will extensively explore in later chapters, is the 2-charge D1-D5 system. The naive solution is obtained by setting $N_p = 0$ in (2.3),

$$ds^2 = \frac{1}{\sqrt{h_1 h_5}}(-dt^2 + dz^2) + \sqrt{h_1 h_5}dx^m dx^m + \sqrt{\frac{h_1}{h_5}}ds^2(M_4), \tag{2.11}$$

and preserves 1/4 of the supersymmetry. The solution is only a toy model for a black hole, since its naive solution has no horizon but only a naked singularity. Only if one includes higher order corrections a small horizon appears, whose associated entropy agrees with the dual CFT calculation.[1] The D1-D5 system can be related by U-duality to the F1-P chiral null model describing a fundamental string winding a compact direction with momentum, whose solution

[1]This is surprising since only a subset of higher order corrections are known and the curvature at the horizon of this small black hole is of order of the string scale. An explanation for black holes involving an AdS_3 factor in their (corrected) near-horizon geometry was given in [32].

has been found in [33, 34]. We will exploit this in chapter 4 to find the most general fuzzball solution.

The D1-D5 fuzzball solutions corresponding to (2.11) are characterized by a curve $F^I(v) = (F^i(v), F^\rho(v))$ extending in the transverse and internal directions, on which D1 and D5 charge is distributed. If there are no internal excitations, $F^\rho(v) = 0$, the solution has a slightly simpler form (3.44) than in the general case (4.51). If $F^I \equiv 0$ the solution collapses to the naive solution; otherwise the size of the curve $F^I(v)$ determines the scale at which the fuzzball geometry starts deviating. If the curve does not intersect with itself and if $\frac{d}{dv}F^I(v) \neq 0$ everywhere, the geometry close to the curve resembles a Kaluza-Klein monopole and remains smooth. Furthermore, since the Killing vector field $\partial/\partial t$ is timelike everywhere, there is no horizon and the solution has the same Penrose diagram as Minkowski space.

Classically, there is an infinite number of solutions parametrized by the curve $F^I(v)$. If one wishes to obtain a statistical entropy out of this ensemble of solutions one has to quantize the phase space by geometric quantization, which has been done for the D1-D5 system with only transverse excitations in [35]. Geometric quantization in this case yields commutation relations, in which the Fourier modes of $F^I(v)$ behave as oscillators. Counting the appropriate subspace yields precisely the fraction of entropy expected for transverse excitations. However one should mention that the counting includes regions in the phase space where higher-derivative corrections to supergravity are non-negligible. It is not clear why in this case the (mostly unknown) higher-derivative corrections do not seem to influence the counting.

(2.4) AdS/CFT supports the fuzzball proposal

If according to the fuzzball proposal there is an explicit representation of the microscopic states of a black hole in terms of geometries, it should be possible to map these states back to the microscopic states in the dual theory which we counted in section 2.2. In fact, this is what most of our discussion in chapter 3 and 4 will be about.

In the first step, we have to bring the fuzzball solutions in a form where we can analyze them with AdS/CFT: We replace the asymptotically flat region by an asymptotically AdS region, which corresponds to the replacement $h_{1,5} \to h_{1,5} - 1$. While the naive solution becomes locally $AdS_3 \times S^3 \times M_4$, the fuzzball geometries will only asymptotically be AdS, differing by normalizable modes from the naive solution. This is due to the fact that the curve $F^I(v)$ spreads out in the transverse directions, generating higher multipole moments in addition to the asymptotic charges. By the AdS/CFT dictionary developed in chapter 1 we can then relate the normalizable modes to the vevs of gauge-invariant operators in the dual CFT. Knowing all such vevs determines in principle the (pure) state of the CFT which corresponds to the fuzzball geometry.

In practice this procedure is however complicated by the fact that we have to use Kaluza-Klein holography to extract holographic data from a ten-dimensional (or in this case six-dimensional, since the M_4 is taken very small) geometry. As discussed in section 1.5 the formula (1.28) for the vevs of the dual operators in terms of the six-dimensional geometry contains in general non-linear contributions which have the same radial falloff as the linear contribution. In our case that means that the vev of a dual operator of dimension k does not only get contributions from the k-th multipole moment but also from the product of lower multipole moments. As we are doing perturbation theory in the order of the multipole moments it follows that for example at cubic order we can only extract the vevs of operators of lowest and second-lowest dimension.

In chapter 3 we will therefore conjecture a specific map between fuzzball geometries and CFT states. The extraction of the dual vevs of lowest and second-lowest dimension operators will serve as a way to perform kinematical and dynamical test of this map.

Nonetheless, on a more general level we can invert above arguments to show how AdS/CFT supports the fuzzball proposal for every black hole whose entropy we believe to be counted by a (strongly coupled) dual QFT. Also in this QFT we would be able to distinguish the dual states of a black hole ensemble by the vevs of gauge invariant operators. By AdS/CFT every such (pure) state maps to a geometry with different normalizable modes and hence with different subleading asymptotics. Replacing again the asymptotically AdS region by an asymptotically flat region we obtain an ensemble of fuzzball solutions which have the same asymptotic geometry as the black hole but differ in the interior.

This procedure however does not imply that the fuzzball geometries obtained in this way are resolvable in supergravity. In fact, as we will see for the D1-D5 system in section 3.11, many states in the dual CFT do not yield geometries which are distinguishable in supergravity.

(2.5) ARE ASTROPHYSICAL BLACK HOLES FUZZBALLS?

Even though we restrict our attention for the rest of this thesis to supersymmetric fuzzballs, we would like to mention that eventually the fuzzball program should also resolve the information loss and entropy problem for astrophysical black holes. These black holes differ from the supersymmetric D1-D5-P black hole in that they are four-dimensional and non-extremal, with an electric charge much smaller than their mass. While counting entropy and constructing fuzzball solutions for four-dimensional black holes are in principle not any more difficult than for five-dimensional black holes, the non-extremality adds an additional challenge. Extremal black holes are particularly easier to handle if there are (sufficiently) supersymmetric. Supersymmetry can not only protect the microstates in the dual QFT as the coupling is changed from weak to strong coupling, a requirement for many black hole counting arguments, but BPS (ie. supersymmetric) supergravity solutions are often given by harmonic functions which can be linearly superposed. For supersymmetric solutions like the D1-D5 system, the coarse-graining of the fuzzball geometries to the naive solution can be be achieved by a simple linear superposition.

For non-extremal black holes however coarse-graining, and particularly how it leads to a horizon in the naive solution, is not yet understood. A few candidate geometries for non-extremal fuzzballs are known [36, 37, 38]. All these geometries possess a superradiant instability which is thought to correspond to black hole evaporation, only that the decay time is much shorter than the evaporation time. This is not a contradiction if these states are atypical in the ensemble of the non-extremal black hole.

In principle, if we assume that also the microstates of a non-extremal black hole are given by a dual QFT, there should be corresponding fuzzball geometries. Following the general argument in the last section, we can invoke AdS/CFT to infer the existence of a large number of geometries with the same ADM charges as the black hole but differing in the interior. Since each of these geometries should correspond to a pure state, they should be horizonless. However the way in which the asymptotic data prevents the presence of a horizon in the interior, which then only appears after coarse-graining, still remains to be understood.

Finally we would like to mention that in all known fuzzball geometries corresponding to macroscopic black holes, typical geometries may contain regions of high curvatures or geometries may not be distinguishable in supergravity. A full gravitational description of an ensemble of fuzzball geometries most likely requires an understanding of these geometries as solutions of the full string theory. Overcoming the technical challenges associated with such a description would be a big progress in the fuzzball program.

Chapter 3

Holographic anatomy of fuzzballs

(3.1) Introduction, summary of results and conclusions

In this chapter we examine the precise relation between the fuzzball solutions and dual microstates for the 2-charge D1-D5 system which we introduced in section 2.3. Recall that the D1-D5 system is a 1/4 supersymmetric system and the "naive" black hole geometry has a near-horizon geometry of the form $AdS_3 \times S^3 \times M$, where M is either T^4 or $K3$. The naive geometry has a naked singularity but one expects that a horizon would emerge from α' corrections. At any rate, the description in terms of D-branes (at weak coupling) is well defined and one can obtain a statistical entropy in much the same way as for the 3 charge geometry which has a finite radius horizon. Indeed, the D1-D5 system can be mapped by dualities to a system of a fundamental string carrying momentum modes and the degeneracy of the system can be computed by standard methods. To be more specific, let us take $M = T^4$; then the degeneracy is the same as that of 8 bosonic and 8 fermionic oscillators at level $N = n_1 n_5$, where n_1 and n_5 are the number of D1 and D5 branes, respectively. The fuzzball proposal in this context is that there should exist an exponential number of horizon free solutions, one for each microstate, each carrying these two D-brane charges.

An exponential number of solutions was constructed by Lunin and Mathur in [27] and proposed to correspond to microstates. These were found by dualizing a subset of the FP solutions [33, 34], namely those that are associated with excitations of four bosonic oscillators. These provide enough solutions to account for a finite fraction of the entropy but one still needs an exponential number of solutions (associated with the additional four bosonic and eight fermionic oscillators in the example of T^4) to account for the total entropy. Such solutions, related to the odd cohomology of T^4 and the middle cohomology of the internal manifold have been discussed in [39] and [40], respectively, and we will complete this program in chapter 4. We thus indeed find that there are an appropriate number of solutions to account for all of the D1-D5 entropy[1].

Do these solutions, however, have the right properties to be associated with D1-D5 microstates, and if yes, what is the precise relation? The aim of this chapter is to address this question for the solutions corresponding to the universal sector of the T^4 and $K3$ compactifications.

As mentioned above the solutions of interest were obtained by dualizing FP solutions so let us briefly review these solutions and their relation to string perturbative states. A more detailed discussion will be given in section 3.2. The FP solutions (which are general chiral null models) involve the metric, B-field and the dilaton and are characterized by a null curve $F^I(x^+)$ with $I = 1, \ldots, 8$ in R^8. The solution describes the long range fields sourced by a string wrapping one compact direction and having a transverse profile given by the null curve $F^I(x^+)$. The ADM conserved charges, i.e. the mass, momentum and angular momentum, associated with this solution are given precisely by the energy, momentum and angular momentum of the classical string that sources the solution.

On general grounds, one would expect that this classical string should be produced by a coherent state of string oscillators. Indeed, we show in section 3.2 that associated to a classical curve $F^I(x^+)$,

$$F^I(x^+) = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\alpha_n^I e^{-in\left(\frac{x^+}{wR_9}\right)} + (\alpha_n^I)^* e^{in\left(\frac{x^+}{wR_9}\right)} \right), \tag{3.1}$$

where $x^+ = x^0 + x^9$, x^9 is the compact direction of radius R_9, w is the winding number and α_n^I are (complex) numerical coefficients, there is a coherent state $|F^I)$ of the first quantized string in an unconventional lightcone gauge with $x^+ = wR_9\sigma^+$, where σ^+ is a worldsheet lightcone coordinate, such that the expectation value of all conserved charges match the conserved charges associated with the solution. More precisely, let

$$X^I = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\hat{a}_n^I e^{-in\sigma^+} + (\hat{a}_n^I)^\dagger e^{in\sigma^+} \right) \tag{3.2}$$

be the 8 transverse left moving coordinates with $\hat{a}_n^I$ the quantum oscillators normalized such that $[\hat{a}_n^I, (\hat{a}_m^J)^\dagger] = \delta^{IJ}\delta_{mn}$. The corresponding coherent state is given by

$$|F^I) = \prod_{n,I} |\alpha_n^I) \tag{3.3}$$

where $|\alpha_n^I)$ is a coherent state of the left-moving oscillator $\hat{a}_n^I$, i.e. it satisfies $\hat{a}_n^I|\alpha_n^I) = \alpha_n^I|\alpha_n^I)$, and the eigenvalues α_n^I are the coefficients appearing in (3.1). By construction

$$(F^I|X^I|F^I) = F^I \tag{3.4}$$

[1]Note however that this is a continuous family of supergravity solutions. To properly count them one needs to appropriately quantize them. Such a quantization has been discussed in [35], see also [41, 42] for a counting using supertubes.

with root mean deviation of order $1/\sqrt{m}$, where $m \equiv (F^I|\hat{m}|F^I)$ the expectation value of the occupation operator[2] $\hat{m} = \sum \hat{a}^I_{-n}\hat{a}^I_n$. In other words, the expectation value is given by the classical string that sources the solution, and this is an accurate description as long as the excitation numbers are high. For low excitation numbers the state produced is fuzzy and the supergravity solution would require quantum corrections (as one would indeed expect). Note that the right-movers are in their ground state throughout this discussion.

Given winding w and momentum p_9 quantum numbers there are also corresponding Fock states

$$\prod(\hat{a}^I_{-n^I})^{m_I}|0\rangle, \qquad N_L = \sum n^I m^I = -wp_9 \tag{3.5}$$

where N_L is the total left-moving excitation level (m_I are integers). It is sometimes stated in the literature that the solutions of [33, 34] represent these states. This cannot be exactly correct as the string coordinates have zero expectation on these states, so semiclassically they do not produce the required source. The statement is however approximately correct since these states strongly overlap with the corresponding coherent state for high excitation numbers. So in the regime where supergravity is valid the coherent state can be approximated by Fock states. Notice that one can organize the Fock states (3.5) into eigenstates of the angular momentum operator by using as building blocks linear combination of oscillators that themselves are eigenstates (e.g. $(\hat{a}^I_{-n} \pm i\hat{a}^{I+1}_{-n})$). The coherent states are however (infinite) superpositions of states with different angular momenta and are thus not eigenstates of the angular momentum operator.

We now return to the discussion of the dual D1-D5 system. The solutions of [27] were obtained by dualizing the FP solutions we just discussed but with a curve that is restricted to lie on R^4. The corresponding underlying states are now R ground states of the CFT associated with the D1-D5 system. This CFT is a deformation of a sigma model with target space the symmetric product of the compactification manifold X, $S^N(X)$ ($N = n_1 n_5$ and n_1, n_5 are the number of D1 and D5 branes). The R ground states can be obtained by spectral flow of the chiral primaries of the NS sector. Recall that the chiral primaries are associated with the cohomology of the internal space. For the discussion at hand only the universal cohomology is relevant and this leads (after spectral flow) to the following R ground states

$$\prod(\mathcal{O}^{R(\pm,\pm)}_{n_l})^{m_l}|0\rangle \qquad \sum n_l m_l = N = n_1 n_5\,, \tag{3.6}$$

where n_l is the twist, m_l are integers and the superscripts denote (twice) the R-charges of the operator. Here the ground states are described in the language of the orbifold CFT; each ground state of the latter will map to a ground state of the deformed CFT. Notice that there is 1-1 correspondence between these states and the Fock states in (3.5). Namely one can map the

[2]Usually the occupation operator is called N but we reserve this letter for the level of the Fock states, $N = \sum n\hat{a}^I_{-n}\hat{a}^I_n$. Note also that after the duality to the D1-D5 system the occupation number becomes the eigenvalue of j_3 which is usually called m.

operators $\mathcal{O}^{R(\pm,\pm)}$ to the harmonic oscillators[3],

$$\hat{a}^{\pm 12}_{-n} \quad \leftrightarrow \quad \mathcal{O}^{R(\mp,\mp)}_{n}, \qquad \hat{a}^{\pm 34}_{-n} \quad \leftrightarrow \quad \mathcal{O}^{R(\pm,\mp)}_{n}. \tag{3.7}$$

where $\hat{a}^{\pm 12}_{-n} \equiv (\hat{a}^{1}_{-n} \pm i\hat{a}^{2}_{-n})/\sqrt{2}$ and $\hat{a}^{\pm 34}_{-n} \equiv (\hat{a}^{3}_{-n} \pm i\hat{a}^{4}_{-n})/\sqrt{2}$. In particular, the frequency n is mapped to the twist of the operator and the R-charge to the angular momentum in the 1-2 and 3-4 plane. However, the underlying algebra of these operators is different from the algebra of the harmonic oscillators.

Motivated by this correspondence it was proposed in [27] that each of the solutions obtained via dualities from the FP solution corresponds to a R ground state and via spectral flow to a chiral primary [28]. One of the original motivations for this work was to understand how such a map might work. Whilst it was clear from these works that the frequencies involved in the Fourier decomposition of the curve should map to twists of operators, it was unclear what the meaning of the amplitudes is in general and moreover a generic curve has far more parameters than an operator of the form (3.6). In our discussion of the FP system we have seen that the geometry is more properly viewed as dual to a coherent state rather than a single Fock state. The coherent state however viewed as linear superposition of Fock states (see (3.32)) contains states that do not satisfy the constraint $N_L = -p_9 w$ and therefore do not map to R ground states after the dualities. This then leads to the following proposal for the map between geometries and states [43] [4]:

Given a curve $F^i(v)$ we construct the corresponding coherent state in the FP system and then find which Fock states in this coherent state satisfy $N_L = -p_9 w$. Applying the map (3.7) then yields the superposition of R ground states that is proposed to be dual to the D1-D5 geometry.

Let us see how this works in some simple examples. The simplest case is that of a circular planar curve that we may take to lie in the 1-2 plane:

$$F^1(v) = \frac{\sqrt{2N}}{n}\cos 2\pi n\frac{v}{L}, \qquad F^2(v) = \frac{\sqrt{2N}}{n}\sin 2\pi n\frac{v}{L}, \qquad F^3 = F^4 = 0, \tag{3.8}$$

where L is the length of the curve and the overall factors are fixed by requiring that the solution has the correct charges (this will be explained in the main text). The corresponding coherent state can immediately be read off from the curve

$$|a_n^{-12}; a_n^{+12}; a_n^{-34}; a_n^{+34}) = |\sqrt{N/n}; 0; 0; 0). \tag{3.9}$$

In this case there is a single state with $N_L = N = -wp_9$ contained in this coherent state, namely

$$|N/n\rangle = (\hat{a}^{-12}_{-n})^{N/n}|0\rangle. \tag{3.10}$$

[3] This correspondence straightforwardly extends to the general case where all R ground states are considered and all bosonic and fermionic oscillators are used in (3.5).

[4] A map between density matrices of the CFT states built from 4 bosonic oscillators and modified fuzzball solutions has been recently discussed in [44]. Here we provide a map between the original fuzzball solutions and superpositions of R ground states of the D1-D5 system.

Using the map (3.7) we get that the D1-D5 solution based on the circle is dual to the R ground state

$$|circle) = \left(\mathcal{O}_n^{R(+,+)}\right)^{N/n} \tag{3.11}$$

which was the proposal in [27].

As soon as one moves to more complicated curves, however, the correspondence becomes more complex, as there is more than one Fock state with $N_L = -wp_9$. For example the next simplest case is the solution based on an ellipse

$$F^1(v) = \frac{\sqrt{2N}}{n} a \cos 2\pi n \frac{v}{L}, \qquad F^2(v) = \frac{\sqrt{2N}}{n} b \sin 2\pi n \frac{v}{L}, \qquad F^3 = F^4 = 0, \tag{3.12}$$

with $a^2 + b^2 = 2$. Following our prescription we obtain the following superposition

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k} (a-b)^k \left(\mathcal{O}_n^{R(+,+)}\right)^{\frac{N}{n}-k} \left(\mathcal{O}_n^{R(-,-)}\right)^k, \tag{3.13}$$

as is explained in section 2.3. The superposition for a general curve will involve a large number of Fock states.

Given such a map from curves to superpositions of states the question is whether the correspondence can be checked quantitatively. The D1-D5 solutions approach $AdS_3 \times S^3$ (times T^4 or $K3$) in the decoupling limit so one can use the AdS/CFT correspondence to make detailed quantitative tests. Recall that the deviations of the solution from $AdS_3 \times S^3$ encode vacuum expectation values of chiral primary operators (and possible deformations of the CFT by such operators), so by analyzing the asymptotics one can in principle completely characterize the ground state of the boundary theory.

Before proceeding to explain this, let us contrast the somewhat different meanings that one attaches to the statement "a geometry is dual to a state $|S\rangle$". In the context of the FP system, the state $|S\rangle$ is meant to provide the source for the supergravity solution and because of that we argued it should be a coherent state. In the context of the D1-D5 system however the same statement means that the ground state of the dual field theory is the state $|S\rangle$ (so $|S\rangle$ need not be approximated by a classical solution) and the vevs of gauge invariant operators on this state, $\langle S|\mathcal{O}|S\rangle$, are encoded in the asymptotics of the solution.

The D1-D5 system is governed by a 1+1 dimensional theory with $\mathcal{N} = (4,4)$ supersymmetry. This theory has Coulomb and Higgs branches (which are distinct even quantum mechanically) [45, 46, 47]. The boundary CFT is the IR limit of the theory on the Higgs branch. Thus the fuzzball solutions should be in correspondence with the Higgs branch. Note that due to strong infrared fluctuations in 1+1 dimensions one usually encounters wavefunctions rather than continuous moduli spaces of the quantum states. So more properly one should view the fuzzball solutions as dual to wavefunctions on the Higgs branch. These wavefunctions, however, may be localized around specific regions in the large N limit and one should view our proposed correspondence in this way.

The vevs of gauge invariant operators in this 1+1 dimensional theory can be computed from the asymptotics of the solution. As we discussed in chapter 1, the existence of such a relationship follows from the basic AdS/CFT dictionary that relates bulk fields to boundary operators and the bulk partition function to boundary correlation functions. The implementation of this program is however very subtle. Precise formulae for the 1-point functions for solutions with asymptotics to $AdS \times S$ were obtained in [22].

Naively the vev of an operator of dimension k is linearly related to coefficients of order z^k in the asymptotic expansion of the solution, where z is a radial coordinate (with the boundary of AdS located at $z = 0$.) The actual map however is more complicated and involves in addition a variety of non-linear contributions from terms of lower order z^l, $l < k$. There are four sources of such non-linear contributions, as we now discuss.

Recall from chapter 1 that the holographic 1-point functions are derived by functionally differentiating the *renormalized* on-shell action w.r.t. the corresponding sources (see, for example, the review [21]). The most transparent way to describe the outcome of this computation is to use a radial Hamiltonian language where the radial coordinate plays the role of time. As we saw in section 1.5 the relationship (1.28) between 1-point functions and asymptotic coefficients is in general non-linear

For the case at hand, the first step is to reduce the 10 dimensional solution over T^4 or $K3$. We show that the fuzzball solutions reduce to solutions of 6-dimensional supergravity coupled to tensor multiplets. These solutions (in the decoupling limit) are asymptotic to $AdS_3 \times S^3$. The next step is to find the non-linear gauge invariant KK map from 6 to 3 dimensions. Following [22], this is done to second order in the fluctuations using (and extending) the results of [24, 48]. The results up to this order are sufficient to derive (after taking into account the subtle issue of extremal correlators) the vevs of all 1/2 BPS operators up to dimension 2. This includes in particular the conserved charges and the stress energy tensor. We emphasize that the non-linear terms are crucial in getting the right physics. We also discuss the vevs of higher dimension operators but these results are only qualitative as we did not compute the non-linear contributions; these could be computed along the lines described above, but the computation becomes very tedious. One point functions for this system have also been discussed in the context of black rings [49], although the non-linear terms (which play a crucial role) were not included there.

The final results for the vevs of the fuzzball solution are given in section 3.6. In particular, the vevs of the stress energy is (non-trivially) zero for all solutions, consistent with the fact that the solutions are supersymmetric. The vevs of the other operators are

$$\begin{aligned}
\left\langle \mathcal{O}_{S^1_i} \right\rangle &= \frac{n_1 n_5}{4\pi}(-4\sqrt{2} f^5_{1i}); \qquad (i{=}1,\dots,4) \\
\left\langle \mathcal{O}_{S^2_I} \right\rangle &= \frac{n_1 n_5}{4\pi}(\sqrt{6}(f^1_{2I} - f^5_{2I})); \qquad (I{=}1,\dots,9) \\
\left\langle \mathcal{O}_{\Sigma^2_I} \right\rangle &= \frac{n_1 n_5}{4\pi}\sqrt{2}(-(f^1_{2I} + f^5_{2I}) + 8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}); \qquad (\alpha{=}1,\dots,3) \\
\left\langle J^{+\alpha} \right\rangle &= \frac{n_1 n_5}{2\pi}a^{\alpha+}(dy - dt); \qquad \left\langle J^{-\alpha} \right\rangle = -\frac{n_1 n_5}{2\pi}a^{\alpha-}(dy + dt),
\end{aligned} \tag{3.14}$$

where $\mathcal{O}_{S_i^1}$ are dimension 1 operators, $\mathcal{O}_{S_I^2}, \mathcal{O}_{\Sigma_I^2}$ are dimension 2 operators,and $J^{\pm\alpha}$ are R-symmetry currents. These operators correspond to the lowest lying KK states, the KK spectrum consisting of two towers of spin 1 supermultiplets, the S and Σ towers, and a tower of spin 2 supermultiplets, which contain the gauge field that is dual to the R-symmetry current. The coefficients $f^5_{1i}, f^1_{2I}, f^5_{2I}, a^{\pm\alpha}$ appear in the asymptotic expansion of the harmonic functions that specify the solution, see (3.68)-(3.83), and $f_{I\alpha\beta}$ is a certain triple overlap of spherical harmonics. Expressed in terms of the defining curve F^i, the degree k coefficients involve symmetric rank k polynomials of F^i, see (3.71). In general, the vev of an operator of dimension k depends linearly on degree k coefficients and non-linearly on lower degree coefficients but such that the sum of degrees is k.

Any proposal for the field theory dual of these geometries should reproduce these vevs. Now, except when the curve is circular, operators charged wrt the R-symmetry acquire a vev. This implies immediately that the ground state of the field theory dual cannot be an eigenstate of R-symmetry since if this were the case only neutral operators would acquire a vev [43]. So none of the fuzzball solutions, except the circular ones, can correspond to a single R-ground state. Indeed, we have argued above (as in [43]) that these solutions should instead be dual to particular superpositions of R-ground states.

To test this proposal we discuss in some detail the case of the ellipse, comparing the vevs extracted from the supergravity solution with those implicit from the corresponding superposition of states in the field theory. We find complete matching for all kinematical properties of these vevs, thus demonstrating the consistency of our proposal. Moreover, the first dynamical test - matching of the R charges - is passed. To match the other vevs would require a knowledge of certain multiparticle three point functions at strong coupling, and is thus not currently possible. However, approximating the required three point functions using free harmonic oscillators leads to vevs which are in remarkable agreement with those extracted from the supergravity solution. This agreement suggests that certain three point functions in the dual CFT may be well approximated by free field computations, a result which in itself merits further investigation. Our proposal therefore passes all kinematical and all accessible dynamical tests, with other dynamical tests requiring going beyond the supergravity approximation.

Given that the original fuzzball solutions do not correspond to single R ground states, one may wonder whether there are other supergravity solutions that do correspond to a given R ground state. A necessary condition for this would be that the vevs of all charged operators are all zero, and this will only be the case if the solution preserves an $SO(2) \times SO(2)$ symmetry (among the original solutions only the circular one had this symmetry). We give the most general asymptotic supergravity solution consistent with these requirements. Different solutions with such asymptotics are parametrized by the vevs of the neutral operators, and to obtain these vevs one needs the complete solutions.

One way to produce solutions with an $SO(2) \times SO(2)$ symmetry is to take appropriate superpositions of the non-symmetric solutions. We discuss how to do such an averaging in general and we work out the details for the ellipse and for a curve that is a straight line followed by a semi-circle. This latter case yields the Aichelburg-Sexl metric namely the metric describing a massless particle moving along a greater circle on S^3 and sitting at the center of AdS_3. Solutions with the same $SO(2) \times SO(2)$ symmetry can also be produced using disconnected circular curves; one would expect that such solutions are related to Coulomb rather than Higgs branch physics.

We then discuss the relationship between such symmetric geometries and R ground states. We argue that the vevs for neutral operators in a particular ground state can be related to three point functions at the conformal point. Thus with knowledge of the latter one can distinguish whether a given geometry corresponds to a particular R ground state. However, we find that implementing this procedure generically requires going beyond the leading supergravity approximation: one would need to know three point functions of multi particle operators, not captured by supergravity, as well as $1/N$ corrections. Thus we cannot currently determine which geometries are indeed dual to R ground states; indeed even the solutions based on disconnected curves (which should be Coulomb branch) could not be ruled out.

So what do hese results imply for the fuzzball program? Firstly, they support the overall picture; the fuzzball solutions can be in correspondence with the black hole microstates in a way that is compatible with the AdS/CFT correspondence and our computations provide the most stringent test to date. The detailed correspondence however is more complicated than anticipated. In particular a generic fuzzball solution corresponds to a superposition of many R ground states, and in general one would need to go beyond the leading supergravity to properly describe the system, even in this simplest 2-charge system. It has long been appreciated that most of the fuzzball solutions, despite being regular, have regions of high curvature so are at best extrapolations of the actual solutions describing the microstates. Here we see that even for solutions with low curvature everywhere, such as the ones based on large ellipses, one needs to go beyond the leading supergravity to test any proposed correspondence.

There has been a lot of interest in finding and analyzing fuzzball geometries in systems with more charges which have classical horizons [50] but a precise matching between these geometries and black hole microstates has not been established. Such a matching is clearly necessary, both to demonstrate that the correct geometries have been identified and to find for what fraction of the total entropy these account. A precise correspondence would also be important in understanding the quantization of the geometries and, most importantly of all, how the black hole properties emerge.

A key result of our work is that the vevs encoded by a given geometry give significant information about the field theory dual, and distinguish between geometries with the same charges (mass, angular momentum). In particular, dipole and higher multipole moments are related to the vevs of operators with dimension two or greater. Vevs determined by kinematics can by themselves rule out proposed correspondences, as shown in [43] and here, and vevs determined by dynamics are strong tests of a given proposal, when they can be computed on both sides. In particular, whilst our solutions based on disconnected curves pass all kinematical tests to correspond to R ground states on the Higgs branch, they should be ruled out by dynamical tests.

Previous work has often focused on computing two point functions and relating them to those in the dual field theory, and vice versa, see for example [51], but extracting vevs is much easier, since one needs only the geometry itself, rather than solving fluctuation equations in the geometry. Thus one can easily extract vevs from geometries with few symmetries, where the corresponding fluctuation equations are intractable. It hence seems worthwhile to explore whether the techniques developed here can give useful information in the context of other fuzzball geometries. One can analyze any fuzzball geometry which has a throat region using AdS/CFT techniques, with the formalism developed here being directly applicable to three charge black strings in six dimensions. Black rings in six dimensions could also be explored using the same formalism; indeed the extracted data should uniquely identify the field theory dual.

The plan of this chapter is as follows. In section 3.2 we discuss the relationship between solitonic string supergravity solutions and coherent states of the fundamental string. In section 3.3 we introduce the dual solutions in the D1-D5 system, and discuss the embedding of their decoupling limit into 6-dimensional supergravity. In section 3.4 we discuss the asymptotic expansion of these six dimensional solutions near the $AdS_3 \times S^3$ boundary. In section 3.5 we explain how the vevs of field theory operators can be extracted from these asymptotics. In section 3.6 we give the explicit values of these vevs for the fuzzball solutions in full generality, and in section 3.7 we specialize to the examples of solutions sourced by circular and elliptical curves. In section 3.8 we recall relevant features of the dual field theory, and discuss how the vevs can be related to three point functions at the conformal point. In section 3.9 we move on to the correspondence between fuzzball geometries and superpositions of chiral primaries, giving evidence for our proposed correspondence in terms of the matching of the vevs for the ellipsoidal case. In section 3.10 we discuss the asymptotics of a geometry dual to a single chiral primary, and give some examples of solutions which have such asymptotics. In section 3.11 we discuss the correspondence between symmetric geometries and chiral primaries, emphasizing that dynamical tests require going beyond the leading supergravity approximation. In section 3.12 we discuss how the asymptotically flat part of the geometry can be included in the field theory description.

Throughout this chapter we use a number of technical results which are contained in appendices. Appendix 3.A.1 contains various properties of S^3 spherical harmonics whilst appendix 3.A.2 proves an addition theorem for harmonic functions on R^4. Appendix 3.A.3 discusses the perturbative expansion of six-dimensional field equations about the $AdS_3 \times S^3$ background. Appendix 3.A.4 discusses the supergravity computation of certain three point functions, whilst appendix 3.A.5 contains a derivation of the one point function for the energy momentum tensor in this system. Appendix 3.A.6 concerns the three point functions in the orbifold CFT; we argue that these differ from those computed in supergravity and that they are therefore not protected by any non-renormalization theorem.

(3.2) FP SYSTEM AND PERTURBATIVE STATES

We begin by discussing solitonic string supergravity solutions and their relation to perturbative string states. The FP solutions are characterized by a curve $F^I(x^+)$ describing the transverse displacement of the string. For later purposes only 4 transverse directions will be excited so the curve is confined to R^4 but for now we keep the discussion general. The supergravity solution describing an oscillating string is given by [33, 34]

$$ds^2 = H^{-1}(-dx^- dx^+ + K(dx^+)^2 - 2A_I dx^I dx^+) + dx_I dx_I$$
$$H = 1 + \frac{Q_f}{|\vec{x} - \vec{F}(x^+)|^6}, \quad K = \frac{Q_f |\dot{F}|^2}{|\vec{x} - \vec{F}(x^+)|^6}, \quad A_I = \frac{Q_f \dot{F}_I}{|\vec{x} - \vec{F}(x^+)|^6} \tag{3.15}$$

with suitable B field and dilaton. Here $x^{\pm} = x^0 \pm x^9$ are lightcone coordinates, $\vec{x}$ are 8 transverse coordinates and $x^9 \equiv x^9 + 2\pi R_9$. $\dot{F}_I$ denotes the derivative with respect to x^+. The fundamental string charge Q_f is proportional to the number of fundamental strings. The ADM mass and momentum along the compact direction are respectively [33, 34]

$$M = kQ_f(1 + |\dot{F}|_0^2); \qquad P^9 = -kQ_f|\dot{F}|_0^2, \tag{3.16}$$

where the subscript denotes the zero mode and $k = 3\omega_7/2\kappa^2$ with ω_7 the volume of the S^7. The angular momenta in the transverse directions are similarly given by

$$J^{IJ} = kQ_f(F^J \dot{F}^I - F^I \dot{F}^J)_0. \tag{3.17}$$

As we will review below, these are exactly the conserved quantities of a string which wraps around the compact direction w times and whose transverse profile is given by F^I.

(3.2.1) STRING QUANTIZATION

To relate the supergravity solutions to perturbative string states, let us consider quantizing a string propagating in a flat background; we discuss this in some detail since the preferred gauge choice is a non standard light cone gauge. The relevant part of the worldsheet action is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_+ X^M \partial_- X_M + \cdots), \tag{3.18}$$

where the worldsheet metric is gauge fixed to $-g_{\tau\tau} = g_{\sigma\sigma} = 1$. Fermions will not play any role in the discussion here and will be suppressed. We will also set $\alpha' = 2$ to simplify formulae. Null worldsheet coordinates are introduced by setting $\sigma^{\pm} = (\tau \pm \sigma)$ and a lightcone gauge can be chosen for V such that

$$X^+ = (w^+ \sigma^+ + \tilde{w}^+ \sigma^-). \tag{3.19}$$

A similar choice of lightcone gauge for open strings has been discussed in [52]. The other fields are then expanded in harmonics as

$$\begin{aligned} X^- &= x^- + (w^-\sigma^+ + \tilde{w}^-\sigma^-) + \sum_n \frac{1}{\sqrt{|n|}}(a_n^- e^{-in\sigma^+} + \tilde{a}_n^- e^{-in\sigma^-}); \\ X^I &= x^I + p^I(\sigma^+ + \sigma^-) + \sum_n \frac{1}{\sqrt{|n|}}(a_n^I e^{-in\sigma^+} + \tilde{a}_n^I e^{-in\sigma^-}). \end{aligned} \tag{3.20}$$

Reality of X^M demands that $a_{-n}^M = (a_n^M)^\dagger$. The Virasoro constraints are

$$T_{++} = \partial_+ X^M \partial_+ X_M = 0; \qquad T_{--} = \partial_- X^M \partial_- X_M = 0. \tag{3.21}$$

At the classical level this enforces

$$\begin{aligned} (-w^+w^- + (p^I)^2)\delta_{m,0} + i\frac{m}{\sqrt{|m|}}(w^+a_m^- - 2p^I a_m^I) + \sum_n \frac{n(n-m)}{\sqrt{|n(n-m)|}} a_n^I a_{m-n}^I = 0; \\ (-\tilde{w}^+\tilde{w}^- + (p^I)^2)\delta_{m,0} + i\frac{m}{\sqrt{|m|}}(\tilde{w}^+\tilde{a}_m^- - 2p^I \tilde{a}_m^I) + \sum_n \frac{n(n-m)}{\sqrt{|n(n-m)|}} \tilde{a}_n^I \tilde{a}_{m-n}^I = 0, \end{aligned}$$

thereby determining the non-dynamical field X^- in terms of the dynamical transverse fields X^I, as in standard lightcone gauge. The conserved momentum and winding charges are given by

$$P^M = \frac{1}{4\pi}\int_0^{2\pi} d\sigma(\partial_\tau X^M); \qquad W^M = \frac{1}{2\pi}\int_0^{2\pi} d\sigma(\partial_\sigma X^M), \tag{3.22}$$

which take the values

$$\begin{aligned} P^M &= \left(\frac{1}{4}(w^- + w^+ + \tilde{w}^- + \tilde{w}^+), \frac{1}{4}(w^+ - w^- + \tilde{w}^+ - \tilde{w}^-), p^I\right); \\ W^M &= \left(\frac{1}{2}(w^- + w^+ - \tilde{w}^- - \tilde{w}^+), \frac{1}{2}(w^+ - w^- - \tilde{w}^+ + \tilde{w}^-), 0\right). \end{aligned} \tag{3.23}$$

In order for the string not to wind the time direction, one thus needs

$$W^0 = \frac{1}{2}(w^- + w^+ - \tilde{w}^- - \tilde{w}^+) = 0. \tag{3.24}$$

We are interested in states with only left moving excitations and no transverse momentum, namely the $\tilde{w}^+ = 0$ sector. For these the momentum and winding charges are

$$\begin{aligned} P^M &= (\tfrac{1}{2}wR_9 - \frac{p_9}{R_9}, \frac{p_9}{R_9}, 0); \qquad W^M = (0, wR_9, 0); \\ w^+ &\equiv wR_9; \qquad w^- \equiv -2\frac{p_9}{R_9}. \end{aligned} \tag{3.25}$$

Restricting to such states the L_0 constraint becomes

$$p_9 w + \sum_{n>0} n a_{-n}^I a_n^I \equiv p_9 w + N_L = 0. \tag{3.26}$$

The angular momenta in the transverse directions are given by the usual expressions

$$J^{IJ} = \frac{1}{4\pi}\int_0^{2\pi} d\sigma(X^J\partial_\tau X^I - X^I\partial_\tau X^J) = -i\sum_{n>0}(a_{-n}^I a_n^J - a_{-n}^J a_n^I). \tag{3.27}$$

Quantization proceeds in the standard way, with the oscillators satisfying the commutation relations

$$\left[\hat{a}^I_n, (\hat{a}^J_m)^\dagger\right] = \delta_{m,n}\delta^{IJ}, \tag{3.28}$$

and states being built out of creation operators $(\hat{a}^I_m)^\dagger$ acting on the vacuum. The classical expressions continue to hold, replacing a^I_m by operators $\hat{a}^I_m$, with appropriate shift in L_0 (which is negligible in the large charge limit).

(3.2.2) Relation to classical curves

On rather general grounds, one expects that the supergravity solution characterized by a null curve corresponds to a coherent state of string oscillators. To be more precise, let us Fourier expand the classical curve

$$F^I(x^+) = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\alpha^I_n e^{-in\sigma^+} + (\alpha^I_n)^* e^{in\sigma^+}\right) \tag{3.29}$$

where α^I_n are (complex) numerical coefficients and $x^+ = wR_9\sigma^+$. Then the coherent state $|F^I)$ of string oscillators that corresponds to this curve is given by

$$|F^I) = \prod_{n,I} |\alpha^I_n) \tag{3.30}$$

where $|\alpha^I_n)$ is a coherent state of the oscillator $\hat{a}^I_n$, i.e. it satisfies,

$$\hat{a}|\alpha) = \alpha|\alpha) \tag{3.31}$$

where we suppress the super and subscripts for clarity. Recall the coherent states are related to the Fock states by

$$|\alpha) = e^{-|\alpha|^2/2} \sum_k \frac{\alpha^k}{\sqrt{k!}} |k\rangle \tag{3.32}$$

and

$$|k\rangle = \frac{1}{\sqrt{k!}} (\hat{a}^\dagger)^k |0\rangle \tag{3.33}$$

is the standard kth excited state. By construction

$$(F^I|\hat{N}_L|F^I) \equiv N_L = \sum_{n>0} n|\alpha^I_n|^2. \tag{3.34}$$

From (3.26) and (3.25) we find that

$$(F^I|\hat{P}^0|F^I) = (\tfrac{1}{2}wR_9 + \frac{1}{wR_9}N_L); \qquad (F^I|\hat{P}^9|F^I) = -\frac{1}{wR_9}N_L. \tag{3.35}$$

Now note that the zero mode of $(\dot{F}^I)^2$ is given by $2N_L/(wR_9)^2$. This means that the mass and momentum of the supergravity solution associated with this curve are, using (3.16),

$$M = kQ_f(1 + \frac{2N_L}{(wR_9)^2}); \qquad P^9 = -kQ_f \frac{2N_L}{(wR_9)^2}, \tag{3.36}$$

which agree with the expressions (3.35) provided that

$$kQ_f = \tfrac{1}{2} w R_9, \tag{3.37}$$

which is the relationship found in [33, 34]. Moreover,

$$(F^I|\hat{J}^{IJ}|F^I) = \frac{1}{2} w R_9 (F^J \dot{F}^I - F^I \dot{F}^J)_0, \tag{3.38}$$

which manifestly agrees with the expression (3.17).

(3.2.3) Examples

Consider an elliptical curve in the 1-2 plane, such that

$$F^1 = \frac{\sqrt{2N}}{n} a \cos(n\sigma^+); \qquad F^2 = \frac{\sqrt{2N}}{n} b \sin(n\sigma^+), \tag{3.39}$$

with $(a^2 + b^2) = 2$; this case was discussed in the introduction around (3.8) and (3.12). The amplitude of the curve is fixed such that the angular momentum in the 1-2 plane is

$$J^{12} = -\frac{N}{n} ab, \tag{3.40}$$

and the total excitation number defined in (3.34) is $N_L = N = -wp_9$. This ensures that the mass and momenta match that of the supergravity solution, as described in the previous subsection.

Introducing the usual combinations of oscillators with definite angular momenta in the 1-2 plane

$$\hat{a}_n^{\pm 12} \equiv \frac{1}{\sqrt{2}} (\hat{a}_n^1 \pm i \hat{a}_n^2), \tag{3.41}$$

the coherent state corresponding to the curve is

$$|a_n^{-12}; a_n^{+12}) = |\frac{\sqrt{N}}{2\sqrt{n}}(a+b); \frac{\sqrt{N}}{2\sqrt{n}}(a-b)), \tag{3.42}$$

which in the case of the circle ($\alpha = \beta$) reduces to (3.9). Extracting from this coherent state those states which satisfy $N_L = N$ gives

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n} - k)! k!}} (a+b)^{\frac{N}{n} - k} (a-b)^k |k_{-12} = (\frac{N}{n} - k); k_{+12} = k\rangle, \tag{3.43}$$

which leads to the corresponding superposition (3.13) in the dual D1-D5 system.

(3.3) The fuzzball solutions

We now consider the two charge fuzzball solutions in the D1-D5 system, obtained from the FP chiral null models by a chain of dualities. These fuzzball solutions were constructed by Lunin and Mathur [53, 27] and are given by

$$\begin{aligned} ds^2 &= f_1^{-1/2} f_5^{-1/2} \left(-(dt - A)^2 + (dy + B)^2 \right) + f_1^{1/2} f_5^{1/2} dx \cdot dx + f_1^{1/2} f_5^{-1/2} dz \cdot dz; \\ e^{2\Phi} &= f_1 f_5^{-1}; \\ C_{ti} &= f_1^{-1} B_i; \qquad C_{ty} = f_1^{-1}; \\ C_{yi} &= f_1^{-1} A_i; \qquad C_{ij} = c_{ij} - f_1^{-1}(A_i B_j - A_j B_i), \end{aligned} \tag{3.44}$$

where i, j are vector indices in the transverse R^4 and the metric is in the string frame. These fields solve the equations of motion following from the type IIB action

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{-2\Phi}(R_{10} + 4(\partial\Phi)^2) - \frac{1}{12} F_3^2 + \cdots \right), \tag{3.45}$$

where F_3 is the curvature of the two form C and $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$ (we set $g_s = 1$ since it plays no role in our discussion), provided the following equations hold

$$\begin{aligned} &dc = *_4 df_5, \qquad dB = *_4 dA, \\ &\Box_4 f_1 = \Box_4 f_5 = \Box_4 A_i = 0, \qquad \partial^i A_i = 0. \end{aligned} \tag{3.46}$$

where the Hodge dual $*_4$ and $\Box_4$ are defined on the four (flat) non-compact overall transverse directions x^i. The compact part of the geometry does not play a role; it could be either T^4 or $K3$.

A solution to the conditions (3.46) based on an arbitrary closed curve $F^i(v)$ of length L in R^4 is given by

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - F|^2}; \qquad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv |\dot{F}|^2}{|x - F|^2}; \qquad A_i = \frac{Q_5}{L} \int_0^L \frac{\dot{F}_i dv}{|x - F|^2}. \tag{3.47}$$

It was argued in [27] that these solutions are related to the R ground states (and via spectral flow to chiral primaries [28]) common to both the T^4 and $K3$ CFTs. The physical interpretation of these solutions is that the D1 and D5 brane sources are distributed on a curve in the transverse R^4. The D5-branes are uniformly distributed along this curve, but the D1-brane density at any point on the curve depends on the tangent to the curve. The total one brane charge is given by

$$Q_1 = \frac{Q_5}{L} \int_0^L |\dot{F}|^2 dv. \tag{3.48}$$

Both the Q_i have dimensions of length squared and are related to the integral charges by

$$Q_1 = \frac{(\alpha')^3 n_1}{V}; \qquad Q_5 = \alpha' n_5, \tag{3.49}$$

where $(2\pi)^4 V$ is the volume of the compact manifold. Furthermore, the length of the curve is given by

$$L = 2\pi Q_5/R, \tag{3.50}$$

where R is the radius of the y circle.

The holographic analysis in this chapter will be done for the general class of solutions (3.44) satisfying (3.46). Results appropriate for the solutions determined by (3.47) will be obtained by specializing the general results to this case and we will indicate how this is done at each step of the analysis.

(3.3.1) COMPACTIFICATION TO SIX DIMENSIONS

Since only the breathing mode of the compact manifold is excited, it is convenient to compactify and work with solutions of six-dimensional supergravity. The effective six-dimensional (Einstein) metric coincides with the six-dimensional part of the (string frame) metric above (because the would be six-dimensional dilaton $\phi_6 = \Phi - \frac{1}{4}\ln \det g_{M^4}$, where g_{M^4} is the metric on the compact space, is constant). Thus the six-dimensional metric

$$ds^2 = f_1^{-1/2} f_5^{-1/2}\left(-(dt-A)^2 + (dy+B)^2\right) + f_1^{1/2} f_5^{1/2} dx\cdot dx \tag{3.51}$$

along with the scalar field and tensor field of (3.44) satisfy the equations of motion following from the reduced action

$$S = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-g}\left(R - (\partial\Phi)^2 - \frac{1}{12}e^{2\Phi}F_3^2\right), \tag{3.52}$$

where R is the six-dimensional curvature and F_3 is the curvature of the antisymmetric tensor field C. These equations of motion are

$$\begin{aligned} R_{MN} &= \tfrac{1}{4}e^{2\Phi}(F_{MPQ}F_N{}^{PQ} - \frac{1}{6}F^2 g_{MN}) + \partial_M\Phi\partial_N\Phi; \\ D_M(e^{2\Phi}F^{MNP}) &= 0; \qquad \Box\Phi = \frac{1}{12}e^{2\Phi}F^2. \end{aligned} \tag{3.53}$$

Note that the six-dimensional scalar field originates from the breathing mode of the compactification manifold.

The equations of motion which follow from the action (3.52) can be embedded into those of $d=6$, $N=4b$ supergravity coupled to n_t tensor multiplets, the covariant field equations for which were constructed in [54]. The bosonic field content of this theory is the graviton and five self-dual tensor fields from the supergravity multiplet, along with n_t anti-self dual tensor fields and $5n_t$ scalars from the tensor multiplets.

Following the notation of [55, 24] the bosonic field equations may be written as

$$R_{MN} = H^m_{MPQ}H^m_N{}^{PQ} + H^r_{MPQ}H^r_N{}^{PQ} + 2P^{mr}_M P^{mr}_N; \tag{3.54}$$

$$D^M P^{mr}_M = \frac{\sqrt{2}}{3}H^{mMNP}H^r_{MNP}, \tag{3.55}$$

along with Hodge duality conditions on the 3-forms

$$H^m_{MNP} = \frac{1}{6}\epsilon_{MNPQRS}H^{mQRS}; \qquad H^r_{MNP} = -\frac{1}{6}\epsilon_{MNPQRS}H^{rQRS}. \tag{3.56}$$

In these equations m, n are $SO(5)$ vector indices running from 1 to 5 whilst r, s are $SO(n_t)$ vector indices running from 6 to $5 + n_t$. The three form field strengths are given by

$$H^m = G^A V^m_A; \qquad H^r = G^A V^r_A, \tag{3.57}$$

where $A \equiv \{n, r\} = 1, \cdots, 5{+}n_t$; $dG^A = 0$ and the vielbein on the coset space $SO(5, n_t)/(SO(5)\times SO(n_t)$ satisfies

$$V^m_A V^m_B - V^r_A V^r_B = \eta_{AB}, \tag{3.58}$$

with $\eta_{AB} = \mathrm{diag}(+++++---\cdots-)$. The associated connection is

$$dVV^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2}P^{ms} \\ \sqrt{2}P^{nr} & Q^{rs} \end{pmatrix}. \tag{3.59}$$

The equations of motion (3.53) can be embedded into this theory using an $SO(1, 1)$ subgroup as follows. Let

$$V_5^{m=5} = \cosh(\Phi); \qquad V_6^{m=5} = \sinh(\Phi); \qquad V_5^{r=6} = \sinh(\Phi); \qquad V_6^{r=6} = \cosh(\Phi), \tag{3.60}$$

so that the connection is $\sqrt{2}P^{56} = d\Phi$. Now let[5]

$$G^5 = \tfrac{1}{4}(F + e^{2\Phi} *_6 F); \qquad G^6 = \tfrac{1}{4}(F - e^{2\Phi} *_6 F), \tag{3.61}$$

which are both closed using the three form equation in (3.53). This implies that

$$H^{m=5} = \tfrac{1}{4}e^{\Phi}(F + *_6F); \qquad H^{r=6} = \tfrac{1}{4}e^{\Phi}(F - *_6F), \tag{3.62}$$

which manifestly have the correct Hodge duality properties to satisfy (3.56). Substituting H and P into (3.54) also correctly reproduces the Einstein and scalar field equations of (3.53).

Since this embedding uses only an $SO(1, 1)$ subgroup it does not depend on the details of the compactification manifold. Thus one can use this six-dimensional supergravity to analyze the fuzzball geometries in both T^4 and $K3$ systems. More generally, the (anomaly free) case of $n_t = 21$ gives the complete six dimensional theory obtained by K3 compactification of type IIB supergravity. For T^4 compactification of type IIB one obtains the maximally supersymmetric non-chiral six-dimensional theory, whose field content is a graviton, eight gravitinos, 5 self-dual and 5 anti-self dual three forms, 16 gauge fields, 40 fermions and 25 scalars. (Bosonic) solutions of this supergravity which do not have gauge fields switched on are solutions of the chiral supergravity given above, with $n_t = 5$.

[5] The field strengths G^5 and G^6 were called $G^{\pm}$ in [43].

(3.3.2) Asymptotically AdS limit

In the appropriate decoupling limit, the solutions (3.44) become asymptotically AdS. This corresponds to harmonic functions with leading behavior r^{-2}. In terms of the harmonic functions in (3.47) the decoupling limit amounts to removing the constant terms in the harmonic functions f_1 and f_5. (Later on in section 3.12 we will discuss the interpretation of these constant terms in the dual CFT.) The solutions are then manifestly asymptotic to $AdS_3 \times S^3$ as $r \to \infty$. Firstly the metric asymptotes to

$$ds_6^2 = \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \sqrt{Q_1 Q_5}\left(\frac{dr^2}{r^2} + d\Omega_3^2\right); \tag{3.63}$$

whilst the three-forms and scalar field from (3.44) asymptote to

$$F_{rty} = \frac{2r}{Q_1}; \qquad F_{\Omega_3} = 2Q_5; \qquad e^{2\Phi_0} = \frac{Q_1}{Q_5}. \tag{3.64}$$

It is convenient to shift the scalar field so that $\Phi \to \Phi - \Phi_0$ and rescale $G^5 \to e^{\Phi_0} G^5$ and same for G^6. Then the relevant background fields of the six-dimensional supergravity are

$$\begin{aligned} g^{o(m=5)} &= H^{o(m=5)} = \frac{r}{\sqrt{Q_1 Q_5}} dr \wedge dt \wedge dy + \sqrt{Q_1 Q_5} d\Omega_3; \\ V_5^{o(m=5)} &= 1; \qquad V_6^{o(r=6)} = 1, \end{aligned} \tag{3.65}$$

with the off-diagonal components of the vielbein vanishing; the anti-self dual field $g^{o(r=6)} = H^{r=6}$ vanishing and Φ being zero also. Note that with the coordinate rescalings $t \to t\sqrt{Q_1 Q_5}$ and $y \to y\sqrt{Q_1 Q_5}$, the curvature radius appears only as an overall scaling factor in both the metric (3.63) and the three form (3.65). When one rescales the coordinates in this way, the new y coordinate will have periodicity $\tilde{R} = R/\sqrt{Q_1 Q_5}$.

The goal is to extract from the subleading asymptotics around the AdS boundary the vevs of chiral primaries in the dual theory, and thus investigate the matching with R vacua. The strategy is as follows. First one expands the solution systematically near the AdS boundary. Then one extracts from the asymptotic solution the values of 6-dimensional gauge invariant fields. These must then be reduced to three dimensional fields using the KK map, and then the vevs can be extracted using holographic renormalization.

(3.4) Harmonic expansion of fluctuations

Let us consider the asymptotic expansion of the solution. The perturbations of the six-dimensional supergravity fields relative to the $AdS_3 \times S^3$ background can be expressed as

$$g_{MN} = g^o_{MN} + h_{MN}; \qquad G^A = g^{oA} + g^A; \qquad \phi^{mr}. \tag{3.66}$$

These fluctuations can then be expanded in spherical harmonics as follows:

$$\begin{aligned}
h_{\mu\nu} &= \sum h^I_{\mu\nu}(x)Y^I(y), \qquad (3.67)\\
h_{\mu a} &= \sum (h^{I_v}_\mu(x)Y^{I_v}_a(y) + h^I_{(s)\mu}(x)D_aY^I(y)),\\
h_{(ab)} &= \sum (\rho^{I_t}(x)Y^{I_t}_{(ab)}(y) + \rho^{I_v}_{(v)}(x)D_aY^{I_v}_b(y) + \rho^I_{(s)}(x)D_{(a}D_{b)}Y^I(y)),\\
h^a_a &= \sum \pi^I(x)Y^I(y),\\
g^A_{\mu\nu\rho} &= \sum 3D_{[\mu}b^{(A)I}_{\nu\rho]}(x)Y^I(y),\\
g^A_{\mu\nu a} &= \sum (b^{(A)I}_{\mu\nu}(x)D_aY^I(y) + 2D_{[\mu}Z^{(A)I_v}_{\nu]}(x)Y^{I_v}_a(y));\\
g^A_{\mu ab} &= \sum (D_\mu U^{(A)I}(x)\epsilon_{abc}D^cY^I(y) + 2Z^{(A)I_v}_\mu D_{[b}Y^{I_v}_{a]});\\
g^A_{abc} &= \sum (-\epsilon_{abc}\Lambda^I U^{(A)I}(x)Y^I(y));\\
\phi^{mr} &= \sum \phi^{(mr)I}(x)Y^I(y),
\end{aligned}$$

Here (μ,ν) are AdS indices and (a,b) are S^3 indices, with x denoting AdS coordinates and y denoting sphere coordinates. The subscript (ab) denotes symmetrization of indices a and b with the trace removed. Relevant properties of the spherical harmonics are reviewed in appendix 3.A.1. We will often use a notation where we replace the index I by the degree of the harmonic k or by a pair of indices (k,I) where k is the degree of the harmonic and I now parametrizes their degeneracy, and similarly for I_v, I_t.

Imposing the de Donder gauge condition $D^Ah_{aM} = 0$ on the metric fluctuations removes the fields with subscripts (s,v). In deriving the spectrum and computing correlation functions, this is therefore a convenient choice. The de Donder gauge choice is however not always a convenient choice for the asymptotic expansion of solutions; indeed the natural coordinate choice in our application takes us outside de Donder gauge. As discussed in [22] this issue is straightforwardly dealt with by working with gauge invariant combinations of the fluctuations; we will present the relevant gauge invariant combinations later.

(3.4.1) Asymptotic expansion of the fuzzball solutions

Now consider the asymptotic expansion at large radius of the fuzzball solutions. The natural radial coordinate in which to expand the solutions is the radial coordinate r of the transverse R^4, even though with this choice it will turn out that the metric is not in de Donder gauge.

The harmonic functions appearing in the solution (3.44) can be expanded as

$$\begin{aligned}
f_5 &= \frac{Q_5}{r^2}\sum_{k,I}\frac{f^5_{kI}Y^I_k(\theta_3)}{r^k};\\
f_1 &= \frac{Q_1}{r^2}\sum_{k,I}\frac{f^1_{kI}Y^I_k(\theta_3)}{r^k}; \qquad (3.68)\\
A_i &= \frac{Q_5}{r^2}\sum_{k,I}\frac{(A_{kI})_iY^I_k(\theta_3)}{r^k},
\end{aligned}$$

for some coefficients f^5_{kI}, f^1_{kI} and $(A_{kI})_i$. There are restrictions on the coefficients $(A_{kI})_i$ because $\partial^i A_i = 0$ which will be given below.

In the case of the (near-horizon) harmonic functions of (3.47), the coefficients $f^5_{kI}, f^1_{kI}, (A_{kI})_i$ are given in terms of the curve $F^i(v)$. To obtain these coefficients we make use of the following addition theorem for harmonic functions on R^4:

$$\frac{1}{(x^i - y^i)^2} = \sum_{k\geq 0} \frac{y^k}{(k+1)r^{2+k}} Y^I_k(\theta^x_3) Y^I_k(\theta^y_3). \tag{3.69}$$

In this expression x^i and y^i are Cartesian coordinates on R^4, with the corresponding polar coordinates being (r, θ^x_3) and (y, θ^y_3) respectively. $Y^I_k(\theta_3)$ are (normalized) spherical harmonics of degree k on S^3 with I labeling their degeneracy; the degeneracy of degree k harmonics is $(k+1)^2$. For the $k = 1$ harmonics of degeneracy four, it is convenient to use the label i, Y^i_1. The addition theorem can also be expressed as

$$\frac{1}{|x-y|^2} = \sum_{k\geq 0} \frac{1}{(k+1)r^{2+k}} Y^I_k(\theta^x_3)(C^I_{i_1\cdots i_k} y^{i_1}\cdots y^{i_k}), \tag{3.70}$$

where $C^I_{i_1\cdots i_k}$ are the orthogonal symmetric traceless rank k tensors on R^4 which are in one-to-one correspondence with the (normalized) spherical harmonics $Y^I_k(\theta_3)$ of degree k on the S^3. This formula is the exact analogue of the well-known addition theorem for electromagnetism (see [56]) and also of the addition theorem for harmonic functions on R^6 discussed in the appendix of [57], and it can be proved in the same way, as we show in appendix 3.A.2.

Using the addition theorem we obtain

$$\begin{aligned} f^5_{kI} &= \frac{1}{(k+1)L}\int_0^L dv C^I_{i_1\cdots i_k} F^{i_1}\cdots F^{i_k}; \\ f^1_{kI} &= \frac{Q_5}{Q_1(k+1)L}\int_0^L dv\, |\dot{F}|^2\, C^I_{i_1\cdots i_k} F^{i_1}\cdots F^{i_k}; \\ (A_{kI})_i &= \frac{1}{(k+1)L}\int_0^L dv \dot{F}_i C^I_{i_1\cdots i_k} F^{i_1}\cdots F^{i_k}. \end{aligned} \tag{3.71}$$

Furthermore, in the final equality of (3.68) the summation is restricted to $k \geq 1$ because of the closure of the curve F^i ($\int dv\dot{F}_i = 0$). Note that we will often suppress implicit summations over the index I in later expressions for compactness.

Before substituting these expressions into the supergravity fields, we need to consider which fluctuations are physical. Suppose we use translational invariance to impose the condition

$$\int_0^L dv F_i = 0, \tag{3.72}$$

which was the choice made in previous literature, for example, in [27]. This corresponds to choosing the origin of the coordinate system to be at the center of mass of the D5-branes. However, the center of mass of the D1-branes does not coincide with that of the D5-branes in general; thus this condition does *not* take one to the center of mass of the whole system.

Indeed with this choice the leading correction to the AdS background derives from the $k = 1$ terms in the harmonic function f_1. The choice (3.72) gives a leading metric deviation

$$h_{\mu\nu} = D_\mu D_\nu \lambda; \qquad h_{ab} = g_{ab}\lambda, \tag{3.73}$$

with

$$\lambda = \sum_i \frac{f^1_{1i} Y^i_1}{2r}, \tag{3.74}$$

which satisfies $\Box\lambda = -\lambda$. Such a perturbation is unphysical because it can be removed by a superconformal transformation (with parameter $-\lambda$). The physical origin of the term is that with the choice (3.72) we are not working in the centre of mass of the system. Instead of imposing that the $k = 1$ term in the D5-brane harmonic function vanishes, we should impose that the $k = 1$ term in $\sqrt{f_1 f_5}$ vanishes, namely

$$f^5_{1i} + f^1_{1i} = 0. \tag{3.75}$$

When the solution is related to a closed curve this reduces to

$$\int_0^L dv F^i (1 + \frac{Q_5}{Q_1} |\dot{F}|^2) = 0. \tag{3.76}$$

Then all unphysical $k = 1$ terms in the metric vanish automatically.

Now consider the asymptotic expansion of A_i. The restriction on the coefficients in the asymptotic expansion imposed by the condition $\partial_i A^i = 0$ is most easily understood as follows. The form A may be written as

$$A = Q_5 \sum_{k,I,i} \frac{(A_{kI})_i}{r^{2+k}} Y^I_k \left(Y^i_1 dr + r dY^i_1 \right), \tag{3.77}$$

using

$$dx^i = dr Y^i_1 + r dY^i_1. \tag{3.78}$$

Projecting the products of spherical harmonics onto the basis of spherical harmonics gives

$$\begin{aligned} A &= Q_5 \sum_{l,L,k,I,i} \frac{(A_{kI})_i}{r^{2+k}} (a_{iIL} Y_l^L dr + \frac{b_{IiL}}{\Lambda^L} r dY_l^L) \\ &+ Q_5 \sum_{k_v, I_v, k, I, i} \frac{(A_{kI})_i}{r^{1+k}} E^{\pm}_{I_v I i} Y^{I_v \pm}_{k_v}, \end{aligned} \tag{3.79}$$

where the spherical harmonic overlaps $(a_{iIJ}, b_{IiJ}, E^{\pm}_{I_v I i})$ are defined in (3.226), (3.225) and (3.229) respectively. The term in A proportional to the vector harmonic is already divergenceless on its own. The first two combine into divergenceless combination iff scalar harmonics with degree $l = (k-1)$ appear in this asymptotic expansion:

$$\begin{aligned} A &= Q_5 \sum_{L,k,I,i} \frac{(A_{kI})_i}{r^{2+k}} a_{IiL} (Y^L_{k-1} dr - \frac{r}{(1+k)} dY^L_{k-1}) \\ &+ Q_5 \sum_{k_v, I_v, k, I, i} \frac{(A_{kI})_i}{r^{1+k}} E^{\pm}_{I_v I i} Y^{I_v \pm}_{k_v}. \end{aligned} \tag{3.80}$$

Vanishing of the other terms requires

$$\sum_{I,i}(A_{kI})_i a_{iIL} = 0 \qquad l \neq (k-1). \tag{3.81}$$

In particular this means that $(A_{1j})_i$ must be antisymmetric (since a_{ijL} is symmetric in i, j). Note that this condition is clearly satisfied for the $(A_{1j})_i$ defined in (3.71).

The leading term in the asymptotic expansion is given in terms of degree one vector harmonics as

$$A = \frac{Q_5}{r^2}(A_{1j})_i Y_1^j dY_1^i \equiv \frac{\sqrt{Q_5 Q_1}}{r^2}(a^{\alpha-}Y_1^{\alpha-} + a^{\alpha+}Y_1^{\alpha+}), \tag{3.82}$$

where $(Y_1^{\alpha-}, Y_1^{\alpha+})$ with $\alpha = 1,2,3$ form a basis for the $k=1$ vector harmonics, which coincide with the Killing one forms of $SU(2)_L$ and $SU(2)_R$ respectively. Here we define

$$a^{\alpha\pm} = \frac{\sqrt{Q_5}}{\sqrt{Q_1}}\sum_{i>j} e^{\pm}_{\alpha ij}(A_{1j})_i \tag{3.83}$$

where the spherical harmonic triple overlap $e^{\pm}_{\alpha ij}$ is defined in (3.227) and explicit values in a particular basis are given in (3.241). For solutions defined by a curve $F^i(v)$, the coefficients $(A_{1j})_i$ are given in (3.71). The dual field to leading order is

$$B = \frac{\sqrt{Q_5 Q_1}}{r^2}(a^{\alpha-}Y_1^{\alpha-} - a^{\alpha+}Y_1^{\alpha+}), \tag{3.84}$$

where we use the Hodge duality property of the vector harmonics given in (3.221).

Putting these results together the leading perturbations of the metric are

$$\begin{aligned}
-h_{tt} &= h_{yy} = \frac{1}{2}\left(-(f^1_{2I} + f^5_{2I})Y_2^I + (f^5_{1i}Y_1^i)^2\right); \\
h_{rr} &= \frac{1}{2r^4}\left((f^1_{2I} + f^5_{2I})Y_2^I - (f^5_{1i}Y_1^i)^2\right); \\
h_{ta} &= \left(a^{\alpha-}Y_1^{\alpha-} + a^{\alpha+}Y_1^{\alpha+}\right); \\
h_{ya} &= \left(a^{\alpha-}Y_1^{\alpha-} - a^{\alpha+}Y_1^{\alpha+}\right); \\
h_{ab} &= \overset{o}{g}_{ab}\frac{1}{2r^2}\left((f^1_{2I} + f^5_{2I})Y_2^I - (f^5_{1i}Y_1^i)^2\right) - \frac{2}{r^2}a^{\alpha-}a^{\beta+}((Y_1^{\alpha-})_a(Y_1^{\beta+})_b + (Y_1^{\alpha-})_b(Y_1^{\beta+})_a).
\end{aligned} \tag{3.85}$$

Note that the condition (3.75) has been used to eliminate f^1_{1i}. Terms quadratic in spherical harmonics will need to be projected back onto the basis of spherical harmonics in order to determine the contributions to each perturbation component in (3.67).

In these expressions we have suppressed the scale factor $\sqrt{Q_1 Q_5}$. As mentioned previously, after rescaling $t \to t\sqrt{Q_1 Q_5}$ and $y \to y\sqrt{Q_1 Q_5}$, the metric has an overall scale factor $\sqrt{Q_1 Q_5}$. Scale factors will similarly be suppressed in the other fields. The overall scaling will be taken into account via the normalization of the three-dimensional action.

Now consider the other supergravity fields; from (3.65) and (3.66) one finds the following three form fluctuations are

$$
\begin{aligned}
g^5_{tya} &= \frac{1}{4}D_a\left(2(f^5_{1i}Y^i_1)^2 - (f^5_{2I}+f^1_{2I})Y^I_2\right); \\
g^5_{tab} &= -(a^{\alpha-}D_{[a}(Y^{\alpha-}_1)_{b]} - a^{\alpha+}D_{[a}(Y^{\alpha+}_1)_{b]}); \\
g^5_{yab} &= -(a^{\alpha-}D_{[a}(Y^{\alpha-}_1)_{b]} + a^{\alpha+}D_{[a}(Y^{\alpha+}_1)_{b]}); \\
g^5_{rab} &= \frac{1}{r^3}\left(\frac{1}{4}\epsilon_{ab}{}^c(f^1_{2I}+f^5_{2I})D_cY^I_2 + 4a^{\alpha-}a^{\beta+}(Y^{\alpha-}_1)_{[a}(Y^{\beta+}_1)_{b]}\right); \\
g^5_{abc} &= \frac{1}{r^2}\epsilon_{abc}(f^1_{2I}+f^5_{2I})Y^I_2 - \frac{6}{r^2}a^{\alpha-}a^{\beta+}D_{[a}(Y^{\alpha-}_1)_b(Y^{\beta+}_1)_{c]}).
\end{aligned} \tag{3.86}
$$

and

$$
\begin{aligned}
g^6_{tyr} &= \frac{1}{2}f^5_{1i}Y^i_1; \\
g^6_{tya} &= \frac{1}{4}D_a\left(2f^5_{1i}Y^i_1 r + (f^5_{2I}-f^1_{2I})Y^I_2\right); \\
g^6_{rab} &= \frac{1}{2r^2}\epsilon_{ab}{}^c f^5_{1i}D_cY^i_1 + \frac{1}{4r^3}\epsilon_{ab}{}^c(f^5_{2I}-f^1_{2I})D_cY^I_2; \\
g^6_{abc} &= \frac{3}{2r}\epsilon_{abc}f^5_{1i}Y^i_1 + \frac{1}{r^2}\epsilon_{abc}(f^5_{2I}-f^1_{2I})Y^I_2.
\end{aligned} \tag{3.87}
$$

Finally the scalar field is expanded as

$$
\phi^{(56)} \equiv \Phi = -\frac{f^5_{1i}}{r}Y^i_1 + \tfrac{1}{2}\frac{f^1_{2I}-f^5_{2I}}{r^2}Y^I_2. \tag{3.88}
$$

All other fluctuations, g^A with $A \neq 5,6$ and ϕ^{mr} with $m \neq 5$, $r \neq 6$ vanish.

(3.4.2) Gauge invariant fluctuations

We now wish to extract gauge invariant combinations of these fluctuations. Gauge invariant means that the fluctuations do not transform under coordinate transformations $\delta x^M = \xi^M$, or, in the case of the three dimensional metric and gauge fields, they have the correct transformation properties. Using the fact that the metric and three forms transform (up to linear order in fluctuations) as

$$
\begin{aligned}
\delta h_{MN} &= D_M\xi_N + D_N\xi_M + D_M\xi^P h_{PN} + D_N\xi^P h_{PM} - \xi^P D_P h_{MN}; \\
\delta g^A_{MNP} &= 3D_{[M}\xi^S g^{oA}_{NP]S} + 3D_{[M}\xi^S g^A_{NP]S} + \xi^S D_S g^A_{MNP},
\end{aligned} \tag{3.89}
$$

one can systematically compute combinations which are gauge invariant to quadratic order in fluctuations. That is, the gauge invariant fluctuations $\hat{\psi}^Q$ are given by the following schematic expression

$$
\hat{\psi}^Q = \sum_R a_{QR}\psi^R + \sum_{R,S} a_{QRS}\psi^R\psi^S, \tag{3.90}
$$

where ψ^Q collectively denotes all fields and the quadratic contributions are rather complicated in general. Note that each gauge invariant field at linearized level should reduce to the corresponding field in de Donder gauge on setting the fields with subscripts (s,v) to zero in (3.67).

Clearly by retaining higher order terms in (3.89) one could compute the invariants to arbitrarily high order in the fluctuations.

For the discussion at hand, however, we do not need the most general expressions. Since we are working perturbatively in the radial coordinate, we need only retain terms in (3.90) with the same radial behavior. In particular, as we discuss only leading order and next to leading order perturbations, we will need at most quadratic invariants. In fact the only combinations which will be needed here are

$$\begin{aligned} \hat{\pi}_2{}^I &= \pi_2^I + \Lambda^2 \rho_{2(s)}^I; \\ \hat{U}_2^{(5)I} &= U_2^{(5)I} - \tfrac{1}{2}\rho_{2(s)}^I; \\ \hat{h}_{\mu\nu}^0 &= h_{\mu\nu}^0 - \sum_{\alpha,\pm} h_\mu^{1\pm\alpha} h_\nu^{1\pm\alpha}. \end{aligned} \tag{3.91}$$

In addition the fluctuations $(\Phi_1^i, \Phi_2^I, U_1^{(6)i}, U_2^{(6)I})$ are by themselves gauge invariant up to the necessary order and the fields $(Z_\mu^{(5)1\pm\alpha}, Z_\mu^{(6)1\pm\alpha}, h_\mu^{1\pm\alpha})$ by themselves transform correctly as gauge fields. Thus only in the metric do we need to take into account a quadratic contribution.

(3.5) EXTRACTING THE VEVS SYSTEMATICALLY

In this section we will compute the vevs following the systematic procedure of [22]. First one should identify the six-dimensional equations of motion that these fields satisfy to appropriate order, in this case quadratic. Secondly one should remove derivative terms in these equations of motion by a field redefinition: this defines the Kaluza-Klein reduction map between six-dimensional and three-dimensional fields. Finally, once one has the three dimensional fields and their equations of motion, one extracts vevs using the by now familiar methods of holographic renormalization.

(3.5.1) LINEARIZED FIELD EQUATIONS

Let us first consider the linearized field equations. As discussed in [22], the equations of motion for the gauge invariant fields at linear order are precisely the same as those in de Donder gauge, provided one replaces all fields with the corresponding gauge invariant field. So now let us briefly review the linearized spectrum in de Donder gauge derived in [55]. Consider first the scalars. It is useful to introduce the following combinations of these fields which diagonalize the linearized equations of motion:

$$\begin{aligned} s_I^{(r)k} &= \frac{1}{4(k+1)}(\phi_I^{(5r)k} + 2(k+2)U_I^{(r)k}), \\ t_I^{(r)k} &= \frac{1}{4}(\phi_I^{(5r)k} - 2kU_I^{(r)k}), \\ \sigma_I^k &= \frac{1}{12(k+1)}(6(k+2)\hat{U}_I^{(5)k} - \hat{\pi}_I^k), \end{aligned} \tag{3.92}$$

$$\tau_I^k = \frac{1}{12(k+1)}(\hat{\pi}_I^k + 6k\hat{U}_I^{(5)k}).$$

Note that these combinations are applicable when the background $AdS_3 \times S^3$ has unit radius. Here the fields $s^{(r)k}$ and σ^k correspond to scalar chiral primaries. In what follows we will need only the $r = 6$ fields and will thus drop the r superscript. The masses of the scalar fields are

$$m_{s^k}^2 = m_{\sigma^k}^2 = k(k-2), \qquad m_{t^k}^2 = m_{\tau^k}^2 = (k+2)(k+4), \qquad m_{\rho^k}^2 = k(k+2). \tag{3.93}$$

Note also that $k \geq 0$ for $(\tau^k, t^{(r)k})$; $k \geq 1$ for $s^{(r)k}$; $k \geq 2$ for (σ^k, ρ^k).

Next consider the vector fields. It is useful to introduce the following combinations which diagonalize the equations of motion:

$$h_{\mu I_v}^{\pm} = \tfrac{1}{2}(C_{\mu I_v}^{\pm} - A_{\mu I_v}^{\pm}), \qquad Z_{\mu I_v}^{(5)\pm} = \pm\tfrac{1}{4}(C_{\mu I_v}^{\pm} + A_{\mu I_v}^{\pm}). \tag{3.94}$$

For general k the equations of motion are Proca-Chern-Simons equations which couple $(A_\mu^{\pm}, C_\mu^{\pm})$ via a first order constraint [55]. The three dynamical fields at each degree k have masses $(k-1, k+1, k+3)$, corresponding to dual operators of dimensions $(k, k+2, k+4)$ respectively. The lowest dimension operators are the R symmetry currents, which couple to the $k = 1$ $A_\mu^{\pm\alpha}$ bulk fields. The latter satisfy the Chern-Simons equation

$$F_{\mu\nu}(A^{\pm\alpha}) = 0, \tag{3.95}$$

where $F_{\mu\nu}(A^{\pm\alpha})$ is the curvature of the connection and the index $\alpha = 1, 2, 3$ is an $SU(2)$ adjoint index. Only these bulk vector fields will be needed in what follows, and therefore the equations of motion for general k discussed in [55] are not given here. There are also the massive vectors $Z_{\mu I_v}^{(6)\pm}$ but their mass is sufficiently high that they are irrelevant for our discussion.

Finally there is a tower of KK gravitons with $m^2 = k(k+2)$ but again only the massless graviton will play a role here. Note that it is the combination $\hat{H}_{\mu\nu} = h_{\mu\nu}^o + \pi^0 g_{\mu\nu}^o$ which satisfies the linearized massless Einstein equation

$$(\mathcal{L}_{\mathcal{E}} + 2)\hat{H}_{\mu\nu} \equiv \tfrac{1}{2}(-\Box\hat{H}_{\mu\nu} + D^\rho D_\mu \hat{H}_{\rho\nu} + D^\rho D_\nu \hat{H}_{\rho\mu} - D_\mu D_\nu \hat{H}_\rho^\rho + 4\hat{H}_{\mu\nu}) = 0. \tag{3.96}$$

That this is the appropriate combination follows from the reduction of the six-dimensional Einstein term in the action over the sphere; keeping terms linear in fluctuations the three dimensional action is

$$S_3 \sim \int d^3x\sqrt{-g}((1 + \tfrac{1}{2}\pi^0)R + \cdots), \tag{3.97}$$

and the Weyl transformation $\hat{H}_{\mu\nu} = h_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^o$ is required to bring the action to Einstein frame.

(3.5.2) FIELD EQUATIONS TO QUADRATIC ORDER

From the asymptotic expansion we now identify the fields of (3.92). In the asymptotic expansion we have retained only terms to quadratic order, that is of order $1/r$ and $1/r^2$ relative to the background. These terms are sufficient to determine vevs for the scalar chiral primaries of dimension one and two; the R symmetry currents and the energy momentum tensor. Using the tables in [55], one finds that the corresponding supergravity fields are $(s^1, s^2, \sigma^2, A_\mu^{1\pm}, H_{\mu\nu})$ respectively. Terms in other supergravity fields at the same order do not capture field theory data: they are simply induced by the non-linearity of the supergravity equations. Therefore we need only consider the above fields.

The next step is to derive the six-dimensional equations satisfied by the fluctuations, at non-linear order. The generic field equation for each field ψ^Q expanded in the number of fields is (schematically)

$$\mathcal{L}_Q \psi^Q = \mathcal{L}_{QRS} \psi^R \psi^S + \mathcal{L}_{QRST} \psi^R \psi^S \psi^T + \cdots , \tag{3.98}$$

where $\mathcal{L}_{Q_1 \cdots Q_n}$ is generically a non-linear differential operator. (Note that each field ψ^Q should be the appropriate diffeomorphism invariant combination.) The complete set of corrections to the field equations involves many terms even to quadratic order.

Fortunately what is required for extracting field theory data is the equations of motion expanded perturbatively near the conformal boundary, where the radial coordinate acts as the perturbation parameter. This means that we need only retain terms on the right hand side which affect the radial expansion at sufficiently low order to impact on the vevs. In practice for our discussion, the relevant quadratic corrections are those involving two s^1 fields or two gauge fields, since all other quadratic terms do not contribute at the required order. (Note that there are no corrections involving one s^1 field and one gauge field.) That all other terms can be neglected will be justified when one carries out the holographic renormalization procedure and considers the perturbative solution of the field equations.

The scalar field corrections to the field equations were computed in [24, 48][6]. These computations along with the corrections quadratic in the gauge field are discussed in detail in appendix 3.A.3. Consider first the scalar field equations. There are no quadratic corrections to the (s^1, s^2) equations from either s^1 fields or gauge fields, and thus the relevant equations remain the linearized equations. The σ^2 field equation does however get corrected by terms quadratic in scalars:

$$\Box \sigma_I^2 = \frac{11}{3}(s_i^1 s_j^1 - (D_\mu s_i^1)(D^\mu s_j^1)) a_{Iij}. \tag{3.99}$$

The coefficient a_{Iij} is the triple overlap of the corresponding spherical harmonics (see appendix 3.A.1). As discussed in the appendix 3.A.3, there are also corrections to this equation quadratic in the gauge fields which involve the field strengths $F_{\mu\nu}(A^{\pm\alpha})$ associated with the connections $A_\mu^{\pm\alpha}$ respectively. However, according to the linearized field equations (3.95) these field strengths vanish and thus these corrections do not play a role.

[6] We thank Gleb Arutyunov for making the latter available to us.

Next consider the corrections to the Einstein equation, which are also discussed in more detail in 3.A.3. Note that these corrections were not computed in [24, 48]. The appropriate three dimensional metric to quadratic order is

$$H_{\mu\nu} = h^0_{\mu\nu} - \sum_{\alpha,\pm} h^{1\pm\alpha}_{\mu} h^{1\pm\alpha}_{\nu} + \pi^0 g^o_{\mu\nu}. \tag{3.100}$$

As discussed previously the quadratic term is necessary in order for the metric to transform correctly under diffeomorphisms. Then the equation satisfied by the metric, up to quadratic order in the scalar fields s^1_i and the gauge fields is

$$(\mathcal{L}_E + 2)H_{\mu\nu} = 16(D_\mu s^1_i D_\nu s^1_i - g^o_{\mu\nu} s^1_i s^1_i), \tag{3.101}$$

where the linearized Einstein operator was defined in (3.96). This equation can be rewritten as

$$\mathcal{G}_{\mu\nu} - g^o_{\mu\nu} = 16\left(D_\mu s^1_i D_\nu s^1_i - \tfrac{1}{2} g^o_{\mu\nu}((D s^1_i)^2 - (s^1_i)^2)\right), \tag{3.102}$$

where $\mathcal{G}_{\mu\nu}$ is the linearized Einstein tensor. The rhs of this equation is the stress energy tensor of s^1. Note that the gauge field contributions to the energy momentum tensor involve the field strengths, and thus are zero when one imposes the lowest order field equation (3.95).

Finally, let us consider the equations for the gauge field. As discussed in [24, 48] the corrections quadratic in the gauge field correct the linearized equation to the non-Abelian Chern-Simons equation. That is, the six-dimensional equation is

$$\epsilon^{\mu\nu\rho}(\partial_\nu A^{\pm\alpha}_\rho + \tfrac{1}{2} A^{\pm\beta}_\nu A^{\pm\gamma}_\rho \epsilon_{\alpha\beta\gamma}) = 0, \tag{3.103}$$

where the $\epsilon_{\alpha\beta\gamma}$ arises from the triple overlap of vector harmonics defined in (3.236). Note that the $SU(2)_L$ and $SU(2)_R$ gauge fields are decoupled from each other. There are also corrections quadratic in the scalars s^1, which provide a source for the field strength:

$$\epsilon^{\mu\nu\rho}(\partial_\nu A^{\pm\alpha}_\rho + \cdots) = \pm 4 s^1_i D^\mu s^1_j e^{\pm}_{\alpha ij}, \tag{3.104}$$

where the ellipses denote the non-linear Chern-Simons terms and the triple overlap is defined in (3.227).

(3.5.3) Reduction to three dimensions

Given the corrected six-dimensional field equations (3.99), (3.101) and (3.103), we now need to determine the corresponding three-dimensional field equations. As discussed in [22], the KK map between six and three dimensional fields is in general non-linear. The non-linear corrections arise from field redefinitions used to remove derivative couplings. From the form of the corrected field equations, it is apparent that only the scalar fields σ^2 are affected (at this order) by such field redefinitions. That is, the derivative couplings in (3.99) can be removed by the field redefinition

$$\Sigma_I^2 = \sqrt{32}(\sigma_I^2 + \frac{11}{6} s_i^1 s_j^1 a_{Iij} + \cdots), \tag{3.105}$$

where Σ_I^2 is the three dimensional field. (The prefactor ensures canonical normalization of the three dimensional field, as we will shortly discuss.) This field redefinition defines the KK reduction map between six and three dimensional fields.

The resulting set of three dimensional field equations can then be integrated to the following three-dimensional bulk action

$$\begin{aligned} &\frac{n_1 n_5}{4\pi} \int d^3x \sqrt{-G}(R_G + 2 - \tfrac{1}{2}(DS_i^1)^2 + \tfrac{1}{2}(S_i^1)^2 - \tfrac{1}{2}(DS_I^2)^2 - \tfrac{1}{2}(D\Sigma_I^2)^2) \qquad (3.106)\\ &+\frac{n_1 n_5}{8\pi} \int (A_\alpha^+ dA^{+\alpha} + \frac{1}{3}\epsilon_{\alpha\beta\gamma} A^{+\alpha} A^{+\beta} A^{+\gamma} - A_\alpha^- dA^{-\alpha} - \frac{1}{3}\epsilon_{\alpha\beta\gamma} A^{-\alpha} A^{-\beta} A^{-\gamma}) + \cdots . \end{aligned}$$

The ellipses denote fields dual to operators of higher dimension not being considered here, along with higher order interactions. The boundary terms in this action will be discussed later in the context of holographic renormalization.

An overall rescaling of the scalar fields arises from demanding that the three-dimensional scalar fields are canonically normalized, up to the overall scaling of the action; it follows from the quadratic actions given in [24]. Thus the three dimensional fields S_I^k and Σ_I^k are related to the six-dimensional fields s_I^k and σ_I^k via

$$S_I^k = 4\sqrt{k(k+1)}(s_I^k + \cdots), \qquad \Sigma_I^k = 4\sqrt{k(k-1)}(\sigma_I^k + \cdots). \tag{3.107}$$

The ellipses denote non-linear terms in the KK map of which only (3.105) will be relevant here; other terms do not contribute to the order we need. The normalization of the gauge field terms also follows from the actions given in [24]. Note that the leading scalar field corrections to the gauge field equation (3.104) are also implicitly contained in the action (3.106), recalling that D is a covariant derivative and the scalar fields are charged under the $SO(4)$ gauge group.

The overall prefactor in the action (3.106) follows from the chain of dimensional reductions

$$\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} e^{-2\Phi}(R_{10} + \cdots) \to \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g}(R + \cdots) \to \frac{1}{2\kappa_3^2} \int d^3x \sqrt{-G}(R_G + 2\cdots). \tag{3.108}$$

Implicitly in the latter expression the curvature scale is contained in the prefactor, so that the background AdS_3 metric G has unit radius. Then

$$2\kappa_{10}^2 = (2\pi)^7(\alpha')^4; \qquad 2\kappa_6^2 = \frac{1}{(2\pi)^4 V} 2\kappa_{10}^2; \qquad 2\kappa_3^2 = \frac{1}{2\pi^2 Q_1 Q_5} 2\kappa_6^2, \tag{3.109}$$

which using (3.49) implies that

$$\frac{1}{2\kappa_3^2} = \frac{n_1 n_5}{4\pi}, \tag{3.110}$$

as in (3.106).

(3.5.4) Holographic renormalization and extremal couplings

Having determined the three-dimensional fields and the equations of motion which they satisfy we are now ready to determine vevs using the procedure of holographic renormalization. We will first briefly review this procedure, using the Hamiltonian formalism developed in [19, 20]. Let $\mathcal{O}_{\Psi^k}$ be the dimension k operator dual to the three dimensional supergravity field Ψ^k, the latter being related to the six dimensional fields ψ^Q by non-linear KK maps. Then its vev can be expressed as

$$\langle \mathcal{O}_{\Psi^k} \rangle = \frac{n_1 n_5}{4\pi} \left((\pi_{\Psi^k})_{(k)} + \cdots \right); \tag{3.111}$$

where we will explain the meaning of the ellipses below. Now π_{Ψ^k} is the radial canonical momentum for the field Ψ^k and $(\pi_{\Psi^k})_{(k)}$ is the kth component in its expansion in terms of eigenfunctions of the dilatation operator. The results of [19, 20] show that there is a one to one correspondence between momentum coefficients and terms in the asymptotic expansion of the fields.

That is, the near boundary expansion of the metric and scalar fields is

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2}\left(g_{(0)uv} + z^2 \left(g_{(2)uv} + \log(z^2) h_{(2)uv} + (\log(z^2))^2 \tilde{h}_{(2)uv} \right) + \cdots \right) dx^u dx^v; \\ \Psi^1 &= z(\log(z^2)\Psi^1_{(0)}(x) + \tilde{\Psi}^1_{(0)}(x) + \cdots); \\ \Psi^k &= z^{2-k}\Psi^k_{(0)}(x) + \cdots + z^k \Psi^k_{(2k-2)}(x) + \cdots, \qquad k \neq 1. \end{aligned} \tag{3.112}$$

In these expressions $(G_{(0)uv}, \Psi^1_{(0)}(x), \Psi^k_{(0)}(x))$ are sources for the stress energy tensor and scalar operators of dimension one and k respectively; as usual one must treat separately the operators of dimension $\Delta = d/2$, where d is the dimension of the boundary. Note that the 2-dimensional boundary coordinates are labeled by (u, v).

The correspondence between the momentum coefficients and these expansion coefficients for the scalar fields is then

$$\begin{aligned} (\pi_{\Psi^k})_{(k)} &= ((2k-2)\Psi^k_{(2k-2)}(x) + \cdots); \\ (\pi_{\Psi^1})_{(1)} &= (2\tilde{\Psi}^1_{(0)} + \cdots). \end{aligned} \tag{3.113}$$

The ellipses denote non-linear terms in the relations that involve the sources and do not play a role here.

The ellipses in (3.111) denote terms non-linear in momenta. Such terms are related to extremal correlators and play a crucial role which we will discuss in detail. Before doing so, however, it is convenient to first discuss the gauge fields.

R symmetry currents

Let us now consider the vevs for R symmetry currents; these were previously discussed in [58, 59] and we will briefly summarize their results here. Given the asymptotic form of the metric (**??**) the Chern-Simons gauge fields have corresponding asymptotic field expansions

$$A^{\pm\alpha} = \mathcal{A}^{\pm\alpha} + z^2 A^{\pm\alpha}_{(2)} + \cdots . \tag{3.114}$$

Here $\mathcal{A}^{\pm\alpha}$ are fixed boundary values which are respectively holomorphic and anti-holomorphic. A key point is that the vev will be obtained from the leading order term in this expansion which is not affected by the other supergravity fields. Supergravity couplings affect only the subleading behavior of the gauge field, and thus we can neglect them. Put differently, the vev for the R symmetry current involves only the gauge field and there are no non-linear contributions.

The following boundary action

$$S_B = \frac{n_1 n_5}{16\pi} \int d^2x \sqrt{-\gamma}\gamma^{uv} (\mathcal{A}^{+\alpha}_u \mathcal{A}^{+\alpha}_v + \mathcal{A}^{-\alpha}_u \mathcal{A}^{-\alpha}_v) \tag{3.115}$$

ensures that the variational problem for the gauge fields is well-defined with these boundary conditions; γ_{uv} is the induced boundary metric. [7]. With these boundary terms the on-shell variation of the action yields the currents

$$\left\langle J^{\pm\alpha}_u \right\rangle = \frac{1}{\sqrt{-\gamma}} \left(\frac{\delta S}{\delta \mathcal{A}^{\pm\alpha u}_{\alpha}} \right) = \frac{n_1 n_5}{8\pi} (g_{(0)uv} \mp \epsilon_{uv}) \mathcal{A}^{\pm\alpha v}. \tag{3.116}$$

As discussed recently in [59] the resulting currents have the desired properties. In particular, momentarily switching to the Euclidean signature and using conformal gauge for the boundary metric so that $g_{(0)uv}dx^u dx^v = dw d\bar{w}$, the currents are

$$\begin{aligned} J^{+\alpha}_w &= \frac{n_1 n_5}{4\pi} \mathcal{A}^{+\alpha}_w; \qquad J^{+\alpha}_{\bar{w}} = 0; \\ J^{-\alpha}_w &= 0; \qquad J^{-\alpha}_{\bar{w}} = \frac{n_1 n_5}{4\pi} \mathcal{A}^{-\alpha}_{\bar{w}}. \end{aligned} \tag{3.117}$$

Thus the $SU(2)_L$ and $SU(2)_R$ right currents are holomorphic and anti-holomorphic respectively, as expected for the boundary CFT. Moreover the current modes defined by

$$J^{+\alpha}_n = \frac{1}{2\pi i} \oint dw w^n J^{+\alpha}_w; \qquad J^{-\alpha}_n = \frac{1}{2\pi i} \oint d\bar{w} \bar{w}^n J^{-\alpha}_{\bar{w}}, \tag{3.118}$$

obey the correct $SU(2)$ current algebras.

SCALAR OPERATORS

Consider next the scalar operators; here the non-linear terms in (3.111) play a crucial role. Just as in [22] we need to take into account the rather subtle issue of extremal couplings. Recall that an extremal correlation function is one for which the dimension of one operator is equal

[7] In [58] the additional boundary term $\Delta S_{\mathcal{A}} = -\frac{n_1 n_5}{16\pi} \int d^2x \sqrt{-\gamma} (\gamma^{uv} + \epsilon^{uv}) \mathcal{A}^{+\alpha}_u \mathcal{A}^{-\alpha}_v$ was added to the action. The variational problem is still consistent, but this term couples left and right movers so it is not appropriate for our purposes.

to the sum of the other operator dimensions. The corresponding bulk couplings in supergravity vanish: this is physically necessary, because such couplings would induce conformal anomalies which are known to be zero (and non-renormalized). In [60] it was appreciated that extremal correlators are obtained not from bulk couplings, but instead from certain finite boundary terms. These would arise from demanding a well posed variational problem in the higher dimensional theory, and then keeping track of all boundary terms when carrying out the KK reduction.

These same extremal couplings play a key role in determining the vevs. Suppose the operator $\mathcal{O}_{\Psi^k}$ has a non-vanishing extremal n-point function with operators $\{\mathcal{O}_{\Psi^{k_a}}\}$, with $a = 1, \cdots (n-1)$. Then this implies an additional term in the holographic renormalization relation

$$\langle \mathcal{O}_{\Psi^k} \rangle = \frac{n_1 n_5}{4\pi} \left((\pi_{\Psi^k})_{(k)} + A_{kk_1 \cdots k_{(n-1)}} \prod_{k_a} (\pi_{\Psi^{k_a}})_{(k_a)} + \cdots \right) \tag{3.119}$$

The coupling $A_{kk_1 \cdots k_{(n-1)}}$ must be such that one obtains the correct n-point function upon functional differentiation.

Now consider how this issue affects the vevs being determined here: there are potentially contributions to vevs of dimension two operators from their couplings to two dimension one operators. The latter include both the operators dual to the scalars S_i^1 and the R-symmetry currents dual to the gauge fields $A_\mu^{\pm\alpha}$. Let us consider first the following extremal three point functions between scalar operators

$$\Sigma^2 : \left\langle \mathcal{O}_{\Sigma_I^2} \mathcal{O}_{S_i^1} \mathcal{O}_{S_j^1} \right\rangle ; \qquad S^2 : \left\langle \mathcal{O}_{S_I^2} \mathcal{O}_{S_i^1} \mathcal{O}_{S_j^1} \right\rangle . \tag{3.120}$$

If these three point functions are non-zero, there will necessarily be additional quadratic contributions to the vevs of the dimension two operators.

In the discussions of [22] one could use the known free field extremal correlators of $\mathcal{N} = 4$ SYM along with non-renormalization theorems to fix the additional terms in (3.119). As we will discuss momentarily comparing with field theory is in this case rather more subtle. From the supergravity side there are two methods to compute these quadratic terms. The first would be to start with the six-dimensional action, demand that the variational problem is well-defined (which fixes boundary terms), and then dimensionally reduce to three dimensions. This is straightforward in principle, but to extract the required coefficient we need boundary terms cubic in the fields, which in turn requires expanding the field equations to cubic order. Thus we choose to use a second method: we compute the extremal correlator in supergravity by computing the corresponding non extremal correlator and then using a careful limiting procedure. This computation of the extremal correlators and hence the non-linear terms (3.119) is presented in appendix 3.A.4.

Since all non-extremal three point functions between three $\mathcal{O}_{S^I}$ operators vanish [61, 24], one also obtains no extremal three point function and therefore no extra contributions to $\left\langle S_I^2 \right\rangle$ beyond the standard term given in (3.111). The cubic coupling between one Σ field and two

S fields is however generically non-vanishing [61, 24] and therefore we do obtain an extremal three point function which leads to the following result for the scalar contributions to the one point function (3.296), (3.298)

$$\langle \mathcal{O}_{\Sigma_I^2} \rangle = \left(\frac{n_1 n_5}{4\pi} \right) \left(\pi_{(2)}^{\Sigma_I^2} - \frac{1}{4\sqrt{2}} a_{Iij} \pi_{(1)}^{S_i^1} \pi_{(1)}^{S_j^1} \right). \quad (3.121)$$

An extremal coupling between the dimension two scalar operators and two R symmetry currents would require a term in the rhs of (3.121) proportional to $\mathcal{A}_u \mathcal{A}^u$. However such term is gauge dependent and thus forbidden. We conclude that there are no additional contributions to (3.121).

Before leaving this section we should note why the extremal correlators were fixed via a limit of the non-extremal supergravity correlators and other indirect arguments rather than from a dual field theory computation. The relevant three point functions of scalar operators in the orbifold CFT were computed in [62] and [63]. There is no known non-renormalization theorem to protect them and thus no justification for extrapolating them to the strong coupling regime. Indeed, as we discuss in appendix 3.A.6, certain correlation functions seem to disagree between supergravity and the orbifold CFT.

STRESS ENERGY TENSOR

Finally we discuss the vev for the stress energy tensor. This being a dimension two operator, we again need to take into account terms quadratic in two dimension one operators. Terms quadratic in the scalar fields S_i^1 and in the gauge fields $A_\mu^{\pm\alpha}$ both contribute. Let us momentarily suppress the gauge field contributions. Then as discussed in the previous section, the three dimensional metric couples at leading order to the scalar field S_i^1 in the three dimensional equations of motion and thus we need to derive the one point functions for this coupled system. This computation is very similar to the Coulomb branch analysis given in [17, 18] and is summarized in appendix 3.A.5.

Next consider the additional contributions to the stress energy tensor quadratic in the gauge field. These immediately follow from the variation of the boundary terms (3.115), since the bulk Chern-Simons terms cannot contribute. Thus the total result for the stress energy tensor follows from (3.312) plus gauge field terms giving:

$$\langle T_{uv} \rangle = \frac{n_1 n_5}{2\pi} \left(g_{(2)uv} + \tfrac{1}{2} R g_{(0)uv} + \frac{1}{4} (\tilde{S}_{(0)}^1)^2 g_{(0)uv} + \tfrac{1}{4} (\mathcal{A}_{(u}^{+\alpha} \mathcal{A}_{v)}^{+\alpha} + \mathcal{A}_{(u}^{-\alpha} \mathcal{A}_{v)}^{-\alpha}) + \cdots \right), \quad (3.122)$$

where the terms in ellipses (source terms for the scalars) are given in (3.312) but do not contribute in our solutions. (Recall that parentheses denote the symmetrised traceless combination of indices.)

Now consider the effect of a large gauge transformation of the form $\mathcal{A}_w^{+3} \to \mathcal{A}_w^{+3} + \eta$. As discussed in [59] (see also [64]) this induces the shifts

$$L_0 \to L_0 + \eta J_0^{+3} + \tfrac{1}{4} k \eta^2; \qquad J_0^{+3} \to J_0^{+3} + \tfrac{1}{2} k \eta, \quad (3.123)$$

where the Virasoro generator is defined as $L_0 = \frac{c}{24} + \oint dw T_{ww}$ and the level of the $SU(2)$ algebra is $k \equiv n_1 n_5$. This is clearly a spectral flow transformation, and shows the relationship between bulk coordinate transformations on the S^3 and spectral flow in the boundary theory.

(3.6) VEVS FOR THE FUZZBALL SOLUTIONS

We are now ready to extract the vevs from the asymptotic expansions of the fields in the fuzzball solutions given in (3.85), (3.86), (3.87) and (3.88). The appropriate (gauge invariant) combinations of six-dimensional scalar and gauge fields are

$$\begin{aligned}
\mathbf{s^1_i} &= \frac{\mathbf{1}}{\mathbf{4r}}(\mathbf{f^1_{1i}} - \mathbf{f^5_{1i}}) + \cdots ; \qquad \mathbf{s^2_I} = \frac{\mathbf{1}}{\mathbf{8r^2}}(\mathbf{f^1_{2I}} - \mathbf{f^5_{2I}}) + \cdots ; \\
\sigma^{\mathbf{2}}_{\mathbf{I}} &= -\frac{\mathbf{1}}{\mathbf{8r^2}}(\mathbf{f^1_{2I}} + \mathbf{f^5_{2I}}) + \frac{1}{24r^2}(f^5_{1i})(f^5_{1j})a_{Iij} + \frac{1}{r^2}a^{\alpha-}a^{\beta+}f_{I\alpha\beta} + \cdots . \\
\mathbf{A^{+\alpha}_t} &= -\mathbf{2a}^{\alpha+} + \cdots ; \qquad \mathbf{A^{+\alpha}_y} = \mathbf{2a}^{\alpha+} + \cdots , \\
\mathbf{A^{-\alpha}_t} &= -\mathbf{2a}^{\alpha-} + \cdots ; \qquad \mathbf{A^{-\alpha}_y} = -\mathbf{2a}^{\alpha-} + \cdots .
\end{aligned} \tag{3.124}$$

The graviton is given by

$$\begin{aligned}
H_{tt} &= f^5_{1i}f^5_{1i} - a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots ; \\
H_{yy} &= -f^5_{1i}f^5_{1i} - a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots ; \\
H_{ty} &= a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots ; \\
H_{rr} &= -\frac{2}{r^4}f^5_{1i}f^5_{1i} + \cdots .
\end{aligned} \tag{3.125}$$

Next we extract the three-dimensional fields, which involves rescaling and shifting the scalar fields as defined in (3.105) and (3.107):

$$\begin{aligned}
S^1_i &= -\frac{2\sqrt{2}}{r}f^5_{1i} + \cdots ; \qquad S^2_I = \frac{\sqrt{3}}{\sqrt{2}r^2}(f^1_{2I} - f^5_{2I}) + \cdots ; \\
\Sigma^I_2 &= \sqrt{32}(-\frac{1}{8r^2}(f^1_{2I} + f^5_{2I}) + \frac{1}{2r^2}(f^5_{1i})(f^5_{1j})a_{Iij} + \frac{1}{r^2}a^{\alpha-}a^{\beta+}f_{I\alpha\beta} + \cdots).
\end{aligned} \tag{3.126}$$

where we used (3.75) in S^1_i. Note that the gauge fields and the metric are not rescaled or shifted upon the dimensional reduction to this order.

Thus for the scalar operators we obtain using (3.111) and (3.121) the vevs

$$\begin{aligned}
\left\langle \mathcal{O}_{S^1_i} \right\rangle &= \frac{n_1 n_5}{4\pi}(-4\sqrt{2}f^5_{1i}); \\
\left\langle \mathcal{O}_{S^2_I} \right\rangle &= \frac{n_1 n_5}{4\pi}(\sqrt{6}(f^1_{2I} - f^5_{2I})); \\
\left\langle \mathcal{O}_{\Sigma^2_I} \right\rangle &= \frac{n_1 n_5}{4\pi}\sqrt{2}(-(f^1_{2I} + f^5_{2I}) + 8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}).
\end{aligned} \tag{3.127}$$

The currents follow from (3.116) as

$$\left\langle J^{+\alpha} \right\rangle = \frac{n_1 n_5}{2\pi}a^{\alpha+}(dy - dt); \qquad \left\langle J^{-\alpha} \right\rangle = -\frac{n_1 n_5}{2\pi}a^{\alpha-}(dy + dt). \tag{3.128}$$

To evaluate the vev of the stress energy tensor using (3.122) we first need to bring the metric into the Fefferman-Graham coordinate system. This requires the following change of radial coordinate

$$z = \frac{1}{r} - \frac{1}{2r^3}(f^5_{1i})^2 + \cdots \tag{3.129}$$

After changing radial coordinate in this way the metric becomes

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2}(1 - 2(f^5_{1i})^2 z^2)(-dt^2 + dy^2) \\ &\quad -a^{\alpha+}a^{\alpha+}(dt - dy)^2 - a^{\alpha-}a^{\alpha-}(dt + dy)^2 + \cdots \end{aligned} \tag{3.130}$$

The metric perturbation in the second line is traceless with respect to the leading order metric. Now applying the formula (3.122) we find that

$$\langle T_{uv} \rangle = 0. \tag{3.131}$$

This is the anticipated answer, since these solutions are supposed to be dual to R vacua. The cancellation is however very non-trivial and needed all the machinery of holographic renormalization.

(3.6.1) HIGHER DIMENSION OPERATORS

Having extracted the vevs for all operators up to dimension two using the systematic procedure developed in [22], it is worth considering whether any predictions can be made for vevs of higher dimension operators. These could of course be determined by the same systematic procedure used above, by retaining all terms to sufficiently high order, but this would involve considerable computation.

It is therefore useful to recall at this point the result obtained in [57] for the vevs extracted from supergravity solutions corresponding to the Coulomb branch of $\mathcal{N} = 4$ SYM. When these solutions are asymptotically expanded in the radial coordinate of the defining harmonic function, non-linear terms in the vevs of CPOs arising from non-linear terms in the higher dimensional fields, non-linear terms in the KK reduction map and non-linear terms in the holographic renormalization relations all cancel out[8]! The vevs are given by the *linear* terms in the higher dimensional fields. "Non-linear" in this context means terms which are non-linear in spherical harmonics.

Now consider what happens here if one retains only the linear terms in the fields, the dimensional reductions and the holographic renormalization relations. Then from (3.124), only the terms in boldface are retained. This means that there is no graviton perturbation to this order, and thus that the three-dimensional mass vanishes, in accordance with the expectation that

[8]Strictly speaking, the cancellation was proven in [57] for operators of dimension four and less for which the corresponding vevs had been extracted using the rigorous procedures of [22]. However, the linearized approach gave results which agreed with the (non-renormalized) weak coupling field theory results for all dimension operators.

these geometries describe R vacua. Furthermore, these terms give precisely the same results as before for the scalar $\mathcal{O}_S$ and current vevs, in which all non-linear contributions canceled. It is an interesting question to understand why the linear terms alone determine the stress energy tensor and $\mathcal{O}_S$ vevs. Note that just as in [57] a priori there is absolutely no justification for neglecting the non-linear terms, given that there is no small parameter. Presumably this question can be answered by understanding holographic renormalization directly in the higher dimension and developing the map between higher-dimensional fields and operators.

However, the linear terms clearly *fail* to give the correct answer for the operators dual to Σ^2. Thus the linearized approximation in this situation fails already at dimension two, which is the first place where non-linear terms can play a role (but note that it still holds for the dimension two operator $\mathcal{O}_{S^2}$).

Nevertheless one may proceed with the linearized procedure in order to get a rough idea of the behavior of the vevs for higher dimension operators. From the asymptotic expansion of the solution we extract the following linear terms for the scalars

$$\begin{aligned} s_I^k &= \frac{1}{4kr^k}(f_{kI}^1 - f_{kI}^5) + \cdots \\ \sigma_I^k &= -\frac{1}{4kr^k}(f_{kI}^1 + f_{kI}^5) + \cdots \end{aligned} \tag{3.132}$$

From these asymptotics the vevs of the dual operators contain the linear terms

$$\begin{aligned} \left\langle \mathcal{O}_{S_I^k} \right\rangle &= \left(\frac{n_1 n_5}{4\pi}\right) 2(k-1)\frac{\sqrt{k+1}}{\sqrt{k}}(f_{kI}^1 - f_{kI}^5 + \cdots); \\ \left\langle \mathcal{O}_{\Sigma_I^k} \right\rangle &= -\left(\frac{n_1 n_5}{4\pi}\right) 2(k-1)\frac{\sqrt{k-1}}{\sqrt{k}}(f_{kI}^1 + f_{kI}^5 + \cdots), \end{aligned} \tag{3.133}$$

where the ellipses denote the non-linear terms. Recall that (f_{kI}^1, f_{kI}^5) are proportional to the kth multipole moments of the D1 and D5 brane charge distributions, respectively. We will argue in the section 3.9 that these linear terms do not give the expected answer for the vevs of operators $\mathcal{O}_{\Sigma_I^k}$, although they seem to be sufficient to give the expected answer for the vevs of operators $\mathcal{O}_{S_I^k}$, at least for circular curves.

Following analogous arguments for the dimension k_v vector chiral primaries $J_{k_v}^{I_v\pm}$ dual to bulk vectors $A_{k_v}^{I_v\pm}$, we get the following structure

$$\left\langle J_{k_v}^{I_v\pm} \right\rangle \propto \left(\frac{n_1 n_5}{4\pi}\right)(A_{kI})_i E_{I_v I i}^{\pm}(dt \mp dy) + \cdots, \tag{3.134}$$

where the ellipses denote again the non-linear terms, the spherical harmonic triple overlap $E_{I_v I i}^{\pm}$ is defined in (3.229) and $(A_{kI})_i$ is defined in terms of the curve $F^i(v)$ in (3.71). To extract the exact coefficient relating the asymptotics of the bulk vector fields to the current vev, one would need to analyze the relevant Proca-Chern-Simons bulk equation and obtain the holographic renormalization relation for this case.

In the discussions of [57], the vevs obtained by the linearized approach gave correctly all the (non-renormalized) field theory vevs. Here the linearized approach does not give correctly vevs for chiral primaries. Moreover, we will also argue that there are additional vevs which are not captured by the linearized approximation at all. For example, when one linearizes the solution following the above procedure the (non primary) scalar fields (t_I^k, τ_I^k) are identically zero, but arguments given in section 3.9 suggest that the corresponding operators should in general have non-zero expectation values. Perhaps these vevs could still be extracted by an appropriate linearized analysis, but it is not apparent what the prescription should be. By contrast, the systematic method of [22] used in earlier sections will certainly give the correct answer for these vevs.

Note also that the linearized approximation manifestly gives different answers in different coordinate systems. For the example of the solution based on a circular curve we discuss in the next section, the linearized approximation in the coordinate system (3.144) actually gives the conjectured answers for scalar vevs, but linearizing in the hatted coordinate system (flat coordinates on R^4) gives different answers. Both in the fuzzball solutions considered here and in the Coulomb branch solutions discussed in [57] there are preferred coordinate systems, those in which the harmonic functions are naturally expressed. For the Coulomb branch this coordinate systems was precisely that in which linearizing gives the correct vevs, but here it does not. In general, however, there will be no preferred coordinate system or it may not be visible (as in (3.144)), and therefore there would be no natural way to linearize; one would have to apply the general methods of [22].

(3.7) EXAMPLES

We discuss in this section the application of the general results to two specific examples: solutions based on circular and ellipsoidal curves, respectively.

(3.7.1) CIRCULAR CURVES

A commonly used example of the fuzzball solutions is that in which the curve $F^i(v)$ is a (multiwound) circle [64, 65, 53],

$$F^1 = \mu_n \cos\frac{2\pi n v}{L}, \qquad F^2 = \mu_n \sin\frac{2\pi n v}{L}, \qquad F^3 = F^4 = 0. \tag{3.135}$$

The ten-dimensional solution in this case is conveniently written as

$$\begin{aligned} ds^2 &= f_1^{-1/2} f_5^{-1/2}\left(-(d\tilde{t} - \frac{\mu_n\sqrt{Q_1Q_5}}{\hat{r}^2+\mu_n^2\cos^2\hat{\theta}}\sin^2\hat{\theta}d\phi)^2 + (d\tilde{y} - \frac{\mu_n\sqrt{Q_1Q_5}}{\hat{r}^2+\mu_n^2\cos^2\hat{\theta}}\cos^2\hat{\theta}d\psi)^2\right) \\ &\quad + f_1^{1/2}f_5^{1/2}\left((\hat{r}^2+\mu_n^2\cos^2\hat{\theta})(\frac{d\hat{r}^2}{\hat{r}^2+\mu_n^2} + d\hat{\theta}^2) + \hat{r}^2\cos^2\hat{\theta}d\psi^2 + (\hat{r}^2+\mu_n^2)\sin^2\hat{\theta}d\phi^2\right) \\ &\quad + f_1^{1/2}f_5^{-1/2}dz\cdot dz; \qquad (3.136) \\ e^{2\Phi} &= f_1 f_5^{-1}, \\ f_{1,5} &= 1 + \frac{Q_{1,5}}{\hat{r}^2+\mu_n^2\cos^2\hat{\theta}}, \end{aligned}$$

whilst the tensor field is as in (3.44) with

$$A = \mu_n \frac{\sqrt{Q_1 Q_5}}{(\hat{r}^2 + \mu_n^2 \cos^2\hat{\theta})} \sin^2\hat{\theta} d\phi; \qquad B = -\mu_n \frac{\sqrt{Q_1 Q_5}}{(\hat{r}^2 + \mu_n^2 \cos^2\hat{\theta})} \cos^2\hat{\theta} d\psi. \tag{3.137}$$

This solution is precisely of the form (3.44), using a non-standard coordinate system on R^4. That is, the hatted coordinates $(\hat{r}, \hat{\theta}, \phi, \psi)$ are related to usual coordinates (r, θ, ϕ, ψ) on R^4 such that the metric is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2), \tag{3.138}$$

via

$$\hat{r}\cos\hat{\theta} = r\cos\theta; \qquad r^2 = \hat{r}^2 + \mu_n^2 \sin^2\hat{\theta}. \tag{3.139}$$

Note that this relation implies

$$\frac{1}{\hat{r}^2 + \mu_n^2\cos^2\hat{\theta}} = \frac{1}{\sqrt{(r^2 + \mu_n^2)^2 - 4\mu_n^2 r^2 \sin^2\theta}}, \tag{3.140}$$

with the latter admitting the following asymptotic expansion

$$\frac{1}{\sqrt{(r^2 + \mu_n^2)^2 - 4\mu_n^2 r^2 \sin^2\theta}} = \sum_{k\in 2Z} (-1)^{k/2} \frac{\mu_n^k Y_k^0(\theta_3)}{\sqrt{k+1} r^{2+k}}, \tag{3.141}$$

where the harmonic function is expanded in normalized spherical harmonics Y_k^0 which are singlets under the $SO(2)^2$ Cartan of $SO(4)$. These harmonics are given in (3.238); there is precisely one such singlet at each even degree. The asymptotic expansion in (3.141) follows from (3.69) upon using the fact that the lhs of (3.141) is equal to

$$\frac{1}{L}\int_0^L \frac{dv}{|x - F|^2}, \tag{3.142}$$

with F^i given in (3.135), so $\theta_3^F = \pi/2$ and $Y_k^0(\pi/2) = (-1)^{k/2}\sqrt{k+1}$.

The parameters (n, μ_n) labeling the curve are related to the charges via

$$n\mu_n = \frac{L}{2\pi}\sqrt{\frac{Q_1}{Q_5}} = \frac{\sqrt{Q_1 Q_5}}{R} \equiv \mu, \tag{3.143}$$

or equivalently $\mu_n = 1/(n\tilde{R})$, where $\tilde{R} = R/\sqrt{Q_1 Q_5}$. In deriving these results we have used (3.48) and (3.50).

The near horizon limit of (3.136) gives the six-dimensional fields

$$\begin{aligned} ds_6^2 &= \sqrt{Q_1 Q_5}\left(-(\hat{r}^2 + \mu_n^2)dt^2 + \hat{r}^2 dy^2 + \frac{d\hat{r}^2}{(\hat{r}^2 + \mu_n^2)}\right) \\ &\quad + \sqrt{Q_1 Q_5}\left(d\hat{\theta}^2 + \sin^2\hat{\theta}(d\phi + \mu_n d\tilde{t})^2 + \cos^2\hat{\theta}(d\psi - \mu_n d\tilde{y})^2\right); \\ G^5 &= \sqrt{Q_1 Q_5}\hat{r} dt \wedge dy \wedge d\hat{r} + \sqrt{Q_1 Q_5}\cos\hat{\theta}\sin\hat{\theta} d\hat{\theta} \wedge (d\phi + \mu_n dt) \wedge (d\psi - \mu_n dy). \end{aligned} \tag{3.144}$$

with the scalar field Φ and the anti-self dual field G^6 vanishing. As previously, it is convenient to use the rescaled coordinates $\tilde{t} = \sqrt{Q_1 Q_5} t$ and $\tilde{y} = \sqrt{Q_1 Q_5} y$ so that the overall scale factor is manifest. Note that the coordinate y has periodicity $2\pi\tilde{R}$. When $n = 1$ there is a coordinate transformation ($\phi \to \phi + \mu_n t$, $\psi \to \psi + \mu_n y$) that makes the metric exactly $AdS_3 \times S^3$. For $n > 1$ one can similarly shift the angular coordinates, but the resulting spacetime metric has a conical defect. As discussed in [64, 40], such a coordinate change is equivalent to carrying out a spectral flow to the NS sector; in the case of $n = 1$ the flow is to the vacuum. One way of seeing this is that under such a shift the Killing spinors change periodicity about the circle direction $\tilde{y}$. In the above coordinate system they are periodic, whilst after the coordinate transformation they are anti-periodic [64].

In the context of this chapter, however, we are interested in R vacua of the CFT, and thus we do not wish to flow to the NS sector. This means we should interpret the solution in the original coordinate system, where the Killing spinors are periodic. From (3.144) we can immediately read off the three dimensional gauge field as

$$A^{-3} = \mu_n(dy + dt); \qquad A^{+3} = \mu_n(dy - dt). \tag{3.145}$$

The superscript indicates that the relevant Killing vectors are those given in the appendix in (3.233), such that A^{+3} and A^{-3} commute. The fact that there is a coordinate transformation where the solution is (locally) $AdS_3 \times S^3$ means that the three dimensional scalar fields $(S_i^1, S_I^2, \Sigma_I^2, \cdots)$ vanish. Note that the latter result is immediately obvious in the hatted coordinate system but it is not manifest in the coordinate system (r, θ, ϕ, ψ). That the S fields vanish in the latter coordinate system follows from (3.124) since $f^1_{kI} = f^5_{kI}$. To see the vanishing of Σ_0^2 one has to use in (3.124) the identity

$$-\frac{1}{8}(f^1_{20} + f^5_{20}) + f_{033} a^{3+} a^{3-} = 0, \tag{3.146}$$

which follows from (3.141) and the identity (3.237).

Now given the three dimensional fields we derive the corresponding vevs,

$$\begin{aligned} \langle T_{uv} \rangle &= \left\langle \mathcal{O}_{S_i^1} \right\rangle = \left\langle \mathcal{O}_{S_I^2} \right\rangle = \left\langle \mathcal{O}_{\Sigma_0^2} \right\rangle = 0; \\ \left\langle J^{+3} \right\rangle &= \frac{n_1 n_5}{4\pi} \mu_n (dy - dt); \qquad \left\langle J^{-3} \right\rangle = \frac{n_1 n_5}{4\pi} \mu_n (dt + dy). \end{aligned} \tag{3.147}$$

Note that the R-symmetry charges

$$\begin{aligned} j^3 &\equiv \int_0^{2\pi\tilde{R}} dy J^{+3}_{\tilde{y}} = \frac{n_1 n_5}{2n}; \\ \bar{j}^3 &\equiv \int_0^{2\pi\tilde{R}} dy J^{-3}_{\tilde{y}} = \frac{n_1 n_5}{2n}, \end{aligned} \tag{3.148}$$

are quantized in half integral units provided that n is a divisor of $n_1 n_5$.

(3.7.2) Ellipsoidal curves

The next simplest case to consider is a solution determined by a planar ellipsoidal curve:

$$F^1 = \mu_n a \cos\frac{2\pi n v}{L}, \qquad F^2 = \mu_n b \sin\frac{2\pi n v}{L}, \qquad F^3 = F^4 = 0, \tag{3.149}$$

with μ_n as in (3.143). The D1-brane charge constraint (3.48) implies that $(a^2 + b^2) = 2$. The vevs for this solution are given by

$$\begin{aligned}
\langle T_{uv}\rangle &= \left\langle \mathcal{O}_{S^1_i}\right\rangle = 0;\\
\langle J^{+3}\rangle &= \frac{N}{4\pi}\mu_n ab(dy - dt); \qquad \langle J^{-3}\rangle = \frac{N}{4\pi}\mu_n ab(dt + dy);\\
\left\langle \mathcal{O}_{S^2_{m,\bar{m}}}\right\rangle &= \left\langle \mathcal{O}_{\Sigma^2_{m,\bar{m}}}\right\rangle = 0; \qquad m \neq \bar{m}\\
\left\langle \mathcal{O}_{S^2_{1,1}}\right\rangle &= \left\langle \mathcal{O}_{S^2_{-1,-1}}\right\rangle = -\frac{N}{8\sqrt{2}\pi}\mu_n^2(a^2 - b^2);\\
\left\langle \mathcal{O}_{S^2_{0,0}}\right\rangle &= \frac{N}{4\sqrt{2}\pi}\mu_n^2(a^2 b^2 - 1);\\
\left\langle \mathcal{O}_{\Sigma^2_{1,1}}\right\rangle &= \left\langle \mathcal{O}_{\Sigma^2_{-1,-1}}\right\rangle = -\frac{\sqrt{3}N}{8\sqrt{2}\pi}\mu_n^2(a^2 - b^2);\\
\left\langle \mathcal{O}_{\Sigma^2_{0,0}}\right\rangle &= \frac{\sqrt{3}N}{4\sqrt{2}\pi}\mu_n^2(a^2 b^2 - 1).
\end{aligned} \tag{3.150}$$

Here we denote by $(m, \bar{m})$ the $(SU(2)_L, SU(2)_R)$ charges. The vanishing of the vevs of operators with charges $m \neq \bar{m}$ follows from the fact that the curve preserves rotational symmetry in the 3-4 plane. The equality of the vevs for operators with charge $(1,1)$ and $(-1,-1)$ follows from the orientation of the ellipse in the 1-2 plane: its axes are orientated with the 1-2 axes. Explicit representations of the corresponding spherical harmonics are given in (3.242).

One can also consider a planar ellipsoidal curve of different orientation, described by the curve

$$F^1 = \mu_n\left(a\cos\frac{2\pi n v}{L} + c \sin\frac{2\pi n v}{L}\right), \qquad F^2 = \mu_n\left(b \sin\frac{2\pi n v}{L} + d\cos\frac{2\pi n v}{L}\right), \tag{3.151}$$

with $F^3 = F^4 = 0$ and μ_n as in (3.143). The D1-brane charge constraint (3.48) in this case requires that $(a^2 + b^2 + c^2 + d^2) = 2$. The non-vanishing vevs are

$$\begin{aligned}
\langle J^{+3}\rangle &= \frac{N}{4\pi}\mu_n(ab - cd)(dy - dt); \qquad \langle J^{-3}\rangle = \frac{N}{4\pi}\mu_n(ab - cd)(dt + dy);\\
\left\langle \mathcal{O}_{S^2_{\pm 1,\pm 1}}\right\rangle &= -\frac{N}{8\sqrt{2}\pi}\mu_n^2((a \pm id)^2 + (c \pm ib)^2);\\
\left\langle \mathcal{O}_{S^2_{0,0}}\right\rangle &= \frac{N}{4\sqrt{2}\pi}\mu_n^2((ab - cd)^2 - 1);\\
\left\langle \mathcal{O}_{\Sigma^2_{\pm 1,\pm 1}}\right\rangle &= -\frac{\sqrt{3}N}{8\sqrt{2}\pi}\mu_n^2((a \pm id)^2 + (c \pm ib)^2);\\
\left\langle \mathcal{O}_{\Sigma^2_{0,0}}\right\rangle &= \frac{\sqrt{3}N}{4\sqrt{2}\pi}\mu_n^2((ab - cd)^2 - 1).
\end{aligned} \tag{3.152}$$

The vevs for operators with charge $(1,1)$ and $(-1,-1)$ are no longer equal, since the axes of the ellipse are no longer orientated with the 1-2 axes. The vevs are however complex conjugate, as they must be since the operators are complex conjugate to each other.

(3.8) DUAL FIELD THEORY

To understand the interpretation of the holographic results it will be useful to review certain aspects of the dual CFT and the ground states of the R sector. The dual CFT is believed to be a deformation of the $\mathcal{N}=(4,4)$ supersymmetric sigma model with target space $S^N(X)$, where $N = n_1 n_5$ and the compact space is either T^4 or $K3$. Most of the discussion below will be for the case of T^4, although the results extend simply to $K3$. The orbifold point is roughly the analogue of the free field limit of $\mathcal{N}=4$ SYM in the context of AdS_5/CFT_4 duality.

The chiral primaries and R ground states can be precisely described at the orbifold point. In particular, there exists a family of chiral primaries in the NS-NS sector associated with the $(0,0)$, $(2,0)$, $(0,2)$, $(1,1)$ and $(2,2)$ cohomology of the internal manifold (we do not discuss the chiral primaries associated with odd cohomology in this chapter). These can be labeled as

$$\begin{aligned}
&\mathcal{O}_n^{(0,0)}, \qquad h=\bar{h}=\tfrac{1}{2}(n-1);\\
&\mathcal{O}_n^{(2,0)}, \qquad h=\bar{h}+1=\tfrac{1}{2}(n+1);\\
&\mathcal{O}_n^{(1,1)q}, \qquad h=\bar{h}=\tfrac{1}{2}n; \qquad q=1,\dots,h^{1,1}\\
&\mathcal{O}_n^{(0,2)}, \qquad h=\bar{h}-1=\tfrac{1}{2}(n-1);\\
&\mathcal{O}_n^{(2,2)}, \qquad h=\bar{h}=\tfrac{1}{2}(n+1),
\end{aligned} \tag{3.153}$$

where n is the twist, $h^{1,1}$ in the dimension of the $(1,1)$ cohomology and $h=j^3$, $\bar{h}=\bar{j}^3$. The operator $\mathcal{O}_1^{(0,0)}$ is the identity operator. The complete set of chiral primaries associated with this cohomology is built from products of the form

$$\prod_{l=1}(\mathcal{O}_{n_l}^{(p_l+1,q_l+1)})^{m_l}, \qquad \sum_{l=1}^{I} n_l m_l = N\,, \tag{3.154}$$

where p_l, q_l take the values ± 1 (so that one gets the product of operators in (3.153); we suppress the index q) and symmetrization over the N copies of the CFT is implicit.

In [66] the spectrum of chiral primary operators of the orbifold CFT was matched with the KK spectrum. One should note however that the matching is not canonical in the sense that the operators at the orbifold point and the fields in supergravity are characterized by additional labels not visible in the other description. In particular, the supergravity spectrum is also organized in representations of an additional[9] $\widetilde{SO}(4)\times SO(n_t)$, as can be seen from the tables of [55], where the $\widetilde{SO}(4)$ is the R-symmetry of the 6D supergravity (not to be confused with the $SO(4)$ R-symmetry of the CFT which is related to the isometries of the S^3) and n_t is the

[9] $\widetilde{SO}(4)$ was called $SO(4)_R$ in [55].

number of tensor multiplets. On the other hand, the chiral spectrum at the orbifold point is characterized by the set of integers n_l, m_l and the type of operator associated with these, as in (3.154). Furthermore, there is an additional $SO(4)_I$ acting on the chiral spectrum, related to global rotations of T^4 (see, for example, [62] or the review [67]). It is not immediately clear how the labels n_l, m_l translate in the supergravity description and what is the relation of $SO(4)_I$ with the supergravity $\widetilde{SO}(4) \times SO(5)$ ($n_t = 5$ for T^4).

To get a more precise mapping let us consider the special case of chiral primaries with $h = \bar{h}$. We see from (3.153) that there are 6 such operators for any $h < N/2$, except when $h = 1/2$ in which case there are only 5 operators ($\mathcal{O}_1^{(0,0)}$ is the identity operator). In all cases 4 of these operators form a vector of $SO(4)_I$. On the supergravity side, the fields $S_k^{(r)I}$ and Σ_k^I have the correct dimensions and charges to correspond to these operators. Note that $k > 1$ for Σ_k^I, so we indeed have only 5 fields corresponding to operators of dimension $(1/2, 1/2)$. These fields are singlets under $\widetilde{SO}(4)$ and $S_k^{(r)I}$ transforms in the vector of $SO(5)$. It thus appears natural to identify $SO(4)_I$ with an $SO(4)$ subgroup of $SO(5)$ and to make the correspondence

$$S_n^{p(q+6)} \quad \leftrightarrow \quad \mathcal{O}_n^{(1,1)q}, \quad q = 1, \ldots, 4, \qquad n \geq 1 \tag{3.155}$$

where here and below the superscript p denotes that the relevant scalar fields are those for which $j^3 = j$ and $\bar{j}^3 = \bar{j}$. The question is then whether $\mathcal{O}_{n+1}^{(0,0)}$ or $\mathcal{O}_{n-1}^{(2,2)}$ corresponds to $S_n^{p(6)}$. The most natural correspondence seems to be

$$\begin{aligned} S_n^{p(6)} \quad &\leftrightarrow \quad \mathcal{O}_{n+1}^{(0,0)}, \qquad n \geq 1; \\ \Sigma_n^p \quad &\leftrightarrow \quad \mathcal{O}_{n-1}^{(2,2)} \qquad n \geq 2. \end{aligned} \tag{3.156}$$

This identification is natural given that there is no Σ_1 in supergravity but is clearly not unique because S_n^p and Σ_n^p have the same charges so it could be that different combinations of them correspond to the operators at the orbifold point.

A similar discussion holds for chiral primaries with $h-\bar{h} = \pm 1/2, \pm 1$. The case of $h-\bar{h} = \pm 1/2$ is not relevant here since we are not considering solutions associated with odd cohomology in this chapter. The case $h - \bar{h} = \pm 1$ is relevant but most of the points we want to make can be made using examples that utilize only chiral primaries with $h = \bar{h}$, so we will not need a detailed discussion of them. We only mention that the corresponding supergravity fields are massive vector fields.

Spectral flow maps these chiral primaries in the NS sector to R ground states, where

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24}; \\ j_3^R &= j_3^{NS} - \frac{c}{12}, \end{aligned} \tag{3.157}$$

where c is the central charge. Each of the operators in (3.154) is mapped by spectral flow to an operator of definite R-charge

$$\prod_{l=1} (\mathcal{O}_{n_l}^{(p_l+1,q_l+1)})^{m_l} \quad \rightarrow \quad \mathcal{O}^{R(2j_3^R, 2\bar{j}_3^R)}, \qquad j_3^R = \tfrac{1}{2}\sum_l p_l m_l, \; \bar{j}_3^R = \tfrac{1}{2}\sum_l q_l m_l. \tag{3.158}$$

In particular, for fixed twist n the operators in (3.153) have the following charges after the flow

$$\begin{aligned} \mathcal{O}_n^{(0,0)} &\to \mathcal{O}_n^{R(-,-)}; \\ \mathcal{O}_n^{(2,0)} &\to \mathcal{O}_n^{R(+,-)}; \\ \mathcal{O}_n^{(0,2)} &\to \mathcal{O}_n^{R(-,+)}; \\ \mathcal{O}_n^{(2,2)} &\to \mathcal{O}_n^{R(+,+)}; \\ \mathcal{O}_n^{(1,1)q} &\to \mathcal{O}_n^{R(0,0)q}, \end{aligned} \tag{3.159}$$

where it is understood that each of these operators is tensored by the appropriate power of the identity operator such that (3.154) holds. For example, $\mathcal{O}_n^{(0,0)}$ should be tensored by $(\mathcal{O}_1^{(0,0)})^{N-n}$, and the R-symmetry charge of the flown operator $\mathcal{O}_n^{R(-,-)}$ follows from (3.157) with $c = 6n$. It follows from (3.159) that the operators $\mathcal{O}_n^{R(\pm,\pm)}$ form a $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)_L \times SU(2)_R$ whilst the operators $\mathcal{O}_n^{R(0,0)q}$ are q singlets. From the form of the operators in the NS sector (3.154) it is clear that $j^R \leq \frac{1}{2}N$, since one can have at most N operators in the product. Symmetrization over the copies of the CFT means that spectral flow in the left and right moving sectors is not quite independent. When one has m copies of the same operator one needs to symmetrize over copies and thus one obtains only states with $j^R = \bar{j}^R = \frac{1}{2}m$ (although the values of j_3^R and $\bar{j}_3^R$ range independently from $-j^R$ to j^R).

We will label by the R-charges the states obtained by the usual operator-state correspondence,

$$|j_3^R, \bar{j}_3^R\rangle = \mathcal{O}^{R(2j_3^R, 2\bar{j}_3^R)}(0)|0\rangle. \tag{3.160}$$

(3.8.1) R GROUND STATES AND VEVS

The R ground states can also be characterized by the expectation value of gauge invariant operators in them. Since the fuzzball solutions are conjectured to be dual to R ground states and the vevs of gauge invariant operators is the information we extracted from the fuzzball solutions we would like to see what one can say about them using the dual CFT. There are two sets of constraints on these vevs: kinematical and dynamical.

KINEMATICAL CONSTRAINTS

The kinematical constraints follow from symmetry considerations and they have been recently discussed in [43]. As discussed above the R ground states in the (usual) basis are eigenstates of the R-symmetry charge. This implies that only neutral operators can have a non-vanishing vev,

$$\langle -j_3^R, -\bar{j}_3^R | O^{(k_1,k_2)} | j_3^R, \bar{j}_3^R \rangle = 0, \qquad \{k_1 \neq 0 \; or \; k_2 \neq 0\} \tag{3.161}$$

where k_1 and k_2 are the R-charges of the operator and we use the fact that the bra state has the opposite R charge to the ket state.

Dynamical constraints and 3-point functions

The vevs of neutral gauge invariant operators are determined dynamically. One way to determine them is using 3-point functions at the conformal point. Let $|\Psi\rangle = O_\Psi(0)|0\rangle$. Then the vev of an operator O_k of dimension k in the this state is given by

$$\langle\Psi|O_k(\lambda^{-1})|\Psi\rangle = \langle 0|(O_\Psi(\infty))^\dagger O_k(\lambda^{-1})O_\Psi(0)|0\rangle, \tag{3.162}$$

where λ is a mass scale. For scalar operators the 3-point function is uniquely determined by conformal invariance and the above computation yields

$$\langle\Psi|O_k(\lambda^{-1})|\Psi\rangle = \lambda^k C_{\Psi k\Psi} \tag{3.163}$$

where $C_{\Psi k\Psi}$ is the fusion coefficient. Similarly, the expectation value of a symmetry current measures the charge of the state

$$\langle\Psi|j(\lambda^{-1})|\Psi\rangle = \langle 0|(O_\Psi(\infty))^\dagger j(\lambda^{-1})O_\Psi(0)|0\rangle = q\lambda\langle\Psi|\Psi\rangle \tag{3.164}$$

where q is the charge of the operator O_Ψ under j.

Let us now apply these principles to the cases of interest here. We will thus need to know the 3-point functions at the conformal point, which can be computed in the NS sector and then flowed to the R sector. A computation of 3-point functions at the orbifold point has been given in [62, 63]. We however need to know the result in the regime where supergravity is valid. For the theory at hand there is no known non-renormalization theorem that would protect the 3-point functions. Moreover, as discussed in appendix 3.A.6, the 3-point functions that can also be computed holographically (i.e. those involving only operators dual to supergravity fields) are different from the 3-point functions computed at the orbifold point.

So the only dynamical tests that one can currently do must involve states created by operators corresponding to single particle states. In our case the fuzzball solutions are meant to correspond to the R ground states connected with universal cohomology, so only states created by the operators $\mathcal{O}_n^{R(\pm,\pm)}$ are relevant. For these cases the corresponding 3-point point functions can be computed by standard holographic methods using the results in [61, 24].

Let $\Phi = (S, A^+, A^-, \Sigma)$ be the fields dual to the operators $\mathcal{O}_n^{R(\pm,\pm)}$. The three point functions involving scalar chiral primaries have the following structure

$$\left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_\Sigma \mathcal{O}_\Phi \right\rangle \neq 0, \qquad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_S \mathcal{O}_\Phi \right\rangle = 0. \tag{3.165}$$

where $\mathcal{O}_\Phi^\dagger$ denotes the conjugate operator with $j^3 = -j$, $\bar{j}^3 = -\bar{j}$. Our results for the vevs include the lowest dimension operators in these towers.

From the results of [24] there are however other non-zero three point functions in supergravity, such as

$$\left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_\tau \mathcal{O}_\Phi \right\rangle \neq 0, \qquad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_{\rho^\pm} \mathcal{O}_\Phi \right\rangle \neq 0, \qquad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_{A^\pm} \mathcal{O}_\Phi \right\rangle \neq 0, \qquad \cdots \tag{3.166}$$

where the ellipses denote other operators, dual to other vectors and KK gravitons. These operators all have sufficiently high dimensions that we did not compute their vevs. Moreover, the vevs of these operators are not captured at all by the linearized approximation.

(3.9) Correspondence between fuzzballs and chiral primaries

(3.9.1) Correspondence with circular curves

Having reviewed the description of the degenerate R ground states in the CFT we now turn to the connection with the fuzzball solutions. The basic proposal is that there is a correspondence between the R ground states and the curves $F^i(v)$ defining the supergravity solutions. Let us consider first states of the specific form

$$(\mathcal{O}_n^{R(\pm,\pm)})^{\frac{N}{n}}|0\rangle, \qquad j_3^R = \pm\frac{N}{2n}; \quad \bar{j}_3^R = \pm\frac{N}{2n}. \tag{3.167}$$

Then such ground states are proposed to be in one to one correspondence with circular curves [27]:

$$(\mathcal{O}_n^{R(+,+)})^{\frac{N}{n}}|0\rangle \quad \leftrightarrow \quad F^1 = \frac{\mu}{n}\cos(\frac{2\pi n v}{L}); \quad F^2 = \frac{\mu}{n}\sin(\frac{2\pi n v}{L}), \tag{3.168}$$

with $F^3 = F^4 = 0$ and where the parameter μ is fixed via (3.48) to be $\sqrt{Q_1 Q_5}/R$, see (3.143). Similarly $(\mathcal{O}_n^{R(-,-)})^{N/n}$ corresponds to a circle of the same radius in the 1-2 plane with the opposite rotation (that is, $F^2 \rightarrow -F^2$) and the operators $(\mathcal{O}_n^{R(+,-)})^{N/n}$, $(\mathcal{O}_n^{R(-,+)})^{N/n}$ correspond to circles in the 3-4 plane.

Note the states (3.167) are generically not dual to supergravity fields. Only the specific states obtained by flowing the NS operators $((\mathcal{O}_1^{(0,0)})^N, \mathcal{O}_N^{(p,q)})$ correspond to supergravity fields. All product operator do not correspond to supergravity fields, with the exception of $(\mathcal{O}_1^{(0,0)})^N$, since this is simply the identity operator in the NS sector. Moreover, whilst the operators $\mathcal{O}_N^{(p,q)}$ are dual to supergravity fields their special properties (following from having maximal dimension) are not visible in supergravity computations which effectively takes $N \rightarrow \infty$.

There are various pieces of evidence for this correspondence between states and circular curves. Firstly the rotation charges match, using the discussions in section 3.7.1, in particular (3.148). Secondly, as first discussed in [27], one can consider absorption processes in the corresponding geometries, and compare the scattering behavior with CFT expectations; they agree. (Note that for a general fuzzball geometry the wave equation for minimal scalars is not separable, so the absorption cross-section cannot be computed, and this comparison cannot be made.)

Our results for the scalar 1-point functions in (3.147) (along with (3.133)) give more data which can be used to test the proposed correspondence. As discussed previously kinematical constraints arise simply from charge conservation: if the R ground state is an eigenstate of both

j_3^R and $\bar{j}_3^R$ then only scalar operators with $j^3 = \bar{j}^3 = 0$ can acquire a vev. These correspond to the Y_0^k harmonics discussed in section 3.7.1. Thus the fact that only such operators appear in (3.147) follows solely from kinematics.

Determining which of the (kinematically allowed) operators actually acquire a vev involves dynamics also and is rather more subtle. Consider first the special case where the operator (3.154) determining the ground state is the product $(\mathcal{O}_1^{(0,0)})^N$, that is, the NS vacuum. Then clearly all three point functions vanish, and thus all 1-point functions (apart from j) in the corresponding R vacuum must vanish.

Moreover the vanishing of all 1-point functions implies that the non-linear terms in the vevs of $\mathcal{O}_{\Sigma_I^k}$ in (3.133) must contribute. The linear terms in (3.133) do give the expected vanishing vev for $\mathcal{O}_{S_I^k}$ since the D1-brane and D5-brane densities are constant along the curve. However, for the circular profile the linear terms in the $\mathcal{O}_{\Sigma_I^k}$ vevs following from (3.133) give

$$\left\langle \mathcal{O}_{\Sigma_0^k} \right\rangle = (-)^{k+1/2} N (\frac{\sqrt{Q_1 Q_5}}{R})^k \frac{(k-1)^{3/2}}{\pi \sqrt{k(k+1)}} + \cdots \tag{3.169}$$

and therefore the non-linear terms denoted by ellipses must contribute, to give the expected zero vev.

Next consider the cases where the operator (3.154) determining the ground state is $(\mathcal{O}_1^{(2,0)})^N$, $(\mathcal{O}_1^{(0,2)})^N$ or $(\mathcal{O}_1^{(2,2)})^N$. The supergravity solutions corresponding to these vacua are clearly closely related to that just discussed: the defining curve is still a circle with radius $a = 1/\tilde{R}$, but the rotation is in the opposite direction or the circle lies in the 3-4 plane. Therefore the one point functions should also vanish in these three cases. This is consistent with the fact that these NS operators are related to the NS vacuum under spectral flow by an integral parameter (i.e. NS to NS). That is, under a spectral flow

$$h' = h - 2\theta j + \frac{c\theta^2}{6}; \qquad j_3' = j_3 - \frac{c\theta}{6} \tag{3.170}$$

with $\theta = 1$ the chiral primary with maximal $j^3 = N$ is mapped to the vacuum.

Now let us move to the more general states of the form (3.167), which are conjectured to correspond to circular curves. Still there are no scalar chiral primary vevs according to (3.169). Kinematics again dictates that only $j^3 = \bar{j}^3 = 0$ operators acquire a vev, but the fact that kinematically allowed vevs are zero follows from dynamical information about three point functions. In particular, one needs to know the three point functions at the conformal point for operators $\mathcal{O}_\Phi$ which are products in the CFT, and which therefore do not correspond to single particle supergravity fluctuations. These are not known, so the results for the vevs provide a prediction for these correlation functions at strong coupling, provided the conjectured correspondence is correct.

(3.9.2) Non-circular curves

Next we consider the curves corresponding to the most general states of the form (3.154); it has been conjectured that these should correspond to connected curves in R^4. For example, a state of the form

$$(\mathcal{O}_n^{R++})^{\gamma N/n}(\mathcal{O}_n^{R--})^{\delta N/n} \qquad \gamma+\delta=1 \qquad j_3^R=\bar{j}_R^3=\tfrac{1}{2}N(\gamma-\delta)/n, \tag{3.171}$$

was conjectured in [27] to correspond to an elliptical curve

$$F^1(v)=\mu\frac{a}{n}\cos(\frac{2\pi nv}{L}); \qquad F^2(v)=\mu\frac{b}{n}\sin(\frac{2\pi nv}{L}), \tag{3.172}$$

with $F^3=F^4=0$ and $\mu=\sqrt{Q_1Q_5}/R$. Provided that

$$a=\frac{1}{\sqrt{2}}(\sqrt{1+(\gamma-\delta)}+\sqrt{1-(\gamma-\delta)}); \qquad b=\frac{1}{\sqrt{2}}(\sqrt{1+(\gamma-\delta)}-\sqrt{1-(\gamma-\delta)}), \tag{3.173}$$

the supergravity solution would have the correct angular momenta to match with the field theory state.

Without any further data to match between supergravity and field theory one could not check the proposed correspondence further. The one point functions of chiral primaries computed here, however, immediately contradict the correspondence between operators of the form (3.154) and connected curves in R^4. The issue is the following. States of the form (3.154) are eigenstates of angular momentum operators j_R^3 and $\bar{j}_R^3$. This means that scalar operators can acquire a vev only if $j_R^3=\bar{j}_R^3=0$, following (3.161). Note that this is again purely kinematical, with dynamical information determining precisely which of these operators actually acquire a vev.

However, the supergravity solution generated by a connected curve will, according to the formulae, give rise to non-zero vevs for operators with $(j_R^3,\bar{j}_R^3)\neq 0$ whenever the curve is not circular. Put differently, a non-circular curve explicitly breaks the $SO(2)\times SO(2)$ symmetries, with the symmetry breaking characterized by the vevs for operators with non-zero $(j_R^3,\bar{j}_R^3)$.

One might wonder whether a non-circular curve could nonetheless give rise to vevs only for $j_R^3=\bar{j}_R^3=0$ operators. That is, although the curve is non-circular in flat coordinates on R^4, it might be circular in another coordinate system, and the vevs might be related to multipole moments in that coordinate system. This however contradicts the explicit formulae for the vevs, exemplified by the case of an ellipsoidal curve, whose vevs are given in (3.150). More generally, the vevs will clearly involve the multipole moments of the charge distribution on the R^4.

(3.9.3) Testing the new proposal

Now consider the proposal made in [43] and here, that the supergravity solution defined by a given curve is dual to a linear superposition of states with coefficients following from those in the coherent state in the dual FP system. In particular, according to (3.13) and (3.43) the ellipse (3.172) would be dual to the linear superposition

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k}(a-b)^k (\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)} (\mathcal{O}_n^{R--})^k; \tag{3.174}$$

note that $(a^2+b^2) = 2$ and that (a,b) are both real.

The issue is whether this proposal is consistent with the vevs extracted from the corresponding geometry in section (3.7.2). Again this question is divided into kinematical and dynamical parts. The fact that operators with equal and opposite J^{12} charge acquire equal values in section (3.7.2) follows from the orientation of the ellipse and is a kinematical constraint which must also be implicit in the dual description. (That operators with non-zero J^{34} charge do not acquire a vev is also a kinematical constraint, of course, but this is automatically satisfied for any proposed dual involving only operators of zero J^{34} charge.) The actual non-zero values for the vevs in section (3.7.2) require dynamical information.

So does the proposed linear superposition satisfy the kinematical constraints? We can prove that it does as follows. Let us write (3.174) as

$$|ellipse) = \sum_{k=0}^{N/n} a_k |(\frac{N}{n}-k);k\rangle, \tag{3.175}$$

where $|(\frac{N}{n}-k);k\rangle$ is shorthand for the state created by $(\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)}(\mathcal{O}_n^{R--})^k$ and a_k are real coefficients (that can be read-off from (3.174)). Now consider a general J^{12} charged operator $\mathcal{O}_{m,m}$. Its vev is given by

$$(ellipse|\mathcal{O}_{m,m}|ellipse) = \sum_{k=0}^{N/n-m} a_k^* a_{m+k} \langle(\frac{N}{n}-k);k|\mathcal{O}_{m,m}|(\frac{N}{n}-k-m);k+m\rangle, \tag{3.176}$$

whilst the corresponding operator with opposite charge $\mathcal{O}_{-m,-m}$ acquires a vev

$$(ellipse|\mathcal{O}_{-m,-m}|ellipse) = \sum_{k=0}^{N/n-m} a_{m+k}^* a_k \left(\langle(\frac{N}{n}-k);k|\mathcal{O}_{m,m}|(\frac{N}{n}-k-m);k+m\rangle\right)^\dagger, \tag{3.177}$$

Given that the coefficients a_m are real, the vevs (3.176) and (3.177) will be the same provided that the overlaps are real; the fusion coefficients for the corresponding extremal three point functions do indeed have this property.

To test the values of the non-zero vevs in (3.150) one needs dynamical information. One can check that the R charges are in agreement with those of the superposition (3.174) as follows. The state $|(\frac{N}{n}-k);k\rangle$ is an eigenstate of both j^3 and $\bar{j}^3$ with (equal) eigenvalues $(N/2n-k)$. Then

$$\begin{aligned}(ellipse|j^3|ellipse) &= \sum_{k=0}^{N/n} \frac{1}{2^{2\frac{N}{n}}} \frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!} (a+b)^{2\frac{N}{n}-k}(a-b)^{2k}(\frac{N}{2n}-k) \qquad (3.178)\\ &= -\frac{(a^2-b^2)^{\frac{N}{n}}}{2^{\frac{2N}{n}+1}} z\frac{\partial}{\partial z}(z+\frac{1}{z})^{\frac{N}{n}} = \frac{N}{2n}ab; \qquad z = \frac{(a-b)}{(a+b)};\end{aligned}$$

with the same result for $\bar{j}^3$. This is in exact agreement with the result of (3.150).

The remaining non-zero vevs of (3.150) are the vevs of the charged operators $\mathcal{O}_{1,1} \equiv \{\mathcal{O}_{S_{1,1}}, \mathcal{O}_{\Sigma_{1,1}}\}$, and the neutral operators, $\mathcal{O}_{0,0} \equiv \{\mathcal{O}_{S_{0,0}}, \mathcal{O}_{\Sigma_{0,0}}\}$, where (m,n) denote the $(SU(2)_L, SU(2)_R)$ charges. To test whether the proposal is consistent with these vevs is far more difficult: we would need to know the three point functions between all operators occurring in (3.174) and the dimension two operators. Given that the former are not dual to supergravity fields, we do not have any information about the relevant three point functions and thus cannot check the vevs. That said, a well motivated guess for the structure of the three point functions leads to vevs which agree remarkably well with those in (3.150).

Note that in (3.150) the vevs of the operators with the same charges are the same up to overall numerical coefficients. We aim here to derive the universal behavior. For simplicity we set $n = 1$. The corresponding state $|N-k;k\rangle$ in the FP system is a multiparticle state, built out of free harmonic oscillators, as in (3.33), containing $(N-k)$ quanta of negative angular momentum and k quanta of positive angular momentum. We will assume that the same picture holds in the D1-D5 system, at least in the large N limit, where the negative (positive) angular momenta quanta are now positive (negative) R-charge quanta.

We now treat $\mathcal{O}_{1,1}$ and $\mathcal{O}_{0,0}$ in similar way. $\mathcal{O}_{1,1}$ creates a quantum of positive R-charge and destroys a quantum of negative R-charge, so

$$\mathcal{O}_{1,1} \sim (a^{-12})^\dagger a^{+12}, \tag{3.179}$$

and $\mathcal{O}_{0,0}$ is the product of number operators for positive and negative R-charge quanta,

$$\mathcal{O}_{0,0} \sim \frac{1}{N}\left((a^{+12})^\dagger a^{+12}\right)\left((a^{-12})^\dagger a^{-12}\right), \tag{3.180}$$

where the normalization factor is introduced for later convenience.

Using standard harmonic oscillator relations then yields

$$\langle N-k;k|\mathcal{O}_{1,1}|N-k-1;k+1\rangle \sim \sqrt{(N-k)(k+1)}\mu^2, \tag{3.181}$$

with the scale μ^2 appropriate to a dimension two operator inserted, as in (3.163). Then the total vev for the ellipse is

$$\begin{aligned}(ellipse|\mathcal{O}_{1,1}|ellipse) &\sim \sum_{k=0}^{N-1}\frac{\mu^2}{2^{2N}}\frac{N!}{(N-1-k)!k!}(a+b)^{2N-2k-1}(a-b)^{2k+1};\\ &= \frac{N\mu^2}{2^{2N}}(2(a^2+b^2))^{N-1}(a^2-b^2) = \tfrac{1}{4}N\mu^2(a^2-b^2),\end{aligned} \tag{3.182}$$

which indeed agrees in form with the vevs of charged operators in (3.150). The fact that such a simple approximation for the three point functions works so well merits further investigation.

For the neutral operators we obtain

$$\langle N-k;k|\mathcal{O}_{0,0}|N-k,k\rangle \sim \frac{1}{N}\mu^2(N-k)k, \tag{3.183}$$

and the corresponding total vev for this neutral operator is

$$\begin{aligned}(ellipse|\mathcal{O}_{0,0}|ellipse) &\sim \sum_{k=1}^{N-1} \frac{\mu^2}{2^{2N}} \frac{(N-1)!}{(k-1)!(N-(1+k))!}(a+b)^{2(N-k)}(a-b)^{2k}; \\ &= \frac{1}{2^{2N}}(N-1)\mu^2(a^2-b^2)^2(2(a^2+b^2))^{N-2} \sim \frac{1}{4}N\mu^2(1-a^2b^2),\end{aligned} \quad (3.184)$$

in agreement with the vevs for uncharged operators given in (3.150). Note that (3.183) also gives zero for $k=0$ and $k=N$, in agreement with the vanishing vevs of the neutral operators for the circular case.

Now consider the more general ellipse of (3.151). The proposed dual in this case would be

$$\begin{aligned}|a,b,c,d) &= \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}}(A_+)^{\frac{N}{n}-k}(A_-)^k(\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)}(\mathcal{O}_n^{R--})^k, \\ A_\pm &= (a\pm b)+i(c\mp d),\end{aligned} \quad (3.185)$$

with $(a^2+b^2+c^2+d^2)=2$. Following the same steps as above, one finds exactly the R charges as in (3.152), supporting the proposal. As discussed below (3.152), charged operators $\mathcal{O}_{1,1}$ and $\mathcal{O}_{-1,-1}$ no longer have equal vevs. Repeating the steps which led to (3.176) and (3.177) one finds that

$$(\frac{A_+}{A_-})^m\langle\mathcal{O}_{m,m}\rangle = (\frac{A_+^*}{A_-^*})^m\langle\mathcal{O}_{-m,-m}\rangle. \quad (3.186)$$

Taking the case $m=1$ this is indeed the relationship between the vevs $\langle\mathcal{O}_{\Sigma^2_{\pm1,\pm1}}\rangle$ and $\langle\mathcal{O}_{S^2_{\pm1,\pm1}}\rangle$ in (3.152), thus demonstrating that the proposal passes kinematical checks. Now let us compute the vevs of the dimension two charged operators using the same approximation (3.181) as before; this gives

$$\langle\mathcal{O}_{\pm1,\pm1}\rangle \sim N\mu^2((a\pm id)^2+(c\pm ib)^2), \quad (3.187)$$

in agreement with (3.152). There is similar agreement for the behavior of the vevs of neutral operators $\mathcal{O}_{0,0}$. Of course, given the agreement for the ellipse above, there must be agreement for the rotated ellipse if the proposed dual captures correctly the orientation of the curve in the 1-2 plane. Nonetheless, this example clearly demonstrates how the parameters of the curve are captured by the (complex) coefficients in the linear superposition.

So to summarize: we have tested the proposed field theory dual in the case of elliptical curves. We find perfect agreement for all kinematically determined quantities, thus demonstrating the consistency of the proposal. We also find exact matching for the R charges and qualitative agreement for the vevs of the scalar operators. To test the correspondence further would require knowledge of three point functions involving multiparticle states at the conformal point.

(3.10) Symmetric supergravity solutions

We next move to the question of whether one can find geometries which are dual to a single chiral primary, rather than a superposition of chiral primaries. As has already been discussed, a geometry which is dual to a chiral primary must preserve the $SO(2) \times SO(2)$ symmetry. This immediately implies that the asymptotics must be of the following form:

$$f_5 = \frac{Q_5}{r^2} \sum_{k=2l} \frac{f^5_{k0}}{r^k} Y^0_k; \tag{3.188}$$
$$f_1 = \frac{Q_1}{r^2} \sum_{k=2l} \frac{f^1_{k0}}{r^k} Y^0_k,$$

where the scalar spherical harmonics Y^0_{2l} which are singlets under $SO(2) \times SO(2)$ are defined in (3.238). The forms (A, B) must similarly admit an asymptotic expansion of the form:

$$A_a = \sum_k \frac{Q_5}{r^{k+1}} (A_{k0+} Y^{0+}_{ka} + A_{k0-} Y^{0-}_{ka}); \tag{3.189}$$
$$B_a = \sum_k \frac{Q_5}{r^{k+1}} (-A_{k0+} Y^{0+}_{ka} + A_{k0-} Y^{0-}_{ka}),$$

where the vector spherical harmonics $Y^{0\pm}_{ka}$ of degree k (k odd) whose Lie derivatives along the $SO(2)^2$ directions are zero are defined in (3.246). Note that these forms have only components along the (ϕ, ψ) directions. We will now give several examples of solutions which have asymptotics of this form, and discuss their interpretations.

(3.10.1) Averaged geometries

Here we discuss a way to construct supergravity solutions based on a general closed curve F^i which are symmetric under $SO(2) \times SO(2)$ and thus have vanishing vevs for all charged operators. Let us first discuss the construction for arbitrary planar curves in the 1-2 plane. Starting from a general curve $(F^1, F^2, 0, 0)$ we construct a rotated curve,

$$\tilde{F}^1 = \cos\alpha F^1 + \sin\alpha F^2, \qquad \tilde{F}^2 = -\sin\alpha F^1 + \cos\alpha F^2, \tag{3.190}$$

and then superimpose the solutions. This leads to a new harmonic function,

$$f_5 = \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{Q_5}{L} \int_0^L \frac{dv}{|x - \tilde{F}|^2} = \frac{Q_5}{L} \int_0^L \frac{dv}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}} \tag{3.191}$$

where we use coordinates on R^4 such that $(x^1)^2 + (x^2)^2 = r^2 \sin^2\theta, (x^3)^2 + (x^4)^2 = r^2 \cos^2\theta$. The harmonic function for f_1 is the same as f_5 in (3.191) but with the numerator on the rhs multiplied by $|\dot{F}|^2$. The non-vanishing part of the gauge field is given by

$$A_\phi = \frac{Q_5}{L} \int_0^L \frac{\dot{F}^{[1} F^{2]} dv}{|F|^2} (1 - \frac{r^2 + |F|^2}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}}), \tag{3.192}$$

where ϕ is a polar coordinate in the 1-2 plane and square brackets indicate antisymmetrization with unit strength. The only non-vanishing component of the dual form B is

$$B_\psi = \frac{Q_5}{L} \int_0^L \frac{\dot{F}^{[1} F^{2]} dv}{|F|^2} (\frac{r^2 - |F|^2}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}} - 1). \tag{3.193}$$

where ψ is a polar coordinate in the 3-4 plane. For a general curve (F^1, F^2, F^3, F^4) we can proceed analogously by considering solutions rotated by angle α in the 1-2 plane and by angle β in 3-4 plane and then averaging over α and β. For example, the function f_5 would be given by

$$\begin{aligned} f_5 &= \int_0^{2\pi} \frac{d\beta}{2\pi} \frac{Q_5}{L} \int_0^L \frac{dv}{\sqrt{(r^2 + |F|^2 - 2r\cos\theta g(\beta))^2 - 4r^2((F^1)^2 + (F^2)^2)\sin^2\theta}}; \\ g(\beta) &= (F^3 \cos(\psi + \beta) + F^4 \sin(\psi + \beta)). \end{aligned} \tag{3.194}$$

This integral can be expressed in terms of elliptic integrals, although we have not obtained the exact form. The asymptotics are however given by:

$$\begin{aligned} f_5 &= \frac{Q_5}{Lr^2} \int_0^L \sum_{l \geq 0} \frac{dv}{r^{2l}} P_l(\cos(2\theta)) P_l(Z(F(v))); \\ f_1 &= \frac{Q_5}{Lr^2} \int_0^L \sum_{l \geq 0} \frac{dv}{r^{2l}} |\partial_v F|^2 P_l(\cos(2\theta)) P_l(Z(F(v))); \\ A &= \frac{Q_5}{L} \int_0^L \sum_k \frac{dv}{\sqrt{2}(k+1) r^{k+1}} \left(p_k(F)(\dot{F}^1 F^2 - \dot{F}^2 F^1)(Y_{ka}^{0+} - Y_{ka}^{0-}) \right. \\ &\qquad \left. + q_k(F)(\dot{F}^3 F^3 - \dot{F}^4 F^3)(Y_{ka}^{0+} + Y_{ka}^{0-}) \right); \\ Z(F) &= (F^3)^2 + (F^4)^2 - (F^1)^2 - (F^2)^2, \end{aligned} \tag{3.195}$$

where $P_l(x)$ are Legendre polynomials of degree l and $p_k(F)$ and $q_k(F)$ are defined in (3.248)-(3.250). These asymptotics are manifestly of the form given in (3.188) and (3.189). Setting $F^3 = F^4 = 0$ gives the asymptotic expansion of the expressions given in (3.191) and (3.192).

EXAMPLE 1: THE AVERAGED ELLIPSE

Consider the case of an ellipse, so that the defining curve is

$$F^1 = \mu a \cos\frac{2\pi v}{L}, \qquad F^2 = \mu b \sin\frac{2\pi v}{L}, \tag{3.196}$$

with $\mu = \sqrt{Q_1 Q_5}/R$ and $(a^2 + b^2) = 2$. (For simplicity we choose the frequency n to be one.) In this example the integral over the curve in (3.191) can be carried out explicitly to give

$$\begin{aligned} f_5 &= \frac{2Q_5}{\pi z} K(w); \\ z^4 &= (C^2 + D^2); \qquad w = \frac{\sqrt{(z^2 - C)}}{\sqrt{2}z}; \\ C &= (r^4 + 2\mu^2 r^2 \cos 2\theta + \mu^4 a^2 b^2); \\ D &= \mu^2 r^2 \sin 2\theta (a^2 - b^2), \end{aligned} \tag{3.197}$$

where $K(w)$ is the complete elliptic integral of the first kind. Then f_5 has poles only where z has zeroes, namely at $\theta = \pi/2$ and $r = \mu a$ or $r = \mu b$. This suggests that any singularities of the metric are confined to these locations, namely circles of radius a and b in the 1-2 plane, and indeed one finds that the other defining functions (f_1, A, B) only have poles at these locations. Thus the geometry is less singular than one might have anticipated. The integrands have singularities in the annular region defined by $\theta = \pi/2$ and $\mu a \leq r \leq \mu b$ (assuming $a \leq b$) but the integrated functions only have singularities on the circles bounding this annulus. Moreover these singularities seem to be such that the only singularities of the resulting metric are conical.

Example 2: Aichelburg-Sexl metric

The Aichelburg-Sexl metric was also obtained by the procedure of averaging over curve orientations in [28]. The defining curve has a section which is constant:

$$\begin{aligned} F^1 &= a\cos(\frac{2\pi v}{\xi L}); \qquad F^2 = a\sin(\frac{2\pi v}{\xi L}), \qquad 0 \leq v \leq \xi L; \\ F^1 &= a, \qquad \xi L \leq v \leq L, \end{aligned} \tag{3.198}$$

with all other $F^i(v) = 0$ and $\xi < 1$. Such a profile gives rise to the following harmonic functions:

$$\begin{aligned} f_5 &= \left(1 + \frac{Q_5 \xi}{\hat{r}^2 + a^2\cos^2\hat{\theta}} + \frac{Q_5(1-\xi)}{(x_1 - a)^2 + x_2^2 + x_3^2 + x_4^2}\right); \\ f_1 &= \left(1 + \frac{Q_1}{\hat{r}^2 + a^2\cos^2\hat{\theta}}\right); \\ A_\phi &= a\sqrt{\xi}\frac{\sqrt{Q_1 Q_5}}{\hat{r}^2 + a^2\cos^2\hat{\theta}}, \end{aligned} \tag{3.199}$$

where $Q_1 = Q_5 a^2 (2\pi/L)^2/\xi$ and as in (3.139) we introduce non-standard polar coordinates on R^4 to simplify the harmonic functions. Now we take the $SO(2)$ orbit of the defining curve, thus averaging over the location of the constant section in the 1-2 plane. This leads to the $SO(2)$ symmetric harmonic functions

$$\begin{aligned} f_5 &= \left(1 + \frac{Q_5}{\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta}}\right); \\ f_1 &= \left(1 + \frac{Q_1}{\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta}}\right); \\ A_\phi &= \frac{\xi\mu\sqrt{Q_1 Q_5}}{\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta}}, \end{aligned} \tag{3.200}$$

which are those of the Aichelburg-Sexl metric

$$\begin{aligned} ds^2 &= f_1^{-1/2} f_5^{-1/2}\left(-(dt - \frac{\xi\mu\sqrt{Q_1Q_5}}{\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta}}\sin^2\hat{\theta}d\phi)^2 + (dy - \frac{\xi\mu\sqrt{Q_1Q_5}}{\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta}}\cos^2\hat{\theta}d\psi)^2\right) \\ &+ f_1^{1/2} f_5^{1/2}\left((\hat{r}^2 + \xi\mu^2\cos^2\hat{\theta})(\frac{d\hat{r}^2}{\hat{r}^2 + \xi\mu^2} + d\hat{\theta}^2) + \hat{r}^2\cos^2\hat{\theta}d\psi^2 + (\hat{r}^2 + \xi\mu^2)\sin^2\hat{\theta}d\phi^2\right). \end{aligned}$$

Here $\mu = \sqrt{Q_1 Q_5}/R$. This solution is clearly very similar to those based on circular curves, discussed in section 3.7.1. The non-zero vevs extracted from the decoupled part of the geometry follow from (3.127) and are given by

$$\begin{aligned} \left\langle J^{+3} \right\rangle &= \frac{N}{4\pi}\mu\xi(dy - dt); \qquad \left\langle J^{-3} \right\rangle = \frac{N}{4\pi}\mu\xi(dy + dt); \\ \left\langle \mathcal{O}_{\Sigma_0^2} \right\rangle &= \frac{N\sqrt{2}\xi\mu^2}{2\pi\sqrt{3}}(1-\xi). \end{aligned} \tag{3.201}$$

These clearly reduce to those for the case of the circular curves when $\xi = 1$. Note that the Aichelburg-Sexl metrics do not have conical singularities, and are therefore actually less singular than the unaveraged geometries. However, whilst the Aichelburg-Sexl metrics do have the correct asymptotics to correspond to chiral primaries, they are based on averaging curves with straight sections. The interpretation of these straight sections from the dual perspective is rather unclear, given the proposed correspondence between frequencies on the curve and twists of the dual operators.

(3.10.2) Disconnected curves

Another way to obtain solutions which preserve the $SO(2) \times SO(2)$ symmetry is to consider curves made up of disconnected circles. There exist supergravity solutions defined by the following functions

$$\begin{aligned} f_5 &= \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^L \frac{dv_l}{|x - F_l|^2}; \\ f_1 &= \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^{L_l} \frac{dv_l (\partial_{v_l} F_l)^2}{|x - F_l|^2}; \\ A_i &= \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^{L_l} \frac{dv_l \partial_{v_l} F_l^i}{|x - F_l|^2}, \end{aligned} \tag{3.202}$$

where the lth curve is parametrized by v_l with $\sum_l N_l = N$ and is circular within either the 1-2 or 3-4 plane. That is, the curve defining the lth circle is given by

$$F_l^1 = \frac{\sqrt{Q_1 Q_5}}{R n_l} \cos\left(\frac{2\pi n_l v_l}{L}\right); \qquad F_l^2 = \pm\frac{\sqrt{Q_1 Q_5}}{R n_l} \sin\left(\frac{2\pi n_l v_l}{L}\right), \tag{3.203}$$

assuming the circle lies in the 1-2 plane; the sign determines the direction of rotation. A curve lying in the 3-4 plane will take an analogous form. Such a linear superposition of sources solves the field equations and is supersymmetric. By construction the total D5-brane and D1-brane charges are Q_5 and Q_1 respectively, with the lth curve sourcing a fraction N_l/N of (both) the total charges. The related radii and frequencies in (3.203) ensure that the D1-brane charge of each curve is a fraction N_l/N of the total. This prescription also reduces to that given for the curves corresponding to the operators (3.167); in that case one lets $I = N/n$ and $N_l = n$ in the supergravity solution above and takes the circles to be coincident. Furthermore the total R-charges will be given by

$$j_3 = \tfrac{1}{2}\sum_{l=1}^{I} \epsilon_l m_l; \qquad \bar{j}_3 = \tfrac{1}{2}\sum_{l=1}^{I} \bar{\epsilon}_l m_l, \tag{3.204}$$

where $m_l = N_l/n_l$. Here $(\epsilon_l, \bar{\epsilon}_l) = (\pm 1, \pm 1)$ depending on the orientation and rotation of the curve.

Since the sources are located on disconnected circles, the singularity structure of these geometries is similar to that discussed in section 3.7.1. Namely, there are conical singularities whenever $n_l \neq 1$. Thus, these solutions are no more singular than the geometries based on a single circle, although they are more singular than a geometry based on a general non-intersecting curve.

(3.10.3) Discussion

These are not the only symmetric geometries. For example, one could consider more general superpositions of curves, superposing not just different orientation curves but also different shape curves. However, the procedure we outlined above does illustrate how symmetric geometries can be obtained from those defined in terms of a single curve. The symmetrization we used is the simplest, in that the measure for each curve is the same. The field theory dual suggests that symmetrizing over shapes of curves should involve a non-trivial measure. That is, if one has an ellipse with parameters (a, b) so that the proposed dual is

$$|ellipse)_{a,b} = \sum_{k=0}^{N/n} (a_k)_{a,b} | \frac{N}{n} - k; k\rangle, \tag{3.205}$$

then one can formally invert the relation to give

$$| \frac{N}{n} - k; k\rangle = \sum_{a,b} (a_k)^{-1}_{a,b} |ellipse)_{a,b}. \tag{3.206}$$

This suggests that to obtain a geometric dual for a given chiral primary one could consider a linear superposition of curves with different parameters (a, b) using a measure which is related to $(a_k)^{-1}_{a,b}$. Precisely what the measure should be is not however immediately apparent, because, as we will discuss below, such a symmetrization via linear superposition may in fact be rather too naive, because of the non-linear relationship between harmonic functions and vevs. To test whether a given symmetric geometry does indeed have the correct properties to correspond to a given chiral primary, one will need to use the actual values of the kinematically allowed vevs, as we will now discuss.

(3.11) Dynamical tests for symmetric geometries

The geometries in sections 3.10.1 and 3.10.2 have the correct asymptotics to correspond to chiral primaries. Since the geometries in section 3.10.2 are based on separated sources, one would not however anticipate that these correspond to Higgs branch vacua; the more natural proposal would be that they relate to Coulomb branch vacua. By extracting all vevs and n-point functions from each geometry one could in principle identify the field theory dual uniquely.

Furthermore, given any proposed correspondence between geometries and field theory vacua, we can use dynamical information for the kinematically allowed vevs to test it. In particular, let us consider the averaged geometries, focusing on the example of the averaged ellipse. In this case, we consider a defining curve (3.196) with corresponding rotated curve $\tilde{F}^1, \tilde{F}^2$ defined in (3.190). The geometry based on the latter is proposed to correspond to the linear superposition (3.185) with

$$A_+ = (a+b)e^{i\alpha}; \qquad A_- = (a-b)e^{-i\alpha}. \tag{3.207}$$

This means that the superposition dual to the rotated ellipse is

$$|ellipse\rangle_\alpha = \sum_{k=0}^{N/n} e^{i\alpha(\frac{N}{n}-2k)} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k}(a-b)^k (\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)} (\mathcal{O}_n^{R--})^k. \tag{3.208}$$

Averaging over the angle α clearly picks out the $k = N/2n$ term in the superposition, which is a state of zero angular momentum. However, the geometry obtained by averaging over rotated ellipses does not have zero angular momentum, but rather the same angular momentum as the original geometry. This suggests that this geometric averaging might actually average over vevs, rather than over states, and thus not pick out a geometry dual to a single chiral primary. Given that the averaging linearly superposes harmonic functions, however, and the vevs are non-linearly related to the harmonic functions, the geometric averaging probably does not lead to just an overall averaging over the vevs. One will have to use the actual vevs for the neutral operators to see what the geometry describes.

So now let us discuss how one would use information about three point functions at the conformal point to test whether a given geometry corresponds to a chiral primary. Let us work with an example: consider the R vacuum corresponding to the operator $(\mathcal{O}_{S_n^p})_R$ obtained by spectrally flowing the operator $\mathcal{O}_{S_n^p}$ dual to the supergravity field $S_n^{p(6)}$. (Recall that the superscript p denotes that it is primary, $j^3 = j$ and $\bar{j}^3 = \bar{j}$.) Next suppose that there is a candidate dual geometry, which has the correct symmetries and R-charges, the latter being $(\frac{1}{2}(n-N), \frac{1}{2}(n-N))$. This means that the holographic vevs for the R symmetry currents must be

$$\langle J^{\pm 3}\rangle = \frac{\mu}{4\pi}(n-N)(dy \mp dt), \tag{3.209}$$

where y has periodicity $2\pi\tilde{R} = 2\pi R/\sqrt{Q_1 Q_5}$ and $\mu = \sqrt{Q_1 Q_5}/R$.

Now let us consider how we can relate this vev to the normalized three point function at the conformal point. That is,

$$\langle J^{\pm 3}\rangle_{\Psi_{S_n}} = \langle (\mathcal{O}_{S^n})_R^\dagger J^{\pm 3}(w_0)(\mathcal{O}_{S^n})_R\rangle \equiv \frac{\langle (\mathcal{O}_{S^n})_R^\dagger(\infty) J^{\pm 3}(w_0)(\mathcal{O}_{S^k})_R(0)\rangle}{\langle (\mathcal{O}_{S^n})_R^\dagger(\infty)(\mathcal{O}_{S^n})_R(0)\rangle}, \tag{3.210}$$

where Ψ_{S_n} denotes that the theory is in the vacuum created by $(\mathcal{O}_{S_n^p})_R$. The scale w_0 at which the current is inserted is found by comparing the vevs (3.209) with the normalized three point functions, computed in (3.309). The latter give

$$\langle J^{+3}\rangle_{\Psi_{S_n}} = \frac{(n-N)}{4\pi w_0}; \qquad \langle J^{-3}\rangle_{\Psi_{S_n}} = \frac{(n-N)}{4\pi \bar{w}_0}, \tag{3.211}$$

which comparing with (3.209) implies that the inserted scale must be $w_0 = \bar{w}_0 = \mu^{-1}$.

We can now use the three point functions between $\mathcal{O}_{S_n^p}$ and neutral dimension two operators to predict the vevs for the latter. This gives

$$\begin{aligned} \langle \mathcal{O}_{S_0^2} \rangle_{\Psi_{S_n}} &= 0; \\ \langle \mathcal{O}_{\Sigma_0^2} \rangle_{\Psi_{S_n}} &= \langle (\mathcal{O}_{S^n})^\dagger_R \mathcal{O}_{\Sigma_0^2}(\mu^{-1}) (\mathcal{O}_{S^n})_R \rangle = \frac{\sqrt{3} n^3 \mu^2}{\sqrt{2}\pi (n-1)^2}. \end{aligned} \tag{3.212}$$

where the normalized three point function is defined in (3.302) and the inserted scale is as before $w_0 = \bar{w}_0 = \mu^{-1}$. Note that $\mu^2 \sim N$, so the vev has the correct large N behavior (for our choice of normalization). From the expressions given in (3.127) for the vevs of these operators in terms of the asymptotics we can determine the degree two coefficients in (3.188). The vanishing of $\langle \mathcal{O}_{S_0^2} \rangle_{\Psi_{S_n}}$ implies that $f^1_{20} = f^5_{20}$ whilst the expression for the vev $\langle \mathcal{O}_{\Sigma_0^2} \rangle_{\Psi_{S_n}}$ in (3.127) implies that

$$\begin{aligned} f^1_{20} &= -\frac{\mu^2}{\sqrt{3}N^2} \left((n-N)^2 + \frac{3n^3 N}{(n-1)^2} \right) \\ &= f^1_{20}(circ) \left(1 + \frac{n}{N} + \cdots \right), \end{aligned} \tag{3.213}$$

where $f^1_{20}(circ) = -\mu^2/\sqrt{3}$ is the value of f^1_{20} for the circular solution. The $(n-N)^2$ contribution on the rhs is due to the non-linear contribution $8a^{\alpha -} a^{\beta +} f_{I\alpha\beta}$ and in the second equality we use $1 \ll n \ll N$. The upper limit on n follows from the fact that the supergravity three point functions are known only to leading order in N and do not apply for operators with dimensions comparable to N. The lower limit is unnecessary and is imposed only to simplify the formula.

By extending the computation of the vevs to higher dimension operators and comparing with those predicted from three point functions at the conformal point, one could in principle extract the higher degree coefficients in (3.188) and resum the asymptotic series to obtain the full geometry.

There is an important caveat, however. In all computations so far we have worked in the $N \to \infty$ limit, retaining only the leading terms. This applies both to the computation of the vevs and to the computation of three point functions. For the computation of the 3-point function to be valid we need $N \gg n$, but then the "holographically engineered" f^1_{20} in (3.213) differs from the answer for the circle only by terms subleading in n/N. In other words, the holographically engineered geometry would be that of the circular solution up to $1/N$ corrections.

Next consider R vacua corresponding to operators obtaining by spectral flow on operators which are either of high dimension (comparable to N) or multiparticle. The latter include operators of the form $(\mathcal{O}_n^{R++})^{N/n-k} (\mathcal{O}_n^{R--})^k$ for which the duals may be related to averaged ellipses. Since there is no information about three point functions of these operators at strong coupling, we have no precise predictions for the vevs of neutral operators and thus cannot currently test whether a given geometry is indeed dual to such a state. Given any future progress on computing the relevant fusion coefficients via string theory, one could however test the correspondence further.

To summarize: a geometry with $SO(2) \times SO(2)$ symmetry can be characterized by its angular momentum and vevs of neutral operators. The latter can in principle be used to determine the corresponding dual, but to implement this program will in general require going beyond the leading supergravity approximation.

(3.12) Including the asymptotically flat region

In this section we will discuss how the asymptotically flat region of the geometry may be interpreted using the AdS/CFT dictionary. Our discussion will parallel an analogous discussion for D3-branes given in section 6 of [57].

The six-dimensional metric of (3.51) along with the scalar and tensor field of (3.44) are characterized by two harmonic functions (f_1, f_5) and a harmonic form A_i. The field equations are satisfied for any choice of harmonic functions. The specific choices in (3.47) correspond to (part of) the (supersymmetric) Higgs branch of the D1-D5 system. Multi-centered harmonic functions for (f_1, f_5) with $A_i = 0$ are also well-known supergravity solutions, corresponding to part of the Coulomb branch.

In (3.68) we gave the most general form for the asymptotic expansions of (f_1, f_5, A_i) under the condition that the solution is asymptotically $AdS_3 \times S^3$. The asymptotically flat region may be included by adding constant terms to the (f_1, f_5) harmonic expansions, namely

$$f_1 = \epsilon_1 + \frac{Q_1}{r^2} \sum_{k,I} \frac{f^1_{kI} Y^I_k(\theta_3)}{r^k}; \qquad f_5 = \epsilon_5 + \frac{Q_5}{r^2} \sum_{k,I} \frac{f^5_{kI} Y^I_k(\theta_3)}{r^k}, \tag{3.214}$$

whilst keeping the large radius expansion for A_i as in (3.68). To include all of the asymptotically flat region, the parameters ϵ_1 and ϵ_5 clearly need to be finite. However, let us take the parameters to be infinitesimal so that the solution remains asymptotically $AdS_3 \times S^3$. Since we have discussed already the terms in the harmonic expansion behaving as r^{-k} with $k \geq 3$, we consider only the new terms as a perturbation to the AdS background. That is, we let

$$f_1 = \epsilon_1 + \frac{Q_1}{r^2}; \qquad f_5 = \epsilon_5 + \frac{Q_5}{r^2}, \tag{3.215}$$

with $A_i = 0$ and then identify the terms induced in the harmonic expansion of the fluctuations (3.67). The field fluctuations are

$$\begin{aligned} -h_{tt} &= h_{yy} = -\tfrac{1}{2} r^4 (\hat{\epsilon}^1 + \hat{\epsilon}^5); \qquad h_{rr} = \tfrac{1}{2}(\hat{\epsilon}^1 + \hat{\epsilon}^5); \\ h_{ab} &= \tfrac{1}{2} r^2 (\hat{\epsilon}^1 + \hat{\epsilon}^5); \qquad \phi^{(56)} = \tfrac{1}{2} r^2 (\hat{\epsilon}^1 - \hat{\epsilon}^5); \\ g^5_{tyr} &= -r^3 (\hat{\epsilon}^1 + \hat{\epsilon}^5); \qquad g^6_{tyr} = -r^3 (\hat{\epsilon}^1 - \hat{\epsilon}^5), \end{aligned} \tag{3.216}$$

where we define $\hat{\epsilon}^1 = \epsilon^1 / Q_1$ and $\hat{\epsilon}^5 = \epsilon^5 / Q_5$. Thus the only non-vanishing dynamical fields are those from (3.92)

$$\tau_0 \equiv \frac{\pi^0}{12} = \frac{1}{8} r^2 (\hat{\epsilon}^1 + \hat{\epsilon}^5); \qquad t_0 \equiv \tfrac{1}{4} \phi^{(56)}_0 = \frac{1}{8} r^2 (\hat{\epsilon}^1 - \hat{\epsilon}^5). \tag{3.217}$$

(The other non-vanishing components are induced by constraint equations and do not correspond to dynamical fields.) Since both τ_0 and t_0 couple respectively to the dimension four operators $\mathcal{O}_{\tau_0}$ and $\mathcal{O}_{t_0}$, the radial dependence of these fields corresponds to source behavior. Thus the CFT lagrangian is deformed by the terms

$$\int d^2 z \left((\hat{\epsilon}^1 + \hat{\epsilon}^5)\mathcal{O}_{\tau_0} + (\hat{\epsilon}^1 - \hat{\epsilon}^5)\mathcal{O}_{t_0} \right) . \tag{3.218}$$

Note that the operators $(\mathcal{O}_{\tau_0}, \mathcal{O}_{t_0})$ are the top components of the short multiplets generated from the chiral primaries $(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{S_2})$ respectively through the action of the supercharges. That is, they are given by

$$G^{1\dagger}_{-1/2} G^2_{-1/2} \tilde{G}^{1\dagger}_{-1/2} \tilde{G}^2_{-1/2} \left| CPO \right\rangle , \tag{3.219}$$

where $(G^a_{\pm 1/2}, \tilde{G}^a_{\pm 1/2})$ with $a = 1, 2$ are left and right supercharges. Here $(G^{1\dagger}_{-1/2}, G^2_{-1/2})$ and corresponding right moving charges act as raising operators on the $\Delta = 2$ chiral primaries. The latter have $h = j = j^3 = \bar{h} = \bar{j} = \bar{j}^3 = 1$. Computing two point functions in the presence of the deformation (3.218) may capture scattering into the asymptotically flat part of the D1-D5 geometry.

(3.A) Appendix

(3.A.1) Properties of spherical harmonics

Scalar, vector and tensor spherical harmonics satisfy the following equations

$$\begin{aligned} \Box Y^I &= -\Lambda_k Y^I, \\ \Box Y^{I_v}_a &= (1 - \Lambda_k) Y^{I_v}_a, \qquad D^a Y^{I_v}_a = 0, \\ \Box Y^{I_t}_{(ab)} &= (2 - \Lambda_k) Y^{I_t}_{(ab)}, \qquad D^a Y^{I_t}_{k(ab)} = 0, \end{aligned} \tag{3.220}$$

where $\Lambda_k = k(k+2)$ and the tensor harmonic is traceless. It will often be useful to explicitly indicate the degree k of the harmonic; we will do this by an additional subscript k, e.g. degree k spherical harmonics will also be denoted by Y^I_k, etc. $\Box$ denotes the d'Alambertian along the three sphere. The vector spherical harmonics are the direct sum of two irreducible representations of $SU(2)_L \times SU(2)_R$ which are characterized by

$$\epsilon_{abc} D^b Y^{c I_v \pm} = \pm (k+1) Y^{I_v \pm}_a \equiv \lambda_k Y^{I_v \pm}_a . \tag{3.221}$$

The degeneracy of the degree k representation is

$$d_{k,\epsilon} = (k+1)^2 - \epsilon, \tag{3.222}$$

where $\epsilon = 0, 1, 2$ respectively for scalar, vector and tensor harmonics. For degree one vector harmonics I_v is an adjoint index of $SU(2)$ and will be denoted by α.

We use normalized spherical harmonics such that

$$\int Y^{I_1} Y^{J_1} = \Omega_3 \delta^{I_1 J_1}; \qquad \int Y^{a I_v} Y_a^{J_v} = \Omega_3 \delta^{I_v J_v}; \qquad \int Y^{(ab) I_t} Y^{J_t}_{(ab)} = \Omega_3 \delta^{I_t J_t}, \tag{3.223}$$

where $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere. Then

$$\int D_a Y^{I_1} D^a Y^{J_1} = \Omega_3 \Lambda^{I_1} \delta^{I_1 J_1}; \qquad \int D^{(a} D^{b)} Y^{I_1} D_a D_b Y^{I_2} = \Omega_3 \frac{2}{3} \Lambda^{I_1} (\Lambda^{I_1} - 3) \delta^{I_1 J_1}. \tag{3.224}$$

The following identities are useful

$$\begin{aligned} \frac{1}{\Omega_3} \int Y^I D^a Y^J D_a Y^K &\equiv b_{IJK} = \tfrac{1}{2}(\Lambda^J + \Lambda^K - \Lambda^I) a_{IJK}; \\ \frac{1}{\Omega_3} \int D^{(a} D^{b)} Y^I D_a D_b Y^J Y^K &\equiv c_{IJK} = (\tfrac{1}{4} \Lambda_{IJK} (\Lambda_{IJK} - 4) - \frac{1}{3} \Lambda^I \Lambda^J) a_{IJK}; \\ \frac{1}{\Omega_3} \int D_{(a} Y^I D_{b)} Y^J D^a D^b Y^K &\equiv d_{IJK} = (\tfrac{1}{4} \Lambda_{IKJ} \Lambda_{JKI} + \frac{1}{6} \Lambda^K \Lambda_{IJK}) a_{IJK}, \end{aligned} \tag{3.225}$$

where $\Lambda_{IJK} = (\Lambda^I + \Lambda^J - \Lambda^K)$. We define the following triple integrals as

$$\int Y^I Y^J Y^K = \Omega_3 a_{IJK}; \tag{3.226}$$

$$\int (Y_1^{\alpha \pm})^a Y_1^j D_a Y_1^i = \Omega_3 e^{\pm}_{\alpha ij}; \tag{3.227}$$

$$\int Y^I (Y_1^{\alpha -})_a (Y_1^{\beta +})^a = \Omega_3 f_{I\alpha\beta}; \tag{3.228}$$

$$\int (Y_{k_v}^{I_v \pm})^a Y_k^I D_a Y_1^i = \Omega_3 E^{\pm}_{I_v I i}; \tag{3.229}$$

$$\int (Y_{k_v}^{I_v \pm})^a Y_k^I D_a Y_l^J = \Omega_3 E^{\pm}_{I_v I J}; \tag{3.230}$$

We also use specific identities for harmonics of low degree. The degree one vector harmonics $Y^{\alpha}_{1\pm}$ transform in the $(1,0)$ and $(0,1)$ representation of $(SU(2)_L, SU(2)_R)$ whilst the degree k scalar harmonics transform in the $(\frac{1}{2}k, \frac{1}{2}k)$ representation. This immediately implies that the following triple overlaps are zero:

$$\int Y_2^I (Y_1^{\alpha +})_a (Y_1^{\beta +})^a) = \int Y_2^I (Y_1^{\alpha -})_a (Y_1^{\beta -})^a) = \int Y_0 (Y_1^{\alpha +})_a (Y_1^{\beta -})^a) = 0. \tag{3.231}$$

Using the following explicit coordinate system on the sphere

$$ds_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2, \tag{3.232}$$

with volume form $\eta_3 = \sin\theta \cos\theta d\theta \wedge d\phi \wedge d\psi$ the following are normalized Killing forms

$$Y_1^{3+} = (\sin^2\theta d\phi + \cos^2\theta d\psi); \qquad Y_1^{3-} = -(\sin^2\theta d\phi - \cos^2\theta d\psi), \tag{3.233}$$

which generate the Cartan of the $SO(4)$ symmetry group. The remaining Killing forms are

$$\begin{aligned} Y_1^{1+} &= (\cos(\psi+\phi) d\theta + \sin(\psi+\phi) \sin\theta \cos\theta d(\psi - \phi)); \\ Y_1^{2+} &= (-\sin(\psi+\phi) d\theta + \cos(\psi+\phi) \sin\theta \cos\theta d(\psi - \phi)); \\ Y_1^{1-} &= (\cos(\psi-\phi) d\theta + \sin(\psi-\phi) \sin\theta \cos\theta d(\phi + \psi)); \\ Y_1^{2-} &= (-\sin(\psi-\phi) d\theta + \cos(\psi-\phi) \sin\theta \cos\theta d(\phi + \psi)). \end{aligned}$$

The $SU(2) \times SU(2)$ algebra realized by the Killing vectors is normalized such that

$$[Y_1^{\alpha+}, Y_1^{\beta+}] = 2\epsilon_{\alpha\beta\gamma} Y_1^{\gamma+}; \qquad [Y_1^{\alpha-}, Y_1^{\beta-}] = 2\epsilon_{\alpha\beta\gamma} Y_1^{\gamma-}; \qquad [Y_1^{\alpha+}, Y_1^{\beta-}] = 0. \tag{3.234}$$

Furthermore

$$Y_1^{\alpha\pm} \wedge Y_1^{\beta\pm} \wedge Y_1^{\gamma\pm} = \mp\epsilon_{\alpha\beta\gamma}\eta_3, \tag{3.235}$$

which implies that

$$\int \epsilon^{abc} Y_a^{\alpha\pm} Y_b^{\beta\pm} Y_c^{\gamma\pm} = \mp\Omega_3 \epsilon_{\alpha\beta\gamma} \tag{3.236}$$

In the same coordinate system $Y_2^0 = \sqrt{3}\cos 2\theta$ is the normalized degree 2 spherical harmonic which is a singlet under the $SO(2)^2$ Cartan, with the following triple overlap

$$\int Y_2^0 (Y_1^{3+})^a (Y_1^{3-})_a = \frac{1}{\sqrt{3}}\Omega_3. \tag{3.237}$$

Thus, $f_{033} = 1/\sqrt{3}$ in this specific case. More generally the normalized spherical harmonics which are singlets under the Cartan can be expressed as

$$Y_{2l}^0 = \sqrt{2l+1} P_l(\cos 2\theta), \tag{3.238}$$

where $P_l(x)$ is a Legendre polynomial of degree l, normalized so that $P_l(1) = 1$ and $P_l(-1) = (-1)^l$.

In this coordinate system normalized degree one spherical harmonics are

$$\begin{aligned} Y_1^1 &= 2\sin\theta\cos\phi; \qquad Y_1^2 = 2\sin\theta\sin\phi; \\ Y_1^3 &= 2\cos\theta\cos\psi; \qquad Y_1^4 = 2\cos\theta\sin\psi. \end{aligned} \tag{3.239}$$

Defining $Y^{ij} \equiv \frac{1}{2}(Y_1^j dY_1^1 - Y_1^1 dY_1^j)$,

$$\begin{aligned} Y^{12} &= (Y_1^{3-} - Y_1^{3+}); \qquad Y^{34} = -(Y_1^{3+} + Y_1^{3-}); \qquad Y^{13} = (Y_1^{1+} + Y_1^{1-}); \\ Y^{34} &= (Y_1^{1-} - Y_1^{1+}); \qquad Y^{14} = -(Y_1^{2+} + Y_1^{2-}); \qquad Y^{23} = (Y_1^{2+} - Y_1^{2-}), \end{aligned} \tag{3.240}$$

and therefore the explicit values for the overlaps $e_{ij}^{\pm\alpha}$ defined in (3.227) are

$$\begin{aligned} e_{12}^{+3} &= -1; \quad e_{12}^{-3} = 1; \quad e_{34}^{+3} = -1; \quad e_{34}^{-3} = -1; \quad e_{13}^{+1} = 1; \quad e_{13}^{-1} = 1; \\ e_{24}^{+1} &= -1; \quad e_{24}^{-1} = 1; \quad e_{14}^{+2} = -1; \quad e_{14}^{-2} = -1; \quad e_{23}^{+2} = 1; \quad e_{23}^{-2} = -1. \end{aligned} \tag{3.241}$$

Note that $e_{ij}^{\pm\alpha} = -e_{ji}^{\pm\alpha}$.

We will also make use of normalized degree k scalar harmonics with maximal $(m, \bar{m})$ $(SU(2)_L, SU(2)_R)$ charges:

$$\begin{aligned} Y_k^{\pm\frac{1}{2}k, \pm\frac{1}{2}k} &= \sqrt{k+1}\sin^k\theta e^{\pm ik\phi}; \\ Y_k^{\pm\frac{1}{2}k, \mp\frac{1}{2}k} &= \sqrt{k+1}\cos^k\theta e^{\pm ik\psi}. \end{aligned} \tag{3.242}$$

The triple overlap between two such harmonics of opposite charges with the neutral harmonic of degree two given in (3.238) is given by

$$\frac{1}{2\pi^2}\int Y_k^{\frac{1}{2}k,\frac{1}{2}k} Y_k^{-\frac{1}{2}k,-\frac{1}{2}k} Y_2^0 = -\frac{\sqrt{3}k}{k+2}. \tag{3.243}$$

We will also need the explicit values of the overlaps between two such harmonics of opposite charges and the commuting Killing vectors:

$$E^{\pm}_{3(--)(++)} \equiv \frac{1}{2\pi^2}\int D^a Y_k^{\frac{1}{2}k,\frac{1}{2}k} Y_k^{-\frac{1}{2}k,-\frac{1}{2}k} Y_a^{3\pm} = \pm ik; \tag{3.244}$$

$$E^{\pm}_{3(+-)(-+)} \equiv \frac{1}{2\pi^2}\int D^a Y_k^{-\frac{1}{2}k,\frac{1}{2}k} Y_k^{\frac{1}{2}k,-\frac{1}{2}k} Y_a^{3\pm} = ik. \tag{3.245}$$

Vector spherical harmonics $Y^{0\pm}_{ka}$ whose Lie derivatives along the $SO(2)$ directions are zero can be expressed as

$$Y_k^{0+} = \frac{1}{\sqrt{2}}(\sin^2\theta p_l(\theta)d\phi + \cos^2\theta q_l(\theta)d\psi); \tag{3.246}$$

$$Y_k^{0-} = \frac{1}{\sqrt{2}}(-\sin^2\theta p_l(\theta)d\phi + \cos^2\theta q_l(\theta)d\psi), \tag{3.247}$$

where $k = 2l+1$ and l is an integer. The functions $p_l(\theta)$ and $q_l(\theta)$ of degree $2l$ are related to degree $k = 2l+1$ scalar harmonics with $SO(2)\times SO(2)$ charges $(\pm\frac{1}{2},\pm\frac{1}{2})$. That is,

$$Y_k^{\pm\frac{1}{2},\pm\frac{1}{2}}(\theta) = e^{\pm i\phi}\sin\theta p_l(\theta); \qquad Y_k^{\pm\frac{1}{2},\mp\frac{1}{2}}(\theta) = e^{\pm i\psi}\cos\theta q_l(\theta), \tag{3.248}$$

are normalized degree k spherical harmonics. Explicit series representation of these functions are

$$p_l(\theta) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m \begin{pmatrix} l \\ m \end{pmatrix}\begin{pmatrix} l+m+1 \\ l+1 \end{pmatrix}(\cos\theta)^{2m}\right); \tag{3.249}$$

$$q_k(\theta) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m \begin{pmatrix} l \\ m \end{pmatrix}\begin{pmatrix} l+m+1 \\ l+1 \end{pmatrix}(\sin\theta)^{2m}\right).$$

Finally, let us make explicit the relation between spherical harmonics and traceless symmetric tensors on R^4. There is a one to one map between scalar spherical harmonics of degree k and rank k symmetric traceless tensors. Given the spherical harmonic, one can read off the associated tensor by lifting it onto a sphere in R^4. For example, for the charged harmonics (3.248), we get

$$Y_k^{\pm\frac{1}{2},\pm\frac{1}{2}}(\theta) \to C_k^{\pm\frac{1}{2},\pm\frac{1}{2}} = (x^1 \pm ix^2)p_l(x); \tag{3.250}$$

$$p_l(x) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m \begin{pmatrix} l \\ m \end{pmatrix}\begin{pmatrix} l+m+1 \\ l+1 \end{pmatrix}((x^1)^2+(x^2)^2)^m(\sum_i (x^i)^2)^{l-m}\right).$$

(3.A.2) Proof of addition theorem for harmonic functions on R^4

To prove the addition theorem one first writes

$$|x-y|^{-2} = \frac{1}{r^2}\sum_{n=0}^{\infty}\sum_{m\geq 0}^{n}(-1)^{n+m}\frac{n!}{m!(n-m)!}\frac{y^{2n-m}}{r^{2n-m}}(2\hat{x}\cdot\hat{y})^m, \tag{3.251}$$

where $x^i = r\hat{x}^i$ and $y^i = y\hat{y}^i$ with $(\hat{x}^i, \hat{y}^i)$ unit vectors. Collecting together terms of the same radial power and summing the finite series one finds

$$|x-y|^{-2} = \sum_{k\geq 0}\frac{y^k}{r^{2+k}}\frac{\sin((k+1)\gamma)}{\sin(\gamma)}, \tag{3.252}$$

where the angle γ is defined as $\cos\gamma = \hat{x}\cdot\hat{y}$.

Now at each degree k there is precisely one $SO(3)$ invariant spherical harmonic and the normalized such harmonic is given by

$$Y_k^0(\gamma) = \sin((k+1)\gamma)/\sin(\gamma). \tag{3.253}$$

One can show this using spherical coordinates adapted to the $SO(3)$ symmetry group, namely

$$ds_3^2 = d\hat{\theta}^2 + \sin^2\hat{\theta} d\Omega_2^2. \tag{3.254}$$

Then $Y_k^0(\hat{\theta})$ satisfies the degree k $SO(3)$ invariant spherical harmonic equation

$$\left(\frac{1}{\sin^2\hat{\theta}}\partial_{\hat{\theta}}(\sin^2\hat{\theta}\partial_{\hat{\theta}}) + k(k+2)\right)Y_k^0(\hat{\theta}) = 0, \tag{3.255}$$

and is normalized as in the previous section. Therefore the addition theorem amounts to proving the following identity

$$Y_k^0(\gamma) = \alpha_k\sum_I Y_k^I(\theta_3^x)Y_k^I(\theta_3^y), \tag{3.256}$$

where $Y_k^I(\theta_3)$ are (normalized) spherical harmonics of degree k on the S^3 and $\alpha_k = 1/(k+1)$. First note that in the coordinate system (3.254) on the sphere

$$\cos\gamma = \cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y(\cos\gamma_2), \tag{3.257}$$

where γ_2 is the angle separating the vectors on the S^2. Thus when $\theta_y = 0$ (it lies on the "axis") $\cos\gamma = \cos\theta_x$. Since the $SO(3)$ singlet harmonic is the only harmonic at level k which is non-vanishing on the axis (3.256) collapses to

$$Y_k^0(\gamma) = \alpha_k Y_k^0(\theta_x)Y_k^0(0), \tag{3.258}$$

which is true if $\alpha_k = 1/(k+1)$ since from (3.253) $Y_k^0(0) = (k+1)$.

Now consider rotating the axes so that θ_y is no longer zero. Then the function $Y_k^0(\gamma)$ still satisfies the covariant version of (3.255), namely

$$(\Box_x + k(k+2))Y_k^0(\gamma) = 0, \tag{3.259}$$

where $\Box_x$ is the Laplacian on the S^3 with coordinates θ_3^x. In other words, the function can always be expanded in spherical harmonics of rank k as

$$Y_k^0(\gamma) = \sum_I \alpha_k^I(\theta_3^y) Y_k^I(\theta_3^x), \tag{3.260}$$

where the coefficients are given by

$$\alpha_k^I(\theta_3^y) = \int_{S^3} d\Omega_3 Y_k^I(\theta_3^x) Y_k^0(\cos\gamma). \tag{3.261}$$

However, a generic function can be expanded in terms of spherical harmonics as

$$f(\theta_3^x) = \sum_{k,I} \beta_{kI} Y_k^I(\theta_3^x), \tag{3.262}$$

where

$$\beta_{kI} = \int_{S^3} d\Omega_3 f(\theta_3^x) Y_k^I(\theta_3^x), \tag{3.263}$$

and in particular for the $SO(3)$ singlet coefficients

$$\beta_k = \int_{S^3} d\Omega_3 f(\theta_3^x) Y_k^0(\theta_x), \tag{3.264}$$

so that $f(\theta_x = 0) = \sum_k \beta_k (k+1)$. Then (3.261) is the $SO(3)$ singlet coefficient in an expansion of the function $Y_k^I(\theta_3^x)$ in a series of $Y_k^I(\gamma, \cdots)$ (i.e. with respect to the rotated axis discussed earlier). One can thus read off the coefficient (3.261) as

$$\alpha_k^I(\theta_3^y) = (k+1)^{-1} Y_k^I(\theta_3(\gamma, \cdots))_{\gamma=0} = (k+1)^{-1} Y_k^I(\theta_3^y), \tag{3.265}$$

since in the limit $\gamma \to 0$ the angles $(\theta, \cdots)$ go over into $(\theta_y, \cdots)$. This completes the proof of (3.256).

(3.A.3) Six dimensional field equations to quadratic order

In this appendix we summarize the computation of the relevant quadratic corrections to the six-dimensional field equations using the results of [24, 48]. Expanding the Einstein equation (3.54) up to second order in fluctuations gives

$$\begin{aligned} R_{MN}^{(1)} + R_{MN}^{(2)} &= H_{MPQ}^A H_N^{A\,PQ} - 2(h^{KL} - h^{KP} h^L{}_P) H_{MKQ}^A H_{NL}^{A\ \ Q} \\ &\quad + h^{KL} h^{PQ} H_{MKP}^A H_{NLQ}^A + D_M \Phi D_N \Phi, \qquad (3.266) \\ &\equiv (E_{MN}^{(1)} + E_{MN}^{(2)}) \qquad (3.267) \end{aligned}$$

where

$$
\begin{aligned}
R^{(1)}_{MN} &= D_K h^K_{MN} - \tfrac{1}{2} D_M D_N (h^L_L); \qquad (3.268)\\
R^{(2)}_{MN} &= -D_K (h^K_L h^L_{MN}) + \tfrac{1}{4} D_M D_N (h^{KL} h_{KL}) + \tfrac{1}{2} h^K_{MN} D_K (h^L_L) - h^K_{ML} h^L_{KN};\\
h^K_{MN} &\equiv \tfrac{1}{2}(D_M h^K_N + D_N h^K_M - D^K h_{MN}).
\end{aligned}
$$

The quantities $(E^{(1)}_{MN}, E^{(2)}_{MN})$ are defined to be linear and quadratic in fluctuations respectively. The expansion of the scalar field and the three forms G^A (3.66) implies the following expansion for the three forms H^A up to quadratic order in fluctuations:

$$
\begin{aligned}
H^5 &= g^o + g^5 + \Phi g^6 + \tfrac{1}{2} g^o \Phi^2; \qquad (3.269)\\
H^6 &= g^6 + g^o \Phi + g^5 \Phi,
\end{aligned}
$$

where (g^5, g^6) are the (closed) three form fluctuations given in (3.66) and g^o is the background three form.

The scalar field equation up to second order is

$$
\begin{aligned}
(\Box + \Box_a)\Phi &\equiv E^{(1)} + E^{(2)}; \qquad (3.270)\\
&= D_K \Phi (D_L h^{KL} - \tfrac{1}{2} D^K (h^L_L) + h^{KL} D_K D_L \Phi + \frac{2}{3} H^5_{KLM} (H^{6KLM} - 3 h^K_S H^{6SLM}),
\end{aligned}
$$

where $E^{(1)}$ is the part linear in fluctuations and $E^{(2)}$ is quadratic part. Recall that $\Box$ is the d'Alambertian on AdS_3 and $\Box_a$ is the d'Alambertian on S^3.

The (anti)-self duality equation is

$$
H \mp *H \pm S^{(1)} \pm S^{(2)} \equiv T^{(1)} + T^{(2)} = 0, \qquad (3.271)
$$

where

$$
\begin{aligned}
S^{(1)}_{KLM} &= \tfrac{1}{2} h (*H)_{KLM} - 3 h^P_{[K} (*H)_{LM]P}; \qquad (3.272)\\
S^{(2)}_{KLM} &= \frac{3}{2} h^P_P h^Q_{[K} (*H)_{LM]Q} - (\frac{1}{8} h^2 + \tfrac{1}{4} h^{PQ} h_{PQ})(*H)_{KLM} - 3 h^P_{[K} h^Q_L (*H)_{M]PQ},
\end{aligned}
$$

and $(T^{(1)}, T^{(2)})$ are the parts linear and quadratic in fluctuations respectively.

We are interested in corrections to the $(s^2, \sigma^2, H_{\mu\nu}, A^{\pm}_\mu)$ field equations quadratic in the scalar field s^1 and the gauge field $A^{\pm}$. Consider first the s^2 field equations. The linearized field equation is given by a combination of the scalar field equation (3.270) and components of the anti-self-duality equation (3.271). That is,

$$
\Box s^2_I \equiv \frac{1}{12} \left((\Box + \Box_a)\Phi - E^{(1)} - \epsilon^{abc} (\tfrac{1}{2} D^\mu D^a T^{(1)}_{\mu bc} + \frac{2}{3} T^{(1)}_{abc}) \right)_{Y^I_2} = 0, \qquad (3.273)
$$

where $A_{Y^I_2}$ denotes the projection of A onto the Y^I_2 harmonic. For the quadratic corrections to this equation first define the following quantities

$$
q_1 = E^{(2)}; \qquad q_{2\mu a} = -\tfrac{1}{2} \epsilon^{abc} T^{(2)6}_{\mu bc}; \qquad q_3 = \frac{1}{6} \epsilon^{abc} T^{(2)6}_{abc}, \qquad (3.274)
$$

then the correction to the s_I^2 equation is given by

$$\Box s_I^2 = \frac{1}{12}((q_1) + D^\mu D^a (q_{2\mu a}) + 4(q_3))_{Y_2^I}. \tag{3.275}$$

Now the explicit computations of [48] show that there are no such correction terms quadratic in S_i^1 and $A^{\pm\alpha}$. Therefore the linearized equation remains uncorrected to quadratic order.

Next consider the σ_I^2 equation. Here the linearized equation is a specific combination of the components of the Einstein equation (3.267) along the sphere with components of the self-duality equation. Namely

$$\Box \sigma_I^2 \equiv \frac{1}{6}\left(\frac{1}{3}(E_a^{(1)a} - R_a^{(1)a}) + \tfrac{1}{4}(E^{(1)}_{(ab)} - R^{(1)}_{(ab)}) - \tfrac{1}{4}\epsilon^{\mu\nu\rho} D_\mu D^a T^{(1)5}_{\nu\rho a} + \frac{2}{3}\epsilon^{abc} T^{(1)5}_{abc}\right)_{Y_2^I} = 0. \tag{3.276}$$

For the quadratic corrections to this equation define

$$\begin{aligned} Q_1 &= \frac{1}{3}(E_a^{(2)a} - R_a^{(2)a}); \qquad Q_{2(ab)} = (E^{(2)}_{(ab)} - R^{(2)}_{(ab)}); \\ Q^\mu_{3a} &= \frac{1}{2}\epsilon^{\mu\nu\rho} T^{(2)5}_{\nu\rho a}; \qquad Q_4 = \frac{1}{3!}\epsilon^{abc} T^{(2)5}_{abc}, \end{aligned} \tag{3.277}$$

and again denote as $(Q)_{Y_k^I}$ the projection of Q onto Y_k^I. Then

$$\Box \sigma^2 = \frac{1}{6}(Q_1 + \tfrac{1}{4} D^a D^b Q_{2(ab)} - \tfrac{1}{2} D^\mu D^a Q_{3a\mu} + 4Q_4)_{Y_2^I}. \tag{3.278}$$

Now the terms quadratic in the scalar fields s^1 were computed in [48]

$$\begin{aligned} (Q_1)_{Y^I} &= -14 s_i^1 s_j^1 a_{Iij} + \frac{2}{3}(D_\mu s_i^1 D^\mu s_j^1 + 2 s_i^1 s_j^1) b_{Iij}; \\ (D^a D^b Q_{2(ab)})_{Y^I} &= 4\left(s_i^1 s_j^1 - D_\mu s_i^1 D^\mu s_j^1\right) d_{ijI}; \\ (D^\mu D^a Q_{3a\mu})_{Y^I} &= -4\left(s_i^1 s_j^1 - D_\mu s_i^1 D^\mu s_j^1\right) b_{iIj}; \\ (Q_4)_{Y^I} &= 4 s_i^1 s_j^1 a_{Iij}. \end{aligned} \tag{3.279}$$

The relevant spherical harmonic triple overlaps are defined in appendix 3.A.1. We should mention here that there are also contributions to (3.278) quadratic in the gauge field which were not explicitly computed in [48]. These are given by

$$\begin{aligned} (Q_1)_{Y^I} &= -\frac{1}{8} F_{\mu\nu}(A^{+\alpha}) F^{\mu\nu}(A^{-\beta}) f_{I\alpha\beta} + \cdots; \\ (D^a D^b Q_{2(ab)})_{Y^I} &= -\frac{5}{2} F_{\mu\nu}(A^{+\alpha}) F^{\mu\nu}(A^{-\beta}) f_{I\alpha\beta} + \cdots; \\ (D^\mu D^a Q_{3a\mu})_{Y^I} &= \frac{3}{4} D_\mu \left(F^{\mu\nu}(A^{+\alpha}) A_\nu^{-\beta} + F^{\mu\nu}(A^{-\beta}) A_\nu^{+\alpha}\right) f_{I\alpha\beta} + \cdots. \end{aligned} \tag{3.280}$$

The spherical harmonic triple overlap $f_{I\alpha\beta}$ is defined in (3.228). Terms quadratic in two $SU(2)_L$ gauge fields or two $SU(2)$ right gauge fields are projected out via the identities (3.231). The ellipses denote terms quadratic in the gauge field rather than its field strength, that is, proportional to $A_\mu^{\pm\alpha} A^{\mu\pm\beta}$. These terms cancel out when combined in (3.278) leaving only a contribution involving field strengths. The latter however vanish when one imposes the leading order field equations, and thus the combination of the corrections (3.279) and (3.280) gives the σ^2 field equation (3.99), containing only scalar field corrections.

Next consider the corrections to the Einstein equation. Recall that the three dimensional metric to quadratic order in the fields is

$$H_{\mu\nu} = h^0_{\mu\nu} + \pi^0 g^o_{\mu\nu} - h^{\pm\alpha}_\mu h^{\pm\alpha}_\nu \equiv \hat{H}_{\mu\nu} - h^{\pm\alpha}_\mu h^{\pm\alpha}_\nu. \quad (3.281)$$

Then one can show that

$$(\mathcal{L}_{\mathcal{E}} + 2)\hat{H}_{\mu\nu} = (E^{(2)}_{\mu\nu} - R^{(2)}_{\mu\nu})_{Y_0} + (3Q_1 + 4Q_4)_{Y_0} g^o_{\mu\nu}, \quad (3.282)$$

where the linearized Einstein operator is defined in (3.96). The following terms which are quadratic in the scalar fields

$$(E^{(2)}_{\mu\nu} - R^{(2)}_{\mu\nu})^0 = (-2s^1_i s^1_j g^o_{\mu\nu} + 16 D_\mu s^1_i D_\nu s^1_j - 6 D_\rho s^1_i D^\rho s^1_j g^o_{\mu\nu})\delta^{ij}, \quad (3.283)$$

in combination with those contained in (3.279) give

$$(\mathcal{L}_{\mathcal{E}} + 2)\hat{H}_{\mu\nu} = 16(D_\mu s^1_i D_\nu s^1_i - g^o_{\mu\nu} s^1_i s^1_i). \quad (3.284)$$

There are also contributions quadratic in the gauge fields to both $(\mathcal{L}_{\mathcal{E}} + 2)\hat{H}_{\mu\nu}$ and $(\mathcal{L}_{\mathcal{E}} + 2)h^{\pm\alpha}_\mu h^{\pm\alpha}_\nu$. These contributions involve both the gauge fields and their field strength, and in particular do not vanish for flat connections. This is unsurprising, since we know from general arguments that $\hat{H}_{\mu\nu}$ on its own does not transform correctly under gauge transformations. However the gauge field contributions to $(\mathcal{L}_{\mathcal{E}} + 2)H_{\mu\nu}$, where $H_{\mu\nu}$ is the three dimensional metric (3.100) that transforms correctly under diffeomorphisms, do vanish for flat connections, as indeed they should, and thus are zero when one imposes the leading order gauge field equations. The corrected Einstein equation is therefore that given in (3.101).

(3.A.4) 3-POINT FUNCTIONS

In this appendix we discuss the supergravity computation of certain 3-point functions.

EXTREMAL SCALAR THREE POINT FUNCTIONS

First we will consider the computation of the 3-point function between two operators of dimension 1 and one operator of dimension k. The operators of dimension 1 may be the same or different and are dual to the fields S^1; there are four such operators corresponding to the four scalar harmonics of degree 1 which are labeled by i, j. The operator $\mathcal{O}_{\Sigma^k_I}$ of dimension k is dual to the field Σ^k_I (there are $(k+1)^2$ such operators labeled by I). The $k = 2$ case is special in that the correlator is extremal [60]. As in the five dimensional case, the computation of extremal correlators is subtle. The bulk coupling vanishes but the spacetime integral diverges when $k \to 2$ in such way that the corresponding 3-point function is finite. We will take this value to be the correct extremal correlator and this will allow us to fix the coefficient of the relevant terms non-linear in momentum in the 1-point function of Σ^2.

The three dimensional field equations to quadratic order were determined in [24] and for the fields of interest and with our normalizations they read

$$\begin{aligned}(\Box - k(k-2))\Sigma^k_I &= w_{Iij} S^1_i S^1_j; \\ (\Box + 1) S^1_i &= w_{Iij} \Sigma^k_I S^1_j; \\ (\Box + 1) S^1_j &= w_{Iij} \Sigma^k_I S^1_i;\end{aligned} \quad (3.285)$$

where

$$w_{Iij} = \frac{k^3(k+2)(k+4)(1-k/2)}{32(k+1)\sqrt{k(k-1)}} a_{Iij}. \quad (3.286)$$

Notice that this coupling vanishes in the extremal case $k = 2$.

The aim is to compute the 3-point $\langle \mathcal{O}_{\Sigma^k_I}(x_1)\mathcal{O}_{S^1_i}(x_2)\mathcal{O}_{S^1_j}(x_3)\rangle$, but we start by discussing 2-point functions. These are obtained by the first variation of the 1-point functions

$$\begin{aligned}\langle \mathcal{O}_{\Sigma^k_I}(x_1)\mathcal{O}_{\Sigma^k_J}(x_2)\rangle &= -\frac{\delta\langle \mathcal{O}_{\Sigma^k_I}(x_1)\rangle}{\delta \Sigma^k_{J(0)}(x_2)} = -\left(\frac{n_1 n_5}{4\pi}\right)(2k-2)\frac{\delta \Sigma^k_{I(2k-2)}(x_1)}{\delta \Sigma^k_{J(0)}(x_2)}; \\ \langle \mathcal{O}_{S^1_i}(x_1)\mathcal{O}_{S^1_j}(x_2)\rangle &= -\frac{\delta\langle \mathcal{O}_{S^1_i}(x_1)\rangle}{\delta S^1_{j(0)}(x_2)} = -\left(\frac{n_1 n_5}{4\pi}\right) 2 \frac{\delta \tilde{S}^1_{i(0)}(x_1)}{\delta S^1_{j(0)}(x_2)},\end{aligned}$$

where we used (3.113). It follows that in order to obtain these 2-point functions we need to solve (3.285) to linear order in the sources (so the r.h.s is set equal to zero) and then extract the appropriate coefficient. The details of this computation can be found in section 6.3 of [68] with the following result

$$\begin{aligned}\langle \mathcal{O}_{\Sigma^k_I}(x_1)\mathcal{O}_{\Sigma^k_J}(x_2)\rangle &= \left(\frac{n_1 n_5}{4\pi}\right)\frac{(2k-2)\Gamma(k)}{\pi\Gamma(k-1)}\left(\frac{1}{x^{2k}}\right)_R \delta_{IJ}, \qquad k \neq 1; \\ \langle \mathcal{O}_{S^1_i}(x_1)\mathcal{O}_{S^1_j}(x_2)\rangle &= \left(\frac{n_1 n_5}{4\pi}\right)\frac{2}{\pi}\left(\frac{1}{x^2}\right)_R \delta_{ij},\end{aligned} \quad (3.287)$$

where the subscript R indicates that these are renormalized correlators.

We now discuss the 3-point function with $k \neq 2$. We can can obtain the 3-point function by the second variation of the 1-point function of O_{Σ^k}:

$$\begin{aligned}\langle \mathcal{O}_{\Sigma^k_i}(x_1)\mathcal{O}_{S^1_i}(x_2)\mathcal{O}_{S^1_j}(x_3)\rangle &= \frac{\delta^2\langle \mathcal{O}_{\Sigma^k_I}(x_1)\rangle}{\delta S^1_{i(0)}(x_2)\delta S^1_{j(0)}(x_3)} \\ &= \left(\frac{n_1 n_5}{4\pi}\right)(2k-2)\frac{\delta^2 \Sigma^k_{I(2k-2)}(x_1)}{\delta S^1_{i(0)}(x_2)\delta S^1_{j(0)}(x_3)}\end{aligned} \quad (3.288)$$

It follows that we need to solve (3.285) to quadratic order in the sources and then extract the coefficient of order z^k. The steps involved in this computation are spelled out in section 5.9 of

[21]. For the case at hand, the result is[10]

$$\langle \mathcal{O}_{\Sigma_I^k}(x_1)\mathcal{O}_{S_i^1}(x_2)\mathcal{O}_{S_j^1}(x_3)\rangle = -\left(\frac{n_1 n_5}{4\pi}\right) w_{Iij}\frac{2\Gamma(k)}{\pi^3\Gamma(k-1)}I_k(x_1,x_2,x_3) \quad (3.289)$$

where

$$I_k(x_1,x_2,x_3) = \int \frac{d^2x dz}{z^3}\left(\frac{z}{z^2+(\vec{x}-\vec{x}_1)^2}\right)^k \left(\frac{z}{z^2+(\vec{x}-\vec{x}_2)^2}\right)\left(\frac{z}{z^2+(\vec{x}-\vec{x}_3)^2}\right). \quad (3.290)$$

This integral was computed in [69] with answer

$$I_k(x_1,x_2,x_3) = \frac{\pi\Gamma(1-k/2)(\Gamma(k/2))^3}{2\Gamma(k)}\frac{1}{|\vec{x}_1-\vec{x}_2|^k|\vec{x}_1-\vec{x}_3|^k|\vec{x}_2-\vec{x}_3|^{2-k}}. \quad (3.291)$$

Notice that this integral diverges in the extremal case $k\to 2$.

The final answer for the correlator is thus

$$\langle \mathcal{O}_{\Sigma_I^k}(x_1)\mathcal{O}_{S_i^1}(x_2)\mathcal{O}_{S_j^1}(x_3)\rangle = \frac{C^k_{Iij}}{|\vec{x}_1-\vec{x}_2|^k|\vec{x}_1-\vec{x}_3|^k|\vec{x}_2-\vec{x}_3|^{2-k}} \quad (3.292)$$

where

$$C^k_{Iij} = -\left(\frac{n_1 n_5}{4\pi}\right)\frac{k^3(k+2)(k+4)\Gamma(k/2)^3\Gamma(2-k/2)}{32\pi^2(k+1)\Gamma(k-1)\sqrt{k(k-1)}}a_{Iij}\,. \quad (3.293)$$

This coefficient has a smooth limit as $k\to 2$; the zero in w_{Iij} cancels against the divergence in I_2, and we get

$$C^2_{Iij} = -\left(\frac{n_1 n_5}{4\pi}\right)\frac{1}{\sqrt{2}\pi^2}a_{Iij}\,. \quad (3.294)$$

We will take this to be the correct extremal 3-point function, i.e.,

$$\langle \mathcal{O}_{\Sigma_I^2}(x_1)\mathcal{O}_{S_i^1}(x_2)\mathcal{O}_{S_j^1}(x_3)\rangle = \frac{C^2_{Iij}}{|\vec{x}_1-\vec{x}_2|^2|\vec{x}_1-\vec{x}_3|^2}\,, \quad (3.295)$$

and use it to deduce the non-linear coupling in the 1-point function of $\langle\mathcal{O}_{\Sigma_I^2}\rangle$. As discussed in [22], the form of the 1-point function is uniquely fixed by general arguments to be

$$\langle\mathcal{O}_{\Sigma_I^2}\rangle = \left(\frac{n_1 n_5}{4\pi}\right)\left(\pi^{\Sigma_I^2}_{(2)} + A_{Iij}\pi^{S_i^1}_{(1)}\pi^{S_j^1}_{(1)}\right) \quad (3.296)$$

The numerical coefficient A_{Iij} should be determined by doing holographic renormalization in 6 (rather than 3) dimensions. We will fix it, however, such that the the extremal correlator is correctly computed directly at $k=2$ (rather than obtained as a limit from $k\neq 2$). Since $w_{Iij}(k=2)$=0 the only contribution comes from the terms non-linear in momenta

$$\begin{aligned}\langle \mathcal{O}_{\Sigma_I^k}(x_1)\mathcal{O}_{S_i^1}(x_2)\mathcal{O}_{S_j^1}(x_3)\rangle &= \left(\frac{n_1 n_5}{4\pi}\right)2A_{Iij}\left(\frac{\delta\pi^{S_i^1}_{(1)}(x_1)}{S^1_{i(0)}(x_2)}\right)\left(\frac{\delta\pi^{S_j^1}_{(1)}(x_1)}{S^1_{j(0)}(x_3)}\right); \\ &= \left(\frac{n_1 n_5}{4\pi}\right)A_{Iij}\frac{8}{\pi^2}\frac{1}{|\vec{x}_1-\vec{x}_2|^2|\vec{x}_1-\vec{x}_3|^2} \quad (3.297)\end{aligned}$$

By comparing with (3.295) we find

$$A_{Iij} = -\frac{1}{4\sqrt{2}}a_{Iij}. \quad (3.298)$$

[10] The normalization of the bulk-to-boundary propagator in (5.52) when $\Delta=1$ is $C_1 = 1/\pi$.

Non-extremal scalar three point functions

We will also need other three-point functions for scalars due to chiral primary operators. The relevant cubic couplings in three dimensions were also computed in [61, 24] and are given by

$$
\begin{aligned}
&-\frac{n_1 n_5}{4\pi}\int d^3x\sqrt{-G}(T_{123}S^1S^2\Sigma^3 + U_{123}\Sigma^1\Sigma^2\Sigma^3); \qquad (3.299)\\
&\equiv -\frac{n_1 n_5}{16\pi}\int d^3x\sqrt{-G}V_{123}\left(\frac{S^1S^2\Sigma^3}{\sqrt{(k_1+1)(k_2+1)}} + \frac{(k_1^2+k_2^2+k_3^2-2)}{(k_1+1)(k_2+1)}\frac{\Sigma^1\Sigma^2\Sigma^3}{6\sqrt{(k_1-1)(k_2-1)}}\right),\\
&V_{123} = \frac{\Sigma(\Sigma+2)(\Sigma-2)\alpha_1\alpha_2\alpha_3 a_{123}}{(k_3+1)\sqrt{k_1k_2k_3(k_3-1)}}
\end{aligned}
$$

where k_a denotes the dimension of the operator dual to the field Ψ^a, $\Sigma = k_1 + k_2 + k_3$, $\alpha_1 = \frac{1}{2}(k_2+k_3-k_1)$ etc and a_{123} is shorthand for the spherical harmonic overlap. It is straightforward to follow the same steps as before to compute the associated three point functions:

$$
\begin{aligned}
\langle \mathcal{O}_{S^1}(x_1)\mathcal{O}_{S^2}(x_2)\mathcal{O}_{\Sigma^3}(x_3)\rangle &= \frac{N}{4\pi^3}\frac{W_{123}T_{123}}{|\vec{x}_1-\vec{x}_2|^{2\alpha_3}|\vec{x}_1-\vec{x}_3|^{2\alpha_2}|\vec{x}_2-\vec{x}_3|^{2\alpha_1}}; \qquad (3.300)\\
\langle \mathcal{O}_{\Sigma^1}(x_1)\mathcal{O}_{\Sigma^2}(x_2)\mathcal{O}_{\Sigma^3}(x_3)\rangle &= \frac{3N}{4\pi^3}\frac{W_{123}U_{123}}{|\vec{x}_1-\vec{x}_2|^{2\alpha_3}|\vec{x}_1-\vec{x}_3|^{2\alpha_2}|\vec{x}_2-\vec{x}_3|^{2\alpha_1}};\\
W_{123} &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\frac{1}{2}(\Sigma-2))}{\Gamma(k_1-1)\Gamma(k_2-1)\Gamma(k_3-1)}.
\end{aligned}
$$

We will be interested in the case where $(S^1, S^2, \Sigma^1, \Sigma^2)$ have dimension k and (S^2, Σ^2) are chiral primary with (S^1, Σ^1) anti-chiral primary. Then charge conservation implies that the correlators are only non-zero when Σ^3 is neutral. In the case where $\mathcal{O}_{\Sigma^3}$ has dimension two the explicit results for the correlators using the spherical harmonic overlap of (3.243) are

$$
\begin{aligned}
\langle (\mathcal{O}_{S^p_k})^\dagger(x_1)\mathcal{O}_{S^p_k}(x_2)\mathcal{O}_{\Sigma^2_0}(x_3)\rangle &= \frac{N\sqrt{3}}{2\sqrt{2}\pi^3}\frac{k^3}{|\vec{x}_1-\vec{x}_2|^{2(k-1)}|\vec{x}_1-\vec{x}_3|^2|\vec{x}_2-\vec{x}_3|^2}; \qquad (3.301)\\
\langle (\mathcal{O}_{\Sigma^p_k})^\dagger(x_1)\mathcal{O}_{\Sigma^p_k}(x_2)\mathcal{O}_{\Sigma^2_0}(x_3)\rangle &= \frac{N\sqrt{3}}{2\sqrt{2}\pi^3(k+2)^3}\frac{k(k-1)(k^4-1)}{|\vec{x}_1-\vec{x}_2|^{2(k-1)}|\vec{x}_1-\vec{x}_3|^2|\vec{x}_2-\vec{x}_3|^2}.
\end{aligned}
$$

It will be useful to define normalized three point functions as

$$
\begin{aligned}
\langle (\mathcal{O}_{S^p_k})^\dagger \mathcal{O}_{\Sigma^2_0}(x)\mathcal{O}_{S^p_k}\rangle &\equiv \frac{\langle (\mathcal{O}_{S^p_k})^\dagger(\infty)\mathcal{O}_{\Sigma^2_0}(x)\mathcal{O}_{S^p_k}(0)\rangle}{\langle (\mathcal{O}_{S^p_k})^\dagger(\infty)\mathcal{O}_{S^p_k}(0)\rangle} = \frac{\sqrt{3}k^3}{\sqrt{2}\pi(k-1)^2}\frac{1}{|\vec{x}|^2}. \qquad (3.302)\\
\langle (\mathcal{O}_{\Sigma^p_k})^\dagger \mathcal{O}_{\Sigma^2_0}(x)\mathcal{O}_{\Sigma^p_k}\rangle &\equiv \frac{\langle (\mathcal{O}_{\Sigma^p_k})^\dagger(\infty)\mathcal{O}_{\Sigma^2_0}(x)\mathcal{O}_{\Sigma^p_k}(0)\rangle}{\langle (\mathcal{O}_{\Sigma^p_k})^\dagger(\infty)\mathcal{O}_{\Sigma^p_k}(0)\rangle};\\
&= \frac{\sqrt{3}k(k+1)(k^2+1)}{\sqrt{2}\pi(k+2)^2}\frac{1}{|\vec{x}|^2}.
\end{aligned}
$$

(Implicitly we assume here that $k \neq 1$.) Note that for $k \gg 1$ these expressions both tend to the same limit, $\sqrt{3}k/\sqrt{2}\pi|\vec{x}|^2$.

Two scalars and R symmetry current

Finally we will need three point functions between two scalars (of the same mass) and the R symmetry current. The relevant cubic couplings were again given in [24]:

$$-\frac{n_1 n_5}{8\pi}\int d^3x\sqrt{-G}A_\mu^{\pm\alpha}(S_I^k D_\mu S_J^k + \Sigma_I^k D_\mu \Sigma_J^k)E_{\alpha IJ}^{\pm}, \tag{3.303}$$

where the triple overlap is defined in (3.230). To compute the corresponding three point functions one again follows the steps given in [21]. This results in

$$\langle \mathcal{O}_{S_I^k}(x_1)J^{\pm\alpha}(x)\mathcal{O}_{S_J^k}(x_2)\rangle = \langle \mathcal{O}_{\Sigma_I^k}(x_1)J^{\pm\alpha}(x)\mathcal{O}_{\Sigma_J^k}(x_2)\rangle = \mp i\frac{N}{8\pi}E_{\alpha IJ}^{\pm}I_{\mp}(x,x_1,x_2), \tag{3.304}$$

where the AdS integral

$$I_{\mp}(x,x_1,x_2) = \int \frac{d^3z}{z^3}K_k(z,\vec{x}_1)D^\mu K_k(z,\vec{x}_2)\mathcal{G}_{\mu\mp}(z,\vec{x}) = \frac{(k-1)^2}{\pi^2}\frac{Z_{\mp}}{|\vec{x}_1-\vec{x}_2|^{2k}}, \tag{3.305}$$

was computed in [69]. In this integral $K_k(z,\vec{x})$ and $\mathcal{G}_{\mu\mp}(z,\vec{x})$ are the standard AdS scalar and vector bulk to boundary propagators respectively and

$$Z_+ = \frac{1}{(w_1-w)} - \frac{1}{(w_2-w)}; \qquad Z_- = \frac{1}{(\bar{w}_1-\bar{w})} - \frac{1}{(\bar{w}_2-\bar{w})}. \tag{3.306}$$

Here we have implicitly switched to Euclidean signature, $t = i\tau$, and introduced complex boundary coordinates $w = y + i\tau$.

In deriving this result we use the standard vector propagator, that following from the field equation $D_\mu F^{\mu\nu} = 0$, although the (linearized) vector equation here is Chern-Simons, $F_{\mu\nu} = 0$. Whilst this step should be justified more rigorously, it can be justified a posteriori by the fact that the three point functions thus obtained are of the standard form for a two dimensional CFT. To see this, consider the case where the scalar operators are chiral primary. Using the specific values for the spherical harmonic overlaps (3.244) in (3.304) gives

$$\begin{aligned}\langle (\mathcal{O}_{S^k})^\dagger(x_1)J^{+3}(w)\mathcal{O}_{S^k}(x_2)\rangle &= \frac{N}{8\pi^3}k(k-1)^2\left(\frac{1}{(w_1-w)} - \frac{1}{(w_2-w)}\right); \\ &= \langle (\mathcal{O}_{S^k})^\dagger(x_1)\mathcal{O}_{S^k}(x_2)\rangle\frac{k}{4\pi}\left(\frac{1}{(w_1-w)} - \frac{1}{(w_2-w)}\right),\end{aligned} \tag{3.307}$$

with the latter being the canonical form for the CFT three point function between the (holomorphic) R current and operators charged under it. An analogous formula holds for the anti-holomorphic current, $J^{-3}(\bar{w})$ and for the correlators involving scalar operators dual to Σ^k. Again it is useful to define normalized three point functions such that

$$\begin{aligned}\langle (\mathcal{O}_{S^k})^\dagger J^{+3}(w)\mathcal{O}_{S^k}\rangle &\equiv \frac{\langle (\mathcal{O}_{S^k})^\dagger(\infty)J^{+3}(w)\mathcal{O}_{S^k}(0)\rangle}{\langle (\mathcal{O}_{S^k})^\dagger(\infty)\mathcal{O}_{S^k}(0)\rangle} = \frac{k}{4\pi w}; \\ \langle (\mathcal{O}_{\Sigma^k})^\dagger J^{+3}(w)\mathcal{O}_{\Sigma^k}\rangle &\equiv \frac{\langle (\mathcal{O}_{\Sigma^k})^\dagger(\infty)J^{+3}(w)\mathcal{O}_{\Sigma^k}(0)\rangle}{\langle (\mathcal{O}_{\Sigma^k})^\dagger(\infty)\mathcal{O}_{\Sigma^k}(0)\rangle} = \frac{k}{4\pi w},\end{aligned} \tag{3.308}$$

with analogous formulae holding for the anti-holomorphic currents. The corresponding normalized three point functions for the spectrally flowed operators in the R sector are then

$$\langle (\mathcal{O}_{S^k})^\dagger_R J^{+3}(w) (\mathcal{O}_{S^k})_R \rangle \equiv \frac{\langle (\mathcal{O}_{S^k})^\dagger_R(\infty) J^{+3}(w) (\mathcal{O}_{S^k})_R(0) \rangle}{\langle (\mathcal{O}_{S^k})^\dagger_R(\infty) (\mathcal{O}_{S^k})_R(0) \rangle} = \frac{k-N}{4\pi w}; \tag{3.309}$$
$$\langle (\mathcal{O}_{\Sigma^k})^\dagger_R J^{+3}(w) (\mathcal{O}_{\Sigma^k})_R \rangle \equiv \frac{\langle (\mathcal{O}_{\Sigma^k})^\dagger_R(\infty) J^{+3}(w) (\mathcal{O}_{\Sigma^k})_R(0) \rangle}{\langle (\mathcal{O}_{\Sigma^k})^\dagger(\infty) \mathcal{O}_{\Sigma^k}(0) \rangle} = \frac{k-N}{4\pi w},$$

where $\mathcal{O}_R$ denotes the spectral flowed operator. Again corresponding formulae hold for the anti-holomorphic currents.

(3.A.5) Holographic 1-point functions

In this appendix we derive the 1-point function for the stress energy tensor and the operators dual to S^1_i. We omit the details of this computation since the analysis is very similar to the Coulomb branch analysis in [17, 18]. The asymptotic analysis of this system is also presented (in a different coordinate system) in [70] and the form of the counterterm was obtained in [68].

The relevant action is given in (3.106), retaining only the graviton and scalar fields S^1_i, and the most general asymptotic solution with Dirichlet boundary conditions is given by the expansion in (3.112) with coefficients given by

$$\begin{aligned} \mathrm{Tr}\, g_{(2)} &= -\tfrac{1}{2}R - \frac{1}{2}\left(2(S^1_{i(0)})^2 + (\tilde{S}^1_{i(0)})^2\right) \\ D^v g_{(2)uv} &= = -D_u\left(\tfrac{1}{2}R + \frac{1}{4}\left((\tilde{S}^1_{(0)i})^2 + 4(S^1_{i(0)})^2 - 2S^1_{i(0)}\tilde{S}^1_{i(0)}\right)\right) - S^1_{i(0)} D_u \tilde{S}^1_{i(0)} \\ h_{(2)uv} &= -\frac{1}{2} S^1_{i(0)} \tilde{S}^1_{i(0)} g_{(0)uv} \\ \tilde{h}_{(2)uv} &= -\frac{1}{4}(S^1_{i(0)})^2 g_{(0)uv} \end{aligned} \tag{3.310}$$

The traceless transverse part of $g_{(2)}$ and $\tilde{S}^1_{i(0)}$ (as well as the sources $g_{(0)uv}$ and $S^1_{i(0)}$) are unconstrained. We will soon see that these coefficients are related to the 1-point functions.

The counterterms needed to render the on-shell action finite are

$$S_{ct} = \frac{n_1 n_5}{4\pi}\int_{z=\epsilon} d^2x\sqrt{-\gamma}\left(2 - \log\epsilon^2 \tfrac{1}{2}R + \tfrac{1}{2}(S^1_i)^2\left(1 + \frac{2}{\log\epsilon^2}\right)\right) \tag{3.311}$$

so the on-shell renormalized action consists of (3.106), the Gibbons-Hawking term and these counterterms (along with additional counterterms for the gauge fields, discussed in the main text). The logarithmic terms determine the holographic conformal anomalies [13].

The renormalized 1-point functions are [11]

$$\begin{aligned} \langle \mathcal{O}_{S_i^1} \rangle &= \frac{n_1 n_5}{4\pi}(2\tilde{S}^1_{i(0)}); \qquad (3.312)\\ \langle T_{uv} \rangle &= \frac{n_1 n_5}{2\pi}\Big(g_{(2)uv} + \tfrac{1}{2}R g_{(0)uv} \\ &\quad + \frac{1}{4}\left((\tilde{S}^1_{i(0)})^2 - 2\tilde{S}^1_{i(0)}S^1_{i(0)} + 4(S^1_{i(0)})^2\right) g_{(0)uv}\Big). \end{aligned}$$

Using the asymptotic solution one may verify that these expressions satisfy the correct Ward identities

$$\langle T^u_u \rangle = = -S^1_{i(0)}\langle \mathcal{O}_{S_i^1} \rangle + \mathcal{A} \qquad (3.313)$$

$$D^v \langle T_{uv} \rangle = -\langle \mathcal{O}_{S_i^1} \rangle D_u S^1_{i(0)}. \qquad (3.314)$$

The first term on the r.h.s. is the standard term due to the coupling of the source $S^1_{i(0)}$ to an operator of dimension one. The conformal anomaly $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{c}{24\pi}R + \frac{n_1 n_5}{2\pi}(S^1_{i(0)})^2 ; \qquad c = 6n_1 n_5 \qquad (3.315)$$

The first term is the standard gravitational conformal anomaly and the second the conformal anomaly induced by the short distance singularities in the 2-point function of $\mathcal{O}_{S_i^1}$ [71].

(3.A.6) THREE POINT FUNCTIONS FROM THE ORBIFOLD CFT

In this appendix we discuss the relationship between three point functions computed in the CFT on the symmetric product $S^N(T^4)$ with those in supergravity. The chiral primary operators are summarized in (3.153); their detailed construction is not important here, but note that they are S_N invariant and orthonormal. The operators (3.153) manifestly have the correct dimensions and charges to correspond to the fields $S_k^{(r)I}$ and Σ_k^I in supergravity. Moreover, as discussed in section 3.8 the most natural correspondence seems to be that given in (3.156) although this choice is not unique.

Extremal three point functions of these operators have the following structure as $N \to \infty$ [62]

$$\begin{aligned} \left\langle \mathcal{O}^{(0,0)\dagger}_{n+k-1}(\infty)\mathcal{O}^{(0,0)}_k(1)\mathcal{O}^{(0,0)}_n(0)\right\rangle &= \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}; \qquad (3.316)\\ \left\langle \mathcal{O}^{(i)\dagger}_{n+k-1}(\infty)\mathcal{O}^{(0,0)}_k(1)\mathcal{O}^{(j)}_n(0)\right\rangle &= \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}\delta^{ij};\\ \left\langle \mathcal{O}^{(2,2)\dagger}_{n+k-1}(\infty)\mathcal{O}^{(0,0)}_k(1)\mathcal{O}^{(2,2)}_n(0)\right\rangle &= \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2};\\ \left\langle \mathcal{O}^{(2,2)\dagger}_{n+k-3}(\infty)\mathcal{O}^{(0,0)}_k(1)\mathcal{O}^{(0,0)}_n(0)\right\rangle &= \frac{2}{\sqrt{N}}\left((n+k-3)nk\right)^{1/2};\\ \left\langle \mathcal{O}^{(2,2)\dagger}_{n+k-1}(\infty)\mathcal{O}^{(i)}_k(1)\mathcal{O}^{(j)}_n(0)\right\rangle &= -\frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}\omega^i * \omega^j;\\ \left\langle \mathcal{O}^{(2,2)\dagger}_{n+k+1}(\infty)\mathcal{O}^{(2,2)}_k(1)\mathcal{O}^{(2,2)}_n(0)\right\rangle &= 0. \end{aligned}$$

[11] In comparing with [68] one should note the factor of 2 difference in the source.

(Here we use $\omega^i_{a\bar{a}}$ as a basis for $H^{(1,1)}(T^4)$).

The cubic couplings between scalars in supergravity were determined in [61, 24]. From (3.299) one sees that the couplings $\Sigma\Sigma\Sigma$ and ΣSS are generically non-zero whereas the couplings SSS and $S\Sigma\Sigma$ are always zero. This implies that the corresponding extremal three point functions between chiral primaries determined in supergravity have the following structures

$$\begin{aligned} &\left\langle \mathcal{O}^\dagger_{\Sigma^p_\Delta} \mathcal{O}_{\Sigma^p_{\Delta_1}} \mathcal{O}_{\Sigma^p_{\Delta_2}} \right\rangle \neq 0; \quad \left\langle \mathcal{O}^\dagger_{\Sigma^p_\Delta} \mathcal{O}_{S^p_{\Delta_1}} \mathcal{O}_{S^p_{\Delta_2}} \right\rangle \neq 0; \quad \left\langle \mathcal{O}^\dagger_{S^p_\Delta} \mathcal{O}_{\Sigma^p_{\Delta_1}} \mathcal{O}_{S^p_{\Delta_2}} \right\rangle \neq 0; \\ &\left\langle \mathcal{O}^\dagger_{S^p_\Delta} \mathcal{O}_{S^p_{\Delta_1}} \mathcal{O}_{S^p_{\Delta_2}} \right\rangle = 0; \quad \left\langle \mathcal{O}^\dagger_{\Sigma^p_\Delta} \mathcal{O}_{\Sigma^p_{\Delta_1}} \mathcal{O}_{S^p_{\Delta_2}} \right\rangle = 0; \quad \left\langle \mathcal{O}^\dagger_{S^p_\Delta} \mathcal{O}_{\Sigma^p_{\Delta_1}} \mathcal{O}_{\Sigma^p_{\Delta_2}} \right\rangle = 0, \end{aligned} \tag{3.317}$$

where $\Delta = \Delta_1 + \Delta_2$. Note that such correlators would be determined in supergravity either by a careful limiting procedure of non-extremal correlators (which uses directly the cubic couplings mentioned above) or by reducing the six-dimensional action including all boundary terms. In the latter case given that there are no bulk couplings SSS and $S\Sigma\Sigma$ it seems that there would be no boundary couplings between such fields, and hence no non-zero extremal correlators.

The correlators (3.316) and (3.317) clearly disagree if one makes the identification proposed in (3.156). Given that this identification was not unique, one might wonder whether there is a different linear map between supergravity and orbifold CFT operators such that the correlators agree. Whilst we have not proved in full generality that this is impossible, the following argument suggests that it is unlikely. Let $\mathcal{O}^a_1 = (\mathcal{O}^{(0,0)}_2, \mathcal{O}^{(i=1)}_1)$ denote two of the dimension one CFT operators and $\mathcal{O}^\alpha_2 = (\mathcal{O}^{(0,0)}_3, \mathcal{O}^{(i=1)}_2, \mathcal{O}^{(2,2)}_1)$ denote three of the dimension two CFT operators. Let $\hat{\mathcal{O}}^a_1 = \mathcal{O}_{S^a_1}$ denote two dimension one operators dual to sugra scalar fields and $\hat{\mathcal{O}}^\alpha_2 = (\mathcal{O}_{S^a_2}, \mathcal{O}_{\Sigma_2})$ denote three of the dimension two operators dual to sugra fields. Next write the fusion coefficients in the corresponding extremal three point functions in the orbifold CFT and supergravity as $C_{\alpha ab}$ and $\hat{C}_{\alpha ab}$ respectively. Since these are symmetric on the last two indices, rewrite them as (square) matrices $D_{\alpha\beta}$ and $\hat{D}_{\alpha\beta}$. Now the key point is that (3.316) and (3.317) imply that $D_{\alpha\beta}$ has non-zero determinant, but $\hat{D}_{\alpha\beta}$ has zero determinant. Any linear maps between $\mathcal{O}^a_1$ and $\hat{\mathcal{O}}^a_1$, and between $\mathcal{O}^\alpha_2$ and $\hat{\mathcal{O}}^\alpha_2$ which preserve the two point functions will not map $D_{\alpha\beta}$ to a zero determinant matrix and therefore one cannot get agreement between (3.316) and (3.317) by making a different identification between operators.

Addendum: This issue was later resolved in [72] after agreement between three-point functions of chiral primaries in orbifold CFT and string theory on $AdS_3 \times S^3 \times T^4$ had been shown in [73]. In general, there is a non-linear map between single particle string and orbifold CFT operators on one side and single particle supergravity operators on the other side. When calculating non-extremal three-point functions, the non-linear terms are suppressed in the large N limit and it is possible to find a non-diagonal matrix which maps the operators $(\mathcal{O}^S_\Delta, \mathcal{O}^\Sigma_\Delta)$ to the CFT operators $(\mathcal{O}^{(0,0)}_{\Delta+1}, \mathcal{O}^{(2,2)}_{\Delta-1})$. For extremal three-point functions however the non-linear terms in the operator map are not suppressed. In converse, extremal three-point functions can be used to fix these terms. Furthermore, a non-renormalization theorem for three-point functions of chiral primaries for AdS_3/CFT_2 has been proven in [74].

CHAPTER 4

FUZZBALLS WITH INTERNAL EXCITATIONS

(4.1) INTRODUCTION

In this chapter we construct and analyze the most general 2-charge D1-D5 fuzzball geometries which involve internal excitations. In the original work of [27], only a subset of the 2-charge fuzzball geometries were constructed using dualities from F1-P solutions. Recall that the D1-D5 system on T^4 is related by dualities to the type II F1-P system, also on T^4, whilst the D1-D5 system on $K3$ is related to the heterotic F1-P system on T^4; the exact duality chains needed will be reviewed in sections 4.2 and 4.3. Now the solution for a fundamental string carrying momentum in type II is characterized by 12 arbitrary curves, eight associated with transverse bosonic excitations and four associated with the bosonization of eight fermionic excitations on the string [39]. The corresponding heterotic string solution is characterized by 24 arbitrary curves, eight associated with transverse bosonic excitations and 16 associated with charge waves on the string.

In the work of [27], the duality chain was carried out for type II F1-P solutions on T^4 for which only bosonic excitations in the transverse R^4 are excited. That is, the solutions are characterized by only four arbitrary curves; in the dual D1-D5 solutions these four curves characterize the blow-up of the branes, which in the naive solutions are sitting in the origin of the transverse R^4, into a supertube. In this chapter we carry out the dualities for generic F1-P solutions in both the T^4 and $K3$ cases, to obtain generic 2-charge fuzzball solutions with internal excitations. Note that partial results for the T^4 case were previously given in the appendix of [40]; we will comment on the relation between our solutions and theirs in section 4.2. The general solutions are then characterized by arbitrary curves capturing excitations along the compact manifold M^4, along with the four curves describing the blow-up in R^4.

They describe a bound state of D1 and D5-branes, wrapped on the compact manifold M^4, blown up into a rotating supertube in R^4 and with excitations along the part of the D5-branes wrapping the M^4.

The duality chain that uses string-string duality from heterotic on T^4 to type II on K3 provides a route for obtaining fuzzball solutions that has not been fully explored. One of the results in this chapter is to make explicit all steps in this duality route. In particular, we work out the reduction of type IIB on K3 and show how S-duality acts in six dimensions. These results may be useful in obtaining fuzzball solution with more charges.

In chapter 3, we made a precise proposal for the relationship between the 2-charge fuzzball geometries characterized by four curves $F^i(v)$ and superpositions of R ground states: *a given geometry characterized by $F^i(v)$ is dual to a specific superposition of R vacua with the superposition determined by the Fourier coefficients of the curves $F^i(v)$*. In particular, note that only geometries associated with circular curves are dual to a single R ground state (in the usual basis, where the states are eigenstates of the R-charge). This proposal has a straightforward extension to generic 2-charge geometries, which we will spell out in section 4.6, and the extended proposal passes all kinematical and accessible dynamical tests, just as in chapter 3.

In particular, we extract one point functions for chiral primaries from the asymptotically AdS region of the fuzzball solutions. We find that chiral primaries associated with the middle cohomology of M^4 acquire vevs when there are both internal and transverse excitations; these vevs hence characterize the internal excitations. Moreover, there are selection rules for these vevs, in that the internal and transverse curves must have common frequencies.

These properties of the holographic vevs follow directly from the proposed dual superpositions of ground states. The vevs in these ground states can be derived from three point functions between chiral primaries at the conformal point. Selection rules for the latter, namely charge conservation and conservation of the number of operators associated with each middle cohomology cycle, lead to precisely the features of the vevs found holographically.

To test the actual values of the kinematically allowed vevs would require information about the three point functions of all chiral primaries which is not currently known and is inaccessible in supergravity. However, as in chapter 3, these vevs are reproduced surprisingly well by simple approximations for the three point functions, which follow from treating the operators as harmonic oscillators. This suggests that the structure of the chiral ring may simplify considerably in the large N limit, and it would be interesting to explore this question further.

An interesting feature of the solutions is that they collapse to the naive geometry when there are internal but no transverse excitations. One can understand this as follows. Geometries with only internal excitations are dual to superpositions of R ground states built from operators associated with the middle cohomology of M^4. Such operators account for a finite fraction of the entropy, but have zero R charges with respect to the $SO(4)$ R symmetry group. This means that they can only be characterized by the vevs of $SO(4)$ singlet operators but the only such operators visible in supergravity are kinematically prevented from acquiring vevs. Thus it is consistent that in supergravity one could not distinguish between such solutions: one would need to go beyond supergravity to resolve them (by, for instance, considering vevs of singlet operators dual to string states).

This brings us to a recurring question in the fuzzball program: can it be implemented consistently within supergravity? As already mentioned, rigorously testing the proposed correspondence between geometries and superpositions of microstates requires information beyond supergravity. Furthermore, the geometric duals of superpositions with very small or zero R charges are not well-described in supergravity. Even if one has geometries which are smooth supergravity geometries, these may not be distinguishable from each other within supergravity: for example, their vevs may differ only by terms of order $1/N$, which cannot be reliably computed in supergravity.

The question of whether the fuzzball program can be implemented in supergravity could first be phrased in the following way. Can one find a complete basis of fuzzball geometries, each of which is well-described everywhere by supergravity, which are distinguishable from each other within supergravity and which together span the black hole microstates? On general grounds one would expect this not to be possible since many of the microstates carry small quantum numbers. We quantify this discussion in the last section of this chapter in the context of both 2-charge and 3-charge systems.

To make progress within supergravity, however, it would suffice to sample the black hole microstates in a controlled way. I.e. one could try to find a basis of geometries which are well-described and distinguishable in supergravity and which span the black hole microstates but for which each basis element is assigned a measure. In this approach, one would deal with the fact that many geometries are too similar to be distinguished in supergravity by picking representative geometries with appropriate measures. In constructing such a representative basis, the detailed matching between geometries and black hole microstates would be crucial, to correctly assign measures and to show that the basis indeed spans all the black hole microstates.

The plan of this chapter is as follows. In section 4.2 we determine the fuzzball geometries for D1-D5 on T^4 from dualizing type II F1-P solutions whilst in section 4.3 we obtain fuzzball geometries for D1-D5 on $K3$ from dualizing heterotic F1-P solutions. The resulting solutions are of the same form and are summarized in section 4.4; readers interested only in the solutions may skip sections 2 and 3. In section 4.5 we extract from the asymptotically AdS regions the dual field theory data, one point functions for chiral primaries. In section 4.6 we discuss the correspondence between geometries and R vacua, extending the proposal of chapter 3 and using the holographic vevs to test this proposal. In section 4.7 we discuss more generally the implications of our results for the fuzzball proposal. Finally there are a number of appendices. In appendix A we state our conventions for the field equations and duality rules, in appendix B we discuss in detail the reduction of type IIB on K3 and appendix C summarizes relevant properties of spherical harmonics. In appendix D we discuss fundamental string solutions with winding along the torus, and the corresponding duals in the D1-D5 system. In appendix E we derive the density of ground states with fixed R charges.

(4.2) Fuzzball solutions on T^4

In this section we will obtain general 2-charge solutions for the D1-D5 system on T^4 from type II F1-P solutions.

(4.2.1) Chiral null models

Let us begin with a general chiral null model of ten-dimensional supergravity, written in the form

$$\begin{aligned} ds^2 &= H^{-1}(x,v)dv(-du + K(x,v)dv + 2A_I(x,v)dx^I) + dx^I dx_I; \\ e^{-2\Phi} &= H(x,v); \qquad B^{(2)}_{uv} = \tfrac{1}{2}(H(x,v)^{-1} - 1); \qquad B^{(2)}_{vI} = H(x,v)^{-1}A_I(x,v). \end{aligned} \tag{4.1}$$

The conventions for the supergravity field equations are given in the appendix 4.A.1. The above is a solution of the equations of motion provided that the defining functions are harmonic in the transverse directions, labeled by x^I:

$$\Box H(x,v) = \Box K(x,v) = \Box A_I(x,v) = (\partial_I A^I(x,v) - \partial_v H(x,v)) = 0. \tag{4.2}$$

Solutions of these equations appropriate for describing solitonic fundamental strings carrying momentum were given in [33, 34]:

$$H = 1 + \frac{Q}{|x - F(v)|^6}, \qquad A_I = -\frac{Q\dot{F}_I(v)}{|x - F(v)|^6}, \qquad K = \frac{Q^2 \dot{F}(v)^2}{Q|x - F(v)|^6}, \tag{4.3}$$

where $F^I(v)$ is an arbitrary null curve describing the transverse location of the string, and $\dot{F}^I$ denotes $\partial_v F^I(v)$. More general solutions appropriate for describing solitonic strings with fermionic condensates were discussed in [39]. Here we will dualise without using the explicit forms of the functions, thus the resulting dual supergravity solutions are applicable for all choices of harmonic functions.

The F1-P solutions described by such chiral null models can be dualised to give corresponding solutions for the D1-D5 system as follows. Compactify four of the transverse directions on a torus, such that x^i with $i = 1, \cdots, 4$ are coordinates on R^4 and x^ρ with $\rho = 5, \cdots, 8$ are coordinates on T^4. Then let $v = (t - y)$ and $u = (t + y)$ with the coordinate y being periodic with length $L_y \equiv 2\pi R_y$, and smear all harmonic functions over both this circle and over the T^4, so that they satisfy

$$\Box_{R^4} H(x) = \Box_{R^4} K(x) = \Box_{R^4} A_I(x) = 0, \qquad \partial_i A^i = 0. \tag{4.4}$$

Thus the harmonic functions appropriate for describing strings with only bosonic condensates are

$$\begin{aligned} H &= 1+\frac{Q}{L_y}\int_0^{L_y}\frac{dv}{|x-F(v)|^2}; \qquad A_i = -\frac{Q}{L_y}\int_0^{L_y}\frac{dv\dot{F}_i(v)}{|x-F(v)|^2}; \\ A_\rho &= -\frac{Q}{L_y}\int_0^{L_y}\frac{dv\dot{F}_\rho(v)}{|x-F(v)|^2}; \qquad K = \frac{Q}{L_y}\int_0^{L_y}\frac{dv(\dot{F}_i(v)^2+\dot{F}_\rho(v)^2)}{|x-F(v)|^2}. \end{aligned} \tag{4.5}$$

Here $|x-F(v)|^2$ denotes $\sum_i(x_i-F_i(v))^2$. Note that neither $\dot{F}_i(v)$ nor $\dot{F}_\rho(v)$ have zero modes; the asymptotic expansions of A_I at large $|x|$ therefore begin at order $1/|x|^3$. Closure of the curve in R^4 automatically implies that $\dot{F}_i(v)$ has no zero modes. The question of whether $\dot{F}_\rho(v)$ has zero modes is more subtle: since the torus coordinate x^ρ is periodic, the curve $F_\rho(v)$ could have winding modes. As we will discuss in appendix 4.A.4, however, such winding modes are possible only when the worldsheet theory is deformed by constant B fields. The corresponding supergravity solutions, and those obtained from them by dualities, should thus not be included in describing BPS states in the original 2-charge systems.

The appropriate chain of dualities to the $D1-D5$ system is

$$\begin{pmatrix} P_y \\ F1_y \end{pmatrix} \stackrel{S}{\rightarrow} \begin{pmatrix} P_y \\ D1_y \end{pmatrix} \stackrel{T5678}{\rightarrow} \begin{pmatrix} P_y \\ D5_{y5678} \end{pmatrix} \stackrel{S}{\rightarrow} \begin{pmatrix} P_y \\ NS5_{y5678} \end{pmatrix} \stackrel{Ty}{\rightarrow} \begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix}, \tag{4.6}$$

to map to the type IIA NS5-F1 system. The subsequent dualities

$$\begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix} \stackrel{T8}{\rightarrow} \begin{pmatrix} F1_y \\ NS5_{y5678} \end{pmatrix} \stackrel{S}{\rightarrow} \begin{pmatrix} D1_y \\ D5_{y5678} \end{pmatrix} \tag{4.7}$$

result in a D1-D5 system. Here the subscripts of $Dp_{a_1\cdots a_p}$ denote the spatial directions wrapped by the brane. In carrying out these dualities we use the rules reviewed in appendix 4.A.1. We will give details of the intermediate solution in the type IIA NS5-F1 system since it differs from that obtained in [40].

(4.2.2) The IIA F1-NS5 system

By dualizing the chiral null model from the F1-P system in IIB to F1-NS5 in IIA one obtains the solution

$$\begin{aligned} ds^2 &= \tilde{K}^{-1}[-(dt-A_idx^i)^2+(dy-B_idx^i)^2]+Hdx_idx^i+dx_\rho dx^\rho \\ e^{2\Phi} &= \tilde{K}^{-1}H, \qquad B^{(2)}_{ty} = \tilde{K}^{-1}-1, \\ B^{(2)}_{\bar{\mu}i} &= \tilde{K}^{-1}\mathcal{B}^{\bar{\mu}}_i, \qquad B^{(2)}_{ij} = -c_{ij}+2\tilde{K}^{-1}A_{[i}B_{j]} \\ C^{(1)}_\rho &= H^{-1}A_\rho, \qquad C^{(3)}_{ty\rho} = (H\tilde{K})^{-1}A_\rho, \qquad C^{(3)}_{\bar{\mu}i\rho} = (H\tilde{K})^{-1}\mathcal{B}^{\bar{\mu}}_iA_\rho, \\ C^{(3)}_{ij\rho} &= (\lambda_\rho)_{ij}+2(H\tilde{K})^{-1}A_\rho A_{[i}B_{j]}, \qquad C^{(3)}_{\rho\sigma\tau} = \epsilon_{\rho\sigma\tau\pi}H^{-1}A^\pi, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \tilde{K} &= 1+K-H^{-1}A_\rho A_\rho, \qquad dc = -*_4dH, \qquad dB = -*_4dA, \\ d\lambda_\rho &= *_4dA_\rho, \qquad \mathcal{B}^{\bar{\mu}}_i = (-B_i,A_i), \end{aligned} \tag{4.9}$$

with $\bar{\mu} = (t, y)$. Here the transverse and torus directions are denoted by (i, j) and (ρ, σ) respectively and $*_4$ denotes the Hodge dual in the flat metric on R^4, with $\epsilon_{\rho\sigma\tau\pi}$ denoting the Hodge dual in flat T^4 metric. The defining functions satisfy the equations given in (4.4).

The RR field strengths corresponding to the above potentials are

$$\begin{aligned} F^{(2)}_{i\rho} &= \partial_i(H^{-1}A_\rho), \qquad F^{(4)}_{tyi\rho} = \tilde{K}^{-1}\partial_i(H^{-1}A_\rho), \\ F^{(4)}_{\bar{\mu}ij\rho} &= 2\tilde{K}^{-1}\mathcal{B}^{\bar{\mu}}_{[i}\partial_{j]}(H^{-1}A_\rho), \qquad F^{(4)}_{i\rho\sigma\tau} = \epsilon_{\rho\sigma\tau\pi}\partial_i(H^{-1}A^\pi), \\ F^{(4)}_{ijk\rho} &= \tilde{K}^{-1}\left(6A_{[i}B_j\partial_{k]}(H^{-1}A_\rho) + H\epsilon_{ijkl}\partial^l(H^{-1}A_\rho)\right). \end{aligned} \tag{4.10}$$

Thus the solution describes NS5-branes wrapping the y circle and the T^4, bound to fundamental strings delocalized on the T^4 and wrapping the y circle, with additional excitations on the T^4. These excitations break the T^4 symmetry by singling out a direction within the torus, and source multipole moments of the RR fluxes; the solution however has no net D-brane charges.

Now let us briefly comment on the relation between this solution and that presented in appendix B of [40][1]. The NS-NS sector fields agree, but the RR fields are different; in [40] they are given as 1, 3 and 5-form potentials. The relation of these potentials to field strengths (and the corresponding field equations) is not given in [40]. As reviewed in appendix 4.A.1, in the presence of both electric and magnetic sources it is rather natural to use the so-called democratic formalisms of supergravity [75], in which one includes p-form field strengths with $p > 5$ along with constraints relating higher and lower form field strengths. Any solution written in the democratic formalism can be rewritten in terms of the standard formalism, appropriately eliminating the higher form field strengths. If one interprets the RR forms of [40] in this way, one does not however obtain a supergravity solution in the democratic formalism; the Hodge duality constraints between higher and lower form field strengths are not satisfied. Furthermore, one would not obtain from the RR fields of [40] the solution written here in the standard formalism, after eliminating the higher forms.

(4.2.3) Dualizing further to the D1-D5 system

The final steps in the duality chain are T-duality along a torus direction, followed by S-duality. When T-dualizing further along a torus direction to a F1-NS5 solution in IIB, the excitations along the torus mean that the dual solution depends explicitly on the chosen T-duality cycle in the torus. We will discuss the physical interpretation of the distinguished direction in section 4.4. In the following the T-duality is taken along the x^8 direction, resulting in the following D1-D5 system:

$$\begin{aligned} ds^2 &= \frac{f_1^{1/2}}{f_5^{1/2}\tilde{f}_1}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} dx_\rho dx^\rho \\ e^{2\Phi} &= \frac{f_1^2}{f_5\tilde{f}_1}, \qquad B^{(2)}_{ty} = \frac{\mathcal{A}}{f_5\tilde{f}_1}, \qquad B^{(2)}_{\bar{\mu}i} = \frac{\mathcal{A}\mathcal{B}^{\bar{\mu}}_i}{f_5\tilde{f}_1}, \end{aligned} \tag{4.11}$$

[1] We thank Samir Mathur for discussions on this issue.

$$
\begin{aligned}
B^{(2)}_{ij} &= \lambda_{ij} + \frac{2\mathcal{A}A_{[i}B_{j]}}{f_5\tilde{f}_1}, \qquad B^{(2)}_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}f_5^{-1}\mathcal{A}^\gamma, \qquad B^{(2)}_{\alpha 8} = f_5^{-1}\mathcal{A}_\alpha, \\
C^{(0)} &= -f_1^{-1}\mathcal{A}, \qquad C^{(2)}_{ty} = 1 - \tilde{f}_1^{-1}, \qquad C^{(2)}_{\bar{\mu}i} = -\tilde{f}_1^{-1}\mathcal{B}^{\bar{\mu}}_i, \\
C^{(2)}_{ij} &= c_{ij} - 2\tilde{f}_1^{-1}A_{[i}B_{j]}, \qquad C^{(4)}_{tyij} = \lambda_{ij} + \frac{\mathcal{A}}{f_5\tilde{f}_1}(c_{ij} + 2A_{[i}B_{j]}), \\
C^{(4)}_{\bar{\mu}ijk} &= \frac{3\mathcal{A}}{f_5\tilde{f}_1}\mathcal{B}^{\bar{\mu}}_{[i}c_{jk]}, \qquad C^{(4)}_{ty\alpha\beta} = -\epsilon_{\alpha\beta\gamma}f_5^{-1}\mathcal{A}^\gamma, \qquad C^{(4)}_{ty\alpha 8} = f_5^{-1}\mathcal{A}_\alpha, \\
C^{(4)}_{\alpha\beta\gamma 8} &= \epsilon_{\alpha\beta\gamma}f_5^{-1}\mathcal{A}, \qquad C^{(4)}_{ij\alpha 8} = (\lambda_\alpha)_{ij} + f_5^{-1}\mathcal{A}_\alpha c_{ij}, \qquad C^{(4)}_{ij\alpha\beta} = -\epsilon_{\alpha\beta\gamma}(\lambda^\gamma{}_{ij} + f_5^{-1}\mathcal{A}^\gamma c_{ij}),
\end{aligned}
$$

where

$$
\begin{aligned}
& f_5 \equiv H, \qquad \tilde{f}_1 = 1 + K - H^{-1}(\mathcal{A}_\alpha\mathcal{A}_\alpha + (\mathcal{A})^2), \qquad f_1 = \tilde{f}_1 + H^{-1}(\mathcal{A})^2, \\
& dc = -*_4\, dH, \qquad dB = -*_4\, dA, \qquad \mathcal{B}^{\bar{\mu}}_i = (-B_i, A_i), \\
& d\lambda_\alpha = *_4 d\mathcal{A}_\alpha, \qquad d\lambda = *_4 d\mathcal{A}.
\end{aligned} \tag{4.12}
$$

Here $\bar{\mu} = (t, y)$ and we denote A_8 as $\mathcal{A}$ with the remaining A_ρ being denoted by $\mathcal{A}_\alpha$ where the index α runs over only $5, 6, 7$. The Hodge dual over these coordinates is denoted by $\epsilon_{\alpha\beta\gamma}$. Explicit expressions for these defining harmonic functions in terms of variables of the D1-D5 system will be given in section 4.4.

The forms with components along the torus directions can be written more compactly as follows. Introduce a basis of self-dual and anti-self dual 2-forms on the torus such that

$$
\omega^{\alpha\pm} = \frac{1}{\sqrt{2}}(dx^{4+\alpha\pm} \wedge dx^8 \pm *_{T^4}(dx^{4+\alpha\pm} \wedge dx^8)), \tag{4.13}
$$

with $\alpha_\pm = 1, 2, 3$. These forms are normalized such that

$$
\int_{T^4} \omega^{\alpha\pm} \wedge \omega^{\beta\pm} = \pm(2\pi)^4 V \delta^{\alpha\pm\beta\pm}, \tag{4.14}
$$

where $(2\pi)^4 V$ is the volume of the torus. Then the potentials wrapping the torus directions can be expressed as

$$
\begin{aligned}
B^{(2)}_{\rho\sigma} &= C^{(4)}_{ty\rho\sigma} = \sqrt{2} f_5^{-1}\mathcal{A}^{\alpha-}\omega^{\alpha-}_{\rho\sigma}, \\
C^{(4)}_{ij\rho\sigma} &= \sqrt{2}\left((\lambda_{ij})^{\alpha-} + f_5^{-1}\mathcal{A}^{\alpha-}c_{ij}\right)\omega^{\alpha-}_{\rho\sigma}, \\
C^{(4)}_{\rho\sigma\tau\pi} &= \epsilon_{\rho\sigma\tau\pi}f_5^{-1}\mathcal{A},
\end{aligned} \tag{4.15}
$$

with $\epsilon_{\rho\sigma\tau\pi}$ being the Hodge dual in the flat metric on T^4. Note that these fields are expanded only in the anti-self dual two-forms, with neither the self dual two-forms nor the odd-dimensional forms on the torus being switched on anywhere in the solution. As we will discuss later, this means the corresponding six-dimensional solution can be described in chiral $N = 4b$ six-dimensional supergravity. The components of forms associated with the odd cohomology of T^4 reduce to gauge fields in six dimensions which are contained in the full $N = 8$ six-dimensional supergravity, but not its truncation to $N = 4b$.

(4.3) Fuzzball solutions on $K3$

In this section we will obtain general 2-charge solutions for the D1-D5 system on $K3$ from F1-P solutions of the heterotic string.

(4.3.1) Heterotic chiral model in 10 dimensions

The chiral model for the charged heterotic F1-P system in 10 dimensions is:

$$\begin{aligned} ds^2 &= H^{-1}(-dudv + (K - 2\alpha' H^{-1} N^{(c)} N^{(c)})dv^2 + 2A_I dx^I dv) + dx_I dx^I \\ \hat{B}^{(2)}_{uv} &= \frac{1}{2}(H^{-1} - 1), \qquad \hat{B}^{(2)}_{vI} = H^{-1} A_I, \\ \hat{\Phi} &= -\frac{1}{2}\ln H, \qquad \hat{V}^{(c)}_v = H^{-1} N^{(c)}, \end{aligned} \tag{4.16}$$

where $I = 1, \cdots, 8$ labels the transverse directions and $\hat{V}^{(c)}_m$ are Abelian gauge fields, with $((c) = 1, \cdots, 16)$ labeling the elements of the Cartan of the gauge group. The fields are denoted with hats to distinguish them from the six-dimensional fields used in the next subsection. The equations of motion for the heterotic string are given in appendix 4.A.1; here again the defining functions satisfy

$$\Box H(x,v) = \Box K(x,v) = \Box A_I(x,v) = (\partial_I A^I(x,v) - \partial_v H(x,v)) = \Box N^{(c)} = 0. \tag{4.17}$$

For the solution to correspond to a solitonic charged heterotic string, one takes the following solutions

$$\begin{aligned} H &= 1 + \frac{Q}{|x - F(v)|^6}, \qquad A_I = -\frac{Q\dot{F}_I(v)}{|x - F(v)|^6}, \qquad N^{(c)} = \frac{q^{(c)}(v)}{|x - F(v)|^6}, \\ K &= \frac{Q^2 \dot{F}(v)^2 + 2\alpha' q^{(c)} q^{(c)}(v)}{Q|x - F(v)|^6}, \end{aligned} \tag{4.18}$$

where $F^I(v)$ is an arbitrary null curve in R^8; $q^{(c)}(v)$ is an arbitrary charge wave and $\dot{F}_I(v)$ denotes $\partial_v F_I(v)$. Such solutions were first discussed in [33, 34], although the above has a more generic charge wave, lying in $U(1)^{16}$ rather than $U(1)$. In what follows it will be convenient to set $\alpha' = \frac{1}{4}$.

These solutions can be related by a duality chain to fuzzball solutions in the D1-D5 system compactified on $K3$. The chain of dualities is the following:

$$\begin{pmatrix} P_y \\ F1_y \end{pmatrix}_{Het,T^4} \rightarrow \begin{pmatrix} P_y \\ NS5_{ty,K3} \end{pmatrix}_{IIA} \xrightarrow{T_y} \begin{pmatrix} F1_y \\ NS5_{ty,K3} \end{pmatrix}_{IIB} \xrightarrow{S} \begin{pmatrix} D1_y \\ D5_{ty,K3} \end{pmatrix}_{IIB} \tag{4.19}$$

The first step in the duality is string-string duality between the heterotic theory on T^4 and type IIA on $K3$. Again the subscripts of $Dp_{a_1 \cdots a_p}$ denote the spatial directions wrapped by the brane.

To use this chain of dualities on the charged solitonic strings given above, the solutions must be smeared over the T^4 and over v, so that the harmonic functions satisfy

$$\Box_{R^4} H = \Box_{R^4} K = \Box_{R^4} A_I = \Box_{R^4} N^{(c)} = \partial_i A^i = 0 \tag{4.20}$$

where $i = 1, \cdots, 4$ labels the transverse R^4 directions. Note that although the chain of dualities is shorter than in the previous case there are various subtleties associated with it, related to the K3 compactification, which will be discussed below.

(4.3.2) Compactification on T^4

Compactification of the heterotic theory on T^4 is straightforward, see [76, 77] and the review [78]. The 10-dimensional metric is reduced as

$$\hat{G}_{mn} = \begin{pmatrix} g_{MN} + G_{\rho\sigma} V_M^{(1)\,\rho} V_N^{(1)\,\sigma} & V_M^{(1)\,\rho} G_{\rho\sigma} \\ V_N^{(1)\,\sigma} G_{\rho\sigma} & G_{\rho\sigma} \end{pmatrix}, \tag{4.21}$$

where $V_M^{(1)\,\rho}$ with $\rho = 1, \cdots 4$, are KK gauge fields. (Recall that the ten-dimensional quantities are denoted with hats to distinguish them from six-dimensional quantities.) The reduced theory contains the following bosonic fields: the graviton g_{MN}, the six-dimensional dilaton Φ_6, 24 Abelian gauge fields $V_M^{(a)} \equiv (V_M^{(1)\,\rho}, V_{M\,\rho}^{(2)}, V_M^{(3)\,(c)})$, a two form B_{MN} and an $O(4,20)$ matrix of scalars M. Note that the index $(a),(b)$ for the $SO(4,20)$ vector runs from $(1, \cdots, 24)$. These six-dimensional fields are related to the ten-dimensional fields as

$$\begin{aligned} \Phi_6 &= \hat{\Phi} - \frac{1}{2} \ln \det G_{\rho\sigma}; \\ V_{M\,\rho}^{(2)} &= \hat{B}_{M\rho}^{(2)} + \hat{B}_{\rho\sigma}^{(2)} V_M^{(1)\,\sigma} + \frac{1}{2} \hat{V}_\rho^{(c)} V_M^{(3)\,(c)}; \\ V_M^{(3)\,(c)} &= \hat{V}_M^{(c)} - \hat{V}_\rho^{(c)} V_M^{(1)\,\rho}; \\ H_{MNP} &= 3(\partial_{[M} \hat{B}_{NP]}^{(2)} - \frac{1}{2} V_{[M}^{(a)} L_{(a)(b)} F(V)_{NP]}^{(b)}), \end{aligned} \tag{4.22}$$

with the metric g_{MN} and $V_M^{(1)\,\rho}$ defined in (4.21). The matrix L is given by

$$L = \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, \tag{4.23}$$

where I_n denotes the $n \times n$ identity matrix. The scalar moduli are defined via

$$M = \Omega_1^T \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}V^T \\ -C^T G^{-1} & G + C^T G^{-1} C + V^T V & C^T G^{-1} V^T + V^T \\ -V G^{-1} & V G^{-1} C + V & I_{16} + V G^{-1} V^T \end{pmatrix} \Omega_1, \tag{4.24}$$

where $G \equiv [\hat{G}_{\rho\sigma}]$, $C \equiv [\frac{1}{2} \hat{V}_\rho^{(c)} \hat{V}_\sigma^{(c)} + \hat{B}_{\rho\sigma}^{(2)}]$ and $V \equiv [\hat{V}_\rho^{(c)}]$ are defined in terms of the components of the 10-dimensional fields along the torus. The constant $O(4,20)$ matrix Ω_1 is given by

$$\Omega_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_4 & I_4 & 0 \\ -I_4 & I_4 & 0 \\ 0 & 0 & \sqrt{2} I_{16} \end{pmatrix}. \tag{4.25}$$

This matrix arises in (4.24) as follows. In [76, 78] the matrix L was chosen to be off-diagonal, but for our purposes it is useful for L to be diagonal. An off-diagonal choice is associated with an off-diagonal intersection matrix for the self-dual and anti-self-dual forms of $K3$, but this is an unnatural choice for our solutions, in which only anti-self-dual forms are active. Thus relative to the conventions of [76, 78] we take $L \to \Omega_1^T L \Omega_1$, which induces $M \to \Omega_1^T M \Omega_1$ and $F \to \Omega_1^T F$. The definitions of this and other constant matrices used throughout the chapter are summarized in appendix 4.A.2.

These fields satisfy the equations of motion following from the action

$$\begin{aligned} S &= \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} e^{-2\Phi_6} [R + 4(\partial\Phi_6)^2 - \frac{1}{12} H_3^2 - \frac{1}{4} F(V)^{(a)}_{MN} (LML)_{(a)(b)} F(V)^{(b)MN} \\ &\quad + \frac{1}{8} \mathrm{tr}(\partial_M ML \partial^M ML)], \end{aligned} \tag{4.26}$$

where α' has been set to $1/4$ and $\kappa_6^2 = \kappa_{10}^2 / V_4$ with V_4 the volume of the torus.

The reduction of the heterotic solution to six dimensions is then

$$\begin{aligned} ds^2 &= H^{-1} \left[-du dv + \left(K - H^{-1} (\tfrac{1}{2} (N^{(c)})^2 + (A_\rho)^2) \right) dv^2 + 2 A_i dx^i dv \right] + dx_i dx^i, \\ B_{uv} &= \tfrac{1}{2} (H^{-1} - 1), \qquad B_{vi} = H^{-1} A_i, \qquad \Phi_6 = -\tfrac{1}{2} \ln H \\ V_v^{(a)} &= \left(0_4, \sqrt{2} H^{-1} A_\rho, H^{-1} N^{(c)} \right), \qquad M = I_{24}, \end{aligned} \tag{4.27}$$

where $i = 1, \cdots, 4$ runs over the transverse R^4 directions and $\rho = 5, \cdots, 8$ runs over the internal directions of the T^4. Thus the six-dimensional solution has only one non-trivial scalar field, the dilaton, with all other scalar fields being constant.

(4.3.3) String-string duality to P-NS5 (IIA) on $K3$

Given the six-dimensional heterotic solution, the corresponding IIA solution in six dimensions can be obtained as follows. Compactification of type IIA on $K3$ leads to the following six-dimensional theory [79]:

$$\begin{aligned} S' &= \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g'} \left(e^{-2\Phi_6'} [R' + 4(\partial\Phi_6')^2 - \frac{1}{12} {H_3'}^2 + \frac{1}{8} \mathrm{tr}(\partial_M M' L \partial^M M' L)] \right. \\ &\quad \left. - \frac{1}{4} F'(V)^{(a)}_{MN} (LM'L)_{(a)(b)} F'(V)^{(b)MN} \right) - 2 \int B_2' \wedge F_2'(V)^{(a)} \wedge F_2'(V)^{(b)} L_{(a)(b)}. \end{aligned} \tag{4.28}$$

The field content is the same as for the heterotic theory in (4.26); note that in contrast to (4.22) there is no Chern-Simons term in the definition of the 3-form field strength, that is, $H'_{MNP} = 3\partial_{[M} B'_{NP]}$.

The rules for string-string duality are [79]:

$$\begin{aligned}\Phi_6' &= -\Phi_6, \qquad g'_{MN} = e^{-2\Phi_6} g_{MN}, \qquad M' = M, \qquad V'^{(a)}_M = V^{(a)}_M, \\ H_3' &= e^{-2\Phi_6} *_6 H_3; \end{aligned} \tag{4.29}$$

these transform the equations of motion derived from (4.26) into ones derived from the action (4.28).

Acting with this string-string duality on the heterotic solutions (4.27) yields, dropping the primes on IIA fields:

$$\begin{aligned} ds^2 &= -dudv + (K - H^{-1}((N^{(c)})^2/2 + (A_\rho)^2))dv^2 + 2A_i dx^i dv + H dx_i dx^i, \\ H_{vij} &= -\epsilon_{ijkl}\partial^k A^l, \qquad H_{ijk} = \epsilon_{ijkl}\partial^l H, \qquad \Phi_6 = \frac{1}{2}\ln H, \\ V^{(a)}_v &= \left(0_4, \sqrt{2}H^{-1}A_\rho, H^{-1}N^{(c)}\right), \qquad M = I_{24}, \end{aligned} \tag{4.30}$$

with ϵ_{ijkl} denoting the dual in the flat R^4 metric. This describes NS5-branes on type IIA, wrapped on K3 and on the circle direction y, carrying momentum along the circle direction.

(4.3.4) T-duality to F1-NS5 (IIB) on $K3$

The next step in the duality chain is T-duality on the circle direction y to give an NS5-F1 solution of type IIB on $K3$. It is most convenient to carry out this step directly in six dimensions, using the results of [80] on T-duality of type II theories on $K3 \times S^1$.

Recall that type IIB compactified on $K3$ gives $d = 6$, $N = 4b$ supergravity coupled to 21 tensor multiplets, constructed by Romans in [54]. The bosonic field content of this theory is the graviton g_{MN}, 5 self-dual and 21 anti-self dual tensor fields and an O(5,21) matrix of scalars $\mathcal{M}$ which can be written in terms of a vielbein $\mathcal{M}^{-1} = V^T V$. Following the notation of [55] the bosonic field equations may be written as

$$\begin{aligned} R_{MN} &= 2P^{nr}_M P^{nr}_N + H^n_{MPQ} H^{n\,PQ}_N + H^r_{MPQ} H^{r\,PQ}_N, \\ \nabla^M P^{nr}_M &= Q^{Mnm} P^{mr}_M + Q^{Mrs} P^{ns}_M + \frac{\sqrt{2}}{3} H^{nMNP} H^r_{MNP}, \end{aligned} \tag{4.31}$$

along with Hodge duality conditions on the 3-forms

$$*_6 H^n_3 = H^n_3, \qquad *_6 H^r_3 = -H^r_3, \tag{4.32}$$

In these equations (m, n) are $SO(5)$ vector indices running from 1 to 5 whilst (r, s) are $SO(21)$ vector indices running from 6 to 26. The 3-form field strengths are given by

$$H^n = G^A V^n_A; \qquad H^r = G^A V^r_A, \tag{4.33}$$

where $A \equiv \{n, r\} = 1, \cdots, 26$; $G^A = db^A$ are closed and the vielbein on the coset space $SO(5,21)/(SO(5) \times SO(21))$ satisfies

$$V^T \eta V = \eta, \qquad V = \begin{pmatrix} V^n{}_A \\ V^r{}_A \end{pmatrix}, \qquad \eta = \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}. \tag{4.34}$$

The associated connection is

$$dVV^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2}P^{ms} \\ \sqrt{2}P^{rn} & Q^{rs} \end{pmatrix}, \tag{4.35}$$

where Q^{mn} and Q^{rs} are antisymmetric and the off-diagonal block matrices P^{ms} and P^{rn} are transposed to each other. Note also that there is a freedom in choosing the vielbein; $SO(5) \times SO(21)$ transformations acting on H_3 and V as

$$V \to OV, \qquad H_3 \to OH_3, \tag{4.36}$$

leave G_3 and $\mathcal{M}^{-1}$ unchanged. Note that the field equations (4.31) can also be derived from the $SO(5,21)$ invariant Einstein frame pseudo-action [81]

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \left(R + \frac{1}{8}\mathrm{tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M}) - \frac{1}{3} G^A_{MNP} \mathcal{M}^{-1}_{AB} G^{BMNP} \right), \tag{4.37}$$

with the Hodge duality conditions (4.32) being imposed independently.

Now let us consider the T-duality relating a six-dimensional IIB solution to a six-dimensional IIA solution of (4.28); the corresponding rules were derived in [80]. Given that the six-dimensional IIA supergravity has only an $SO(4,20)$ symmetry, relating IIB to IIA requires explicitly breaking the $SO(5,21)$ symmetry of the IIB action down to $SO(4,20)$. That is, one defines a conformal frame in which only an $SO(4,20)$ subgroup is manifest and in which the action reads

$$\begin{aligned} S &= \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \left\{ e^{-2\Phi} \left(R + 4(\partial\Phi)^2 + \frac{1}{8}\mathrm{tr}(\partial M^{-1}\partial M) \right) + \frac{1}{2}\partial l^{(a)} M^{-1}_{(a)(b)} \partial l^{(b)} \right. \\ &\quad \left. -\frac{1}{3} G^A_{MNP} \mathcal{M}^{-1}_{AB} G^{BMNP} \right\}. \end{aligned} \tag{4.38}$$

The $SO(5,21)$ matrix $\mathcal{M}^{-1}$ has now been split up into the dilaton Φ, an $SO(4,20)$ vector $l^{(a)}$ and an $SO(4,20)$ matrix $M^{-1}_{(a)(b)}$, and we have chosen the parametrization

$$\mathcal{M}^{-1}_{AB} = \Omega_3^T \begin{pmatrix} e^{-2\Phi} + l^T M^{-1} l + \frac{1}{4} e^{2\Phi} l^4 & -\frac{1}{2} e^{2\Phi} l^2 & (l^T M^{-1})_{(b)} + \frac{1}{2} e^{2\Phi} l^2 (l^T L)_{(b)} \\ -\frac{1}{2} e^{2\Phi} l^2 & e^{2\Phi} & -e^{2\Phi} (l^T L)_{(b)} \\ (M^{-1} l)_{(a)} + \frac{1}{2} e^{2\Phi} l^2 (Ll)_{(a)} & -e^{2\Phi}(Ll)_{(a)} & M^{-1}_{(a)(b)} + e^{2\Phi} (Ll)_{(a)} (l^T L)_{(b)} \end{pmatrix} \Omega_3, \tag{4.39}$$

where $l^2 = l^{(a)} l^{(b)} L_{(a)(b)}$, $L_{(a)(b)}$ was defined in (4.23) and Ω_3 is a constant matrix defined in appendix 4.A.2.

The fields Φ, $l^{(a)}$ and M^{-1} and half of the 3-forms can now be related to the IIA fields of section 4.3.3 by the following T-duality rules (given in terms of the 2-form potentials b^A) [80]:

$$\begin{aligned} \tilde{g}_{yy} &= g_{yy}^{-1}, & \tilde{b}^1_{yM} + \tilde{b}^{26}_{yM} &= \tfrac{1}{2} g_{yy}^{-1} g_{yM}, \\ \tilde{g}_{yM} &= g_{yy}^{-1} B_{yM}, & \tilde{b}^1_{MN} + \tilde{b}^{26}_{MN} &= \tfrac{1}{2} g_{yy}^{-1} (B_{MN} + 2(g_{y[M} B_{N]y})), \\ \tilde{g}_{MN} &= g_{MN} - g_{yy}^{-1}(g_{yM} g_{yN} - B_{yM} B_{yN}), & \tilde{l}^{(a)} &= V_y^{(a)}, \\ \tilde{\Phi} &= \Phi - \frac{1}{2} \log |g_{yy}|, & \tilde{M}^{-1}_{(a)(b)} &= M^{-1}_{(a)(b)}, \\ \tilde{b}^{(a)+1}_{yM} &= \frac{1}{\sqrt{8}} (V_M^{(a)} - g_{yy}^{-1} V_y^{(a)} g_{yM}), & & (1 \leq (a) \leq 24), \end{aligned} \tag{4.40}$$

Here y is the T-duality circle, the six-dimensional index M excludes y and IIB fields are denoted by tildes to distinguish them from IIA fields. The other half of the tensor fields, that is $\Big((\tilde{b}^{1}_{yM} - \tilde{b}^{26}_{yM}), (\tilde{b}^{1}_{MN} - \tilde{b}^{26}_{MN}), \tilde{b}^{(a)+1}_{MN}, \tilde{b}^{(a)+1}_{MN}\Big)$, can then be determined using the Hodge duality constraints (4.32).

We now have all the ingredients to obtain the T-dual of the IIA solution (4.30) along $y \equiv \frac{1}{2}(u - v)$. The IIA solution is expressed in terms of harmonic functions which also depend on the null coordinate v, and thus one needs to smear the solutions before dualizing. Note that it is the harmonic functions $(H, K, A^{I}, N^{(c)})$ which must be smeared over v, rather than the six-dimensional fields given in (4.30), since it is the former that satisfy linear equations and can therefore be superimposed.

The Einstein frame metric and three forms are given by

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H\tilde{K}}}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2)] + \sqrt{H\tilde{K}} dx_i dx^i, \\ G^{A}_{tyi} &= \partial_i \left(\frac{n^A}{H\tilde{K}}\right), \qquad G^{A}_{\bar{\mu}ij} = -2\partial_{[i}\left(\frac{n^A}{H\tilde{K}}\mathcal{B}^{\bar{\mu}}_{j]}\right), \\ G^{A}_{ijk} &= \epsilon_{ijkl}\partial^{l} n^{A} + 6\partial_{[i}\left(\frac{n^A}{H\tilde{K}} A_j B_{k]}\right), \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} n^m &= \frac{1}{4}(H + K + 1, 0_4), \qquad n^r = \frac{1}{4}\left(-2A_\rho, -\sqrt{2}N^{(c)}, H - K - 1\right), \\ \tilde{K} &= 1 + K - H^{-1}(\tfrac{1}{2}(N^{(c)})^2 + (A_\rho)^2), \qquad dB = - *_4 dA, \qquad \mathcal{B}^{\mu}_{i} = (-B_i, A_i). \end{aligned} \tag{4.42}$$

Recall that $n = 1, \cdots, 5$ and $r = 6, \cdots, 26$ and $*_4$ denotes the dual on flat R^4; $\bar{\mu} = (t, y)$. The $SO(4, 20)$ scalars are given by

$$\Phi = \frac{1}{2}\ln\frac{H}{\tilde{K}}, \qquad l^{(a)} = \left(0_4, \sqrt{2}H^{-1}A_\rho, H^{-1}N^{(c)}\right), \qquad M = I_{24}. \tag{4.43}$$

The $SO(5, 21)$ scalar matrix $\mathcal{M}^{-1} = V^T V$ in (4.39) can then conveniently be expressed in terms of the vielbein

$$V = \Omega_3^T \begin{pmatrix} \sqrt{H^{-1}\tilde{K}} & 0 & 0 \\ -(\sqrt{H^3\tilde{K}})^{-1}(A_\rho^2 + \frac{1}{2}(N^{(c)})^2) & \sqrt{H\tilde{K}^{-1}} & -\sqrt{H\tilde{K}^{-1}}l^{(b)} \\ l^{(a)} & 0 & I_{24} \end{pmatrix} \Omega_3. \tag{4.44}$$

(4.3.5) S-DUALITY TO D1-D5 ON $K3$

One further step in the duality chain is required to obtain the D1-D5 solution in type IIB, namely S duality. However, in the previous section the type II solutions have been given in six rather than ten dimensions. To carry out S duality one needs to specify the relationship between six and ten dimensional fields. Whilst the ten-dimensional $SL(2, R)$ symmetry is part of the six-dimensional symmetry group, its embedding into the full six-dimensional symmetry group is only defined once one specifies the uplift to ten dimensions. The details of the dimensional reduction are given in appendix 4.A.2, with the six-dimensional S duality rules being given in (4.157); the S duality leaves the Einstein frame metric invariant, and acts as a constant rotation and similarity transformation on the three forms G^A and the matrix of scalars $\mathcal{M}$ respectively. The S-dual solution is thus

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{f_5\tilde{f}_1}}[-(dt-A_i dx^i)^2+(dy-B_i dx^i)^2)]+\sqrt{f_5\tilde{f}_1}dx_i dx^i, \\ G^A_{tyi} &= \partial_i\left(\frac{m^A}{f_5\tilde{f}_1}\right), \qquad G^A_{\bar{\mu}ij} = -2\partial_{[i}\left(\frac{m^A}{f_5\tilde{f}_1}\mathcal{B}^{\bar{\mu}}_{j]}\right), \\ G^A_{ijk} &= \epsilon_{ijkl}\partial^l m^A + 6\partial_{[i}\left(\frac{m^A}{f_5\tilde{f}_1}A_j B_{k]}\right), \end{aligned} \tag{4.45}$$

with

$$\begin{aligned} m^n &= \left(0_4, \tfrac{1}{4}(f_5+F_1)\right), \\ m^r &= \frac{1}{4}\left((f_5-F_1), -2A_\alpha, -\sqrt{2}N^{(c)}, 2A_5\right) \\ &\equiv \frac{1}{4}\left((f_5-F_1), -2\mathcal{A}^{\alpha-}, 2\mathcal{A}\right). \end{aligned} \tag{4.46}$$

Here the index $\alpha = 6,7,8$. Note that the specific reduction used here, see appendix 4.A.2, distinguished A_5 from the other A_ρ and $N^{(c)}$. A different embedding would single out a different harmonic function, and hence a different vector, and it is thus convenient to introduce $(\mathcal{A}, \mathcal{A}^{\alpha-})$ to denote the choice of splitting more abstractly. Also as in (4.12) it is convenient to introduce the following combinations of harmonic functions:

$$\begin{aligned} f_5 &= H, \qquad \tilde{f}_1 = 1+K-H^{-1}(\mathcal{A}^2+\mathcal{A}^{\alpha-}\mathcal{A}^{\alpha-}), \\ F_1 &= 1+K, \qquad f_1 = \tilde{f}_1 + H^{-1}\mathcal{A}^2. \end{aligned} \tag{4.47}$$

The vielbein of scalars is given by

$$V = \Omega_4^T \begin{pmatrix} \sqrt{f_1^{-1}\tilde{f}_1} & 0 & 0 & 0 & 0 \\ G\mathcal{A}^2 & \sqrt{\tilde{f}_1^{-1}f_1} & -G\mathcal{A}F_1 & (\sqrt{f_1\tilde{f}_1})^{-1}\mathcal{A} & -G\mathcal{A}\,k^\gamma \\ -F\mathcal{A} & 0 & \sqrt{f_5^{-1}f_1} & 0 & 0 \\ F\mathcal{A} & 0 & -\frac{1}{2}f_5^{-1}F(k^\gamma)^2 & \sqrt{f_5 f_1^{-1}} & -Fk^\gamma \\ 0 & 0 & f_5^{-1}k^\gamma & 0 & I_{22} \end{pmatrix}\Omega_4, \tag{4.48}$$

where to simplify notation quantities (F, G) are defined as

$$F = (f_1 f_5)^{-1/2}, \qquad G = (f_1\tilde{f}_1 f_5^2)^{-1/2}. \tag{4.49}$$

We also define the 22-dimensional vector k^γ as

$$k^\gamma = (0_3, \sqrt{2}\mathcal{A}^{\alpha-}). \tag{4.50}$$

Here $\gamma = 1, \cdots, b^2$ where the second Betti number is $b^2 = 22$ for K3. Using the reduction formulae (4.154) and (4.155), the six-dimensional solution (4.45), (4.48) can be lifted to ten dimensions, resulting in a solution with an analogous form to the T^4 case (4.11). We will thus summarize the solution for both cases in the following section.

(4.4) D1-D5 FUZZBALL SOLUTIONS

In this section we will summarize the D1-D5 fuzzball solutions with internal excitations, for both the $K3$ and T^4 cases. In both cases the solutions can be written as

$$\begin{aligned}
ds^2 &= \frac{f_1^{1/2}}{\tilde{f}_1 f_5^{1/2}}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds^2_{M^4}, \\
e^{2\Phi} &= \frac{f_1^2}{f_5 \tilde{f}_1}, \qquad B^{(2)}_{ty} = \frac{\mathcal{A}}{f_5 \tilde{f}_1}, \qquad B^{(2)}_{\bar{\mu}i} = \frac{\mathcal{A}\mathcal{B}^{\bar{\mu}}_i}{f_5 \tilde{f}_1}, \\
B^{(2)}_{ij} &= \lambda_{ij} + \frac{2\mathcal{A} A_{[i} B_{j]}}{f_5 \tilde{f}_1}, \qquad B^{(2)}_{\rho\sigma} = f_5^{-1} k^\gamma \omega^\gamma_{\rho\sigma}, \qquad C^{(0)} = -f_1^{-1}\mathcal{A}, \\
C^{(2)}_{ty} &= 1 - \tilde{f}_1^{-1}, \qquad C^{(2)}_{\bar{\mu}i} = -\tilde{f}_1^{-1}\mathcal{B}^{\bar{\mu}}_i, \qquad C^{(2)}_{ij} = c_{ij} - 2\tilde{f}_1^{-1} A_{[i}B_{j]}, \\
C^{(4)}_{tyij} &= \lambda_{ij} + \frac{\mathcal{A}}{f_5 \tilde{f}_1}(c_{ij} + 2A_{[i}B_{j]}), \qquad C^{(4)}_{\bar{\mu}ijk} = \frac{3\mathcal{A}}{f_5 \tilde{f}_1}\mathcal{B}^{\bar{\mu}}_{[i} c_{jk]}, \\
C^{(4)}_{ty\rho\sigma} &= f_5^{-1} k^\gamma \omega^\gamma_{\rho\sigma}, \qquad C^{(4)}_{ij\rho\sigma} = (\lambda^\gamma_{ij} + f_5^{-1} k^\gamma c_{ij})\omega^\gamma_{\rho\sigma}, \qquad C^{(4)}_{\rho\sigma\tau\pi} = f_5^{-1}\mathcal{A}\epsilon_{\rho\sigma\tau\pi},
\end{aligned} \tag{4.51}$$

where we introduce a basis of self-dual and anti-self-dual 2-forms $\omega^\gamma \equiv (\omega^{\alpha+}, \omega^{\alpha-})$ with $\gamma = 1, \cdots, b^2$ on the compact manifold M^4. For both T^4 and $K3$ the self-dual forms are labeled by $\alpha_+ = 1, 2, 3$ whilst the anti-self-dual forms are labeled by $\alpha_- = 1, 2, 3$ for T^4 and $\alpha_- = 1, \cdots 19$ for $K3$. The intersections and normalizations of these forms are defined in (4.13), (4.14) and (4.145). The solutions are expressed in terms of the following combinations of harmonic functions $(H, K, A_i, \mathcal{A}, \mathcal{A}^{\alpha-})$

$$\begin{aligned}
f_5 &= H; \qquad \tilde{f}_1 = 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-}\mathcal{A}^{\alpha-}); \qquad f_1 = \tilde{f}_1 + H^{-1}\mathcal{A}^2; \\
k^\gamma &= (0_3, \sqrt{2}\mathcal{A}^{\alpha-}); \qquad dB = - *_4 dA; \qquad dc = - *_4 df_5; \\
d\lambda^\gamma &= *_4 dk^\gamma; \qquad d\lambda = *_4 d\mathcal{A}; \qquad \mathcal{B}^{\bar{\mu}}_i = (-B_i, A_i),
\end{aligned} \tag{4.52}$$

where $\bar{\mu} = (t, y)$ and the Hodge dual $*_4$ is defined over (flat) R^4, with the Hodge dual in the Ricci flat metric on the compact manifold being denoted by $\epsilon_{\rho\sigma\tau\pi}$. The constant term in $C^{(2)}_{ty}$ is chosen so that the potential vanishes at asymptotically flat infinity. The corresponding RR field strengths are

$$\begin{aligned}
F^{(1)}_i &= -\partial_i\left(f_1^{-1}\mathcal{A}\right), \qquad F^{(3)}_{tyi} = (f_1 \tilde{f}_1 f_5^2)^{-1}\left(f_5^2 \partial_i \tilde{f}_1 + f_5 \mathcal{A}\partial_i \mathcal{A} - \mathcal{A}^2 \partial_i f_5\right), \\
F^{(3)}_{\bar{\mu}ij} &= (f_5^2 f_1 \tilde{f}_1)^{-1}\left(2\mathcal{B}^{\bar{\mu}}_{[i}(f_5 \partial_{j]}\tilde{f}_1 + f_5 \mathcal{A}\partial_{j]}\mathcal{A} - \mathcal{A}^2 \partial_{j]} f_5) + 2\tilde{f}_1 f_5^2 \partial_{[i}\mathcal{B}^{\bar{\mu}}_{j]}\right), \\
F^{(3)}_{ijk} &= -\epsilon_{ijkl}(\partial^l f_5 - f_1^{-1}\mathcal{A}\partial^l \mathcal{A}) - 6 f_1^{-1}\partial_{[i}(A_j B_{k]}) \\
&\quad +(f_5^2 f_1 \tilde{f}_1)^{-1}\left(6A_{[i}B_j(f_5 \partial_{k]}\tilde{f}_1 + f_5 \mathcal{A}\partial_{k]}\mathcal{A} - \mathcal{A}^2 \partial_{k]} f_5)\right), \\
F^{(3)}_{i\rho\sigma} &= f_1^{-1}\mathcal{A}\partial_i(f_5^{-1}k^\gamma)\omega^\gamma_{\rho\sigma}, \\
F^{(5)}_{i\rho\sigma\tau\pi} &= \epsilon_{\rho\sigma\tau\pi}\partial_i(f_5^{-1}\mathcal{A}), \qquad F^{(5)}_{tyijk} = \epsilon_{ijkl}\tilde{f}_1^{-1} f_5 \partial^l(f_5^{-1}\mathcal{A}), \\
F^{(5)}_{\bar{\mu}ijkl} &= -\epsilon_{ijkl} f_5 \tilde{f}_1^{-1}\mathcal{B}^{\bar{\mu}}_m \partial^m(f_5^{-1}\mathcal{A}), \\
F^{(5)}_{tyi\rho\sigma} &= \tilde{f}_1^{-1}\partial_i(k^\gamma/f_5)\omega^\gamma_{\rho\sigma}, \qquad F^{(5)}_{\bar{\mu}ij\rho\sigma} = 2\tilde{f}_1^{-1}\mathcal{B}^{\bar{\mu}}_{[i}\partial_{j]}(f_5^{-1}k^\gamma)\omega^\gamma_{\rho\sigma}, \\
F^{(5)}_{ijk\rho\sigma} &= \left(6\tilde{f}_1^{-1} A_{[i}B_j \partial_{k]}(f_5^{-1}k^\gamma) + \epsilon_{ijkl} f_5 \partial^l(f_5^{-1}k^\gamma)\right)\omega^\gamma_{\rho\sigma}.
\end{aligned} \tag{4.53}$$

It has been explicitly checked that this is a solution of the ten-dimensional field equations for any choices of harmonic functions $(H, K, A_i, \mathcal{A}, \mathcal{A}^{\alpha-})$ with $\partial_i A^i = 0$. Note that in the case of $K3$ one needs the identity (4.156) for the harmonic forms to check the components of the Einstein equation along $K3$.

We are interested in solutions for which the defining harmonic functions are given by

$$\begin{aligned} H &= 1 + \frac{Q_5}{L}\int_0^L \frac{dv}{|x - F(v)|^2}; \qquad A_i = -\frac{Q_5}{L}\int_0^L \frac{dv \dot{F}_i(v)}{|x - F(v)|^2}, \\ \mathcal{A} &= -\frac{Q_5}{L}\int_0^L \frac{dv \dot{\mathcal{F}}(v)}{|x - F(v)|^2}; \qquad \mathcal{A}^{\alpha-} = -\frac{Q_5}{L}\int_0^L \frac{dv \dot{\mathcal{F}}^{\alpha-}(v)}{|x - F(v)|^2}, \\ K &= \frac{Q_5}{L}\int_0^L \frac{dv(\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2)}{|x - F(v)|^2}. \end{aligned} \tag{4.54}$$

In these expressions Q_5 is the 5-brane charge and L is the length of the defining curve in the D1-D5 system, given by

$$L = 2\pi Q_5/R, \tag{4.55}$$

where R is the radius of the y circle. Note that Q_5 has dimensions of length squared and is related to the integral charge via

$$Q_5 = \alpha' n_5 \tag{4.56}$$

(where g_s has been set to one). Assuming that the curves $(\dot{\mathcal{F}}(v), \dot{\mathcal{F}}^{\alpha-}(v))$ do not have zero modes, the D1-brane charge Q_1 is given by

$$Q_1 = \frac{Q_5}{L}\int_0^L dv(\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2), \tag{4.57}$$

and the corresponding integral charge is given by

$$Q_1 = \frac{n_1(\alpha')^3}{V}, \tag{4.58}$$

where $(2\pi)^4 V$ is the volume of the compact manifold. The mapping of the parameters from the original F1-P systems to the D1-D5 systems was discussed in [27] and is unchanged here. The fact that the solutions take exactly the same form, regardless of whether the compact manifold is T^4 or $K3$, is unsurprising given that only zero modes of the compact manifold are excited.

The solutions defined in terms of the harmonic functions (4.54) describe the complete set of two-charge fuzzballs for the D1-D5 system on $K3$. In the case of T^4, these describe fuzzballs with only bosonic excitations; the most general solution would include fermionic excitations and thus more general harmonic functions of the type discussed in [39]. Solutions involving harmonic functions with disconnected sources would be appropriate for describing Coulomb branch physics. Note that, whilst the solutions obtained by dualities from supersymmetric F1-P solutions are guaranteed to be supersymmetric, one would need to check supersymmetry explicitly for solutions involving other choices of harmonic functions.

In the final solutions one of the harmonic functions $\mathcal{A}$ describing internal excitations is singled out from the others. In the original F1-P system, the solutions pick out a direction in the internal space. For the type II system on T^4, the choice of A_ρ singles out a direction in the torus whilst in the heterotic solution the choice of $(A_\rho, N^{(c)})$ singles out a direction in the 20d internal space. Both duality chains, however, also distinguish directions in the internal space. In the T^4 case one had to choose a direction in the torus, whilst in the $K3$ case the choice is implicitly made when one uplifts type IIB solutions from six to ten dimensions. In particular, the uplift splits the 21 anti-self-dual six-dimensional 3-forms into $19+1+1$ associated with the ten-dimensional $(F^{(5)}, F^{(3)}, H^{(3)})$ respectively.

When there are no internal excitations, the final solutions must be independent of the choice of direction made in the duality chains but this does not remain true when the original solution breaks the rotational symmetry in the internal space. $\mathcal{A}$ is the component of the original vector along the direction distinguished in the duality chain, whilst $\mathcal{A}^{\alpha-}$ are the components orthogonal to this direction. When there are no excitations along the direction picked out by the duality, i.e. $\mathcal{A}=0$, the solution considerably simplifies, becoming

$$\begin{aligned}
ds^2 &= \frac{1}{(f_1 f_5)^{1/2}}[-(dt - A_i dx^i)^2 + (dy - B_i dx^i)^2] + f_1^{1/2} f_5^{1/2} dx_i dx^i + f_1^{1/2} f_5^{-1/2} ds^2_{M^4}, \\
e^{2\Phi} &= \frac{f_1}{f_5}, \qquad B^{(2)}_{\rho\sigma} = f_5^{-1} k^\gamma \omega^\gamma_{\rho\sigma}, \qquad C^{(2)}_{ty} = 1 - f_1^{-1}, \qquad C^{(2)}_{\bar{\mu} i} = -f_1^{-1} \mathcal{B}^{\bar{\mu}}_i, \\
C^{(2)}_{ij} &= c_{ij} - 2 f_1^{-1} A_{[i} B_{j]}, \qquad C^{(4)}_{ty\rho\sigma} = f_5^{-1} k^\gamma \omega^\gamma_{\rho\sigma}, \qquad C^{(4)}_{ij\rho\sigma} = (\lambda^\gamma_{ij} + f_5^{-1} k^\gamma c_{ij}) \omega^\gamma_{\rho\sigma}.
\end{aligned}$$

In this solution the internal excitations induce fluxes of the NS 3-form and RR 5-form along anti-self dual cycles in the compact manifold (but no net 3-form or 5-form charges). By contrast the excitations parallel to the duality direction induce a field strength for the RR axion, NS 3-form field strength in the non-compact directions and RR 5-form field strength along the compact manifold (but again no net charges).

Let us also comment on the M^4 moduli in our solutions. The solutions are expressed in terms of a Ricci flat metric on M^4 and anti-self dual harmonic two forms. The forms satisfy

$$\omega^\gamma_{\rho\sigma} \omega^{\delta\rho\sigma} = D^\epsilon{}_\delta d_{\gamma\epsilon} \equiv \delta_{\gamma\delta}, \tag{4.59}$$

where the intersection matrix $d_{\delta\gamma}$ and the matrix $D^\gamma{}_\delta$ relating the basis of forms and dual forms are defined in (4.145) and (4.147) respectively. The latter condition on $D^\gamma{}_\epsilon$ arose from the duality chain, and followed from the fact that in the original F1-P solutions the internal manifold had a flat square metric. Thus, the final solutions are expressed at a specific point in the moduli space of M^4 because the original F1-P solutions have specific fixed moduli. It is straightforward to extend the solutions to general moduli: one needs to change

$$\tilde{f}_1 = 1 + K - H^{-1}(\mathcal{A}^2 + \mathcal{A}^{\alpha-}\mathcal{A}^{\alpha-}) \to 1 + K - H^{-1}(\mathcal{A}^2 + \tfrac{1}{2} k^\gamma k^\delta D^\epsilon{}_\delta d_{\gamma\epsilon}), \tag{4.60}$$

with k^γ as defined in (4.52), to obtain the solution for more general $D^\gamma{}_\delta$.

Given a generic fuzzball solution, one would like to check whether the geometry is indeed smooth and horizon-free. For the fuzzballs with no internal excitations this question was discussed in [40], the conclusion being that the solutions are non-singular unless the defining curve $F^i(v)$ is non-generic and self-intersects. In the appendix of [40], the smoothness of fuzzballs with internal excitations was also discussed. However, their D1-D5 solutions were incomplete: only the metric was given, and this was effectively given in the form (4.45) rather than (4.51). Nonetheless, their conclusion remains unchanged: following the same discussion as in [40] one can show that a generic fuzzball solution with internal excitations is non-singular provided that the defining curve $F^i(v)$ does not self-intersect and $\dot{F}_i(v)$ only has isolated zeroes. In particular, if there are no transverse excitations, $F^i(v) = 0$, the solution will be singular as discussed in section 4.6.6.

One can show that there are no horizons as follows. The harmonic function f_5 is clearly positive definite, by its definition. The functions $(f_1, \tilde{f}_1)$ are also positive definite, since they can be rewritten as a sum of positive terms as

$$\begin{aligned} f_5\tilde{f}_1 &= \left(1+\frac{Q_5}{L}\int_0^L \frac{dv}{|x-F|^2}\right)\left(1+\frac{Q_5}{L}\int_0^L \frac{dv\dot{F}^2}{|x-F|^2}\right) \\ &+\frac{Q_5}{L}\int_0^L \frac{dv(\dot{\mathcal{F}}(v))^2+(\dot{\mathcal{F}}^{\alpha-}(v))^2}{|x-F|^2} \\ &+\tfrac{1}{2}(\frac{Q_5}{L})^2\int_0^L\int_0^L dvdv'\frac{(\dot{\mathcal{F}}(v)-\dot{\mathcal{F}}(v'))^2+(\dot{\mathcal{F}}^{\alpha-}(v)-\dot{\mathcal{F}}^{\alpha-}(v'))^2}{|x-F(v)|^2\,|x-F(v')|^2}, \end{aligned} \tag{4.61}$$

and a corresponding expression for $f_5 f_1$. Note that in the decoupling limit only the terms proportional to Q_5^2 remain, and these are also manifestly positive definite. Given that the defining functions have no zeroes anywhere, the geometry therefore has no horizons.

Now let us consider the conserved charges. From the asymptotics one can see that the fuzzball solutions have the same mass and D1-brane, D5-brane charges as the naive solution; the latter are given in (4.56) and (4.58) whilst the ADM mass is

$$M = \frac{\Omega_3 L_y}{\kappa_6^2}(Q_1+Q_5), \tag{4.62}$$

where $L_y = 2\pi R$, $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere, and $2\kappa_6^2 = (2\kappa^2)/(V(2\pi)^4)$ with $2\kappa^2 = (2\pi)^7(\alpha')^4$ in our conventions. The fuzzball solutions have in addition angular momenta, given by

$$J^{ij} = \frac{\Omega_3 L_y}{\kappa_6^2 L}\int_0^L dv(F^i\dot{F}^j - F^j\dot{F}^i). \tag{4.63}$$

These are the only charges; the fields $F^{(1)}$ and $F^{(5)}$ fall off too quickly at infinity for the corresponding charges to be non-zero. One can compute from the harmonic expansions of the fields dipole and more generally multipole moments of the charge distributions. A generic solution breaks completely the $SO(4)$ rotational invariance in R^4, and this symmetry breaking is captured by these multipole moments.

However, the multipole moments computed at asymptotically flat infinity do not have a direct interpretation in the dual field theory. In contrast, the asymptotics of the solutions in the decoupling limit do give field theory information: one-point functions of chiral primaries are expressed in terms of the asymptotic expansions (and hence multipole moments) near the $AdS_3 \times S^3$ boundary. Thus it is more useful to compute in detail the latter, as we shall do in the next section.

(4.5) VEVS FOR THE FUZZBALL SOLUTIONS

Similarly to the analysis in section 3.6 we now take the decoupling limit of the fuzzball solutions and extract the vevs using Kaluza-Klein holography.

For fuzzball solutions on $K3$, the relevant solution of six-dimensional $N = 4b$ supergravity coupled to 21 tensor multiplets was given explicitly in (4.45). For the case of T^4, we obtained the solution in ten dimensions, but there is a corresponding six-dimensional solution of $N = 4b$ supergravity coupled to 5 tensor multiplets. This solution is of exactly the same form as the $K3$ solution given in (4.45), but with the index $\alpha_- = 1, 2, 3$. Thus in what follows we will analyze both cases simultaneously. As mentioned earlier, the T^4 solution reduces to a solution of $d = 6, N = 4b$ supergravity rather than a solution of $d = 6, N = 8$ supergravity because forms associated with the odd cohomology of T^4 (and hence six-dimensional vectors) are not present in our solutions.

(4.5.1) HOLOGRAPHIC RELATIONS FOR VEVS

Consider an $AdS_3 \times S^3$ solution of the six-dimensional field equations (4.31), such that

$$\begin{aligned} ds_6^2 &= \sqrt{Q_1 Q_5}\left(\frac{1}{z^2}(-dt^2 + dy^2 + dz^2) + d\Omega_3^2\right); \\ G^5 &= H^5 \equiv g^{o5} = \sqrt{Q_1 Q_5}(r dr \wedge dt \wedge dy + d\Omega_3), \end{aligned} \tag{4.64}$$

with the vielbein being diagonal and all other three forms (both self-dual and anti-self dual) vanishing. In what follows it is convenient to absorb the curvature radius $\sqrt{Q_1 Q_5}$ into an overall prefactor in the action, and work with the unit radius $AdS_3 \times S^3$. Now express the perturbations of the six-dimensional supergravity fields relative to the $AdS_3 \times S^3$ background as

$$\begin{aligned} g_{MN} &= g^o_{MN} + h_{MN}; \qquad G^A = g^{oA} + g^A; \\ V^n_A &= \delta^n_A + \phi^{nr}\delta^r_A + \tfrac{1}{2}\phi^{nr}\phi^{mr}\delta^m_A; \\ V^r_A &= \delta^r_A + \phi^{nr}\delta^n_A + \tfrac{1}{2}\phi^{nr}\phi^{ns}\delta^s_A. \end{aligned} \tag{4.65}$$

These fluctuations can then be expanded in spherical harmonics as follows:

$$\begin{aligned}
h_{\mu\nu} &= \sum h^I_{\mu\nu}(x) Y^I(y), \\
h_{\mu a} &= \sum (h^{I_v}_{\mu}(x) Y^{I_v}_a(y) + h^I_{(s)\mu}(x) D_a Y^I(y)), \\
h_{(ab)} &= \sum (\rho^{I_t}(x) Y^{I_t}_{(ab)}(y) + \rho^{I_v}_{(v)}(x) D_a Y^{I_v}_b(y) + \rho^I_{(s)}(x) D_{(a} D_{b)} Y^I(y)), \\
h^a_a &= \sum \pi^I(x) Y^I(y), \\
g^A_{\mu\nu\rho} &= \sum 3 D_{[\mu} b^{(A)I}_{\nu\rho]}(x) Y^I(y), \\
g^A_{\mu\nu a} &= \sum (b^{(A)I}_{\mu\nu}(x) D_a Y^I(y) + 2 D_{[\mu} Z^{(A)I_v}_{\nu]}(x) Y^{I_v}_a(y)); \\
g^A_{\mu ab} &= \sum (D_\mu U^{(A)I}(x) \epsilon_{abc} D^c Y^I(y) + 2 Z^{(A)I_v}_{\mu} D_{[b} Y^{I_v}_{a]}); \\
g^A_{abc} &= \sum (-\epsilon_{abc} \Lambda^I U^{(A)I}(x) Y^I(y)); \\
\phi^{mr} &= \sum \phi^{(mr)I}(x) Y^I(y),
\end{aligned} \tag{4.66}$$

Here (μ, ν) are AdS indices and (a, b) are S^3 indices, with x denoting AdS coordinates and y denoting sphere coordinates. The subscript (ab) denotes symmetrization of indices a and b with the trace removed. Relevant properties of the spherical harmonics are reviewed in appendix 4.A.3. We will often use a notation where we replace the index I by the degree of the harmonic k or by a pair of indices (k, I) where k is the degree of the harmonic and I now parametrizes their degeneracy, and similarly for I_v, I_t.

Imposing the de Donder gauge condition $D^A h_{aM} = 0$ on the metric fluctuations removes the fields with subscripts (s, v). In deriving the spectrum and computing correlation functions, this is therefore a convenient choice. The de Donder gauge choice is however not always a convenient choice for the asymptotic expansion of solutions; indeed the natural coordinate choice in our application takes us outside de Donder gauge. As discussed in [22] this issue is straightforwardly dealt with by working with gauge invariant combinations of the fluctuations.

Next let us briefly review the linearized spectrum derived in [55], focusing on fields dual to chiral primaries. Consider first the scalars. It is useful to introduce the following combinations which diagonalize the linearized equations of motion:

$$\begin{aligned}
s^{(r)k}_I &= \frac{1}{4(k+1)} (\phi^{(5r)k}_I + 2(k+2) U^{(r)k}_I), \\
\sigma^k_I &= \frac{1}{12(k+1)} (6(k+2) \hat{U}^{(5)k}_I - \hat{\pi}^k_I),
\end{aligned} \tag{4.67}$$

The fields $s^{(r)k}$ and σ^k correspond to scalar chiral primaries, with the masses of the scalar fields being

$$m^2_{s^{(r)k}} = m^2_{\sigma^k} = k(k-2), \tag{4.68}$$

The index r spans $6 \cdots 5 + n_t$ with $n_t = 5, 21$ respectively for T^4 and $K3$. Note also that $k \geq 1$ for $s^{(r)k}$; $k \geq 2$ for σ^k. The hats $(\hat{U}^{(5)k}_I, \hat{\pi}^k_I)$ denote the following. As discussed in [22], the equations of motion for the gauge invariant fields are precisely the same as those in de Donder gauge, provided one replaces all fields with the corresponding gauge invariant field.

The hat thus denotes the appropriate gauge invariant field, which reduces to the de Donder gauge field when one sets to zero all fields with subscripts (s, v). For our purposes we will need these gauge invariant quantities only to leading order in the fluctuations, with the appropriate combinations being

$$\begin{aligned} \hat{\pi_2}^I &= \pi_2^I + \Lambda^2 \rho_{2(s)}^I; \\ \hat{U}_2^{(5)I} &= U_2^{(5)I} - \tfrac{1}{2}\rho_{2(s)}^I; \\ \hat{h}_{\mu\nu}^0 &= h_{\mu\nu}^0 - \sum_{\alpha,\pm} h_\mu^{1\pm\alpha} h_\nu^{1\pm\alpha}. \end{aligned} \tag{4.69}$$

Next consider the vector fields. It is useful to introduce the following combinations which diagonalize the equations of motion:

$$h_{\mu I_v}^\pm = \tfrac{1}{2}(C_{\mu I_v}^\pm - A_{\mu I_v}^\pm), \qquad Z_{\mu I_v}^{(5)\pm} = \pm\tfrac{1}{4}(C_{\mu I_v}^\pm + A_{\mu I_v}^\pm). \tag{4.70}$$

For general k the equations of motion are Proca-Chern-Simons equations which couple $(A_\mu^\pm, C_\mu^\pm)$ via a first order constraint [55]. The three dynamical fields at each degree k have masses $(k-1, k+1, k+3)$, corresponding to dual operators of dimensions $(k, k+2, k+4)$ respectively; the operators of dimension k are vector chiral primaries. The lowest dimension operators are the R symmetry currents, which couple to the $k = 1$ $A_\mu^{\pm\alpha}$ bulk fields. The latter satisfy the Chern-Simons equation

$$F_{\mu\nu}(A^{\pm\alpha}) = 0, \tag{4.71}$$

where $F_{\mu\nu}(A^{\pm\alpha})$ is the curvature of the connection and the index $\alpha = 1, 2, 3$ is an $SU(2)$ adjoint index. We will here only discuss the vevs of these vector chiral primaries.

Finally there is a tower of KK gravitons with $m^2 = k(k+2)$ but only the massless graviton, dual to the stress energy tensor, will play a role here. Note that it is the combination $\hat{H}_{\mu\nu} = \hat{h}_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^o$ which satisfies the Einstein equation; moreover one needs the appropriate gauge covariant combination $\hat{h}_{\mu\nu}^0$ given in (4.69).

Let us denote by $(\mathcal{O}_{S_I^{(r)k}}, \mathcal{O}_{\Sigma_I^k})$ the chiral primary operators dual to the fields $(s_I^{(r)k}, \sigma_I^k)$ respectively. The vevs of the scalar operators with dimension two or less can then be expressed in terms of the coefficients in the asymptotic expansion as

$$\begin{aligned} \left\langle \mathcal{O}_{S_i^{(r)1}} \right\rangle &= \frac{2N}{\pi}\sqrt{2}[s_i^{(r)1}]_1; \qquad \left\langle \mathcal{O}_{S_I^{(r)2}} \right\rangle = \frac{2N}{\pi}\sqrt{6}[s_I^{(r)2}]_2; \\ \left\langle \mathcal{O}_{\Sigma_I^2} \right\rangle &= \frac{N}{\pi}\left(2\sqrt{2}[\sigma_I^2]_2 - \frac{1}{3}\sqrt{2}a_{Iij}\sum_r [s_i^{(r)1}]_1[s_j^{(r)1}]_1\right). \end{aligned} \tag{4.72}$$

Here $[\psi]_n$ denotes the coefficient of the z^n term in the asymptotic expansion of the field ψ. The coefficient a_{Iij} refers to the triple overlap between spherical harmonics, defined in (4.167). Note that dimension one scalar spherical harmonics have degeneracy four, and are thus labeled by $i = 1, \cdots 4$.

Now consider the stress energy tensor and the R symmetry currents. The three dimensional metric and the Chern-Simons gauge fields admit the following asymptotic expansions

$$\begin{aligned} ds_3^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2}\left(g_{(0)\bar{\mu}\bar{\nu}} + z^2\left(g_{(2)\bar{\mu}\bar{\nu}} + \log(z^2)h_{(2)\bar{\mu}\bar{\nu}} + (\log(z^2))^2\tilde{h}_{(2)\bar{\mu}\bar{\nu}}\right) + \cdots\right) dx^{\bar{\mu}}dx^{\bar{\nu}}; \\ A^{\pm\alpha} &= \mathcal{A}^{\pm\alpha} + z^2 A^{\pm\alpha}_{(2)} + \cdots \end{aligned} \quad (4.73)$$

The vevs of the R symmetry currents $J_u^{\pm\alpha}$ are then given in terms of terms in the asymptotic expansion of $A_\mu^{\pm\alpha}$ as

$$\langle J_{\bar{\mu}}^{\pm\alpha}\rangle = \frac{N}{4\pi}\left(g_{(0)\bar{\mu}\bar{\nu}} \pm \epsilon_{\bar{\mu}\bar{\nu}}\right)\mathcal{A}^{\pm\alpha\bar{\nu}}. \quad (4.74)$$

The vev of the stress energy tensor $T_{\bar{\mu}\bar{\nu}}$ is given by

$$\langle T_{\bar{\mu}\bar{\nu}}\rangle = \frac{N}{2\pi}\left(g_{(2)\bar{\mu}\bar{\nu}} + \tfrac{1}{2}Rg_{(0)\bar{\mu}\bar{\nu}} + 8\sum_r [\tilde{s}_i^{(r)1}]_1^2 g_{(0)\bar{\mu}\bar{\nu}} + \tfrac{1}{4}(\mathcal{A}^{+\alpha}_{(\bar{\mu}}\mathcal{A}^{+\alpha}_{\bar{\nu})} + \mathcal{A}^{-\alpha}_{(\bar{\mu}}\mathcal{A}^{-\alpha}_{\bar{\nu})})\right) \quad (4.75)$$

where parentheses denote the symmetrized traceless combination of indices.

This summarizes the expressions for the vevs of chiral primaries with dimension two or less which were derived in chapter 3. Note that these operators correspond to supergravity fields which are at the bottom of each Kaluza-Klein tower. The supergravity solution of course also captures the vevs of operators dual to the other fields in each tower. Expressions for these vevs were not derived in chapter 3, the obstruction being the non-linear terms: in general the vev of a dimension p operator will include contributions from terms involving up to p supergravity fields. Computing these in turn requires the field equations (along with gauge invariant combinations, KK reduction maps etc) up to pth order in the fluctuations.

Now (apart from the stress energy tensor) none of the operators whose vevs are given above is an $SO(4)$ (R symmetry) singlet. For later purposes it will be useful to review which other operators are $SO(4)$ singlets. The computation of the linearized spectrum in [55] picks out the following as $SO(4)$ singlets:

$$\tau^0 \equiv \frac{1}{12}\pi^0; \qquad t^{(r)0} \equiv \tfrac{1}{4}\phi^{5(r)0}, \quad (4.76)$$

along with $\phi^{0i(r)}$ with $i = 1, \cdots, 4$. Recall ψ^0 denotes the projection of the field ψ onto the degree zero harmonic. The fields $(\tau^0, t^{(r)0})$ are dual to operators of dimension four, whilst the fields $\phi^{0i(r)}$ are dual to dimension two (marginal) operators. The former lie in the same tower as $(\sigma^2, s^{(r)2})$ respectively, whilst the latter are in the same tower as $s^{(r)1}$. In total there are $(n_t + 1)$ $SO(4)$ singlet irrelevant operators and $4n_t$ $SO(4)$ singlet marginal operators, where $n_t = 5, 21$ for T^4 and $K3$ respectively.

Consider the $SO(4)$ singlet marginal operators dual to the supergravity fields $\phi^{i(r)}$. These operators have been discussed previously in the context of marginal deformations of the CFT, see the review [67] and references therein. Suppose one introduces a free field realization for the T^4 theory, with bosonic and fermionic fields $(x^i_I(z), \psi^i_I(z))$ where $I = 1, \cdots, N$. Then some of the marginal operators can be explicitly realized in the untwisted sector as bosonic bilinears

$$\partial x^i_I(z) \bar{\partial} x^j_I(\bar{z}); \tag{4.77}$$

there are sixteen such operators, in correspondence with sixteen of the supergravity fields. The remaining four marginal operators are realized in the twisted sector, and are associated with deformation from the orbifold point.

(4.5.2) Application to the fuzzball solutions

The six-dimensional metric of (4.45) in the decoupling limit manifestly asymptotes to

$$ds^2 = \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \sqrt{Q_1 Q_5}\left(\frac{dr^2}{r^2} + d\Omega_3^2\right). \tag{4.78}$$

where

$$Q_1 = \frac{Q_5}{L}\int_0^L dv(\dot{F}(v)^2 + \dot{\mathcal{F}}(v)^2 + \dot{\mathcal{F}}^{\alpha-}(v)^2). \tag{4.79}$$

Note that the vielbein (4.48) is asymptotically constant

$$V^o = \Omega_4^T \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & \sqrt{Q_1/Q_5} & 0 & 0 \\ 0 & 0 & \sqrt{Q_5/Q_1} & 0 \\ 0 & 0 & 0 & I_{22} \end{pmatrix} \Omega_4, \tag{4.80}$$

but it does not asymptote to the identity matrix. Thus one needs the constant $SO(5,21)$ transformation

$$V \to V(V^o)^{-1}, \qquad G_3 \to V^o G_3. \tag{4.81}$$

to bring the background into the form assumed in (4.64).

The fields are expanded about the background values, by expanding the harmonic functions defining the solution in spherical harmonics as

$$\begin{aligned} H &= \frac{Q_5}{r^2}\sum_{k,I}\frac{f^5_{kI} Y^I_k(\theta_3)}{r^k}, \qquad K = \frac{Q_1}{r^2}\sum_{k,I}\frac{f^1_{kI} Y^I_k(\theta_3)}{r^k}, \\ A_i &= \frac{Q_5}{r^2}\sum_{k\geq 1,I}\frac{(A_{kI})_i Y^I_k(\theta_3)}{r^k}, \qquad \mathcal{A} = \frac{\sqrt{Q_1 Q_5}}{r^2}\sum_{k\geq 1,I}\frac{(\mathcal{A}_{kI}) Y^I_k(\theta_3)}{r^k}, \\ \mathcal{A}^{\alpha-} &= \frac{\sqrt{Q_1 Q_5}}{r^2}\sum_{k\geq 1,I}\frac{\mathcal{A}^{\alpha-}_{kI} Y^I_k(\theta_3)}{r^k}. \end{aligned} \tag{4.82}$$

The polar coordinates here are denoted by (r, θ_3) and $Y^I_k(\theta_3)$ are (normalized) spherical harmonics of degree k on S^3 with I labeling the degeneracy. Note that the restriction $k \geq 1$ in the last three lines is due to the vanishing zero mode, see section 3.4.1. As in section 3.4.1, the coefficients in the expansion can be expressed as

$$\begin{aligned} f^5_{kI} &= \frac{1}{L(k+1)}\int_0^L dv(C^I_{i_1\cdots i_k}F^{i_1}\cdots F^{i_k}), \\ f^1_{kI} &= \frac{Q_5}{L(k+1)Q_1}\int_0^L dv\left(\dot F^2+\dot{\mathcal{F}}^2+(\dot{\mathcal{F}}^{\alpha-})^2\right)C^I_{i_1\cdots i_k}F^{i_1}\cdots F^{i_k}, \\ (A_{kI})_i &= -\frac{1}{L(k+1)}\int_0^L dv\dot F_i C^I_{i_1\cdots i_k}F^{i_1}\cdots F^{i_k}, \\ (\mathcal{A}_{kI}) &= -\frac{\sqrt{Q_5}}{\sqrt{Q_1}L(k+1)}\int_0^L dv\dot{\mathcal{F}}C^I_{i_1\cdots i_k}F^{i_1}\cdots F^{i_k}, \\ \mathcal{A}^{\alpha-}_{kI} &= -\frac{\sqrt{Q_5}}{\sqrt{Q_1}L(k+1)}\int_0^L dv\dot{\mathcal{F}}^{\alpha-}C^I_{i_1\cdots i_k}F^{i_1}\cdots F^{i_k}. \end{aligned} \tag{4.83}$$

Here the $C^I_{i_1\cdots i_k}$ are orthogonal symmetric traceless rank k tensors on $\mathbb{R}^4$ which are in one-to-one correspondence with the (normalized) spherical harmonics $Y^I_k(\theta_3)$ of degree k on S^3. Fixing the center of mass of the whole system implies that

$$(f^1_{1i}+f^5_{1i})=0. \tag{4.84}$$

The leading term in the asymptotic expansion of the transverse gauge field A_i can be written in terms of degree one vector harmonics as

$$A=\frac{Q_5}{r^2}(A_{1j})_iY^j_1dY^i_1\equiv\frac{\sqrt{Q_1Q_5}}{r^2}(a^{\alpha-}Y^{\alpha-}_1+a^{\alpha+}Y^{\alpha+}_1), \tag{4.85}$$

where $(Y^{\alpha-}_1,Y^{\alpha+}_1)$ with $\alpha=1,2,3$ form a basis for the $k=1$ vector harmonics and we have defined

$$a^{\alpha\pm}=\frac{\sqrt{Q_5}}{\sqrt{Q_1}}\sum_{i>j}e^{\pm}_{\alpha ij}(A_{1j})_i, \tag{4.86}$$

where the spherical harmonic triple overlap $e^{\pm}_{\alpha ij}$ is defined in 4.168. The dual field is given by

$$B=-\frac{\sqrt{Q_1Q_5}}{r^2}(a^{\alpha-}Y^{\alpha-}_1-a^{\alpha+}Y^{\alpha+}_1). \tag{4.87}$$

Now given these asymptotic expansions of the harmonic functions one can proceed to expand all the supergravity fields, and extract the appropriate combinations required for computing the vevs defined in (4.72), (4.74) and (4.75). Since the details of the computation are very similar to those in chapter 3, we will simply summarize the results as follows. Firstly the vevs of the stress energy tensor and of the R symmetry currents are the same as in section 3.6, namely

$$\langle T_{\bar\mu\bar\nu}\rangle = 0; \tag{4.88}$$

$$\left\langle J^{\pm\alpha}\right\rangle = \pm\frac{N}{2\pi}a^{\alpha\pm}(dy\pm dt). \tag{4.89}$$

The vanishing of the stress energy tensor is as anticipated, since these solutions should be dual to R vacua. Again, the cancellation is very non-trivial. The vevs of the scalar operators dual to the fields $(s_I^{(6)k}, \sigma_I^k)$ are also unchanged from section 3.6:

$$\begin{aligned}
\left\langle \mathcal{O}_{S_i^{(6)1}} \right\rangle &= \frac{N}{4\pi}(-4\sqrt{2}f_{1i}^5); \\
\left\langle \mathcal{O}_{S_I^{(6)2}} \right\rangle &= \frac{N}{4\pi}(\sqrt{6}(f_{2I}^1 - f_{2I}^5)); \\
\left\langle \mathcal{O}_{\Sigma_I^2} \right\rangle &= \frac{N}{4\pi}\sqrt{2}(-(f_{2I}^1 + f_{2I}^5) + 8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}).
\end{aligned} \tag{4.90}$$

The internal excitations of the new fuzzball solutions are therefore captured by the vevs of operators dual to the fields $s_I^{(r)k}$ with $r > 6$:

$$\begin{aligned}
\left\langle \mathcal{O}_{S_i^{(5+n_t)1}} \right\rangle &= -\frac{N}{\pi}\sqrt{2}(\mathcal{A}_{1i}); \quad \left\langle \mathcal{O}_{S_i^{(6+\alpha_-)1}} \right\rangle = \frac{N}{\pi}\sqrt{2}\mathcal{A}_{1i}^{\alpha-}; \\
\left\langle \mathcal{O}_{S_I^{(5+n_t)2}} \right\rangle &= -\frac{N}{2\pi}\sqrt{6}(\mathcal{A}_{2I}); \quad \left\langle \mathcal{O}_{S_I^{(6+\alpha_-)2}} \right\rangle = \frac{N}{2\pi}\sqrt{6}\mathcal{A}_{2I}^{\alpha-}.
\end{aligned} \tag{4.91}$$

Here $n_t = 5, 21$ for T^4 and $K3$ respectively, with $\alpha_- = 1, \cdots, b^{2-}$ with $b^{2-} = 3, 19$ respectively. Thus each curve $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$ induces corresponding vevs of operators associated with the middle cohomology of M^4. Note the sign difference for the vevs of operators which are related to the distinguished harmonic function $\mathcal{F}(v)$.

(4.6) PROPERTIES OF FUZZBALL SOLUTIONS

In this section we will discuss various properties of the fuzzball solutions, including the interpretation of the vevs computed in the previous section.

(4.6.1) DUAL FIELD THEORY

Let us start by briefly reviewing aspects of the dual CFT and the ground states of the R sector; a more detailed review of the issues relevant here is contained in chapter 3. Consider the dual CFT at the orbifold point; there is a family of chiral primaries in the NS sector associated with the cohomology of the internal manifold, T^4 or $K3$. For our discussions only the chiral primaries associated with the even cohomology are relevant; let these be labeled as $\mathcal{O}_n^{(p,q)}$ where n is the twist and (p, q) labels the associated cohomology class. The degeneracy of the operators associated with the $(1, 1)$ cohomology is $h^{1,1}$. The complete set of chiral primaries associated with the even cohomology is then built from products of the form

$$\prod_l (\mathcal{O}_{n_l}^{p_l,q_l})^{m_l}, \qquad \sum_l n_l m_l = N, \tag{4.92}$$

where symmetrization over the N copies of the CFT is implicit. The correspondence between (scalar) supergravity fields and chiral primaries is [2]

$$\begin{array}{llll} \sigma_n & \leftrightarrow & \mathcal{O}^{(2,2)}_{(n-1)}, & n \geq 2; \\ s^{(6)}_n & \leftrightarrow & \mathcal{O}^{(0,0)}_{(n+1)}, & s^{(6+\tilde{\alpha})}_n \leftrightarrow \mathcal{O}^{(1,1)}_{(n)\tilde{\alpha}}, \qquad \tilde{\alpha} = 1, \cdots h^{1,1}, \qquad n \geq 1. \end{array} \tag{4.93}$$

Spectral flow maps these chiral primaries in the NS sector to R ground states, where

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24}; \\ j_3^R &= j_3^{NS} - \frac{c}{12}, \end{aligned} \tag{4.94}$$

where c is the central charge. Each of the operators in (4.92) is mapped by spectral flow to a (ground state) operator of definite R-charge

$$\begin{aligned} \prod_{l=1} (\mathcal{O}^{(p_l,q_l)}_{n_l})^{m_l} \quad &\rightarrow \quad \prod_{l=1} (\mathcal{O}^{R(p_l,q_l)}_{n_l})^{m_l}, \\ j_3^R \; = \; \tfrac{1}{2}\sum_l (p_l - 1) m_l, \quad & \quad \bar{j}_3^R = \tfrac{1}{2}\sum_l (q_l - 1) m_l. \end{aligned} \tag{4.95}$$

Note that R operators which are obtained from spectral flow of those associated with the $(1,1)$ cohomology have zero R charge.

(4.6.2) Correspondence between geometries and ground states

In chapter 3 we discussed the correspondence between fuzzball geometries characterized by a curve $F^i(v)$ and R ground states (4.95) with $(p_l, q_l) = 1 \pm 1$. The latter are related to chiral primaries in the NS sector built from the cohomology common to both T^4 and $K3$, namely the $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$ cohomology.

The following proposal was made for the precise correspondence between geometries and ground states; see also [44]. Given a curve $F^i(v)$ we construct the corresponding coherent state in the FP system and then find which Fock states in this coherent state have excitation number N_L equal to nw, where n is the momentum and w is the winding. Applying a map between FP oscillators and R operators then yields the superposition of R ground states that is proposed to be dual to the D1-D5 geometry.

This proposal can be straightforwardly extended to the new geometries, which are characterized by the curve $F^i(v)$ along with $h^{1,1}$ additional functions $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$. Consider first the T^4 system, for which the four additional functions are $F^\rho(v)$. Then the eight functions $F^I(v) \equiv (F^i(v), F^\rho(v))$ can be expanded in harmonics as

$$F^I(v) = \sum_{n>0} \frac{1}{\sqrt{n}} (\alpha_n^I e^{-in\sigma^+} + (\alpha_n^I)^* e^{in\sigma^+}), \tag{4.96}$$

[2]As discussed in chapter 3, the dictionary between $(\sigma_n, s_n^{(6)})$ and $(\mathcal{O}^{(2,2)}_{(n-1)}, \mathcal{O}^{(0,0)}_{(n+1)})$ may be more complicated, since their quantum numbers are indistinguishable, but this subtlety will not play a role here.

where $\sigma^+ = v/wR_9$. The corresponding coherent state in the FP system is

$$\left|F^I\right) = \prod_{n,I} \left|\alpha_n^I\right) , \tag{4.97}$$

where $\left|\alpha_n^I\right)$ is a coherent state of the left moving oscillator $\hat{a}_n^I$, satisfying $\hat{a}_n^I \left|\alpha_n^I\right) = \alpha_n^I \left|\alpha_n^I\right)$. Contained in this coherent state are Fock states, such that

$$\prod (\hat{a}_{n_I}^I)^{m_I} |0\rangle , \qquad N = \sum n_I m_I . \tag{4.98}$$

Now retain only the terms in the coherent state involving these Fock states, and map the FP oscillators to CFT R operators via the dictionary

$$\begin{aligned} \frac{1}{\sqrt{2}}(\hat{a}_n^1 \pm i\hat{a}_n^2) \quad &\leftrightarrow \quad \mathcal{O}_n^{R(\pm 1+1),(\pm 1+1)}; \\ \frac{1}{\sqrt{2}}(\hat{a}_n^3 \pm i\hat{a}_n^4) \quad &\leftrightarrow \quad \mathcal{O}_n^{R(\pm 1+1),(\mp 1+1)}; \\ \hat{a}_n^\rho \quad &\leftrightarrow \quad \mathcal{O}_{(\rho-4)n}^{R(1,1)}. \end{aligned} \tag{4.99}$$

The dictionary for the case of $K3$ is analogous. Here one has four curves $F^i(v)$ describing the transverse oscillations and twenty curves $\mathcal{F}^{\tilde{\alpha}}(v)$ describing the internal excitations. The oscillators associated with the former are mapped to operators associated with the universal cohomology as in (4.99) whilst the oscillators associated with the latter are mapped to operators associated with the $(1,1)$ cohomology as

$$\hat{a}_n^{\tilde{\alpha}} \leftrightarrow \mathcal{O}_{\tilde{\alpha}n}^{R(1,1)}. \tag{4.100}$$

This completely defines the proposed superposition of R ground states to which a given geometry corresponds. Note that below we will suggest that a slight refinement of this dictionary may be necessary, taking into account that one of the internal curves is distinguished by the duality chain. For the distinguished curve the mapping may include a negative sign, namely $\hat{a}_n \leftrightarrow -\mathcal{O}_n^{R(1,1)}$; this mapping would explain the relative sign between the vevs found in (4.91) associated with the distinguished curve $\mathcal{F}$ and the remaining curves $\mathcal{F}^\alpha$ respectively.

Note that there is a direct correspondence between the frequency of the harmonic on the curve and the twist label of the CFT operator. The latter is strictly positive, $n \geq 1$, and thus in the dictionary (4.99) there are no candidate CFT operators to correspond to winding modes of the curves $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$. In the case of T^4 such candidates might be provided by the additional chiral primaries associated with the extra T^4 in the target space of the sigma model, discussed in [82]. However the latter is related to the degeneracy of the right-moving ground states in the dual F1-P system, rather than to winding modes. For $K3$ all chiral primaries have been included (except for the additional primaries which appear at specific points in the $K3$ moduli space). Thus one confirms that winding modes of the curves $(\mathcal{F}(v), \mathcal{F}^{\alpha-}(v))$ should not be included in constructing geometries dual to the R ground states. As discussed in appendix 4.A.4 these winding modes may describe geometric duals of states in deformations of the CFT.

(4.6.3) Matching with the holographic vevs

In this section we will see how the general structure of the vevs given in (4.91) can be reproduced using the proposed dictionary. The holographic vevs take the form

$$\left\langle \mathcal{O}^{(1,1)}_{\tilde{\alpha}kI} \right\rangle \approx \frac{N\sqrt{Q_5}}{\sqrt{Q_1}L} \int_0^L dv \dot{\mathcal{F}}^{\tilde{\alpha}} C^I_{i_1\cdots i_k} F^{i_1} \cdots F^{i_k}. \tag{4.101}$$

Thus the vevs of the operators $\mathcal{O}^{(1,1)}_{\tilde{\alpha}kI}$ are zero unless the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-vanishing and at least one of the $F^i(v)$ is non-vanishing. Moreover, the dimension one operators will not acquire a vev unless the transverse and internal curves have excitations with the same frequency. Analogous selection rules for frequencies of curve harmonics apply for the vevs of higher dimension operators.

These properties of the vevs follow directly from the proposed superpositions, along with selection rules for three point functions of chiral primaries. The superposition dual to a given set of curves is built from the R ground states

$$\mathcal{O}^{R\mathcal{I}}|0\rangle = \prod_l (\mathcal{O}^{R(p_l,q_q)}_{n_l})^{m_l}|0\rangle, \tag{4.102}$$

with $\sum_l n_l m_l = N$ and $\mathcal{I}$ labeling the degeneracy of the ground states. So this superposition can be denoted abstractly as $|\Psi) = \sum_{\mathcal{I}} a_{\mathcal{I}} \mathcal{O}^{R\mathcal{I}}|0\rangle$ with certain coefficients $a_{\mathcal{I}}$. In particular, if the curve $\mathcal{F}^{\tilde{\alpha}}(v) = 0$ the superposition does not contain any R ground states built from $\mathcal{O}^{R(1,1)}_{\tilde{\alpha}n}$ operators. Moreover, if there are no transverse excitations, the superposition will contain only states with zero R charge.

Now consider evaluating the vev of a dimension k operator $\mathcal{O}^{(1,1)}_{\tilde{\alpha}k}$ in such a superposition. This is determined by three point functions between this operator and the chiral primary operators occurring in the superposition. More explicitly, the operator vev is related to three point functions via

$$(\Psi_{NS}|\mathcal{O}^{(1,1)}_{\tilde{\alpha}k}|\Psi_{NS}) = \sum_{\mathcal{I},\mathcal{J}} a^*_{\mathcal{I}} a_{\mathcal{J}} \langle (\mathcal{O}^{\mathcal{I}})^\dagger(\infty) \mathcal{O}^{(1,1)}_{\tilde{\alpha}k}(\mu)(\mathcal{O}^{\mathcal{J}})(0)\rangle. \tag{4.103}$$

Here $\mathcal{O}^{\mathcal{I}}$ is the NS sector operator which flows to $\mathcal{O}^{R\mathcal{I}}$ in the R sector and $|\Psi_{NS})$ is the flow of the superposition back to the NS sector, namely $\sum_{\mathcal{I}} a_{\mathcal{I}} \mathcal{O}^{\mathcal{I}}|0\rangle$. The quantity μ is a mass scale. Note we are evaluating the relevant three point function in the NS sector, and have hence flowed the ground states back to NS sector chiral primaries. We would get the same answer by flowing the operator whose vev we wish to compute, $\mathcal{O}^{(1,1)}_{\tilde{\alpha}k}$, into the Ramond sector and computing the three point function there. Recall that the R charges of these operators are related by the spectral flow formula (4.94) as $j_3^{NS} = j_3^R + \frac{1}{2}N$. In particular, NS sector chiral primaries built only from operators associated with the middle cohomology all have the same R charges, namely $\frac{1}{2}N$.

There are two basic selection rules for the three point functions (4.103). Firstly, as usual one has to impose conservation of the R charges. Secondly, a basic property of such three point functions is that they are only non-zero when the total number of operators $\mathcal{O}^{(1,1)}_{\tilde{\alpha}}$ with a given index $\tilde{\alpha}$ in the correlation function is even [3]. From a supergravity perspective one can see this selection rule arising as follows. One computes n-point correlation functions using n-point couplings in the three dimensional supergravity action, with the latter following from the reduction of the ten-dimensional action on $S^3 \times M^4$. Since a $(1,1)$ form integrates to zero over M^4, the three dimensional action only contains terms with an even number of fields $s^{\tilde{\alpha}}$ associated with a given $(1,1)$ cycle $\tilde{\alpha}$ on M^4. Therefore non-zero n-point functions must contain an even number of operators $\mathcal{O}^{(1,1)}_{\tilde{\alpha}}$, and so do corresponding multi-particle 3-point functions obtained by taking coincident limits.

Expressed in terms of cohomology, allowed three point functions contain an even number of $(1,1)_{\tilde{\alpha}}$ cycles labeled by $\tilde{\alpha}$. Thus in single particle correlators one can have processes such as $\mathcal{O}^{(0,0)} + \mathcal{O}^{(1,1)}_{\tilde{\alpha}} \rightarrow \mathcal{O}^{(1,1)}_{\tilde{\alpha}}$ and $\mathcal{O}^{(1,1)}_{\tilde{\alpha}} + \mathcal{O}^{(1,1)}_{\tilde{\alpha}} \rightarrow \mathcal{O}^{(2,2)}$, but processes such as $\mathcal{O}^{(0,0)} + \mathcal{O}^{(1,1)}_{\tilde{\alpha}} \rightarrow \mathcal{O}^{(0,0)}$ which involve an odd number of $\tilde{\alpha}$ cycles are kinematically forbidden. This kinematical selection rule for $(1,1)$ cycle conservation immediately explains why the operator $\mathcal{O}^{(1,1)}_{\tilde{\alpha}k}$ can only acquire a vev when the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-vanishing: only then does the ground state superposition contain operators $\mathcal{O}^{R(1,1)}_{\tilde{\alpha}}$ such that the selection rule can be satisfied.

One can also easily see why the operator only acquires a vev if there are transverse excitations as well. All Ramond ground states associated with the middle cohomology have zero R charge, with the corresponding chiral primaries in the NS sector having the same charge $j_3^{NS} = \frac{1}{2}N$. Thus a superposition involving only $\mathcal{O}^{(1,1)}$ operators has a definite R charge, and a charged operator cannot acquire a vev. Including transverse excitations means that the superposition of Ramond ground states contains charged operators, associated with the universal cohomology, and does not have definite R charge. Therefore a charged operator can acquire a vev.

Thus, to summarize, the proposed map between curves and superpositions of R ground states indeed reproduces the principal features of the holographic vevs. Using basic selection rules for three point functions we have explained why the operators $\mathcal{O}^{(1,1)}_{\tilde{\alpha}k}$ acquire vevs only when the curve $\mathcal{F}^{\tilde{\alpha}}(v)$ is non-zero and when there are excitations in R^4. We will see below that using reasonable assumptions for the three point functions we can also reproduce the selection rules for vevs relating to frequencies on the curves. Before discussing the general case, however, it will be instructive to consider a particular example.

(4.6.4) A SIMPLE EXAMPLE

Consider a fuzzball geometry characterized by a circular curve in the transverse R^4 and one additional internal curve, with only one harmonic of the same frequency:

$$F^1(v) = \frac{\mu A}{n}\cos(2\pi n\frac{v}{L}); \qquad F^2(v) = \frac{\mu A}{n}\sin(2\pi n\frac{v}{L}); \qquad \mathcal{F}(v) = \frac{\mu B}{n}\cos(2\pi n\frac{v}{L}), \quad (4.104)$$

[3] Note that this selection rule was used for the computation of three point functions of single particle operators in the orbifold CFT in [62].

where $\mu = \sqrt{Q_1 Q_5}/R$ and the D1-brane charge constraint (4.79) enforces

$$(A^2 + \tfrac{1}{2}B^2) = 1. \tag{4.105}$$

The corresponding dual superposition of R ground states is then given by

$$\begin{aligned} |\Psi) &= \sum_{l=0}^{N/n} C_l (\mathcal{O}_n^{R(2,2)})^l (\mathcal{O}_{1n}^{R(1,1)})^{\frac{N}{n}-l} |0\rangle , \\ C_l &= \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-l)!l!}} A^l (\frac{B}{\sqrt{2}})^{\frac{N}{n}-l}, \end{aligned} \tag{4.106}$$

with the operators being orthonormal in the large N limit. In the case that either A or B are zero the superposition manifestly collapses to a single term. In the general case, this superposition gives the following for the expectation values of the R charges:

$$\begin{aligned} \Big(\Psi|j_3^R|\Psi\Big) &= \Big(\Psi|\bar{j}_3^R|\Psi\Big) = \tfrac{1}{2}\sum_{l=0}^{N/n} C_l^2 l; \\ &= \frac{N}{2n}\sum_{l=0}^{N/n-1} \frac{(\frac{N}{n}-1)!}{l!(\frac{N}{n}-(l+1))!} A^{2(l+1)} (\frac{B}{\sqrt{2}})^{2(\frac{N}{n}-(l+1))} = \frac{N}{2n}A^2. \end{aligned} \tag{4.107}$$

Evaluating (4.89) for (4.104) gives

$$\left\langle J^{\pm 3}\right\rangle = \frac{N}{2nR}A^2(dy \pm dt), \tag{4.108}$$

and thus the integrated R charges defined in our conventions as

$$\langle j_3 \rangle = \frac{1}{2\pi}\int dy \left\langle J^{+3}\right\rangle ; \qquad \langle \bar{j}_3 \rangle = \frac{1}{2\pi}\int dy \left\langle J^{-3}\right\rangle , \tag{4.109}$$

agree with those of the superposition of R ground states.

The kinematical properties also match between the geometry and the proposed superposition. In particular, when $B \neq 0$ the $SO(2)$ symmetry in the 1-2 plane is broken: the harmonic functions $(K, \mathcal{A})$ depend explicitly on the angle ϕ in this plane. The asymptotic expansions of these functions involve charged harmonics, and therefore charged operators acquire vevs characterizing the symmetry breaking. More explicitly, the relevant terms in (4.83) are

$$\begin{aligned} f_{kI}^1 &\propto \int_0^L dv (A^2 + B^2 \sin^2(\frac{2\pi n v}{L})) C^I_{i_1\cdots i_k} F^{i_1}\cdots F^{i_k}; \\ \mathcal{A}_{kI} &\propto \int_0^L dv B \sin(\frac{2\pi n v}{L})) C^I_{i_1\cdots i_k} F^{i_1}\cdots F^{i_k}. \end{aligned} \tag{4.110}$$

Now the symmetric tensor of rank k and $SO(2)$ charge in the 1-2 plane of $\pm m$ behaves as

$$((F^1)^2 + (F^2)^2)^{k-m}(F^1 \pm iF^2)^m = (\frac{\mu A}{n})^k e^{\pm 2\pi i n m \frac{v}{L}}. \tag{4.111}$$

Note that m is related to $(j_3, \bar{j}_3)$ via $m = j_3 + \bar{j}_3$. Thus, when $B \neq 0$, harmonics in the expansion of f^1 with charges $|m| = 2$ are excited, and terms with $|m| = 1$ are excited in the expansion of $\mathcal{A}$. Following (4.101) the latter implies that the dimension k operators $\mathcal{O}^{(1,1)}_{1(km)}$ only acquire vevs when their $SO(2)$ charge m in the 1-2 plane is ± 1. In particular using (4.91) the vevs of the dimension one operators are

$$\langle \mathcal{O}^{(1,1)}_{1(1\pm 1)} \rangle = \mp i \frac{N}{2\pi n} \mu AB, \tag{4.112}$$

where the normalized degree one symmetric traceless tensors are $\sqrt{2}(F^1 \pm iF^2)$.

These properties are implied by the superposition (4.106). The latter is a superposition of states with different R charge, and therefore it does break the $SO(2)$ symmetry, with the symmetry breaking being characterized by the vevs of charged operators. Moreover following (4.103) the vev of $\mathcal{O}^{(1,1)}_{1(km)}$ is given by

$$\sum_{l,l'} C^*_l C_{l'} \langle (\mathcal{O}^{(2,2)}_n)^l (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}-l} | \mathcal{O}^{(1,1)}_{1(km)}(\mu) | (\mathcal{O}^{(2,2)}_n)^{l'} (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}-l'} \rangle. \tag{4.113}$$

For the dimension one operators, charge conservation reduces this to

$$\sum_{l} C^*_{l\pm 1} C_l \langle (\mathcal{O}^{(2,2)}_n)^{l\pm 1} (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}\mp 1-l} | \mathcal{O}^{(1,1)}_{1(1\pm 1)}(\mu) | (\mathcal{O}^{(2,2)}_n)^{l} (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}-l} \rangle. \tag{4.114}$$

Thus there are contributions only from neighboring terms in the superposition. Computing the actual values of these vevs is beyond current technology: one would need to know three point functions for single and multiple particle chiral primaries at the conformal point. However, as in chapter 3, the behavior of the vevs as functions of the curve radii (A, B) can be captured by remarkably simple approximations for the correlators, motivated by harmonic oscillators. Suppose one treats the operators as harmonic oscillators, with the operator $\mathcal{O}^{(1,1)}_{1(11)}$ destroying one $\mathcal{O}^{(1,1)}_{1n}$ and creating one $\mathcal{O}^{(2,2)}_n$. For harmonic oscillators such that $[\hat{a}, \hat{a}^\dagger] = 1$ the normalized state with p quanta is given by $|p\rangle = (\hat{a}^\dagger)^p/\sqrt{p!}|0\rangle$ and therefore $\hat{a}^\dagger|p\rangle = \sqrt{p+1}|p+1\rangle$. Using harmonic oscillator algebra for the operators gives

$$\langle (\mathcal{O}^{(2,2)}_n)^{l+1} (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}-1-l} | \mathcal{O}^{(1,1)}_{1(11)}(\mu) | (\mathcal{O}^{(2,2)}_n)^{l} (\mathcal{O}^{(1,1)}_{1n})^{\frac{N}{n}-l} \rangle \approx \mu \sqrt{(\frac{N}{n} - l)(l+1)}. \tag{4.115}$$

Then the corresponding vev in the superposition $|\Psi)$ is

$$\langle \mathcal{O}^{(1,1)}_{1(11)} \rangle_\Psi = \mu \sum_{l=0}^{N/n-1} c^*_{l+1} c_l \sqrt{(\frac{N}{n} - l)(l+1)} = \mu \frac{N}{n} AB, \tag{4.116}$$

which has exactly the structure of (4.112). Given that such simple approximations (and factorizations) of the correlators reproduce the structure of the vevs so well, it would be interesting to explore whether this relates to simplifications in the structure of the chiral ring in the large N limit.

Next consider the vevs of dimension k operators. Using charge conservation and $(1,1)$ cycle conservation in (4.113) implies that only operators with m odd can acquire a vev. To reproduce the holographic result, that vevs are non-zero only when $m = \pm 1$, requires the assumption that only nearest neighbor terms in the superposition contribute to one point functions. This would follow from a stronger selection rule for $(1, 1)$ cycle conservation, that the number of $(1, 1)$ cycles in the in and out states differ by at most one. In particular, multi-particle processes such as $(\mathcal{O}^{(1,1)}_{\tilde{a}n})^3 + \mathcal{O}^{(1,1)}_{\tilde{\alpha}n} \to (\mathcal{O}^{(2,2)}_{n})^3$ would be forbidden. The selection rules for holographic vevs suggest that there is indeed such cycle conservation, and it would be interesting to explore this issue further.

Let us now return to the comment made below (4.100), that one may need to include a minus sign in the dictionary for the distinguished curve. Such a minus sign would introduce factors of $(-1)^{N/n-l}$ into the superposition (4.106), and thence an overall sign in the vevs of the associated operators $\mathcal{O}^{(1,1)}_{1(kI)}$. This would naturally account for the relative sign difference between the vevs associated with the distinguished curve and those associated with the remaining curves. It is not conclusive that one needs such a minus sign without knowing the exact three point functions and hence vevs. However such a sign change for oscillators associated with the direction distinguished by the duality would not be surprising. Recall that under T-duality of closed strings right moving oscillators associated with the duality direction switch sign, whilst the left moving oscillators and oscillators associated with orthogonal directions do not.

(4.6.5) Selection rules for curve frequencies

Selection rules for charge and $(1, 1)$ cycles are sufficient to reproduce the general structure of the vevs. In the particular example discussed above, these rules also implied the selection rules for the curve frequencies: operators acquire vevs only when the transverse and internal curves have related frequencies.

Here we will note how, with reasonable assumptions, one can motivate the selection rules for frequencies in the general case. Consider the computation of the vev of a dimension one operator $\mathcal{O}^{(1,1)}_{\tilde{\alpha}1}$ for a general superposition $|\Psi)$ using (4.103). Using the selection rules for charge and $(1, 1)$ cycles, the contributions to (4.103) involve only certain pairs of operators $(\mathcal{O}^{\mathcal{I}}, \mathcal{O}^{\mathcal{J}})$. Their $SO(2)$ charges must differ by $(\pm 1/2, \pm 1/2)$ and they must differ by an odd number of $\mathcal{O}^{(1,1)}_{\tilde{\alpha}}$ operators.

Now let us make the further assumption that there are contributions to (4.103) only from pairs of operators $(\mathcal{O}^{\mathcal{I}}, \mathcal{O}^{\mathcal{J}})$ which differ by only one term, the relevant operators taking the form

$$\mathcal{O}^{\mathcal{J}} = \mathcal{O}^{(p,q)}_{n} \mathcal{O}^{\tilde{\mathcal{J}}}, \tag{4.117}$$

with $\mathcal{O}^{\tilde{\mathcal{J}}}$ being the same for in and out states, but the single operator $\mathcal{O}^{(p,q)}_{n}$ differing between in and out states. Thus we are assuming that the relevant three point functions factorize, with the non-trivial part of the correlator arising from a single particle process.

This is indeed the structure of the three point functions arising in our example. Only nearest neighbor terms in the superposition contribute in the computation of the vev of the dimension one operator in (4.114). Moreover the $m = \pm 1$ charge selection rule for the vevs of higher dimension operators immediately follows from restricting to nearest neighbor terms in the three-point functions. Note further that this factorization structure is present in the orbifold CFT computation of the three point functions. The operator $\mathcal{O}^{(1,1)}_{\tilde{\alpha}1} \equiv \mathcal{O}^{(1,1)}_{\tilde{\alpha}1} I^{N-1}$ is the identity operator in $(N-1)$ copies of the CFT and thus only acts non-trivially in one copy of the CFT.

Consider the case of the vev of the operator with $SO(2)$ charges $(1/2, 1/2)$; it would take the form

$$\sum_{\mathcal{I},\mathcal{J},\tilde{\mathcal{I}}} a^*_{\mathcal{I}} a_{\mathcal{J}} \mathcal{N}_{\tilde{\mathcal{I}}} \Big(\langle (\mathcal{O}^{(2,2)}_n)^\dagger(\infty) \mathcal{O}^{(1,1)}_{\tilde{\alpha}1}(\mu)(\mathcal{O}^{(1,1)}_{\tilde{\alpha}n})(0)\rangle \\ + \langle (\mathcal{O}^{(1,1)}_n)^\dagger(\infty) \mathcal{O}^{(1,1)}_{\tilde{\alpha}1}(\mu)(\mathcal{O}^{(0,0)}_{\tilde{\alpha}n})(0)\rangle \Big) , \tag{4.118}$$

where $\mathcal{N}_{\tilde{\mathcal{I}}}$ is the norm of $\mathcal{O}^{\tilde{\mathcal{I}}}$. Analogous expressions would hold for the dimension one operators with other charge assignments. Such a factorization would immediately explain the frequency selection rule found in the holographic vevs obtained from supergravity (4.101). The superposition contains operators of the form (4.117) with both $(p,q) = (1,1)$ and $(p,q) \neq (1,1)$ only when the internal curve and the transverse curves share a frequency. Extending these arguments to vevs of higher dimension operators would be straightforward, and would imply selection rules for curve frequencies.

(4.6.6) Fuzzballs with no transverse excitations

Consider the case where the fuzzball geometry has only internal excitations, $F^i(v) = 0$. Then the corresponding dual superposition of ground states can involve only states built from the operators $\mathcal{O}^{R(1,1)}_{\alpha n}$. Any such state will be a zero eigenstate of both j^R_3 and $\bar{j}^R_3$. Furthermore, such ground states associated with the middle cohomology account for a finite fraction of the entropy of the D1-D5 system. In the case of $K3$ the total entropy behaves as

$$S = 2\pi\sqrt{\frac{c}{6}}, \tag{4.119}$$

with $c = 24N$. The ground states associated with the middle cohomology account for a central charge $c = 20N$. In the case of T^4 the entropy behaves as (4.119) with $c = 12N$. The states associated with the universal cohomology account for $c = 4N$, the odd cohomology accounts for another $c = 4N$ and the middle cohomology accounts for the final $c = 4N$.

Now let us consider the properties of the corresponding fuzzball geometry. When there are no transverse excitations and no winding modes of the internal curves, the $SO(4)$ symmetry in R^4 is unbroken, and the defining harmonic functions (4.54) reduce to

$$H = 1 + \frac{Q_5}{r^2}; \qquad K = \frac{Q_1}{r^2}; \tag{4.120}$$

with $A_i = 0$ and where Q_1 is defined in (4.79). The solutions manifestly all collapse to the standard (singular) D1-D5 solution and so, whilst one would need an exponential number of geometries (upon quantization) to account for dual ground states build from operators associated with the middle cohomology, one has only one singular geometry. Therefore *the relevant fuzzball solutions are not visible in supergravity*: one needs to take into account higher order corrections.

One can understand this from several perspectives. Firstly, as discussed above, R ground states associated with the middle cohomology have zero R charge; they do not break the $SO(4)$ symmetry. A geometry which is asymptotically $AdS_3 \times S^3$ for which the $SO(4)$ symmetry is exact can be characterized by the vevs of $SO(4)$ singlet operators. The only such operators in supergravity are the stress energy tensor, and the scalar operators listed in (4.76). Since the vev of the stress energy tensor must be zero for the D1-D5 ground states, the geometry would have to be distinguished by the vevs of the singlet operators given in (4.76).

Our results imply that these operators do not acquire vevs, and therefore within supergravity (without higher order corrections) geometries dual to different R ground states associated with the middle cohomology cannot be distinguished. The reason is the following. The $SO(4)$ singlet operators dual to supergravity fields are related to chiral primaries by the action of supercharge raising operators; they are the top components of the multiplets. Thus these $SO(4)$ singlet operators cannot acquire vevs in states built from the chiral primaries. $SO(4)$ singlet operators associated with stringy excitations would be needed to characterize the different ground states.

A heuristic argument based on the supertube picture also indicates that geometries dual to these ground states are not to be found in the supergravity approximation. The geometries with transverse excitations in R^4 can be viewed as a bound state of D1-D5 branes, blown up by their angular momentum in the R^4. Indeed, the characteristic size of the fuzzball geometry is directly related to this angular momentum. The simplest example, related to a circular supertube, is to take a geometry characterized by a circular curve; this is obtained by setting $B = 0$ in (4.104). The characteristic scale of the geometry is

$$r_c \sim g_s \mu / n, \tag{4.121}$$

where g_s is the string coupling and μ has dimensions of length, whilst the (dimensionless) angular momentum behaves as $j^{12} = N/n$, and thus $r_c \sim g_s \mu (j^{12}/N)$. Hence the size of the D1-D5 bound state increases linearly with the angular momentum. A general fuzzball geometry will of course not be as symmetric but nonetheless the characteristic scale averaged over the R^4 is still related to the total angular momentum. In chapter 3 we noted that fuzzball geometries dual to vacua for which the R charge is very small are not well described by supergravity. Here we have found that this implies that an exponential number of geometries dual to a finite fraction of the Ramond ground states, with strictly zero R charge, cannot be described at all in the supergravity approximation.

(4.7) Implications for the fuzzball program

In this section we will consider the implications of our results for the fuzzball program, focusing in particular on whether one can find a set of smooth weakly curved supergravity geometries which span the black hole microstates.

We have seen in the previous sections that the geometric duals of superpositions of R vacua with small or zero R charge are not well-described in supergravity. The natural basis for R ground states (4.95) uses states of definite R-charges, and it is therefore straightforward to work out the density of ground states with given R-charges, $d_{N,j_3,\bar{j}_3}$, with the total number of ground states being given by $d_N = \sum_{N,j_3,\bar{j}_3} d_{N,j_3,\bar{j}_3}$. This computation is discussed in appendix 4.A.5 with the resulting density in the large N limit being

$$d_{N,j^1,j^2} \cong \frac{1}{4(N+1-j)^{31/4}} \exp\left[\frac{2\pi(2N-j)}{\sqrt{N+1-j}}\right] \frac{1}{\cosh^2(\frac{\pi j^1}{\sqrt{N+1-j}})\cosh^2(\frac{\pi j^2}{\sqrt{N+1-j}})}, \quad (4.122)$$

where $j^1 = (j_3 + \bar{j}_3)$ and $j^2 = (j_3 - \bar{j}_3)$ and $j = |j^1| + |j^2|$. The key feature is that the number of states with zero R charge differs from the total number of R ground states given in (4.203) only by a polynomial factor:

$$d_{N,0,0} \cong d_N/N. \quad (4.123)$$

The geometries dual to such ground states are unlikely to be well-described in supergravity, and therefore the basis of black hole microstates labeled by R charges is not a good basis for the geometric duals. This argument reinforces the discussion of chapter 3, where we showed in detail that the geometric duals of specific states (in this basis) must be characterized by very small vevs which cannot be reliably distinguished in supergravity; they are comparable in magnitude to higher order corrections.

The geometries that are smooth in supergravity correspond to specific superpositions of the R charge eigenstates, for which some vevs are atypically large. The natural basis for the field theory description of the microstates is thus *not* the natural basis for the geometric duals. This issue is likely to persist in other black hole systems. For example, the microstates of the D1-D5-P system are also most naturally described as $(j_3, \bar{j}_3)$ eigenstates, with a relation analogous to (4.123) holding, so the number of states with zero R-charge is suppressed only polynomially compared to the total number of black hole microstates. Just as in the 2-charge system discussed here, the geometric duals are related to supertubes whose radii depend on the R-charges. States or superpositions of states which have small or zero R-charges are unlikely to be well-described by supergravity solutions. Thus a given smooth supergravity geometry should be described by a specific superposition of the black hole microstates. Identifying the specific superpositions for known 3-charge geometries is an open and important question.

The issue is whether there exist enough such geometries, well-described and distinguishable in supergravity, to span the entire set of black hole microstates. It seems unlikely that a basis exists which simultaneously satisfies all three requirements. Firstly, on general grounds microstates with small quantum numbers will not be well-described in supergravity. Even when considering superpositions that are well described by supergravity, to span the entire basis, one will have to include superpositions which can only be distinguished by these small vevs. I.e. in choosing a basis of geometries for which some vevs are sufficiently large for the supergravity description to be valid one will find that some of these geometries cannot be distinguished among themselves in supergravity.

We have already seen several examples of this problem in the 2-charge system. Let us parameterize the curves as

$$\begin{aligned} F^i(v) &= \mu \sum_n (\alpha^i_n e^{2\pi i n v/L} + (\alpha^i_n)^* e^{-2\pi i n v/L}); \\ \mathcal{F}^{\tilde{\beta}}(v) &= \mu \sum_n (\alpha^{\tilde{\beta}}_n e^{2\pi i n v/L} + (\alpha^{\tilde{\beta}}_n)^* e^{-2\pi i n v/L}), \end{aligned} \tag{4.124}$$

where $\mu = \sqrt{Q_1 Q_5}/R$ and $\tilde{\beta}$ runs from 1 to $h^{1,1}(M^4)$. The D1-brane charge constraint (4.57) limits the total amplitude of these curves as

$$\sum_n n^2 (|\alpha^i_n|^2 + |\alpha^{\tilde{\beta}}_n|^2) = 1. \tag{4.125}$$

Thus in general increasing the amplitude in one mode, to make certain quantum numbers large, decreases the amplitudes in the others. Moreover, the amplitude in a given mode is bounded via $|\alpha_n|^2 \leq 1/n^2$, and is thus is intrinsically very small for high frequency modes, which sample vacua with large twist labels in the CFT. Note also that the vevs of R-charges are given in terms of

$$j^{ij} = iN \sum_n n(\alpha^i_n (\alpha^j_n)^* - \alpha^j_n (\alpha^i_n)^*) \tag{4.126}$$

As we have seen, to be describable in supergravity, geometries must have transverse R^4 excitations, and thus some large R-charges, requiring $j^{ij} \gg 1$. Combining (4.126) and (4.125) one sees that this restricts the amplitudes of the internal excitations, and thus of the sampling of the black hole microstates associated with the middle cohomology of M^4.

Another way to understand the limitations of supergravity is to go back to the F1-P system where the corresponding state is the coherent state $|\{\alpha^i_n\}, \{\alpha^{\tilde{\beta}}_m\})$. These states form a complete basis of states, so we know that there is an F1-P geometry corresponding to every 1/2 BPS state. However, only when all $\alpha^i_n, \alpha^{\tilde{\beta}}_m$ are large are the geometries well-described and distinguishable within supergravity. Indeed, the amplitudes $\alpha^i_n, \alpha^{\tilde{\beta}}_m$ are also the root mean deviations of the distribution around the mean (which is described by the classical curve), so only for large $\alpha^i_n, \alpha^{\tilde{\beta}}_m$ is the classical string that sources the supergravity solution a good approximation of the quantum state. Putting it differently, when some of the amplitudes are small the difference in the solutions for different amplitudes is comparable with the error in the solutions due to the approximation of the source by a classical string, so one cannot reliably distinguish them within this approximation.

If one could not find a basis of distinguishable supergravity geometries spanning the microstates, one might ask whether a sufficiently representative basis exists. That is, suppose one chooses a single representative of the indistinguishable geometries, and assigns a measure to this geometry. Then is the corresponding basis of weighted geometries sufficiently representative to obtain the black hole properties? In the 2-charge system, the now complete set of fuzzball geometries along with the precise mapping between these geometries and R vacua allows these questions to be addressed at a quantitative level and we hope to return to this issue elsewhere.

(4.A) APPENDIX

(4.A.1) CONVENTIONS

The following table summarises the indices used throughout this chapter. In some cases an index is used more than once, with different meanings, in separate sections.

Index	Range	Usage
(m,n)	$0,\cdots,9$	10d sugra fields
(M,N)	$0,\cdots,5$	6d sugra fields
(μ,ν)	$0,1,2$	3d fields
(a,b)	$1,2,3$	S^3 indices
(i,j)	$1,2,3,4$	R^4 indices
(ρ,σ)	$1,2,3,4$	M^4 indices
$(\bar{\mu},\bar{\nu})$	$0,1$	2d fields
(α,β)	$1,2,3$	$SU(2)$ vector index
(γ,δ)	$1,\cdots,b^2$	$H_2(M^4)$
$(\tilde{\alpha},\tilde{\beta})$	$1,\cdots,h^{1,1}$	$H_{1,1}(M^4)$
(I,J)	$1,\cdots,8$	$SO(8)$ vector
$((c),(d))$	$1,\cdots,16$	heterotic vector fields
$((a),(b))$	$1,\cdots,24$	$SO(4,20)$ vector
(A,B)	$1,\cdots,26$	$SO(5,21)$ vector
(m,n)	$1,\cdots,5$	$SO(5)$ vector
(r,s)	$6,\cdots,(n_t+1)$	$SO(n_t)$ vector

FIELD EQUATIONS

The equations of motion for IIA supergravity are:

$$
\begin{aligned}
&e^{-2\Phi}(R_{mn}+2\nabla_m\nabla_n\Phi-\frac{1}{4}H^{(3)}_{mpq}H^{(3)pq}_n)-\frac{1}{2}F^{(2)}_{mp}F^{(2)p}_n-\frac{1}{2\cdot 3!}F^{(4)}_{mpqr}F^{(4)pqr}_n\\
&\quad+\frac{1}{4}G_{mn}(\frac{1}{2}(F^{(2)})^2+\frac{1}{4!}(F^{(4)})^2)=0, \qquad (4.127)\\
&4\nabla^2\Phi-4(\nabla\Phi)^2+R-\frac{1}{12}(H^{(3)})^2=0,
\end{aligned}
$$

$$
\begin{aligned}
&dH^{(3)} = 0, \qquad dF^{(2)} = 0, \qquad \nabla_m F^{(2)mn} - \frac{1}{6} H^{(3)}_{pqr} F^{(4)npqr} = 0, \\
&\nabla_m (e^{-2\Phi} H^{(3)mnp}) - \frac{1}{2} F^{(2)}_{qr} F^{(4)qrnp} - \frac{1}{2 \cdot (4!)^2} \epsilon^{npm_1 \cdots m_4 n_1 \cdots n_4} F^{(4)}_{m_1 \cdots m_4} F^{(4)}_{n_1 \cdots n_4} = 0, \\
&dF^{(4)} = H^{(3)} \wedge F^{(2)}, \qquad \nabla_m F^{(4)mnpq} - \frac{1}{3! \cdot 4!} \epsilon^{npqm_1 \cdots m_3 n_1 \cdots n_4} H^{(3)}_{m_1 \cdots m_3} F^{(4)}_{n_1 \cdots n_4} = 0.
\end{aligned}
$$

The corresponding equations for type IIB are:

$$
\begin{aligned}
&e^{-2\Phi}(R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H^{(3)}_{mpq} H_n^{(3)pq}) - \frac{1}{2} F^{(1)}_m F^{(1)}_n - \frac{1}{4} F^{(3)}_{mpq} F_n^{(3)pq} - \frac{1}{4 \cdot 4!} F^{(5)}_{mpqrs} F_n{}^{(5)pqrs} \\
&\quad + \frac{1}{4} G_{mn} ((F^{(1)})^2 + \frac{1}{3!} (F^{(3)})^2) = 0, \\
&4\nabla^2 \Phi - 4(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 = 0, \\
&dH^{(3)} = 0, \qquad \nabla_m (e^{-2\Phi} H^{(3)mnp}) - F^{(1)}_m F^{(3)mnp} - \frac{1}{3!} F^{(3)}_{mqr} F^{(5)mqrnp} = 0, \\
&dF^{(1)} = 0, \qquad \nabla_m F^{(1)m} + \frac{1}{6} H^{(3)}_{pqr} F^{(3)pqr} = 0, \\
&dF^{(3)} = H^{(3)} \wedge F^{(1)}, \qquad \nabla_m F^{(3)mnp} + \frac{1}{6} H^{(3)}_{mqr} F^{(5)mqrnp} = 0, \\
&dF^{(5)} = d(*F^{(5)}) = H^{(3)} \wedge F^{(3)},
\end{aligned}
\tag{4.128}
$$

where the Hodge dual of a p-form ω_p in d dimensions is given by

$$
(*\,\omega_p)_{i_1 \cdots i_{d-p}} = \frac{1}{p!} \epsilon_{i_1 \cdots i_{d-p} j_1 \cdots j_p} \omega_p^{j_1 \cdots j_p}, \tag{4.129}
$$

with $\epsilon_{01 \cdots d-1} = \sqrt{-g}$. The RR field strengths are defined as

$$
F^{(p+1)} = dC^{(p)} - H^{(3)} \wedge C^{(p-2)}. \tag{4.130}
$$

The equations of motion for the heterotic theory are:

$$
\begin{aligned}
&4\nabla^2 \Phi - 4\,(\nabla \Phi)^2 + R - \frac{1}{12} (H^{(3)})^2 - \alpha' (F^{(c)})^2 = 0, \\
&\nabla_m \left(e^{-2\Phi} H^{(3)mnr} \right) = 0, \\
&R^{mn} + 2\nabla^m \nabla^n \Phi - \frac{1}{4} H^{(3)mrs} H^{(3)n}_{rs} - 2\alpha' F^{(c)mr} F^{(c)n}_r = 0, \\
&\nabla_m \left(e^{-2\Phi} F^{(c)mn} \right) + \tfrac{1}{2} e^{-2\Phi} H^{(3)nrs} F^{(c)}_{rs} = 0.
\end{aligned}
$$

$F^{(c)}_{mn}$ with $(c) = 1, \cdots 16$ are the field strengths of Abelian gauge fields $V^{(c)}_m$; we consider here only supergravity backgrounds with Abelian gauge fields. This restriction means that the gauge field part of the Chern-Simons form in H_3,

$$
H^{(3)} = dB^{(2)} - 2\alpha' \omega_3(V) + \cdots, \tag{4.131}
$$

does not play a role in the supergravity solutions, nor does the Lorentz Chern-Simons term denoted by the ellipses.

Duality rules

The T-duality rules for RR fields were derived in [83] by reducing type IIA and type IIB supergravities on a circle and relating the respective RR potentials in the 9-dimensional theory. However, for calculations involving magnetic sources, it is more convenient to work with T-duality rules for RR field strengths, since potentials can only be defined locally. In the following we will rederive the T-duality rules in terms of RR field strengths.

It is slightly easier although not necessary to use the democratic formalism of IIA and IIB supergravity introduced in [75]. In this formalism one includes p-form field strengths for $p > 5$ with Hodge dualities relating higher and lower-form field strengths being imposed in the field equations. This formalism is natural when both magnetic and electric sources are present; moreover there is no need for Chern-Simons terms in the field equations. The RR part of the (pseudo)-action is simply

$$S_{RR} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\sum_q \frac{1}{4q!}(F^{(q)})^2, \tag{4.132}$$

where $q = 2,4,6,8$ is even in the IIA case and $q = 1,3,5,7,9$ is odd in the IIB case. The field strengths are defined as $F^{(q)} = dC^{(q-1)} - H^{(3)} \wedge C^{(q-3)}$ for $q \geq 3$ and $F_q = dC^{(q-1)}$ for $q < 3$. The Hodge duality relation between higher and lower form field strengths in our conventions is

$$* F^{(q)} = (-1)^{\lfloor \frac{q}{2} \rfloor} F^{(10-q)}, \tag{4.133}$$

where $\lfloor n \rfloor$ denotes the largest integer less or equal to n.

Now to compactify on a circle the ten-dimensional metric can be parameterized as

$$ds^2 = e^{2\psi}(dy - A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu, \tag{4.134}$$

where y denotes the compact direction, and 9-dimensional quantities will be denoted as hatted. An economic way to derive the T-duality rules for the field strengths is the following. Choose the vielbein to be

$$e^{\underline{y}} = e^{\psi}(dy - A_\mu dx^\mu); \qquad e^{\underline{\mu}} = \hat{e}^{\underline{\mu}}, \tag{4.135}$$

where $\underline{\mu}$ denotes a tangent space index, and $\hat{e}^{\underline{\mu}}$ is the 9-dimensional vielbein. Now reduce the field strengths (in the tangent frame) as

$$\hat{F}^{(q)}_{\underline{\mu}_1\cdots\underline{\mu}_q} = F^{(q)}_{\underline{\mu}_1\cdots\underline{\mu}_q}, \qquad \hat{F}^{(q-1)}_{\underline{\mu}_1\cdots\underline{\mu}_{q-1}} = F^{(q)}_{\underline{\mu}_1\cdots\underline{\mu}_{q-1}\underline{y}}. \tag{4.136}$$

The corresponding 9-dimensional action for the field strengths is given by

$$S_{RR} = -\frac{2\pi R}{2\kappa_{10}^2}\int d^9x\sqrt{-\hat{g}}\sum_{q=1}^{9}\frac{1}{4q!}e^{\psi}\hat{F}_q^2. \tag{4.137}$$

Since $\psi_{IIA} = -\psi_{IIB}$ under T-duality, one can read from this action the transformation rules for field strengths in 10d:

$$\begin{aligned} \tilde{F}^{(q+1)}_{\underline{\mu}_1\cdots\underline{\mu}_q\underline{y}} &= e^{\psi}F^{(q)}_{\underline{\mu}_1\cdots\underline{\mu}_q}, \\ \tilde{F}^{(q+1)}_{\underline{\mu}_1\cdots\underline{\mu}_{q+1}} &= e^{\psi}F^{(q+2)}_{\underline{\mu}_1\cdots\underline{\mu}_{q+1}\underline{y}}. \end{aligned} \tag{4.138}$$

Here q even defines IIB fields in terms of IIA fields and q odd defines IIA in terms of IIB. Note that the field strengths on both sides are in the tangent frame. Given the T-duality rules for NSNS fields

$$\begin{aligned} e^{\tilde{\psi}} &= e^{-\psi}, \qquad \tilde{A}_\mu = B^{(2)}_{y\mu}, \qquad \tilde{B}^{(2)}_{ym} = A_m, \\ \tilde{B}^{(2)}_{mn} &= B^{(2)}_{mn} + 2A_{[m}B^{(2)}_{n]y}, \qquad \tilde{\Phi} = \Phi - \psi, \end{aligned} \tag{4.139}$$

with the metric g_{mn} invariant, one can easily convert (4.138) back into

$$\begin{aligned} F^{(q)}_{m_1\ldots m_q} &= F^{(q+1)}_{m_1\ldots m_q y} - q(-1)^q B^{(2)}_{y[m_1}F^{(q-1)}_{m_2\ldots m_q]} + q(q-1)B^{(2)}_{y[m_1}A_{m_2}F^{(q-1)}_{m_3\ldots m_q]y} \\ F^{(q)}_{m_1\ldots m_{q-1}y} &= F^{(q-1)}_{m_1\ldots m_{q-1}} - (q-1)(-1)^q A_{[m_1}F^{(q-1)}_{m_2\ldots m_{q-1}]y}. \end{aligned} \tag{4.140}$$

Strictly speaking, this gives the duality rules in the democratic formalism. However we can obtain the usual rules by simply dropping the $(p > 5)$-form field strengths as long as we make sure to self-dualise $F^{(5)}$ in each IIB solution.

The S duality rules for type IIB are

$$\begin{aligned} \tilde{\tau} &= -\frac{1}{\tau}, \qquad \tilde{B}^{(2)} = C^{(2)}, \qquad \tilde{C}^{(2)} = -B^{(2)}, \\ \tilde{F}^{(5)} &= F^{(5)}, \qquad \tilde{G}_{mn} = |\tau|G_{mn}, \end{aligned} \tag{4.141}$$

where $\tau = C^{(0)} + ie^{-\Phi}$.

(4.A.2) Reduction of type IIB solutions on $K3$

The reduction of type IIB on $K3$ is very similar to the reduction of type IIA, which was discussed in some detail in [84]. In the following we will use the reduction of the NS-NS sector fields given in [84], and derive the reduction of the type IIB RR fields. Let us first review the reduction of the NS-NS sector. Starting from the ten-dimensional action

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\hat{g}}\left(e^{-2\hat{\Phi}}(\hat{R} + 4(\partial\hat{\Phi})^2 - \frac{1}{12}\hat{H}_3^2)\right), \tag{4.142}$$

where ten-dimensional fields are denoted by hats, the corresponding six-dimensional field equations can be derived from the action [84]

$$S = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-g}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H_3^2 + \frac{1}{8}\mathrm{tr}(\partial M^{-1}\partial M)\right), \tag{4.143}$$

where the six-dimensional fields are defined as follows. Firstly the 10-dimensional 2-form potential is reduced as

$$\hat{B}^{(2)}(x,y) = B_2(x) + b^\gamma(x)\omega_2^\gamma(y), \tag{4.144}$$

where (x, y) are six-dimensional and $K3$ coordinates respectively and the two forms ω_2^γ with $\gamma = 1, \cdots 22$ span the cohomology $H^2(K3, \mathbb{R})$. The 2-forms ω_2^γ transform under an $O(3, 19)$ symmetry, with a metric defined by the 22-dimensional intersection matrix

$$d_{\gamma\delta} = \frac{1}{(2\pi)^4 V}\int_{K3}\omega_2^\gamma \wedge \omega_2^\delta, \tag{4.145}$$

where $(2\pi)^4 V$ is the volume of $K3$. A natural choice for $d_{\gamma\delta}$ is

$$d_{\gamma\delta} = \begin{pmatrix} I_3 & 0 \\ 0 & -I_{19} \end{pmatrix}, \tag{4.146}$$

corresponding to a diagonal basis for the 3 self-dual and 19 anti-self dual two forms of $K3$. Furthermore, there is a matrix $D^{\delta}{}_{\gamma}$ defined by the action of the Hodge operator

$$*_4^{K3}\,\omega_2^{\gamma} = \omega_2^{\delta} D^{\delta}{}_{\gamma}, \tag{4.147}$$

which is dependent on the $K3$ metric and satisfies

$$D^{\gamma}{}_{\delta} D^{\delta}{}_{\epsilon} = \delta^{\gamma}{}_{\epsilon}, \qquad D^{\epsilon}{}_{\delta} d_{\epsilon\zeta} D^{\zeta}{}_{\gamma} = d_{\delta\zeta}. \tag{4.148}$$

The $SO(4,20)$ matrix of scalars $M^{-1}_{(a)(b)}$ was derived in [84] to be

$$M^{-1} = \Omega_2^T \begin{pmatrix} e^{-\rho} + b^{\gamma} b^{\delta} d_{\gamma\epsilon} D^{\epsilon}_{\delta} + \frac{1}{4} e^{\rho} b^4 & \frac{1}{2} e^{\rho} b^2 & \frac{1}{2} e^{\rho} b^2 b^{\gamma} d_{\gamma\delta} + b^{\gamma} d_{\gamma\epsilon} D^{\epsilon}_{\delta} \\ \frac{1}{2} e^{\rho} b^2 & e^{\rho} & e^{\rho} b^{\gamma} d_{\gamma\delta} \\ \frac{1}{2} e^{\rho} b^2 b^{\gamma} d_{\gamma\delta} + b^{\gamma} d_{\gamma\epsilon} D^{\epsilon}_{\delta} & e^{\rho} b^{\gamma} d_{\gamma\delta} & e^{\rho} b^{\epsilon} d_{\epsilon\gamma} b^{\zeta} d_{\zeta\delta} + d_{\gamma\epsilon} D^{\epsilon}_{\delta} \end{pmatrix} \Omega_2, \tag{4.149}$$

with $b^2 \equiv b^{\gamma} b^{\delta} d_{\gamma\delta}$. Here ρ is the breathing mode of $K3$, $e^{-\rho} = \frac{1}{(2\pi)^4 V} \int_{K3} *_4 1$. The six-dimensional dilaton is related to the 10-dimensional dilaton via $\Phi = \hat{\Phi} + \rho/2$.

The dimensional reduction of the NS sector makes manifest only an $SO(4,20)$ subgroup of the full $SO(5,21)$ symmetry. Including the reduction of the RR sector should thus give the equations of motion following from the six-dimensional string frame action, which for IIB was given in (4.38)

$$\begin{aligned} S &= \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \Big\{ e^{-2\Phi} \Big(R + 4(\partial\Phi)^2 + \frac{1}{8} \mathrm{tr}(\partial M^{-1} \partial M) \Big) + \frac{1}{2} \partial l^{(a)} M^{-1}_{(a)(b)} \partial l^{(b)} \\ &\quad - \frac{1}{3} G^{A}_{MNP} \mathcal{M}^{-1}_{AB} G^{BMNP} \Big\}, \end{aligned}$$

and in which only an $SO(4,20)$ subgroup of the total $SO(5,21)$ symmetry is manifest; recall that $\mathcal{M}^{-1}_{AB}$ here is an $SO(5,21)$ matrix, with $M^{-1}_{(a)(b)}$ being $SO(4,20)$. Note that the six-dimensional coupling is related to the ten-dimensional coupling via $(2\pi)^4 V(2\kappa_6^2) = 2\kappa_{10}^2$, where $(2\pi)^4 V$ is the volume of $K3$.

Following the same steps as [84] the RR potentials can be reduced as

$$\begin{aligned} \hat{C}^{(0)}(x,y) &= C_0(x), \qquad \hat{C}^{(2)}(x,y) = C_2(x) + c^{\gamma}_{(0,2)}(x) \omega_2^{\gamma}(y), \\ \hat{C}^{(4)}(x,y) &= C_4(x) + c^{\gamma}_{(2,4)}(x) \wedge \omega_2^{\gamma}(y) + c_{(0,4)}(x)(e^{\rho} *_{K3} 1)(y), \end{aligned} \tag{4.150}$$

where $*_{K3}$ denotes the Hodge dual in the $K3$ metric and the corresponding field strengths are

$$\begin{aligned}
\hat{F}^{(1)}(x,y) &= F_1(x), \\
\hat{F}^{(3)}(x,y) &= dC_2(x) - C_0(x)H_3(x) + \left(dc^{\gamma}_{(0,2)}(x) - C_0(x)db^{\gamma}(x)\right)\omega_2(y) \equiv F_3 + K_1^{\gamma}\wedge\omega_2^{\gamma}, \\
\hat{H}^{(3)}(x,y) &= dB_2(x) + db^{\gamma}(x)\wedge\omega_2^{\gamma}(y) \equiv H_3 + db^{\gamma}\wedge\omega_2^{\gamma}, \\
\hat{F}^{(5)}(x,y) &= dC_4(x) - C_2(x)\wedge H_3(x) + \left(dc^{\gamma}_{(2,4)}(x) - C_2(x)db^{\gamma}(x) - c^{\gamma}_{(0,2)}(x)H_3(x)\right)\wedge\omega_2^{\gamma}(y) \\
&\quad + \left(dc_{(0,4)}(x) - c^{\gamma}_{0,2}(x)db^{\delta}(x)d_{\gamma\delta}\right)\wedge(e^{\rho}(x) *_{K3} 1)(y) \\
&\equiv F_5 + K_3^{\gamma}\wedge\omega_2^{\gamma} + \tilde{F}_1\wedge e^{\rho} *_{K3} 1.
\end{aligned} \tag{4.151}$$

The reduction of the potentials thus gives two three form field strengths H_3 and F_3, 3 self-dual and 19 anti-self dual three form field strengths K_3^{γ} and 46 scalars b^{γ}, $c^{\gamma}_{(0,2)}$, $c_{(0,4)}$ and C_0. After splitting the three forms H_3 and F_3 into their self-dual and anti-self-dual parts, we obtain 5 self-dual and 21 anti-self dual tensors in total, as described in [85].

It is then straightforward to obtain the map relating six and ten-dimensional fields by inserting the expressions (4.150) and (4.151) into the ten-dimensional field equations (4.128). The additional RR scalars are contained in

$$l^{(a)} = \Omega_2^T \begin{pmatrix} C_0 \\ \tilde{c}_{(0,4)} \\ \tilde{c}^{\gamma}_{(0,2)} \end{pmatrix}, \tag{4.152}$$

with Ω_2 defined in the appendix 4.A.2 and the shifted fields defined as

$$\begin{aligned}
\tilde{c}^{\gamma}_{(0,2)} &= c^{\gamma}_{(0,2)} - C_0 b^{\gamma}, \\
\tilde{c}_{(0,4)} &= c_{(0,4)} - b^{\gamma}c^{\delta}_{(0,2)}d_{\gamma\delta} + \frac{1}{2}b^2 C_0.
\end{aligned} \tag{4.153}$$

The fields Φ, $l^{(a)}$ and the $SO(4,20)$ matrix M^{-1} given in (4.149) can be recombined into the $SO(5,21)$ matrix $\mathcal{M}^{-1} = V^T V$, with the latter conveniently expressed in terms of the vielbein

$$V = \Omega_4^T \begin{pmatrix} e^{-\Phi} & 0 & 0 & 0 & 0 \\ -e^{\Phi}(C_0 c_{(0,4)} - \frac{1}{2}c^2_{(0,2)}) & e^{\Phi} & -e^{\Phi}\tilde{c}_{(0,4)} & -e^{\Phi}C_0 & e^{\Phi}\tilde{c}^{\gamma}_{(0,2)}d_{\gamma\delta} \\ e^{-\rho/2}C_0 & 0 & e^{-\rho/2} & 0 & 0 \\ e^{\rho/2}c_{(0,4)} & 0 & \frac{1}{2}e^{\rho/2}b^2 & e^{\rho/2} & e^{\rho/2}b^{\gamma}d_{\gamma\delta} \\ \tilde{V}_{\delta\gamma}c^{\gamma}_{(0,2)} & 0 & \tilde{V}_{\delta\gamma}b^{\gamma} & 0 & \tilde{V}_{\gamma\delta} \end{pmatrix}\Omega_4. \tag{4.154}$$

Here the $SO(3,19)$ vielbein $\tilde{V}_{\alpha\beta}$ is defined by $d_{\alpha\beta}D^{\beta}{}_{\gamma} = \tilde{V}_{\alpha\beta}\tilde{V}_{\beta\gamma}$, $c^2_{(0,2)} \equiv c^{\gamma}_{(0,2)}c^{\delta}_{(0,2)}d_{\gamma\delta}$ and the matrix Ω_4 is defined in the appendix 4.A.2. The six-dimensional tensor fields are related to the ten-dimensional fields as

$$\begin{aligned}
H_3^1 &= \frac{e^{-\Phi}}{4}(1 + *_6)H_3, \qquad H_3^{\alpha_+ + 1} = -\frac{1}{\sqrt{8}}(\tilde{V}K_3)^{\alpha_+}, \\
H_3^5 &= -\frac{e^{-\rho/2}}{4}(1 + *_6)F_3, \qquad H_3^6 = -\frac{e^{-\rho/2}}{4}(1 - *_6)F_3, \\
H_3^{\alpha_- + 3} &= -\frac{1}{\sqrt{8}}(\tilde{V}K_3)^{\alpha_-}, \qquad H_3^{26} = \frac{e^{-\Phi}}{4}(1 - *_6)H_3.
\end{aligned} \tag{4.155}$$

Here $\alpha_+ = 1,2,3$ and $\alpha_- = 4,\cdots 22$, labeling the self dual and anti-self dual forms respectively. Note that using formulas (4.154) and (4.155) to lift a six-dimensional solution to ten dimensions requires a specific choice of six-dimensional vielbein.

The solutions we find have $D^\gamma_\delta = d_{\gamma\delta}$; this implies the identity

$$(\omega_2^{\alpha_-})_{\rho\sigma}(\omega_2^{\beta_-})_\tau{}^\sigma = \tfrac{1}{2} g_{\rho\tau}\delta^{\alpha_-\beta_-}, \tag{4.156}$$

where (ρ,τ) are $K3$ coordinates and $g_{\rho\tau}$ is the K3 metric. As discussed in [86], a choice of $D^\gamma{}_\epsilon$ fixes the complex structure completely and implies $(\omega_2^\gamma)_{\rho\sigma}(\omega_2^\delta)^{\rho\sigma} = D^\epsilon{}_\delta d_{\gamma\epsilon}$. Varying this identity with respect to the metric results in (4.156).

S-duality in 6 dimensions

Given the map between 10-dimensional and 6-dimensional fields, we can now obtain the action of S-duality on 6-dimensional fields as part of the $SO(5,21)$ symmetry:

$$G_3 \to O_S G_3, \qquad \mathcal{M}^{-1} \to O_S \mathcal{M}^{-1} O_S^T, \tag{4.157}$$

where

$$(O_S)_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_3 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (O_S)_{rs} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{19} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{4.158}$$

Moreover one can perform an $SO(5)\times SO(21)$ transformation to bring the vielbein of the S-dual solution back to the form used by the 10-dimensional lift. Including this transformation, H_3 and V transform as

$$H_3 \to O_G H_3, \qquad V \to O_G V O_S^T, \tag{4.159}$$

with

$$(O_G)_{ij} = \frac{1}{|\tau|}\begin{pmatrix} C_0 & 0 & -e^{\hat\Phi} \\ 0 & I_3 & 0 \\ e^{\hat\Phi} & 0 & C_0 \end{pmatrix}, \qquad (O_G)_{rs} = \frac{1}{|\tau|}\begin{pmatrix} C_0 & 0 & -e^{\hat\Phi} \\ 0 & I_{19} & 0 \\ e^{\hat\Phi} & 0 & C_0 \end{pmatrix}, \tag{4.160}$$

where $\tau = C_0 + ie^{-\hat\Phi}$, $\hat\Phi = \Phi - \rho/2$ is the 10-dimensional dilaton and the fields C_0 and $e^{\hat\Phi}$ are the original ones taken before the S-duality.

Basis change matrices

In defining six-dimensional supergravities there are implicit choices of constant $SO(p,q)$ matrices. When discussing the compactification from the ten to six dimensions, the most convenient choices for these matrices are certain off-diagonal forms, see for example [34, 79, 77, 78, 81, 80]. When one is interested in specific solutions of the six-dimensional supergravity equations, such as $AdS_3\times S^3$ solutions, and deriving the spectrum in such backgrounds, it is rather more convenient to use diagonal choices for these matrices, see for example [55, 24]. In this chapter we both compactify from ten to six dimensions, and expand six-dimensional solutions about a given background. We therefore find it most convenient to use diagonal choices for the constant matrices. To use previous results on compactification and T-duality, we need to apply certain similarity transformations. For the most part these may be implicitly written in terms of basis change matrices, so that compactification and duality formulas remain as simple as possible. Thus let us define matrices Ω_1 and Ω_2 for $SO(4,20)$, and Ω_3 and Ω_4 for $SO(5,21)$ via:

$$\Omega_1^T \begin{pmatrix} v^\rho \\ w^\rho \\ x^{(c)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v^\rho - w^\rho) \\ \frac{1}{\sqrt{2}}(v^\rho + w^\rho) \\ x^{(c)} \end{pmatrix}, \qquad \Omega_3^T \begin{pmatrix} v \\ w \\ x^{(a)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v - w) \\ x^{(a)} \\ \frac{1}{\sqrt{2}}(v + w) \end{pmatrix} \tag{4.161}$$

$$\Omega_2^T \begin{pmatrix} v \\ w \\ x^\alpha \\ y^{\alpha_-} \end{pmatrix} = \begin{pmatrix} x^\alpha \\ \frac{1}{\sqrt{2}}(v - w) \\ \frac{1}{\sqrt{2}}(v + w) \\ y^{\alpha_-} \end{pmatrix}, \qquad \Omega_4^T \begin{pmatrix} v_1 \\ w_1 \\ v_2 \\ w_2 \\ x^\alpha \\ y^{\alpha_-} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_1 - w_1) \\ x^\alpha \\ \frac{1}{\sqrt{2}}(v_2 - w_2) \\ \frac{1}{\sqrt{2}}(v_2 + w_2) \\ y^{\alpha_-} \\ \frac{1}{\sqrt{2}}(v_1 + w_1) \end{pmatrix},$$

where $\rho = 1, \cdots 4$, $(c) = 1, \cdots 16$, $(a) = 1, \cdots 24$, $\alpha = 1,2,3$ and $\alpha_- = 1, \cdots 19$. These satisfy the conditions:

$$\begin{aligned} \Omega_1 \begin{pmatrix} 0 & -I_4 & 0 \\ -I_4 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \Omega_1^T &= \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, \\ \Omega_2 \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & -I_{19} \end{pmatrix} \Omega_2^T &= \begin{pmatrix} I_4 & 0 \\ 0 & -I_{20} \end{pmatrix}, \\ \Omega_3 \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & -I_{20} \end{pmatrix} \Omega_3^T &= \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}, \\ \Omega_4 \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & o & -I_{19} \end{pmatrix} \Omega_4^T &= \begin{pmatrix} I_5 & 0 \\ 0 & -I_{21} \end{pmatrix}. \end{aligned} \tag{4.162}$$

Here σ_1 is the Pauli matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(4.A.3) Properties of spherical harmonics

Scalar, vector and tensor spherical harmonics satisfy the following equations

$$\Box Y^I = -\Lambda_k Y^I, \tag{4.163}$$

$$\Box Y_a^{I_v} = (1-\Lambda_k) Y_a^{I_v}, \qquad D^a Y_a^{I_v} = 0,$$
$$\Box Y_{(ab)}^{I_t} = (2-\Lambda_k) Y_{(ab)}^{I_t}, \qquad D^a Y_{k(ab)}^{I_t} = 0,$$

where $\Lambda_k = k(k+2)$ and the tensor harmonic is traceless. It will often be useful to explicitly indicate the degree k of the harmonic; we will do this by an additional subscript k, e.g. degree k spherical harmonics will also be denoted by Y_k^I, etc. $\Box$ denotes the d'Alambertian along the three sphere. The vector spherical harmonics are the direct sum of two irreducible representations of $SU(2)_L \times SU(2)_R$ which are characterized by

$$\epsilon_{abc} D^b Y^{cI_v \pm} = \pm(k+1) Y_a^{I_v \pm} \equiv \lambda_k Y_a^{I_v \pm}. \tag{4.164}$$

The degeneracy of the degree k representation is

$$d_{k,\epsilon} = (k+1)^2 - \epsilon, \tag{4.165}$$

where $\epsilon = 0, 1, 2$ respectively for scalar, vector and tensor harmonics. For degree one vector harmonics I_v is an adjoint index of $SU(2)$ and will be denoted by α. We use normalized spherical harmonics such that

$$\int Y^{I_1} Y^{J_1} = \Omega_3 \delta^{I_1 J_1}; \qquad \int Y^{aI_v} Y_a^{J_v} = \Omega_3 \delta^{I_v J_v}; \qquad \int Y^{(ab)I_t} Y_{(ab)}^{J_t} = \Omega_3 \delta^{I_t J_t}, \tag{4.166}$$

where $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere. We define the following triple integrals as

$$\int Y^I Y^J Y^K = \Omega_3 a_{IJK}; \tag{4.167}$$
$$\int (Y_1^{\alpha\pm})^a Y_1^j D_a Y_1^i = \Omega_3 e_{\alpha ij}^{\pm}; \tag{4.168}$$

(4.A.4) INTERPRETATION OF WINDING MODES

In the fundamental string supergravity solutions (4.1) the null curves describing the motion of the string along a torus direction x^ρ (whose periodicity is $2\pi R_\rho$) could have winding modes such that $F_\rho(v) = w_\rho R_\rho v / R_y$, with w_ρ integral. Consider now the correspondence with quantum string states. Such winding modes are not consistent with both supersymmetry and momentum and winding quantization for a string propagating in flat space, with no B field. Recall that the zero modes of a worldsheet compact boson field can be written as

$$X(\sigma^+, \sigma^-) = x + \frac{1}{2}(\alpha' \frac{p}{R} + nR)\sigma^+ + \frac{1}{2}(\alpha' \frac{p}{R} - nR)\sigma^- \equiv x + \tilde{w}\sigma^+ + w\sigma^-, \tag{4.169}$$

where R is the radius and (p, n) are the quantized momentum and winding respectively; note that we define $\sigma^\pm = (\tau \pm \sigma)$. BPS left-moving states with no right-moving excitations have $w = 0$ and hence $\alpha' p = -nR^2$. However the latter condition has no solutions at generic radius and so states with winding along the torus directions cannot be BPS. Therefore winding modes should not be included to describe the F1-P states and corresponding dual D1-D5 ground states of interest here.

Now consider switching on constant $B^{(2)}_{\rho v} \equiv b_\rho$ on the worldsheet. The constant B field shifts the momentum charges, and thus there are BPS left-moving states with winding around the torus directions. To be more precise, following the discussion of chapter 3, one can describe a string with left-moving excitations using a null lightcone gauge. The relevant terms in the worldsheet fields are then

$$\begin{aligned} V &= w^v \sigma^-; \qquad U = w^u \sigma^- + \tilde{w}^u \sigma^+ + \sum_n \frac{1}{\sqrt{n}} a_n^- e^{-in\sigma^-}; \\ X^I &= \delta_{I\rho} w^I \sigma^- + \sum_n \frac{1}{\sqrt{n}} a_n^I e^{-in\sigma^-}, \end{aligned} \tag{4.170}$$

where winding modes are included only along torus directions, labeled by ρ. The L_0 constraint implies

$$w^v w^u = (w^\rho)^2 + 2\sum_{n>0} |n| a_n^I a_{-n}^I \equiv (w^v)^2 |\partial_V X^I|_0^2, \tag{4.171}$$

where $|A|_0$ denotes the projection onto the zero mode. The momentum and winding charges are given by

$$P^m = \frac{1}{4\pi}\int d\sigma (\partial_\tau X^m + B^{(2)}_{mn} \partial_\sigma X^n); \qquad W^m = \frac{1}{2\pi}\int d\sigma \partial_\sigma X^m, \tag{4.172}$$

respectively, where $\alpha' = 2$. Requiring no winding in the time direction and no momentum along the x^ρ directions imposes $\tilde{w}^u = w^u + w^v$ and $w^\rho = b_\rho w^v$. The conserved momentum and winding charges are then

$$P^M = \tfrac{1}{2} w^v \left((1 + |\partial_V X^I|_0^2 + b_\rho^2), (|\partial_V X^I|_0^2 - b_\rho^2), 0 \right); \qquad W^M = w^v (0, 1, 0, b_\rho). \tag{4.173}$$

Note that the integral quantized momentum charge p_y along the y direction is therefore

$$p_y = R_y (w^u - (w^v)^{-1} (w^\rho)^2). \tag{4.174}$$

Now consider the solitonic string supergravity solution (4.1) with defining curves $F^I(v)$ where $F^\rho(v) = b_\rho v + \bar{F}^\rho(v)$, with $\bar{F}^\rho(v)$ having no zero mode. The ADM charges of this solitonic string were computed in [34], and are given by

$$P^M_{\rm ADM} = kQ \left((1 + |\partial_v F^I|_0^2), |\partial_v F^I|_0^2, 0, b_\rho \right), \tag{4.175}$$

where the effective Newton constant is $k = \Omega_3 L_y / 2\kappa_6^2$. When $b_\rho = 0$ these charges match the worldsheet charges (4.173) provided that $w^v = 2kQ$ as in [34] but when $b_\rho \neq 0$ they do not quite agree with the worldsheet charges. The reason is that in the supergravity solution $B^{(2)}_{\rho v}$ approaches zero at infinity, but to match with the constant $B^{(2)}_{\rho v}$ background on the worldsheet, $B^{(2)}_{\rho v}$ should approach b_ρ at infinity. This can be achieved via a constant gauge transformation $A_\rho \to A_\rho - b_\rho$, combined with a coordinate shift $u \to u + 2 b_\rho x^\rho$. The ADM charges of this shifted background indeed exactly match the worldsheet charges (4.173). The harmonic functions A_ρ then take the form

$$A_\rho = -b_\rho H - \frac{Q}{L_y}\int_0^{L_y} dv \frac{\partial_v \bar{F}^\rho}{|x - F|^2}, \tag{4.176}$$

where in the latter expression $|x-F|^2$ denotes $\sum_i (x^i - F^i(v))^2$; the harmonic function has been smeared over the T^4 and the y circle. Note that when $F^i(v) = 0$ the supergravity solution collapses to

$$\begin{aligned} ds^2 &= H^{-1} dv(-du + K dv) + dx^I dx_I; \qquad K = (1 + \frac{Q|\partial_v F^\rho|_0^2}{r^2}), \\ e^{-2\Phi} &= H \equiv (1 + \frac{Q}{r^2}); \qquad B^{(2)}_{uv} = \tfrac{1}{2}(H^{-1} - 1); \qquad B^{(2)}_{v\rho} = -b_\rho. \end{aligned} \tag{4.177}$$

This is the naive $SO(4)$ invariant F1-P solution, with an additional constant B field. Finally let us note that one can similarly switch on winding modes for the curves $q^{(c)}(v)$ characterizing the charge waves in the heterotic solution (4.16) by including constant $A^{(c)}_v$ on the worldsheet.

Now let us consider solutions in the D1-D5 system, and the interpretation of including winding modes of the internal curves. In particular, it is interesting to note that the general $SO(4)$ invariant solutions include harmonic functions

$$\mathcal{A} = a_o + \frac{a}{r^2}; \qquad \mathcal{A}^{\alpha-} = a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2}, \tag{4.178}$$

in addition to the harmonic functions (H, K) given in (4.120). The non-constant terms in these harmonic functions are related to the winding modes of the internal curves, with the quantities $a^{\tilde{\alpha}} = (a, a^{\alpha-})$ being given by

$$a = -\frac{Q_5}{L}\int_0^L dv \dot{\mathcal{F}}(v); \qquad a^{\alpha-} = -\frac{Q_5}{L}\int_0^L dv \dot{\mathcal{F}}^{\alpha-}(v). \tag{4.179}$$

Following the duality chain, these constants are given by $a^{\tilde{\alpha}} = -Q_5 b^{\tilde{\alpha}}$ where for the T^4 case $b^{\tilde{\alpha}} \equiv B^{(2)}_{\rho v} = b_\rho$ and for the $K3$ case $b^{\tilde{\alpha}} \equiv (B^{(2)}_{\rho v} = b_\rho, A^{(c)}_v = b^{(c)})$. The constant terms $(a_o, a_o^{\alpha-})$ are related to the boundary conditions at asymptotically flat infinity, as we will discuss below.

When these functions $(\mathcal{A}, \mathcal{A}^{\alpha-})$ are non-zero, the geometry generically differs from the naive D1-D5 geometry. The functions $(f_1, \tilde{f}_1)$ appearing in the metric behave as

$$\begin{aligned} \tilde{f}_1 &= 1 + \frac{Q_1}{r^2} - (1 + \frac{Q_5}{r^2})^{-1}\left((a_o + \frac{a}{r^2})^2 + (a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2})^2\right) \\ f_1 &= 1 + \frac{Q_1}{r^2} - (1 + \frac{Q_5}{r^2})^{-1}\left((a_o^{\alpha-} + \frac{a^{\alpha-}}{r^2})^2\right). \end{aligned} \tag{4.180}$$

In the decoupling limit these functions become

$$\tilde{f}_1 \to r^{-2}(Q_1 - Q_5^{-1}(a^2 + a^{\alpha-}a^{\alpha-})) \equiv \frac{\tilde{q}_1}{r^2}; \qquad f_1 \to r^{-2}(Q_1 - Q_5^{-1}(a^{\alpha-}a^{\alpha-})) \equiv \frac{q_1}{r^2}, \tag{4.181}$$

and thus $(a_o, a_o^{\alpha-})$ drop out. Note that $\tilde{q}_1$ corresponds to the conserved momentum charge in the F1-P system (4.174). Substituting the decoupling region functions into (4.51), one finds that the near horizon region of the solution is $AdS_3 \times S^3 \times M^4$, supported by both $F^{(3)}$ and

$H^{(3)}$ flux:

$$
\begin{aligned}
ds^2 &= \frac{r^2\sqrt{q_1}}{\tilde{q}_1\sqrt{Q_5}}(-dt^2+dy^2)+\sqrt{q_1Q_5}(\frac{dr^2}{r^2}+d\Omega_3^2)+\frac{\sqrt{q_1}}{\sqrt{Q_5}}ds^2_{M^4}; \\
e^{2\Phi} &= \frac{q_1^2}{Q_5\tilde{q}_1}, \qquad F^{(3)}_{tyr}=-\frac{2r}{\tilde{q}_1}, \qquad F^{(3)}_{\Omega_3}=2q_1^{-1}\tilde{q}_1Q_5; \\
H^{(3)}_{tyr} &= 2aQ_5^{-1}\tilde{q}_1^{-1}r, \qquad H^{(3)}_{\Omega_3}=-2a.
\end{aligned} \tag{4.182}
$$

The field strengths $F^{(1)}$ and $F^{(5)}$ vanish, but there are non-vanishing potentials:

$$
\begin{aligned}
B^{(2)}_{\rho\sigma} &= \sqrt{2}Q_5^{-1}a^{\alpha-}\omega^{\alpha-}_{\rho\sigma}, \qquad C^{(0)}=-q_1^{-1}a, \qquad C^{(4)}_{\rho\sigma\tau\pi}=Q_5^{-1}a\epsilon_{\rho\sigma\tau\pi}; \\
C^{(4)}_{ty\alpha\beta} &= a(1+\tilde{q}_1^{-1}r^2)\epsilon_{\alpha\beta}, \qquad C^{(4)}_{\alpha\beta\rho\sigma}=2\sqrt{2}\epsilon_{\alpha\beta}a^{\alpha-}\omega^{\alpha-}_{\rho\sigma}, \qquad C^{(4)}_{ty\rho\sigma}=\sqrt{2}Q_5^{-1}a^{\alpha-}\omega^{\alpha-}_{\rho\sigma},
\end{aligned} \tag{4.183}
$$

where ϵ is a 2-form such that $d\epsilon$ is the volume form of the unit 3-sphere. The conserved charges therefore include Chern-Simons terms; using the equations of motion (4.128) one finds that they are given by

$$
\begin{aligned}
D5 &: \quad Q_5=\tfrac{1}{2}\int_{S^3}(F^{(3)}+H^{(3)}C^{(0)}); \\
D1 &: \quad \tilde{q}_1=\tfrac{1}{2}\int_{S^3\times M^4}(*F^{(3)}+H^{(3)}\wedge C^{(4)}); \\
D3 &: \quad a^{\alpha-}=\frac{1}{2\sqrt{2}}\int_{S^3\times\omega^{\alpha-}}B^{(2)}\wedge(F^{(3)}+H^{(3)}C^{(0)}); \\
NS5 &: \quad a=-\tfrac{1}{2}\int_{S^3}H^{(3)},
\end{aligned} \tag{4.184}
$$

where we drop terms which do not contribute to the charges. The curvature radius of the $AdS_3\times S^3$ is $l=(q_1Q_5)^{1/4}$, and the three-dimensional Newton constant is

$$
\frac{1}{2G_3}=\frac{8\pi V_4\Omega_3}{\kappa_{10}^2}\frac{\tilde{q}_1}{q_1}(q_1Q_5)^{3/4}, \tag{4.185}
$$

with the volume of M^4 being $(2\pi)^4V$ and $2\kappa_{10}^2=(2\pi)^7(\alpha')^4$. Then using [30, 13] the central charge of the dual CFT is

$$
c=\frac{3l}{2G_3}=6\frac{V}{(\alpha)'^4}\tilde{q}_1Q_5\equiv 6\tilde{n}_1n_5 \tag{4.186}
$$

where the integral charges $(\tilde{n}_1,n_5)$ are given by

$$
Q_5=\alpha' n_5; \qquad \tilde{q}_1=\frac{(\alpha')^3\tilde{n}_1}{V}. \tag{4.187}
$$

Now consider the relation between this system and the F1-P system discussed previously. The conserved charges here are $(Q_5,\tilde{q}_1,a,a^{\alpha-})$, which correspond to the winding, momentum along the y circle and winding along the internal manifold in the original system. The fact that $(a,a^{\alpha-})$ measure NS5-brane and D3-brane charges in the final system is consistent with the duality chains from the F1-P systems: applying the standard duality rules along the chains given in (4.6),(4.7) and (4.19), one indeed finds that the original winding charges become NS5-brane and D3-brane charges.

Finally let us comment on the constant terms in the harmonic functions, $(a_o, a_o^{\alpha-})$. These clearly determine the behavior of the solution at asymptotically flat infinity: the B field and RR potentials at infinity depend on them. Now consider how these constant terms can be described in the CFT. In the context of the pure D1-D5 system it was noted in chapter 3 that (infinitesimal) constant terms in the harmonic functions (f_1, f_5) can be reinstated by making (infinitesimal) irrelevant deformations of the CFT by $SO(4)$ singlet operators. See also [57] for a related discussion in the context of the AdS_5/CFT_4 correspondence. It seems probable that a similar interpretation would hold here: the $(n_t - 1)$ parameters $(a_o, a_o^{\alpha-})$ (where $n_t = 5, 21$ for T^4 and $K3$ respectively) would be related to the parameters of deformations of the CFT by irrelevant $SO(4)$ singlet operators. In total taking into account these $(n_t - 1)$ zero modes, plus the two constant terms in the (f_1, f_5) harmonic functions, one gets $(n_t + 1)$ parameters. This agrees exactly with the count of the number of irrelevant $SO(4)$ singlet operators[4]. How to describe these deformations in the field theory beyond the infinitesimal level is not known, however.

(4.A.5) Density of ground states with fixed R charges

In this appendix we will derive an asymptotic formula for the number of R ground states with given R charges. Our derivation follows closely that of [87] for the density of fundamental string states with a given mass and angular momentum. In fact, we will consider the case of $K3$, so the relevant counting is precisely that of the density of left moving heterotic string states with a given excitation level N and (commuting) angular momenta (j^{12}, j^{34}) in the transverse R^4. For this purpose we can consider the following Hamiltonian

$$H = \sum_{n=1}^{\infty} \left(\sum_{(a)=1}^{24} \alpha_{-n}^{(a)} \alpha_n^{(a)} \right) + \lambda_1 j^1 + \lambda_2 j^2, \tag{4.188}$$

where (λ_1, λ_2) are Lagrange multipliers and

$$j^1 = j^{12} = -i \sum_{n=1}^{\infty} n^{-1} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1); \qquad j^2 = j^{34} = -i \sum_{n=1}^{\infty} n^{-1} (\alpha_{-n}^3 \alpha_n^4 - \alpha_{-n}^4 \alpha_n^3). \tag{4.189}$$

Here the oscillators satisfy the standard commutation relations, namely $\left[\alpha_n^{(a)}, \alpha_m^{(b)}\right] = n\delta_{n+m}\delta^{(a)(b)}$. In [87] the partition function was computed in the case $\lambda_2 = 0$, and thus the partition function of interest here can be computed by generalizing their results. The first step is to diagonalize the Hamiltonian by introducing combinations

$$a_n^{12} = \frac{1}{\sqrt{2n}} (\alpha_n^1 + i\alpha_n^2); \qquad b_n^{12} = \frac{1}{\sqrt{2n}} (\alpha_n^1 - i\alpha_n^2) \tag{4.190}$$

[4]Such deformations may also be related to the attractor flow of moduli; this idea was used for the non-renormalization theorem of [74].

and analogously (a_n^{34}, b_n^{34}). Then the Hamiltonian takes the form

$$H = \sum_{n=1}^{\infty}\left(\sum_{(a)=5}^{24}\alpha_{-n}^{(a)}\alpha_n^{(a)} + (n-\lambda_1)(a_n^{12})^\dagger a_n^{12} + (n+\lambda_1)(b_n^{12})^\dagger b_n^{12}\right.$$
$$\left. +(n-\lambda_2)(a_n^{34})^\dagger a_n^{34} + (n+\lambda_2)(b_n^{34})^\dagger b_n^{34}\right) \tag{4.191}$$

The partition function $Z = \mathrm{Tr}(\mathrm{e}^{-\beta \mathrm{H}})$ is then

$$Z = \prod_{n=1}^{\infty}\left[(1-w^n)^{-20}(1-c_1w^n)^{-1}(1-c_1^{-1}w^n)^{-1}(1-c_2w^n)^{-1}(1-c_2^{-1}w^n)^{-1}\right] \tag{4.192}$$

with $w = e^{-\beta}$ and $c_1 = e^{\beta\lambda_1}$, $c_2 = e^{\beta\lambda_2}$. To estimate the asymptotic density of states, one as usual expresses the partition function in terms of modular functions and then uses the modular transformation properties. Here one needs the Jacobi theta function

$$\theta_1(z|\tau) = 2f(q^2)q^{1/4}\sin(\pi z)\prod_{n=1}^{\infty}(1-2q^{2n}\cos(2\pi z)+q^{4n}), \tag{4.193}$$

with

$$f(q^2) = \prod_{n=1}^{\infty}(1-q^{2n}), \qquad q = e^{i\pi\tau}, \tag{4.194}$$

and the modular transformation property

$$\theta_1(-\frac{z}{\tau}|-\frac{1}{\tau}) = e^{i\pi/4}\sqrt{\tau}e^{i\pi z^2/\tau}\theta_1(z|\tau) \tag{4.195}$$

Rewriting the partition function in terms of the modular functions, applying this modular transformation and then taking the high temperature limit results in

$$Z(\beta,\lambda_1,\lambda_2) = C\beta^{12}e^{4\pi^2/\beta}\frac{\lambda_1\lambda_2}{\sin(\pi\lambda_1)\sin(\pi\lambda_2)}, \tag{4.196}$$

with C a constant. From this expression one can extract the density of states with level N and angular momenta (j^1, j^2) by expanding

$$Z(w,k_1,k_2) = \sum_{N,j} d_{N,j^1,j^2}w^N e^{ik_1j^1+ik_2j^2}, \tag{4.197}$$

where $k_1 = -i\beta\lambda_1$ and $k_2 = -i\beta\lambda_2$, and projecting out the d_{N,j^1,j^2}. Integrating over (k_1,k_2) can be done exactly, since

$$\int_{-\infty}^{\infty} dk e^{iky}\frac{k}{\sinh(\pi k/\beta)} = \tfrac{1}{2}\beta^2\frac{1}{\cosh^2(\beta y/2)}, \tag{4.198}$$

resulting in the following contour integral over a circle around $w=0$ for d_{N,j^1,j^2}:

$$d_{N,j^1,j^2} = C'\oint\frac{dw}{w^{N+1}}\beta^{14}e^{4\pi^2/\beta}\frac{1}{\cosh^2(\beta j^1/2)\cosh^2(\beta j^2/2)}. \tag{4.199}$$

Assuming N is large the integral can be approximated by a saddle point evaluation, with the saddle point defined by the solution of

$$\frac{4\pi^2}{\beta^2} = N + 1 - j^1 \tanh(\tfrac{1}{2}j^1\beta) - j^2 \tanh(\tfrac{1}{2}j^2\beta). \tag{4.200}$$

For small angular momenta, which is the case of primary interest here, the solution is $\beta \cong 2\pi/\sqrt{N+1}$. For $(\left|j^1\right|, \left|j^2\right|) = \mathcal{O}(N)$ the stationary point is at

$$\beta \cong \frac{2\pi}{\sqrt{N+1-|j^1|-|j^2|}}. \tag{4.201}$$

Note that $\left|j^1\right| + \left|j^2\right| \leq N$. This latter stationary point is equally applicable to small angular momenta, and thus one can write the asymptotic density of states as

$$d_{N,j^1,j^2} \cong \frac{1}{4(N+1-j)^{31/4}} \exp\left[\frac{2\pi(2N-j)}{\sqrt{N+1-j}}\right] \frac{1}{\cosh^2(\frac{\pi j^1}{\sqrt{N+1-j}})\cosh^2(\frac{\pi j^2}{\sqrt{N+1-j}})}, \tag{4.202}$$

where $j = \left|j^1\right| + \left|j^2\right|$. The constant of proportionality is fixed by the state with $j^1 = N$, $j^2 = 0$ being unique. Note that the commuting generators $(j_3, \bar{j}_3)$ of $(SU(2)_L, SU(2)_R)$ respectively are related to the rotations in the 1-2 and 3-4 planes via $j_3 = \frac{1}{2}(j^1 + j^2)$ and $\bar{j}_3 = \frac{1}{2}(j^1 - j^2)$. The total number of states at level N is

$$d_N \cong \frac{1}{N^{27/4}} \exp(4\pi\sqrt{N}), \tag{4.203}$$

and thus the density of states with zero angular momenta differs from the total number of states only by a factor of $1/N$; the exponential growth with N is the same.

Chapter 5

Precision holography of non-conformal branes

The last two chapters of this thesis apply the techniques of precision holography in a slightly different context: In chapter 5 we set up the holographic dictionary for non-conformal branes. We use these results in chapter 6 to extend the correspondence between gravitational fluctations of the black D3 brane geometry and the hydrodynamics of the dual field theory to non-conformal branes.

(5.1) Introduction

Recall from the introductory sections 1.3 that, in order to promote the bulk/boundary correspondence from a formal relation to a framework in which one can calculate, one needs to specify how divergences on both sides are treated. In the boundary theory, these are the UV divergences, which are dealt with by standard techniques of renormalization. In the bulk, the divergences are due to the infinite volume, and are thus IR divergences, which need to be dealt with by holographic renormalization, the precise dual of standard QFT renormalization [13, 14, 15, 16, 17, 18, 19, 20]; for a review see [21]. The procedure of holographic renormalization in asymptotically AdS spacetimes allows one to extract the renormalized one point functions for local gauge invariant operators from the asymptotics of the spacetime; these can then be functionally differentiated in the standard way to obtain higher correlation functions.

By now there are many other conjectured examples of gravity/gauge theory dualities in string theory, which involve backgrounds with different asymptotics. The case of interest for us is the dualities involving non-conformal branes [88, 89] which follow from decoupling limits, and are thus believed to hold, although rather few quantitative checks of the dualities have been carried out. It is important to develop our understanding of these dualities for a number of reasons. First of all, a primary question in quantum gravity is whether the theory is holographic. Examples such as AdS/CFT indicate that the theory is indeed holographic for certain spacetime asymptotics, but one wants to know whether this holds more generally. Exploring cases where the asymptotics are different but one has a proposal for the dual field theory is a first step to addressing this question.

Secondly, the cases mentioned are interesting in their own right and have many useful applications. For example, one of the major aims of work in gravity/gauge dualities is to find holographic models which capture features of QCD. A simple model which includes confinement and chiral symmetry breaking can be obtained from the decoupling limit of a D4-brane background, with D8-branes added to include flavor, the Witten-Sakai-Sugimoto model [90, 91, 92]. This model has been used extensively to extract strong coupling behavior as a model for that in QCD. More generally, non-conformal p-brane backgrounds with $p = 0, 1, 2$ may have interesting unexploited applications to condensed matter physics; the conformal backgrounds have proved useful in modeling strong coupling behavior of transport properties and the non-conformal examples may be equally useful.

The non-conformal brane dualities have not been extensively tested, although some checks of the duality can be found in [93, 94, 95, 96] whilst the papers [97, 98, 99] discuss the underlying symmetry structure on both sides of the correspondence. Recently, there has been progress in using lattice methods to extract field theory quantities, particularly for the D0-branes [100]. Comparing these results to the holographic predictions serves both to test the duality, and conversely to test lattice techniques (if one assumes the duality holds).

Given the increasing interest in these gravity/gauge theory dualities, one would like to develop precision holography for the non-conformal branes, following the same steps as in AdS: one wants to know exactly how quantum field theory data is encoded in the asymptotics of the spacetime. Precision holography has not previously been extensively developed for non-conformal branes (see however [101, 102, 103, 104, 105]), although as we will see the analysis is very close to the analysis of the Asymptotically AdS case. The reason is that the non-conformal branes admit a generalized conformal symmetry [97, 98, 99]: there is an underlying conformal symmetry structure of the theory, provided that the string coupling (or in the gauge theory, the Yang-Mills coupling) is transformed as a background field of appropriate dimension under conformal transformations. Whilst this is not a symmetry in the strict sense of the word, the underlying structure can be used to derive Ward identities and perhaps even prove non-renormalization theorems.

In this chapter we develop in detail how quantum field theory data can be extracted from the asymptotics of non-conformal brane backgrounds. We begin in section 5.2 by recalling the correspondence between non-conformal brane backgrounds and quantum field theories. We also introduce the dual frame, in which the near horizon metric is $AdS_{p+2} \times S^{8-p}$. In section 5.3 we give the field equations in the dual frame for both D-brane and fundamental string solutions.

In the near horizon region of the supergravity solutions conformal symmetry is broken only by the dilaton profile. This means that the background admits a generalized conformal structure: it is invariant under generalized conformal transformations in which the string coupling is also transformed. This generalized conformal structure and its implications are discussed in section 5.4.

Next we proceed to set up precision holography. The basic idea is to obtain the most general asymptotic solutions of the field equations with appropriate Dirichlet boundary conditions. Given such solutions, one can identify the divergences of the onshell action, find the corresponding counterterms and compute the holographic 1-point functions, in complete generality and at the non-linear level. This is carried out in section 5.5. In particular, we give renormalized one point functions for the stress energy tensor and the gluon operator, in the presence of general sources, for all cases.

In section 5.6 we proceed to develop a radial Hamiltonian formulation for the holographic renormalization. As in the asymptotically AdS case, the Hamiltonian formulation is more elegant and exhibits clearly the underlying generalized conformal structure. In the following sections, 5.7 and 5.8, we give a number of applications of the holographic formulae. In particular, in section 5.7 we compute two point functions and in section 5.8 we compute condensates in Witten's model of holographic QCD and the renormalized action, mass etc. in a non-extremal D1-brane background.

In section 5.9 we give conclusions and a summary of our results. The appendices 5.A.1, 5.A.2, 5.A.3 and 5.A.4 contain a number of useful formulae and technical details. Appendix 5.A.1 summarizes useful formulae for the expansion of the curvature whilst appendix 5.A.2 discusses the holographic computation of the stress energy tensor for asymptotically AdS_{D+1}, with $D = 4, 6$; in the latter the derivation is streamlined, relative to earlier discussions, and the previously unknown traceless, covariantly constant contributions to the stress energy tensor in six dimensions are determined. Appendix 5.A.3 contains the detailed relationship between the M5-brane and D4-brane holographic analysis whilst appendix 5.A.4 gives explicit expressions for the asymptotic expansion of momenta.

(5.2) NON-CONFORMAL BRANES AND THE DUAL FRAME

Let us begin by recalling the brane solutions of supergravity, see for example [106] for a review. The relevant part of the supergravity action in the string frame is

$$S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} \left[e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12} H_3^2) - \frac{1}{2(p+2)!} F_{p+2}^2 \right]. \tag{5.1}$$

The Dp-brane solutions can be written in the form:

$$\begin{aligned} ds^2 &= (H^{-1/2}ds^2(E^{p,1}) + H^{1/2}ds^2(E^{9-p})); \\ e^{\phi} &= g_s H^{(3-p)/4}; \\ C_{0\cdots p} &= g_s^{-1}(H^{-1}-1) \qquad \text{or} \qquad F_{8-p} = g_s^{-1} *_{9-p} dH, \end{aligned} \tag{5.2}$$

where the latter depends on whether the brane couples electrically or magnetically to the field strength. Here g_s is the string coupling constant. We are interested in the simplest supersymmetric solutions, for which the defining function H is harmonic on the flat space E^{9-p} transverse to the brane. Choosing a single-centered harmonic function

$$H = 1 + \frac{Q_p}{r^{7-p}}, \tag{5.3}$$

then the parameter Q_p for the brane solutions of interest is given by $Q_p = d_p N g_s l_s^{7-p}$ with the constant d_p equal to $d_p = (2\sqrt{\pi})^{5-p}\Gamma(\frac{7-p}{2})$, whilst $l_s^2 = \alpha'$ and N denotes the integral quantized charge.

Soon after the AdS/CFT duality was proposed [5], it was suggested that an analogous correspondence exists between the near-horizon limits of non-conformal D-brane backgrounds and (non-conformal) quantum field theories [88]. More precisely, one considers the field theory (or decoupling) limit to be:

$$g_s \to 0, \qquad \alpha' \to 0, \qquad U \equiv \frac{r}{\alpha'} = \text{fixed}, \qquad g_d^2 N = \text{fixed}, \tag{5.4}$$

where g_d^2 is the Yang-Mills coupling, related to the string coupling by

$$g_d^2 = g_s (2\pi)^{p-2} (\alpha')^{(p-3)/2}. \tag{5.5}$$

Note that N can be arbitrary for $p < 3$ but (5.4) requires that $N \to \infty$ when $p > 3$. The decoupling limit implies that the constant part in the harmonic function is negligible:

$$H = 1 + \frac{D_p g_d^2 N}{\alpha'^2 U^{7-p}} \quad \Rightarrow \quad \frac{1}{\alpha'^2} \frac{D_p g_d^2 N}{U^{7-p}}, \tag{5.6}$$

where $D_p \equiv d_p (2\pi)^{2-p}$.

The corresponding dual $(p+1)$-dimensional quantum field theory is obtained by taking the low energy limit of the $(p+1)$-dimensional worldvolume theory on N branes. In the case of the *Dp*-branes this theory is the dimensional reduction of $\mathcal{N} = 1$ SYM in ten dimensions. Recall that the action of ten-dimensional SYM is given by

$$S_{10} = \int d^{10}x \sqrt{-g} \text{Tr} \left(-\frac{1}{4g_{10}^2} F_{mn}F^{mn} + \frac{i}{2}\bar{\psi}\Gamma^m [D_m, \psi] \right), \tag{5.7}$$

with $D_m = \partial_m - iA_m$. The dimensional reduction to d dimensions gives the bosonic terms

$$S_d = \int d^d x \sqrt{-g} \text{Tr} \left(-\frac{1}{4g_d^2} F_{ij}F^{ij} - \frac{1}{2} D_i X D^i X + \frac{g_d^2}{4} [X, X]^2 \right) \tag{5.8}$$

where $i = 0, \cdots (d-1)$ and there are $(9-p)$ scalars X. The fermionic part of the action will not play a role here. Note that the Yang-Mills coupling in $d = (p+1)$ dimensions, g_d^2, has (length) dimension $(p-3)$, and thus the theory is not renormalizable for $p > 3$. Since the coupling constant is dimensionful, the effective dimensionless coupling constant $g_{eff}^2(E)$ is

$$g_{eff}^2(E) = g_d^2 N E^{p-3}. \tag{5.9}$$

at a given energy scale E.

This discussion of the decoupling limit applies to D-branes, but we will also be interested in fundamental strings. The fundamental string solutions can be written in the form:

$$\begin{aligned} ds^2 &= (H^{-1} ds^2(E^{1,1}) + ds^2(E^8)); \\ e^{\phi} &= g_s H^{-1/2}; \\ B_{01} &= (H^{-1} - 1), \end{aligned} \tag{5.10}$$

where the harmonic function $H = 1 + Q_{F1}/r^6$ with $Q_{F1} = d_1 N g_s^2 l_s^6$. For completeness, let us also mention that the NS5-brane solutions can be written in the form:

$$\begin{aligned} ds^2 &= (ds^2(E^{1,5}) + H ds^2(E^4)); \\ e^{\phi} &= g_s H^{1/2}; \\ H_3 &= *_4 dH, \end{aligned} \tag{5.11}$$

where the harmonic function $H = 1 + Q_{NS5}/r^2$ with $Q_{NS5} = N l_s^2$.

Whilst the fundamental string solutions have a near string region which is conformal to $AdS_3 \times S^7$ with a linear dilaton, they do not appear to admit a decoupling limit like the one in (5.4) which decouples the asymptotically flat region of the geometry and has a clear meaning from the worldsheet point of view. Nonetheless one can discuss holography for such conformally $AdS_3 \times S^7$ linear dilaton backgrounds, using S duality and the relation to M2-branes: IIB fundamental strings can be included in the discussion by applying S duality to the D1 brane case, and IIA fundamental strings by using the fact they are related to M2 branes wrapped on the M-theory circle.

In the cases of Dp-branes the decoupled region is conformal to $AdS_{p+2} \times S^{8-p}$ and there is a non-vanishing dilaton. The same holds for the near string region of the fundamental string solutions. This implies that there is a Weyl transformation such that the metric is exactly $AdS_{p+2} \times S^{8-p}$. This Weyl transformation brings the string frame metric g_{st} to the so-called *dual frame* metric g_{dual} [89] and is given by

$$ds_{dual}^2 = (N e^{\phi})^c ds_{st}^2, \tag{5.12}$$

with

$$c = -\frac{2}{(7-p)} \qquad \text{Dp}. \tag{5.13}$$

In this frame the action is

$$S = \frac{N^2}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} (Ne^{\phi})^{\gamma} (R + 4\frac{(p-1)(p-4)}{(7-p)^2} (\partial\phi)^2 - \frac{1}{2(8-p)!N^2} F_{8-p}^2). \quad (5.14)$$

with $\gamma = 2(p-3)/(7-p)$. It is convenient to express the field strength magnetically; for $p < 3$ this should be interpreted as $F_{p+2} = *F_{8-p}$, with the Hodge dual being taken in the string frame metric. The terminology dual frame has the following origin. Each p-brane couples naturally to a $(p+1)$ potential. The corresponding (Hodge) dual field strength is an $(8-p)$ form. In the dual frame this field strength and the graviton couple to the dilaton in the same way. For example the dual frame of the NS5 branes is the string frame: the dual $(8-p)$ form is H_3 and the metric and H_3 couple the same way to the dilaton in the string frame, as can be seen from (5.1).[1]

The D5-brane behaves qualitatively differently, as the solution in the dual frame is a linear dilaton background with metric $E^{5,1} \times R \times S^3$:

$$\begin{aligned} ds^2_{dual} &= ds^2(E^{5,1}) + Q\left(\frac{dr^2}{r^2} + d\Omega_3^2\right); \\ e^{\phi} &= \frac{r}{\sqrt{Q}}; \qquad F_3 = Q d\Omega_3. \end{aligned} \quad (5.15)$$

Holography for both D5 and NS5 branes involves such linear dilaton background geometries, and will not be discussed further in this thesis.

Here we will interested in precision holography for the cases where the geometry is conformal to $AdS_{p+2} \times S^{8-p}$; this encompasses Dp-branes with $p = 0, 1, 2, 3, 4, 6$. In all such cases the dual frame solution takes the form

$$\begin{aligned} ds^2_{dual} &= \alpha' d_p^{\frac{2}{(7-p)}} \left(D_p^{-1} (g_d^2 N)^{-1} U^{5-p} ds^2(E^{p,1}) + \frac{dU^2}{U^2} + d\Omega_{8-p}^2 \right); \\ e^{\phi} &= \frac{1}{N} (2\pi)^{2-p} D_p^{(3-p)/4} \left((g_d^2 N) U^{p-3} \right)^{(7-p)/4}, \end{aligned} \quad (5.16)$$

with the field strength being

$$F_{8-p} = (7-p) d_p N (\alpha')^{(7-p)/2} d\Omega_{8-p}. \quad (5.17)$$

Note that the factors of α' cancel in the effective supergravity action, with only dependence on the dimensionful 't Hooft coupling and N remaining.

[1]The dual frame was originally introduced in [107] and the rational behind its introduction was the following. If one has a formulation where the fundamental degrees of freedom are p-branes that couple electrically to a p-form, then one expects there to exist non-singular magnetic solitonic solutions. For example, for perturbative strings, where the elementary objects are strings, the corresponding magnetic objects, the NS5 branes, indeed appear as solitonic objects. Moreover, the target space metric and the B field couple to the the dilaton in the same way, so the low energy effective action is in the string frame. In a formulation where the elementary degrees of freedom are p-branes one would anticipate that there exist smooth solitonic $(6-p)$-brane solutions of the effective action in the p-frame, which is precisely the dual frame. Indeed, the spacetime metric of Dp-branes when expressed in the dual frame is non-singular. We should note though that there is currently no formulation of string theory where p-branes appear to be the elementary degrees of freedom. Other special properties of the dual frame solutions are discussed in [108, 109].

Changing the variable,

$$u^2 = \mathcal{R}^{-2}(D_p g_d^2 N)^{-1} U^{5-p}, \qquad \mathcal{R} = \frac{2}{5-p}, \tag{5.18}$$

brings the AdS metric into the standard form

$$\begin{aligned} ds^2_{dual} &= \alpha' d_p^{\frac{2}{7-p}} \left[\mathcal{R}^2 \left(\frac{du^2}{u^2} + u^2 ds^2(E^{p,1}) \right) + d\Omega^2_{8-p} \right], \\ e^{\phi} &= \frac{1}{N}(2\pi)^{2-p}(g_d^2 N)^{\frac{(7-p)}{2(5-p)}} D_p^{\frac{(3-p)}{2(p-5)}} \left(\mathcal{R}^2 u^2\right)^{\frac{(p-3)(p-7)}{4(p-5)}} . \end{aligned} \tag{5.19}$$

with the field strength being (5.17). Note that by rescaling the metric, dilaton and field strength as

$$ds^2_{dual} = \alpha' d_p^{\frac{2}{7-p}} \tilde{ds}^2; \quad Ne^{\phi} = (2\pi)^{2-p}(g_d^2 N)^{\frac{(7-p)}{2(5-p)}} D_p^{\frac{(3-p)}{2(p-5)}} e^{\tilde{\phi}}; \quad F_{8-p} = d_p N(\alpha')^{(7-p)/2} \tilde{F}_{8-p}.$$

the factors of D_p, N and the 't Hooft coupling can be absorbed into the overall normalization of the action.

It has been argued in [89] that the dual frame is the holographic frame in the sense that the radial direction u in this frame is identified with the energy scale of the boundary theory,

$$u \sim E. \tag{5.20}$$

More properly, as we will discuss later, the dilatations of the boundary theory are identified with rescaling of the u coordinate. Using (5.20) and (5.9) the dilaton in (5.19) and for the case of D-branes becomes

$$e^{\phi} = \frac{1}{N} c_d \left(g^2_{eff}(u)\right)^{\frac{7-p}{2(5-p)}}, \qquad c_d = (2\pi)^{2-p} D_p^{\frac{(p-3)}{2(5-p)}} \mathcal{R}^{\frac{(p-3)(7-p)}{2(5-p)}}. \tag{5.21}$$

The validity of the various approximations was discussed in [88, 110, 89]. In particular, we consider the large N limit, keeping fixed the effective coupling constant g^2_{eff}, so the dilaton is small in all cases (recall that the decoupling limit when $p > 3$ requires $N \to \infty$). If $g^2_{eff} \ll 1$ then the perturbative SYM description is valid, whereas in the opposite limit $g^2_{eff} \gg 1$ the supergravity approximation is valid.

As a consistency check, one can also derive (5.21) using the open string description. The low energy description in the string frame is given by

$$S_{st} = -\frac{1}{(2\pi)^{p-2}(\alpha')^{(p-3)/2}} \int d^{p+1}x \sqrt{-g_{st}} e^{-\phi} \frac{1}{4} \mathrm{Tr}(F_{ij}F_{kl}) g^{ik}_{st} g^{jl}_{st} + \cdots, \tag{5.22}$$

where we indicate explicitly that the metric involved is the string frame metric. In the case of flat target spacetime, g_{st} is the Minkowski metric and $e^{\phi} = g_s$ and we recover (5.5) by identifying the overall prefactor of $\mathrm{Tr}F^2$ with $1/(4g_d^2)$. In our case, transforming to the dual frame and using the form of the metric in (5.19) we get

$$S_{dual} = -\frac{\mathcal{R}^{p-3} d_p^{\frac{(p-3)}{(7-p)}}}{(2\pi)^{p-2}} \int d^{p+1}x (Ne^{\phi})^{\frac{2(p-5)}{(7-p)}} (Nu^{p-3}) \frac{1}{4} (\mathrm{Tr}F^2) + \cdots \tag{5.23}$$

where now the Lorentz index contractions in $\mathrm{Tr}F^2$ are with the Minkowski metric. Identifying now the overall prefactor of $\mathrm{Tr}F^2$ with $1/(4g_d^2)$ is indeed equivalent to (5.21).

As mentioned above, we will also include fundamental strings in our analysis, exploiting the relation to D1-branes and M2-branes. In this case we focus on the near string geometry, dropping the constant term in the harmonic function, and introduce a dual frame metric $ds^2_{dual} = (Ne^{\phi})^c ds^2_{st}$ with

$$c = -\frac{2}{3} \qquad \mathrm{F1}, \tag{5.24}$$

with the dual frame metric being $AdS_3 \times S^7$. The detailed form of the effective action in the dual frame will be given in the next section.

The aim of this chapter will be to consider solutions which asymptote to the decoupled non-conformal brane backgrounds and show how renormalized quantum field theory information can be extracted from the geometry. It may be useful to recall first how the conformal case of $p = 3$ works. Given the $AdS_5 \times S^5$ background, the spectrum of supergravity fluctuations about this background corresponds to the spectrum of single trace gauge invariant chiral primary operators in the dual $\mathcal{N} = 4$ SYM theory. The spectrum includes stringy modes and D-branes, which correspond to other non primary, high dimension and non-local operators in the dual $\mathcal{N} = 4$ SYM theory. Encoded in the asymptotics of any asymptotically $AdS_5 \times S^5$ supergravity background are one point functions of the chiral primary operators. These allow one to extract the vacuum structure of the dual theory (its vevs and deformation parameters), and if one switches on sources one can also extract higher correlation functions.

The sphere in this background has a radius which is of the same order as the AdS radius, so the higher KK modes are not suppressed relative to the zero modes and one cannot ignore them. It is nevertheless possible to only keep a subset of modes when the equations of motion admit solutions with all modes except the ones kept set equal to zero, i.e. there exist consistent truncations. The existence of such truncations signify the existence of a subset of operators of the dual theory that are closed under OPEs. The resulting theory is a $(d+1)$-dimensional gauged supergravity and such gauged supergravity theories have been the starting point for many investigations in AdS/CFT. Gauged supergravity retains only the duals to low dimension chiral primaries in SYM, those in the same multiplet as the stress energy tensor. More recently, the method of Kaluza-Klein holography [22, 57] has been developed to extract systematically one point functions of all other single trace chiral operators.

The goal here is to take the first step in holographic renormalization for non-conformal branes. We will consistently truncate the bulk theory to just the $(p+2)$-dimensional graviton and the dilaton, and compute renormalized correlation functions in this sector. Unlike the $p = 3$ case one must retain the dilaton as it is running: the gauge coupling of the dual theory is dimensionful and runs. Such a truncation was considered already in [89] and we will recall the resulting $(p+2)$-dimensional action in the next section. Given an understanding of holographic renormalization in this truncated sector, it is straightforward to generalize this setup to include fields dual to other gauge theory operators.

(5.3) Lower dimensional field equations

The supergravity solutions for Dp-branes and fundamental strings in the decoupling limit can be best analyzed by going to the *dual frame* reviewed in the previous section, (5.12) and (5.24). The dual frame is defined as $ds^2_{dual} = (Ne^{\phi})^c ds^2$, with $c = -2/(7-p)$ for Dp-branes and $c = -2/3$ for fundamental strings. The Weyl transformation to the dual frame in ten dimensions results in the following action:

$$S = -\frac{N^2}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{g} N^{\gamma} e^{\gamma\phi} [R + \beta(\partial\phi)^2 - \frac{1}{2(8-p)!N^2} |F_{8-p}|^2] \tag{5.25}$$

where the constants (β, γ) are given below in (5.29) for Dp-branes and (5.30) for fundamental strings respectively. Note that it is convenient to express the field strength magnetically; for $p < 3$ this should be interpreted as $F_{p+2} = *F_{8-p}$. From here onwards we will also work in Euclidean signature.

For $p \neq 5$, the field equations in this frame admit $AdS_{p+2} \times S^{8-p}$ solutions with linear dilaton. One can reduce the field equations over the sphere, truncating to the $(p+2)$-dimensional graviton $\tilde{g}_{\mu\nu}$ and scalar $\tilde{\phi}$. For the Dp-branes the reduction ansatz is

$$\begin{aligned} ds^2_{dual} &= \alpha' d_p^{-c} (\mathcal{R}^2 \tilde{g}_{\mu\nu}(x^{\rho}) dx^{\mu} dx^{\nu} + d\Omega^2_{8-p}); \\ F_{8-p} &= (7-p) g_s^{-1} Q_p d\Omega_{8-p}; \\ e^{\phi} &= g_s (r_o^2 \mathcal{R}^2)^{(p-3)(7-p)/4(5-p)} e^{\tilde{\phi}}, \end{aligned} \tag{5.26}$$

with $r_o^{7-p} \equiv Q_p$ and $\mathcal{R} = 2/(5-p)$. The ten-dimensional metric is in the dual frame and prefactors are chosen to absorb the radius and overall metric and dilaton prefactors of the AdS_{p+2} solution. For the fundamental string one reduces the near horizon geometry as:

$$\begin{aligned} ds^2_{dual} &= \alpha' (d_1 N^{-1})^{1/3} (\mathcal{R}^2 \tilde{g}_{\mu\nu}(x^{\rho}) dx^{\mu} dx^{\nu} + d\Omega^2_7); \\ H_7 &= 6 Q_{F1} d\Omega_7; \\ e^{\phi} &= g_s (r_o \mathcal{R})^{3/2} e^{\tilde{\phi}}, \end{aligned} \tag{5.27}$$

where $H_7 = *H_3$, $r_o^6 \equiv Q_{F1}$ and $\mathcal{R} = 2/(5-p)$. It is then straightforward to show that the equations of motion for the lower-dimensional fields for both Dp-branes and fundamental strings follow from an action of the form:

$$S = -L \int d^{d+1}x \sqrt{\tilde{g}} e^{\gamma\tilde{\phi}} [\tilde{R} + \beta(\partial\tilde{\phi})^2 + C]. \tag{5.28}$$

Here $d = p+1$ and the constants (L, β, γ, C) depend on the case of interest; since from here onwards we are interested only in $(d+1)$-dimensional fields we suppress their tilde labeling. For Dp-branes the constants are given by

$$\begin{aligned} \gamma &= \frac{2(p-3)}{7-p}, \qquad \beta = \frac{4(p-1)(p-4)}{(7-p)^2}, \\ \mathcal{R} &= \frac{2}{5-p}, \qquad C = \frac{1}{2}(9-p)(7-p)\mathcal{R}^2, \\ L &= \frac{\Omega_{8-p} r_o^{(7-p)^2/(5-p)} \mathcal{R}^{(9-p)/(5-p)}}{(2\pi)^7 \alpha'^4} = \frac{(d_p N)^{(7-p)/(5-p)} g_d^{2(p-3)/(5-p)} \mathcal{R}^{(9-p)/(5-p)}}{64\pi^{(5+p)/2} (2\pi)^{(p-3)(p-2)/(5-p)} \Gamma(\frac{9-p}{2})}. \end{aligned} \tag{5.29}$$

For the fundamental string one gets instead:

$$\begin{aligned}\gamma &= \frac{2}{3}, \qquad \beta = 0, \qquad C = 6, \\ L &= \frac{\Omega_7 r_o^9}{4(2\pi)^7 g_s^2 (\alpha')^4} = \frac{g_s N^{3/2} (\alpha')^{1/2}}{6\sqrt{2}},\end{aligned} \tag{5.30}$$

This expression is related to that for the D1-brane background by $g_s \to 1/g_s$ with $\alpha' \to \alpha' g_s$, as one would expect from S duality. The truncation is consistent, as one can show that any solution of the lower-dimensional equations of motion also solves the ten-dimensional equations of motion, using the reduction given in (5.26). Note that more general reductions of type II theories on spheres to give gauged supergravity theories were discussed in [111]. These reductions would be relevant if one wants to include additional operators in the boundary theory, beyond the stress energy tensor and scalar operator.

In both cases the equations of motion admit an AdS_{d+1} solution

$$\begin{aligned}ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{dx_i dx^i}{\rho}; \\ e^{\phi} &= \rho^{\alpha},\end{aligned} \tag{5.31}$$

where $i = 1, \dots, d$. Note that ρ is related to the radial coordinate u used earlier by $\rho = 1/u^2$. The constant α again depends on the case of interest:

$$\begin{aligned}\alpha &= -\frac{(p-7)(p-3)}{4(p-5)}; \qquad \text{Dp} \\ \alpha &= -\frac{3}{4}; \qquad \text{F1}.\end{aligned} \tag{5.32}$$

Note that for computational convenience the metric and dilaton have been rescaled relative to [89] to set the AdS radius to one and to pull all factors of N and g_s into an overall normalization factor. The radial variable ρ then has length dimension 2 and e^{ϕ} has length dimension 2α.

For arbitrary d, β and γ, the field equations for the metric and scalar field following from (5.28) are [2]

$$\begin{aligned}-R_{\mu\nu} + (\gamma^2 - \beta)\partial_\mu\phi\partial_\nu\phi + \gamma\nabla_\mu\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}[R + (\beta - 2\gamma^2)(\partial\phi)^2 - 2\gamma\nabla^2\phi + C] = 0, \\ \gamma R - \beta\gamma(\partial\phi)^2 + C\gamma - 2\beta\nabla^2\phi = 0.\end{aligned} \tag{5.33}$$

These equations admit an AdS solution with linear dilaton provided that α and C satisfy

$$\alpha = -\frac{\gamma}{2(\gamma^2 - \beta)}, \qquad C = \frac{(d(\gamma^2 - \beta) + \gamma^2)(d(\gamma^2 - \beta) + \beta)}{(\gamma^2 - \beta)^2}. \tag{5.34}$$

We can thus treat both Dp-brane and fundamental string cases simultaneously, by processing the field equations for arbitrary (d, β, γ) and writing (α, C) in terms of these parameters. It

[2]Our conventions for the Riemann and Ricci tensor are $R^{\sigma}{}_{\mu\nu\rho} = -2\Gamma^{\sigma}_{\mu[\nu,\rho]} - 2\Gamma^{\tau}_{\mu[\nu}\Gamma^{\sigma}_{\rho]\tau}$, $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$.

might be interesting to consider whether other choices of (d, β, γ) admit interesting physical interpretations.

By taking the trace of the first equation in (5.33) and combining it with the second one can obtain the more convenient three equations

$$\begin{aligned} -R_{\mu\nu} + (\gamma^2 - \beta)\partial_\mu\phi\partial_\nu\phi + \gamma\nabla_\mu\partial_\nu\phi - \frac{\gamma^2 + d(\gamma^2 - \beta)}{\gamma^2 - \beta} g_{\mu\nu} &= 0, \\ \nabla^2\phi + \gamma(\partial\phi)^2 - \frac{\gamma(d(\gamma^2 - \beta) + \gamma^2)}{(\gamma^2 - \beta)^2} &= 0, \\ R + \beta(\partial\phi)^2 + \frac{(d(\gamma^2 - \beta) + \gamma^2)(d(\gamma^2 - \beta) - \beta)}{(\gamma^2 - \beta)^2} &= 0, \end{aligned} \tag{5.35}$$

where the last line follows from the first two.

The type IIA fundamental strings and D4-branes are related to the M theory M2-branes and M5-branes respectively under dimensional reduction along a worldvolume direction. The M brane theories fall within the framework of AdS/CFT, with the correspondence being between $AdS_4 \times S^7$ and $AdS_7 \times S^4$ geometries, respectively, and the still poorly understood conformal worldvolume theories. Reducing on the spheres gives four and seven dimensional gauged supergravity, respectively, which can be truncated to Einstein gravity with negative cosmological constant. That is, the effective actions are simply

$$S_M = -L_M \int d^{d+2}x\sqrt{G}\,(R(G) + d(d+1))\,, \tag{5.36}$$

where $d = 2$ for the M2-brane and $d = 5$ for the M5-brane. The normalization constant is

$$L_{M2} = \frac{\sqrt{2}N^{3/2}}{24\pi}; \qquad L_{M5} = \frac{N^3}{3\pi^3}. \tag{5.37}$$

and the action clearly admits an AdS_{d+2}-dimensional space with unit radius as a solution:

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(dx_i dx^i + dy^2), \tag{5.38}$$

where $i = 1, \cdots, d$.

Now consider a diagonal dimensional reduction of the $(d+2)$-dimensional solution over y, i.e. let the metric be

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{4\phi(x)/3}dy^2. \tag{5.39}$$

Substituting into the $(d+2)$-dimensional field equations gives precisely the field equations following from the action (5.28); note that $\gamma = 2/3, \beta = 0$ for both the fundamental string and D4-branes. It may be useful to recall here that the standard dimensional reduction of an M theory metric to a (string frame) type IIA metric g_{MN} is

$$ds_{11}^2 = e^{-2\phi/3}g_{MN}dx^M dx^N + e^{4\phi/3}dy_{11}^2. \tag{5.40}$$

The relation between dual frame and string frame metrics given in (5.12) leads to (5.39). Note that

$$L = L_M(2\pi R_y) = 2\pi g_s l_s L_M, \tag{5.41}$$

where we use the standard relation for the radius of the M theory circle.

The other Dp-branes of type IIA are of course also related to M theory objects: the D0-brane background uplifts to a gravitational wave background, the D6-brane background uplifts to a Kaluza-Klein monopole background whilst the D2-branes are related to the reduction of M2-branes transverse to the worldvolume. These connections will not play a role in this thesis. The uplifts reviewed above are useful here as holographic renormalization for the conformal branes is well understood, but holography for gravitational wave backgrounds and Kaluza-Klein monopoles is less well understood than that for the non-conformal branes.

One could use a different reduction and truncation of the theory in the $AdS_4 \times S^7$ background to obtain the action (5.28) for D2-branes. In this case one would embed the M theory circle into the S^7, and then truncate to only the four-dimensional graviton, along with the scalar field associated with this M theory circle. This reduction will not however be used here.

(5.4) Generalized conformal structure

In this section we will discuss the underlying generalized conformal structure of the non-conformal brane dualities. Recall that the corresponding worldvolume theory is SYM$_{p+1}$. We will be interesting in computing correlation functions of gauge invariant operators in this theory. Recall that gauge/gravity duality maps bulk fields to boundary operators. In our discussion in the previous section we truncated the bulk theory to gravity coupled to a scalar field in $(d+1)$ dimensions. The bulk metric corresponds to the stress energy tensor as usual, while as we will see the scalar field corresponds to a scalar operator of dimension four. As usual the fields that parametrize their boundary conditions are identified with sources that couple to gauge invariant operators.

Consider the following $(p+1)$-dimensional (Euclidean) action,

$$\begin{aligned} S_d[g_{(0)ij}(x), \Phi_{(0)}(x)] \quad = \quad -\int d^dx\sqrt{g_{(0)}}\Bigg(-\Phi_{(0)}\frac{1}{4}\mathrm{Tr}F_{ij}F^{ij} + \frac{1}{2}\mathrm{Tr}\left(X(D^2 - \frac{(d-2)}{4(d-1)}R)X\right) \\ + \frac{1}{4\Phi_{(0)}}\mathrm{Tr}[X,X]^2\Bigg). \end{aligned} \tag{5.42}$$

where $g_{(0)ij}$ is a background metric $\Phi_{(0)}(x)$ is a scalar background field. Setting

$$g_{(0)ij} = \delta_{ij}, \qquad \Phi_{(0)} = \frac{1}{g_d^2}, \tag{5.43}$$

the action (5.42) becomes equal to the action of the SYM$_{p+1}$ given in (5.8) (here and it what follows we suppress the fermionic terms). The action (5.42) is invariant under the following

Weyl transformations

$$g_{(0)} \to e^{2\sigma} g_{(0)}, \quad X \to e^{(1-\frac{d}{2})\sigma} X, \quad A_i \to A_i, \quad \Phi_{(0)} \to e^{-(d-4)\sigma} \Phi_{(0)} \tag{5.44}$$

Note that the combination $P_1 = D^2 - \frac{d-2}{4(d-1)} R$, is the conformal Laplacian in d dimensions, which transforms under Weyl transformations as $P_1 \to e^{-(d/2+1)\sigma} P_1 e^{(d/2-1)\sigma}$.

Let us now define,

$$T_{ij} = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_d}{\delta g_{(0)}^{ij}}, \qquad \mathcal{O} = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_d}{\delta \Phi_{(0)}} \tag{5.45}$$

They are given by

$$\begin{aligned} T_{ij} &= \mathrm{Tr}\left(\Phi_{(0)} F_{ik} F_j{}^k + D_i X D_j X + \frac{d-2}{4(d-1)} (X^2 R_{ij} - D_i D_j X^2 + g_{(0)ij} D^2 X^2) \right. \\ &\quad \left. - g_{(0)ij} \left(\frac{1}{4} \Phi_{(0)} F^2 + \frac{1}{2} (DX)^2 + \frac{(d-2)}{8(d-1)} R X^2 - \frac{1}{4\Phi_{(0)}} [X,X]^2 \right) \right) \qquad (5.46) \\ \mathcal{O} &= \mathrm{Tr}\left(\frac{1}{4} F^2 + \frac{1}{4\Phi_{(0)}^2} [X,X]^2 \right). \qquad (5.47) \end{aligned}$$

Using standard manipulations, see for example [17, 18], we obtain the standard diffeomorphism and trace Ward identities,

$$\nabla^j \langle T_{ij} \rangle_J + \langle \mathcal{O} \rangle_J \partial_i \Phi_{(0)} = 0, \tag{5.48}$$

$$\langle T_i^i \rangle_J + (d-4) \Phi_{(0)} \langle \mathcal{O} \rangle_J = 0, \tag{5.49}$$

where $\langle B \rangle_J$ denotes an expectation value of B in the presence of sources J. One can verify that these relations are satisfied at the classical level, i.e. by using (5.46) and the equations of motion that follow from (5.42). Setting $g_{(0)ij} = \delta_{ij}, \Phi_{(0)} = g_d^{-2}$ one recovers the conservation of the energy momentum tensor of the SYM$_d$ theory and the fact that conformal invariance is broken by the dimensionful coupling constant. Note that the kinetic part of the scalar field does not contribute to the breaking of conformal invariance because this part of the action is conformally invariant in any dimension (using the conformal Laplacian). This also dictates the position of the coupling constant in (5.8). In a flat background one can change the position of the coupling constant by rescaling the fields. For example, by rescaling $X \to X/g_d$ the coupling constant becomes an overall constant. This is the normalization one gets from worldvolume D-brane theory in the string frame. This action however does not generalize naturally to a Weyl invariant action. Instead it is (5.8) (with the coupling constant promoted to a background field) that naturally couples to a metric in a Weyl invariant way.

The Ward identities (5.48) lead to an infinite number of relations for correlation functions obtained by differentiating with respect to the sources and setting the sources to $g_{(0)ij} = \eta_{ij}$, where η_{ij} is the Minkowski metric and $\Phi_{(0)} = 1/g_d^2$. The first non-trivial relations are at the level of 2-point functions ($x \neq 0$).

$$\partial_x^j \langle T_{ij}(x) T_{kl}(0) \rangle = 0, \qquad \partial_x^j \langle T_{ij}(x) \mathcal{O}(0) \rangle = 0 \tag{5.50}$$
$$\langle T_i^i(x) T_{kl}(0) \rangle + (p-3) \frac{1}{g_d^2} \langle \mathcal{O}(x) T_{kl}(0) \rangle = 0$$
$$\langle T_i^i(x) \mathcal{O}(0) \rangle + (p-3) \frac{1}{g_d^2} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle = 0.$$

The Ward identities (5.48) were derived by formal path integral manipulations and one should examine whether they really hold at the quantum level. Firstly, for the case of the D4 brane the worldvolume theory is non-renormalizable, so one might question whether the correlators themselves are meaningful. At weak coupling, renormalizing the correlators would require introducing new higher dimension operators in the action, as well as counterterms that depend on the background fields. This process should preserve diffeomorphism and supersymmetry, but it may break the Weyl invariance. Introducing a new source $\Phi^j_{(0)}$ for every new higher dimension operator $\mathcal{O}_j$ added in the process of renormalization would then modify the trace Ward identity as

$$\langle T_i^i \rangle - \sum_{j \geq 0} (d - \Delta_j) \Phi^j_{(0)} \langle \mathcal{O}_j \rangle = \mathcal{A}, \tag{5.51}$$

where Δ_j is the dimension of the operator $\mathcal{O}_i$ (with $\Phi^0_{(0)} = \Phi_{(0)}, \mathcal{O}_0 = \mathcal{O}, \Delta_0 = 4$). Due to supersymmetry one would anticipate that Δ_i are protected. One would also anticipate that these operators are dual to the KK modes of the reduction over the sphere S^{8-p}. As discussed in the previous section, one can consistently truncate these modes at strong coupling, so the gravitational computation should lead to Ward identities of the form (5.49), up to a possible quantum anomaly $\mathcal{A}$. $\mathcal{A}$ originates from the counterterms that depend on the background fields only ($g_{(0)}, \Phi_{(0)}, \ldots$). In general, $\mathcal{A}$ would be restricted by the Wess-Zumino consistency and therefore should be built from generalized conformal invariants. *We will show the extracted holographic Ward identities, (5.141), indeed agree with (5.48)-(5.49)) with a quantum anomaly only for $p = 4$.*

In a $(p+1)$-dimensional conformal field theory, the entropy S at finite temperature T_H necessarily scales as

$$S = c(g_{YM}^2 N, N, \cdots) V_p T_H^p \tag{5.52}$$

where V_p is the spatial volume, g_{YM} is the coupling, N is the rank of the gauge group, $g_{YM}^2 N$ is the 't Hooft coupling constant and the ellipses denote additional dimensionless parameters. $c(g_{YM}^2 N, N, \cdots)$ denotes an arbitrary function of these dimensionless parameters. In the cases of interest here, scaling indicates that the entropy behaves as

$$S = \tilde{c}((g_{eff}^2(T_H), N, \cdots) V_p T_H^p, \tag{5.53}$$

where $g_{eff}^2(T_H) = g_d^2 N T_H^{p-3}$ is the effective coupling constant and $\tilde{c}((g_d^2 N T_H^{p-3}), N, \cdots)$ denotes a generic function of the dimensionless parameters.

Next let us consider correlation functions, in particular of the gluon operator $\mathcal{O} = -\frac{1}{4}\mathrm{Tr}(F^2 + \cdots)$. In a theory which is conformally invariant the two point function of any operator of dimension Δ behaves as

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = f(g_{YM}^2 N, N, \cdots)\frac{1}{|x-y|^{2\Delta}}, \tag{5.54}$$

where $f(g_{YM}^2 N, N, \cdots)$ denotes an arbitrary function of the dimensionless parameters. Now consider the constraints on a two point function in a theory with generalized conformal invariance; these are far less restrictive, with the correlator constrained to be of the form:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \tilde{f}(g_{eff}^2(x), N, \cdots)\frac{1}{|x|^{2\Delta}}. \tag{5.55}$$

where $g_{eff}^2(x) = g_d^2 N|x|^{3-p}$ and $\tilde{f}(g_{eff}^2(x), N, \cdots)$ is an arbitrary function of these (dimensionless) variables. Note that the scaling dimension of the gluon operator as defined above is 4. Both (5.54) and (5.55) are over-simplified as even in a conformal field theory the renormalized correlators can depend on the renormalization group scale μ. For example, for $p = 3$ the renormalized two point function of the dimension four gluon operator is

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = f(g_{YM}^2 N, N)\Box^3\left(\frac{1}{|x|^2}\log(\mu^2 x^2)\right), \tag{5.56}$$

where note that the renormalized version $\mathcal{R}\frac{1}{|x|^8}$ of $\frac{1}{|x|^8}$ is given by:

$$\mathcal{R}\left(\frac{1}{|x|^8}\right) = -\frac{1}{3\cdot 2^8}\Box^3\left(\frac{1}{|x|^2}\log(\mu^2 x^2)\right). \tag{5.57}$$

$\mathcal{R}(\frac{1}{|x|^8})$ and $\frac{1}{|x|^8}$ are equal when $x \neq 0$ but they differ by infinite renormalization at $x = 0$. In particular, it is only $\mathcal{R}\frac{1}{|x|^8}$ that has a well defined Fourier transform, given by $p^4\log(p^2/\mu^2)$, which may be obtained using the identity

$$\int d^4x e^{ipx}\frac{1}{|x|^2}\log(\mu^2 x^2) = -\frac{4\pi^2}{p^2}\log(p^2/\mu^2). \tag{5.58}$$

(see appendix A, [112]). Thus the correlator in a theory with generalized conformal invariance is

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \mathcal{R}\left(\tilde{f}(g_{eff}^2(x), \mu|x|, N, \cdots)\frac{1}{|x|^{2\Delta}}\right). \tag{5.59}$$

Note that this is of the same form as a two point function of an operator with definite scaling dimension in any quantum field theory; the generalized conformal structure does not restrict it further, although as discussed above the underlying structure does relate two point functions via Ward identities.

The general form of the two point function (5.59) is compatible with the holographic results discussed later. One can also compute the two point function to leading (one loop) order in perturbation theory, giving:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \langle :\mathrm{Tr}(F^2)(x) :: \mathrm{Tr}(F^2)(0):\rangle \sim \mathcal{R}\left(\frac{g_{eff}^4(x)}{|x|^8}\right), \tag{5.60}$$

which is also compatible with the general form. (Note that although the complete operator includes in addition other bosonic and fermionic terms the latter do not contribute to the two point function at one loop, whilst the former contribute only to the overall normalization.) One shows this result as follows. The gauge field propagator for $SU(N)$ in Feynman gauge in momentum space is

$$\langle A^a_{b\mu}(k) A^c_{d\nu}(-k)\rangle = i g_d^2 (\delta^a_d \delta^c_b - \frac{1}{N}\delta^a_b \delta^c_d)\frac{\eta_{\mu\nu}}{|k|^2}, \tag{5.61}$$

where (a, b) are color indices. Then the one loop contribution to the correlation function in momentum space reduces (at large N) to

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim N^2(d-1)|k|^4 \int d^d q \frac{1}{|q|^2|k-q|^2}. \tag{5.62}$$

Using the integral

$$\begin{aligned} I &= \int d^d q \frac{1}{|q|^{2\alpha}|k-q|^{2\beta}} \\ &= \frac{\Gamma(\alpha+\beta-d/2)\Gamma(d/2-\beta)\Gamma(d/2-\alpha)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}|k|^{d-2\alpha-2\beta}, \end{aligned} \tag{5.63}$$

one finds that

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim N^2(g_d^2)^2(d-1)|k|^d \frac{\Gamma(2-d/2)(\Gamma(d/2-1))^2}{\Gamma(d-2)}. \tag{5.64}$$

This is finite for d odd, as expected given the general result that odd loops are finite in odd dimensions; dimensional regularization when d is even results in a two point function of the form $N^2 g_d^4 |k|^d \log(|k^2|)$. Fourier transforming back to position space results in

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \mathcal{R}\left(\frac{g^4_{eff}(x)}{|x|^8}\right), \tag{5.65}$$

where again in even dimensions the renormalized expression is of the type given in (5.57). This is manifestly consistent with the form (5.59).

The structure that we find at weak coupling is also visible at strong coupling. The gravitational solution is the linear dilaton AdS_{d+1} solutions in (5.31) and conformal symmetry is broken only by the dilaton profile. Therefore the background is invariant under generalized conformal transformations in which one also transforms the string coupling g_s appropriately. This *generalized conformal structure* was discussed in [97, 98, 99], particularly in the context of D0-branes.

(5.5) HOLOGRAPHIC RENORMALIZATION

In this section we will determine how gauge theory data is extracted from the asymptotics of the decoupled non-conformal brane backgrounds, following the same steps as in the asymptotically

AdS case. In particular, one first fixes the non-normalizable part of the asymptotics: we will consider solutions which asymptote to a linear dilaton asymptotically locally AdS background. Next one needs to analyze the field equations in the asymptotic region, to understand the asymptotic structure of these backgrounds near the boundary.

Given this analysis, one is ready to proceed with holographic renormalization. Recall that the aim of holographic renormalization is to render well-defined the definition of the correspondence: the onshell bulk action with given boundary values $\Phi_{(0)}$ for the bulk fields acts as the generating functional for the dual quantum field theory in the presence of sources $\Phi_{(0)}$ for operators $\mathcal{O}$. The asymptotic analysis allows one to isolate the volume divergences of the onshell action, which can then be removed with local covariant counterterms, leading to a renormalized action. The latter allows one to extract renormalized correlators for the quantum field theory.

(5.5.1) Asymptotic expansion

In determining how gauge theory data is encoded in the asymptotics of the non-conformal brane backgrounds the first step is to understand the asymptotic structure of these backgrounds in the asymptotic region near $\rho = 0$ where the solution becomes a linear dilaton locally AdS background. Let us expand the metric and dilaton as:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x,\rho)dx^i dx^j}{\rho}, \\ \phi(x,\rho) &= \alpha \log \rho + \frac{\kappa(x,\rho)}{\gamma}, \end{aligned} \tag{5.66}$$

where we expand $g(x,\rho)$ and $\kappa(x,\rho)$ in powers of ρ:

$$\begin{aligned} g(x,\rho) &= g_{(0)}(x) + \rho g_{(2)}(x) + \cdots \\ \kappa(x,\rho) &= \kappa_{(0)}(x) + \rho\kappa_{(2)}(x) + \cdots \end{aligned} \tag{5.67}$$

For $p = 3$ we should instead expand the scalar field as

$$\phi(x,\rho) = \kappa_{(0)}(x) + \rho\kappa_{(2)}(x) + \cdots , \tag{5.68}$$

since $\alpha = \gamma = 0$. Note that by allowing $(g_{(0)}, \kappa_{(0)})$ to be generic the spacetime is only asymptotically locally AdS.

Consider first the case of $p = 3$, so that the action is Einstein gravity in the presence of a negative cosmological constant, and a massless scalar. The latter couples to the dimension four operator $\mathrm{Tr}(F^2)$. The metric is expanded in the Fefferman-Graham form, with the scalar field expanded accordingly. By the standard rules of AdS/CFT $g_{(0)}$ acts as the source for the stress energy tensor and $\kappa_{(0)}$ acts as the source for the dimension four operator, i.e. it corresponds to the Yang-Mills coupling. The vevs of these operators are captured by subleading terms in the asymptotic expansion.

For general p an analogous relationship should hold: $g_{(0)}$ sources the stress energy tensor and the scalar field determines the (dimensionful) gauge coupling. More precisely, the bulk field that is dual to the operator $\mathcal{O}$ in (5.46) is

$$\Phi(x,\rho) = \exp\left(\chi\phi(x,\rho)\right) = \rho^{-\frac{1}{2}(p-3)}\left(\Phi_{(0)}(x) + \rho\Phi_{(2)}(x) + \cdots\right) \qquad (5.69)$$

$$\Phi_{(0)}(x) = \exp\left(-\frac{(p-5)}{(p-3)}\kappa_{(0)}(x)\right) \qquad (5.70)$$

The $\Phi_{(0)}$ appearing here is identified with $\Phi_{(0)}$ in (5.42). It will be convenient however to work on the gravitational side with $\phi(x,\rho)$ instead of $\Phi(x,\rho)$.

In the asymptotic expansion we fix the non-normalizable part of the asymptotics, and the vevs should be captured by subleading terms. One now needs to show that such an expansion is consistent with the equations of motion, and what terms occur in the expansion for given (α,β,γ).

Substituting the scalar and the metric given in (5.66) into the field equations (5.35) gives

$$-\frac{1}{4}\mathrm{Tr}(g^{-1}g')^2 + \frac{1}{2}\mathrm{Tr}g^{-1}g'' + \kappa'' + (1-\frac{\beta}{\gamma^2})(\kappa')^2 = 0, \qquad (5.71)$$

$$-\frac{1}{2}\nabla^i g'_{ij} + \frac{1}{2}\nabla_j(\mathrm{Tr}g^{-1}g') + (1-\frac{\beta}{\gamma^2})\partial_j\kappa\kappa' + \partial_j\kappa' - \frac{1}{2}g_j'^{\,k}\partial_k\kappa = 0, \qquad (5.72)$$

$$\left[-Ric(g) - (d-2-2\alpha\gamma)g' - \mathrm{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g')\right]_{ij}$$
$$+\nabla_i\partial_j\kappa + (1-\frac{\beta}{\gamma^2})\partial_i\kappa\partial_j\kappa - 2(g_{ij} - \rho g'_{ij})\kappa' = 0, \qquad (5.73)$$

$$4\rho(\kappa'' + (\kappa')^2) + (8\alpha\gamma + 2(2-d))\kappa' + \nabla^2\kappa + (\partial\kappa)^2 + 2\mathrm{Tr}(g^{-1}g')(\alpha\gamma + \rho\kappa') = 0, \qquad (5.74)$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g and $d = p+1$ is the dimension of the space orthogonal to ρ. Note that coefficients in these equations are polynomials in ρ implying that this system of equations admits solutions with $g(x,\rho)$ and $\kappa(x,\rho)$ being regular functions of ρ and this justifies (5.67). To solve these equations one may successively differentiate the equations w.r.t. ρ and then set $\rho = 0$.

Let us first recall how these equations are solved in the pure gravity, asymptotically locally AdS_{d+1} case, i.e. when the scalar is trivial. Then the equations become

$$-\frac{1}{4}\mathrm{Tr}(g^{-1}g')^2 + \frac{1}{2}\mathrm{Tr}g^{-1}g'' = 0; \qquad -\frac{1}{2}\nabla^i g'_{ij} + \frac{1}{2}\nabla_j(\mathrm{Tr}g^{-1}g') = 0 \qquad (5.75)$$
$$\left[-Ric(g) - (d-2)g' - \mathrm{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g')\right]_{ij} = 0,$$

The structure of the expansions depends on whether d is even or odd. For d odd, the expansion is of the form

$$g(x,\rho) = g_{(0)}(x) + \rho g_{(2)}(x) + \cdots + \rho^{d/2}g_{(d)}(x) + \cdots. \qquad (5.76)$$

Terms with integral powers of ρ in the expansion are determined locally in terms of $g_{(0)}$ but $g_{(d)}(x)$ is not determined by $g_{(0)}$, except for its trace and divergence, i.e. $g^{ij}_{(0)}g_{(d)ij}$ and $\nabla^i g_{(d)ij}$, which are forced by the field equations to vanish. In this case $g_{(d)}(x)$ determines the vev of the dual stress energy tensor, whose trace must vanish as the theory is conformal and there is no conformal anomaly in odd dimensions. The fact that $g_{(d)}$ is divergenceless leads to the conservation of the stress energy tensor.

For d even, the structure is rather different:

$$g(x,\rho) = g_{(0)}(x) + \rho g_{(2)}(x) + \cdots + \rho^{d/2}(g_{(d)}(x) + h_{(d)}(x)\log\rho) + \cdots . \tag{5.77}$$

In this case one needs to include a logarithmic term to satisfy the field equations; the coefficient of this term is determined by $g_{(0)}$ whilst only the trace and divergence of $g_{(d)}(x)$ are determined by $g_{(0)}$. This structure reflects the fact that the trace of the stress energy tensor of an even-dimensional conformal field theory on a curved background is non-zero and picks up an anomaly determined in terms of $g_{(0)}$; the explicit expression for the stress energy tensor in terms of $(g_{(0)}, g_{(d)})$ is rather more complicated than in the other case but it is such that the divergence of $g_{(d)}$ leads again to conservation of the stress energy tensor.

Let us return now to the cases of interest. As mentioned above, the field equations are solved by successively differentiating the equations w.r.t. ρ and then setting ρ to zero. This procedure leads to equations of the form

$$c(n,d) g_{(2n)ij} = f(g_{(2k)ij}, \kappa_{(2k)}), \qquad k < n \tag{5.78}$$

where the right hand side depends on the lower order coefficients and $c(n,d)$ is a numerical coefficient that depends on n and d. If this coefficient is non-zero, one can solve this equation to determine $g_{(n)ij}$. However, in some cases this coefficient is zero and one has to include a logarithmic term at this order for the equations to have a solution. An example of this is the case of pure gravity with d even, where $c(d/2, d) = 0$. Furthermore, note that since in (5.73) -(5.74) only integral powers of ρ enter, likewise only integral powers in (5.67) will depend on $g_{(0)}$ and $\kappa_{(0)}$. In general however non-integral powers can also appear at some order and one must determine these terms separately. An example of this is the case of pure gravity with d odd reviewed above, where a half integral power of ρ appears at order $\rho^{d/2}$.

Let us first consider when one needs to include non-integral powers in the expansion. Let us assume that ρ^σ is the lowest non-integral power that appears in the asymptotic expansion

$$\begin{aligned} \kappa(x,\rho) &= \kappa_{(0)} + \rho\kappa_{(2)} + \cdots + \rho^\sigma \kappa_{(2\sigma)} + \cdots \\ g_{ij}(x,\rho) &= = g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^\sigma g_{(2\sigma)ij} + \cdots \end{aligned} \tag{5.79}$$

Differentiating the scalar equation (5.74) $[\sigma]$ times, where $[\sigma]$ is the integer part of σ, and taking $\rho \to 0$ after multiplying with $\rho^{1+[\sigma]-\sigma}$ one obtains

$$(2\sigma + 4\alpha\gamma - d)\kappa_{(2\sigma)} + \alpha\gamma \mathrm{Tr} g_{(2\sigma)} = 0, \tag{5.80}$$

Similarly, equation (5.73) yields,

$$(2\sigma - d + 2\alpha\gamma) g_{(2\sigma)ij} - (\mathrm{Tr} g_{(2\sigma)} + 2\kappa_{(2\sigma)}) g_{(0)ij} = 0. \tag{5.81}$$

which upon taking the trace becomes

$$-d\kappa_{(2\sigma)} + (\sigma - d + \alpha\gamma)\mathrm{Tr}g_{(2\sigma)} = 0, \tag{5.82}$$

If the determinant of the coefficients of the system of equation (5.80)-(5.82) is non-zero,

$$D = (2\sigma + 4\alpha\gamma - d)(\sigma - d + \alpha\gamma) + \alpha\gamma d \neq 0 \tag{5.83}$$

the only solution of these equations is

$$\mathrm{Tr}g_{(2\sigma)} = \kappa_{(2\sigma)} = 0 \tag{5.84}$$

which then using (5.81) implies

$$g_{(2\sigma)ij} = 0 \tag{5.85}$$

i.e. in these cases no non-integral power appears in the expansion.

On the other hand, when $D = 0$ equations (5.82)-(5.80) admit a non-trivial solution. The two solution of $D = 0$ are $\sigma_1 = d/2 - \alpha\gamma$ and $\sigma_2 = 2(d/2 - \alpha\gamma)$. Clearly, $\sigma_2 > \sigma_1$ and when σ_2 in non-integer so is σ_1, so a non-integer power first appears at:

$$\sigma = \frac{d}{2} - \alpha\gamma \tag{5.86}$$

When this holds equations (5.80)-(5.82) reduce to

$$\mathrm{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)} = 0. \tag{5.87}$$

and the coefficient of $g_{(2\sigma)ij}$ in (5.81) vanishes, so apart from its trace, these equations leave $g_{(2\sigma)ij}$ undetermined. The remaining Einstein equation (5.72) also imposes a constraint on the divergence of the terms occurring at this order, as will be discussed later. To summarize, the expansion contains a non-integer power of ρ^σ in the following cases

$$\sigma = \frac{p-7}{p-5} \Rightarrow \qquad D0 : \sigma = 7/5; \qquad D1, F1 : \sigma = 3/2; \qquad D2 : \sigma = 5/3, \tag{5.88}$$

and the coefficient multiplying this power in only partly constrained. As we will see, this category is the analogue of even dimensional asymptotically AdS backgrounds, which are dual to odd dimensional boundary theories.

The second case to discuss is the case of only integral powers. In this case the undetermined term occurs at an integral power ρ^σ with

$$\sigma = \frac{p-7}{p-5} \Rightarrow \qquad D3 : \sigma = 2; \qquad D4 : \sigma = 3, \tag{5.89}$$

and logarithmic terms need to be included in the expansions. In these cases the combination $(\mathrm{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)})$ is determined by $g_{(0)}$ and $\kappa_{(0)}$. This category is analogous to odd-dimensional asymptotically AdS backgrounds, which are dual to even-dimensional boundary theories. The remaining Einstein equation (5.72) also imposes a constraint on the divergence of the terms occurring at this order.

Actually one can see on rather general grounds why the undetermined terms occur at these powers: the undetermined terms will relate to the vev of the stress energy tensor, which is of dimension $(p+1)$ for a $(p+1)$-dimensional field theory. However, the overall normalization of the action behaves as $l_s^{(p-3)^2/(5-p)}$, and therefore on dimensional grounds the vev should sit in the $g_{(2\sigma)}\rho^{\sigma}$ term where

$$\sigma = (p+1) + \frac{(p-3)^2}{(5-p)} = \frac{(p-7)}{(p-5)}, \tag{5.90}$$

which agrees with the discussion above. Put differently we can compare the power of the first undetermined term to pure AdS and notice that it is shifted by $-\alpha\gamma = -\frac{(p-3)^2}{2(p-5)}$ (for both Dp-branes and the fundamental string). This is just what is needed to offset the background value of the $e^{\gamma\phi}$ term multiplying the Einstein-Hilbert action in (5.28), in order to ensure that all divergent terms in the action are still determined by the asymptotic field equations.

One should note here that the case of $p = 6$ is outside the computational framework discussed above. In this case the prefactor in the action is of positive mass dimension nine, whilst the stress energy tensor in the dual seven-dimensional theory must be of dimension seven. Therefore one finds a (meaningless) negative value for σ, indicating that one is not making the correct asymptotic expansion. In other words, one finds that the "subleading terms" are more singular than the leading term.

(5.5.2) Explicit expressions for expansion coefficients

In all cases of interest $2\sigma > 2$ and thus there are $g_{(2)}$ and $\kappa_{(2)}$ terms. Evaluating (5.74) and (5.73) at $\rho = 0$ gives in the case of $\beta = 0$ and $2\alpha\gamma = -1$ (relevant for D1-branes, fundamental strings and D4-branes):

$$\begin{aligned} \kappa_{(2)} &= \frac{1}{2d}(\nabla^2\kappa_{(0)} + g^{ij}_{(0)}\partial_i\kappa_{(0)}\partial_j\kappa_{(0)} + \frac{1}{2(d-1)}R_{(0)}), \\ g_{(2)ij} &= \frac{1}{d-1}(-R_{(0)ij} + \frac{1}{2d}R_{(0)}g_{(0)ij} + (\nabla_{\{i}\partial_{j\}}\kappa)_{(0)} + \partial_{\{i}\kappa_{(0)}\partial_{j\}}\kappa_{(0)}) \end{aligned} \tag{5.91}$$

Here the parentheses in a quantity $A_{\{ab\}}$ denote the traceless symmetric tensor and ∇_i is the covariant derivative in the metric $g_{(0)ij}$.

If $\beta \neq 0$, as for $p = 0, 2$, the expressions are slightly more involved:

$$\begin{aligned} \kappa_{(2)} &= -\frac{1}{M}\left(2\alpha\gamma R_{(0)} - 2(d-1)\nabla^2\kappa_{(0)} + (\frac{2\alpha\beta}{\gamma} - 2d + 2)(g^{ij}_{(0)}\partial_i\kappa_{(0)}\partial_j\kappa_{(0)})\right), \\ g_{(2)ij} &= \frac{1}{d - 2\alpha\gamma - 2}\left(-R_{(0)ij} + \nabla_i\partial_j\kappa_{(0)} + (1 - \frac{\beta}{\gamma^2})\partial_i\kappa_{(0)}\partial_j\kappa_{(0)}\right. \\ &\quad \left. + \frac{\gamma^2 - \beta}{2(\gamma^2 d - \beta d + \beta)}g_{(0)ij}\left(R_{(0)} - 2\nabla^2\kappa_{(0)} - 2(1 - \frac{\beta}{2\gamma^2})(g^{ij}_{(0)}\partial_i\kappa_{(0)}\partial_j\kappa_{(0)})\right)\right), \\ M &\equiv 16\alpha^2\beta - 2(d-1)(8\alpha\gamma + 4 - 2d) = \frac{16(9-p)}{(5-p)^2}. \end{aligned} \tag{5.92}$$

The final equality, expressing the coefficient M in terms of p, holds for the Dp-branes of interest here.

CATEGORY 1: UNDETERMINED TERMS AT NON-INTEGRAL ORDER

Let us first consider the case where the undetermined terms occur at non-integral order.

In the cases of $p = 0, 1, 2$ the terms given above in (5.92) are the only determined terms. The underdetermined terms appear at order $\rho^{(p-7)/(p-5)}$ and satisfy the constraints

$$2\kappa_{(2\sigma)} + \mathrm{Tr} g_{(2\sigma)} = 0, \qquad \sigma = \frac{p-7}{p-5} \tag{5.93}$$

$$\nabla^i g_{(2\sigma)ij} - 2(1 - \frac{\beta}{\gamma^2})\partial_j \kappa_{(0)} \kappa_{(2\sigma)} + g_{(2\sigma)ij}\partial^i \kappa_{(0)} = 0. \tag{5.94}$$

We will see that the trace and divergent constraints translate into conformal and diffeomorphism Ward identities respectively.

CATEGORY 2: UNDETERMINED TERMS AT INTEGRAL ORDER

Let us next consider the case where the undetermined terms occur at integral order: this includes the D3 and D4 branes. Explicit expressions for the conformal cases, including the case of D3-branes, are given in [15]. For the D4-branes, the equations at next order can be solved to determine $\kappa_{(4)}$ and $g_{(4)ij}$:

$$\begin{aligned} \kappa_{(4)} &= \frac{1}{8}((\nabla^2\kappa)_{(2)} + 6\kappa_{(2)}^2 + (\partial\kappa)_{(2)}^2 + \tfrac{1}{2}\mathrm{Tr} g_{(2)}^2 + 2\kappa_{(2)}\mathrm{Tr} g_{(2)}), \\ g_{(4)ij} &= \frac{1}{4}[(2\kappa_{(2)}^2 + \tfrac{1}{2}\mathrm{Tr} g_{(2)}^2) g_{(0)ij} - R_{(2)ij} - 2(g_{(2)}^2)_{ij} + (\nabla_i \partial_j \kappa)_{(2)} + 2\partial_i \kappa_{(2)} \partial_j \kappa_{(0)}]. \end{aligned} \tag{5.95}$$

where we introduce the notation

$$A[g(x,\rho), \kappa(x,\rho)] = A_{(0)}(x) + \rho A_{(2)}(x) + \rho^2 A_{(4)}(x) + \cdots \tag{5.96}$$

for composite quantities $A[g,\kappa]$ of $g(x,\rho)$ and $\kappa(x,\rho)$. For (5.95) we need the coefficients of $A = \{\nabla^2\kappa, (\partial\kappa)^2, R_{ij}\}$. The explicit expression for these coefficients can be worked out straightforwardly using the asymptotic expansion of $g(x,\rho)$ and $\kappa(x,\rho)$ and we give these expressions for the Christoffel connections and curvature coefficients in appendix 5.A.1. Note also that we use the compact notation

$$(g_{(2)}^2)_{ij} \equiv (g_{(2)} g_{(0)}^{-1} g_{(2)})_{ij}, \qquad \mathrm{Tr}(g_{(2n)}) \equiv \mathrm{Tr}(g_{(0)}^{-1} g_{(2n)}). \tag{5.97}$$

Proceeding to the next order, one finds that the expansion coefficients $\kappa_{(6)}$ and $g_{(6)ij}$ cannot be determined independently in terms of lower order coefficients because after further differentiating the highest derivative terms in (5.73) *and* (5.74) both vanish. Only the combination $(2\kappa_{(6)} + \mathrm{Tr} g_{(6)})$ is fixed, along with a constraint on the divergence. Furthermore one has to introduce logarithmic terms in (5.67) for the equations to be satisfied, namely

$$\begin{aligned} g(x,\rho) &= g_{(0)}(x) + \rho g_{(2)}(x) + \rho^2 g_{(4)}(x) + \rho^3 g_{(6)}(x) + \rho^3 \log(\rho) h_{(6)}(x) + \cdots \\ \kappa(x,\rho) &= \kappa_{(0)}(x) + \rho\kappa_{(2)}(x) + \rho^2\kappa_{(4)}(x) + \rho^3\kappa_{(6)}(x) + \rho^3 \log(\rho)\tilde{\kappa}_{(6)}(x) + \cdots \end{aligned} \tag{5.98}$$

For the logarithmic terms one finds

$$\begin{aligned}\tilde{\kappa}_{(6)} &= -\frac{1}{12}[(\nabla^2\kappa)_{(4)} + (\partial\kappa)^2_{(4)} + 20\kappa_{(2)}\kappa_{(4)} - \tfrac{1}{2}\mathrm{Tr}g^3_{(2)} + \mathrm{Tr}g_{(2)}g_{(4)} \\ &\quad +2\kappa_{(2)}(-\mathrm{Tr}g^2_{(2)} + 2\mathrm{Tr}g_{(4)}) + 4\kappa_{(4)}\mathrm{Tr}g_{(2)}], \\ h_{(6)ij} &= -\frac{1}{12}[-2R_{(4)ij} + (-\mathrm{Tr}g^3_{(2)} + 2\mathrm{Tr}g_{(2)}g_{(4)} + 8\kappa_{(2)}\kappa_{(4)})g_{(0)ij} + 2\mathrm{Tr}g_{(2)}g_{(4)ij} \\ &\quad -8(g_{(4)}g_{(2)})_{ij} - 8(g_{(2)}g_{(4)})_{ij} + 4g^3_{(2)ij} + 2(\nabla_i\partial_j\kappa)_{(4)} + 2(\partial_i\kappa\partial_j\kappa)_{(4)} + 4\kappa_{(2)}g_{(4)ij}],\end{aligned} \tag{5.99}$$

Note that these coefficients satisfy the following identities

$$\begin{aligned}&\mathrm{Tr}h_{(6)} + 2\tilde{\kappa}_{(6)} = 0, \\ &g^{ki}_{(0)}(\nabla_k h_{(6)ij} + h_{(6)ij}\partial_k\kappa_{(0)}) - 2\partial_j\kappa_{(0)}\tilde{\kappa}_{(6)} = 0.\end{aligned} \tag{5.100}$$

Furthermore, $\kappa_{(6)}, \mathrm{Tr}g_{(6)}$ and $\nabla^i g_{(6)ij}$ are constrained by the following equations,

$$\begin{aligned}&2\kappa_{(6)} + \mathrm{Tr}g_{(6)} = -\frac{1}{6}(-4\mathrm{Tr}g_{(2)}g_{(4)} + \mathrm{Tr}g^3_{(2)} + 8\kappa_{(2)}\kappa_{(4)}), \\ &\nabla^i g_{(6)ij} - 2\partial_j\kappa_{(0)}\kappa_{(6)} + g_{(6)ij}\partial^i\kappa_{(0)} = T_j,\end{aligned} \tag{5.101}$$

where T_j is locally determined in terms of $(g_{(2n)}, \kappa_{(2n)})$ with $n \leq 2$,

$$\begin{aligned}T_j &= \nabla^i A_{ij} - 2\partial_j\kappa_{(0)}(A - \frac{2}{3}\kappa^3_{(2)} - 2\kappa_{(2)}\kappa_{(4)}) + A_{ij}\partial^i\kappa_{(0)} \\ &\quad +\frac{1}{6}\mathrm{Tr}(g_{(4)}\nabla_j g_{(2)}) + \frac{2}{3}(\kappa_{(4)} + \kappa^2_{(2)})\partial_j\kappa_{(2)},\end{aligned} \tag{5.102}$$

with

$$\begin{aligned}A_{ij} &= \frac{1}{3}\Big((2g_{(2)}g_{(4)} + g_{(4)}g_{(2)})_{ij} - (g^3_{(2)})_{ij} \\ &\quad +\frac{1}{8}(\mathrm{Tr}(g^2_{(2)}) - \mathrm{Tr}g_{(2)}(\mathrm{Tr}g_{(2)} + 4\kappa_{(2)}))g_{(2)ij} \\ &\quad -(\mathrm{Tr}g_{(2)} + 2\kappa_{(2)})(g_{(4)ij} - \tfrac{1}{2}(g^2_{(2)})_{ij}) \\ &\quad -(\frac{1}{8}\mathrm{Tr}g_{(2)}\mathrm{Tr}g^2_{(2)} - \frac{1}{24}(\mathrm{Tr}g_{(2)})^3 - \frac{1}{6}\mathrm{Tr}g^3_{(2)} + \frac{1}{2}\mathrm{Tr}g_{(2)}g_{(4)})g_{(0)ij} \\ &\quad + \left(\frac{1}{4}\kappa_{(2)}((\mathrm{Tr}g_{(2)})^2 - \mathrm{Tr}g^2_{(2)}) - \frac{4}{3}\kappa^3_{(2)} - 2\kappa_{(2)}\kappa_{(4)}\right)g_{(0)ij}\Big) \\ A &= \frac{1}{6}\left(-\left(\frac{1}{8}\mathrm{Tr}g_{(2)}\mathrm{Tr}g^2_{(2)} - \frac{1}{24}(\mathrm{Tr}g_{(2)})^3 - \frac{1}{6}\mathrm{Tr}g^3_{(2)} + \frac{1}{2}\mathrm{Tr}g_{(2)}g_{(4)}\right)\right. \\ &\quad \left. -\frac{32}{3}\kappa^3_{(2)} - 6\kappa_{(2)}\kappa_{(4)} - \kappa^2_{(2)}\mathrm{Tr}g_{(2)} - 2\kappa_{(4)}\mathrm{Tr}g_{(2)}\right).\end{aligned} \tag{5.103}$$

We would now like to integrate the equations (5.101). Following the steps in [15], it is convenient to express $g_{(6)ij}$ and $\kappa_{(6)}$ as

$$\begin{aligned}g_{(6)ij} &= A_{ij} - \frac{1}{24}S_{ij} + t_{ij}; \\ \kappa_{(6)} &= A - \frac{1}{24}S - 2\kappa_{(2)}\kappa_{(4)} - \frac{2}{3}\kappa^3_{(2)} + \varphi,\end{aligned} \tag{5.104}$$

where (S_{ij}, S) are local functions of $g_{(0)}, \kappa_{(0)}$,

$$
\begin{aligned}
S_{ij} &= (\nabla^2 + \partial^m \kappa_{(0)} \nabla_m) I_{ij} - 2\partial^m \kappa_{(0)} \partial_{(i} \kappa_{(0)} I_{j)m} + 4\partial_i \kappa_{(0)} \partial_j \kappa_{(0)} I \qquad (5.105)\\
&+2R_{kilj} I^{kl} - 4I(\nabla_i \partial_j \kappa_{(0)} + \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}) + 4(g_{(2)} g_{(4)} - g_{(4)} g_{(2)})_{ij}\\
&+\frac{1}{10}(\nabla_i \partial_j B - g_{(0)ij}(\nabla^2 + \partial^m \kappa_{(0)} \partial_m) B)\\
&+\frac{2}{5} B + g_{(0)ij}(-\frac{2}{3}\mathrm{Tr} g_{(2)}^3 - \frac{4}{15}(\mathrm{Tr} g_{(2)})^3 + \frac{3}{5}\mathrm{Tr} g_{(2)} \mathrm{Tr} g_{(2)}^2\\
&-\frac{8}{3}\kappa_{(2)}^3 - \frac{8}{5}\kappa_{(2)}(\mathrm{Tr} g_{(2)})^2 - \frac{4}{5}\kappa_{(2)}^2 \mathrm{Tr} g_{(2)} + \frac{6}{5}\kappa_{(2)} \mathrm{Tr} g_{(2)}^2),\\
S &= (\nabla^2 + \partial^m \kappa_{(0)} \partial_m) I + \partial_i \kappa_{(0)} \partial_j \kappa_{(0)} I^{ij} - 2(\partial\kappa)_{(0)}^2 I \qquad (5.106)\\
&-(\nabla_k \partial_l \kappa_{(0)} + \partial_k \kappa_{(0)} \partial_l \kappa_{(0)}) I^{kl} - \frac{1}{20}(\nabla^2 + \partial^m \kappa_{(0)} \partial_m) B\\
&+\frac{2}{5} B\kappa_{(2)} - \frac{4}{3}\kappa_{(2)}^3 - \frac{4}{5}\kappa_{(2)}(\mathrm{Tr} g_{(2)})^2 - \frac{2}{5}\kappa_{(2)}^2 \mathrm{Tr} g_{(2)} + \frac{3}{5}\kappa_{(2)} \mathrm{Tr} {g_{(2)}}^2,\\
I_{ij} &= (g_{(4)} - \frac{1}{2} g_{(2)}^2 + \frac{1}{4} g_{(2)}(\mathrm{Tr} g_{(2)} + 2\kappa_{(2)}))_{ij} + \frac{1}{8} g_{(0)ij} B,\\
I &= \kappa_{(4)} + \frac{1}{2}\kappa_{(2)}^2 + \frac{1}{4}\kappa_{(2)} \mathrm{Tr} g_{(2)} + \frac{B}{16},\\
B &= \mathrm{Tr} g_{(2)}^2 - \mathrm{Tr} g_{(2)}(\mathrm{Tr} g_{(2)} + 4\kappa_{(2)}).
\end{aligned}
$$

Note that these definitions imply the following identities

$$
\begin{aligned}
\nabla^i S_{ij} - 2\partial_j \kappa_{(0)} S + S_{ij} \partial^i \kappa_{(0)} &= -4\left(\mathrm{Tr}(g_{(4)} \nabla_j g_{(2)}) + 4(\kappa_{(4)} + \kappa_{(2)}^2) \partial_j \kappa_{(2)}\right); \qquad (5.107)\\
\mathrm{Tr}(S_{ij}) + 2S &= -8\mathrm{Tr}(g_{(2)} g_{(4)} - 32\kappa_{(2)}(\kappa_{(2)}^2 + \kappa_{(4)}).
\end{aligned}
$$

Now, these definitions imply that t_{ij} defined in (5.104) is a symmetric tensor: A_{ij} contains an antisymmetric part but this is canceled by a corresponding antisymmetric part in S_{ij}. Inserting (5.104) in (5.101) one finds that the quantities (t_{ij}, φ) satisfy the following divergence and trace constraints:

$$
\begin{aligned}
\nabla^i t_{ij} &= 2\partial_j \kappa_{(0)} \varphi - t_{ij} \partial^i \kappa_{(0)}; \qquad (5.108)\\
\mathrm{Tr} t + 2\varphi &= -\frac{1}{3}\left(\frac{1}{8}(\mathrm{Tr} g_{(2)})^3 - \frac{3}{8}\mathrm{Tr} g_{(2)} \mathrm{Tr} g_{(2)}^2 + \frac{1}{2}\mathrm{Tr} g_{(2)}^3 - \mathrm{Tr} g_{(2)} g_{(4)}\right.\\
&\left.-\frac{3}{4}\kappa_{(2)}(\mathrm{Tr} g_{(2)}^2 - (\mathrm{Tr} g_{(2)})^2) - 4\kappa_{(2)}\kappa_{(4)} + 2\kappa_{(2)}^3\right).
\end{aligned}
$$

We will find that the one point functions are expressed in terms of (t_{ij}, φ) and these constraints translate into the conformal and diffeomorphism Ward identities.

(5.5.3) Reduction of M-branes

The D4-brane and type IIA fundamental string solutions are obtained from the reduction along a worldvolume direction of the M5 and M2 brane solutions respectively. The boundary conditions for the supergravity solutions also descend directly from dimensional reduction: diagonal reduction on a circle of an asymptotically (locally) AdS_{d+2} spacetime results in an asymptotically (locally) AdS_{d+1} spacetime with linear dilaton. Therefore the rather complicated results for the asymptotic expansions in the D4 and fundamental string cases should follow directly from the previously derived results for AdS_7 and AdS_4 given in [15], and we show that this is indeed the case in this subsection.

As discussed in section 5.3, solutions of the field equations of (5.36) are related to solutions of the field equations of the action (5.28) via the reduction formula (5.39). In the cases of F1 and D4 branes this means in particular

$$e^{4\phi/3} = \frac{1}{\rho}e^{2\kappa}, \tag{5.109}$$

where in comparing with (5.66) one should note that $\alpha = -3/4, \gamma = 2/3$ for both F1 and D4. This implies that the $(d+2)$ solution is automatically in the Fefferman-Graham gauge:

$$ds^2_{d+2} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(g_{ij}dx^i dx^j + e^{2\kappa}dy^2). \tag{5.110}$$

Recall that for an asymptotically AdS_{d+2} Einstein manifold, the asymptotic expansion in the Fefferman-Graham gauge is

$$ds^2_{d+2} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}G_{ab}dx^a dx^b \tag{5.111}$$

where $a = 1, \dots, (d+1)$ and

$$G = G_{(0)}(x) + \rho G_{(2)}(x) + \cdots + \rho^{(d+1)/2}G_{(d+1)/2}(x) + \rho^{(d+1)/2}\log(\rho)H_{(d+1)/2}(x) + \cdots, \tag{5.112}$$

with the logarithmic term present only when $(d+1)$ is even. The explicit expression for $G_{(2)}(x)$ in terms of $G_{(0)}(x)$ is[3]

$$G_{(2)ab} = \frac{1}{d-1}\left(-R_{ab} + \frac{1}{2d}RG_{(0)ab}\right). \tag{5.113}$$

where the R_{ab} is the Ricci tensor of $G_{(0)}$, etc.

Comparing (5.110) with (5.111) one obtains

$$G_{ij} = g_{ij}; \qquad G_{yy} = e^{2\kappa}. \tag{5.114}$$

In particular $G_{(0)ij} = g_{(0)ij}$ and $G_{(0)yy} = e^{2\kappa_{(0)}}$, so

$$\begin{aligned} R[G_{(0)}]_{ij} &= R_{(0)ij} - \nabla_i\partial_j\kappa_{(0)} - \partial_i\kappa_{(0)}\partial_j\kappa_{(0)}; \\ R[G_{(0)}]_{yy} &= e^{2\kappa_{(0)}}(-\nabla^i\partial_i\kappa_{(0)} - \partial_i\kappa_{(0)}\partial^i\kappa_{(0)}), \end{aligned} \tag{5.115}$$

with $R[G_{(0)}]_{yi} = 0$. Substituting into (5.113) gives

$$\begin{aligned} G_{(2)ij} &= \frac{1}{d-1}\left(-R_{(0)ij} + \frac{1}{2d}R_{(0)}g_{(0)ij} + (\nabla_{\{i}\partial_{j\}}\kappa)_{(0)} + \partial_{\{i}\kappa_{(0)}\partial_{j\}}\kappa_{(0)}\right); \\ G_{(2)yy} &= e^{2\kappa_{(0)}}\left(\frac{1}{2d(d-1)}R_{(0)} + \frac{1}{d}(\nabla^2\kappa_{(0)} + (\partial\kappa_{(0)})^2)\right), \end{aligned} \tag{5.116}$$

[3]Note that the conventions for the curvature used here differ by an overall sign from those in [15].

with $G_{(2)yi} = 0$. We thus find exact agreement between $G_{(2)ij}$ and $g_{(2)ij}$ in (5.91). Now using

$$G_{yy} = e^{2\kappa} = e^{(2\kappa_{(0)} + 2\rho\kappa_{(2)} + \cdots)} = e^{2\kappa_{(0)}}(1 + 2\rho\kappa_{(2)} + \cdots) \tag{5.117}$$

one determines $\kappa_{(2)}$ to be exactly the expression given in (5.91).

Now restrict to the asymptotically AdS_4 case; the next coefficient in the asymptotic expansion occurs at order $\rho^{3/2}$, in $G_{(3)ab}$, and is undetermined except for the vanishing of its trace and divergence:

$$G_{(0)}^{ab} G_{(3)ab} = 0; \qquad D^a G_{(3)ab} = 0. \tag{5.118}$$

Reducing these constraints leads immediately to

$$\begin{aligned} & g_{(0)}^{ij} g_{(3)ij} + 2\kappa_{(3)} = 0; \\ & \nabla^i g_{(3)ij} - 2\partial_j \kappa_{(0)} \kappa_{(3)} + g_{(3)ij} \partial^i \kappa_{(0)} = 0, \end{aligned} \tag{5.119}$$

in agreement with (5.93) and (5.94).

Similarly if one considers the asymptotically AdS_7 case, the determined coefficients $G_{(4)}$ and $H_{(6)}$ reduce to give $(g_{(4)}, \kappa_{(4)})$ and $(h_{(6)}, \tilde{\kappa}_{(6)})$ respectively. Furthermore, the trace of $G_{(6)}$ fixes the combination $(2\kappa_{(6)} + \mathrm{Tr} g_{(6)})$. One can show that all explicit formulae agree precisely with the dimensional reduction of the formulae in [15]; the details are discussed in appendix 5.A.3.

(5.5.4) Renormalization of the action

Having derived the general form of the asymptotic expansion one can now proceed to holographic renormalization, following the discussion in [15]. In this method one substitutes the asymptotic expansions back into the regulated action and then introduces local covariant counterterms to cancel the divergences and renormalise the action. Whilst this method is conceptually very simple, in practice it is rather cumbersome for explicit computations. A more efficient method based on a radial Hamiltonian formalism [19, 20] will be discussed in the next section.

Let us choose an illustrative yet simple example to demonstrate this method of holographic renormalization: we will work out the renormalised on-shell action and compute the one-point function of the energy-momentum tensor and the operator $\mathcal{O}$ for the case $p = 1$, both fundamental strings and D1-branes.

Since in this case $\beta = 0$, $\hat{\Phi} \equiv e^{\gamma\phi}$ behaves like a Lagrange multiplier and the bulk part of the action vanishes on-shell. The only non-trivial contribution comes then from the Gibbons-Hawking boundary term:

$$S_{boundary} = -L \int_{\rho=\epsilon} d^2x \sqrt{h} 2\hat{\Phi} K, \tag{5.120}$$

where h_{ij} is the induced metric on the boundary and K is the trace of the extrinsic curvature. Since (5.120) is divergent we regularise the action by evaluating it at $\rho = \epsilon$.

We would like now to find counterterms to remove the divergences in (5.120). From the discussion in section 5.5.1 we know the asymptotic expansion for Φ and $h_{ij}(x,\rho) = g_{ij}(x,\rho)/\rho$:

$$\begin{aligned} \hat{\Phi} &= \frac{e^{\kappa_{(0)}}}{\sqrt{\rho}}(1 + \rho\kappa_{(2)} + \rho^{3/2}\kappa_{(3)} + \cdots), \\ h &= \frac{1}{\rho}(g_{(0)} + \rho g_{(2)} + \rho^{3/2} g_{(3)} + \cdots), \end{aligned} \tag{5.121}$$

where $\kappa_{(3)}$ and $g_{(3)}$ are the lowest undetermined coefficients. Note that the expansions are the same for both fundamental strings and D1-branes, since in both cases $\alpha\gamma = -1/2$. Inserting the expansion (5.121) in (5.120) we find for the divergent part

$$S_{div} = -4L \int_{\rho=\epsilon} d^2x e^{\kappa_{(0)}} \sqrt{g_{(0)}}(\epsilon^{-3/2} + \epsilon^{-1/2}\kappa_{(2)}), \tag{5.122}$$

using the formula

$$K = d - \rho \mathrm{Tr}(g^{-1}g') \tag{5.123}$$

for the trace of the extrinsic curvature in the asymptotically AdS_{d+1} background. The trace term here cancels against the one in the expansion of the determinant.

From (5.121) and (5.91) we find

$$\sqrt{g_{(0)}} = \rho\sqrt{h}(1 + \frac{1}{4(d-1)}R[h]), \tag{5.124}$$

which allows us to write the counterterms in a gauge-invariant form:

$$S_{ct} = -S_{div} = 4L \int_{\rho=\epsilon} d^2x \sqrt{h}\hat{\Phi}(1 + \frac{1}{4}R[h]). \tag{5.125}$$

The renormalised action is then

$$S_{ren}[g_{(0)}, \kappa_{(0)}] = \lim_{\epsilon\to 0} S_{sub}[h(x,\epsilon), \hat{\Phi}(x,\epsilon)); \epsilon] \tag{5.126}$$

where

$$\begin{aligned} S_{sub} &= S_{bulk} + S_{boundary} + S_{ct} \\ &= -L[\int_{\rho\geq\epsilon} d^3x\sqrt{g}\hat{\Phi}(R + C) + \int_{\rho=\epsilon} d^2x\sqrt{h}\hat{\Phi}(2K - 4 - R[h])]. \end{aligned} \tag{5.127}$$

This allows us to compute the renormalised vevs of the operator dual to $\hat{\Phi}$ and the stress-energy tensor. For the former, only the boundary part contributes, since $R + C = 0$ from the equation of motion for $\hat{\Phi}$. It can be easily checked that the divergent parts cancel and we obtain the finite result

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{sub}}{\delta \Phi_{(0)}} = -\frac{1}{2}e^{3\kappa_{(0)}} \lim_{\epsilon\to 0}(\frac{1}{\epsilon^{3/2}\sqrt{h}}\frac{\delta S_{sub}}{\delta\hat{\Phi}}) = \frac{3}{2}e^{3\kappa_{(0)}} L\mathrm{Tr} g_{(3)} = -3e^{3\kappa_0} L\kappa_{(3)}. \tag{5.128}$$

where we used (5.69) and the definition of $\hat{\Phi}$. The vev of the stress-energy tensor $\langle T_{ij}\rangle = \lim_{\epsilon\to 0} T_{ij}[h]$ gets a contribution from the bulk term as well. We can split it into the contribution of the regularised action and the counterterms

$$T_{ij}[h] = T_{ij}^{reg} + T_{ij}^{ct}, \tag{5.129}$$

where

$$\begin{aligned} T_{ij}^{reg}[h] &= 2L[\hat{\Phi}(Kh_{ij} - K_{ij}) - 2\rho\partial_\rho\hat{\Phi}h_{ij}], \\ T_{ij}^{ct}[h] &= 2L[\hat{\Phi}(R_{ij} - \frac{1}{2}Rh_{ij} - 2h_{ij}) + \nabla^2\hat{\Phi}h_{ij} - \nabla_i\partial_j\hat{\Phi}]. \end{aligned} \tag{5.130}$$

One can again check that the divergent terms cancel and obtain the finite contribution

$$\langle T_{ij}\rangle = \lim_{\epsilon\to 0}(\frac{2}{\sqrt{h}}\frac{\delta S_{ren}}{\delta h^{ij}}) = 3Le^{\kappa_{(0)}}g_{(3)ij}. \tag{5.131}$$

Note that the expressions for the vevs take the same form for both D1-brane and fundamental string cases. The one point functions satisfy the following Ward identities:

$$\begin{aligned} \langle T_i^i\rangle - 2\Phi_{(0)}\langle\mathcal{O}\rangle &= 0. \\ \nabla^i\langle T_{ij}\rangle + \partial_j\Phi_{(0)}\langle\mathcal{O}\rangle &= 0. \end{aligned} \tag{5.132}$$

To derive these one needs the trace and divergence identities given in (5.93) and (5.94) and the relation $\Phi_{(0)} = e^{-2\kappa_{(0)}}$ (see (5.69)). These Ward identities indeed agree exactly with what we derived on the QFT side, (5.48)-(5.49).

The first variation of the renormalized action yields the relation between the 1-point functions and non-linear combinations of the asymptotic coefficients. The one point functions are obtained in the presence of sources, so higher point functions can be obtained by further functional differentiation with respect to sources.

One should note here that the local boundary counterterms are required, irrespectively of the issue of finiteness, by the more fundamental requirement of the well-posedness of the appropriate variational problem [113]. The conformal boundary of asymptotically AdS spacetimes has a well-defined conformal class of metric rather than an induced metric. This means that the appropriate variational problem involves keeping fixed a conformal class and not an induced metric as in the usual Dirichlet problem for gravity in a spacetime with a boundary. The new variational problem requires the addition of further boundary terms, on top of the Gibbons-Hawking term. In the context of asymptotically AdS spacetimes (with no linear dilaton) these turn out to be precisely the boundary counterterms, see [113] for the details and a discussion of the subtleties related to conformal anomalies.

(5.5.5) RELATION TO M2 THEORY

In the case of fundamental strings these formulae again follow directly from dimensional reduction of the AdS_4 case, since for the latter the renormalized stress energy tensor is [15]

$$\langle T_{ab}\rangle = 3L_M G_{(3)ab}. \tag{5.133}$$

Recalling the dimensional reduction formula (5.114), and noting that

$$L_M = Le^{\kappa_0}, \tag{5.134}$$

one finds immediately that

$$\langle T_{ij} \rangle = 3Le^{\kappa_0} g_{(3)ij}, \tag{5.135}$$

in agreement with (5.131). Noting that $G_{yy} = e^{4\phi/3}\rho = \hat{\Phi}^2 \rho$ one finds

$$\langle T_{yy} \rangle = 6Le^{3\kappa_0}\kappa_{(3)} = -2\langle \mathcal{O} \rangle, \tag{5.136}$$

in agreement with (5.128). The first Ward identity in (5.132) is thus an immediate consequence of the conformal Ward identity of the M2 brane theory, i.e. the tracelessness of the stress energy tensor. The second Ward identity in (5.132) similarly follows from the vanishing divergence of the stress energy tensor in the M2-brane theory.

(5.5.6) Formulae for other Dp-branes

It is straightforward to derive analogous formulae for the other Dp-branes. Note that in general there is also a bulk contribution to the on-shell action

$$S_{on-shell} = L\frac{4\alpha\beta(d-2\alpha\gamma)}{h}\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{g}e^{\gamma\phi} + L\int_{\rho=\epsilon} d^d x\sqrt{h}e^{\gamma\phi}2K \tag{5.137}$$

where h_{ij} is the induced metric on the boundary, K is the trace of the extrinsic curvature and the action is regularised at $\rho = \epsilon$. Focusing first on the cases $p < 3$ the divergent terms are:

$$\begin{aligned} S_{div} &= -L\int_{\rho=\epsilon} d^d x\sqrt{g_{(0)}}e^{\kappa_{(0)}}\epsilon^{-d/2+\alpha\gamma}\Big(2d - \frac{4\alpha\beta}{\gamma} + (-\frac{4\alpha\beta(d-2\alpha\gamma)}{\gamma(d-2\alpha\gamma-2)} + 2d)\rho\kappa_{(2)} \\ &\quad +(-\frac{2\alpha\beta(d-2\alpha\gamma)}{\gamma(d-2\alpha\gamma-2)} + d - 2)\rho \mathrm{Tr} g_{(2)}\Big), \end{aligned} \tag{5.138}$$

which can be removed with the counterterm action

$$\begin{aligned} S_{ct} &= L\int_{\rho=\epsilon} d^d x\sqrt{h}e^{\gamma\phi}(2d - \frac{4\alpha\beta}{\gamma} + C_R(\hat{R}[h] + \beta(\partial_i\phi)^2)) \\ &= L\int_{\rho=\epsilon} d^d x\sqrt{h}e^{\gamma\phi}\left(\frac{2(9-p)}{5-p} + \frac{5-p}{4}(\hat{R}[h] + \beta(\partial_i\phi)^2)\right) \\ C_R &\equiv \frac{\gamma^2-\beta}{d\gamma^2 - d\beta - \gamma^2 + 2\beta} = \frac{5-p}{4}. \end{aligned} \tag{5.139}$$

Again for convenience we give the formulae both in terms of (α, β, γ) and for the specific cases of interest here, the Dp-branes. The renormalised vevs of the operator[4] $\mathcal{O}_\phi$ dual to ϕ and the stress-energy tensor can now be computed giving:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= 2\sigma Le^{\kappa_{(0)}}\frac{1}{\alpha}\kappa_{(2\sigma)}, \\ \langle T_{ij} \rangle &= 2\sigma Le^{\kappa_{(0)}} g_{(2\sigma)ij}. \end{aligned} \tag{5.140}$$

[4] Note that $\langle \mathcal{O}_\phi \rangle = \chi\Phi_{(0)}\langle \mathcal{O} \rangle$. This is obtained using (5.69) and the chain rule.

Using (5.93) and (5.94) one obtains

$$0 = \langle T^i_i \rangle + 2\alpha \langle \mathcal{O}_\phi \rangle = \langle T^i_i \rangle + (p-3)\Phi_{(0)} \langle \mathcal{O} \rangle \tag{5.141}$$

$$0 = \nabla^i \langle T_{ij} \rangle - \frac{1}{\gamma} \partial_j \kappa_{(0)} \langle \mathcal{O}_\phi \rangle = \nabla^i \langle T_{ij} \rangle + \partial_j \Phi_{(0)} \langle \mathcal{O} \rangle, \tag{5.142}$$

where in the second equality we use the relation between $\kappa_{(0)}$ and $\Phi_{(0)}$ in (5.69) which implies in particular that $\langle \mathcal{O}_\phi \rangle = \chi \Phi_{(0)} \langle \mathcal{O} \rangle$. These are the anticipated dilatation and diffeomorphism Ward identities.

Next let us consider the case of D4-branes, for which one needs more counterterms:

$$\begin{aligned} S_{ct} &= L \int_{\rho=\epsilon} d^5x \sqrt{h} e^{\gamma\phi} (10 + \frac{1}{4}\hat{R}[h] + \frac{1}{32}(\hat{R}[h]_{ij} - \gamma(\hat{\nabla}_i \partial_j \phi + \partial_i \phi \partial_j \phi))^2 \\ & + \frac{1}{32}\gamma^2 (\hat{\nabla}^2 \phi + (\partial_i \phi)^2)^2 - \frac{3}{320}(\hat{R}[h] - 2\gamma(\hat{\nabla}^2\phi + (\partial_i\phi)^2))^2 + a_{(6)} \log \epsilon), \end{aligned} \tag{5.143}$$

where the coefficient of the logarithmic term $a_{(6)}$ is given by

$$\begin{aligned} a_{(6)} &= 6 \mathrm{Tr} h_{(6)}; \\ &= \frac{1}{8}(\mathrm{Tr} g_{(2)})^3 - \frac{3}{8} \mathrm{Tr} g_{(2)} \mathrm{Tr} g^2_{(2)} + \frac{1}{2} \mathrm{Tr} g^3_{(2)} - \mathrm{Tr} g_{(2)} g_{(4)} \\ & - \frac{3}{4}\kappa_{(2)} \mathrm{Tr} g^2_{(2)} + \frac{3}{4}\kappa_{(2)} (\mathrm{Tr} g_{(2)})^2 - 4\kappa_{(2)}\kappa_{(4)} - 2\kappa^3_{(2)}. \end{aligned} \tag{5.144}$$

Note that in cases such as the D4-brane, where one needs to compute many counterterms, it is rather more convenient to use the Hamiltonian formalism, which will be discussed in the next section. We will also discuss the structure of this anomaly further in the following section.

The renormalised vevs of the operator dual to ϕ and the stress-energy tensor can now be computed giving:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= -L e^{\kappa_{(0)}} (8\varphi + \frac{44}{3} \tilde{\kappa}_{(6)}), \\ \langle T_{ij} \rangle &= L e^{\kappa_{(0)}} (6 t_{ij} + 11 h_{(6)ij}), \end{aligned} \tag{5.145}$$

where (t_{ij}, φ) are defined in (5.104). Note that the contributions proportional to $\tilde{\kappa}_{(6)}, h_{(6)ij}$ are scheme dependent; one can remove these contributions by adding finite local boundary terms.

The dilatation Ward identity is

$$\langle T^i_i \rangle + \Phi_{(0)} \langle \mathcal{O} \rangle = -2 L e^{\kappa_{(0)}} a_{(6)}, \tag{5.146}$$

whilst the diffeomorphism Ward identity is

$$\nabla^i \langle T_{ij} \rangle + \partial_j \Phi_{(0)} \langle \mathcal{O} \rangle = 0. \tag{5.147}$$

The terms involving $(h_{(6)ij}, \tilde{\kappa}_{(6)})$ drop out of the Ward identities because of the trace and divergence identities given in (5.100).

These formulae are as expected consistent with the reduction of the M5 brane formulae given in [15]. This computation of the renormalized stress energy tensor for the M5-brane case is reviewed in appendix 5.A.2. In fact in [15] the renormalized stress energy tensor for the AdS_7 case was given only up to scheme dependent traceless, covariantly constant terms, proportional to the coefficient $H_{(6)ab}$ of the logarithmic term in the asymptotic expansion. In appendix 5.A.2 we determine these contributions to the stress energy tensor, with the resulting stress energy tensor being (5.330):

$$\langle T_{ab} \rangle = \frac{N^3}{3\pi^3}(6t_{ab} + 11H_{(6)ab}). \tag{5.148}$$

The streamlined method of derivation of the renormalized stress energy tensor given in appendix 5.A.2 is also useful in the explicit derivation of the D4-brane formulae given in (5.145). Dimensional reduction of the t_{ab} term in the stress energy tensor results in the (t_{ij}, φ) terms in the D4-brane operator vevs, whilst reduction of the $H_{(6)ab}$ term gives the terms involving $(h_{(6)ij}, \tilde{\kappa}_{(6)})$. The details of this dimensional reduction are discussed in appendix 5.A.3.

(5.6) HAMILTONIAN FORMULATION

In the previous section we showed how correlation functions can be computed using the basic holographic dictionary that relates the on-shell gravitational action to the generating functional of correlators, and we renormalized the action with counterterms to obtain finite expressions. This method of holographic renormalization is conceptually very simple but does not exploit all the structure of the theory.

The underlying structure of the correlators is best exhibited in the radial Hamiltonian formalism, which is a Hamiltonian formulation with the radius playing the role of time. The Hamilton-Jacobi theory, introduced in this context in [114], relates the variation of the on-shell action w.r.t. boundary conditions, thus the holographic 1-point functions, to radial canonical momenta. It follows that one can bypass the on-shell action and directly compute renormalized correlators using radial canonical momenta π, as was developed for asymptotically AdS spacetimes in [19, 20].

A fundamental property of asymptotically (locally) AdS spacetimes is that dilatations are part of their asymptotic symmetries. This implies that all covariant quantities can be decomposed into a sum of terms each of which has definite scaling. These coefficients are in 1-1 correspondence with the asymptotic coefficients in (5.66) with the exact relation being in general non-linear. The advantage of working with dilatation eigenvalues rather than with asymptotic coefficients is that the former are manifestly covariant while the latter in general are not: the asymptotic expansion (5.66) singles out one coordinate so it is not covariant. Holographic 1-point functions can be expressed most compactly in terms of eigenfunctions of the dilatation operator, and this explains the non-linearities found in explicit computations of 1-point functions.

(5.6.1) HAMILTONIAN METHOD FOR NON-CONFORMAL BRANES

We now develop a Hamiltonian version of the holographic renormalization of these backgrounds following closely the steps of [19, 20]. We consider the action (5.28) with the Gibbons-Hawking boundary term added to ensure that the action depends only on first radial derivatives (as we will see shortly), so a radial Hamiltonian formalism can be set up:

$$S = -L\int_{AdS_{d+1}} d^{d+1}x\sqrt{g}e^{\gamma\phi}[R+\beta(\partial\phi)^2+C] - 2L\int_{\partial AdS_{d+1}} d^dx\sqrt{h}e^{\gamma\phi}K. \tag{5.149}$$

Note that we are again working in Euclidean signature. Next we introduce a radial Hamiltonian formulation. In the usual Hamiltonian formulation of gravity in the ADM formalism one foliates spacetime by hypersurfaces of constant time. Here analogously we introduce a family of hypersurfaces Σ_r of constant radius r near the boundary and denote by n^μ their unit normal. For asymptotically locally AdS manifolds there always exists a radial function normal to the boundary which can be used to foliate the space in such radial slices, at least in a neighborhood of the boundary.

In order to give a Hamiltonian description of the dynamics, one needs to express the action (5.28) in terms of quantities on Σ_r. In particular, this means that the Ricci scalar in the action (5.28) should be expressed in terms of expressions which only contain first derivatives in the radial variable. The induced metric on the hypersurface Σ_r can be expressed as $h_{\sigma\mu} = g_{\sigma\mu} - n_\sigma n_\mu$, with $h^\rho_\mu \equiv g^{\rho\sigma}h_{\sigma\mu}$. Now let us define the radial flow vector field r^μ by the relation $r^\mu\partial_\mu r = 1$, such that the components of r^μ tangent and normal to Σ_r define shift and lapse functions respectively:

$$r^\mu_\parallel = h^\mu_\rho r^\rho \equiv N^\mu; \qquad r^\mu_\perp = Nn^\mu. \tag{5.150}$$

Thus the metric is decomposed as

$$ds^2 = (N^2+N_\mu N^\mu)dr^2 + 2N_\mu dx^\mu dr + h_{\mu\nu}dx^\mu dx^\nu, \tag{5.151}$$

analogously to the usual ADM decomposition.

A useful tool in our analysis is the extrinsic curvature $K_{\mu\nu}$ of the hypersurface given by the covariant derivative of the unit normal

$$K_{\mu\nu} = h_{\sigma(\mu}\nabla^\sigma n_{\nu)}. \tag{5.152}$$

The geometric *Gauss-Codazzi* equations (in the contracted form of [19, 20]) can be used to express the curvature of the embedding space in terms of extrinsic and intrinsic curvatures on the hypersurface[5] :

$$\begin{aligned} &K^2 - K_{\mu\nu}K^{\mu\nu} = \hat{R} + 2G_{\mu\nu}n^\mu n^\nu, \\ &\hat{\nabla}_\mu K^\mu_\nu - \partial_\nu K = G_{\rho\sigma}h^\rho_\nu n^\sigma, \\ &\pounds_n K_{\mu\nu} + KK_{\mu\nu} - 2K^\rho_\mu K_{\rho\nu} = \hat{R}_{\mu\nu} - h^\rho_\mu h^\sigma_\nu R_{\rho\sigma}, \end{aligned} \tag{5.153}$$

[5]The Lie derivative in our conventions is defined as $\pounds_n K_{\mu\nu} = n^\sigma K_{\mu\nu,\sigma} - 2n^\sigma{}_{,(\mu}K_{\nu)\sigma}$.

where $G_{\mu\nu}$ is the Einstein tensor in the embedding spacetime, K is the trace of the extrinsic curvature, $\hat{R}_{\mu\nu}$ is the intrinsic curvature and $\hat{\nabla}$ is the covariant derivative on the hypersurface.

Combining the first equation in (5.153) with the Ricci identity $R_{\mu\nu}n^\mu n^\nu = n^\nu(\nabla_\sigma\nabla_\nu - \nabla_\nu\nabla_\sigma)n^\sigma$ the Ricci scalar can be expressed as

$$R = K^2 - K_{\mu\nu}K^{\mu\nu} + \hat{R} - 2\nabla_\mu(n^\mu\nabla_\nu n^\nu) + 2\nabla_\nu(n^\mu\nabla_\mu n^\nu), \tag{5.154}$$

Inserting this expression into the action (5.28), the last two terms cancel the Gibbons-Hawking boundary term in (5.28) after partial integration and the remaining term is

$$\begin{aligned} S &= -L\int d^{d+1}x\sqrt{g}e^{\gamma\phi}[\hat{R} + K^2 - K_{\mu\nu}K^{\mu\nu} + \beta(\partial\phi)^2 + C \\ &\quad +2\gamma\partial_\mu\phi n^\mu\nabla_\nu n^\nu - 2\gamma\partial_\nu\phi n^\mu\nabla_\mu n^\nu]. \end{aligned} \tag{5.155}$$

Note that the extrinsic curvature can be expressed as

$$K_{\mu\nu} = \frac{1}{2N}(\partial_r h_{\mu\nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu), \tag{5.156}$$

and thus the action can be expressed entirely in terms of the fields $(h_{\mu\nu}, N^\mu, N)$ and the scalar field ϕ, and their derivatives. The canonical momenta conjugate to these fields are given by

$$\pi^{\mu\nu} \equiv \frac{\delta L}{\delta\dot{h}_{\mu\nu}}, \qquad \pi_\phi \equiv \frac{\delta L}{\delta\dot{\phi}}, \tag{5.157}$$

where $\dot{f} \equiv \partial_r f$ and the momenta conjugate to the lapse and shift functions vanish identically. The corresponding equations of motion in the canonical formalism become constraints, which are precisely those obtained from the first two equations in (5.153) and are the Hamiltonian and momentum constraints respectively.

The diffeomorphism gauge is most naturally fixed by choosing Gaussian normal coordinates ($N^\mu = 0$ and $N = 1$), such that

$$\begin{aligned} ds^2 &= dr^2 + h_{ij}(r,x)dx^i dx^j, \qquad K_{ij} = \frac{1}{2}\dot{h}_{ij} \\ n^\mu &= \delta^\mu_r, \qquad \nabla_\mu n^\mu = K, \qquad n^\mu\nabla_\mu n^\nu = 0, \end{aligned} \tag{5.158}$$

where the dot denotes differentiation with respect to r. The action becomes

$$S = -L\int d^{d+1}x\sqrt{h}e^{\gamma\phi}[\hat{R} + K^2 - K_{ij}K^{ij} + \beta(\dot{\phi}^2 + (\partial_i\phi)^2) + C + 2\gamma\dot{\phi}K]. \tag{5.159}$$

and the canonical momenta are given by

$$\begin{aligned} \pi_\phi &= 2B\,(\beta\dot{\phi} + \gamma K), \qquad B \equiv -Le^{\gamma\phi}\sqrt{h}. \\ \pi^{ij} &= B\,(Kh^{ij} - K^{ij} + \gamma\dot{\phi}h^{ij}), \end{aligned} \tag{5.160}$$

The Gauss-Codazzi identities in this gauge become:

$$\begin{aligned} K^2 - K_{ij}K^{ij} &= \hat{R} + 2G_{rr}, \\ D_i K^i_j - D_j K &= G_{jr}, \\ \dot{K}^i_j + KK^i_j &= \hat{R}^i_j - R^i_j, \end{aligned} \tag{5.161}$$

Now consider the regulated manifold $\mathcal{M}_{r_0}$ defined as the submanifold of $\mathcal{M}$ bounded by the hypersurface Σ_{r_0}. The values of the induced fields on Σ_{r_0} become boundary conditions for the action, and therefore the momenta on the regulating surface can be obtained from variations of the on-shell action with respect to the boundary values of the induced fields. The Hamilton-Jacobi identities thus allow the momenta (5.160) on the regulating surface to be expressed in terms of the on-shell action by

$$\pi^{ij}(r_0,x) = \frac{\delta S_{on-shell}}{\delta h_{ij}(r_0,x)}, \qquad \pi_\phi(r_0,x) = \frac{\delta S_{on-shell}}{\delta \phi(r_0,x)}. \tag{5.162}$$

Since the choice of the regulator r_0 is arbitrary, the equations (5.165) and (5.162) can be used not just to compute the on-shell action and momentum on the regulating surface Σ_{r_0} but on any radial surface Σ_r.

Now to calculate the regulated on-shell action one uses the first of the Gauss-Codazzi identities, together with the field equations (5.35):

$$S_{on-shell} = -2L \int_{\mathcal{M}_{r_0}} d^{d+1}x\sqrt{h}e^{\gamma\phi}[\hat{R} + \beta(\partial_i\phi)^2 + C]. \tag{5.163}$$

However, since the field equations follow from the variation of the bulk part of the action, the radial derivative of the on-shell action can be expressed as a purely boundary term,

$$\dot{S}_{on-shell} = -2L \int_{\Sigma_{r_0}} d^dx\sqrt{h}e^{\gamma\phi}[\hat{R} + \beta(\partial_i\phi)^2 + C]. \tag{5.164}$$

From this expression follows that the regulated on-shell action can itself also be written as a d-dimensional integral by introducing a covariant variable λ,

$$S_{on-shell} = -2L \int_{\Sigma_{r_0}} d^dx\sqrt{h}e^{\gamma\phi}[K - \lambda], \tag{5.165}$$

where λ satisfies the equation

$$\dot{\lambda} + \lambda(K + \gamma\dot{\phi}) + E = 0, \tag{5.166}$$

$$E = \frac{(\gamma^2 + d(\gamma^2 - \beta))\beta}{(\gamma^2-\beta)^2} = -\frac{2(p-1)(p-4)(p-7)}{(p-5)^2},$$

and the trace of the third equation in (5.161) is used, along with the field equations (5.35). Note that since Σ_{r_0} is compact λ is defined only up to a total divergence.

The Hamilton-Jacobi identities then imply that:

$$\pi^{ij}\delta h_{ij} + \pi_\phi \delta\phi = -2L\delta[\sqrt{h}e^{\gamma\phi}(K-\lambda)], \tag{5.167}$$

up to a total derivative. One can always use the total divergence ambiguity in λ to ensure that this expression holds without integrating it over Σ_r. First one chooses any λ satisfying (5.166), and then one calculates the variation $\delta[\sqrt{h}e^{\gamma\phi}(K-\lambda)]$. This variation necessarily gives the left hand side of (5.167), up to total derivative terms, which can be absorbed into the definition of λ. (Strictly speaking, this argument applies only to the local terms in λ; the finite part of λ as $r \to \infty$ is actually non-local in the sources, and only the integrated identity holds for this part.)

(5.6.2) HOLOGRAPHIC RENORMALIZATION

We next turn to the formulation of a Hamiltonian method of holographic renormalization. In the earlier sections, we discussed holographic renormalization by solving asymptotically the field equations, as a function of sources. Here we will instead use the equations of motion to determine the asymptotic form of the momenta as functionals of induced fields. Such a procedure is manifestly covariant at all stages, with the Ward identities being manifest and the one point functions of dual operators being naturally expressed in terms of the momenta.

An important tool in the Hamiltonian method is the dilatation operator, whose eigenfunctions are covariant expressions on the hypersurface Σ_r, and which asymptotically behaves like the radial derivative. The radial derivative acting on the on-shell action and on the momenta can be represented as a functional derivative, since by means of the field equations the on-shell action and the momenta are given as functionals of h_{ij} and ϕ:

$$\partial_r = \int d^d x (2K_{ij}[h,\phi]\frac{\delta}{\delta h_{ij}} + \dot{\phi}[h,\phi]\frac{\delta}{\delta \phi}) \tag{5.168}$$

where we used (5.158). Now, recall that the dilatation operator for a d-dimensional theory on a curved background containing sources for operators of dimension Δ is given by

$$\delta_D \equiv \int d^d x (2h_{ij}\frac{\delta}{\delta h_{ij}} + (\Delta - d)\Phi\frac{\delta}{\delta \Phi}) \tag{5.169}$$

In our case, the field $\Phi = \exp \frac{2(p-5)}{(7-p)}\phi$ couples to $\mathcal{O}$ which has dimension 4. Using the chain rule we obtain

$$\delta_D \equiv \int d^d x (2h_{ij}\frac{\delta}{\delta h_{ij}} - 2\alpha\frac{\delta}{\delta \phi}) = \partial_r + \mathcal{O}(e^{-2r}), \tag{5.170}$$

so indeed the radial derivative can be asymptotically identified with the dilatation operator since asymptotically $\dot{\phi} \to -2\alpha$ and $\dot{h}_{ij} \to 2h_{ij}$.

The next key observation is that the momenta and on-shell action can be expanded asymptotically in terms of eigenfunctions of the dilatation operator δ_D. The structure one expects in these expansions of K^i_j, λ and $\dot{\phi}$ in terms of weights of the dilatation operator is similar to the radial expansions (5.67), except that every term in the expansion also contains terms subleading in e^{-2r}:

$$\begin{aligned} K^i_j[h,\phi] &= K_{(0)}{}^i_j + K_{(2)}{}^i_j + \cdots + K_{(d-2\alpha\gamma)}{}^i_j + \tilde{K}_{(d-2\alpha\gamma)}{}^i_j \log e^{-2r}, \\ \lambda[h,\phi] &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d-2\alpha\gamma)} + \tilde{\lambda}_{(d-2\alpha\gamma)} \log e^{-2r}, \\ \dot{\phi}[h,\phi] &= p^{\phi}_{(0)} + p^{\phi}_{(2)} + \cdots + p^{\phi}_{(d-2\alpha\gamma)} + \tilde{p}^{\phi}_{(d-2\alpha\gamma)} \log e^{-2r}. \end{aligned} \tag{5.171}$$

(We will see that the logarithmic terms appear only if $(d-2\alpha\gamma)$ is an even integer, i.e. for $p = 3, 4$.) The transformation properties of these terms under the dilatation operator are:

$$\begin{aligned} \delta_D K_{(n)}{}^i_j &= -nK_{(n)}{}^i_j, \qquad \delta_D \tilde{K}_{(d-2\alpha\gamma)}{}^i_j = -(d-2\alpha\gamma)\tilde{K}_{(d-2\alpha\gamma)}{}^i_j, \\ \delta_D K_{(d-2\alpha\gamma)}{}^i_j &= -(d-2\alpha\gamma)K_{(d-2\alpha\gamma)}{}^i_j - 2\tilde{K}_{(d-2\alpha\gamma)}{}^i_j, \end{aligned} \tag{5.172}$$

and similarly for λ_k and p^ϕ_k. Thus terms with weight $n < (d - 2\alpha\gamma)$ transform homogeneously, whilst terms with weight $n = (d - 2\alpha\gamma)$ transform inhomogeneously, indicating that these terms depend non-locally on the induced fields. As we will see below, the terms with weight $n < d-2\alpha\gamma$ are algebraically (locally) determined in terms of the asymptotics, while the weight $(d - 2\alpha\gamma)$ terms are undetermined up to some constraints. The latter will be identified with the renormalized one point functions and the on-shell action, which are non-local functionals of the sources. Given a solution from which one wishes to extract the 1-point function dual to a given field, one simply subtracts the lower weight terms in the dilatation expansion of the corresponding momentum. We will show below how these lower weight terms can be determined recursively in terms of the asymptotic data.

Although it is as mentioned above not necessary to compute the renormalised action to obtain renormalised 1-point functions, the Hamiltonian method is more efficient at determining the counterterms. The divergences in the on-shell action can be expressed as terms in the expansions which are divergent as $r_0 \to \infty$. These divergences can be removed by a counterterm action which consists of these divergent terms in the expansions, namely:

$$I_{ct} = 2L \int_{\Sigma_{r_0}} \sqrt{h} e^{\gamma\phi} (\sum_{0 \leq n < d-2\alpha\gamma} (K_{(n)} - \lambda_{(n)}) + (\tilde{K}_{(n)} - \tilde{\lambda}_{(n)}) \log e^{-2r_0}). \tag{5.173}$$

This counterterm action also leads through the Hamilton-Jacobi relations to the covariant counterterms of the momenta. The renormalised action is then given by the terms of appropriate weight in the on-shell action (5.165):

$$I_{ren} = -2L \int_{\Sigma_{r_0}} d^d x \sqrt{h} e^{\gamma\phi} [K_{(d-2\alpha\gamma)} - \lambda_{(d-2\alpha\gamma)}]. \tag{5.174}$$

The gravity/gauge theory prescription identifies this with the generating functional in the dual field theory, and so, in particular, the first derivatives of this action with respect to the sources correspond to the one point functions of the dual operators. Since the Hamilton-Jacobi relations identify these first derivatives with the non-local terms in the expansions of the momenta one obtains immediately the relations:

$$\langle T_{ij} \rangle = \pi_{(d-2\alpha\gamma)ij}; \qquad \langle \mathcal{O}_\phi \rangle = (\pi_\phi)_{(d-2\alpha\gamma)}. \tag{5.175}$$

From (5.160) one sees that the one-point functions are given by:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= -2Le^{\gamma\phi}(\beta p^{\phi}{}_{(d-2\alpha\gamma)} + \gamma K_{(d-2\alpha\gamma)}), \\ \langle T_{ij} \rangle &= 2Le^{\gamma\phi}((K_{(d-2\alpha\gamma)} + \gamma p^{\phi}{}_{(d-2\alpha\gamma)})h_{ij} - K_{(d-2\alpha\gamma)ij}). \end{aligned} \tag{5.176}$$

Thus to obtain both the counterterms and the one-point functions one needs to solve for the terms in the dilatation expansions.

(5.6.3) WARD IDENTITIES

The diffeomorphism Ward identity can be derived from the momentum constraint, the second Gauss-Codazzi equation in (5.161):

$$\hat{\nabla}_i K^i_j - \hat{\nabla}_j K = G_{jr} = (\gamma^2 - \beta)\partial_j\phi\dot{\phi} + \gamma\partial_j\dot{\phi} - \gamma K^i_j\partial_i\phi. \tag{5.177}$$

Using (5.160) this can easily be expressed in terms of momenta:

$$\hat{\nabla}_i(\frac{\pi^{ij}}{\sqrt{h}}) = \frac{1}{2\sqrt{h}}\partial^j\phi\pi_\phi. \tag{5.178}$$

Expressing this identity at weight $(d - 2\alpha\gamma)$ in terms of one-point functions yields the Ward identity

$$\hat{\nabla}_i\langle T^{ij}\rangle - \gamma^{-1}\langle\mathcal{O}_\phi\rangle\partial^j\kappa_{(0)} = 0. \tag{5.179}$$

which becomes of the standard QFT form (5.48) upon expressing it in terms of $\langle O\rangle$ and $\Phi_{(0)}$. To determine the dilatation Ward identity one computes the infinitesimal Weyl transformation of the renormalised action (5.174)

$$\delta_\sigma I_{ren} = 4L\int_{\Sigma_r} d^dx\sqrt{h}(Ne^\phi)^\gamma[\tilde{K}_{(d-2\alpha\gamma)} - \tilde{\lambda}_{(d-2\alpha\gamma)}]\delta\sigma, \tag{5.180}$$

where one uses the non-diagonal behaviour of $K_{(d-2\alpha\gamma)}$ and $\lambda_{(d-2\alpha\gamma)}$ under the dilatation operator exhibited in (5.172). However, this infinitesimal Weyl transformation is also given by the renormalised version of the Hamilton-Jacobi relations (5.162) given by[6]

$$\delta_\sigma I_{ren} = -\int_{\Sigma_r} d^dx\sqrt{h}[2\pi_{(d-2\alpha\gamma)}{}^i_i - 2\alpha\pi_{\phi(d-2\alpha\gamma)}]\delta\sigma. \tag{5.181}$$

Since these identities hold for arbitrary $\delta\sigma$ we can infer the conformal Ward identity

$$\langle T^i_i\rangle + 2\alpha\langle\mathcal{O}_\phi\rangle = \mathcal{A}, \tag{5.182}$$

where the anomaly is given by

$$\mathcal{A} = -4L[\tilde{K}_{(d-2\alpha\gamma)} - \tilde{\lambda}_{(d-2\alpha\gamma)}]. \tag{5.183}$$

The anomaly for the D4-brane will be computed below. Again this becomes the standard Ward identity (5.49) (with an anomaly) upon replacing $\langle O_\phi\rangle$ by $\chi\Phi_{(0)}\langle O\rangle$ (see footnote 4).

(5.6.4) EVALUATION OF TERMS IN THE DILATATION EXPANSION

Let us now discuss how to evaluate the local terms in the dilatation expansion. In the previous section we have derived a number of identities which can be solved recursively to determine

[6] We define e.g. $\pi_{\phi(d-2\alpha\gamma)}$ to be the weight $(d - 2\alpha\gamma)$ part of $\pi_\phi/\sqrt{h}$.

terms in the expansions. In particular, applying the Hamilton-Jacobi identity (5.167) to dilatations gives

$$(1+\delta_D)K - (d - 2\alpha\gamma + \delta_D)\lambda - (d\gamma - 2\alpha\beta)\dot{\phi} = 0. \tag{5.184}$$

The Hamilton-Jacobi relations (5.162) and (5.160) also imply expressions for the extrinsic curvature and scalar field momenta:

$$\begin{aligned}(Kh^{ij} - K^{ij} + \gamma\dot{\phi}h^{ij}) &= \frac{2}{e^{\gamma\phi}\sqrt{h}}\frac{\delta}{\delta h_{ij}}\int_{\Sigma_{r_0}} d^dx\sqrt{h}e^{\gamma\phi}(K-\lambda); \\ (\beta\dot{\phi} + \gamma K) &= \frac{1}{e^{\gamma\phi}\sqrt{h}}\frac{\delta}{\delta\phi}\int_{\Sigma_{r_0}} d^dx\sqrt{h}e^{\gamma\phi}(K-\lambda).\end{aligned} \tag{5.185}$$

Next one has the Einstein equations, rewritten as the Gauss-Codazzi equations (5.161). Note that the Hamiltonian constraint in (5.161) can be written as

$$K^2 - K_{ij}K^{ij} = \hat{R} - \beta\dot{\phi}^2 + (\beta - 2\gamma^2)(\partial_i\phi)^2 - 2\gamma\hat{\nabla}^2\phi - 2\gamma K\dot{\phi} + C, \tag{5.186}$$

where the field equations (5.35) are used on the right hand side, and the double radial derivative terms $\ddot{\phi}$ are eliminated using the scalar equation of motion. One can also use the scalar equation of motion (the second equation in (5.35)), which in Gaussian normal coordinates reads

$$\ddot{\phi} + \hat{\nabla}^2\phi + K\dot{\phi} + \gamma\dot{\phi}^2 + \gamma(\partial_i\phi)^2 - \frac{\gamma(d(\gamma^2-\beta)+\gamma^2)}{(\gamma^2-\beta)^2} = 0. \tag{5.187}$$

as well as the differential equation for λ (5.166). Not all of these identities are necessary in order to recursively determine the lower terms in the dilatation expansion.

In practice it is convenient to first use the Hamilton-Jacobi identity (5.184) to express the local coefficients of λ in terms of those in K and $\dot{\phi}$:

$$\lambda_{(2n)} = \frac{(1-2n)K_{(2n)} - (d\gamma - 2\alpha\beta){p^{\phi}}_{(2n)}}{d - 2\alpha\gamma - 2n}. \tag{5.188}$$

Thus this identity ensures that all counterterms are expressed in terms of the momenta.

Next one needs to solve for the momenta, using the Hamilton-Jacobi relations, Gauss-Codazzi relations and the scalar equation of motion. Consider first the Hamilton constraint (5.186); this equation can be expanded into terms of given dilatation weight, and solving at each weight yields a recursion relation for terms in the expansion of K_{ij} and $\dot{\phi}$. At dilatation weight zero this constraint yields merely a check of the background solution. Noting that $K_{(0)} = K_{(0)ij}K^{ij}_{(0)} = d$ the zero weight constraint is

$$d(d-1) = -\beta(p^{\phi}_{(0)})^2 - 2\gamma d{p^{\phi}}_{(0)} + C, \tag{5.189}$$

which is satisfied given that ${p^{\phi}}_{(0)} = -2\alpha$ and the definition of α in terms of (β, d, C).

At higher dilatation weight one obtains a recursion relation for a linear combination for $K_{(2n)}$

and ${p^{\phi}}_{(2n)}$ at a given weight $2n$:

$$K_{(2)} + \gamma {p^{\phi}}_{(2)} = \frac{1}{2(d-2\alpha\gamma-1)}[\hat{R} + (\beta - 2\gamma^2)(\partial_i\phi)^2 - 2\gamma\hat{\nabla}^2\phi], \tag{5.190}$$

$$\begin{aligned} K_{(2n)} + \gamma {p^{\phi}}_{(2n)} &= \frac{1}{2(d-2\alpha\gamma-1)}[\sum_{m=1}^{n-1}(K_{(2m)}{}^i_j K_{(2n-2m)}{}^j_i - K_{(2m)}K_{(2n-2m)}) \\ &- \sum_{m=1}^{n-1}(\beta {p^{\phi}}_{(2m)}{p^{\phi}}_{(2n-2m)} + 2\gamma K_{(2m)}{p^{\phi}}_{(2n-2m)})]. \end{aligned}$$

Note that if $(d-2\alpha\gamma)$ is not an even integer one immediately finds the relation

$$K_{(d-2\alpha\gamma)} + \gamma {p^{\phi}}_{(d-2\alpha\gamma)} = 0, \tag{5.191}$$

since no terms on the right hand side can contribute at this weight. This relation precisely corresponds to (5.87) in the old formalism, in the case where the undetermined term appears at a non-integral power of ρ.

Consider next the scalar equation of motion; to express this in terms of terms of given dilatation weight, it is necessary to expand $\ddot{\phi}$ in terms of eigenfunctions of the dilatation operator. (Note that eliminating $\ddot{\phi}$ using the other field equations does not give an identity which is independent of (5.186).) The additional radial derivative in $\ddot{\phi}$ can be expressed in terms of the dilatation operator by keeping higher terms in the expansion of the radial derivative:

$$\begin{aligned} \partial_r &= \int d^dx(2K_{ij}\frac{\delta}{\delta h_{ij}} + \dot{\phi}\frac{\delta}{\delta\phi}) \\ &= \delta_D + \sum_{n\geq 1}\int d^dx(2K_{(2n)ij}\frac{\delta}{\delta h_{ij}} + {p^{\phi}}_{(2n)}\frac{\delta}{\delta\phi}) \equiv \delta_D + \sum_{n\geq 1}\delta_{(2n)}. \end{aligned} \tag{5.192}$$

Given the transformation properties (5.172) of the expansion coefficients of the momenta, the subleading terms in the expansion of ∂_r must satisfy the commutation relation $[\delta_D, \delta_{(2n)}] = -2n\delta_{(2n)}$.

Solving the scalar field equation at weight zero, (5.187) is automatically satisfied given the leading asymptotic behavior. At higher weights $2n$ with $n > 1$ a recursion relation for a distinct linear combination of $K_{(2n)}$ and ${p^{\phi}}_{(2n)}$ is obtained:

$$(d-2-4\alpha\gamma){p^{\phi}}_{(2)} - 2\alpha K_{(2)} = -\hat{\nabla}^2\phi - \gamma(\partial_i\phi)^2, \tag{5.193}$$

$$(d-2n-4\alpha\gamma){p^{\phi}}_{(2n)} - 2\alpha K_{(2n)} = -\sum_{m=1}^{n-1}(\delta_{(2m)}{p^{\phi}}_{(2n-2m)} + K_{(2m)}{p^{\phi}}_{(2n-2m)}).$$

In the case that $(d-2\alpha\gamma)$ is not an even integer, the relevant term in the recursion relation becomes

$$-2\alpha(K_{(d-2\alpha\gamma)} + \gamma {p^{\phi}}_{(d-2\alpha\gamma)}) = 0, \tag{5.194}$$

since no terms on the right hand side can contribute at this weight, and thus reproduces the trace constraint (5.191).

The Hamiltonian constraint (5.190) together with the scalar equation (5.193) thus constitutes a linear system of equations which allows one to express $K_{(2n)}$ and $p^{\phi}{}_{(2n)}$ in terms of lower order coefficients. One can then determine $\lambda_{(2n)}$ from (5.188), and use the Hamilton-Jacobi relations (5.185) to determine the extrinsic curvature $K_{(2n)}{}^{i}_{j}$. This is all information needed to proceed in the recursion.

It is useful to recall here the equation (5.166) for the variable λ, which determines the on-shell action. Here again the radial derivative can be expressed in terms of the dilatation operator, giving:

$$(\delta_D + \sum_{n=1}^{d/2-\alpha\gamma} \delta_{(2n)})\lambda + \lambda(K + \gamma\dot{\phi}) + E = 0. \tag{5.195}$$

Note that in the case of $E = 0$, i.e. for F1,D1 and D4 branes $\lambda = 0$ solves the differential equation, and thus the coefficients $\lambda_{(2n)}$ consist only of total derivative terms which are determined by (5.188).

Category 1: Undetermined terms at non-integral order

Let us consider first the case where the undetermined terms occur at non-integral order, namely $p < 3$, and obtain the counterterms and one point functions.

The Hamiltonian constraint (5.190) together with the scalar equation (5.193) can be solved at first order to give:

$$\begin{aligned} K_{(2)} &= \frac{1}{2(d-2\alpha\gamma-1)(d-2\alpha\gamma-2)}\Big((d-2-4\alpha\gamma)(\hat{R}+\beta(\partial\phi)^2) + 2(1+2\alpha\gamma)e^{-\gamma\phi}\hat{\nabla}^2(e^{\gamma\phi})\Big); \\ p^{\phi}{}_{(2)} &= \frac{1}{\gamma(d-2\alpha\gamma-1)(d-2\alpha\gamma-2)}\Big(\gamma\alpha(\hat{R}+\beta(\partial\phi)^2) - (d-1)e^{-\gamma\phi}\hat{\nabla}^2(e^{\gamma\phi})\Big); \end{aligned} \tag{5.196}$$

Next note that the counterterms $\lambda_{(2n)}$ follow from (5.188), and are given by

$$\begin{aligned} \lambda_{(0)} &= -\frac{2\alpha\beta}{\gamma}; \\ \lambda_{(2)} &= -\frac{K_{(2)} + (d\gamma - 2\alpha\beta)p^{\phi}{}_{(2)}}{(d-2\alpha\gamma-2)}. \end{aligned} \tag{5.197}$$

For the cases $p < 3$ one only needs to solve up to this order to obtain all counterterms, with the counterterm action being:

$$\begin{aligned} I_{ct} &= L\int_{\Sigma_{r_0}} \sqrt{h}e^{\gamma\phi}\left(2d - \frac{4\alpha\beta}{\gamma} + \frac{\gamma^2-\beta}{(d-1)\gamma^2+\beta(2-d)}(\hat{R}+\beta(\partial\phi)^2)\right) \\ &\quad -L\int_{\Sigma_{r_0}} \sqrt{h}\frac{d}{(d-2\alpha\gamma-2)}\hat{\nabla}^2(e^{\gamma\phi}). \end{aligned} \tag{5.198}$$

This coincides with the counterterm action found earlier in (5.139), up to the (irrelevant) total derivative term in the second line.

Next consider the one point functions. To apply the general formula (5.176), one needs to relate the momentum coefficients with terms in the asymptotic expansion of the metric and the scalar field. In the case that $(d-2\alpha\gamma)$ is not an even integer, this identification turns out to be very simple. Recall that in the original method of holographic renormalization one expanded the induced metric asymptotically in the radial coordinate $\rho = e^{-2r}$ as

$$h_{ij} = \frac{1}{\rho}(g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^{\frac{1}{2}(d-2\alpha\gamma)} g_{(d-2\alpha\gamma)ij} + \rho^{\frac{1}{2}(d-2\alpha\gamma)} \ln \rho h_{(d-2\alpha\gamma)ij} + \cdots), \quad (5.199)$$

where the logarithmic term is included when $(d-2\alpha\gamma)$ is an even integer. Differentiating with respect to r gives

$$\begin{aligned} K_{ij} &= \tfrac{1}{2}\dot{h}_{ij} = \frac{1}{\rho} g_{(0)ij} - \rho g_{(4)ij} + \cdots + \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \left((1-\tfrac{1}{2}(d-2\alpha\gamma))g_{(d-2\alpha\gamma)ij} - h_{(d-2\alpha\gamma)ij}\right) \\ &\quad + \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \ln \rho (1-\tfrac{1}{2}(d-2\alpha\gamma)) h_{(d-2\alpha\gamma)ij} + \cdots \end{aligned} \quad (5.200)$$

However, each term in the covariant expansion of the extrinsic curvature is a functional of h_{ij} and can be expanded as:

$$\begin{aligned} K_{(0)ij}[h] &= h_{ij} = \frac{1}{\rho}\Big(g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^{\frac{1}{2}(d-2\alpha\gamma)} g_{(d-2\alpha\gamma)ij} \\ &\qquad + \rho^{\frac{1}{2}(d-2\alpha\gamma)} \ln \rho h_{(d-2\alpha\gamma)ij} + \cdots\Big); \\ K_{(2)ij}[h] &= K_{(2)ij}[g_{(0)}] + \rho \int d^d x g_{(2)kl} \frac{\delta K_{(2)ij}}{\delta g_{(0)kl}} + \cdots; \\ K_{(d-2\alpha\gamma)ij}[h] &= \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} K_{(d-2\alpha\gamma)ij}[g_{(0)}] + \cdots; \\ \tilde{K}_{(d-2\alpha\gamma)ij}[h] &= \rho^{\frac{1}{2}(d-2\alpha\gamma-2)} \tilde{K}_{(d-2\alpha\gamma)ij}[g_{(0)}] + \cdots. \end{aligned} \quad (5.201)$$

Inserting these expressions into the expansion and comparing with (5.200) implies:

$$\begin{aligned} K_{(0)ij}[g_{(0)}] &= g_{(0)ij}; \\ K_{(2)ij}[g_{(0)}] &= -g_{(2)ij}; \\ K_{(d-2\alpha\gamma)ij}[g_{(0)}] &= -\tfrac{1}{2}(d-2\alpha\gamma) g_{(d-2\alpha\gamma)ij} - h_{(d-2\alpha\gamma)ij} + \cdots; \\ \tilde{K}_{(d-2\alpha\gamma)ij}[g_{(0)}] &= -\tfrac{1}{2}(d-2\alpha\gamma) h_{(d-2\alpha\gamma)ij}. \end{aligned} \quad (5.202)$$

Here the ellipses denote terms involving functional derivatives with respect to $g_{(0)ij}$ of lower order coefficients $g_{(2n)ij}[g_{(0)}]$.

The formulae are thus simplified in the case where $(d-2\alpha\gamma)$ is not an even integer, since no lower weight terms can contribute and we obtain $K_{(d-2\alpha\gamma)ij} = -(\frac{d}{2} - \alpha\gamma) g_{(d-2\alpha\gamma)ij}$. Similarly treating the scalar field expansion, one finds that

$$\gamma p^{\phi}{}_{(d-2\alpha\gamma)} = -(d-2\alpha\gamma)\kappa_{(d-2\alpha\gamma)}, \quad (5.203)$$

which yields for the one point functions:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= (d-2\alpha\gamma)(\gamma - \frac{\beta}{\gamma}) L e^{\kappa_{(0)}} \mathrm{Tr} g_{(d-2\alpha\gamma)}, \\ \langle T_{ij} \rangle &= (d-2\alpha\gamma) L e^{\kappa_{(0)}} g_{(d-2\alpha\gamma)ij}, \end{aligned} \quad (5.204)$$

where we used the constraint (5.191) in the last equation. Note that the mixing of K and $\dot{\phi}$ in the momenta conspires to ensure that the expectation value of the energy-momentum tensor is proportional to just $g_{(d-2\alpha\gamma)ij}$, without involving $\mathrm{Tr}g_{(d-2\alpha\gamma)}$. These formulas exactly agree with the ones in (5.140) we derived earlier (upon use of (5.86) and (5.34)).

The D4-branes are the only case under consideration where $(d-2\alpha\gamma)$ is an even integer; here the lower weight terms do contribute and the expressions for the vevs are considerably more complicated. We thus turn next to the evaluation of the momentum coefficients in this case.

Category 2: The D4-brane

In this section we will consider the case of the D4-branes, where $(d-2\alpha\gamma)$ is an even integer, and derive the counterterms; the anomaly term $\mathcal{A}$ in the dilatation Ward identity (5.182) and the one point functions. Note that the anomaly appears only if $(d-2\alpha\gamma)$ is an even integer, since only then do we need nonzero coefficients $\tilde{K}_{(d-2\alpha\gamma)}$ and $\tilde{p}^{\phi}{}_{(d-2\alpha\gamma)}$ of the logarithmic terms in (5.171) to fulfill the field equations. For the branes of interest, only the cases of $p=3$ and $p=4$ have anomalies, and the coefficients can be calculated from the counterterms. The case $p=3$ was discussed already in [13, 15] and will not be discussed further here.

The counterterms and the anomaly are found by recursively computing the momentum coefficients. The Hamiltonian constraint (5.190) along with the scalar equation (5.193) provides a system of equations to determine $K_{(2n)}$ and $p^{\phi}{}_{(2n)}$, whilst the uncontracted Hamilton-Jacobi identity (5.185) can be used to obtain $K_{(2n)}{}^{i}_{j}$. Recall that in this case $E=0$, and thus λ is zero, up to total derivatives. This means in particular that the dilatation equation (5.188) can always be written as

$$(1-2n)K_{(2n)} - d\gamma\dot{\phi}_{(2n)} = (d-2\alpha\gamma-2n)\lambda_{(2n)} \equiv \hat{\Phi}^{-1}\hat{\nabla}_l Y^l_{(2n)}, \qquad (5.205)$$

where $\hat{\Phi} \equiv e^{\gamma\phi}$. As λ is zero, up to these total derivatives, the only counterterms needed are the $K_{(2n)}$, along with the logarithmic counterterm $\tilde{K}_{(6)}$. Explicit expressions for the momenta found by solving the recursion relations are given in appendix 5.A.4, with the terms $K_{(2)}$ and $K_{(4)}$ agreeing with the (non-logarithmic) counterterms found previously, see (5.143).

At weight $(d-2\alpha\gamma)=6$ the dilatation equation (5.188) breaks down and only a linear combination of $K_{(6)}$ and $p^{\phi}{}_{(6)}$ can thus be determined. This however is sufficient to determine the anomaly

$$\langle T^i_i\rangle - \frac{3}{2}\langle\mathcal{O}_\phi\rangle = \mathcal{A}, \qquad (5.206)$$

which is given by

$$\mathcal{A} = -4L\tilde{K}_{(6)} = 2Ld(K_{(6)} + \gamma p^{\phi}{}_{(6)}), \qquad (5.207)$$

where the right hand side is the combination of $K_{(6)}$ and $p^{\phi}{}_{(6)}$ which is determined by (5.190) in terms of lower counterterms. The anomaly in terms of the momentum coefficients is therefore:

$$\begin{aligned} \mathcal{A} &= 10L(K_{(6)} + \gamma {p^{\phi}}_{(6)}) \\ &= 2L(K_{(2)}{}^i_j K_{(4)}{}^j_i - K_{(2)}K_{(4)} - K_{(2)}\gamma {p^{\phi}}_{(4)} - K_{(4)}\gamma {p^{\phi}}_{(2)}). \end{aligned} \tag{5.208}$$

Explicit expressions for each of these terms are given in appendix 5.A.4; the total anomaly can then be written as

$$\begin{aligned} \mathcal{A} = & -\frac{N^2}{192\pi^4}(g_5^2 N)\Big[-R^{ljki}\mathcal{R}_{lk}\mathcal{R}_{ij} - 2\hat{\Phi}^{-2}\nabla^2\hat{\Phi}\nabla_i\partial_j\hat{\Phi}\mathcal{R}^{ij} \\ & +\frac{1}{2}\mathcal{R}(\mathcal{R}_{ij}\mathcal{R}^{ij} + \hat{\Phi}^{-2}(\nabla^2\hat{\Phi})^2) - \frac{3}{50}\mathcal{R}^3 + \frac{1}{5}\mathcal{R}_{ij}\nabla^i\partial^j\mathcal{R} \\ & +\frac{1}{20}\mathcal{R}(\nabla^2 + \hat{\Phi}^{-1}\partial^i\hat{\Phi}\partial_i)\mathcal{R} \\ & -\frac{1}{2}\mathcal{R}_{ij}[(\nabla^2 + \hat{\Phi}^{-1}\partial^l\hat{\Phi}\nabla_l)\mathcal{R}^{ij} - 2\hat{\Phi}^{-2}\partial_l\hat{\Phi}\partial^{(i}\hat{\Phi}\mathcal{R}^{j)l} - 2\hat{\Phi}^{-3}\partial^i\hat{\Phi}\partial^j\hat{\Phi}\nabla^2\hat{\Phi}] \\ & +\frac{1}{2}\hat{\Phi}^{-1}\nabla^2\hat{\Phi}[-(\nabla^2 + \hat{\Phi}^{-1}\partial^i\hat{\Phi}\partial_i)(\hat{\Phi}^{-1}\nabla^2\hat{\Phi}) + 2\hat{\Phi}^{-2}\partial_i\hat{\Phi}\partial_j\hat{\Phi}\mathcal{R}^{ij} + 2\hat{\Phi}^{-3}\partial_i\hat{\Phi}\partial^i\hat{\Phi}\nabla^2\hat{\Phi}]\Big], \end{aligned} \tag{5.209}$$

where ∇ is the covariant derivative in the five-dimensional metric and

$$\begin{aligned} \mathcal{R} &\equiv R - 2\hat{\Phi}^{-1}\nabla^2\hat{\Phi}, \\ \mathcal{R}_{ij} &\equiv R_{ij} - \hat{\Phi}^{-1}\nabla_i\partial_j\hat{\Phi}. \end{aligned} \tag{5.210}$$

Note that for notational simplicity we dropped the hats for the covariant derivative and curvature of the boundary metric. Here the anomaly has been expressed in such a way to demonstrate that it agrees with the dimensional reduction of the anomaly of the M5-brane theory found in [13, 15]. The latter is given in terms of the six-dimensional curvature $R_{abcd}(G)$ of the six-dimensional metric G_{ab} by

$$\begin{aligned} \langle T^a_a \rangle = & \frac{N^3}{96\pi^3}\Big(R^{ab}R^{cd}R_{abcd} - \tfrac{1}{2}RR^{ab}R_{ab} + \frac{3}{50}R^3 \\ & +\frac{1}{5}R^{ab}D_a D_b R - \tfrac{1}{2}R^{ab}\Box R_{ab} + \frac{1}{20}R\Box R\Big). \end{aligned} \tag{5.211}$$

In particular, the anomaly vanishes for a Ricci flat manifold (more generally it vanishes for conformally Einstein manifolds). Now recall that on diagonal reduction the six-dimensional Ricci tensor $R(G)_{ab}$ can be written as:

$$R(G)_{ij} = R_{ij} - \hat{\Phi}^{-1}\nabla_i\partial_j\hat{\Phi}; \qquad R(G)_{yy} = -\hat{\Phi}^{-1}\nabla^2\hat{\Phi}. \tag{5.212}$$

Clearly,

$$R_{ij} = \nabla_i\partial_j\hat{\Phi} = 0, \tag{5.213}$$

in the reduced theory implies that the six dimensional manifold is Ricci flat. Comparing with (5.209) one sees that indeed the anomaly vanishes under these conditions.

The anomaly of the six-dimensional theory can be expressed in terms of conformal invariants, such that it is of the form

$$\mathcal{A} = aN^3(E_{(6)} + I_{(6)} + D_a J^a_{(5)}), \tag{5.214}$$

where a is an appropriate constant, $E_{(6)}$ is proportional to the six-dimensional Euler density (type A anomaly), $I_{(6)}$ is a conformal invariant (type B anomaly) and the $D_a J^a_{(5)}$ terms are scheme dependent, as they can always be canceled by adding finite counterterms.

The D4 anomaly can necessarily be expressed in terms of invariants of the generalized conformal structure: dimensional reduction of each of the six-dimensional conformal invariants gives a generalized conformal invariant. Note however that the reduction of the six-dimensional Euler density will give an invariant which is not topological with respect to the five-dimensional background. It is also not clear that the basis of generalized conformal invariants obtained by dimensional reduction would be irreducible; it would be interesting to explore this issue further.

The general one point functions in this case are given by evaluating the expressions:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= -2Le^{\gamma\phi}(\gamma K_{(d-2\alpha\gamma)}), \\ \langle T_{ij} \rangle &= 2Le^{\gamma\phi}\left((K_{(d-2\alpha\gamma)} + \gamma p^{\phi}_{(d-2\alpha\gamma)})h_{ij} - K_{(d-2\alpha\gamma)ij}\right). \end{aligned} \tag{5.215}$$

The resulting expressions are as found before, see (5.145):

$$\langle \mathcal{O}_\phi \rangle = -Le^{\kappa_{(0)}}(8\varphi + \frac{44}{3}\tilde{\kappa}_{(6)}); \qquad \langle T_{ij} \rangle = Le^{\kappa_{(0)}}(6t_{ij} + 11h_{(6)ij}), \tag{5.216}$$

where (φ, t_{ij}) are given in (5.104).

(5.7) Two-point functions

In this section we will discuss the computation of 2-point functions for backgrounds with the asymptotics of the non-conformal branes. Transforming to the dual frame, these become Asymptotically locally AdS backgrounds with a linear dilaton and this implies that their analysis is essentially the same as the analysis of the more familiar holographic RG flows with conformal asymptotics [17, 18, 20]. In the next subsection we briefly review the basic principles of the computation of 2-point functions, mostly following the discussion in [17]. Then we compute the 2-point functions for the D-branes in subsection 5.7.2 and finally we will discuss the computation for the general case in subsection 5.7.3.

(5.7.1) Generalities

Let us start by recalling the basic formula relating bulk and boundary quantities:

$$\langle \exp(-S_{QFT}[g_{(0)}, \Phi_{(0)}]) \rangle = \exp(-S_{SG}[g_{(0)}, \Phi_{(0)}]). \tag{5.217}$$

The left hand side denotes the functional integration involving the field theory action S_{QFT} coupled to background metric $g_{(0)}$ and sources $\Phi_{(0)}$ that couple to composite operators. For the case of Dp-branes the action S_{QFT} is given in (5.42). On the right hand side $S_{SG}[g_{(0)}, \Phi_{(0)}]$ is the bulk supergravity action evaluated on classical solutions with boundary data $g_{(0)}, \Phi_{(0)}$. For the cases at hand this action is given in (5.28). As discussed extensively in previous sections, this relation needs to be renormalized and we have determined the renormalized action S_{ren} for all cases. By definition the variation of the renormalized action is given by

$$\delta S_{ren}[g_{(0)}, \Phi_{(0)}] = \int d^{d+1}x \sqrt{g_{(0)}} \left(\frac{1}{2} \langle T_{ij} \rangle \delta g^{ij}_{(0)} + \langle \mathcal{O} \rangle \delta \Phi_{(0)} \right). \tag{5.218}$$

Higher point functions are determined by further differentiation of the 1-point functions, e.g. for the case of Dp-branes

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = - \left. \frac{1}{\sqrt{g_{(0)}}} \frac{\delta \langle \mathcal{O}(x) \rangle}{\delta \Phi_{(0)}(y)} \right|_{g_{(0)ij} = \delta_{ij}, \Phi_{(0)} = g_d^{-2}}. \tag{5.219}$$

As we have shown in earlier sections, the 1-point functions in the presence of sources are expressed in terms of the asymptotic coefficients in the near-boundary expansion of the bulk solution. In particular, they depend on the coefficients that the asymptotic analysis does not determine. To obtain those we need exact regular solutions with prescribed boundary conditions. On general grounds, regularity in the interior should fix the relation between the asymptotically undetermined coefficients and the boundary data. Having obtained such relations one can then proceed to compute the holographic n-point functions. To date, this program has only been possible to carry out perturbatively around given solutions. In particular, linearized solutions determine 2-point functions, second order perturbations determine 3-point functions etc. Here we will discuss the 2-point functions involving the stress energy tensor T_{ij} and the scalar operator $\mathcal{O}$.

Let us decompose the metric perturbation as,

$$\delta g_{(0)ij}(x) = \delta h^T_{(0)ij} + \nabla_{(i} \delta h^L_{(0)j)} + g_{(0)ij} \frac{1}{d-1} \delta f_{(0)} - \nabla_i \nabla_j \delta H_{(0)} \tag{5.220}$$

where

$$\nabla^i h^T_{(0)ij} = 0, \qquad h^{T\ i}_{(0)i} = 0, \qquad \nabla^i h^L_{(0)i} = 0. \tag{5.221}$$

All covariant derivatives are that of $g_{(0)}$. Then the different components source different irreducible components of the stress energy tensor,

$$\begin{aligned} \delta S_{ren}[g_{(0)}, \Phi_{(0)}] \quad = \quad & \int d^{d+1}x \sqrt{g_{(0)}} \Big(\langle \mathcal{O} \rangle \delta \Phi_{(0)} - \frac{1}{2} \langle T_{ij} \rangle \delta h^{T\ ij}_{(0)} - \frac{1}{2(d-1)} \langle T^i_i \rangle \delta f_{(0)} \\ & + \nabla^i \langle T_{ij} \rangle \delta h^{L\ j}_{(0)} + \nabla^i \nabla^j \langle T_{ij} \rangle \delta H_{(0)} \Big) \end{aligned} \tag{5.222}$$

Now, recall that in the cases we discuss here we have already established that the holographic Ward identities,

$$\nabla^j \langle T_{ij} \rangle_J + \langle \mathcal{O} \rangle_J \partial_i \Phi_{(0)} = 0, \tag{5.223}$$

$$\langle T^i_i \rangle_J + (d-4) \Phi_{(0)} \langle \mathcal{O} \rangle_J = \mathcal{A}, \tag{5.224}$$

where there is an anomaly only for $p = 4$. These and the fact that $\Phi_{(0)}$ in the background solution is a constant imply that the second line in (5.222) does not contribute to 2-point functions. Note also that the source for the trace of stress energy tensor is $-f_{(0)}/(2(d-1))$.

We will be interested in cases with $g_{(0)ij} = \delta_{ij}$ (or somewhat more generally the cases with $g_{(0)}$ being conformally flat). The two-point functions of T_{ij} and $\mathcal{O}$ have the following standard representation in momentum space,

$$\begin{aligned} \langle T_{ij}(q)T_{kl}(-q)\rangle &= \Pi^{TT}_{ijkl}A(q^2) + \pi_{ij}\pi_{kl}B(q^2) \\ \langle T_{ij}(q)\mathcal{O}(-q)\rangle &= \pi_{ij}C(q^2) \\ \langle \mathcal{O}(q)\mathcal{O}(-q)\rangle &= D(q^2) \end{aligned} \tag{5.225}$$

where A, B, C, D are functions of q^2 and

$$\begin{aligned} \pi_{ij} &= \delta_{ij} - \frac{q_i q_j}{q^2} \\ \Pi^{TT}_{ijkl} &= -\frac{\delta h^{TT}_{(0)ij}}{\delta h^{TT\ kl}_{(0)}} = \frac{1}{2}(\pi_{ik}\pi_{jl} + \pi_{il}\pi_{jk}) - \frac{1}{d-1}\pi_{ij}\pi_{kl} \end{aligned} \tag{5.226}$$

are transverse and transverse traceless projectors, respectively. The trace Ward identity implies

$$\begin{aligned} \langle T_{ij}(q)T^k_k(-q)\rangle &= -\frac{1}{g_d^2}(d-4)\langle T_{ij}(q)\mathcal{O}(-q)\rangle \\ \langle T^i_i(q)\mathcal{O}(-q)\rangle &= -\frac{1}{g_d^2}(d-4)\langle \mathcal{O}(q)\mathcal{O}(-q)\rangle \end{aligned} \tag{5.227}$$

which then leads to the relations,

$$B(q^2) = -\frac{1}{g_d^2}\frac{(d-4)}{(d-1)}C(q^2) = \left(\frac{1}{g_d^2}\frac{(d-4)}{(d-1)}\right)^2 D(q^2) \tag{5.228}$$

Furthermore, the coefficient $D(q^2)$ is also constrained by the generalized conformal invariance as discussed in section 5.4.

(5.7.2) Holographic 2-point functions for the brane backgrounds

We next discuss the computation of the 2-point functions in the backgrounds of the non-conformal branes. Earlier discussions of the 2-point functions in the D0-brane background can be found in [102] and for Dp-brane backgrounds they were discussed in [93, 94, 103].

We need to solve for small fluctuations around the background solution given in (5.31). We thus consider a solution of the form

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x,\rho)dx^i dx^j}{\rho}, \\ \phi(x,\rho) &= \alpha\log\rho + \varphi(x,\rho), \qquad \varphi(x,\rho) \equiv \frac{\kappa(x,\rho)}{\gamma}, \end{aligned} \tag{5.229}$$

with

$$g_{ij}(x,\rho) = \delta_{ij} + \gamma_{ij}(x,\rho). \tag{5.230}$$

and φ, γ_{ij} considered infinitesimal. The background metric is translationally invariant, so it is convenient to Fourier transform. The fluctuation $\gamma_{ij}(q,\rho)$ can be decomposed into irreducible pieces as

$$\gamma_{ij}(q,\rho) = e_{ij}(q,\rho) + \frac{d}{d-1}\left(\frac{1}{d}\delta_{ij} - \frac{q_i q_j}{q^2}\right) f(q,\rho) + \frac{q_i q_j}{q^2} S(q,\rho), \tag{5.231}$$

Let us also express the transverse traceless part as $e_{ij}(q,\rho) \equiv h^T_{(0)ij}(q)h(q,\rho)$, where $h(q,\rho)$ is normalized to go to 1 as $\rho \to 0$. The field theory sources $h^T_{(0)ij}(q), f_{(0)}(q), S_{(0)}(q)$ are the leading ρ independent parts of $e_{ij}(q,\rho), f(q,\rho), S(q,r)$. Relative to the discussion in the previous subsection, we have gauged away the longitudinal vector perturbation h^L_i and traded H for $S = \frac{d}{d-1}f + p^2 H$.

The linearized equations are now obtained by inserting (5.230)-(5.231) into (5.71)-(5.74) and treating κ, h, f, S as infinitesimal variables. This leads to the following equations:

$$\frac{1}{2}S'' + \kappa'' = 0; \tag{5.232}$$

$$\frac{1}{2}f' + \kappa' = 0; \tag{5.233}$$

$$2\rho h'' - (d-2-2\alpha\gamma)h' - \frac{1}{2}q^2 h = 0; \tag{5.234}$$

$$2\rho S'' + (2\alpha\gamma + 2 - 2d)S' - 2d\kappa' - q^2(\kappa + f) = 0; \tag{5.235}$$

$$4\rho\kappa'' + (8\alpha\gamma + 4 - 2d)\kappa' + 2\alpha\gamma S' - q^2\kappa = 0, \tag{5.236}$$

where the equations are listed in the same order as in (5.71)-(5.74) with (5.234) and (5.235) being the transverse traceless and trace part of (5.73). Equation (5.234) is already diagonal. The remaining equations can be diagonalized by elementary manipulations leading to the following expressions,

$$\begin{aligned} \kappa(q,\rho) &= 2\alpha\gamma v_0(q) + v_1(q)\chi(q,\rho) \\ f(q,\rho) &= -2(d-1)v_0(q) - 2v_1(q)\chi(q,\rho), \\ S(q,\rho) &= v_2(q) + \rho q^2 v_0(q) - 2v_1(q)\chi(q,\rho) \end{aligned} \tag{5.237}$$

where v_0, v_1, v_2 are integration constants, which can be expressed in terms of the sources as

$$v_0 = \frac{2\gamma\phi_{(0)} + f_{(0)}}{2(1-2\sigma)}, \quad v_1 = \frac{(d-1)\gamma\phi_{(0)} + \alpha\gamma f_{(0)}}{2\sigma - 1}, \quad v_2 = S_{(0)} + 2v_1, \tag{5.238}$$

where $\sigma = d/2 - \alpha\gamma = (p-7)/(p-5)$ and $\phi_{(0)} = \kappa_{(0)}/\gamma$ with $\kappa_{(0)}$ the ρ independent part of $\kappa(q,\rho)$. $\chi(q,r)$ is normalized to go to 1 as $\rho \to 0$ and satisfies the same differential equation as the transverse traceless mode, namely

$$2\rho\chi'' - 2(\sigma-1)\chi' - \frac{1}{2}q^2\chi = 0 \tag{5.239}$$

The solution of this equation that is regular in the interior is given in terms of the modified Bessel function of the second kind,

$$\chi_\sigma(q,\rho) = c(\sigma)x^\sigma K_\sigma(x), \qquad x = \sqrt{q^2\rho}, \qquad \sigma = \frac{p-7}{p-5}, \tag{5.240}$$

where the normalization coefficient $c(\sigma)$ is chosen such that $\chi(q,\rho)$ approaches 1 as $\rho \to 0$. In our case, $\sigma = \{7/5, 3/2, 5/3, 3\}$ for $p = \{0,1,2,4\}$.

NON-INTEGRAL CASES

The asymptotic expansion for non-integer values of σ is

$$\chi_\sigma(q,\rho) = 1 + \frac{1}{4(1-\sigma)}q^2\rho + \cdots + \tilde{\chi}_{(2\sigma)}(q)\rho^\sigma + \cdots \qquad (\nu \text{ non}-\text{integer}) \tag{5.241}$$

where

$$\tilde{\chi}_{(2\sigma)}(q) = -\frac{\Gamma(1-\sigma)}{2^{2\sigma}\Gamma(1+\sigma)}(q^2)^\sigma. \tag{5.242}$$

One can verify that the leading order terms in the exact linearized solution indeed agree with the linearization of the asymptotic coefficients derived earlier and furthermore one can obtain the coefficient that the asymptotic analysis left undetermined. Combining the previous formulas we obtain,

$$\begin{aligned} \kappa_{(2\sigma)} &= v_1(q)\tilde{\chi}_{(2\sigma)}(q^2) \\ g_{(2\sigma)ij} &= \left(h^T_{(0)ij}(q) - \frac{2}{(d-1)}v_1(q)\pi_{ij}\right)\tilde{\chi}_{(2\sigma)}(q^2) \end{aligned} \tag{5.243}$$

which indeed satisfy the linearization of (5.93)-(5.94). Thus the 1-point functions (5.140) to linear order in the sources are then given by

$$\langle \mathcal{O}_\phi \rangle = \frac{2\sigma L\gamma(d-1)}{\alpha(2\sigma-1)}\left(\phi_{(0)} - 2\alpha\left(-\frac{f_{(0)}}{2(d-1)}\right)\right)\tilde{\chi}_{(2\sigma)}(q^2), \tag{5.244}$$

$$\langle T_{ij} \rangle = 2\sigma L\left(h^T_{(0)ij} - \frac{2\gamma}{(2\sigma-1)}\left(\phi_{(0)} - 2\alpha\left(-\frac{f_{(0)}}{2(d-1)}\right)\right)\pi_{ij}\right)\tilde{\chi}_{(2\sigma)}(q^2). \tag{5.245}$$

It follows that the 2-point functions are given by

$$\begin{aligned} \langle T_{ij}(q)T_{kl}(-q)\rangle &= \Pi^{TT}_{ijkl}\left(4\sigma L\tilde{\chi}_{(2\sigma)}(q^2)\right) + \pi_{ij}\pi_{kl}\left(-\frac{2\alpha}{(d-1)}\right)^2\langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q)\rangle \\ \langle T_{ij}(q)\mathcal{O}_\phi(-q)\rangle &= \pi_{ij}\left(-\frac{2\alpha}{(d-1)}\right)\langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q)\rangle \\ \langle \mathcal{O}_\phi(q)\mathcal{O}_\phi(-q)\rangle &= -\frac{2\sigma L\gamma(d-1)}{\alpha(2\sigma-1)}\tilde{\chi}_{(2\sigma)}(q^2) \end{aligned} \tag{5.246}$$

These relations are of the form (5.225) with the coefficients B, C related to the D coefficient as dictated by the trace Ward identity (with the relation becoming (5.228) once we pass from

$\mathcal{O}_\phi$ to $\mathcal{O}$). Thus we only need discuss the transverse traceless part of the 2-point function of T_{ij} and the scalar 2-point function.

We now Fourier transform to position space using

$$\int d^d q e^{-iqx}(q^2)^\sigma = \pi^{d/2} 2^{d+2\sigma} \frac{\Gamma(d/2-\sigma)}{\Gamma(-\sigma)} \frac{1}{|x|^{d+2\sigma}}, \tag{5.247}$$

which is valid when $\sigma \neq -(d/2+k)$, where k is an integer. Let us first discuss the case of Dp-branes. The scalar two function becomes

$$\begin{aligned} \langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(0)\rangle &= C_\phi N^{(7-p)/(5-p)} (g_d^2)^{(p-3)/(5-p)} |x|^{\frac{p^2-19-2p}{5-p}}, \\ &= C_\phi N^2 \frac{(g_{eff}^2(x))^{\frac{(p-3)}{(5-p)}}}{|x|^{2d}} \end{aligned} \tag{5.248}$$

where C_ϕ is a positive numerical constant (obtained by collecting all numerical constants in previous formulas). Note that the characteristic scale in this case is x and therefore the effective coupling constant is $g_{eff}^2(x) = g_d^2 N |x|^{3-p}$. The g_d and x dependence is consistent with the constraints of generalized conformal invariance discussed in section 5.4. Recall also that the operator $\mathcal{O}_\phi$ at weak coupling has dimension d (and $\mathcal{O}$ has dimension 4). So going from weak to strong coupling we find that the dimension is protected but the 2-point function itself gets corrections. The overall factor of N^2 reflects the fact that this is a tree level computation. Similarly, the transverse traceless part of the 2-point function of the stress energy tensor is given by

$$\langle T_{ij}(x)T_{kl}(0)\rangle_{TT} = C_T \Pi^{TT}_{ijkl} \frac{N^2 (g_{eff}^2(x))^{\frac{(p-3)}{(5-p)}}}{|x|^{2d}} \tag{5.249}$$

with C_T a positive constant. In this case the dimension is protected because T_{ij} is conserved. We can trust these results provided

$$g_{eff}^2(x) \gg 1 \quad \Rightarrow \quad |x| \gg (g_d^2 N)^{-1/(3-p)} \tag{5.250}$$

For the fundamental string background we obtain

$$\begin{aligned} \langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(0)\rangle &\sim N^{3/2} g_s (\alpha')^{1/2} \frac{1}{|x|^5}, \\ \langle T_{ij}(x)T_{kl}(0)\rangle_{TT} &\sim N^{3/2} g_s (\alpha')^{1/2} \Pi^{TT}_{ijkl} \frac{1}{|x|^5} \end{aligned} \tag{5.251, 5.252}$$

In the IIB case S-duality relates the fundamental string solution to the D1 brane solution. Indeed, the 2-point function (5.251) becomes equal the $p=1$ case in (5.248) under S-duality, $g_s \to 1/g_s, \alpha' \to \alpha' g_s$.

In the IIA case the fundamental string lifts to the M2 brane. As discussed in section 5.5.3, the source for the stress energy tensor of the M2 theory is simply related to the sources for the stress energy tensor of the string and the operator $\mathcal{O}_\phi$, see (5.114). Taking into account that the worldvolume theories are related by reduction over the M-theory circle and so their actions are related by the factor of R_{11}, the radius of the M-theory circle, we find (up to numerical constants)

$$T_{ij}^{M2} \sim R_{11}^{-1} T_{ij}, \qquad T_{yy}^{M2} \sim R_{11}^{-1} \mathcal{O}_\phi \tag{5.253}$$

Using $R_{11} = g_s l_s$ we get

$$\langle T_{yy}^{M2}(x) T_{yy}^{M2}(0) \rangle = \frac{1}{R_{11}^2} \langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle \sim \frac{N^{3/2}}{R_{11}|x|^5} \tag{5.254}$$

with similar results for the other correlators. The stress energy tensor of the M2 theory has dimension 3, so one expects the correlator to scale as $|x|^{-6}$. However, one of the worldvolume directions is compactified with radius R_{11}. Smearing out over the compactified direction indeed results in the fall off in (5.254). Finally the N scaling is the well-known $N^{3/2}$ scaling of the M2 theory.

The D4 case

For the $\sigma = 3$ case corresponding to D4 branes we have

$$\chi_3(q, \rho) = 1 - \frac{1}{8} q^2 \rho + \cdots + \rho^3 (\tilde{\chi}_{(6)}(q) + \frac{1}{768} q^6 \log \rho) + \cdots \tag{5.255}$$

where

$$\tilde{\chi}_{(6)}(q) = \frac{1}{384} q^6 (\frac{1}{2} \log q^2 - \log 2 + \gamma - \frac{11}{12}) \tag{5.256}$$

and γ is the Euler constant (not to be confused with the γ used in other parts of this chapter). The terms without $\log q^2$ are scheme dependent and will be omitted in what follows. The one point functions and two point functions are then given by (5.244), (5.245) and (5.246) respectively. In particular,

$$\langle \mathcal{O}_\phi(q) \mathcal{O}_\phi(-q) \rangle = \frac{L}{180} q^6 \ln q^2. \tag{5.257}$$

Fourier transforming back to position space, the scalar two function becomes

$$\langle \mathcal{O}_\phi(x) \mathcal{O}_\phi(0) \rangle = C_\phi N^2 \mathcal{R} \left(\frac{g_{eff}^2(x)}{|x|^{10}} \right), \tag{5.258}$$

where C_ϕ is a positive numerical constant (obtained by collecting all numerical constants) and, as in section 5.4, $\mathcal{R}(1/|x|^a)$ denotes the renormalised version of $(1/|x|^a)$. The effective coupling constant is $g_{eff}^2(x) = g_d^2 N/|x|$, and the g_d and x dependence is consistent with the constraints of generalized conformal invariance discussed in section 5.4.

This result is also consistent with the uplift to the M5-brane results. The source for the stress energy tensor of the M5 theory is simply related to the sources for the stress energy tensor of the D4-brane and the operator $\mathcal{O}_\phi$. Taking into account that the worldvolume theories are related by reduction over M-theory circle and so their actions are related by the factor of R_{11}, the radius of the M-theory circle, we find (up to numerical constants)

$$T_{ij}^{M5} \sim R_{11}^{-1} T_{ij}, \qquad T_{yy}^{M5} \sim R_{11}^{-1} \mathcal{O}_\phi \tag{5.259}$$

Using $R_{11} = g_s l_s$ we then get

$$\langle T^{M5}_{yy}(x) T^{M5}_{yy}(0)\rangle = \frac{1}{R_{11}^2}\langle \mathcal{O}_\phi(x)\mathcal{O}_\phi(0)\rangle \sim \frac{N^3}{R_{11}}\mathcal{R}\left(\frac{1}{|x|^{11}}\right) \tag{5.260}$$

with similar results for the other correlators. The stress energy tensor of the M5 theory has dimension six, and the correlator of the six-dimensional theory behaves as $\mathcal{R}|x|^{-12}$. Here one of the worldvolume directions is compactified with radius R_{11} and smearing out over the compactified direction indeed results in the fall off in (5.260). Note that the N scaling is the well-known N^3 scaling of the M5-brane theory.

(5.7.3) General case

In the simple case discussed above, it was straightforward to solve the equations for linear perturbations, but in more general backgrounds the diagonalisation of the fluctuation equations is more involved. To treat the general case, it is convenient to use the analysis [115, 17, 20] of linear fluctuations around background solutions of a single scalar field coupled to gravity; in these papers the fluctuation equations were diagonalised for a general domain wall scalar system.

In this section we will explain a general method for computing the two point functions which exploits this analysis. As discussed in section 5.7.1 we need to determine the one point functions to linear order in the sources and in the Hamiltonian method this corresponds to determining the momenta to linear order in the sources. So, as in the previous section, let us begin by considering linear fluctuations around the background of interest in the dual frame:

$$\begin{aligned} h_{ij} &= h^B_{ij}(r) + \gamma_{ij}(r,x) = e^{2A(r)}\delta_{ij} + \gamma_{ij}(r,x), \\ \phi &= \phi_B(r) + \varphi(r,x). \end{aligned} \tag{5.261}$$

Note that the metric fluctuation has already been put into axial gauge. Next we will express the canonical momenta in terms of these fluctuations. To do this, first note that the extrinsic curvature of constant r hypersurfaces can be expressed as:

$$K^i_j = \dot{A}\delta^i_j + \frac{1}{2}\dot{S}^i_j, \tag{5.262}$$

where $S^i_j \equiv h^{ik}_B \gamma_{kj}$. S^i_j can be decomposed into irreducible components as

$$S^i_j = e^i_j + \frac{d}{d-1}\left(\frac{1}{d}\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2_B}\right)f + \frac{\partial^i\partial_j}{\nabla^2_B}S, \tag{5.263}$$

where $\partial_i e^i_j = e^i_i = 0$, $S = S^i_i$, indices are raised with the inverse background metric $e^{-2A}\delta^{ij}$ and $\nabla^2_B = e^{-2A}\nabla^2 = e^{-2A}\delta^{ij}\partial_i\partial_j$. Here the diffeomorphism invariance of the transverse space was used to set the vector component to zero.

The momenta (5.160) up to linear order in the fluctuations are then given by

$$\begin{aligned} \pi_\phi &= 2B(\beta\partial_r\phi + \gamma K) = \pi^B_\phi + B(2\beta\partial_r\varphi + \gamma\partial_r S), \\ \pi^i_i &= \pi^{i,B}_i - \tfrac{1}{2}B(d-1)\partial_r S + Bd\gamma\partial_r\varphi, \qquad \pi^i_{j,TT} = \pi^{i,B}_{j,TT} - \tfrac{1}{2}B\partial_r e^i_j, \end{aligned} \tag{5.264}$$

where $\pi^B_{\tilde{\phi}}$, $\pi^{i,B}_i$ and $\pi^{i,B}_{j,TT}$ are the background values, in the absence of fluctuations, and TT stands for transverse and traceless. The one point functions are obtained by extracting the components of appropriate dilatation weight from these momenta. So we need to determine $\partial_r\varphi, \partial_r S, \partial_r e^i_j$.

To obtain these momenta, however, we would need to diagonalise the equations of motion for the linear fluctuations, and solve for $\partial_r\varphi$ etc. Diagonalising such fluctuation equations is in general rather difficult, and thus it is convenient to exploit the analysis of [17, 20], where the fluctuation equations were diagonalised for a generic domain wall dilaton background. In the latter work, however, an Einstein frame bulk action was used, so we will first need to transform our backgrounds to the Einstein frame, and then map our fluctuation equations to the set of equations which were diagonalised in full generality in [17, 20].

The analysis of [17, 20] begins with an Einstein frame bulk action:

$$S = -\int d^{d+1}x\sqrt{G_E}(\frac{1}{2\kappa^2}R_E - \frac{1}{2}(\partial\tilde{\phi})^2 - V(\tilde{\phi})). \tag{5.265}$$

and then one considers domain wall solutions of the form

$$ds^2_B = d\tilde{r}^2 + e^{2A(\tilde{r})}dx_i dx^i, \qquad \tilde{\phi} = \tilde{\phi}_B(\tilde{r}), \tag{5.266}$$

which preserve Poincaré symmetry in the transverse directions. Here the subscript B denotes that this is the background solution around which linear fluctuation equations will be solved.

Substituting the ansatz (5.266) into the field equations gives:

$$\begin{aligned} \dot{A}^2 - \frac{\kappa^2}{d(d-1)}(\dot{\tilde{\phi}}_B^2 - 2V(\tilde{\phi}_B)) &= 0, \\ \ddot{A} + d\dot{A}^2 + \frac{2\kappa^2}{d-1}V(\tilde{\phi}_B) &= 0, \\ \ddot{\tilde{\phi}}_B + d\dot{A}\dot{\tilde{\phi}}_B - V'(\tilde{\phi}_B) &= 0, \end{aligned} \tag{5.267}$$

where the dot denotes differentiation with respect to $\tilde{r}$ and the prime denotes differentiation with respect to $\tilde{\phi}$. In explicitly solving these equations one can use the fact that these second order equations are solved by any solution of the first order flow equations [116, 117]:

$$\begin{aligned} \dot{A} &= -\frac{\kappa^2}{d-1}W(\tilde{\phi}_B), \\ \dot{\tilde{\phi}}_B &= W'(\tilde{\phi}_B), \end{aligned} \tag{5.268}$$

with the potential expressed in terms of a superpotential W as:

$$V(\tilde{\phi}_B) = \frac{1}{2}[W'^2 - \frac{d\kappa^2}{d-1}W^2]. \tag{5.269}$$

Conversely, given an explicit solution of (5.267), which may not be asymptotically AdS but $\tilde{\phi}_B$ should have at most isolated zeros, one can use (5.268) to define a superpotential $W(\tilde{\phi}_B)$ [118].

Now let us consider the backgrounds of interest here, which are asymptotic to Dp-brane backgrounds. In these cases, the action (5.28) in the dual frame can be transformed to the Einstein frame using the transformation $g_E = \exp(2\gamma\phi/(d-1))g_{dual}$, giving

$$S = -L\int d^{d+1}x\sqrt{G_E}[R_E - \frac{1}{2}(\partial\tilde{\phi})^2 + Ce^{-2\gamma\tilde{\phi}/\nu(d-1)}]. \tag{5.270}$$

Here the scalar has been rescaled as

$$\tilde{\phi} \equiv \nu\phi, \qquad \nu \equiv \sqrt{2(\frac{d\gamma^2}{d-1} - \beta)}, \tag{5.271}$$

so that $\tilde{\phi}$ is canonically normalized. The metric and dilaton for the decoupled Dp-brane background can then be written in Einstein frame as

$$\begin{aligned} ds_E^2 &= d\tilde{r}^2 + (\mu\tilde{r})^{2(\mu+1)/\mu}dx_i dx^i, \\ \tilde{\phi} &= -\frac{2\alpha\nu}{\mu}\log(\mu\tilde{r}), \\ \tilde{r} &= \frac{\rho^{-\mu/2}}{\mu} = \frac{e^{\mu r}}{\mu}, \qquad \mu = -\frac{2\alpha\gamma}{d-1} = \frac{(p-3)^2}{p(5-p)}. \end{aligned} \tag{5.272}$$

From this solution one can extract the parameters and functions abstractly defined in (5.266), (5.268) and (5.269):

$$\begin{aligned} \kappa^2 &= \frac{1}{2}, \qquad A(\tilde{r}) = \frac{\mu+1}{\mu}\log(\mu\tilde{r}), \qquad \tilde{\phi}_B = \sqrt{\frac{2(\mu+1)(d-1)}{\mu}}\log(\mu\tilde{r}) \\ V(\tilde{\phi}_B) &= -C\exp(-\sqrt{\frac{2\mu}{(\mu+1)(d-1)}}\tilde{\phi}_B), \\ W(\tilde{\phi}_B) &= -2(d-1)(\mu+1)\exp(-\sqrt{\frac{\mu}{2(\mu+1)(d-1)}}\tilde{\phi}_B). \end{aligned}$$

Given a more general solution in the dual frame, which asymptotes to an AdS linear dilaton background, one can similarly transform it into Einstein frame and extract the corresponding superpotential etc.

Suppose the fluctuations in the Einstein frame are given by:

$$g_{E\mu\nu} = g^B_{E\mu\nu} + \tilde{\gamma}_{\mu\nu}; \qquad \tilde{\phi} = \tilde{\phi}^B + \tilde{\varphi}, \tag{5.273}$$

where $\tilde{S}^i_j \equiv h^{ik}_B\tilde{\gamma}_{kj}$ is:

$$\tilde{S}^i_j = \tilde{e}^i_j + \frac{d}{d-1}(\frac{1}{d}\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2_B})\tilde{f} + \frac{\partial^i\partial_j}{\nabla^2_B}\tilde{S}, \tag{5.274}$$

Then these fluctuations in Einstein frame are related to those in the dual frame defined in (5.261) via:

$$\begin{aligned} \tilde{e}^i_j &= e^i_j, \qquad \tilde{f} = 2\gamma\varphi + f, \\ \tilde{S} &= \frac{2\gamma d}{(d-1)}\varphi + S, \qquad \nu\tilde{\varphi} = \varphi, \qquad \tilde{\gamma}_{rr} = \frac{2\gamma d}{(d-1)}\varphi. \end{aligned} \tag{5.275}$$

Note in particular that the Weyl transformation to the Einstein frame takes the fluctuations outside axial gauge: $\tilde{\gamma}_{rr} \neq 0$.

Using [17, 20], one can write down the diagonalised equations of motion for the linear fluctuations in Einstein frame:

$$
\begin{aligned}
&(\partial_{\tilde{r}}^2 + d\dot{A}\partial_{\tilde{r}} - e^{-2A}q^2)\tilde{e}^i_j = 0, \\
&(\partial_{\tilde{r}}^2 + [d\dot{A} + 2W\partial_{\tilde{\phi}}^2 \log W]\partial_{\tilde{r}} - e^{-2A}q^2)\omega = 0, \\
&\partial_{\tilde{r}}\tilde{S} = \frac{1}{(d-1)\dot{A}}\left(e^{-2A}q^2\tilde{f} + 2\kappa^2(\partial_{\tilde{r}}\tilde{\phi}_B\partial_{\tilde{r}}\varphi - V'(\tilde{\phi}_B)\tilde{\varphi} - V(\tilde{\phi}_B)\tilde{\gamma}_{rr})\right),
\end{aligned} \tag{5.276}
$$

where

$$
\omega \equiv \frac{W}{W'}\tilde{\varphi} + \frac{1}{2\kappa^2}\tilde{f}, \tag{5.277}
$$

and we have Fourier transformed to momentum space, with q being the momentum.

To derive the two point functions we will need to obtain the functional dependence of the one-point functions on the sources. The one-point functions are given in terms of the canonical momenta, with the parts dependent on the fluctuations being given by linear combinations of radial derivatives of fluctuations. Hence we write the radial derivatives of the fluctuations $\tilde{e}^i_j$ and ω as functionals of the background fields A and $\tilde{\phi}_B$:

$$
\partial_{\tilde{r}}\tilde{e}^i_j = E(A, \tilde{\phi}_B)\tilde{e}^i_j, \qquad \partial_{\tilde{r}}\omega = \Omega(A, \tilde{\phi}_B)\omega. \tag{5.278}
$$

The first two equations in (5.276) then become first order equations for E and Ω:

$$
\begin{aligned}
&\dot{E} + E^2 + d\dot{A}E - e^{-2A}q^2 = 0, \\
&\dot{\Omega} + \Omega^2 + [d\dot{A} + 2W\partial_{\tilde{\phi}}^2 \log W]\Omega - e^{-2A}q^2 = 0.
\end{aligned} \tag{5.279}
$$

Note that in the case of the Dp-brane backgrounds these equations actually coincide since $\partial_{\tilde{\phi}}^2 \log W = 0$. Given the solutions for E and Ω and omitting terms that contribute to contact terms one can obtain the required expressions for the radial derivatives of other fluctuations:

$$
\begin{aligned}
\partial_{\tilde{r}}\tilde{e}^i_j &= E\tilde{e}^i_j, \\
\partial_{\tilde{r}}\tilde{\varphi} &= \Omega\tilde{\varphi} + \frac{1}{2\kappa^2}\frac{W'}{W}\Omega\tilde{f}, \\
\partial_{\tilde{r}}\tilde{S} &= -\frac{1}{\kappa^2}\left[\left(\frac{W'}{W}\right)^2\Omega + \frac{e^{-2A}}{W}q^2\right]\tilde{f} - 2\frac{W'}{W}\Omega\tilde{\varphi}.
\end{aligned} \tag{5.280}
$$

This completes the diagonalisation of the fluctuation equations in the Einstein frame. Next one can rewrite these relations in terms of the fluctuations and radial derivative in the dual frame as:

$$\begin{aligned} \partial_r \tilde{e}^i_j &= e^{\gamma\phi_B/(d-1)} E e^i_j, \\ \nu\partial_r \tilde{\varphi} &= e^{\gamma\phi_B/(d-1)} \left(\nu^2 \Omega (1 + \frac{\gamma}{\nu\kappa^2}\frac{W'}{W})\varphi + \frac{\nu}{2\kappa^2}\frac{W'}{W}\Omega f \right), \\ \partial_r \tilde{S} &= e^{\gamma\phi_B/(d-1)} \left(-\frac{1}{\kappa^2}[(\frac{W'}{W})^2\Omega + \frac{e^{-2A}}{W}q^2] f \right. \\ & \left. -2\nu[(\frac{W'}{W} + \frac{\gamma}{\nu\kappa^2}(\frac{W'}{W})^2)\Omega + \frac{\gamma}{\nu\kappa^2}\frac{e^{-2A}}{W}q^2]\varphi \right). \end{aligned} \tag{5.281}$$

Using (5.275) in (5.264), and applying (5.176) one finds that the expressions for the one point functions to linear order in the fluctuations are:

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= \langle \mathcal{O}_\phi \rangle_B - B(\nu\partial_r\tilde{\varphi} - \gamma\partial_r\tilde{S})_{(2\sigma)}, \\ \langle T^i_i \rangle &= \langle T^i_i \rangle_B - B(d-1)(\partial_r\tilde{S})_{(2\sigma)}, \\ \langle T^i_{j,TT} \rangle &= \langle T^i_{j,TT} \rangle_B + B(\partial_r \tilde{e}^i_j)_{(2\sigma)}, \end{aligned} \tag{5.282}$$

where $X_{(2\sigma)}$ denotes the term of dilatation weight $2\sigma \equiv (d - 2\alpha\gamma)$ in X.

To explicitly evaluate these one point functions with linear sources we now need to determine exact regular solutions for E and Ω. Up to this point, we have given completely general expressions, applicable for all solutions asymptotic to the Dp-brane backgrounds. The actual background determines the defining differential equations for E and Ω. Next we will solve these equations for the specific case of the decoupled Dp-brane background; as mentioned before, the equations for E and Ω become identical in this case since $\partial^2_\phi \log W = 0$. The only equation to be solved is thus:

$$(\partial^2_{\tilde{r}} + \frac{d(\mu+1)}{\mu\tilde{r}}\partial_{\tilde{r}} - (\mu\tilde{r})^{-2(\mu+1)/\mu)} q^2)\omega = 0. \tag{5.283}$$

The solution which is regular in the interior, $\tilde{r} \to 0$, is given by

$$\begin{aligned} \omega(\tilde{r}) &= (\mu\tilde{r})^{-c} K_{\mu c}(\frac{q}{(\mu\tilde{r})^{1/\mu}}) \equiv e^{-\sigma r} K_\sigma(q e^{-r}), \\ \mu c &= \tfrac{1}{2}(d - 2\alpha\gamma) \equiv \sigma, \end{aligned} \tag{5.284}$$

where K is the modified Bessel function of the second kind; these are exactly the same functions found in the previous section. The solution for Ω is then

$$\Omega = \partial_{\tilde{r}} \ln((\mu\tilde{r})^{-c} K_{\mu c}(\frac{q}{(\mu\tilde{r})^{1/\mu}})) \equiv e^{-\mu r}\partial_r \ln(\chi_\sigma(q, e^{-2r})), \tag{5.285}$$

where $\chi_\sigma(q,\rho)$ was given in (5.240), and is normalized to approach one as $\rho \equiv e^{-2r} \to 0$. The terms appearing in the one point functions (5.282) follow from taking the projections onto appropriate dilatation weight:

$$(e^{\gamma\phi_B/(d-1)}\Omega)_{(2\sigma)} \equiv (e^{\mu r}\Omega)_{(2\sigma)} = -2\sigma\tilde{\chi}_{(2\sigma)}(q). \tag{5.286}$$

where we have used the expansions of $\chi_\sigma(q,\rho)$ given in (5.240) and the terms of appropriate dilatation weight, $\tilde{\chi}_{(2\sigma)}(q)$, in these asymptotic expansions, see (5.242) and (5.256).

Using (5.282) one obtains the renormalised one point functions to linear order in the sources:

$$\begin{aligned}
\langle \mathcal{O}_\phi(q)\rangle &= L\tilde{\chi}_{(2\sigma)}(q)\nu(d-2\alpha\gamma)\left(-\nu\varphi(q)[1-\frac{2\gamma^2}{(d-1)\nu^2}]^2\right. \\
&\left. \qquad +f(q)[\frac{\gamma}{\nu(d-1)}-\frac{2\gamma^3}{\nu^3(d-1)^2}]\right) \\
\langle T_i^i(q)\rangle &= 2L(d-2\alpha\gamma)\tilde{\chi}_{(2\sigma)}(q)\left([-\frac{\gamma}{\nu}+\frac{2\gamma^3}{\nu^3(d-1)}]\nu\varphi(q)+\frac{\gamma^2}{\nu^2(d-1)}f(q)\right), \\
\langle T_j^i(q)\rangle_{TT} &= L(d-2\alpha\gamma)\tilde{\chi}_{(2\sigma)}(q)e_j^i(q),
\end{aligned} \tag{5.287}$$

where we have used $W'/W = -\gamma/\nu(d-1)$ and $\kappa^2 = 1/2$. The first two expressions can be rewritten as:

$$\begin{aligned}
\langle \mathcal{O}_\phi(q)\rangle &= L\gamma\frac{(d-2\alpha\gamma)}{\alpha(d-1-2\alpha\gamma)}\tilde{\chi}_{(2\sigma)}(q)\left((d-1)\phi_{(0)}(q)+\alpha f_{(0)}(q)\right) \\
\langle T_i^i(q)\rangle &= -2L\gamma\frac{(d-2\alpha\gamma)}{(d-1-2\alpha\gamma)}\tilde{\chi}_{(2\sigma)}(q)\left((d-1)\phi_{(0)}(q)+\alpha f_{(0)}(q)\right),
\end{aligned} \tag{5.288}$$

where we have renamed the sources as $\varphi(q) \equiv \phi_{(0)}(q)$ and $f(q) \equiv f_{(0)}(q)$ to demonstrate agreement with the expressions obtained previously in (5.244) and (5.245). The two point functions are given as before by (5.246).

(5.8) Applications

In this section we will present a number of applications of the holographic methods.

(5.8.1) Non-extremal D1 branes

Let us first consider non-extremal D1-branes, and derive the renormalized vevs and onshell action. The ten-dimensional solution for non-extremal D1-branes is:

$$\begin{aligned}
ds^2 &= H^{-1/2}(-fdt^2+dx^2)+H^{1/2}(\frac{dr^2}{f}+r^2d\Omega_7^2); \\
e^\phi &= g_sH^{1/2}; \qquad F_{01r} = g_s^{-1}\partial_r(1-\frac{Q}{r^6}H^{-1}),
\end{aligned} \tag{5.289}$$

with

$$H = 1+\frac{\mu^6\sinh^2\alpha}{r^6}; \qquad f = (1-\frac{\mu^6}{r^6}); \qquad Q \equiv r_o^6 = \mu^6\sinh\alpha\cosh\alpha. \tag{5.290}$$

The extremal limit is reached by taking $\mu \to 0$ and $\alpha \to \infty$ with $\mu^3\sinh\alpha$ fixed. In the near extremal limit, for which $\mu \ll 1$, the decoupled dual frame metric is

$$ds^2_{dual} = (g_sN)^{-1/3}\left(\left(\frac{r}{r_o}\right)^4(-fdt^2+dx^2)+r_o^2(\frac{dr^2}{r^2f}+d\Omega_7^2)\right). \tag{5.291}$$

Applying the reduction formulae (5.26) gives an asymptotically AdS_3 solution of the three-dimensional action:

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2 f} + \frac{1}{\rho}(-fdt^2 + dx^2); \\ e^{-4\phi/3} &= \frac{1}{\rho}, \qquad f = \left(1 - \frac{8\mu^6}{r_o^9}\rho^{3/2}\right). \end{aligned} \tag{5.292}$$

The inverse Hawking temperature β_H and the area of the horizon A are respectively given by

$$\beta_H = \frac{2\pi r_o^3}{3\mu^2}; \qquad A = \frac{8\pi R_x \mu^4}{r_o^6}, \tag{5.293}$$

where the x direction is taken to be periodic with period $2\pi R_x$.

Next one can read off the vevs for the stress energy tensor and scalar operator by bringing the metric into Fefferman-Graham form:

$$\begin{aligned} ds^2 &= \frac{dz^2}{4z^2} + \frac{1}{z}\left(-dt^2(1 - \frac{16\mu^6}{3r_o^9}z^{3/2}) + dx^2(1 + \frac{8\mu^6}{3r_o^9}z^{3/2})\right); \\ e^{-2\phi/3} &\equiv \frac{1}{\sqrt{z}}e^{\kappa} = \frac{1}{\sqrt{z}}(1 + \frac{4\mu^6}{3r_o^9}z^{3/2}). \end{aligned} \tag{5.294}$$

Then applying (5.128) and (5.131) (analytically continued back to the Lorentzian) the vevs of the stress energy tensor are:

$$\langle T_{tt}\rangle = 16L\frac{\mu^6}{r_o^9}; \qquad \langle T_{yy}\rangle = 8L\frac{\mu^6}{r_o^9}; \qquad \langle \mathcal{O}\rangle = -4L\frac{\mu^6}{r_o^9}, \tag{5.295}$$

with the conformal Ward identity (5.132) manifestly satisfied. Note that the mass is given by

$$M = \int dx \langle T_{tt}\rangle = LR_x\frac{32\pi\mu^6}{r_o^9}. \tag{5.296}$$

The renormalized onshell (Euclidean) action I_E is given by

$$I_E = -L[\int_{\rho\geq\epsilon} d^3x\sqrt{g}\Phi(R + C) - \int_{\rho=\epsilon} d^2x\sqrt{h}\Phi(2K - 4 - R[h])]. \tag{5.297}$$

Evaluating this action on the solution gives

$$I_E = -2\pi\beta_H R_x L\frac{8\mu^6}{r_o^9}, \tag{5.298}$$

whilst the entropy is

$$S = 4\pi LA = L\frac{32\pi^2 R_x\mu^4}{r_o^6}, \tag{5.299}$$

and thus the expected relation

$$I_E = \beta_H M - S \tag{5.300}$$

is satisfied. Note that $M/T_H S = 2/3$. This result is in agreement with the results found in [119] for the entropy of non-extremal Dp-branes. The entropy can be rewritten as

$$S = \frac{2^4\pi^{5/2}}{3^3}\frac{N^2}{g_{eff}(T_H)}(V_1 T_H), \tag{5.301}$$

where $V_1 = 2\pi R_H$ is the spatial volume of the D1 brane and $g_{eff}^2 = g_2^2 N T_H^{-2}$ is the dimensionless effective coupling (with $g_2^2 = g_s/(2\pi\alpha')$ the dimensionful Yang-Mills coupling constant). This is indeed of the form (5.53) dictated from the generalized conformal structure. The overall N^2 is due to the fact that the bulk computation is a tree-level computation.

(5.8.2) The Witten model of holographic YM_4 theory

As the next application of the formalism let us discuss Witten's holographic model for four dimensional Yang-Mills theory [90]. An early discussion of holographic computations in this model can be found in [120]. In this model one considers D4 branes wrapping a circle of size L_τ with anti-periodic boundary conditions for the fermions, which breaks the supersymmetry. This system at low energies looks like a four-dimensional $SU(N)$ gauge theory, with Yang-Mills coupling $g_4^2 = g_5^2/L_\tau$. In the limit that $\lambda_4 = g_4^2 N \gg 1$ there is an effective supergravity description given by the D4 brane soliton solution, which (in the string frame) is [90, 121]:

$$\begin{aligned} ds_{st}^2 &= \left(\frac{r}{r_o}\right)^{3/2}[\eta_{\alpha\beta}dx^\alpha dx^\beta + f(r)d\tau^2] + \left(\frac{r_o}{r}\right)^{3/2}(\frac{dr^2}{f(r)} + r^2 d\Omega_4^2), \\ e^\phi &= g_s\left(\frac{r}{r_o}\right)^{3/4}, \qquad F_4 = 3g_s^{-1}r_o^3 d\Omega_4, \\ f(r) &= 1 - \frac{r_{KK}^3}{r^3}, \end{aligned} \tag{5.302}$$

where $d\Omega_4$ is the volume form of the S^4 and r_o was defined below (5.26). Then r_{KK} is the minimum value of the radial coordinate and the circle direction τ must have periodicity $L_\tau = 4\pi r_o^{3/2}/(3r_{KK}^{1/2})$ to prevent a conical singularity.

By wrapping D8-branes around the S^4, and along the four flat directions, one can model chiral flavors in the gauge theory [91, 92] and the resulting Witten-Sakai-Sugimoto model has attracted considerable attention as a simple holographic model for a non-supersymmetric four-dimensional gauge theory. The methods developed in this chapter immediately allow one to extract holographic data from this background, and to quantify the features of QCD which are well or poorly modeled.

Starting from the ten-dimensional string frame solution, one can move to the dual frame $ds_{dual}^2 = (Ne^\phi)^{-2/3}ds^2$ in which the metric becomes asymptotically $AdS_6 \times S^4$:

$$\begin{aligned} ds_{dual}^2 &= (Ne^\phi)^{-2/3}ds_{st}^2 = \pi^{2/3}\alpha'\left(4[\frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{\eta_{\alpha\beta}dx^\alpha dx^\beta + f(\rho)d\tau^2}{\rho}] + d\Omega_4^2\right), \\ f(\rho) &\equiv 1 - \frac{\rho^3}{\rho_{KK}^3} = f(r), \end{aligned} \tag{5.303}$$

with changed variable $\rho = 4r_o^3/r$. Comparing with the reduction given in (5.26), one obtains the following six-dimensional background:

$$
\begin{aligned}
ds^2 &= \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{\eta_{\alpha\beta}dx^\alpha dx^\beta + f(\rho)d\tau^2}{\rho}; \\
e^\phi &= \frac{1}{\rho^{3/4}},
\end{aligned}
\tag{5.304}
$$

which is asymptotically AdS_6 with a linear dilaton.

The gauge theory operators dual to the metric and the scalar field are the five-dimensional stress energy tensor T_{ij} and the gluon operator $\mathcal{O}$ respectively, which satisfy the dilatation Ward identity (see (5.146) or (5.182)):

$$
\langle T^i_i \rangle + \frac{1}{g_5^2}\langle \mathcal{O} \rangle = 0. \tag{5.305}
$$

(There is no anomaly in this case, as both $g_{(0)}$ and $\kappa_{(0)}$ are constant.) This Ward identity can be rewritten in terms of operators in the four-dimensional theory obtained via reduction over the circle: the four-dimensional stress energy tensor $T^{(4)}_{ab} = L_\tau T_{ab}$ and the scalar operator $\mathcal{O}_\tau = L_\tau T_{\tau\tau}$. This gives

$$
\langle T^a_a \rangle + \langle \mathcal{O}_\tau \rangle + \frac{1}{g_4^2}\langle \mathcal{O} \rangle = 0. \tag{5.306}
$$

Consider the dimensional reduction of the stress energy tensor and gluon operator defined in (5.46) from five to four dimensions. When the reduction over the circle preserves supersymmetry, the operator $\mathcal{O}_\tau$ coincides with $-\frac{1}{g_4^2}\mathcal{O}$ and the four-dimensional stress energy tensor is traceless. With non-supersymmetric boundary conditions, this is not the case anymore, since as we will see shortly the vacuum expectation value of the trace of the stress energy tensor is not zero and the vevs of the two operators are different. With the proper identification of the relation between $\mathcal{O}_\tau$ and $\mathcal{O}$, the trace Ward identity would lead to the identification of the beta function.

Next one can extract the one point functions for the stress energy tensor and gluon operators from the coefficients in the asymptotic expansion of this solution near the boundary. To apply the formulae for the holographic vevs, the metric should first be brought into Fefferman-Graham form by changing the radial variable:

$$
\begin{aligned}
\tilde{\rho} &= (1 + \frac{\rho^3}{6\rho_{KK}^3})\rho + \mathcal{O}(\rho^5), \\
ds^2 &= \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \tilde{\rho}^{-1}(1 + \frac{\tilde{\rho}^3}{6\rho_{KK}^3})\eta_{\alpha\beta}dx^\alpha dx^\beta + \tilde{\rho}^{-1}(1 - \frac{5\tilde{\rho}^3}{6\rho_{KK}^3})d\tau^2 + \cdots .
\end{aligned}
\tag{5.307}
$$

Using (5.176) the one-point function of the scalar operator is thus:

$$
\langle \mathcal{O}_\phi \rangle = -12L\gamma\kappa_{(6)} = -\frac{2L}{3\rho_{KK}^3}, \tag{5.308}
$$

with the vev of the stress energy tensor being:

$$
\langle T_{\alpha\beta} \rangle = \frac{L}{\rho_{KK}^3}\eta_{\alpha\beta}; \qquad \langle T_{\tau\tau} \rangle = -5\frac{L}{\rho_{KK}^3}. \tag{5.309}
$$

The gluon condensate can be reexpressed as:

$$\langle \mathcal{O}_\phi \rangle = -\frac{2^6 \pi^2}{3^8} N^2 \frac{\lambda_4}{L_\tau^5}, \tag{5.310}$$

where recall that $\lambda_4 = g_4^2 N$ is the four-dimensional 't Hooft coupling and L_τ is the radius of the circle. In terms of the dimension four operator $\mathcal{O}$ the condensate is

$$\langle \mathcal{O} \rangle = \frac{2^5 \pi^2}{3^7} N \frac{\lambda_4^2}{L_\tau^4}. \tag{5.311}$$

In comparing results for this holographic model with those of QCD, it would be natural to match the condensate values, and thus fix L_τ.

(5.9) DISCUSSION

In this chapter we have developed precision holography for the non-conformal branes. We found that all holographic results that were developed earlier in the context of holography for the conformal branes can be extended to this more general setup. All branes under consideration have a near-horizon limit with non-vanishing dilaton and a metric that (in the string frame) is conformal to $AdS_{p+2} \times S^{8-p}$. This implies that there is a frame, the dual frame, where the metric is exactly $AdS_{p+2} \times S^{8-p}$ (one can cancel the overall conformal factor by multiplying the metric with the appropriate power of the dilaton).

There are a number of reasons why this frame is distinguished. Firstly, it is manifest in this frame that there is an effective $(p+1)$-dimensional gravitational description, obtained by reducing over S^{8-p}, as required by holography. Secondly, the setup becomes the same as that of holographic RG flows studied earlier. Actually the bulk solutions do describe an RG flow, albeit a trivial one driven by the dimension of the coupling constant. Recall that in the holographic RG flows studied in the past the bulk solution asymptotically becomes AdS, corresponding to the fact that the dual QFT approaches a fixed point in the UV. The scalar fields vanish asymptotically, and from the asymptotic fall off one can infer whether the bulk solution corresponds to a deformation of the UV Lagrangian by the addition of the operator dual to the corresponding field or the conformal theory in a non-trivial state characterized (in part) by the vev of the dual operator. The coefficients in the asymptotic expansion of the solution determine the coupling constant multiplying the dual operator in the case of deformations, or the vev of the dual operator in the case of non-trivial states.

The non-conformal branes are analogous to the case of deformations: the asymptotic value of the dilaton determines the value of the coupling constant, which is the (dimensionful) Yang-Mills coupling constant in the case of Dp branes. The main difference is that in the current context the theory does not flow in the UV to a $(p+1)$-dimensional fixed point. Rather in the regime where the various approximations are valid, the theory runs trivially due to the dimensionality of the coupling constant.

In some cases however we know that a new dimension, the M-theory dimension, opens up at strong coupling and the theory flows to a $(p+2)$-dimensional fixed point. This is the case for the IIA fundamental string and the D4 brane which uplift to the M2 and M5 brane theories, respectively. Here is another instance that illustrates the preferred status of the dual frame: the general solution in the dual frame

$$ds_d^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij} dx^i dx^j \tag{5.312}$$

$$e^{4\phi/3} = \frac{1}{\rho} e^{2\kappa}, \tag{5.313}$$

lifts to

$$ds_{d+1}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(g_{ij} dx^i dx^j + e^{2\kappa} dy^2). \tag{5.314}$$

In other words, the dual frame metric in the Fefferman-Graham gauge in d-dimensions is equal to the d-dimensional part of the metric in $(d+1)$ dimensions in the Fefferman-Graham gauge, with the dilaton providing the additional dimension. It was already observed in [89] that the radial coordinate in the dual frame is identified with the energy of the dual theory via the UV-IR connection and here we see a more precise formulation of this statement. The radial direction of the M5 and M2 branes is also the radial direction in the dual frame of the D4 and F1 branes, respectively. In more covariant language, the dilatation operator of the boundary theory is to leading order equal to the radial derivative of the dual frame metric.

Working in the dual frame, we have systematically developed holographic renormalization for all non-conformal branes. In particular, we obtained the general solutions of the field equations with the appropriate Dirichlet boundary conditions. This allowed us to identify the volume divergences of the action, and then remove these divergences with local covariant counterterms. Having defined the renormalized action, we then proceeded to calculate the holographic one-point functions which, by further functional differentiation w.r.t sources, yield the higher point functions. The counterterm actions can be found in (5.139) and (5.143), whilst the holographic one point functions are given in (5.140) and (5.145). Note that the result for the stress energy tensor properly defines the notion of mass for backgrounds with these asymptotics.

We developed holographic renormalization both in the original formulation, described in the previous paragraph, and in the radial Hamiltonian formalism (in section 5.6). In the latter, Hamilton-Jacobi theory relates the variation of the on-shell action w.r.t. boundary conditions, thus the holographic 1-point functions, to radial canonical momenta. It follows that one can bypass the on-shell action and directly compute renormalized correlators using radial canonical momenta π, as was developed for asymptotically AdS spacetimes in [19, 20]. For explicit calculations, the Hamiltonian method is more efficient and powerful, as it exploits to the full the underlying symmetry structure.

Throughout the existence of an underlying generalized conformal structure plays a crucial role. As we discussed in section 5.4, SYM in d dimensions admits a generalized conformal structure, in which the action is invariant under Weyl transformations provided that the coupling constant is also promoted to a background field $\Phi_{(0)}$ which transforms appropriately. This background field can be thought of as a source for a gauge invariant operator $\mathcal{O}$. Then diffeomorphism and Weyl invariance imply Ward identities for the correlators of the stress energy tensor and the operator $\mathcal{O}$. This generalized conformal structure is preserved at strong coupling, and governs the holographic Ward identities. In particular, the Dirichlet boundary conditions for the dilaton are determined by the field theory source $\Phi_{(0)}$.

In the cases of the type IIA fundamental string and D4-branes, all the holographic results we find are manifestly compatible with the M theory uplift. In particular, we showed in detail how the asymptotic solutions, counterterms, one point functions and anomalies descend from those of M2 and M5 branes. The generalized conformal structure is also inherited from the higher dimensional conformal symmetry in these cases. This is exactly analogous to the case of the more familiar holographic RG flows, which also have a similar generalized conformal structure inherited from the UV fixed point.

Having set up the formalism in full generality, we then proceeded to discuss a number of examples and applications. In section 5.7 we calculated two point functions of the stress energy tensor and gluon operator. We computed these two point functions for the supersymmetric backgrounds, and showed that the results were consistent with the underlying generalized conformal structure. In section 5.7.3 we developed a general method for computing two point functions in any background which asymptotes to the non-conformal brane background.

In section 5.8 we gave several more applications. One was the explicit evaluation of the mass and action in a non-extremal brane background. The second was Witten's model for a non-supersymmetric four-dimensional gauge theory: we computed the dimension four condensates in this model. One would anticipate that there are many further interesting applications of the formalism developed here, to be explored in future work.

(5.A) APPENDIX

(5.A.1) USEFUL FORMULAE

In this appendix we collect some useful formulae for the asymptotic expansions. Given the expansion of the d-dimensional metric g_{ij} as

$$g_{ij} = g_{(0)ij} + \rho g_{(2)ij} + \rho^2 g_{(4)ij} + \cdots \tag{5.315}$$

the inverse metric is given by

$$g^{-1} = g^{-1}_{(0)} - \rho g^{-1}_{(0)} g_{(2)} g^{-1}_{(0)} + \rho^2 (g^{-1}_{(0)} g_{(2)} g^{-1}_{(0)} g_{(2)} g^{-1}_{(0)} - g^{-1}_{(0)} g_{(4)} g^{-1}_{(0)}) + \cdots \tag{5.316}$$

Next we compute the expansion of the Christoffel connection,

$$\Gamma^i_{ij} = \Gamma^i_{(0)ij} + \rho \Gamma^i_{(2)ij} + \rho^2 \Gamma^i_{(4)ij} + \cdots \tag{5.317}$$

Here $\Gamma^i_{(0)ij}$ is the Christoffel connection of the metric $g_{(0)}$ and

$$\begin{aligned}\Gamma^i_{(2)ij} &= \frac{1}{2} g^{il}_{(0)} (\nabla_j g_{(2)kl} + \nabla_k g_{(2)jl} - \nabla_l g_{(2)jk}) \qquad (5.318)\\ \Gamma^i_{(4)ij} &= \frac{1}{2} \left(g^{il}_{(0)} (\nabla_j g_{(4)kl} + \nabla_k g_{(4)jl} - \nabla_l g_{(4)jk}) - g^{il}_{(2)} (\nabla_j g_{(2)kl} + \nabla_k g_{(2)jl} - \nabla_l g_{(2)jk}) \right),\end{aligned}$$

where ∇ is the covariant derivative in the metric $g_{(0)}$.

From here we then compute the expansion of the associated curvature

$$R_{ij} = R_{(0)ij} + \rho R_{(2)ij} + \rho^2 R_{(4)ij} + \cdots \qquad (5.319)$$

with $R_{(0)ij}$ the Ricci tensor of $g_{(0)}$ and

$$\begin{aligned}R_{(2)ij} &= \frac{1}{2} \Big(\nabla^k \nabla_j g_{(2)ik} - \nabla^i \nabla_j \mathrm{Tr}(g_{(2)}) + R_{(0)kijl} g^{kl}_{(2)} + R_{(0)im} g^m_{(2)j} \\ &\quad - \nabla^2 g_{(2)ij} + \nabla_i \nabla^k g_{(2)jk} \Big), \qquad (5.320)\\ R_{(4)ij} &= \frac{1}{2} \Big(\frac{1}{2} \nabla^l \mathrm{Tr} g_{(2)} \nabla_l g_{(2)ij} + g^{kl}_{(2)} (R_{kimj} g^m_{(2)l} + R_{kiml} g^m_{(2)j}) \qquad (5.321)\\ &\quad + g^{kl}_{(2)} (R_{kjmi} g^m_{(2)l} + R_{kjml} g^m_{(2)i}) + g^{kl}_{(2)} \nabla_k \nabla_l g_{(2)ij} + \frac{1}{2} \nabla_j g_{(2)lm} \nabla^l g^m_{(2)i} \\ &\quad - \frac{1}{2} g^{kl}_{(2)} (\nabla_i \nabla_k g_{(2)jl} + \nabla_j \nabla_k g_{(2)il}) - 2R_{limj} g^{lm}_{(4)} + R_{im} g^m_{(4)j} + R_{jm} g^m_{(4)i} \\ &\quad + \frac{1}{4} g^l_{(2)j} \nabla_i \nabla_l \mathrm{Tr} g_{(2)} + \frac{1}{4} g^l_{(2)i} \nabla_j \nabla_l \mathrm{Tr}(g_{(2)}) + \frac{1}{2} \nabla_i g_{(2)lm} \nabla^l g^m_{(2)j} \\ &\quad - \nabla^2 g_{(4)ij} - \frac{1}{2} \nabla_i g_{(2)lm} \nabla_j g^{lm}_{(2)} - \nabla_m g_{(2)il} \nabla^l g^m_{(2)j} - \nabla_m g_{(2)il} \nabla^m g^l_{(2)j} \Big).\end{aligned}$$

(5.A.2) THE ENERGY MOMENTUM TENSOR IN THE CONFORMAL CASES

In this section we streamline the derivation of the vev of the energy-momentum tensor in terms of the asymptotic coefficients for the conformal cases $D = 4$ and $D = 6$ given in [15]. The starting point is the expression of the stress energy tensor as sum of two contributions, one originating from the bulk action and the other from the counterterms, eqns (3.5)-(3.6)-(3.7) of [15]:

$$\begin{aligned}\langle T_{ab} \rangle &= 2L_{D+1} \lim_{\rho \to 0} \Big(\frac{1}{\rho^{D/2-1}} T_{ab}[G] \Big), \qquad (5.322)\\ T_{ab}[G] &= T^{reg}_{ab} + T^{ct}_{ab},\\ T^{reg}_{ab} &= G'_{ab} - G_{ab} \mathrm{Tr}[G^{-1} G'] - \frac{1-D}{\rho} G_{ab},\\ T^{ct}_{ab} &= -\frac{D-1}{\rho} G_{ab} + \frac{1}{(D-2)} \Big(R(G)_{ab} - \frac{1}{2} R(G) G_{ab} \Big) \\ &\quad + \frac{\rho}{(D-4)(D-2)^2} \Big[\Box R(G)_{ab} + 2R(G)_{acbd} R(G)^{cd} - \frac{D-2}{2(D-1)} D_a D_b R(G) \\ &\quad - \frac{D}{2(D-1)} R(G) R(G)_{ab} - \frac{1}{2} G_{ab} \Big(R(G)_{cd} R(G)^{cd} - \frac{D}{4(D-1)} R(G)^2 \\ &\quad + \frac{1}{D-1} \Box R(G) \Big) \Big] + \frac{1}{2} T^{log}_{ab} \log \rho,\end{aligned}$$

where $L_{D+1} = \frac{1}{16\pi G_{D+1}}$ with G_{D+1} the Newton constant and T^{log}_{ab} is the stress energy tensor of the action given by the conformal anomaly[7]. Note that for $D = 2$ only the first term in T^{ct}_{ab} applies; for $D = 4$ only the first line applies plus the logarithmic terms, whilst for $D = 6$ all terms listed are needed and for $D > 6$ one would need to include additional terms.

For $D = 2$ one immediately obtains the answer

$$\langle T_{ab}\rangle = 2L_{D+1}\left(G_{(2)ab} - G_{(0)ab}\mathrm{Tr}G_{(2)}\right) \tag{5.323}$$

For $D > 2$ one can simplify the evaluation of (5.322) by using the equation of motions (5.75) to obtain

$$\begin{aligned} R_{ab} - \frac{1}{2}RG_{ab} &= -(D-2)G'_{ab} + (D-2)\mathrm{Tr}(G^{-1}G')G_{ab} \\ &\quad +\rho[2G'' - 2G'G^{-1}G' + \mathrm{Tr}(G^{-1}G')G' \\ &\quad +(-\mathrm{Tr}(G^{-1}G'') + \mathrm{Tr}(G^{-1}G')^2 - \frac{1}{2}(\mathrm{Tr}G^{-1}G')^2)G]_{ab}. \end{aligned} \tag{5.324}$$

Using this identity in (5.322) we see that T^{reg}_{ab} cancels the first line of T^{ct}_{ab} up to the terms proportional to ρ in (5.324), so $T_{ab}[G]$ is manifestly linear in ρ. It follows that we only need to set $\rho = 0$ in the remaining terms to obtain the vev for $D = 4$:

$$\begin{aligned} \langle T_{ab}\rangle &= 2L_{D+1}\Bigg(2G_{(4)ab} - G_{(2)}{}^2_{ab} + \frac{1}{2}\mathrm{Tr}G_{(2)}G_{(2)ab} \\ &\quad +\frac{1}{4}G_{(0)ab}(\mathrm{Tr}(G^{-1}G_{(2)})^2 - (\mathrm{Tr}G^{-1}G_{(2)})^2) + 3H_{(4)ab}\Bigg). \end{aligned} \tag{5.325}$$

For $D = 6$ one can check straightforwardly that order ρ terms in $T_{ab}[G]$ cancel, so there is indeed a finite limit. To obtain the vev one needs to extract the order ρ^2 terms. To simplify this computation we differentiate the field equations (5.75) to obtain a formula for the radial derivative of the Ricci tensor,

$$\begin{aligned} R'_{ab} &= R^c_{(a}G'_{b)c} - R_{acbd}G'^{cd} + D_{(a}D^bG'_{b)c} - \frac{1}{2}\Box G'_{ab} + D_a\partial_b\mathrm{Tr}G' \\ &= \frac{1}{D-2}[-R_{acbd}R^{cd} + \frac{D-2}{4(D-1)}D_aD_bR + \frac{1}{2}\Box R_{ab} + \frac{1}{4(D-1)}\Box RG_{ab} \\ &\quad +R^c_aR_{bc} - \rho[4R_{(0)}{}^c_{(a}\tilde{C}_{b)c} - 4R_{(0)acbd}\tilde{C}^{cd} - \frac{D-2}{4(D-1)}D_a\partial_bB \\ &\quad -2\Box\tilde{C}_{ab} - \frac{1}{4(D-1)}G_{(0)ab}]] + \mathcal{O}(\rho^2), \\ \tilde{C}_{ab} &= (G_{(4)} - \frac{1}{2}G_{(2)}{}^2 + \frac{1}{4}G_{(2)}\mathrm{Tr}G_{(2)})_{ab}, \qquad B = \mathrm{Tr}G_{(2)}{}^2 - (\mathrm{Tr}G_{(2)})^2. \end{aligned} \tag{5.326}$$

Then we note that the terms involving the Riemann tensor and covariant derivatives enter with the same relative factors as in T^{ct}_{ab}, so we can use (5.326) to express T^{ct}_{ab} in terms of $R'_{ab} = R_{(2)ab} + 2\rho R_{(4)ab} + \cdots$, which is easier to relate to higher expansion coefficients. Indeed, as is discussed in the next appendix, the coefficient $R_{(2)ab}, R_{(4)ab}$ can be expressed in terms of $G_{(2)ab}$, $G_{(4)ab}$ and $H_{(6)ab}$.

[7] The factor of 1/2 in front of T^{log}_{ab} corrects a typo in [15].

Combining these results and setting $D = 6$ we obtain

$$\langle T_{ab}\rangle = 2L_7\left(3G_{(6)ab} - 3A_{(6)ab} + \frac{1}{8}S_{ab} + \frac{11}{2}H_{(6)ab}\right), \quad (5.327)$$

where $A_{(6)ab}$ and S_{ab} are given by [15]

$$\begin{aligned}
S_{ab} &= \Box C_{ab} + 2R_{acbd}C^{cd} + 4(G_{(2)}G_{(4)} - G_{(4)}G_{(2)})_{ab} \\
&+\frac{1}{10}(D_aD_bB - G_{(0)ab}\Box B) + \frac{2}{5}G_{(2)ab}B \\
&+G_{(0)ab}(-\frac{2}{3}\mathrm{Tr}G_{(2)}^3 - \frac{4}{15}(\mathrm{Tr}G_{(2)})^3 + \frac{3}{5}\mathrm{Tr}G_{(2)}\mathrm{Tr}G_{(2)}^2), \\
A_{(6)ab} &= \frac{1}{3}\Big((2G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab} - G_{(2)ab}^3 + \frac{1}{8}[\mathrm{Tr}G_{(2)}^2 - (\mathrm{Tr}G_{(2)})^2]G_{(2)ab} \\
&-\mathrm{Tr}G_{(2)}[G_{(4)ab} - \frac{1}{2}G_{(2)ab}^2] - [\frac{1}{8}\mathrm{Tr}G_{(2)}^2\mathrm{Tr}G_{(2)} - \frac{1}{24}(\mathrm{Tr}G_{(2)})^3 \\
&-\frac{1}{6}\mathrm{Tr}G_{(2)}^3 + \frac{1}{2}\mathrm{Tr}(G_{(2)}G_{(4)})]G_{(0)ab}\Big), \\
C_{ab} &= (G_{(4)} - \frac{1}{2}{G_{(2)}}^2 + \frac{1}{4}G_{(2)}\mathrm{Tr}G_{(2)})_{ab} + \frac{1}{8}G_{(0)ab}B, \qquad B = \mathrm{Tr}{G_{(2)}}^2 - (\mathrm{Tr}G_{(2)})^2.
\end{aligned} \quad (5.328)$$

Noting that $L_7 = N^3/(3\pi^3)$ and introducing the combination

$$t_{ab} = G_{(6)ab} - A_{(6)ab} + \frac{1}{24}S_{ab} \quad (5.329)$$

the stress energy tensor may be expressed as

$$\langle T_{ab}\rangle = \frac{N^3}{3\pi^3}(6t_{ab} + 11H_{(6)ab}). \quad (5.330)$$

This result includes the term in $H_{(6)ab}$ which was not given in [15].

(5.A.3) Reduction of M5 to D4

The expansion coefficients for an asymptotically local AdS_{D+1} metric were given in [15]. We will be interested in the case where $D = d+1$, for which the first expansion coefficients are:

$$\begin{aligned}
G_{(2)ab} &= \frac{1}{d-1}\left(-R_{(0)ab} + \frac{1}{2d}R_{(0)}G_{(0)ab}\right); \\
G_{(4)ab} &= \frac{1}{2(d-3)}\left(-R_{(2)ab} - 2(G_{(2)}^2)_{ab} + \frac{1}{2}\mathrm{Tr}(G_{(2)}^2)G_{(0)ab}\right).
\end{aligned} \quad (5.331)$$

Using the explicit form of $G_{(2)ab}$ and the D-dimensional analogue of (5.320) we obtain:

$$\begin{aligned}
R_{(2)ab} &= -\frac{1}{2(d-1)}\Big(2R_{(0)ac}R_{(0)b}^c - 2R_{(0)cadb}R_{(0)}^{cd} - \frac{d-1}{2d}D_aD_bR_{(0)} \\
&+D^2R_{(0)ab} - \frac{1}{2d}D^2R_{(0)}G_{(0)ab}\Big),
\end{aligned} \quad (5.332)$$

$$\begin{aligned} G_{(4)ab} &= -\frac{1}{d-3}\Big(-\frac{1}{8d}D_aD_bR + \frac{1}{4(d-1)}D_cD^cR_{ab} \\ &-\frac{1}{8d(d-1)}D_cD^cRG_{(0)ab} + \frac{1}{2(d-1)}R^{cd}R_{acbd} \\ &-\frac{d-3}{2(d-1)^2}R_a^cR_{cb} - \frac{1}{d(d-1)^2}RR_{ab} \\ &-\frac{1}{4(d-1)^2}R^{cd}R_{cd}G_{(0)ab} + 3\frac{(d+1)}{16d^2(d-1)^2}R^2G_{(0)ab}\Big), \end{aligned}$$

where D_a is the covariant derivative in the metric $G_{(0)}$. Note that $R_{(2)} = 0$, and thus

$$\mathrm{Tr}G_{(4)} = \frac{1}{4}\mathrm{Tr}(G_{(2)}^2). \tag{5.333}$$

At next order one finds that the trace and the divergence of $G_{(6)}$ are determined via

$$\begin{aligned} \mathrm{Tr}(G_{(6)}) &= \frac{2}{3}\mathrm{Tr}(G_{(2)}G_{(4)}) - \frac{1}{6}\mathrm{Tr}(G_{(2)}^3); \\ D^aG_{(6)ab} &= D^aA_{(6)ab} + \frac{1}{6}\mathrm{Tr}(G_{(4)}D_bG_{(2)}); \\ A_{(6)ab} &= \frac{1}{3}\Big((2G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab} - (G_{(2)}^3)_{ab} + \frac{1}{8}[\mathrm{Tr}G_{(2)}^2 - (\mathrm{Tr}G_{(2)})^2]G_{(2)ab} \\ &-\mathrm{Tr}G_{(2)}[G_{(4)ab} - \frac{1}{2}(G_{(2)}^2)_{ab}] - [\frac{1}{8}\mathrm{Tr}G_{(2)}^2\mathrm{Tr}G_{(2)} - \frac{1}{24}(\mathrm{Tr}G_{(2)})^3 \\ &-\frac{1}{6}\mathrm{Tr}G_{(2)}^3 + \frac{1}{2}\mathrm{Tr}(G_{(2)}G_{(4)})]G_{(0)ab}\Big). \end{aligned} \tag{5.334, 5.335}$$

The logarithmic term in the expansion $H_{(6)}$ is given by

$$\begin{aligned} H_{(6)ab} &= \frac{1}{6}(R_{(4)ab} + (-\mathrm{Tr}(G_{(2)}G_{(4)}) + \tfrac{1}{2}\mathrm{Tr}(G_{(2)}^3))G_{(0)ab}) \\ &-\frac{1}{6}\mathrm{Tr}(G_{(2)})G_{(4)ab} - \frac{1}{3}(G_{(2)}^3)_{ab} + \frac{2}{3}(G_{(2)}G_{(4)} + G_{(4)}G_{(2)})_{ab}. \end{aligned} \tag{5.336}$$

Note that $H_{(6)}$ is traceless and divergence free.

For the dimensional reduction it is useful to note that the non-vanishing components of the Riemann tensor can be expressed as

$$\begin{aligned} R(G)_{ijkl} &= R_{ijkl}; \\ R(G)_{yiyj} &= -e^{2\kappa}(\nabla_i\partial_j\kappa + (\partial_i\kappa)(\partial_j\kappa)), \end{aligned} \tag{5.337}$$

and similarly the non-vanishing components of the Ricci tensor are

$$\begin{aligned} R(G)_{ij} &= R_{ij} - \nabla_i\partial_j\kappa - \partial_i\kappa\partial_j\kappa; \\ R(G)_{yy} &= e^{2\kappa}(-\nabla^i\partial_i\kappa - \partial_i\kappa\partial^i\kappa). \end{aligned} \tag{5.338}$$

Let furthermore S be a scalar and C_{ab} a symmetric tensor with $C_{iy} = 0$. Then the Laplacian reduces as

$$\begin{aligned} D^2S &= (\nabla^2 + \partial^i\kappa\partial_i)S, \\ D^2C_{ij} &= (\nabla^2 + \partial^l\kappa\nabla_l)C_{ij} - 2\partial_l\kappa\partial_{(i}\kappa C_{j)}^l + 2\partial_i\kappa\partial_j\kappa C_y^y, \\ D^2C_y^y &= (\nabla^2 + \partial^i\kappa\partial_i)C_y^y + 2\partial_i\kappa\partial_j\kappa C^{ij} - 2\partial_i\kappa\partial^i\kappa C_y^y. \end{aligned} \tag{5.339}$$

Letting $G_{(0)ij} = g_{(0)ij}$ and $G_{(0)yy} = e^{2\kappa_{(0)}}$ one finds that

$$\begin{aligned} R(G)_{(0)ij} &= R_{(0)ij} - \nabla_i \partial_j \kappa_{(0)} - \partial_i \kappa_{(0)} \partial_j \kappa_{(0)}; \\ R(G)_{(0)yy} &= e^{2\kappa_{(0)}}(-\nabla^i \partial_i \kappa_{(0)} - \partial_i \kappa_{(0)} \partial^i \kappa_{(0)}), \end{aligned} \tag{5.340}$$

with $R(G)_{(0)yi} = 0$. Substituting into (5.113) gives[8]

$$\begin{aligned} G_{(2)ij} &= \frac{1}{d-1}\left(-R_{(0)ij} + \frac{1}{2d} R_{(0)} g_{(0)ij} + (\nabla_{\{i}\partial_{j\}}\kappa)_{(0)} + \partial_{\{i}\kappa_{(0)}\partial_{j\}}\kappa_{(0)})\right); \\ G_{(2)yy} &= e^{2\kappa_{(0)}}\left(\frac{1}{2d(d-1)} R_{(0)} + \frac{1}{d}(\nabla^2 \kappa_{(0)} + (\partial\kappa_{(0)})^2)\right), \end{aligned} \tag{5.341}$$

with $G_{(2)yi} = 0$. Now using

$$G_{yy} = e^{2\kappa} = e^{(2\kappa_{(0)} + 2\rho\kappa_{(2)} + \cdots)} = e^{2\kappa_{(0)}}(1 + 2\rho\kappa_{(2)} + \cdots) \tag{5.342}$$

one determines $\kappa_{(2)}$ to be exactly the expression given in (5.91).

One next shows that $G_{(4)ab}$ in (5.331) reduces as

$$G_{(4)ij} = g_{(4)ij}; \qquad G_{(4)yy} = e^{2\kappa_{(0)}}(2\kappa_{(2)}^2 + 2\kappa_{(4)}), \tag{5.343}$$

with $g_{(4)ij}$ and $\kappa_{(4)}$ given in (5.95). This follows from the expansion of the six-dimensional curvatures at second order:

$$\begin{aligned} R(G)_{(2)ij} &= R_{(2)ij} - (\nabla_i \partial_j \kappa)_{(2)} - (\partial_i \kappa \partial_j \kappa)_{(2)}; \\ R(G)_{(2)yy} &= -e^{2\kappa_{(0)}}(\nabla^i \partial_i \kappa + \partial_i \kappa \partial^i \kappa)_{(2)} \\ &\quad - e^{2\kappa_{(0)}} 2\kappa_{(2)} (\nabla^i \partial_i \kappa + \partial_i \kappa \partial^i \kappa)_{(0)}. \end{aligned} \tag{5.344}$$

Reducing (5.334) gives

$$\begin{aligned} \mathrm{Tr}(G_{(6)}) &= \mathrm{Tr}(g_{(6)}) + 2\kappa_{(6)} + \frac{4}{3}\kappa_{(2)}^3 + 4\kappa_{(2)}\kappa_{(4)}; \\ &= \frac{2}{3}\mathrm{Tr}(g_{(2)}g_{(4)}) + \frac{4}{3}\kappa_{(2)}(\kappa_{(2)}^2 + 2\kappa_{(4)}) - \frac{1}{6}\mathrm{Tr}(g_{(2)}^3), \end{aligned} \tag{5.345}$$

and thus gives

$$\mathrm{Tr}(g_{(6)}) + 2\kappa_{(6)} = \frac{2}{3}\mathrm{Tr}(g_{(2)}g_{(4)}) - \frac{4}{3}\kappa_{(2)}\kappa_{(4)} - \frac{1}{6}\mathrm{Tr}(g_{(2)}^3). \tag{5.346}$$

The reduction of (5.336) gives

$$H_{(6)ij} = h_{(6)ij}; \qquad H_{(6)yy} = e^{2\kappa_0} 2\tilde{\kappa}_{(6)}, \tag{5.347}$$

with

$$\begin{aligned} h_{(6)ij} &= -\frac{1}{12}[-2R_{(4)ij} + (-\mathrm{Tr}g_{(2)}^3 + 2\mathrm{Tr}g_{(2)}g_{(4)} + 8\kappa_2\kappa_4)g_{(0)ij} + 2(\mathrm{Tr}g_{(2)})g_{(4)ij} \\ &\quad -8(g_{(4)}g_{(2)})_{ij} - 8(g_{(2)}g_{(4)})_{ij} + 4g_{(2)ij}^3 + 2(\nabla_i\partial_j\kappa)_{(4)} + 2(\partial_i\kappa\partial_j\kappa)_{(4)} + 4\kappa_{(2)}g_{(4)ij}] \\ \tilde{\kappa}_{(6)} &= -\frac{1}{12}[(\nabla^2\kappa)_{(4)} + (\partial\kappa)_{(4)}^2 + \mathrm{Tr}g_{(2)}g_{(4)} - \tfrac{1}{2}\mathrm{Tr}g_{(2)}^3 \\ &\quad -\kappa_{(2)}\mathrm{Tr}g_{(2)}^2 + 4\kappa_{(4)}\mathrm{Tr}g_{(2)} - 4\kappa_{(2)}^3 + 12\kappa_{(2)}\kappa_{(4)}], \end{aligned} \tag{5.348}$$

[8] Round brackets (ij) denote symmetrisation and curly brackets $\{ij\}$ traceless symmetrisation of indices.

which agree with the expressions (5.99). In reducing the curvature term $R(G)_{(4)yy}$ one should use the identities:

$$\begin{aligned} -((\nabla^2\kappa)+(\partial\kappa)^2)_{(0)} &= -10\kappa_{(2)} - \mathrm{Tr}g_{(2)}; \\ -((\nabla^2\kappa)+(\partial\kappa)^2)_{(2)} &= -8\kappa_{(4)} + 6\kappa_{(2)}^2 + 2\kappa_{(2)}\mathrm{Tr}g_{(2)} + \frac{1}{2}\mathrm{Tr}g_{(2)}^2. \end{aligned} \tag{5.349}$$

(5.A.4) Explicit expressions for momentum coefficients

In the following we give explicit expressions for the terms in the expansions of the momenta in eigenfunctions of the dilatation operator. The expressions given below are applicable for $\beta = 0$ in (5.28) and $d \geq 3$, although in this chapter we will use only the case of $d = 5$ (the D4-branes). Here we give $K_{(2n)ij}$ and $p^{\phi}{}_{(2n)}$ up to $n = 2$; note that $\hat{\Phi} = e^{\gamma\phi}$. These expressions are needed to compute the anomaly and one point functions for the D4-brane in the Hamiltonian formalism in section 5.6.4[9]:

$$\begin{aligned}
\gamma p^{\phi}{}_{(2)} &= -\frac{1}{d}[\frac{1}{2(d-1)}\hat{R} + \hat{\Phi}^{-1}\hat{\nabla}^2\hat{\Phi}], \\
K_{(2)} &= \frac{1}{2(d-1)}\hat{R}, \\
K_{(2)ij} &= \frac{1}{d-1}[\hat{R}_{ij} - \frac{1}{2d}\hat{R}h_{ij} - \hat{\Phi}^{-1}\hat{\nabla}_{\{i}\partial_{j\}}\hat{\Phi}]; \\
\gamma p^{\phi}{}_{(4)} &= \frac{1}{2d(d-1)^2(d-3)}[-3\hat{R}_{ij}\hat{R}^{ij} + \frac{3(d+1)}{4d}\hat{R}^2 - \frac{3}{d}\hat{\nabla}^2\hat{R} - 3(\hat{\Phi}^{-1}\hat{\nabla}_{\{i}\partial_{j\}}\hat{\Phi})^2 \\
&\quad -2(d-3)(\hat{\Phi}^{-1}\hat{\nabla}_i(\hat{R}^{ij}\partial_j\hat{\Phi}) - \frac{d+2}{2d}\hat{\Phi}^{-1}\hat{\nabla}^j(\hat{R}\partial_j\hat{\Phi}) + \frac{1}{2d}\hat{\Phi}^{-1}\hat{\nabla}^2(\hat{\Phi}\hat{R})) \\
&\quad -2d(\hat{\Phi}^{-1}\hat{\nabla}^i\hat{\nabla}^j\hat{\nabla}_{\{i}\hat{\nabla}_{j\}}\hat{\Phi} - 2\hat{\Phi}^{-1}\nabla^i(\hat{\Phi}^{-1}\partial^j\hat{\Phi}\hat{\nabla}_{\{i}\hat{\nabla}_{j\}}\hat{\Phi}))], \\
K_{(4)} &= -\frac{1}{2(d-3)(d-1)^2}[-\hat{R}_{ij}\hat{R}^{ij} + \frac{d+1}{4d}\hat{R}^2 - \frac{1}{d}\hat{\nabla}^2\hat{R} - (\hat{\Phi}^{-1}\hat{\nabla}_{\{i}\partial_{j\}}\hat{\Phi})^2 \\
&\quad -2\hat{\Phi}^{-1}\hat{\nabla}^i\hat{\nabla}^j\hat{\nabla}_{\{i}\hat{\nabla}_{j\}}\hat{\Phi} + 4\hat{\Phi}^{-1}\nabla^i(\hat{\Phi}^{-1}\partial^j\hat{\Phi}\hat{\nabla}_{\{i}\hat{\nabla}_{j\}}\hat{\Phi})], \\
K_{(4)}{}^{ij} &= \gamma p^{\phi}{}_{(4)}h^{ij} - \frac{1}{(d-1)^2(d-3)}[-2R^{ik}\hat{R}^j_k + \frac{d+1}{2d}\hat{R}\hat{R}^{ij} - 2\hat{\Phi}^{-2}\hat{\nabla}^i\partial_k\hat{\Phi}\hat{\nabla}^{\{j}\partial^{k\}}\hat{\Phi} \\
&\quad -\frac{1}{d}(\hat{\nabla}^i\partial^j\hat{R} + \hat{\nabla}^2\hat{R}^{ij}) + \hat{\Phi}^{-1}\hat{\nabla}_l X^{ijl}], \\
X^{ijl} &= -2\hat{\nabla}_k(\hat{\Phi}\hat{R}^{kl})h^{ij} + 2\hat{\nabla}^{(i}(\hat{\Phi}\hat{R}^{j)l}) - \hat{\nabla}^l(\hat{\Phi}\hat{R}^{ij}) \\
&\quad +\frac{d+1}{2d}[\hat{\nabla}^l(\hat{\Phi}\hat{R})h^{ij} - h^{l(i}\hat{\nabla}^{j)}(\hat{\Phi}\hat{R})] + 2\hat{\Phi}^{-1}\hat{\nabla}^l\partial^{(i}\hat{\Phi}\partial^{j)}\hat{\Phi} - \hat{\Phi}^{-1}\hat{\nabla}^{\{i}\partial^{j\}}\hat{\Phi}\partial^l\hat{\Phi} \\
&\quad -\frac{2}{d}\hat{\Phi}^{-1}h^{l(i}\hat{\nabla}^2\hat{\Phi}\partial^{j)}\hat{\Phi} + \frac{1}{d}[h^{l(i}\hat{\Phi}\partial^{j)}\hat{R} + \frac{d-1}{2}\hat{\Phi}\partial^l\hat{R}h^{ij} - \hat{\nabla}^l(\hat{\Phi}\hat{R}^{ij}) \\
&\quad +2\hat{\Phi}\hat{\nabla}^l\hat{R}^{ij} - d\hat{\nabla}^l\hat{\nabla}^2\hat{\Phi}h^{ij} + h^{l(i}\hat{\nabla}^{j)}\hat{\nabla}^2\hat{\Phi}].
\end{aligned} \tag{5.350}$$

Note that the terms $K_{(2)}$ and $K_{(4)}$ correspond to the (non-logarithmic) counterterms in the action.

[9] Round brackets (ij) denote symmetrisation and curly brackets $\{ij\}$ traceless symmetrisation of indices.

Chapter 6

Hydrodynamics of non-conformal branes

(6.1) Introduction

The AdS/CFT correspondence provides not only a powerful tool to study black hole physics, but also strongly coupled quantum field theories. In particular, it has become possible to explore strongly coupled finite temperature conformal field theories by analyzing asymptotically AdS black hole backgrounds, identifying the Hawking temperature of the black hole with the temperature of the dual field theory.

Since any interacting field theory locally equilibrates at high enough densities, it is expected that the evolution of long-wavelength fluctuations of strongly coupled field theories is governed by fluid dynamics. Recently, it was shown that solutions to the long-wavelength fluctuation equations around the boosted black D3 brane geometry can be mapped to solutions to non-linear equations of hydrodynamics in the dual strongly coupled conformal field theory [122]. Using the well-known AdS/CFT dictionary, the fluctuations of the metric were shown to be dual to a fluid configuration which is determined by a conserved hydrodynamic stress tensor. Demanding the bulk fluctuations to be smooth in the interior constrained the transport coefficients of the dual stress tensor.

As the hydrodynamic limit is a limit of long wavelength fluctuations, one proceeds hereby in a derivative expansion of the velocity and temperature field of the fluid. In [122] the hydrodynamical energy-momentum tensor was computed to second derivative order. In [123, 124, 125, 126], this connection was further explored for pure gravity in arbitrary dimension, in [127, 128, 129] for Einstein-Maxwell theory in 4+1 dimensions, and in [130] for gravity coupled to a scalar in 4+1 dimensions. All these cases dealt with geometries which are asymptotically AdS.

In this chapter, we generalize this discussion to non-conformal branes. As in chapter 5 we will consider solutions that asymptote locally to the near-horizon limit of Dp-brane ($p \neq 5$) and fundamental string solutions. We will call these backgrounds *asymptotically non-conformal brane* backgrounds. Using holographic renormalization for such backgrounds set up in chapter 5 makes it possible to generalize the map of bulk gravity equations to boundary hydrodynamic equations. Earlier computations of transport coefficients of non-conformal brane backgrounds using linear response theory, following the original work [131, 132] for conformal backgrounds, include [133, 134, 135, 136] (see also [137, 138, 139] for computations of transport coefficients of other non-conformal backgrounds).

Recall that in the near-horizon limit, the supergravity solutions of Dp-branes ($p \neq 5$) and fundamental strings are conformal to $AdS_{p+2} \times S^{8-p}$ and exhibit a running dilaton. The universal sector of these backgrounds is obtained by dimensionally reducing on the sphere and truncating to $(p+2)$-dimensional gravity coupled to a scalar. The action and solution can be best analyzed in the *dual frame* in which the equations of motion admit a linear dilaton AdS_{p+2} solution [89]. Actually there is a family of $(d+1)$-dimensional bulk actions parametrized by the positive parameter σ whose equations of motion admit a linear dilaton AdS solution, namely the metric is exactly AdS_{d+1} and the dilaton is given by a power of the radial coordinate, with the parameter σ determining the power. When expanding solutions which asymptotically (locally) approach such solutions in a Fefferman-Graham expansion, σ turns out to also correspond to the radial power of the normalizable mode.

The main observation of this chapter is that for half-integer σ, the $(d+1)$-dimensional action can be obtained by dimensionally reducing $(2\sigma+1)$-dimensional pure gravity with cosmological constant on a torus. In particular, any $(d+1)$-dimensional solution which is asymptotically locally linear dilaton AdS_{d+1} in the dual frame can be lifted to an asymptotically locally $AdS_{2\sigma+1}$ pure gravity solution. Furthermore, the lower dimensional equations of motion depend smoothly on σ, which implies that every solution can be generalized to arbitrary positive σ. In our case we use this fact to obtain a black brane solution which solves the equations of motion with arbitrary σ.

More generally, we show that all holographic results derived in chapter 5, namely counterterms, 1-point functions and Ward identities, can be obtained from their well-known counterparts [15] using this procedure. Note that in chapter 5 we already saw such a relationship between the holographic results for IIA fundamental strings and M2 branes and D4/M5 branes. In these cases however this was a manifestation of the M-theory unlift, whereas it is unclear what is the underlying reason for the more general relation we uncover here.

Recall also from section 5.4 that non-conformal branes admit a generalized conformal structure both at weak and at strong coupling [97, 98, 99]. For the case of Dp branes, the low-energy world-volume theory, namely maximally supersymmetric Yang-Mills theory in $(p+1)$ dimensions, is Weyl invariant when coupled to a background metric provided the coupling constant is promoted to a background field that transforms appropriately [1]. This invariance leads to a dilatation Ward identity, which is of exactly the same form as the dilatation Ward identity for RG flows induced by a relevant operator. The main difference is that in the latter case the theory flows in the UV to a fixed point and the generalized conformal structure is inherited from the conformal structure of the fixed point but in the non-conformal theories the running, within the regime of validity of the corresponding descriptions (weak or strong), is due to dimensionality of the coupling constant. For IIA fundamental strings and D4 branes, which have $\sigma = 3/2, 3$, respectively, a new dimension, the M-theory dimension, opens up at strong coupling and the theories indeed flow to a fixed point. In these cases one can understand the generalized conformal structure as descending from the conformal symmetry of the corresponding M-theory system. The results of this chapter connect the generalized conformal structure of all cases to a conformal structure of a theory in higher (albeit non-integral!) dimensions.

Applied to hydrodynamics we can draw the following conclusion. According to [122] and their generalizations, every smooth metric fluctuations around the black brane solution in $(2\sigma + 1)$-dimensional pure gravity can be mapped to a solution of conformal hydrodynamics with specific transport coefficients in (2σ) dimensions. In a similar manner, metric and scalar fluctuations around the non-conformal black brane with given σ in $(d + 1)$ bulk dimensions will be dual to a solution to non-conformal hydrodynamics in d dimensions. Since every solution of the non-conformal gravity/scalar system can after continuation in σ be uplifted to a solution of the higher dimensional pure gravity system, we conclude that the non-conformal hydrodynamics in d dimensions can be obtained by dimensional reduction of conformal hydrodynamics in (2σ) dimensions and continuation in σ.

An immediate consequence of the fact that the non-conformal hydrodynamic stress tensor can be obtained by dimensional reduction from the conformal stress tensor is that the ratio between bulk and shear viscosity ζ/η is fixed. A different ratio in the non-conformal fluid would uplift to a non-vanishing bulk viscosity in the conformal fluid, which is forbidden by conformal symmetry. A related argument was presented in [139]. Hence the ratio of bulk and shear viscosity in the non-conformal fluid is dictated by the generalized conformal structure. Furthermore the ratio we find, and which was found earlier for Dp-branes in [133, 134, 135], saturates the bound proposed in [135], $\zeta/\eta \geq 2(1/(d-1) - c_s^2)$, where c_s is the speed of sound in the fluid. This bound was proposed to be universal for strongly coupled gauge theory plasmas, similar to the KSS conjecture [140], $\eta/s \geq 1/4\pi$, for the the ratio between shear viscosity and entropy density. However, there is an important qualitative difference between the two cases. In the latter case, the ratio that saturates the bound, $\eta/s = 1/4\pi$, is obtained by requiring smoothness of the bulk solution in the interior, so it has a dynamics origin, whereas in the latter case, the ratio that saturates the bound, $\zeta/\eta = 2(1/(d-1) - c_s^2)$, follows from the generalized conformal structure so it is of kinematical origin.

In [122] and generalizations, the transport coefficients of the conformal hydrodynamic stress

[1] In a flat background, the generalized conformal structure implies that the theory is invariant under generalized conformal transformations which act not only on the fields in the Lagrangian but also on the coupling constant.

tensor were computed by demanding smoothness in the interior. We use the map between conformal hydrodynamics and hydrodynamics of generalized non-conformal branes to predict the form of the stress tensor to second derivative order. In the sequel, we confirm the first order form by an independent bulk calculation. However, instead of using the framework of [122], in which the bulk equations were analyzed in Eddington-Finkelstein coordinates, we use the method of [141] where the black D3 brane fluctuations were analyzed in Fefferman-Graham coordinates. This is advantageous since it allows for a Lorentz covariant expansion, the constraint equations become trivial and reading off the stress tensor from the bulk metric is completely straightforward. Only to fix the coefficients of the dual stress tensor one switches to Eddington-Finkelstein coordinates, which is necessary to ensure the absence of singularities at the horizon of the black brane. We first generalize the discussion of [141] to arbitrary dimension (2σ) and compactify to obtain the non-conformal case for general σ.

This chapter is organized as follows. We begin in section 6.2 by showing that the gravity/scalar system relevant for the non-conformal branes can be obtained by dimensional reduction from pure gravity in $(2\sigma + 1)$ dimensions on a torus and continuation in σ, and discussing the implications of this for holography. In section 6.3, we apply this reasoning to hydrodynamics and conclude that the hydrodynamics of non-conformal branes can be obtained by dimensional reduction from conformal hydrodynamics. We predict the form of the non-conformal hydrodynamic energy-momentum tensor up to second derivative order, confirm that the KSS bound [140] is saturated and comment on Buchel's bound [135] for the ratio between shear and bulk viscosity. Sections 6.4 to 6.6 are devoted to explicitly checking the first order coefficients of the non-conformal energy-momentum tensor by a bulk calculation for the case of flat boundary metric. In section 6.4, we set up the conformal black brane solution and its non-conformal generalization. In section 6.5, we bring the black brane solution to Fefferman-Graham coordinates and calculate its first order correction in the derivative expansion. From that we extract the energy-momentum tensor to first derivative order. In section 6.6, we transform the first order solution to Eddington-Finkelstein coordinates to examine the singularity at the horizon of the unperturbed black brane. Analogously to [141], we find that the solution is only smooth in Eddington-Finkelstein coordinates if the shear viscosity given in the stress tensor saturates the KSS bound. Throughout sections 6.4 to 6.6, we often make use of the fact that at any point in the calculation we can obtain the non-conformal case by dimensional reduction and continuation from the conformal case. Finally we end with conclusions and prospects for future research. In appendix 6.A.1 we discuss the Fefferman-Graham expansion beyond the normalizable mode order and the dependence of the coefficients on the vev of the energy-momentum tensor.

(6.2) LOWER DIMENSIONAL FIELD EQUATIONS

In the near-horizon limit, the supergravity solutions of Dp-branes and fundamental strings are conformal to $AdS_{p+2} \times S^{8-p}$ and exhibit a running dilaton. There is a Weyl transformation to the *dual frame*, in which the metric is exactly $AdS_{p+2} \times S^{8-p}$ [89]. As in chapter 5, reducing the action in the dual frame on the sphere yields

$$S = -L \int d^{d+1}x \sqrt{g} e^{\gamma\phi} [R + \beta(\partial\phi)^2 + C], \tag{6.1}$$

where the constants (L, β, γ, C) depend on the case of interest. The equations of motions admit a linear dilaton AdS_{d+1} solution

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{dz_i dz^i}{\rho}; \\ e^\phi &= \rho^\alpha, \end{aligned} \tag{6.2}$$

where $i = 1, \ldots, d$, provided that α and C satisfy

$$\alpha = -\frac{\gamma}{2(\gamma^2 - \beta)}, \qquad C = \frac{(d(\gamma^2 - \beta) + \gamma^2)(d(\gamma^2 - \beta + \beta)}{(\gamma^2 - \beta)^2}. \tag{6.3}$$

Taking α instead of β as fundamental parameter we obtain the simpler form

$$\beta = \gamma^2(1 + \frac{1}{2\alpha\gamma}), \qquad C = (d - 2\alpha\gamma)(d - 2\alpha\gamma - 1). \tag{6.4}$$

After rescaling the scalar $\phi \to \phi/\gamma$ the action (6.1) takes the form

$$S = -L \int d^{d+1}x \sqrt{-\hat{G}} e^\phi [R + (1 + \frac{1}{2\alpha\gamma})(\partial\phi)^2 + (d - 2\alpha\gamma)(d - 2\alpha\gamma - 1)], \tag{6.5}$$

with solution (6.2) becoming

$$\begin{aligned} ds^2 &\equiv \hat{G}_{MN} dx^M dx^N = \frac{d\rho^2}{4\rho^2} + \frac{dz_i dz^i}{\rho}; \\ e^\phi &= \rho^{\alpha\gamma}. \end{aligned} \tag{6.6}$$

We observe that in any dimension d we have a family of actions of the form (6.5) which depend on the parameter $\alpha\gamma$ and whose equation of motions each admit a linear dilaton AdS_{d+1} solution of the form (6.6). The limit $\alpha\gamma \to 0$[2] and the choice $\alpha\gamma = -(d-4)^2/2(6-d)$ correspond to Einstein frame pure gravity with cosmological constant and decoupled Dp-branes with $d = p + 1$, respectively. For further reference let us also comment on the slightly non-standard dimensions of solution (6.6) and action (6.5). The AdS radius in (6.6) is absorbed in Newton's constant in the prefactor L of the action (6.5) and furthermore we have length dimensions:

$$\begin{aligned} [\rho] &= 2, \qquad [z^i] = 1, \qquad [ds^2] = -2, \\ [R] &= 0, \qquad [\sqrt{-\hat{G}}] = -d - 2, \\ [\int d^{d+1}x] &= [\int d\rho\, d^d z] = d + 2, \\ [e^\phi] &= -[L] = 2\alpha\gamma. \end{aligned} \tag{6.7}$$

[2] To be precise this is only a well-defined limit of the action if we first rescale the scalar $\phi \to \alpha\gamma\phi$ before taking $\alpha\gamma \to 0$.

Extracting precise boundary theory data from the asymptotics requires using holographic renormalization. In chapter 5, we developed a framework of holographic renormalization for Dp backgrounds and noted that this framework could be generalized to arbitrary values of $\alpha\gamma$. With the ansatz for metric and scalar

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(z,\rho)dz^i dz^j}{\rho}, \\ \phi(z,\rho) &= \alpha\gamma\log\rho + \kappa(z,\rho), \end{aligned} \tag{6.8}$$

the field equations following from (6.5) become

$$-\frac{1}{4}\mathrm{Tr}(g^{-1}g')^2 + \frac{1}{2}\mathrm{Tr}g^{-1}g'' + \kappa'' - \frac{1}{2\alpha\gamma}(\kappa')^2 = 0, \tag{6.9}$$

$$-\frac{1}{2}\nabla^i g'_{ij} + \frac{1}{2}\partial_j(\mathrm{Tr}g^{-1}g') - \frac{1}{2\alpha\gamma}\partial_j\kappa\kappa' + \partial_j\kappa' - \frac{1}{2}g_j'^{\,k}\partial_k\kappa = 0, \tag{6.10}$$

$$\begin{aligned} &\left[-Ric(g) - 2(\sigma-1)g' - \mathrm{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g')\right]_{ij} \\ &\quad +\nabla_i\partial_j\kappa - \frac{1}{2\alpha\gamma}\partial_i\kappa\partial_j\kappa - 2(g_{ij} - \rho g'_{ij})\kappa' = 0, \end{aligned} \tag{6.11}$$

$$4\rho(\kappa'' + (\kappa')^2) + 2(d - 4\sigma + 2)\kappa' + \nabla^2\kappa + (\partial\kappa)^2 + 2\mathrm{Tr}(g^{-1}g')(\alpha\gamma + \rho\kappa') = 0, \tag{6.12}$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g and $\sigma \equiv d/2 - \alpha\gamma$. Since the field equations are polynomials in ρ we can conclude that $g(z,\rho)$ and $\kappa(z,\rho)$ are regular functions of ρ and expand them in powers of ρ. Inserting these expressions in the equations of motion yields algebraic expressions for the subleading terms $g_{(2n>0)}$ and $\kappa_{(2n>0)}$ in terms of the sources $g_{(0)}$ and $\kappa_{(0)}$, until order $n = \sigma \equiv d/2 - \alpha\gamma$, at which only the divergence of $g_{(2\sigma)}$ and the combination $\mathrm{Tr}g_{(2\sigma)} + 2\kappa_{(2\sigma)}$ is determined. Furthermore, if σ is integer, we have to introduce logarithmic terms at order σ to fulfill the equations of motion:

$$\begin{aligned} g(z,\rho) &= g_{(0)}(z) + \rho g_{(2)}(z) + \ldots + \rho^\sigma(g_{(2\sigma)}(z) + h_{(2\sigma)}(z)\log\rho) + \ldots, \\ \kappa(z,\rho) &= \kappa_{(0)}(z) + \rho\kappa_{(2)}(z) + \ldots + \rho^\sigma(\kappa_{(2\sigma)}(z) + \tilde{\kappa}_{(2\sigma)}(z)\log\rho) + \ldots. \end{aligned} \tag{6.13}$$

Compared to pure gravity AdS, where $\sigma = d/2$, the power of the non-local term is shifted by $-\alpha\gamma$; this corresponds precisely to the additional factor e^ϕ in the action (6.5) ensuring that all counterterms can still be defined as local functionals of the sources and that the non-local terms $g_{(2\sigma)}$ and $\kappa_{(2\sigma)}$ contribute only to the finite part of the regularized action. The presence of the logarithmic terms in (6.13) appearing for σ integer corresponds precisely to the presence of an anomaly [13] in the generalized conformal Ward identity of the dual theory:

$$\langle T_i^i\rangle + 2\alpha\gamma\langle\mathcal{O}_\phi\rangle = \mathcal{A}. \tag{6.14}$$

We will now present a new and much simpler derivation of the holographic results for the non-conformal branes. Let us first consider the case of half integer $\sigma > d/2$. In this case the action (6.5) can be obtained by reducing $(2\sigma+1)$-dimensional gravity with cosmological constant $\Lambda = -\sigma(2\sigma-1)$ on a $(2\sigma-d)$-dimensional torus with the reduction ansatz

$$ds^2 = ds^2_{(d+1)}(\rho,z) + e^{\frac{2\phi(\rho,z)}{2\sigma-d}}dy_a dy^a, \tag{6.15}$$

where $a = 1, \ldots, (2\sigma - d)$ runs over the torus directions. The Ricci scalar and the action reduce as

$$\begin{aligned} R_{2\sigma+1} &= R_{d+1} - 2\nabla^2\phi - \frac{2\sigma - d + 1}{2\sigma - d}(\partial\phi)^2, \qquad (6.16)\\ S &= -L_{AdS}\int d^{2\sigma+1}x\sqrt{-g_{2\sigma+1}}(R_{2\sigma+1} + 2\sigma(2\sigma-1))\\ &= -L_{AdS}(2\pi R_y)^{2\sigma-d}\int d^{d+1}x\sqrt{-g_{d+1}}e^{\phi}(R_{d+1} + \frac{2\sigma-d-1}{2\sigma-d}(\partial\phi)^2 + 2\sigma(2\sigma-1)), \end{aligned}$$

where the (in our conventions) dimensionless prefactor L_{AdS} of the $(2\sigma+1)$-dimensional pure gravity action is given by

$$L_{AdS} = \frac{l_{AdS}^{2\sigma-1}}{16\pi G_{2\sigma+1}}, \qquad (6.17)$$

with l_{AdS} the radius of the $(2\sigma+1)$-dimensional AdS space, $G_{2\sigma+1}$ Newton's constant in $(2\sigma+1)$ dimensions and R_y the radius of the torus. Given that $\sigma = d/2 - \alpha\gamma$ the last line in (6.16) can easily seen to be proportional to (6.5) and thus lead to the same equation of motions. Furthermore one can make the prefactors match by choosing a torus radius R_y so that

$$L = L_{AdS}(2\pi R_y)^{2\sigma-d}. \qquad (6.18)$$

Thus, since for half-integer $\sigma > d/2$ the action can be obtained by dimensional reduction, local counterterms for the action (6.5) can be obtained by reducing the local $AdS_{(2\sigma+1)}$ counterterms.

Furthermore, the generalized conformal Ward identity (6.14) can also be shown to be the dimensional reduction of the conformal Ward identity of $AdS_{(2\sigma+1)}$. In the conformal case, the vev of the energy-momentum tensor is given by [15],

$$\langle T_{\mu\nu}\rangle_{2\sigma} = \frac{2}{\sqrt{-g_{(0),2\sigma}}}\frac{\delta S_{ren}}{\delta g^{\mu\nu}_{(0)}} = 2\sigma L_{AdS}\, g_{(2\sigma)\mu\nu} + \ldots, \qquad (6.19)$$

where S_{ren} denotes the renormalized on-shell action and the dots denote terms that locally depend on $g_{(0)\mu\nu}$. These terms are present when $g_{(0)\mu\nu}$ is curved and there is a conformal anomaly, i.e. when σ is an integer. They do not play an important role in the discussion here and so they will be suppressed. When relating the vev in (6.19) to the vev of the dimensionally reduced theory, we have to account for the additional prefactor $(2\pi R_y)^{2\sigma-d}$ of the lower-dimensional action in (6.16) which results from the integration over the torus and for the change in the determinant of the metric in the definition of the vev, $\sqrt{g_{(0),d}} = e^{-\kappa_{(0)}}\sqrt{g_{(0),2\sigma}}$. One obtains

$$\begin{aligned} e^{\kappa_{(0)}}(2\pi R_y)^{2\sigma-d}\langle T_{ij}\rangle_{2\sigma} &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ij} + \ldots = \langle T_{ij}\rangle_d, \qquad (6.20)\\ e^{\kappa_{(0)}}(2\pi R_y)^{2\sigma-d}\langle T_{ab}\rangle_{2\sigma} &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ab} + \ldots\\ &= 2\sigma L e^{\kappa_{(0)}}\left(e^{2\kappa/(2\sigma-d)}\right)_{(2\sigma)}\delta_{ab} + \ldots\\ &= \frac{4\sigma L}{2\sigma-d}e^{(1+2/(2\sigma-d))\kappa_{(0)}}\kappa_{(2\sigma)}\delta_{ab} + \ldots\\ &= -\langle\mathcal{O}_\phi\rangle_d e^{2\kappa_{(0)}/(2\sigma-d)}\delta_{ab}, \end{aligned}$$

where the dots again contain curvatures of the boundary metric $g_{(0)ij}$ and derivatives of $\kappa_{(0)}$ and we used in the last line formula (5.140) for the vev of the scalar operator,

$$\langle \mathcal{O}_\phi \rangle_d = -\frac{4\sigma L}{2\sigma - d} e^{\kappa_{(0)}} \kappa_{(2\sigma)} + \dots . \tag{6.21}$$

The conformal Ward identity $\langle T^\mu_\mu \rangle_{2\sigma} = \mathcal{A}_{2\sigma}$ then reduces to

$$\begin{aligned} & e^{-\kappa_{(0)}} (2\pi R_y)^{d-2\sigma} \left(\langle T^i_i \rangle_{2\sigma} + g^{ab}_{(0)} \langle T_{ab} \rangle_{2\sigma} \right) = \langle T^i_i \rangle_d - (2\sigma - d) \langle \mathcal{O}_\phi \rangle_d \\ & = e^{-\kappa_{(0)}} (2\pi R_y)^{d-2\sigma} \mathcal{A}_{2\sigma} \equiv \mathcal{A}_d, \end{aligned} \tag{6.22}$$

which is indeed equal to (6.14). The most efficient way to incorporate all local terms in the analysis (which are denoted by dots here) is to use the Hamiltonian formulation of holographic renormalization [19, 20] and dimensionally reduce the results. This has been discussed in detail for the case of D4 brane (which is related to M5 by the M-theory lift) in section 5.5.3 and appendix 5.A.3.

Thus we find that local counterterms, 1-point functions and the generalized conformal Ward identity for half-integer $\sigma > d/2$ can be obtained by dimensional reduction. From the lower-dimensional point of view however, σ is just a parameter of the theory on which the equations of motion depend smoothly. Therefore local counterterms, 1-point functions and generalized conformal Ward identities should also exist for positive, but non-integer $\sigma > d/2$.

The reduction argument yields the following prescription to obtain the counterterms to (6.5) with $\sigma > d/2$ from AdS-counterterms. Choose any half-integer $\tilde{\sigma} > \sigma$ and determine the $[\sigma] + 1$ most singular $AdS_{(2\tilde{\sigma}+1)}$-counterterms as a function of $\tilde{\sigma}$, where $[\sigma]$ denotes the largest integer less than or equal to σ (when σ is an integer one of these counterterms is logarithmic) . Reducing these $AdS_{(2\tilde{\sigma}+1)}$-counterterms on a $(2\tilde{\sigma} - d)$-dimensional torus and replacing $\tilde{\sigma}$ by σ yields the counterterms appropriate for (6.5).

As an example we rederive the counterterm action found in (5.139) for $1 < \sigma < 2$, which encompasses the cases of D0/1/2 branes and of the fundamental string, for which $\sigma = \{7/5, 3/2, 5/3, 3/2\}$ and $d = \{1, 2, 3, 2\}$ respectively. Since $\sigma < 2$ we only need two counterterms. The two most singular counterterms in $AdS_{2\tilde{\sigma}+1}$ defined on a regulating hypersurface are given by (see appendix B of [15])[3]

$$S^{ct} = L_{AdS} \int_{\rho=\epsilon} d^{2\tilde{\sigma}} x \sqrt{-\gamma_{2\tilde{\sigma}}} \left[2(2\tilde{\sigma} - 1) + \frac{1}{2\tilde{\sigma} - 2} \hat{R}[\gamma_{2\tilde{\sigma}}] \right], \tag{6.23}$$

where $\gamma_{2\tilde{\sigma}ij}$ is the induced metric on the $(2\tilde{\sigma})$-dimensional hypersurface and $\hat{R}[\gamma_{2\tilde{\sigma}}]$ the corresponding curvature. The curvature on the hypersurface reduces to d dimensions as

$$\hat{R}_{2\tilde{\sigma}} = \hat{R}_d[\gamma] - 2\hat{\nabla}^2 \phi - \frac{2\tilde{\sigma} - d + 1}{2\tilde{\sigma} - d} (\partial_i \phi)^2. \tag{6.24}$$

The counterterm action to (6.5) for $1 < \sigma < 2$ is then given by reducing (6.23) to d dimensions and replacing $\tilde{\sigma}$ with σ,

$$S^{ct} = L \int_{\rho=\epsilon} d^d x \sqrt{-\gamma_d}\, e^{\phi} \left[2(2\sigma - 1) + \frac{1}{2\sigma - 2} (\hat{R}_d + \frac{2\sigma - d - 1}{2\sigma - d} (\partial_i \phi)^2) \right], \tag{6.25}$$

[3] Note that convention for the curvature tensor used in [15] has the opposite sign.

which agrees with (5.139). The remaining case, i.e. the case of D4 branes has $\sigma = 3$ and the counterterm action also follows in the same manner (i.e. from the gravitational counterterms for AdS_7), as discussed in detail in chapter 5. Finally, let us comment on the restriction $\sigma > d/2$. At $\sigma = d/2$ the action (6.16) has a pole and the kinetic term of the scalar becomes negative in the interval $(d-1)/2 < \sigma < d/2$ so one should use the reduction argument when $\sigma < d/2$ with caution. Note also that for D6 branes, which do not have a sensible decoupling limit, $\sigma = -1$.

For later convenience let us finally mention that one can always formally recover the $(2\sigma+1)$-dimensional equation of motion for the metric in the conformal case from the non-conformal case by setting the scalar to zero. The conformal version of (6.9) - (6.11) reads

$$-\frac{1}{4}\mathrm{Tr}(g^{-1}g')^2 + \frac{1}{2}\mathrm{Tr}g^{-1}g'' = 0, \tag{6.26}$$
$$-\frac{1}{2}\nabla^{\mu}g'_{\mu\nu} + \frac{1}{2}\partial_{\nu}(\mathrm{Tr}g^{-1}g') = 0, \tag{6.27}$$
$$\left[-Ric(g) - 2(\sigma-1)g' - \mathrm{Tr}(g^{-1}g')g + \rho(2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g')\right]_{\mu\nu} = 0, \tag{6.28}$$

where from now on we use transverse indices $\mu, \nu, \ldots$ for the conformal case and transverse indices $i, j, \ldots$ for the non-conformal case.

(6.3) UNIVERSAL HYDRODYNAMICS

The hydrodynamic energy-momentum tensor for a conformal fluid at first-derivative order in (2σ) dimensions on a curved manifold with metric $g_{(0)\mu\nu}$ is

$$\begin{aligned} T_{\mu\nu} &= L_{AdS}\left(\frac{2\pi T}{\sigma}\right)^{2\sigma}(g_{(0)\mu\nu} + 2\sigma u_{\mu}u_{\nu}) - 2\eta_{2\sigma}(T)\sigma_{\mu\nu}, \\ \sigma_{\mu\nu} &= P_{\mu}^{\kappa}P_{\nu}^{\lambda}\nabla_{(\kappa}u_{\lambda)} - \frac{1}{2\sigma-1}P_{\mu\nu}(\nabla\cdot u), \qquad P_{\mu\nu} = g_{(0)\mu\nu} + u_{\mu}u_{\nu}, \end{aligned} \tag{6.29}$$

where T, u_{μ} and $\eta_{2\sigma}(T)$ denote the temperature, velocity and shear viscosity respectively of the fluid and ∇_i is the covariant derivative corresponding to the metric $g_{(0)ij}$. For given $\eta_{2\sigma}(T)$ the evolution of the fluid is determined by the conservation of the energy-momentum tensor,

$$\nabla^{\mu}T_{\mu\nu} = 0. \tag{6.30}$$

Furthermore, the conformal Ward identity $T_{\mu}^{\mu} = 0$ constrains energy density ϵ and pressure p to be related by the equation of state

$$p = L_{AdS}\left(\frac{2\pi T}{\sigma}\right)^{2\sigma} = \frac{1}{2\sigma-1}\epsilon. \tag{6.31}$$

Since we saw above that the bulk equations of motion for a non-conformal geometry with given σ can be obtained by dimensional reduction of $(2\sigma+1)$-dimensional gravity on a $(2\sigma-d)$-dimensional torus, we can perform the same procedure on the boundary to obtain the hydrodynamic energy-momentum tensor dual to a non-conformal black brane solution with given σ

in d dimensions. Demanding that $T_{\mu\nu}$ in (6.29) only depends on non-compact directions and that the fluid velocity $u^\mu = (u^i, 0)$ only has non-zero non-compact components yields

$$\begin{aligned} T_{ij} &= Le^{\kappa_{(0)}} \left(\frac{2\pi T}{\sigma}\right)^{2\sigma} (g_{(0)ij} + 2\sigma u_i u_j) - 2\eta_d \sigma_{ij} - \zeta_d P_{ij}(\nabla \cdot u), \\ \langle \mathcal{O}_\phi \rangle &= -Le^{\kappa_{(0)}} \left(\frac{2\pi T}{\sigma}\right)^{2\sigma} - \frac{2}{2\sigma - 1}\eta_d(\nabla \cdot u), \end{aligned} \tag{6.32}$$

where

$$\begin{aligned} \sigma_{ij} &= P_i^k P_j^l \nabla_{(k} u_{l)} - \frac{1}{d-1} P_{ij}(\nabla \cdot u), \qquad P_{ij} = g_{(0)ij} + u_i u_j, \\ \eta_d &= (2\pi R_y)^{2\sigma - d} e^{\kappa_{(0)}} \eta_{2\sigma}, \\ \zeta_d &= \frac{2(2\sigma - d)}{(d-1)(2\sigma - 1)} \eta_d, \end{aligned} \tag{6.33}$$

with η_d and ζ_d shear and bulk viscosity respectively of the d-dimensional fluid. The conformal conservation equation (6.30) reduces to

$$\nabla^i T_{ij} - \partial_j \kappa_{(0)} \langle \mathcal{O}_\phi \rangle = 0, \tag{6.34}$$

where $\langle \mathcal{O}_\phi \rangle$ is again the expectation value of the operator dual to ϕ. Since we would like the evolution of the d-dimensional fluid to be described purely by a divergence equation, we demand that $\kappa_{(0)}$ is constant or without loss of generality zero. Moreover note that the d-dimensional non-conformal fluid obeys the same equation of state (6.31) as the (2σ)-dimensional conformal fluid.

In (6.33) we observe that $\eta_d(T)$ as function of the temperature in the non-conformal theory is proportional to $\eta_{2\sigma}(T)$ in the higher dimensional conformal theory. As we will check below, smoothness of the bulk solution forces the viscosity to saturate the KSS bound [140]

$$\frac{\eta_d}{s_d} = \frac{\eta_{2\sigma}}{s_{2\sigma}} \geq \frac{1}{4\pi}, \tag{6.35}$$

where the entropy density s_d corresponding to (6.32) is given by

$$s_d = 2\sigma L \left(\frac{2\pi}{\sigma}\right)^{2\sigma} T^{2\sigma - 1}, \tag{6.36}$$

and the dimensionful L is related to the dimensionless L_{AdS} via (6.18). More generally, we see that any fluid which is related to a conformal fluid satisfying the KSS bound by dimensional reduction will satisfy the KSS bound as well.

Furthermore we note that in (6.33), the bulk viscosity ζ_d is determined by the shear viscosity η_d. In [135] it was conjectured that the ratio of bulk to shear viscosity of a strongly coupled gauge theory plasma satisfies the bound[4]

$$\frac{\zeta_d}{\eta_d} \geq 2\left(\frac{1}{d-1} - c_s^2\right), \tag{6.37}$$

[4] Note that unlike the η/s bound, this bound has known counterexamples in weakly coupled systems, e.g. monatomic gases [135].

where c_s denotes the speed of sound. In our case c_s can be calculated from the equation of state (6.31) to be

$$c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} = \frac{1}{\sqrt{2\sigma - 1}}. \tag{6.38}$$

Hence we see that the last line in (6.33) implies that the bound (6.37) is saturated for arbitrary σ. This confirms the calculation of [134, 133, 135], in which this result was obtained for black Dp-branes for $p = 2, \ldots, 6$ and their toroidal compactifications. Note however that (6.37) will be saturated for any fluid which arises from dimensional reduction and continuation in dimension of a conformal fluid, irrespective of the value of η/s. In particular we did not have to assume that the dual bulk solution is smooth. Thus in this case the ratio ζ/η is fixed kinematically. This indicates that this case is qualitatively different than the case of η/s.

Combining (6.33) and (6.35) we obtain the bound

$$\frac{\zeta_d}{s_d} \geq \frac{2\sigma - d}{2\pi(d-1)(2\sigma - 1)}. \tag{6.39}$$

which should hold for all non-conformal fluids that can be related to a (2σ)-dimensional conformal fluid (with (2σ) non necessarily integral), as discussed above.

We can also obtain to second order the coefficients of the non-conformal energy-momentum tensor from the coefficients of the conformal energy-momentum tensor. In was argued in [142] that the second order contribution to the conformal energy-momentum tensor is given by a linear combination of all possible Weyl invariants containing two derivatives,

$$\begin{aligned} T_{2\mu\nu} &= 2\eta_{2\sigma}\tau_M \left[(u \cdot \nabla)\sigma_{\mu\nu} + \frac{1}{2\sigma - 1}\sigma_{\mu\nu}(\nabla \cdot u) \right] \\ &\quad + \tilde{\kappa} \left[R_{\mu\nu} - (2\sigma - 2) u^\kappa u^\lambda R_{\kappa\langle\mu\nu\rangle\lambda} \right] \\ &\quad + 4\lambda_1 \sigma_{\kappa\langle\mu}\sigma_{\nu\rangle}{}^\kappa + 2\lambda_2 \sigma_{\kappa\langle\mu}\Omega_{\nu\rangle}{}^\kappa + \lambda_3 \Omega_{\kappa\langle\mu}\Omega_{\nu\rangle}{}^\kappa, \end{aligned} \tag{6.40}$$

where $R_{\mu\nu\kappa\lambda}$ and $R_{\mu\nu}$ are Riemann and Ricci tensor of the metric $g_{(0)\mu\nu}$, angle brackets denote the transverse traceless part of a second rank tensor $A_{\mu\nu}$,

$$A_{\langle\mu\nu\rangle} = \frac{1}{2} P_\mu^\kappa P_\nu^\lambda (A_{\kappa\lambda} + A_{\lambda\kappa}) - \frac{1}{2\sigma - 1} P_{\mu\nu} P^{\kappa\lambda} A_{\kappa\lambda}, \tag{6.41}$$

and the vorticity $\Omega_{\mu\nu}$ is given by

$$\Omega_{\mu\nu} = \frac{1}{2} P_\mu^\kappa P_\nu^\lambda (\nabla_\kappa u_\lambda - \nabla_\lambda u_\kappa). \tag{6.42}$$

Note also that with notation (6.41) the shear tensor $\sigma_{\mu\nu}$ can be written as

$$\sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}. \tag{6.43}$$

Again, we can obtain the non-conformal second order energy-momentum tensor with given σ by reducing (6.40) on a $(2\sigma - d)$-dimensional torus. The result can be obtained by replacing all tensors with angle brackets by

$$\begin{aligned} A_{\langle\mu\nu\rangle} &\rightarrow A_{\langle ij\rangle} + \frac{2\sigma - d}{(d-1)(2\sigma-1)} P_{ij} P^{kl} A_{kl}, \\ \sigma_{\mu\nu} &\rightarrow \sigma_{ij} + \frac{2\sigma - d}{(d-1)(2\sigma-1)} P_{ij} (\nabla \cdot u), \end{aligned} \tag{6.44}$$

where $A_{\langle ij\rangle}$ in the first line on the right hand side is defined as transverse traceless part in the lower dimensional theory,

$$A_{\langle ij\rangle} = \frac{1}{2} P_i^k P_j^l (A_{kl} + A_{lk}) - \frac{1}{d-1} P_{ij} P^{kl} A_{kl}. \tag{6.45}$$

The Riemann tensor, Ricci tensor, Ricci scalar and vorticity reduce trivially,

$$R^{2\sigma}_{ijkl} = R^{d}_{ijkl}, \qquad R^{2\sigma}_{ij} = R^{d}_{ij}, \qquad R^{2\sigma} = R^{d}, \qquad \Omega^{2\sigma}_{ij} = \Omega^{d}_{ij}, \tag{6.46}$$

since we demanded that $\kappa_{(0)} = 0$. The components of the Riemann and Ricci tensor in the internal directions of the torus do not contribute to the d-dimensional energy-momentum tensor. Finally the d-dimensional energy momentum tensor gets multiplied by the overall factor $(2\pi R_y)^{2\sigma - d}$ which stems from the torus volume factor multiplying the $(d+1)$-dimensional bulk action.

(6.4) Generalized black branes

In [122] it was shown that the long wavelength fluctuation equations around the boosted black D3 brane geometry in Eddington-Finkelstein coordinates can be mapped to the non-linear equations of hydrodynamics of the dual strongly coupled conformal field theory. In [141] it was pointed out that it can be advantageous to perform the same analysis in Fefferman-Graham coordinates, since it allows for a Lorentz covariant expansion, the constraint equations become trivial and reading off the stress tensor from the bulk metric is completely straightforward. Furthermore, one can construct bulk solutions dual to an arbitrary hydrodynamic boundary stress tensor. On the other hand, irrespectively of the precise values of the coefficients of the energy-momentum tensor, the Fefferman-Graham coordinates will have a singularity at the (unperturbed) horizon. To find out whether this singularity is a coordinate singularity or a real one requires to transform to Eddington-Finkelstein coordinates. Only requiring smoothness in Eddington-Finkelstein coordinates away from the singularity of the static black brane fixes the coefficients in the boundary stress tensor to the values found in [122].

Here we will generalize the analysis of [141] to non-conformal geometries with AdS-solution in the dual frame and arbitrary positive σ. As a first step, we generalize it to pure gravity in arbitrary dimension $(2\sigma + 1)$ for half-integer σ and then invoke the reduction argument of section 6.2 to obtain the case of a non-conformal geometry with arbitrary positive σ in dimension d. Throughout the rest of this chapter we assume the boundary metric $g_{(0)ij} = \eta_{ij}$ to be flat and $\kappa_{(0)}$ to be constant or without loss of generality zero.

For half-integer σ, the $(2\sigma+1)$-dimensional pure gravity action in (6.16) has the black brane solution

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2 f_b(\rho)} + \frac{-f_b(\rho)dt^2 + dz_r dz^r}{\rho}, \\ f_b(\rho) &= 1 - \frac{\rho^\sigma}{b^{2\sigma}}, \end{aligned} \tag{6.47}$$

where r runs over spatial transverse coordinates and b is related to the black brane temperature by

$$b = \frac{\sigma}{2\pi T}. \tag{6.48}$$

After boosting the geometry (6.47) with the boost parameter u_μ we obtain the metric

$$ds^2 = \frac{d\rho^2}{4\rho^2 f_b(\rho)} + \frac{[\eta_{\mu\nu} + (1 - f_b(\rho))u_\mu u_\nu]dz^\mu dz^\nu}{\rho}, \tag{6.49}$$

which solves the equation of motions as long as b and u_μ are constants, with b and u_μ mapped to the to the dual (inverse) temperature and velocity of the fluid. However, once we allow the temperature in the definition of b in (6.48) and u_μ to become z-dependent,

$$ds^2 = \frac{d\rho^2}{4\rho^2 f_{b(z)}(\rho)} + \frac{[\eta_{\mu\nu} + (1 - f_{b(z)}(\rho))u_\mu(z)u_\nu(z)]dz^\mu dz^\nu}{\rho}, \tag{6.50}$$

we have to correct the metric (6.50) at each order in the derivative expansion to still fulfill the equations of motions. The corrections to the metric then determine the dissipative part of the hydrodynamic energy-momentum tensor.

The non-conformal generalization of (6.50) can again be obtained by compactification. We split the transverse coordinates $z^\mu = (z^i, y^a)$ in non-compact and torus directions and demand that the metric only depends on non-compact directions and that the fluid velocity $u^\mu = (u^i, 0)$ has only non-zero non-compact components. This enables us to reduce using the reduction ansatz (6.15) to obtain for metric and scalar

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2 f_b(\rho)} + \frac{[\eta_{ij} + (1 - f_{b(z)}(\rho))u_i(z)u_j(z)]dz^i dz^j}{\rho}, \\ e^\phi &= \rho^{\alpha\gamma}. \end{aligned} \tag{6.51}$$

It can be checked explicitly that this is a solution of the equations of motion following from the $(d+1)$-dimensional action (6.5) for arbitrary σ.

(6.5) GENERALIZED BLACK BRANES IN FEFFERMAN-GRAHAM COORDINATES

Before computing the derivative corrections to the boosted brane solution (6.51) by perturbing around equations (6.9) - (6.12), we change to Fefferman-Graham coordinates, in which the solution takes the form (6.8). Again, to keep the discussion as concise as possible, we first discuss the conformal case in arbitrary dimension and compactify to obtain the non-conformal case. In both cases, we obtain Fefferman-Graham coordinates by a redefinition of the radial coordinate:

$$\tilde{\rho}(\rho) = \left(\frac{2}{1+\sqrt{f_b(\rho)}}\right)^{2/\sigma} \rho, \tag{6.52}$$

whose inverse transformation is

$$\rho(\tilde{\rho}) = \left(1 + \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}\right)^{-2/\sigma} \tilde{\rho}. \tag{6.53}$$

The metric (6.50) corresponding to the conformal fluid becomes

$$\begin{aligned} ds^2 &= \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \frac{g(z,\tilde{\rho})_{\mu\nu} dz^\mu dz^\nu}{\tilde{\rho}}, \\ g(z,\tilde{\rho})_{\mu\nu} &= A(\tilde{\rho})\eta_{\mu\nu} + B(\tilde{\rho}) u_\mu u_\nu, \end{aligned} \tag{6.54}$$

where

$$\begin{aligned} A(\tilde{\rho}) &= \frac{\tilde{\rho}}{\rho(\tilde{\rho})} = \left(1 + \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}\right)^{2/\sigma}, \\ B(\tilde{\rho}) &= \frac{\tilde{\rho}[1 - f_b(\rho(\tilde{\rho}))]}{\rho(\tilde{\rho})} = \frac{\tilde{\rho}^\sigma}{b^{2\sigma}} \left(1 + \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}\right)^{2/\sigma - 2}. \end{aligned} \tag{6.55}$$

According to (6.19) we obtain the perfect fluid part of the energy-momentum tensor (6.29) by reading off the $\tilde{\rho}^\sigma$ coefficient of $g(z,\tilde{\rho})$,

$$T_{0\mu\nu} = 2\sigma L_{AdS} g_{(2\sigma)\mu\nu} = L_{AdS} b^{-2\sigma}(\eta_{\mu\nu} + 2\sigma u_\mu u_\nu), \tag{6.56}$$

using the definition of b in (6.48). The horizon in Fefferman-Graham coordinates is at $\tilde{\rho} = \tilde{\rho}_h \equiv 2^{2/\sigma} b^2$, where $g(\tilde{\rho}_h, z)_{\mu\nu}$ becomes non-invertible since $A(\tilde{\rho}_h) = B(\tilde{\rho}_h)$.

If $u_\mu(z)$ and $b(z)$ in (6.54) become dependent on the boundary coordinates z^μ we have to introduce corrections to the metric at each order in the derivative expansion to still satisfy the equations of motion. At first order we perturb the metric as

$$g(z,\tilde{\rho}) = g_0(z,\tilde{\rho}) + g_1(z,\tilde{\rho}), \tag{6.57}$$

where $g_0(z,\tilde{\rho})$ is given by (6.54),

$$g_0(z,\tilde{\rho})_{ij} = A(b(z),\tilde{\rho})\eta_{ij} + B(b(z),\tilde{\rho}) u_i(z) u_j(z). \tag{6.58}$$

The equations of motion (6.26) and (6.28) become

$$-\frac{1}{2}\mathrm{Tr} g_0^{-1} g_0' g_0^{-1} g_1' + \frac{1}{2}\mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0' g_0^{-1} g_0' + \frac{1}{2}(\mathrm{Tr} g_0^{-1} g_1'' - \mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0'') = 0 \tag{6.59}$$

$$\begin{aligned} &2\tilde{\rho}(g_1'' - g_1' g_0^{-1} g_0' - g_0' g_0^{-1} g_1' + g_0' g_0^{-1} g_1 g_0^{-1} g_0') - 2(\sigma - 1) g_1' \\ &+ \mathrm{Tr} g_0^{-1} g_0' (\tilde{\rho} g_1' - g_1) + (\mathrm{Tr} g_0^{-1} g_1' - \mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0')(\tilde{\rho} g_0' - g_0) = 0, \end{aligned} \tag{6.60}$$

where now the prime denotes differentiation with respect to the Fefferman-Graham radial variable $\tilde{\rho}$. At first order in the derivative expansion, $Ric(g)$ in (6.28) does not contribute since it contains at least two derivatives of z.

At every order in the derivative expansion, the constraint equation (6.27) at small $\tilde{\rho}$ is equivalent to the conservation of the dual energy-momentum tensor. However, if this equation is fulfilled on a radial hypersurface close to the boundary, the evolution equations (6.26) and (6.28) ensure that it remains fulfilled in the interior. Only the equations (6.59) and (6.60) constrain the form of the metric perturbation further and with it also the form of the dual hydrodynamic stress tensor. In section 6.6 though, where we transform the perturbed metric to Eddington-Finkelstein coordinates, it will be convenient to use the conservation equation of the (perfect fluid) energy-momentum tensor to relate derivatives of the temperature field to derivatives of the velocity field.

The perturbations $g_{1\mu\nu}$ will contain first derivatives of u_μ and its order σ term will correct the energy momentum tensor by $T_{\mu\nu} = T_{0\mu\nu} + T_{1\mu\nu}$, where $T_{0\mu\nu}$ is the perfect fluid energy-momentum tensor (6.56) and $T_{1\mu\nu}$ the dissipative part at first derivative order. Since $T^\mu_\mu = 0$ in the conformal case and since we can always go to Landau gauge $u^\mu T_{1\mu\nu} = 0$ by a redefinition of the temperature and velocity field, $T_{1\mu\nu}$ will be given by

$$T_{1\mu\nu} = -2\eta_{2\sigma}\sigma_{\mu\nu}, \tag{6.61}$$

where the parameter $\eta_{2\sigma}$ is the shear viscosity. Only for a specific value of the shear viscosity, the bulk solution will be smooth at the horizon of the black brane. However, in Fefferman-Graham coordinates the metric becomes non-invertible at the horizon. Fixing the value of $\eta_{2\sigma}$ will require changing to Eddington-Finkelstein coordinates, which we do in section 6.6 below. In the meantime we parametrize $\eta_{2\sigma}$ as

$$\eta_{2\sigma} = L_{AdS}\gamma b^{1-2\sigma}, \tag{6.62}$$

where $\eta_{2\sigma}$ fulfilling $\eta_{2\sigma}/s_{2\sigma} = 1/4\pi$ corresponds to $\gamma = 1$.

The form of the metric perturbation $g_{1\mu\nu}$ can now be determined using the the following argument [141]. As is shown in appendix 6.A.1, the derivatives in the $\tilde{\rho}$ expansion of the metric always enter in pairs, see (6.95), which implies that the $\tilde{\rho}$-expansion of the metric perturbation $g_{1\mu\nu}$ at first derivative order will only contain non-derivative terms of the form $(T_0^p T_1 T_0^q)$. Due to the Landau gauge condition $u^\mu T_{1\mu\nu} = 0$ and the tracelessness condition $T^\mu_{1\mu} = 0$ only the $\eta_{\mu\nu}$ part inside $(T_0^p)_{\mu\nu}$ contributes to the coefficients of $g_{1\mu\nu}$ at each order in $\tilde{\rho}$. Thus, each coefficient in the expansion of $g_{1\mu\nu}$ will be proportional to $T_{1\mu\nu}$. Hence also $g_{1\mu\nu}$ as a whole will be proportional to $T_{1\mu\nu}$, which in the conformal case only contains a shear part,

$$g_{1\mu\nu} = \lambda(\tilde{\rho})\sigma_{\mu\nu}. \tag{6.63}$$

Extracting the transverse, traceless mode proportional to $\sigma_{\mu\nu}$ out of (6.60) we obtain a second order ordinary differential equation in $\lambda(\tilde{\rho})$

$$2\tilde{\rho}(\lambda'' - 2\frac{A'}{A}\lambda' + \frac{A'^2}{A^2}\lambda) - 2(\sigma-1)\lambda' + \mathrm{Tr} g_0^{-1} g_0'(\tilde{\rho}\lambda' - \lambda) = 0, \tag{6.64}$$

whose asymptotically vanishing solution is given by

$$\lambda(\tilde{\rho}) = C_\lambda \left(1 + \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}\right)^{2/\sigma} \log \frac{1 - \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}}{1 + \frac{\tilde{\rho}^\sigma}{4b^{2\sigma}}} = C_\lambda A(\tilde{\rho}) \log \frac{2 - A(\tilde{\rho})^{\sigma/2}}{A(\tilde{\rho})^{\sigma/2}}. \qquad (6.65)$$

To fix the integration constant C_λ we demand that the order σ term of $\lambda(\tilde{\rho})$ in the Taylor expansion in $\tilde{\rho}$ reproduces $T_{1\mu\nu}$,

$$C_\lambda = \frac{2\eta_{2\sigma}}{\sigma L_{AdS}} b^{2\sigma} = \frac{2\gamma b}{\sigma}. \qquad (6.66)$$

The metric in Fefferman-Graham coordinates in the conformal case up to first order is then

$$\begin{aligned} ds^2 &= \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \frac{g_{\mu\nu}(z,\tilde{\rho}) dz^\mu dz^\nu}{\tilde{\rho}}, \qquad (6.67) \\ g_{\mu\nu}(z,\tilde{\rho}) &= A(\tilde{\rho})\eta_{\mu\nu} + B(\tilde{\rho}) u_\mu u_\nu + \lambda(\tilde{\rho})\sigma_{\mu\nu}. \end{aligned}$$

The whole discussion can be straightforwardly generalized to the nonconformal case. Starting from (6.51) we change to Fefferman-Graham coordinates, perturb metric and scalar as

$$\begin{aligned} g(z,\tilde{\rho}) &= g_0(z,\tilde{\rho}) + g_1(z,\tilde{\rho}), \qquad (6.68) \\ \kappa(z,\tilde{\rho}) &= \kappa_0(z,\tilde{\rho}) + \kappa_1(z,\tilde{\rho}), \end{aligned}$$

to obtain perturbation equations around (6.9), (6.11) and (6.12):

$$\begin{aligned} &-\frac{1}{2}\mathrm{Tr} g_0^{-1} g_0' g_0^{-1} g_1' + \frac{1}{2}\mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0' g_0^{-1} g_0' \qquad (6.69) \\ &\quad + \frac{1}{2}(\mathrm{Tr} g_0^{-1} g_1'' - \mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0'') + \kappa_1'' - \frac{1}{\alpha\gamma}\kappa_0'\kappa_1' = 0, \\ &2\tilde{\rho}(g_1'' - g_1' g_0^{-1} g_0' - g_0' g_0^{-1} g_1' + g_0' g_0^{-1} g_1 g_0^{-1} g_0') - 2(\sigma-1) g_1' \qquad (6.70) \\ &\quad + (\mathrm{Tr} g_0^{-1} g_0' + 2\kappa_0')(\tilde{\rho} g_1' - g_1) \\ &\quad + (\mathrm{Tr} g_0^{-1} g_1' - \mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0' + 2\kappa_1')(\tilde{\rho} g_0' - g_0) = 0 \\ &4\tilde{\rho}(\kappa_1'' + 2\kappa_1'\kappa_0') + 2(d - 4\sigma + 2)\kappa_1' \qquad (6.71) \\ &\quad + (\mathrm{Tr} g_0^{-1} g_1' - \mathrm{Tr} g_0^{-1} g_1 g_0^{-1} g_0')(2\alpha\gamma + 2\tilde{\rho}\kappa_0') + 2\tilde{\rho}\kappa_1' \mathrm{Tr} g_0^{-1} g_0' = 0. \end{aligned}$$

However, as we know the first order solution (6.67) in the conformal case, we can again obtain the first order solution for metric and scalar in the non-conformal case by dimensional reduction using the ansatz (6.15):

$$\begin{aligned} g_{ij}(z,\tilde{\rho}) &= A(\tilde{\rho})\eta_{ij} + B(\tilde{\rho}) u_i u_j + \lambda(\tilde{\rho})\sigma_{ij} + \frac{2\sigma - d}{(d-1)(2\sigma-1)}\lambda(\tilde{\rho}) P_{ij}(\partial\cdot u), \\ \exp\left(\frac{2\kappa(z,\tilde{\rho})}{2\sigma - d}\right) &= A(\tilde{\rho}) - \frac{\lambda(\tilde{\rho})}{2\sigma - 1}(\partial\cdot u). \qquad (6.72) \end{aligned}$$

From (6.72) we can read off

$$\begin{aligned} g_0(z,\tilde{\rho})_{ij} &= A(\tilde{\rho})\eta_{ij} + B(\tilde{\rho}) u_i(z) u_j(z), \qquad (6.73) \\ \kappa_0(z,\tilde{\rho}) &= \frac{2\sigma - d}{2}\log A(\tilde{\rho}), \\ g_1(z,\tilde{\rho})_{ij} &= \lambda(\tilde{\rho})\sigma_{ij} + \frac{2\sigma - d}{(d-1)(2\sigma-1)}\lambda(\tilde{\rho}) P_{ij}(\partial\cdot u), \\ \kappa_1(z,\tilde{\rho}) &= -\frac{2\sigma - d}{2(2\sigma-1)A(\tilde{\rho})}\lambda(\tilde{\rho})(\partial\cdot u), \end{aligned}$$

which can be straightforwardly shown to be a solution of (6.69) - (6.71). Finally, by extracting the order σ term, it can be checked that the metric in (6.72) gives rise to the non-conformal energy momentum tensor and scalar vev given in (6.32),

$$\begin{aligned} T_{ij} &= \frac{L}{b^{2\sigma}}(g_{(0)ij} + 2\sigma u_i u_j) - 2\eta_d \sigma_{ij} - \zeta_d P_{ij}(\partial \cdot u), \\ \langle \mathcal{O}_\phi \rangle &= -\frac{L}{b^{2\sigma}} - \frac{2}{2\sigma - 1}\eta_d(\partial \cdot u). \end{aligned} \tag{6.74}$$

(6.6) Transformation to Eddington-Finkelstein coordinates

We have thus found that the first order perturbation results in a hydrodynamic stress energy tensor and vev for the operator $\mathcal{O}_\phi$ that are parametrized by the shear viscosity η_d, which at this point is unconstrained. The bulk viscosity ζ_d is fixed in a way prescribed by the dilation Ward identity. Now recall that the source and the vev are a conjugate pair with the vev being the (renormalized) radial canonical momentum [19, 20] so specifying them yields in principle a unique bulk solution. Not all these solutions however will be non-singular. Regularity in the interior in general leads to additional restrictions.

We now discuss the constraints imposed by the smoothness of the gravity solution in the bulk. This requires changing to Eddington-Finkelstein coordinates, which are well-defined beyond the horizon. In the conformal case the metric in the Eddington-Finkelstein coordinates will be of the form

$$ds^2 = -2u_\mu(x) dr dx^\mu + G_{\mu\nu}(x, r) dx^\mu dx^\nu, \tag{6.75}$$

and the transformation equations for the metric between Fefferman-Graham of the form (6.8) and Eddington-Finkelstein coordinates are given by

$$\begin{aligned} &(\partial_r \tilde{\rho})^2 + 4\tilde{\rho}\, g_{\mu\nu}(z, \tilde{\rho})\, \partial_r z^\mu\, \partial_r z^\nu = 0, \\ &\partial_r \tilde{\rho}\, \partial_\mu \tilde{\rho} + 4\tilde{\rho}\, \partial_r z^\kappa\, \partial_\mu z^\lambda\, g_{\kappa\lambda}(z, \tilde{\rho}) = -4\tilde{\rho}^2 u_\mu, \\ &\partial_\mu \tilde{\rho}\, \partial_\nu \tilde{\rho} + 4\tilde{\rho}\, \partial_\mu z^\kappa\, \partial_\nu z^\lambda\, g_{\kappa\lambda}(z, \tilde{\rho}) = 4\tilde{\rho}^2 G_{\mu\nu}(x, \rho), \end{aligned} \tag{6.76}$$

where $\tilde{\rho}(x, r)$ and $z^\mu(x, r)$ encode the dependence of the Fefferman-Graham coordinates on the Eddington-Finkelstein coordinates. Given a solution in Fefferman-Graham coordinates we use (6.76) to solve for $\tilde{\rho}(x, r)$, $z^\mu(x, r)$ and $G_{\mu\nu}(x, \rho)$. At zeroth order in the derivative expansion, the transformation is given by

$$\begin{aligned} \tilde{\rho}_0(r) &= \tilde{\rho}(\rho = 1/r^2) = \left(\frac{2}{1 + \sqrt{f_b(r)}} \right)^{2/\sigma} \frac{1}{r^2}, \\ z_0^\mu(r) &= x^\mu + u^\mu k_b(r), \end{aligned} \tag{6.77}$$

where

$$f_b(r) \equiv f_b(\rho = 1/r^2) = 1 - (br)^{-2\sigma}, \qquad k_b(r) \equiv \frac{1}{r}\, {}_2F_1(1, \frac{1}{2\sigma}; 1 + \frac{1}{2\sigma}; (br)^{-2\sigma}), \tag{6.78}$$

with ${}_2F_1(a,b;c;w)$ a hypergeometric function. Note that the radial coordinate r in the Eddington-Finkelstein coordinates is related to the radial coordinate ρ in the original black brane solution (6.51) simply by $\rho = 1/r^2$. Furthermore $\tilde{\rho}(r)$ and $k_b(r)$ obey the first order differential equations

$$\begin{aligned} \partial_r \tilde{\rho} &= -\frac{2\tilde{\rho}}{r\sqrt{f_b(r)}}, \\ \partial_r k_b &= -\frac{1}{r^2 f_b(r)}. \end{aligned} \tag{6.79}$$

$G_{0\mu\nu}(x,r)$ is given by

$$G_{0\mu\nu} = r^2[\eta_{\mu\nu} + (f_b(r)-1)u_\mu u_\nu]. \tag{6.80}$$

At first order in the derivative expansion we perturb the Fefferman-Graham coordinates in the transformation equations (6.76) by

$$\begin{aligned} \tilde{\rho}(x,r) &= \tilde{\rho}_0(x,r) + \tilde{\rho}_1(x,r), \\ z^\mu(x,r) &= z_0^\mu(x,r) + z_1^\mu(x,r), \end{aligned} \tag{6.81}$$

while $G_{\mu\nu}(x,r)$ is expanded as

$$G_{\mu\nu}(x,r) = G_{0\mu\nu}(x,r) + G_{1\mu\nu}(x,r). \tag{6.82}$$

At the same time, we use for the Fefferman-Graham metric the full first order expression $g_{\mu\nu}(z,\tilde{\rho}) = g_{0\mu\nu}(z,\tilde{\rho}) + g_{1\mu\nu}(z,\tilde{\rho})$ in (6.67). Note that the zeroth order expressions $\tilde{\rho}_0$, z_0^μ, $G_{0\mu\nu}$ and $g_{0\mu\nu}(z,\tilde{\rho})$ depend also on x through their dependence on $b(z)$ and $u_\mu(z)$, which now have been made dependent on z. Furthermore, we have to account for the transverse coordinates change from z^μ to x^μ in (6.77) and Taylor expand $b(z)$ and $u_\mu(z)$ as

$$\begin{aligned} b(z) &= b(x) + u^\mu(x)k_b(r)\,\partial_\mu b(x), \\ u_\mu(z) &= u_\mu(x) + u^\nu(x)k_b(r)\,\partial_\nu u_\mu(x). \end{aligned} \tag{6.83}$$

The tensor $g_{\mu\nu}(z,\tilde{\rho})$ we expand both in transverse coordinates and in $\tilde{\rho}$

$$g_{\mu\nu}(z,\tilde{\rho}) = g_{\mu\nu}(x,r) + u^\lambda(x)k_b(r)\,\partial_\lambda g_{\mu\nu}(x,r) + \tilde{\rho}_1(x,r)\,\partial_r g_{\mu\nu}(x,r).$$

As mentioned above, the derivatives $\partial_\mu b(x)$ can be converted into derivatives of the fluid velocity $\partial_\mu u_\nu(x)$ by the continuity equation

$$\partial_\mu b = b\left(-\frac{1}{2\sigma-1}u_\mu(\partial\cdot u) + (u\cdot\partial)u_\mu\right), \tag{6.84}$$

which follows from the conservation of the perfect fluid energy-momentum tensor (6.56), $\partial^\mu T_{0\mu\nu} = 0$, or equivalently from the divergence equation (6.27) at order σ.

Putting everything together we obtain the transformation to first order in derivatives,

$$\begin{aligned} \tilde{\rho}(x,r) &= \tilde{\rho}_0(1 + k_b\frac{\partial\cdot u}{2\sigma-1}), \\ z^\mu(x,r) &= x^\mu + u^\mu k_b + u^\mu\frac{\partial\cdot u}{2\sigma-1}l(r) + (u\cdot\partial)u^\mu m(r), \end{aligned} \tag{6.85}$$

where $l(r)$, $m(r)$ satisfy the differential equations

$$\begin{aligned} \frac{dl}{dr} &= -\frac{k_b(r)}{r^2 f_b(r)} - \frac{1}{r^3 (f_b(r))^{3/2}}, \\ \frac{dm}{dr} &= -\frac{k_b(r)}{r^2 f_b(r)} + \frac{1}{r^3 \sqrt{f_b(r)}}, \end{aligned} \tag{6.86}$$

with the boundary condition that they vanish for $r \to \infty$. The metric in Eddington-Finkelstein coordinates up to first derivative order is then given by

$$\begin{aligned} G_{\mu\nu}(x,r) &= r^2[\eta_{\mu\nu} + (f_b(r) - 1)u_\mu u_\nu] \\ &\quad -\chi_b(r)\sigma_{\mu\nu} - \frac{2r}{2\sigma - 1} u_\mu u_\nu (\partial \cdot u) + r(u \cdot \partial)(u_\mu u_\nu), \end{aligned} \tag{6.87}$$

where

$$\chi_b(r) = \frac{2A(\tilde{\rho}_0(r))k_b(r) + \lambda(\tilde{\rho}(r))}{\tilde{\rho}_0(r)} = r^2 \left[2k_b(r) + \frac{2\gamma b}{\sigma} \log \frac{2 - A^{\sigma/2}}{A^{\sigma/2}} \right]. \tag{6.88}$$

Near the horizon $r \to 1/b$, the hypergeometric function in the definition of $k_b(r)$ in (6.78) develops a logarithmic divergence of the form [143]

$${}_2F_1(x, y; x + y; w) = -\frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \log(w - 1) + finite, \tag{6.89}$$

and $\chi_b(r)$ becomes

$$\chi_b \to \frac{(\gamma - 1)}{\sigma b} \log(r - \frac{1}{b}) + finite. \tag{6.90}$$

Hence the divergence in $\chi_b(r)$ cancels precisely if $\gamma = 1$, ie. the shear viscosity to entropy density bound $\eta_{2\sigma}/s_{2\sigma} \geq 1/4\pi$ is saturated.

In the non-conformal case, the transformation from Fefferman-Graham to Eddington-Finkelstein coordinates is given by the same coordinate transformations (6.77) and (6.85). By either transforming (6.72) or by dimensionally reducing (6.87) according to the reduction ansatz (6.15) we obtain the metric and scalar to first derivative order in Eddington-Finkelstein coordinates:

$$\begin{aligned} G_{ij} &= r^2[\eta_{ij} + (f_b(r) - 1)u_i u_j] - \chi_b(r) \left[\sigma_{ij} + \frac{2\sigma - d}{(d-1)(2\sigma - 1)} P_{ij}(\partial \cdot u) \right] \\ &\quad - \frac{2r}{2\sigma - 1} u_i u_j (\partial \cdot u) + r(u \cdot \partial)(u_i u_j), \end{aligned} \tag{6.91}$$

$$\phi = \frac{2\sigma - d}{2} \left[\log r^2 - \frac{1}{2\sigma - 1} \frac{\chi_b(r)}{r^2} (\partial \cdot u) \right]. \tag{6.92}$$

In particular, we see that the condition for the scalar and metric of the non-conformal solution to be smooth at the horizon is identical to the smoothness condition in the conformal case, namely that $\gamma = 1$ in the definition of $\chi_b(r)$ in (6.88). Hence as expected also in the non-conformal case the bound $\eta_d/s_d \geq 1/4\pi$ is saturated for arbitrary σ.

In contrast, the fixed value of the ratio of the bulk to shear viscosity $\zeta_d/\eta_d = 2(1/(d-1) - c_s^2)$ does not follow from a smoothness condition but instead it follows from the equation of motions away from the horizon of the black brane. As mentioned in section 6.3 and in the introduction, it is a consequence of the generalized conformal structure established by the Ward identity (6.14).

(6.7) DISCUSSION

In this note we have shown that the universal sector of solutions asymptotic to non-conformal brane geometries can be obtained by dimensionally reducing asymptotically AdS pure gravity solutions and continuing the number of dimensions of the higher-dimensional geometry. As a consequence, the hydrodynamics dual to non-conformal black branes is fully determined in terms of the hydrodynamics dual to conformal black branes. We used this relation to rederive the first order contribution to the non-conformal hydrodynamic stress tensor and predict the second order contributions. As expected, the KSS bound [140] for the ratio between shear viscosity and entropy density is always saturated. Furthermore we reconfirm that also the bound between shear and bulk viscosity proposed by [135] is saturated. We show however that the saturation of this bound for non-conformal brane geometries follows from the generalized conformal symmetry, which indicates that it is of kinematical origin, unlike the KSS bound.

It would be interesting to explore whether the relation between conformal and non-conformal brane backgrounds also holds at the higher derivative level. Corrections to the KSS ratio for higher derivative bulk actions dual to conformal fluids have been investigated in [144]. The generalized conformal structure in non-conformal brane geometries is expected to hold for arbitrary coupling, although it is not clear whether it would always descend from a higher dimensional conformal structure. For the cases of D4 branes and fundamental strings, this would be the case since this is just the M-theory uplift, so at least in these cases the ratio of bulk to shear viscosity should not receive any corrections at the higher derivative level.

On a more general level, one might wonder which further generalization of the AdS/CFT dictionary can be found by compactifying asymptotically AdS spaces on other manifolds or more general tori. In such a setup, the lower-dimensional geometry will automatically inherit many holographic results from the higher-dimensional asymptotically AdS case. Applied to hydrodynamics, one might obtain in this way non-conformal fluids with interesting properties.

Finally, it would be interesting to explore the hydrodynamics of non-conformal non-relativistic field theories. Hydrodynamics of conformal non-relativistic field theories have been explored by [145]. By compactification one might be able to obtain non-conformal generalizations.

(6.A) APPENDIX

(6.A.1) THE ASYMPTOTIC EXPANSION OF METRIC AND SCALAR BEYOND THE NON-LOCAL MODE

A general result that is most easily seen using the radial Hamiltonian formalism [19, 20] is that a bulk solution is uniquely specified by the holographic vevs. The reason is that the vevs are the radial canonical momenta and the sources the corresponding coordinates. Thus specifying the source and the vev is equivalent to specifying a point in the phase space of the theory, which is equivalent to specifying a full solution.

A special case that is relevant for us is the gravity/scalar system for the non-conformal branes with the boundary metric taken to be flat, $g_{(0)ij} = \eta_{ij}$, and $\kappa_{(0)} = 0$. Then all subleading terms in the expansion (6.13) up to order σ including the logarithmic term vanish, since they depend on derivatives of the sources $g_{(0)}$ and $\kappa_{(0)}$. The order σ terms will be given by

$$g_{(2\sigma)ij} = \frac{1}{2\sigma L} T_{ij}, \qquad \kappa_{(2\sigma)} = -\frac{1}{4\sigma L} T_i^i, \tag{6.93}$$

where T_{ij} and T_i^i denote the vev of the dual energy-momentum tensor and its trace, and $\kappa_{(2\sigma)}$ is determined by the requirement that the generalized conformal Ward identity (6.14) is satisfied. The higher order terms in Fefferman-Graham expansion are then determined in terms of T_{ij}. We will need the schematic form of these coefficients. The non-linear equations of motion induce the expansion

$$\begin{aligned} g(z,\rho)_{ij} &= \eta_{ij} + \sum_{\tau = n\sigma + m} \rho^{\tau} g_{(2\tau)ij}, \\ \kappa(z,\rho) &= \kappa_{(0)} + \sum_{\tau = n\sigma + m} \rho^{\tau} \kappa_{(2\tau)}, \end{aligned} \tag{6.94}$$

where $n > 1$ and $m > 0$ in the summation are positive integers. Suppressing the index structure, the higher order terms are schematically of the form

$$g_{(2\tau)ij} \propto \sum_{n\sigma + m = \tau} a_{n,m} (\partial^{2m} T^n)_{ij}, \tag{6.95}$$

and similarly for $\kappa_{(2n\sigma+2m)}$, due to dimensional considerations. In particular, if T_{ij} is constant, only the coefficients with $\tau = n\sigma$ are non-zero. Once T_{ij} becomes dependent on the boundary coordinates, the transverse derivatives in (6.95) always enter in pairs. This fact is used in section 6.5 to restrict the form of the first order derivative correction to the metric.

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SUMMARY

One of the most inspiring conjectures of the last fifteen years in search of a theory of quantum gravity is the conjecture of a *holographic principle*. In analogy to an optical hologram, which stores a three-dimensional picture on a a two-dimensional photographic plate, this conjectures states that all gravitational phenomena in a $(d+1)$-dimensional spacetime can be described by a d-dimensional quantum field theory without gravity. The best elaborated example of such a holographic duality is the so-called AdS/CFT correspondence.

This thesis discusses two different aspects of the AdS/CFT correspondence: On the one hand we apply the correspondence in order to examine a microscopic theory of black holes which was proposed about ten years ago and which contains promising features to solve long-standing black hole paradoxa, the *fuzzball proposal*. This application of a holographic duality is discussed in chapter 3 and 4. Before that, we provide an introduction to holography in chapter 1 and an introduction to the fuzzball proposal in chapter 2. On the other hand, the last part of the thesis, chapter 5 and 6, discusses a generalization of the AdS/CFT correspondence to cases in which the d-dimensional quantum field theory does not have conformal symmetry. We lay down the basics in chapter 5 and look at applications on the hydrodynamic limit of the quantum field theory in chapter 6.

THE INFORMATION LOSS PARADOXON AND THE FUZZBALL PROPOSAL

A black hole is an object whose mass density is so high that according to General Relativity not even light can escape its event horizon. An observer outside the event horizon has no possibility to explore what is happening inside the horizon and different black hole geometries are according to the no-hair theorem only distinguishable with respect to their overall mass, charge and angular momentum. As a consequence, all information about an object falling into a black hole is lost for the outside observer.

This by itself would not be a paradox, since it could be possible that the information is somehow contained near the singularity of the black hole, the area in the center in which the spacetime curvature is so high that General Relativity ceases to be valid and which then can only be described by a full theory of quantum gravity. At this point though Stephen Hawking's discovery

comes into play that, due to quantum field theory arguments, black holes emit radiation and, as a consequence, lose mass. The spectrum of this radiation is purely thermal, which means that it only depends on the mass, charge and angular momentum of the black hole and not of the detailed consistency of the previously infallen objects. Black holes radiate more the smaller they are, losing more and more mass, until they finally evaporate. But with the evaporation of the black hole also the singularity disappears and all information about infallen objects seems irretrievably lost. This loss of information is in sharp contradiction with quantum theory according to which all the information in the universe has to be conserved. Quantum theory namely says that given the exact knowledge of a later state it should always be possible to reconstruct an earlier state, including the earlier state of an object which has fallen into the black hole.

One possibility to overcome the information loss paradox is to conjecture that a full quantum gravity description of Hawking radiation would contain minuscule deviations from thermality which allow information about infallen objects to escape. But even if such a description could be successfully formulated, the information transfer from the singularity to the horizon would be necessarily non-local. Locality however is a principle physicists give up only unwillingly since it guarantees causality, i.e. the separation of cause and effect.

The fuzzball proposal, which has been formulated in the context of string theory, tries to resolve the information loss problem by assuming that the usual spacetime of a black hole is only an effective description of a sufficiently distant observer. According to the fuzzball proposal a microscopic description of a black hole does neither contain an event horizon nor a singularity. Hence, an object falling into the black hole can in principle escape (after a very long time) or its information can escape through non-thermal deviations of the Hawking radiation, without violating locality and causality.

In chapter 3 and 4 we discuss a simplified, supersymmetric toy model of a black hole in which all microscopic states are known. These microscopic states can also be described by means of a dual theory, which has been used fifteen years ago to count for the first time microscopically the entropy of a black hole. It turns out however that this dual theory is just the AdS/CFT dual of the usual gravitational description and we use the AdS/CFT correspondence to examine the precise map between the microscopic states in both descriptions.

Nonconformal branes and their hydrodynamics

Although it seems that the AdS/CFT correspondence can be applied to many spacetimes it could only be formulated explicitly for few spacetimes with enough precision to allow comparison between detailed calculations on the gravity side and on the quantum field theory side. Prime examples of precisely formulated correspondences are the spacetimes close to a large number of D3 branes or close to a bound state of D1 and D5 branes. (Dp branes, with p integer, are extended massive objects in String Theory with $p+1$ spacetime dimensions.)

An important step towards a precise formulation of an AdS/CFT correspondence is a careful handling of infinities which would arise in a naïve formulation. On the gravity side, there appear for example integrals over the whole spacetime which diverge due to the infinite volume of the latter. These infinities correspond to infinities in the quantum field theory which arise in the so-called renormalization and which require a careful redefinition of the theory. The according redefinition of the correspondence is therefore called *holographic renormalization*.

Holographic renormalization for the spacetimes of D3 branes and the D1/D5 system is already well-known. Both of these examples however correspond to quantum field theories which have a so-called conformal symmetry, at least at high energies. In chapter 5 we develop holographic renormalization for Dp branes with $p \neq 3$, which do not possess conformal symmetry and hence are called nonconformal.

An interesting application of the AdS/CFT correspondence are spacetimes whose dual quantum field theory describes a plasma. The plasma corresponding to the quantum field theory of (black) D3 branes for example has been proven useful as a toy model of the quark gluon plasma, which is examined by experimental physicists in accelerators like the Relativistic Heavy Ion Collider in New York and soon the Large Hadron Collider in Geneva. These plasmas can often be described by a fluid which obeys the laws of relativistic hydrodynamics. Using the AdS/CFT correspondence one can map the equations of motion of hydrodynamics to the gravitational fluctuation equations around the dual spacetime.

Even though the D3 brane plasma provides a reasonable model of the quark gluon plasma it differs from the latter in that the quantum field theory proper on which the quark gluon plasma is based, Quantum chromodynamics, is not conformal. For this reason it is interesting to study non-conformal plasmas using the AdS/CFT correspondence. In chapter 6 we therefore apply the results of chapter 5 to examine the hydrodynamics of nonconformal Dp branes.

ZUSAMMENFASSUNG

Einer der inspirierendsten Vermutungen der letzten fünfzehn Jahren auf der Suche nach einer Theorie der Quantengravitation ist die Vermutung der Existenz eines *holografischen Prinzips*. In Analogie zum Hologramm in der Optik, welches ein dreidimensionales Bild auf einer zweidimensionalen Fotoplatte speichert, besagt diese Vermutung, dass alle Gravitationsphänomene in einer (d+1)-dimensionalen Raumzeit durch eine d-dimensionale Quantenfeldtheorie ohne Gravitation beschrieben werden können. Das am besten ausgearbeitete Beispiel einer solchen holografischen Dualität ist die sog. AdS/CFT Korrespondenz.

Diese Dissertation behandelt zwei unterschiedliche Aspekte der AdS/CFT Korrespondenz: Zum einen verwenden wir diese, um eine mikroskopische Theorie Schwarzer Löcher zu untersuchen, die vor knapp zehn Jahren vorgeschlagen wurde und vielversprechende Ansätze zur Überwindung hartnäckiger Paradoxa beinhaltet, die *Fuzzball-Vermutung*. Dieser Anwendung einer holografischen Dualität ist Kapitel 3 und 4 gewidmet. Zuvor enthält Kapitel 1 eine Einführung in die Holografie und Kapitel 2 eine Einführung in die Fuzzball-Vermutung. Zum anderen behandelt der letzte Teil der Dissertation, Kapitel 5 und 6, eine Verallgemeinerung der AdS/CFT Korrespondenz für Fälle, in denen die d-dimensionale Quantenfeldtheorie keine konforme Symmetrie mehr aufweist. In Kapitel 5 entwickeln wir dazu die Grundlagen und betrachten in Kapitel 6 Anwendungen auf das hydrodynamische Limit der Quantenfeldtheorie.

DAS INFORMATIONSVERLUSTPARADOXON UND DIE FUZZBALL-VERMUTUNG

Ein Schwarzes Loch ist ein Objekt, dessen Massendichte so hoch ist, das der Allgemeinen Relativitätstheorie zu Folge nicht einmal mehr Licht seinem Ereignishorizont entweichen kann. Ein Beobachter außerhalb des Ereignishorizontes hat keine Möglichkeit Aufschluss zu erhalten, was sich innerhalb des Ereignishorizontes abspielt, und die Raumzeiten verschiedener Schwarzer Löcher unterscheiden sich dem No-hair-Theorem zu Folge nur gemäß deren Gesamtmasse, -ladung und -drehimpuls. Also ist jede Information über ein Objekt, welches in ein Schwarzes Loch fällt für den äußeren Beobachter unwiederbringlich verloren.

Dies allein würde noch kein Paradoxon darstellen, denn man könnte annehmen, dass diese Information statt dessen in irgendeiner Weise in der Nähe der Singularität des Schwarzen Loches enthalten ist, jenem Gebiet im Inneren, in dem die Raum-Zeit-Krümmung so hoch ist, dass die Allgemeine Relativitätstheorie ihre Gültigkeit verliert und das dann nur durch eine vollständige Theorie der Quantengravitation beschrieben werden kann. Nun jedoch kommt die Entdeckung Stephen Hawkings ins Spiel, der aus quantenfeldtheoretischen Überlegungen folgerte, dass Schwarze Löcher Strahlung abgeben und dabei gleichzeitig Masse verlieren. Das Spektrum dieser Strahlung ist rein thermisch, das bedeutet lediglich von Masse, Ladung und Drehimpuls des Schwarzen Loches abhängig und nicht etwa von der detaillierten Beschaffenheit der Objekte, die zuvor in das Schwarze Loch gefallen sind. Schwarze Löcher strahlen umso mehr je kleiner sie sind und verlieren damit immer mehr Masse, bis sie irgendwann vollständig verdampfen. Doch nach der Verdampfung des Schwarzen Loches gibt es auch keine Singularität mehr, und somit scheint jegliche Information über die hineingefallenen Objekte endgültig verloren. Ein solcher Informationsverlust ist jedoch im krassen Widerspruch zur Quantentheorie, der zu Folge die gesamte Information im Universum erhalten bleiben muss. Die Quantentheorie besagt nämlich, dass man theoretisch jederzeit einen früheren Zustand mit exakter Kenntnis eines späteren Zustands rekonstruieren kann, in diesem Fall also den früheren Zustands eines Objekts, das in das Schwarze Loch gefallen ist.

Eine Möglichkeit zur Überwindung des Informationsverlustparadoxon wäre anzunehmen, dass eine volle quantengravitative Beschreibung der Hawking-Strahlung winzige nicht-thermische Abweichungen beinhaltet, über die die Information der hineingefallenen Objekte nach außen gelangen kann. Aber selbst wenn so eine Beschreibung gelänge, würde die Informationsübertragung von der Singularität bis an den Horizont notwendigerweise nicht-lokal sein. Lokalität aber geben Physiker nur ungern auf, da dieses Prinzip ein Garant für Kausalität, also die Trennung von Ursache und Wirkung ist.

Die Fuzzball-Vermutung, die im Kontext der Stringtheorie formuliert wurde, versucht das Informationsverlustparadoxon dadurch aufzulösen, dass es davon ausgeht, dass die übliche Raumzeit eines Schwarzen Loches nur eine effektive Beschreibung eines genügend weit entfernten Beobachters darstellt. Der Fuzzball-Vermutung zu Folge besitzt eine mikroskopische Beschreibung eines Schwarzen Loches weder einen Ereignishorizont noch eine Singularität. Ein Objekt, das in das Schwarze Loch fällt kann also im Prinzip (nach sehr langer Zeit) wieder entweichen oder dessen Information kann über nicht-termische Abweichungen der Hawking Strahlung nach außen gelangen, ohne Lokalität und Kausalität zu verletzen.

In Kapitel 3 und 4 betrachten wir ein vereinfachtes, supersymmetrisches Modellsystem eines Schwarzen Loches, in dem alle mikroskopischen Zustände bekannt sind. Diese mikroskopischen Zustände können auch mit Hilfe einer dualen Theorie beschrieben werden, mit deren Hilfe vor fünfzehn Jahren erstmals die Entropie eines Schwarzen Loches mikroskopisch abgezählt werden konnte. Diese duale Theorie ist mit der üblichen gravitativen Beschreibung aber gerade über die AdS/CFT-Korrespondenz verbunden, und wir verwenden letztere, um die präzise Abbildung zwischen diesen beiden Beschreibungen zu untersuchen.

Nichtkonforme Branen und ihre Hydrodynamik

Obwohl es den Anschein hat, dass die AdS/CFT Korrespondenz im Prinzip auf viele unterschiedliche Raumzeiten angewandt werden kann, konnte sie bisher nur für wenige Raumzeiten in genügendem Detail formuliert werden, dass damit detaillierte Berechnungen auf der gravitativen Seite mit detaillierten Berechnungen auf der Seite der Quantenfeldtheorie verglichen werden konnten. Paradebeispiele präzis formulierter Korrespondenzen sind die Raumzeit in der Nähe einer großen Anzahl von sog. D3 Branen oder in der Nähe eines gebunden Zustandes aus D1 und D5 Branen. (Dp Branen, wobei p eine ganze Zahl ist, sind ausgedehnte, massive Objekte in der Stringtheorie mit $p+1$ Raumzeit-Dimensionen.)

Ein wichtiger Schritt zu einer präzisen Formulierung einer AdS/CFT Korrespondenz ist ein sorgsamer Umgang mit Unendlichkeiten, die in einer naiven Formulierung auftreten würden. Auf der gravitativen Seite sind das beispielsweise Integrale über die ganze Raumzeit, die auf Grund deren unendlichen Volumens divergieren. Diesen Unendlichkeiten entsprechen die Unendlichkeiten in der Quantenfeldtheorie die bei der sog. Renormalisierung auftreten und die eine sorgsame Neudefinition der Quantenfeldtheorie erfordern. Die entsprechende Neudefinition der Korrespondenz wird deshalb *Holografische Renormalisierung* genannt.

Die Holografische Renormalisierung für die Raumzeiten von D3 Branen und dem D1/D5-System sind schon länger bekannt. Beide dieser Beispiele entsprechen aber Quantenfeldtheorien, die zumindest bei hohen Energien eine sog. konforme Symmetrie aufweisen. In Kapitel 5 entwickeln wir die Holografische Renormalisierung für Dp Branen mit $p \neq 3$, die keine konforme Symmetrie besitzen, also nichtkonform sind.

Eine interessante Anwendung der AdS/CFT Korrespondenz sind Raumzeiten, deren duale Quantenfeldtheorie ein Plasma beschreibt. Das Plasma, das der Quantenfeldtheorie (schwarzer) D3 Branen entspricht, hat sich beispielsweise als Modellsystem für das Quark-Gluon Plasma bewährt, das von Experimentalphysikern in Beschleunigern wie dem Relativistic Heavy Ion Collider in New York und bald dem Large Hadron Collider in Genf untersucht wird. Diese Plasmen können häufig als ein Fluid beschrieben werden, das den Gesetzen der relativistischen Hydrodynamik unterworfen ist. Mit Hilfe der AdS/CFT-Korrespondenz können die Bewegungsgleichungen der Hydrodynamik auf gravitative Fluktuationsgleichungen um die duale Raumzeit abgebildet werden.

Wenn auch das D3-Branen Plasma eine recht gutes Modell des Quark-Gluon Plasmas darstellt, so unterscheidet es sich dahingehend, dass die eigentliche Quantenfeldtheorie, auf dem das Quark-Gluon Plasma beruht, nämlich die Quantenchromodynamik, keine konforme Symmetrie hat. Aus diesem Grund ist es interessant mit Hilfe der AdS/CFT Korrespondenz nicht-konforme Plasmen zu untersuchen. In Kapitel 6 wenden wir darum die Ergebnisse von Kapitel 5 an, um die Hydrodynamik von nicht-konformen Dp Branen zu untersuchen.

SAMENVATTING

Een van de meest inspirerende onderwerpen van de laatste vijftien jaar bij de zoektocht naar een theorie van de kwantumzwaartekracht is het vermoeden van het bestaan van een *holografisch principe*. Net als een hologram in de optica, dat een driedimensionaal beeld op een tweedimensionale fotoplaat opslaat, zegt dit vermoeden dat alle gravitationele verschijnselen in een $(d+1)$-dimensionale ruimtetijd beschreven kunnen worden door een d-dimensionale kwantumveldentheorie zonder zwaartekracht. Het best uitgewerkte voorbeeld van een dergelijke holografische dualiteit is de zogenaamde AdS/CFT correspondentie.

Dit proefschrift richt zich op twee verschillende aspecten van de AdS/CFT correspondentie: Enerzijds gebruiken we deze om een microscopische theorie van zwarte gaten te onderzoeken, die bijna tien jaar geleden werd voorgesteld en veelbelovende aspecten voor de oplossing van hardnekkige paradoxen inhoudt, de *fuzzball-stelling*. Deze toepassing van een holografische dualiteit is in hoofdstuk 3 en 4 behandeld. Voorafgaand bevat hoofdstuk 1 een inleiding in de holografie en hoofdstuk 2 een inleiding in de fuzzball-stelling. Anderzijds behandelt het laatste deel van de proefschrift, hoofdstuk 5 en 6, een generalisatie van de AdS/CFT correspondentie voor gevallen waarin de d-dimensionale kwantumveldentheorie geen conforme symmetrie meer heeft. In hoofdstuk 5 ontwikkelen we hiervoor de basis en in hoofdstuk 6 beschouwen we toepassingen op de hydrodynamische limiet van de kwantumveldentheorie.

DE INFORMATIEVERLIESPARADOX EN DE FUZZBALL-STELLING

Een zwart gat is een object waarvan de massadichtheid zo groot is dat volgens de Allgemene Relativiteitstheorie zelfs licht zijn waarnemingshorizon niet kan ontwijken. Een waarnemer buiten de horizon heeft geen mogelijkheid vast te stellen wat er binnen de horizon gebeurt, en de ruimtetijden van verschillenden zwarte gaten schelen volgens het no-hair theorem alleen maar met betrekking tot hun gezamenlijke massa, lading en impulsmoment. Daarom is iedere informatie over een object dat in een zwart gat valt voor de externe waarnemer onherroepelijk verloren.

Dit alleen zou nog geen paradox zijn, omdat men zou kunnen aannemen dat deze informatie in plaats daarvan op een of andere manier opgenomen is in de buurt van de singulariteit van het zwarte gat, in dat gebied binnenin, waarin de ruimte-tijd kromming zo groot is dat de Algemene Relativiteitstheorie zijn geldigheid verliest en dat vervolgens alleen maar door een volledige theorie van de kwantumzwaartekracht kan worden beschreven. Nu komt echter de ontdekking van Stephen Hawking in het spel, die uit kwantumveldentheoretische overwegingen afleidde dat zwarte gaten straling afgeven en daarbij tegelijkertijd massa verliezen. Het spectrum van deze straling is puur thermisch, wat betekent dat het alleen maar van massa, lading en impulsmoment van het zwarte gat afhankelijk is en niet van de gedetailleerde geaardheid van de objecten die vooraf in het zwarte gat zijn gevallen. Zwarte gaten stralen meer naarmate ze kleiner zijn en verliezen zo steeds meer massa, tot ze uiteindelijk volledig verdampen. Maar na de verdamping van een zwart gat is er ook geen singulariteit meer en is alle informatie over de binnengevallenen objecten verloren. Een dergelijk verlies van informatie staat in schril contrast tot de kwantumtheorie, die zegt dat alle informatie in het universum behouden moet zijn. Volgens de kwantumtheorie kan men namelijk altijd een eerdere toestand met exacte kennis van een latere toestand reconstrueren, ook zoals hier de eerdere toestand van een object dat in het zwarte gat is gevallen.

Een mogelijkheid ter omzeiling van de informatieverliesparadox is te veronderstellen dat een volledig kwantumgravitationele beschrijving van de Hawking straling minuscule niet-thermische afwijkingen bevat, waardoor informatie van de binnengevallen objecten naar buiten kan lekken. Maar zelfs als zo'n beschrijving mogelijk zou zijn, zou de informatieoverdracht van de singulariteit naar de horizon noodzakelijkerwijs niet-lokaal zijn. Natuurkundigen aarzelen om localiteit prijs te geven omdat dit principe een garantie voor causaliteit, dus de scheiding tussen oorzaak en werking is.

De fuzzball-stelling, die in de context van de snaartheorie werd geformuleerd, probeert de informatieverliesparadox op te lossen door ervan uit te gaan, dat de gewone ruimtetijd van een zwart gat alleen maar een effectieve beschrijving van een waarnemer op voldoende afstand weergeeft. Volgens de fuzzball-stelling heeft een micoscopische beschrijving van een zwart gat noch een waarnemingshorizon noch een singulariteit. Een object dat in een zwart gat valt kan dus in principe (na zeer lange tijd) weer ontsnappen, of de informatie ervan kan door niet-thermische afwijkingen in de Hawking-straling naar buiten lekken, zonder localiteit en causaliteit te schenden.

In hoofdstuk 3 en 4 beschouwen we een vereenvoudigd, supersymmetrisch modelsysteem van een zwart gat waarbij alle microscopische toestanden bekend zijn. Deze microscopische toestanden kunnen ook met behulp van een duale theorie worden beschreven die vijftien jaar geleden gebruikt werd om voor het eerst de entropie van zwarte gaten microscopisch te tellen. Echter het blijkt dat deze duale theorie niets anders is dan het AdS/CFT-equivalent van de normale zwaartekracht-beschrijving en we gebruiken de AdS/CFT-correspondentie om de precieze afbeelding tussen deze twee beschrijvingen te onderzoeken.

Niet-conforme branen en hun hydrodynamica

Hoewel blijkt dat de AdS/CFT correspondentie in principe op veel verschillende ruimtetijden toegepast kan worden, kon ze tot nu toe alleen maar voor weinig ruimtetijden in genoeg detail worden geformuleerd om daarmee gedetailleerde berekeningen op het vlak van de zwaartekracht met gedetailleerde berekeningen op het vlak van de kwantumveldentheorie te vergelijken. Schoolvoorbeelden van precies geformuleerde correspondenties zijn de ruimtetijd vlakbij een groot aantal van zogenaamde D3 branen en vlakbij een gebonden toestand van D1 en D5 branen. (Dp branen, waar p een geheel getal is, zijn uitgebreide, massale objecten in de snaartheorie met $p+1$ dimensies.)

Een belangrijke stap op weg naar een precieze formulering van een AdS/CFT correspondentie is een zorgvuldige behandeling van oneindigheden, die in een naive formulering zouden optreden. Op het vlak van de zwaartekracht zijn dit bijvoorbeeld integralen over de gehele ruimtetijd die op grond van diens oneindige volume divergeren. Deze oneindigheden komen overeen met oneindigheden die in de kwantumveldentheorie bij de zogenaamde renormalisatie optreden en die een zorgvuldige herdefinitie van de kwantumveldentheorie vereisen. De navenante herdefinitie van de correspondentie wordt daarom *holografische renormalisatie* genoemd.

De holografisch renormalisatie van de ruimtetijden van D3 branen en van het D1/D5 systeem zijn al langer bekend. Maar beide voorbeelden komen overeen met kwantumveldentheorieën, die tenminste bij hoge energieën een zogenaamde conforme symmetrie vertonen. In hoofdstuk 5 ontwikkelen we holografische renormalisatie voor Dp branen met $p \neq 3$, die geen conforme symmetrie bezitten en dus niet-conform zijn.

Een interessante toepassing van de AdS/CFT correspondentie zijn ruimtetijden waarvan de duale kwantumveldentheorie een plasma beschrijft. Het plasma dat met de kwantumveldentheorie van (zwarte) D3 branen overeenkomt, heeft zich bijvoorbeeld bewezen als een modelsysteem voor het Quark-Gluon-Plasma dat door experimentele natuurkundigen in versnellers als de Relativistic Heavy Ion Collider in New York onderzocht wordt en straks de Large Hadron Collider in Genève onderzocht gaat worden. Deze plasma's kunnen vaak worden beschreven als een fluïdum dat onderworpen is aan de wetten van relativistische hydrodynamica. Met behulp van de AdS/CFT correspondentie kunnen de bewegingsvergelijkingen van de hydrodynamica vertaald worden in vergelijkingen van gravitationele fluctuaties rondom de duale ruimtetijd.

Hoewel het D3 branen plasma een vrij goed model van het Quark-Gluon-Plasma vormt, verschilt het in die zin dat de eigenlijke kwantumveldentheorie waarop het Quark-Gluon-Plasma berust, namelijk de Kwantumchromodynamica, geen conforme symmetrie heeft. Om deze reden is het interessant met behulp van de AdS/CFT correspondentie niet-conforme plasma's te onderzoeken. In hoofdstuk 6 gebruiken we daarom de resultaten van hoofdstuk 5 om de hydrodynamica van niet-conforme Dp branen te onderzoeken.

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