On the Instability of Pope-Warner Solutions

Dissertation

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1 Introduction

The research presented in this dissertation is of interest due to the important result in string theory and M-theory known as the AdS/CFT correspondence. The AdS/CFT correspondence first came to light in 1998 [1], and has since been a very active area of research.

In [1] the AdS/CFT correspondence was proposed in the context of D3-branes in type IIB superstring theory. In this case of the AdS/CFT correspondence, type IIB superstring theory in the background geometry $\mathrm{AdS}_5 \times \mathrm{S}^5$ is conjectured to be dual to the four-dimensional U(N) $\mathcal{N} = 4$ super-Yang Mills theory that is known to live on the worldvolume of N coincident D3-branes. When $g_s N$, where g_s is the string coupling constant, is very large, it can be shown that the gauge theory becomes strongly coupled, and the string theory can be well-approximated as a classical gravity theory, which means that stringy effects need not be included. This case of the AdS/CFT correspondence is reviewed in detail in [2].

A concrete realization of the AdS/CFT correspondence in the context of M2 branes in M-theory was given in [3]. This correspondence is known as ABJM theory. In this case of the AdS/CFT correspondence, M-theory in the geometry $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_\mathrm{k}$ is conjectured to be dual to the three-dimensional $\mathrm{U(N)} \times \mathrm{U(N)}$ Chern-Simons-matter theory at level k with $\mathcal{N} = 6$ or 8 supersymmetry, that is known to live on the worldvolume of N coincident M2-branes. This duality holds in the limit where N is very large. Additionally, when $N \gg k^5$, M-theory can be well-approximated as a classical gravity theory.

In light of ABJM theory, a natural question to ask was whether the correspondence could be generalized to the case of Chern-Simons theories with less supersymmetry. See, e.g. [4, 5].

In the case of Chern-Simons theories with $\mathcal{N} \geq 2$ supersymmetry it is known that if such a theory has an M-theory dual, then the geometry of such a dual theory must be of the form $\mathrm{AdS}_4 \times \mathrm{SE}_7$, where SE_7 is a type of compact manifold known as a 'Sasaki-Einstein manifold'. The definition of a Sasaki-Einstein manifold and some important facts about them are given in Appendix B.

It is known that supergravity theory on the background geometry $\mathrm{AdS}_4 \times \mathrm{SE}_7$, which is called the ('skew-whiffed') 'Freund-Rubin' background, can be continuously deformed to supergravity theory on another AdS product space background known as the 'Pope-Warner' background, see, e.g., [6, 11]. In chapter 3 we give the Freund-Rubin, skew-whiffed Freund-Rubin, and Pope-Warner background solutions explicitly. Whereas the supergravity theory on the Freund-Rubin background was known to be stable [44], it was unknown whether the theory on the Pope-Warner background was stable. The purpose of the research in this dissertation is to study the stability of Pope-Warner solutions on Sasaki-Einstein manifolds, which in light of the AdS/CFT correspondence should correspond to vacua of $2+1$-dimensional field theories.

A major motivation for studying the AdS/CFT correspondence is its possible application to condensed matter physics, see e.g. [7, 8, 9]. In this vein, it was found that in 'top-down' constructions of holographic superconductors, the Pope-Warner solution corresponds to a zero-temperature quantum critical phase of a $2+1$-dimensional superconductor [10, 11]. In light of this promising find, it was of strong interest to determine the stability of Pope-Warner solutions on Sasaki-Einstein manifolds. In section 1.1 of this introductory chapter we further discuss the relevance of the Pope-Warner solution to superconductor solutions.

Having discussed the broader context in which the research presented here is of interest, we now go more directly into the research itself. The field equations of eleven-dimensional supergravity [14, 15] in the bosonic sector are:[1]

$$\mathcal{R}_{MN} + \mathfrak{g}_{MN}\mathcal{R} = \frac{1}{3}\mathcal{F}_{MPQR}\mathcal{F}_N{}^{PQR}\,, \tag{1.1}$$

$$d \star \mathcal{F}_{(4)} + \mathcal{F}_{(4)} \wedge \mathcal{F}_{(4)} = 0\,, \tag{1.2}$$

where $\mathfrak{g}_{MN}$ is the metric, $\mathcal{F}_{(4)} = d\,\mathcal{A}_{(3)}$ is the four form flux, and $\star$ denotes the Hodge dual in eleven dimensions. A simple and important class of solutions are the ones in which the eleven-dimensional space time is a product $AdS_4 \times M_7$, where M_7 is a seven-dimensional Sasaki-Einstein (SE) manifold. Those manifolds are characterized by the existence of two real Killing spinors (see, e.g., [23, 24, 26], Appendix B) and the corresponding Freund-Rubin (FR) solutions [18] are $\mathcal{N} \geq 2$ supersymmetric. Solutions in which M_7 is one of the homogeneous SE manifolds:

$$S^7\,, \qquad N^{1,1}\,, \qquad M^{3,2}\,, \qquad Q^{1,1,1}\,, \qquad V^{5,2}\,, \tag{1.3}$$

were classified in the 1980s [22], but it is only quite recently that new solutions with nonhomogeneous SE metrics have been discovered [27, 28, 29].

It has also been known since the 1980s that given a SE manifold, M_7, there are, in addition to the supersymmetric FR solution, three non-supersymmetric solutions: the skew-whiffed FR solution [18] obtained by the change of orientation on M_7, and the Englert [19]

[1] We summarize our conventions in appendix A.

and Pope-Warner (PW) [20, 21] solutions with nonvanishing internal fluxes constructed from the geometric data on M_7.

Quite generally, non-supersymmetric solutions in gauged supergravity tend to be unstable. Indeed, while the stability of an AdS-type solution is guaranteed if there are some unbroken supersymmetries [36, 37], in non-supersymmetric backgrounds one expects to find scalar fluctuations,

$$(\Box_{AdS_4} - m^2)\varphi = 0\,, \tag{1.4}$$

whose masses violate the Breitenlohner-Freedman (BF) bound [38]

$$m^2 L^2 \geq -\frac{9}{4}\,, \tag{1.5}$$

where L here is the radius of AdS_4.

For the solutions above, the perturbative stability of the skew-whiffed FR solution in eleven-dimensional supergravity was proved in [39]. It follows by a simple observation that the mass spectrum of fluctuations that might produce an instability is invariant under the change of orientation of M_7 and hence is the same for the skew-whiffed and the supersymmetric backgrounds.

The Englert solutions are more difficult to analyze because the background flux couples the scalar and pseudoscalar fluctuations. The resulting perturbative instability for any SE background, M_7, can be shown by an explicit construction of unstable modes in terms of the two Killing spinors [42]. The same instability is also visible in the massive truncation of the eleven-dimensional supergravity on M_7 [31, 11], and when M_7 is the round seven-sphere, S^7, it corresponds to the instability of the $\mathrm{SO}(7)^-$ critical point of $\mathcal{N}=8$, $d=4$ gauged supergravity [40, 41].

Prior to the research presented in this dissertation, it was known from [32] that the PW solution on S^7 is in fact unstable. The question of the stability of the PW solution on other SE manifolds is the main concern of the research presented here.

In the rest of this introductory chapter, we will do three things. First, we will discuss the motivation for looking at the stability of PW solutions. Given that the PW solution was first constructed in 1984, one may ask why its stability is a concern now, many years later. The answer to this question lies in the context of AdS/CFT [2] and "top down" constructions of holographic superconductors [10, 11, 12], as will be discussed in the following section. Second, we will discuss prior results that are relevant to the research presented here, and finally, we will discuss and summarize the main results of this research.

1.1 Motivation from holographic superconductors

AdS/CFT

It was proposed in [1] that supergravity in the background geometry AdS_4 is dual to a $d = 3$ dimensional CFT in flat space. As previously mentioned, such a duality was concretely realized in [3]. Here, we discuss a basic aspect of the AdS/CFT correspondence. Namely, we identify the space-time of the CFT theory with the radial slices of the AdS geometry, and we identify the value of the radial coordinate in the AdS geometry as the energy scale of the CFT.

The AdS_4 metric can be written as

$$ds^2 = -r^2 dt^2 + \frac{1}{r^2} dr^2 + r^2(dx^2 + dy^2). \tag{1.6}$$

Under the change of variable $r = \frac{1}{z}$ it becomes

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dx^2 + dy^2). \tag{1.7}$$

With this form of the metric it is easy to see that for each fixed value of the radial variable z there is a copy of flat $d = 3$ Minkowski space. So AdS_4 can be regarded as copies of $d = 3$ Minkowski space along a radial variable z. The Minkowski space variables t, x, and y of the AdS_4 space can be identified with the time and space variables of the $d = 3$ CFT. The radial variable z of the AdS_4 space is to be identified as the energy scale of the $d = 3$ CFT [16, 17].

This identification of z as the energy scale of the CFT simply follows from the fact that the $d = 4$ and $d = 3$ theories are dual, and from the fact that the $d = 3$ theory is conformally invariant [17]. To see how this identification follows from these facts, consider what happens when changing the length scale of the $d = 3$ theory. Changing the length scale of the $d = 3$ theory amounts to making the transformation $(t, x, y) \to \lambda(t, x, y)$. Since the $d = 3$ theory is conformally invariant this transformation has no effect. However this transformation will clearly change the AdS_4 metric. In order to maintain the equivalence, i.e. duality, of the $d = 4$ and $d = 3$ theories, it is necessary to also simultaneously make the transformation $z \to \lambda z$. With this additional transformation it is clear that the AdS_4 metric stays the same. Therefore, since energy goes as inverse length, changing the energy (length) scale of the $d = 3$ theory amounts to moving along the radial direction as $z \to \lambda z$. And hence, the radial direction of the $d = 4$ theory should be identified with the energy scale of the $d = 3$ theory.

Since energy goes as inverse length, larger λ should correspond to a smaller energy scale and a smaller λ should correspond to a larger energy scale. In conjunction, taking the limit

$z \to \infty$ (or $r \to 0$) corresponds to flowing to the IR of the $d = 3$ theory, and taking the limit $z \to 0$ (or $r \to \infty$) corresponds to flowing to the UV of the $d = 3$ theory. For more discussion on the identification of the radial variable as the energy scale, see section 12.3 of [17].

It is important to note that in general, the supergravity solution to the kind of set-up discussed below is only asymptotically AdS_4, with the geometry in the bulk of the space-time being more complicated. In this case it is difficult to rigorously prove that the radial direction is to be identified with the field theory energy scale, however, it is nonetheless taken to be so.

Holographic superconductors

Here, we discuss the basic idea of holographic superconductors. For more detailed discussions of this topic, see [7, 8, 9].

In the IR, many condensed matter systems of interest become strongly coupled and therefore difficult to study using standard condensed matter techniques. The AdS/CFT correspondence provides a way to possibly obtain valuable information about such systems. Even though the AdS/CFT correspondence is only valid for a large number of gauge degrees of freedom N, it is nonetheless hoped that by working at very large N it wil be possible to obtain valuable information about strongly-coupled systems that is independent of N, or that it will be possible to at least gain some hints as to how to proceed for small N.

Using the AdS/CFT correspondence one can consider the dual gravity theory of the system of interest. In the dual gravity theory one can derive the equations of motion for the relevant fields from an action and solve them, at least numerically.

Since the energy scale of the condensed matter sytem is identified with the radial variable z in the dual gravity system, to obtain the IR stongly coupled behavior of the condensed matter system, one need only take the $z \to \infty$ limit of the gravity solution that was found, and then use the AdS/CFT correspondence to obtain the IR strongly coupled condensed matter system.

A holographic superconductor setup involves at least a U(1) gauge field and a complex scalar field that is charged with respect to it. Above a critical temperature, the scalar field has no expectation value, and is said to have 'no hair'. If the scalar field develops a non-zero expectation value, i.e. hair, below the critical temperature, then it comes to possess a definite phase, thus breaking the U(1) symmetry. In such a case the gravity solution describing this behavior is said to be a 'holographic superconductor'.

In order to have a non-zero temperature and finite chemical potential, an electrically charged black hole is placed at the center of the space-time. This black hole solution describes the unbroken phase of the superconductor.

The gravity theory at the near-horizon limit of the black hole solution corresponds to the IR of the field theory, and the $z \to 0$ or $r \to \infty$ limit of the black hole solution corresponds to the UV of the field theory. The space-time at the $z \to 0$ or $r \to \infty$ limit is AdS_4, and the gravity theory at this limit is dual to the UV of the field theory.

The goal is to obtain the behavior of the system as the temperature is decreased from above the critical temperature, where the superconductor is in an unbroken phase, to below it, where the superconductor is in a broken phase. One is especially interested in what happens at the $z \to \infty$ or $r \to 0$ limit. The gravity theory at this limit corresponds to the

IR of the field theory, which is difficult to study using standard condensed matter techniques.

The system is set up in such a way that for all temperatures the spacetime at the $z \to 0$ or $r \to \infty$ limit, which corresponds to the UV of the field theory, is AdS_4. The system is set up in this manner so that the AdS/CFT correspondence can be used.

Holographic superconductors from M-theory

In [8] the authors showed that many M-theory vacua corresponding to Freund-Rubin compactifications on seven-dimensional Sasaki-Einstein manifolds provide holographic gravity duals of $d = 3$ CFTs that exhibit superconductivity. Holographic superconductor solutions are given in [8] for the linearized equations of motion of $d = 11$ supergravity. Much information can be obtained from solutions to the linearized supergravity equations of motion, e.g., critical temperatures [8], but it is of course desireable to construct solutions for the full nonlinear equations of motion.

In order to construct a holographic superconductor solution, one needs at least a metric, a U(1) gauge field, and a charged scalar that can condense and break the U(1) symmetry. However, finding holographic superconductor solutions for the full supergravity equations of motion, involving at least these three fields, is in general a difficult task. Finding such solutions is in general difficult because of the many types of couplings that can occur between the few 'desired' fields and various other 'undesireable' fields that exist in the theory.

A way to avoid this difficulty is by working within a consistent truncation of the full theory. Working within a consistent truncation of the full theory guarantees that the many fields in the theory that are 'undesireable' can be consistently set to 0, without being sourced in the course of the evolution of the system.

Nonlinear $D = 11$ superfluid black brane solutions of [10] and [11]

Indeed, using the universal Sasaki-Einstein consistent truncation found in [31], the authors of [10] and [11] were able to construct nonlinear black brane solutions of $d = 11$ supergravity whose corresponding four-dimensional gravity theories are holographic superconductors. These black-brane solutions are particularly elegant because they apply universally for all Sasaki-Einstein manifolds.

A notable feature of the holographic superconductor solutions found in [10] and [11] is that the $T \to 0$ limit of these solutions are charged domain wall solutions that interpolate between the skew-whiffed Freund-Rubin vacuum in the UV and the Pope-Warner vacuum in the IR. Since the Pope-Warner solution is a compactification to AdS_4, it follows that this $T = 0$ domain wall solution corresponds to a $d = 3$ CFT that has emergent conformal symmetry in the far IR. However, in the case that the Pope-Warner vacuum is unstable for a particular Sasaki-Einstein manifold, the viability of the corresponding superconductor solution is put into question.

In such a case, it is reasonable to conclude that the Pope-Warner vacuum can not be used as a viable ground state for a CFT at $T = 0$. Indeed, one would expect that a quantum fluctuation of an unstable mode would grow exponentially and cause the system to flow to a stable vacuum. However, it is not clear what to conclude for the superconductor solutions at $T > 0$, whose $T \to 0$ limits are the Pope-Warner vacuum. Perhaps it is possible that thermal fluctuations could serve to stabilize the vacuum.

Instability of the Pope-Warner solution and its implications for holographic superconductors

At the time [11] was written it was unknown whether the Pope-Warner solution on any Sasak-Einstein manifold is unstable. It was later found in [32] that the Pope-Warner solution on S^7 is unstable, and more recently in [33] it was shown that, in fact, the Pope-Warner solution on any of the homogeneous Sasaki-Einstein spaces is unstable.

A consequence of the results of [32] and [33] on the program of constructing non-linear superfluid black brane solutions is clear: the Pope-Warner solution likely cannot be used as a viable $T = 0$ ground state if the compactifying manifold is taken to be a homogenous Sasaki-Einstein manifold or an orbifold of one that is discussed in [33].

In conjuction, the results of [32] and [33] indicate that in constructing superfluid black brane solutions, especially for $T \to 0$, one should also utilize consistent truncations other than the universal Sasaki-Einstein truncation, and perhaps focus on particular compactification manifolds or restricted classes of them.

Nonlinear $D = 11$ superfluid black brane solutions of [13]

Interestingly, in [13] it was pointed out that the critical temperatures that were obtained in [10] and [11] from using the universal Sasaki-Einstein truncation are not as high as those discussed in [8], and that, therefore, the superconductor solutions of [10] and [11] are not thermodynamically relevant. Motivated, at least in part, by this fact, and perhaps also by the instability results of [32], the authors of [13] used several different consistent truncations specific to S^7 to construct a variety of superfluid black brane solutions.

Among these solutions are ones that in the $T = 0$ limit are domain wall solutions that interpolate between the SO(8) AdS_4 fixed point in the UV and the $\mathrm{SU}(3) \times \mathrm{U}(1)$ AdS_4 fixed point in the IR. Unlike the $\mathrm{SU}(4)^-$ fixed point that uplifts to the Pope-Warner solution on

S^7, the SU(3) × U(1) fixed point in the IR is stable because it is supersymmetric.

1.2 Prior relevant results

Having discussed why the stability of the PW solution became a topic of interest, we now want to discuss prior results that were relevant in carrying out the research presented in this dissertation.

The first major result on the stability of the PW solution was given in [32]. In this paper, the authors showed that the PW solution on S^7 was in fact unstable. The authors showed that in the S^7 case the minimal sector of the SE truncation of [11, 31] coincides with the $\mathrm{SU}(4)^-$ sector of $\mathcal{N} = 8$, $d = 4$ gauged supergravity. Then, expanding the $\mathcal{N} = 8$ potential to quadratic order about the critical point corresponding to the PW point, the authors found that there are unstable scalars that transform in $\mathbf{20'}$ of $\mathrm{SU}(4)^-$.

The authors were able to uplift these unstable modes to eleven-dimensional supergravity, where the SU(4) symmetry becomes the isometry of $\mathbb{CP}^3$, which is the KE base of S^7. They give a metric perturbation and 3-form perturbations that yield the unstable scalars under reduction to four dimension. It is the discussion of the $D = 11$ picture in [32] that most pertains to the research presented in this dissertation. In particular, looking carefully at the structure of the metric perturbation given in [32] helped guide us toward a way to generalize the results of [32], as will be discussed in chapter 3.

Given the results of [32], it was natural to ask what happens for PW solutions in the case of SE manifolds other than S^7. It is in fact this question that is the main topic of the research presented here. A possible way to generalize the results of [32] was hinted at by the contents of the paper [42], on the instability of the Englert solution.

The Englert solution on the round S^7 was first shown to be unstable in [41]. The results of [41] were then generalized to Englert solutions on internal manifolds with two or more Killing spinors in [42]. The key idea of [42] was to construct metric and 3-form perturbations about the Englert background using two or more Killing spinors, and to see what the masses of the resulting scalars were after dimensionally reducing to AdS_4. Carrying out this procedure yielded scalar masses that violate the BF bound. Since the construction of [42] only utilizes two or more Killing spinors, its results apply universally to all SE manifolds.

In the case of the PW solution, we carried out an analogous construction using three or more Killing spinors, and we likewise found that the construction yielded scalars whose masses violate the BF bound. In this way we were able to generalize the results of [32] to all tri-Sasaki manifolds, which are SE manifolds with three or more Killing spinors (see Appendix B). This calculation is presented in chapter 2. It should be mentioned that this calculation was carried out before the release of [34], which contained the same results.

In [34] the authors carried out a universal consistent truncation of eleven-dimensional supergravity on tri-Sasakian manifolds. After presenting a solution to eleven-dimensional supergravity based on seven-dimensional tri-Sasakian structure, the authors dimensionally reduced the theory to four dimensions. The process of dimensional reduction yields a potential for the four-dimensional theory that has the PW solution as a critical point. Expanding the potential about the PW critical point to quadratic order, the authors found that a scalar contained in the truncation has a mass-value that violates the BF bound. As mentioned, we were able to obtain the same unstable scalar using a construction analogous to that in [42].

Having established that the PW solution on tri-Sasaki manifolds is unstable, one would like to know what the situation is for non-tri-Sasaki SE manifolds, i.e., for SE manifolds with exactly $\mathcal{N} = 2$ supersymmetry. A possible way to proceed is by looking at consistent truncations on SE manifolds, and expanding the corresponding potentials about the PW point to see whether there exist any unstable modes. Indeed, additional consistent truncations on $\mathcal{N} = 2$ SE manifolds that generalize the consistent truncation of [11, 31] were carried out in [35]. These truncations, however, do not yield unstable modes at the PW point.

Another possible way to proceed for the $\mathcal{N} = 2$ case was provided by the key observations we made that the metric perturbation that led to instability in the S^7 case had components only along the KE base, and furthermore, that it could be expressed in terms of a transverse, primitive (1,1)-form and a certain canonical SE object. These observations led us to focus our attention on transverse, primitive (1,1)-forms that are eigenfunctions of the Hodge-de Rham Laplacian. In particular, we used such objects together with canonical SE objects to construct metric and 3-form perturbations on SE manifolds. See chapter 4 for the details of our construction.

An analogous construction is found in [58]. In this paper the authors examined the stability of AdS_5 solutions of eleven-dimensional supergravity compactified on six-dimensional Kähler-Einstein (KE) spaces. The main result of the paper is that the solution suffers a bosonic instability if and only if there exists a transverse, primitive (1,1)-form that is an eigenfunction of the Hodge-deRham Laplacian with eigenvalue within a certain given range of values. In particular, such a (1,1)-form can be used to construct metric and 4-form perturbations that reduce to unstable AdS_5 scalars.

Even though [58] deals with compactification on six-dimensional KE spaces, it contains results that are applicable to the case of interest to us because regular SE manifolds can be seen as U(1)-bundles over KE bases, see e.g. Appendix B. In particular, [58] contains explicit (1,1)-forms that also exist on SE manifolds that we look at, and that are eigenfunctions of the Hodge-de Rham Laplacian with eigenvalues that lead to instability. See subsection 1.3 and chapter 5 for details.

1.3 Summary of main results

In this research we identify a potential source of perturbative instability of the PW solution on an arbitrary (regular) SE manifold. We show that starting with a basic, primitive, transverse (1,1)-form ω on M_7, which is an eigenform of the Hodge-de Rham Laplacian, Δ_2, with the eigenvalue $\lambda_\omega \geq 0$, one can construct explicitly one metric and two flux harmonics, which after diagonalization of the linearized equations of motion give rise to three modes in the scalar spectrum with the following masses:

(i) supersymmetric FR

$$m^2L^2 \ : \qquad \frac{\lambda_\omega}{4} - 2\,, \qquad \frac{\lambda_\omega}{4} + \sqrt{\lambda_\omega + 1} - 1\,, \qquad \frac{\lambda_\omega}{4} - \sqrt{\lambda_\omega + 1} - 1\,, \tag{1.8}$$

(ii) skew-whiffed FR

$$m^2L^2 \ : \qquad \frac{\lambda_\omega}{4} - 2\,, \qquad \frac{\lambda_\omega}{4} + 2\sqrt{\lambda_\omega + 1} + 2\,, \qquad \frac{\lambda_\omega}{4} - 2\sqrt{\lambda_\omega + 1} + 2\,, \tag{1.9}$$

(iii) PW

$$m^2L^2 \ : \qquad \frac{3}{8}\,\lambda_\omega\,, \qquad \frac{3}{8}\,\lambda_\omega + 3\sqrt{\lambda_\omega + 1} + 3\,, \qquad \frac{3}{8}\,\lambda_\omega - 3\sqrt{\lambda_\omega + 1} + 3\,. \tag{1.10}$$

M_7	λ_ω	m^2L^2	# of modes	KK spectra
S^7	24	-3	20	[43, 44]
$N^{1,1}$	24	-3	1	[45, 46, 47]
$M^{3,2}$	16	$9-3\sqrt{17}$	8	[48, 49, 50]
$Q^{1,1,1}$	16	$9-3\sqrt{17}$	9	[51]
$V^{5,2}$	32/3	$7-\sqrt{105}$	5	[52]

Table 1: Unstable modes for the PW solution on homogeneous SE manifolds.

For the first two solutions, all modes in (1.8) and (1.9) are stable with the lowest possible masses saturating the BF-bound (1.5) when $\lambda_\omega = 3$ and $\lambda_\omega = 15$, respectively. However, for the PW solution, the last mode in (1.10) becomes unstable when λ_ω lies in the range

$$2(9-4\sqrt{3}) < \lambda_\omega < 2(9+4\sqrt{3})\,. \tag{1.11}$$

In principle, all that remains then is to determine which SE manifolds admit such stability violating $(1,1)$-forms. Unfortunately, this appears to be a difficult problem since no general bounds on the low lying eigenvalues of Δ_2 on an arbitrary SE manifold are known.

In the absence of general results, we look at the homogeneous SE manifolds (1.3) for which the spectra of the Hodge-de Rham Laplacians, Δ_k, and of the Lichnerowicz operator, Δ_L, have been calculated in the references listed in Table 1, either as part of the Kaluza-Klein program in the 1980s,[2] or, more recently, to test the AdS/CFT correspondence for

[2]For a review, see, e.g., [44] and [49].

M_2-branes at conical singularities [53, 54, 55]. Specifically, the eigenvalues of the Hodge-de Rham Laplacian, Δ_2, can be read-off from the masses of Z-vector fields that arise from the Kaluza-Klein reduction of the three-form potential along two-form harmonics.

By examining the mass spectra of Z-vector fields, we conclude that on each homogeneous SE manifold there are two-forms with the eigenvalues of Δ_2 within the instability range (1.11). One must then determine whether any of those forms are basic, transverse and primitive. We found that, given the KK data for the two-form harmonics, which include the representation and the R-charge, it is actually the easiest to construct those forms explicitly and then verify that they indeed satisfy all the required properties. Our results are summarized in Table 1, which shows that there are unstable modes for the PW solution on all homogeneous SE manifolds.

The three harmonics for the scalar fields in (1.8)-(1.10) are related to the master $(1,1)$-form by operations (contractions and exterior products) that involve canonical objects of the SE geometry: the metric and the forms, which can be expressed in terms of Killing spinors on the SE manifold. From a general analysis of harmonics on coset spaces with Killing spinors [56], it is reasonable to expect that, at the supersymmetric solution, the three scalar fields and the Z-vector field should lie in the same $\mathcal{N}=2$ supermultiplet. Indeed, the pattern of masses in (1.8) and the presence of the Z-vector field with the correct mass, and their R-charges, suggest that it is a long Z-vector supermuliplet [57]. Ultimately, this observation explains why we can diagonalize the mass operator for fluctuations around the PW solution on such a small set of modes – the mixing due to the background flux involves only harmonics within a single supermultiplet. It also suggests where to look for an instability of the PW solution on a general SE manifold.

A regular SE manifold, M_7, is a U(1) fibration over its KE base, B_6, so any (1,1)-form, ω, as above is a pull-back of a transverse, primitive (1,1)-form on B_6 with the same eigenvalue of the corresponding Hodge-de Rham Laplacian, $\Delta_{(1,1)}$. This shows that the potential instability of the PW solution that we have identified resides in the spectrum of $\Delta_{(1,1)}$ on KE manifolds. It also provides a link to a different class of solutions whose stability has been analyzed recently. As discussed in the previous section, precisely the same type (1,1)-forms, albeit with a different "window of instability," were shown in [58] to destabilize the $AdS_5 \times B_6$ solutions [59, 60] of eleven-dimensional supergravity.

Two results in [58] are directly applicable to our analysis. The first one is an explicit construction of a $(1,1)$-form ω, with $\lambda_\omega = 16$, on $S^2 \times S^2 \times S^2$, which is the KE base for $Q^{1,1,1}$. The second one is more general and concerns the spectrum of $\Delta_{(1,1)}$ on a product of two Kähler manifolds, $B_6 = B_2 \times B_4$. It is shown that if B_4 admits a continous symmetry, then there exists a transverse, primitive $(1,1)$-form ω on B_6 with the eigenvalue $\lambda_\omega = 16$. In particular, the unstable modes on $M^{3,2}$, which is a U(1) fibration over $S^2 \times \mathbb{CP}^2$, arise in this way. Another KE manifold that is covered by this construction is $S^2 \times dP_3$, where dP_3 is the del Pezzo surface. This gives us an example of an inhomogeneous SE manifold with an unstable PW solution.

The rest of the dissertation is organized as follows. In chapter 2 we present the calculation analogous to what was done in [42] showing that the PW solution on tri-Sasakian manifolds is unstable. In chapter 3, we review the FR and PW solutions together with some pertinent SE geometry. Even though the PW solution is given in chapter 2, we present it again in chapter 3, because the conventions in chapter 2 are different than they are in the rest of this dissertation. We then in chapter 4 present the details of our calculation leading to the mass formulae (1.8)-(1.10). In chapter 5 we construct explicitly the unstable modes for all homogeneous examples. We conclude with some comments in chapter 6. Our conventions and some useful identities are summarized in appendices.

2 Page-Pope-like construction

2.1 Introduction

In [42] Killing spinors on S^7 were used to construct linearized modes about the Englert solution of the bosonic field equations of $d = 11$ supergravity. In the Englert solution, the 4-form flux has two parts. One part is taken to be the volume form of AdS_4, and the other part is an internal flux that has components only along the compact S^7 directions. The internal 4-form flux is constructed as a spinor bilinear with four legs, using one of the eight Killing spinors on the round S^7. This single spinor is invariant under an SO(7) subgroup of SO(8), the symmetry group of the round S^7, while the other seven spinors transform as the **7** of this SO(7). Since the internal 4-form is constructed from an SO(7)-invariant spinor, it itself is an SO(7)-invariant form. Therefore, since the round S^7 is also SO(7)-invariant, the Englert solution is SO(7)-invariant.

The S^7 in the Englert solution is the round S^7, which has SO(8) symmetry. Therefore, all the spinors on the S^7 are Killing spinors, satisfying $\nabla_a \eta = \alpha \Gamma_a \eta$, where α is a constant. Perturbations of the metric and 4-form are constructed from the 8 Killing spinors and $d = 7$

gamma matrices. Since Killing spinors are used, the $d = 11$ linearized field equations reduce to a much simpler set of differential equations on AdS_4, from which the masses of AdS_4 scalars can be easily obtained.

Here, we carry out an analogous procedure on the Pope-Warner solution on the stretched S^7 [20]. The Pope-Warner solution has SU(4) (SO(6)) invariance. Like the Englert solution it has an internal 4-form flux, however in the Pope-Warner case it is constructed from two, rather than one Killing spinor. These two spinors are singlets of the SU(4) invariance group, while the other six spinors transform as the **6** of this SU(4). Analogously to the Englert case, perturbations to the metric and 4-form flux are constructed from the eight spinors and $d = 7$ gamma matrices, and the $d = 11$ linearized field equations are reduced to a simple set of differential equations on AdS_4, from which masses of AdS_4 scalars are easily obtained.

However, the procedure in the Pope-Warner case is complicated by the fact that spinors on the stretched S^7 do not satisfy the equation $\nabla_a \eta = \alpha \Gamma_a \eta$. We are able to deal with this complication by rearranging the spinor covariant derivative in a convenient way. This rearrangement of the spinor covariant derivative is carried out in section 2.2.

In section 2.3 the $d = 11$ bosonic field equations and their linearization are given. In section 2.4 we discuss spinors and spinor bilinears. In section 2.5 we obtain the Pope-Warner background solution. In section 2.6 we present the perturbation ansatz. In section 2.7 we plug the Pope-Warner solution and our perturbation ansatz into the linearized field equations and obtain simple differential equations in AdS_4 for scalar fields. In section 2.8 we diagonalize the AdS_4 equations to obtain the masses of the scalar fields. Finally, in section 2.9 we compare our results to known results.

It is important to mention that the conventions used in this chapter are different from those used in the rest of this work. The Dirac matrices are taken to be real and antisymmetric, satisfying

$$\{\Gamma_a, \Gamma_b\} = -2\delta_{ab}.$$

Also, in the metric for the $d = 11$ compactified solution, the size of the KE base is held fixed in going from the FR to the PW points. In the rest of this work, the size of the KE base varies, whereas the AdS radius squared is taken to be L^2.

2.2 Spinor covariant derivative

The spinor covariant derivative is [44]

$$\nabla_m \eta = (\partial_m - \frac{1}{4}\omega_m{}^{bc}\Gamma_{bc})\eta. \tag{2.1}$$

The index 'm' is for the curved coordinates, and the $\omega_m{}^{ab}$ are the spin connections.

The spinor covariant derivative can be expressed in terms of the frame coordinates simply by contracting with the inverse frame $e^m{}_a$. So

$$\nabla_a \eta = (\partial_a - \frac{1}{4}\omega_a{}^{bc}\Gamma_{bc})\eta, \tag{2.2}$$

where

$$\partial_a = e^m{}_a \partial_m \tag{2.3}$$

$$\omega_a{}^{bc} = e^m{}_a \omega_m{}^{bc}. \tag{2.4}$$

We are interested in S^7 as a U(1) fibration over the KE space $\mathbb{CP}^3$. In this case the metric can be written as [44]

$$ds^2 = d\bar{s}^2 + c^2(d\tau - A)^2, \tag{2.5}$$

where $d\bar{s}^2$ is the metric on $\mathbb{CP}^3$, and A is a 1-form potential on $\mathbb{CP}^3$ that gives rise to the complex structure J. In the case that $c = 1$, the sphere is round. Otherwise the sphere is said to be 'stretched'.

For frames we take

$$e^i = \bar{e}^i, \;\; i = 1, \ldots, 6 \tag{2.6}$$

$$e^7 = c(d\tau - A) \tag{2.7}$$

$$= c\breve{e}^7, \tag{2.8}$$

where the $\bar{e}^i$ are frames on the $\mathbb{C}P^3$, and $\breve{e}^7$ is the frame for the fiber in the case that the sphere is round. With this choice of frames, it is found that the spin connections are given by

$$\omega^{ij} = \bar{\omega}^{ij} + cJ^{ij}e^7 \tag{2.9}$$

$$\omega^{7i} = -cJ^i{}_j e^j, \tag{2.10}$$

where $J_{ij} = (dA)_{ij}$ is the complex structure on $\mathbb{C}P^3$.

Rearranging the spinor covariant derivative for the stretched sphere

In the case that the sphere is stretched, i.e. $c \neq 1$, we would like to rearrange the spinor covariant derivative in such a way that the contribution made to it from stretching is manifest.

First, we express the inverse frames on the stretched sphere in terms of those of the round sphere. To do so we write

$$e^m{}_a e_m{}^b = \delta^b_i e^m{}_a e_m{}^i + c\delta^b_7 e^m{}_a \breve{e}_m{}^7 \tag{2.11}$$

$$= \delta^b_a. \tag{2.12}$$

From the above equation, it must be that

$$e^m{}_i = \breve{e}^m{}_i \tag{2.13}$$

$$e^m{}_7 = \frac{1}{c}\breve{e}^m{}_7, \tag{2.14}$$

where $\breve{e}^m{}_a$ are the inverse frames on the round sphere.

Next, we express the partial derivatives on the stretched sphere in terms of those on the round sphere. To do so we write

$$\partial_a = e^m{}_a\partial_m \tag{2.15}$$

$$= \delta^i_a e^m{}_i\partial_m + \delta^7_a e^m{}_7\partial_m \tag{2.16}$$

$$= \delta^i_a \breve{e}^m{}_i\partial_m + \frac{1}{c}\delta^7_a \breve{e}^m{}_7\partial_m. \tag{2.17}$$

From the above equation we see that

$$\partial_i = \breve{\partial}_i \tag{2.18}$$

$$\partial_7 = \frac{1}{c}\breve{\partial}_7, \tag{2.19}$$

where $\breve{\partial}_a$ is the partial derivative for the round sphere, i.e. for $c = 1$.

Now, we express the spin connections on the stretched sphere in terms of those on the round sphere. The spin connections are

$$\omega^{ij} = \bar{\omega}^{ij} + cJ^{ij}e^7 \tag{2.20}$$

$$\omega^{7i} = -cJ^i{}_je^j. \tag{2.21}$$

From these we see that

$$\omega_k{}^{ij} = \bar{\omega}_k{}^{ij} \tag{2.22}$$

$$\omega_k{}^{7j} = -cJ^j{}_k \tag{2.23}$$

$$\omega_7{}^{ij} = cJ^{ij}. \tag{2.24}$$

So along the $\mathbb{CP}^3$ we have

$$\begin{aligned}
\frac{1}{4}\omega_k{}^{bc}\Gamma_{bc} &= \frac{1}{4}\bar{\omega}_k{}^{ij}\Gamma_{ij} + \frac{1}{2}\omega_k{}^{7j}\Gamma_{7j} && (2.25)\\
&= \frac{1}{4}\bar{\omega}_k{}^{ij}\Gamma_{ij} - \frac{1}{2}cJ^j{}_k\Gamma_{7j} && (2.26)\\
&= \frac{1}{4}\bar{\omega}_k{}^{ij}\Gamma_{ij} - \frac{1}{2}J^j{}_k\Gamma_{7j} + \frac{1}{2}(1-c)J^j{}_k\Gamma_{7j} && (2.27)\\
&= \frac{1}{4}\breve{\omega}_k{}^{bc}\Gamma_{bc} + \frac{1}{2}(1-c)J^j{}_k\Gamma_{7j}, && (2.28)
\end{aligned}$$

giving

$$\frac{1}{4}\omega_k{}^{bc}\Gamma_{bc} = \frac{1}{4}\breve{\omega}_k{}^{bc}\Gamma_{bc} + \frac{1}{2}(1-c)J^j{}_k\Gamma_{7j}. \tag{2.29}$$

Along the fiber we have

$$\begin{aligned}
\frac{1}{4}\omega_7{}^{bc}\Gamma_{bc} &= \frac{1}{4}cJ^{ij}\Gamma_{ij} && (2.30)\\
&= \frac{1}{4}J^{ij}\Gamma_{ij} + \frac{1}{4}(c-1)J^{ij}\Gamma_{ij} && (2.31)\\
&= \frac{1}{4}\breve{\omega}_7{}^{bc}\Gamma_{bc} + \frac{1}{4}(c-1)J^{ij}\Gamma_{ij}, && (2.32)
\end{aligned}$$

giving

$$\frac{1}{4}\omega_7{}^{bc}\Gamma_{bc} = \frac{1}{4}\breve{\omega}_7{}^{bc}\Gamma_{bc} + \frac{1}{4}(c-1)J^{ij}\Gamma_{ij}, \tag{2.33}$$

where $\breve{\omega}^{bc}$ are the spin connections on the round sphere.

The rearranged spinor covariant derivative

Finally, putting together equations (2), (18), and (29) we have along the $\mathbb{CP}^3$

$$\begin{aligned} \nabla_k &= \partial_k - \frac{1}{4}\omega_k{}^{bc}\Gamma_{bc} && (2.34) \\ &= \breve{\nabla}_k + \frac{1}{2}(c-1)J^j{}_k\Gamma_{7j}, && (2.35) \end{aligned}$$

giving

$$\nabla_k = \breve{\nabla}_k + \frac{1}{2}(c-1)J^j{}_k\Gamma_{7j}, \tag{2.36}$$

where $\breve{\nabla}_a$ is the spinor covariant derivative for the round sphere. And putting together equations (2), (19), and (33) we have along the fiber

$$\begin{aligned} \nabla_7 &= \partial_7 - \frac{1}{4}\omega_7{}^{bc}\Gamma_{bc} && (2.37) \\ &= \frac{1}{c}\breve{\partial}_7 - \frac{1}{4}\breve{\omega}_7{}^{bc}\Gamma_{bc} - \frac{1}{4}(c-1)J^{ij}\Gamma_{ij} && (2.38) \\ &= \frac{1}{c}\breve{\nabla}_7 - \frac{1}{4}(1-\frac{1}{c})\breve{\omega}_7{}^{ij}\Gamma_{ij} - \frac{1}{4}(c-1)J^{ij}\Gamma_{ij} && (2.39) \\ &= \frac{1}{c}\breve{\nabla}_7 - \frac{1}{4}\left(\frac{c^2-1}{c}\right)J^{ij}\Gamma_{ij}, && (2.40) \end{aligned}$$

giving

$$\nabla_7 = \frac{1}{c}\breve{\nabla}_7 - \frac{1}{4}\left(\frac{c^2-1}{c}\right)J^{ij}\Gamma_{ij}. \tag{2.41}$$

In summary we have

$$\nabla_a = \lambda_a\breve{\nabla}_a + E_a, \tag{2.42}$$

where

$$\begin{aligned} \lambda_i &= 1 && (2.43) \\ \lambda_7 &= \frac{1}{c}, && (2.44) \end{aligned}$$

$$E_i = \beta J^j{}_i \Gamma_{7j} \tag{2.45}$$

$$E_7 = \mu J^{jk} \Gamma_{jk}, \tag{2.46}$$

and

$$\beta = \frac{1}{2}(c-1) \tag{2.47}$$

$$\mu = -\frac{1}{4}\left(\frac{c^2-1}{c}\right). \tag{2.48}$$

So if η is a Killing spinor on the S^7, then

$$\breve{\nabla}_a \eta = \alpha \Gamma_a \eta, \tag{2.49}$$

where $\alpha = \frac{1}{2}$ for unit radius.

2.3 The $D = 11$ field equations and their linearization

The bosonic sector of $d = 11$ supergravity consists of a metric g_{AB} and a 3-form potential A_{ABC}. The exterior derivative of the 3-form potential gives a 4-form flux F_{ABCD}. Classically these fields must satisfy the $d = 11$ supergravity bosonic field equations. These field equations consist of an Einstein equation, a Maxwell equation, and the Bianchi identity for F_{ABCD}.

The Einstein equation is

$$R_{AB} = \frac{1}{3} F_{ACDE} F_B{}^{CDE} - \frac{1}{36} g_{AB} F_{CDEF} F^{CDEF}, \tag{2.50}$$

the Maxwell equation is

$$\nabla_A F^{ABCD} = -\frac{1}{576} \epsilon^{BCDEFGHIJKL} F_{EFGH} F_{IJKL}, \tag{2.51}$$

and the Bianchi identity is

$$\nabla_{[A}F_{BCDE]} = 0. \tag{2.52}$$

We would like to perturb the fields g_{AB} and F_{ABCD}, in such a way that the perturbed fields still satisfy the equations of motion. Let h_{AB} and f_{MNPQ} be the perturbations to the metric and flux, respectively. The perturbed fields are then

$$\mathsf{g}_{AB} = g_{AB} + h_{AB} \tag{2.53}$$

$$\mathcal{F}_{ABCD} = F_{ABCD} + f_{MNPQ}. \tag{2.54}$$

We would like to put these perturbed fields into the equations of motion and determine the equations the perturbations h_{AB} and f_{ABCD} must satisfy to first order in order for g_{AB} and $\mathcal{F}_{ABCD}$ to be solutions. The equations h_{AB} and f_{ABCD} must satisfy to first order are the 'linearized field equations'.

The d=11 linearized field equations

The linearized bosonic field equations of $d = 11$ supergravity are derived in Appendix F. The linearized Einstein equation is

$$\begin{aligned}\frac{1}{2}\hat{\Delta}h_{AB} + \breve{\nabla}_{(A}\breve{\nabla}^{C}h_{B)C} - \frac{1}{2}\breve{\nabla}_{A}\breve{\nabla}_{B}h^{C}_{C} &= -F_{A}{}^{CNP}F_{B}{}^{M}{}_{NP}h_{MC} - \frac{1}{36}h_{AB}F_{CDEF}F^{CDEF} \\ &\quad + \frac{1}{9}g_{AB}h^{CM}F_{CDEF}F_{M}^{DEF} + \frac{2}{3}F_{(A}{}^{MNP}f_{B)MNP} \\ &\quad - \frac{1}{18}g_{AB}F^{MNPQ}f_{MNPQ},\end{aligned} \tag{2.55}$$

the linearized Maxwell equation is

$$\begin{aligned}\breve{\nabla}_{A}f^{ABCD} + 4\breve{\nabla}_{A}(F^{M[ABC}h^{D]}{}_{M}) - \frac{1}{2}F^{BCDR}\breve{\nabla}_{R}h_{A}{}^{A} &= -\frac{1}{288}\epsilon^{BCDEFGHIJKL}F_{EFGH}f_{IJKL} \\ &\quad - \frac{1}{1152}Tr(g^{-1}h)\epsilon^{BCDEFGHIJKL}F_{EFGH}F_{IJKL},\end{aligned} \tag{2.56}$$

and the linearized Bianchi identity is

$$\nabla_{[A}f_{BCDE]} = 0. \tag{2.57}$$

We want to plug the Pope-Warner background solution obtained in section 2.5 and the perturbation ansatz given in section 2.6 into the linearized field equations and obtain field equations for scalars in AdS_4.

2.4 Spinor bilinears

The round S^7, i.e. that with $c = 1$ in the metric (5), has symmetry group SO(8). When the sphere is stretched, so that $c \neq 1$, the symmetry group SO(8) is broken to the symmetry group of the $\mathbb{CP}^3$, which is SU(4). The group SO(8) has two 8-dimensional spinor irreps. Under the subgroup SU(4) that is the symmetry group of the $\mathbb{CP}^3$ one of these spinor irreps breaks as [80]

$$\mathbf{8} \to \mathbf{6} + \mathbf{1} + \mathbf{1}. \tag{2.58}$$

Let the singlet spinors be denoted by ζ and ψ, and the spinors that transform in the $\mathbf{6}$ be denoted by η^i, $i = 1, \ldots, 6$. Since ζ and ψ are invariant under SU(4), spinor bilinear forms constructed from these spinors are invariant under SU(4).

Due to the antisymmetry of the gamma matrices in 7 dimensions, there is only one spinor bilinear 1-form and only one spinor bilinear 2-form that can be constructed from ζ and ψ, namely $\bar{\zeta}\Gamma_a\psi$ and $\bar{\zeta}\Gamma_{ab}\psi$, respectively. In fact the former is $-e^7$, where e^7 is the seventh frame in section (I), and the latter is $-J$, where J is the complex structure.

The 3-forms $\bar{\zeta}\Gamma_{abc}\zeta$ and $\bar{\psi}\Gamma_{abc}\psi$ are used in the construction of the Pope-Warner background solution in the next section. Their exterior derivatives $\sim \bar{\zeta}\Gamma_{abcd}\zeta$ and $\sim \bar{\psi}\Gamma_{abcd}\psi$ along with the volume form of AdS_4 make up the background 4-form flux.

Spinor bilinear forms can also be constructed from the spinors η^i that transform in the **6**. These forms are clearly not SU(4)-invariant. It is convenient to define

$$K^{ij}_{abcd} = \bar{\eta}^i\Gamma_{abcd}\eta^j,\ i,j = 1,\ldots,6. \tag{2.59}$$

By antisymmetry of the gamma matrices, the K^{ij} are symmetric under interchange of i and j.

It is also convenient to define the 4-forms

$$K^a_{abcd} = \bar{\zeta}\Gamma_{abcd}\zeta + \bar{\psi}\Gamma_{abcd}\psi \tag{2.60}$$

$$K^{ij}_{7,abcd} = 4\bar{\zeta}\Gamma_{[a}\psi K^{ij}_{bcd]7}. \tag{2.61}$$

The latter is simply K^{ij}_{abcd} with all component set equal to 0, except for those components that have a direction along the fiber, i.e. along the '7' direction.

The covariant derivatives of the various spinor bilinears are needed. They are obtained using the Leibnitz rule with the spinor covariant derivative.

2.5 Pope-Warner background solution

The Pope-Warner solution is a compactification solution of the $d = 11$ supergravity bosonic field equations with $SU(4)$-invariance [20]. The metric part of it is a product of the AdS_4 metric and of the stretched S^7 metric:

$$ds^2 = l^2 ds^2(AdS_4) + ds^2(S^7), \tag{2.62}$$

where $ds^2(AdS_4)$ is the metric for unit radius AdS_4 and $ds^2(S^7)$ is the metric (5) with the size of the $\mathbb{CP}^3$ set so that $R(\mathbb{CP}^3)_{ij} = 8\delta_{ij}$.

The background 4-form fluxes are taken to be

$$F_{\mu\nu\rho\sigma} = -2m\epsilon_{\mu\nu\rho\sigma} \tag{2.63}$$

$$F_{abcd} = s\left(\bar{\zeta}\Gamma_{abcd}\zeta - \bar{\psi}\Gamma_{abcd}\psi\right), \tag{2.64}$$

where Greek indices are used to label coordinates of AdS_4 and Latin indices are used to label coordinates of S^7. All other components of the 4-form flux are 0, i.e. there is no mixing of the AdS_4 and S^7 components.

The Einstein equation is

$$R_{AB} = \frac{1}{3}F_{ACDE}F_B{}^{CDE} - \frac{1}{36}g_{AB}F_{CDEF}F^{CDEF}. \tag{2.65}$$

The Riemann tensor for the stretched S^7 is given in [3]. It is

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + (1-c^2)(J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl}) \tag{2.66}$$

$$R_{7i7j} = c^2 J_i{}^k J_{jk} = c^2\delta_{ij}. \tag{2.67}$$

Contracting the Riemann tensor gives the Ricci tensor, which can then be input into the Einstein equation. It is

$$R_{ij} = (8-2c^2)\bar{\delta}_{ij} \tag{2.68}$$

$$R_{77} = 6c^2. \tag{2.69}$$

Putting the above metric and fluxes into the Einstein equation gives the equations

$$\frac{3}{l^2} = \frac{8}{3}(2m^2 + 4s^2) \tag{2.70}$$

$$8 - 2c^2 = \frac{4}{3}(2m^2 + 4s^2) \tag{2.71}$$

$$6c^2 = \frac{8}{3}(m^2 + 8s^2), \tag{2.72}$$

where l is the AdS radius.

The Maxwell equation is

$$\nabla_a F^{abcd} = -\frac{1}{6} m \epsilon^{bcdefgh} F_{efgh}. \tag{2.73}$$

Putting the flux (32) into the Maxwell equation gives, using the spinor covariant derivative,

$$sB\left(\bar{\zeta}\Gamma_{bcd}\zeta - \bar{\psi}\Gamma_{bcd}\psi\right) = 2ms\left(\bar{\zeta}\Gamma_{bcd}\zeta - \bar{\psi}\Gamma_{bcd}\psi\right), \tag{2.74}$$

where

$$B = \alpha(\lambda_7 + 3) + 3(\beta + 2\mu). \tag{2.75}$$

So the Maxwell equation gives

$$B = 2m. \tag{2.76}$$

Solving the Einstein and Maxwell equations gives:

$$c = \sqrt{2} \tag{2.77}$$

$$m = \frac{1}{\sqrt{2}} \tag{2.78}$$

$$s = \pm\frac{1}{\sqrt{2}} \tag{2.79}$$

$$l = \sqrt{\frac{3}{8}}. \tag{2.80}$$

2.6 Perturbation ansatz

We want to perturb the Pope-Warner background solution obtained in the previous section. We use the following perturbation ansatz:

$$h_{ab} = 2G^{ij}(x)W^{ij}_{ab} \tag{2.81}$$

$$a_{abc} = \chi^{ij}(x)X^{ij}_{abc} + \xi(x)Y_{abc} + \omega^{ij}(x)Z^{ij}_{abc}, \tag{2.82}$$

where

$$W^{ij}_{ab} = \bar{\zeta}\Gamma_m\eta^{(i}\bar{\zeta}\Gamma_n\eta^{j)} - \bar{\psi}\Gamma_m\eta^{(i}\bar{\psi}\Gamma_n\eta^{j)} \tag{2.83}$$

$$X^{ij}_{abc} = 3\left(\bar{\zeta}\Gamma_{[m}\eta^{(i}\bar{\zeta}\Gamma_{np]}\eta^{j)} + \bar{\psi}\Gamma_{[m}\eta^{(i}\bar{\psi}\Gamma_{np]}\eta^{j)}\right) \tag{2.84}$$

$$Y_{abc} = 3\bar{\zeta}\Gamma_{[m}\psi\bar{\zeta}\Gamma_{np]}\psi \tag{2.85}$$

$$Z^{ij}_{abc} = 6\left(\bar{\zeta}\Gamma_{[m}\eta^{(i}\bar{\psi}\Gamma_n\eta^{j)}\bar{\zeta}\Gamma_{p]}\psi\right). \tag{2.86}$$

Taking the exterior derivative of a gives

$$\begin{aligned} f_{abcd} &= (da)_{abcd} \\ &= \chi^{ij}(x)(dX^{ij})_{abcd} + \xi(x)(dY)_{abcd} + \omega^{ij}(x)(dZ^{ij})_{abcd} \end{aligned} \tag{2.87}$$

$$\begin{aligned} f_{\alpha bcd} &= \nabla_\alpha a_{bcd} \\ &= \nabla_\alpha\chi^{ij}(x)X^{ij}_{abc} + \nabla_\alpha\xi(x)Y_{abc} + \nabla_\alpha\omega^{ij}(x)Z^{ij}_{abc}, \end{aligned} \tag{2.88}$$

where

$$dX^{ij} = -4(\beta - 2\alpha)K^{ij} + 2\left[\alpha(\lambda - 1) + \beta - 2\mu\right]K^{ij}_7 + 2(2\alpha + \beta)\delta^{ij}K^a \tag{2.89}$$

$$dY = 4(\alpha + \beta)K^a \tag{2.90}$$

$$dZ^{ij} = 2(\alpha + \beta)K^{ij} - 2(3\alpha + \beta)K^{ij}_7 - (\alpha + \beta)\delta^{ij}K^a. \tag{2.91}$$

We want to plug this perturbation ansatz, together with the Pope-Warner background solution, into the linearized field equations given in section 2.3. To begin with, the linearized field equations are unwieldy and contain many terms. However, due to the following facts, which are straightforward to verify, they simplify considerably.

1. The metric fluctuation ansatz h_{ab} is traceless, i.e. $h^a{}_a = 0$.

2. $F^{abcd} f_{abcd} = 0$.

3. $F_n{}^{abc} a_{abc} = 0$.

4. $\left(\bar{\zeta}\Gamma^{efg}\zeta - \bar{\psi}\Gamma^{efg}\psi\right) a_{efg} = 0$.

5. $\nabla_m f^{mnp\sigma} = 0$. This is because the divergence operator is $-\star d \star$, and $d^2 = 0$.

6. $\nabla_n h^n_{\ a} = 0$. h_{ab} is transverse.

7. $F_a{}^{cnp} F_b{}^m{}_{np} h_{mc} = 0$.

8. $F_{cdef} F_m{}^{def} h^{cm} = 0$. This follows from 7 by contracting the indices a and b.

2.7 AdS equations of motion

Using the perturbation ansatz and the facts given above, the linearized $d = 11$ field equations reduce to

$$\frac{1}{2}\hat{\Delta} h_{mn} = \frac{2}{3} F_{(m}{}^{abc} f_{n)abc} - \frac{1}{36}(384 s^2 - 96 m^2) h_{mn} \tag{2.92}$$

and

$$\nabla_\mu f^{\mu npq} + \nabla_m f^{mnpq} + 4\nabla_m (F^{a[mnp} h^{q]}{}_a) = -\frac{1}{6} m \epsilon^{abcdnpq} f_{abcd}. \tag{2.93}$$

These equations come from the Einstein and Maxwell field equations, respectively. The linearized Bianchi identity is trivially satisfied by the fact that $f = da$ in the perturbation ansatz, so it yields no new information.

We want to further simplify the linearized field equations so that all dependence on the internal 7-dimensional coordinates disappers and we are left with equations that are only on the AdS space.

Einstein equation

The Einstein equation is

$$\frac{1}{2}\hat{\Delta}h_{mn} = \frac{2}{3}F_{(m}{}^{abc}f_{n)abc} - \frac{1}{36}(384s^2 - 96m^2)h_{mn}. \tag{2.94}$$

$\hat{\Delta}$ is the Lichnerowitz operator and is defined as [44]

$$\hat{\Delta}h_{mn} = -\Box h_{mn} - 2R_{mpnq}h^{pq} + 2R_{(m}{}^{p}h_{n)p}, \tag{2.95}$$

where

$$\Box h_{mn} = \Box_4 h_{mn} + \Box_7 h_{mn}, \tag{2.96}$$

and

$$\Box_7 = \nabla_a \nabla^a. \tag{2.97}$$

The various terms in $\hat{\Delta}h_{mn}$ are found to be

$$\begin{aligned} \Box_7 h_{mn} &= -8[\alpha^2(2\lambda^2+5) + 2\alpha(\beta+4\lambda\mu) + \beta^2 + 8\mu^2]h_{mn} && (2.98)\\ R_{mpnq}h^{pq} &= (3c^2-4)h_{mn} && (2.99)\\ R_{(m}{}^{p}h_{n)p} &= -2(c^2-4)h_{mn}, && (2.100) \end{aligned}$$

giving

$$\hat{\Delta}h_{mn} = -\Box_4 h_{mn} + l\,h_{mn}, \tag{2.101}$$

where

$$l = 8[\alpha^2(2\lambda^2 + 5) + 2\alpha(\beta + 4\lambda\mu) + \beta^2 + 8\mu^2] - 10c^2 + 24. \tag{2.102}$$

The other term in the Einstein equation is found to be

$$F_{(m}{}^{abc} f_{n)abc} = -48s \left[(\alpha(\lambda + 3) - \beta - 2\mu)\chi^{ij} - 2\alpha\omega^{ij} \right] W^{ij}_{mn}. \tag{2.103}$$

Now the Einstein equation is

$$\Box_4 h_{mn} = \kappa_G h_{mn} + 64s \left[(\alpha(\lambda + 3) - \beta - 2\mu)\chi^{ij} - 2\alpha\omega^{ij} \right] W^{ij}_{mn}, \tag{2.104}$$

giving

$$\Box_4 G^{ij} = \kappa_G G^{ij} + 32s \left[(\alpha(\lambda + 3) - \beta - 2\mu)\chi^{ij} - 2\alpha\omega^{ij} \right], \tag{2.105}$$

where

$$\kappa_G = l + \frac{1}{18}(384s^2 - 96m^2). \tag{2.106}$$

So the linearized Einstein equation is

$$\Box_4 G^{ij} = M_{11} G^{ij} + M_{12}\chi^{ij} + M_{14}\omega^{ij}, \tag{2.107}$$

where the coefficients M_{1j} are constants that will be part of a 4×4 matrix $\mathbf{M}$ called the 'mass matrix'. The rest of the elements of $\mathbf{M}$ will come from the linearized Maxwell equation.

Maxwell equation

The Maxwell equation is

$$\nabla_\mu f^{\mu npq} + \nabla_m f^{mnpq} + 4\nabla_m (F^{a[mnp} h^{q]}{}_a) = -\frac{1}{6} m \epsilon^{abcdnpq} f_{abcd}. \tag{2.108}$$

Expanded, the first term on the left hand side of the Maxwell equation is

$$\nabla_\mu f^\mu{}_{npq} = (\Box_4 \chi^{ij}) X^{ij}_{npq} + (\Box_4 \xi) Y_{npq} + (\Box_4 \omega^{ij}) Z^{ij}_{npq}, \tag{2.109}$$

and the second term on the left hand side is

$$\nabla_m f^m{}_{npq} = -(\star d \star f)_{npq} \tag{2.110}$$

$$= g_1^{ij} X_{npq}^{ij} + g_2 Y_{npq} + g_3^{ij} Z_{npq}^{ij}, \tag{2.111}$$

where

$$g_1^{ij} = -4\left[\alpha^2(\lambda(\lambda+2)+13) - 2\alpha\beta(\lambda+5) - 4\alpha(\lambda+1)\mu + (\beta+2\mu)^2\right]\chi^{ij}$$

$$+8\alpha(\alpha(\lambda-3) - 3\beta - 2\mu)\omega^{ij} \tag{2.112}$$

$$g_2 = -16\left(\alpha^2(-\lambda) + \alpha^2 + 7\alpha\beta + 2\alpha\mu + 2\beta^2\right)\mathrm{Tr}\,\chi - 64(\alpha+\beta)^2\xi$$

$$+16\left(-\alpha^2 + 2\alpha\beta + \beta^2\right)\mathrm{Tr}\,\omega \tag{2.113}$$

$$g_3^{ij} = 8(\alpha(\lambda-1) + \beta - 2\mu)(-\alpha(\lambda-3) + 3\beta + 2\mu)\chi^{ij}$$

$$-16\left(-\alpha^2(\lambda-10) + \alpha(5\beta+2\mu) + \beta^2\right)\omega^{ij}. \tag{2.114}$$

In g_2, $\mathrm{Tr}\,\chi = \sum_{\mathrm{i}=1}^{6} \chi^{\mathrm{ii}}$ and $\mathrm{Tr}\,\omega = \sum_{\mathrm{i}=1}^{6} \omega^{\mathrm{ii}}$.

One finds that in the third term on the left hand side of the Maxwell equation

$$4F_{a[mnp}h_{q]}{}^a = 4sG^{ij}K_{7,mnpq}^{ij}. \tag{2.115}$$

So,

$$4\nabla^m(F_{a[mnp}h_{q]}{}^a) = 4sG^{ij}(-\star d \star K_7^{ij})_{npq} \tag{2.116}$$

$$= (\kappa_1^{ij} X_{npq}^{ij} + \kappa_2 Y_{npq} + \kappa_3^{ij} Z_{npq}^{ij}), \tag{2.117}$$

$$\tag{2.118}$$

where

$$\kappa_1^{ij} = -8s(\alpha(\lambda-1)-\beta-2\mu)G^{ij}, \tag{2.119}$$

$$\kappa_2 = 32s\alpha \mathrm{Tr}\,\mathrm{G}, \tag{2.120}$$

$$\kappa_3^{ij} = -16s(\alpha(\lambda-7)-\beta-2\mu)G^{ij}. \tag{2.121}$$

The term on the right hand side is

$$\epsilon^{abcdnpq} f_{abcd} = 4!(\star f)^{npq} \tag{2.122}$$

$$= 4!\left(h_1^{ij} X_{npq}^{ij} + h_2 Y_{npq} + h_3^{ij} Z_{npq}^{ij}\right), \tag{2.123}$$

where

$$h_1^{ij} = 2\left(\alpha(\lambda+3)-\beta-2\mu\right)\chi^{ij} - 4\alpha\omega^{ij} \tag{2.124}$$

$$h_2 = 8\beta \mathrm{Tr}\,\chi + 8(\alpha+\beta)\xi - 4(\alpha+\beta)\mathrm{Tr}\,\omega \tag{2.125}$$

$$h_3^{ij} = 4\left(\alpha(\lambda-1)+\beta-2\mu\right)\chi^{ij} - 4(3\alpha+\beta)\omega^{ij}. \tag{2.126}$$

Plugging the expansions of each of the terms into the linearized Maxwell equation yields three equations for the scalar fields, one equation for each of X^{ij}, Y, and Z^{ij}. They are

$$\Box\chi^{ij} = -g_1^{ij} - \kappa_1 G^{ij} - 4mh_1^{ij} \tag{2.127}$$

$$\Box\xi = -g_2 - \kappa_2 \mathrm{Tr}\,\mathrm{G} - 4\mathrm{mh}_2 \tag{2.128}$$

$$\Box\omega^{ij} = -g_3^{ij} - \kappa_3 G^{ij} - 4mh_3^{ij}. \tag{2.129}$$

Expanding these equations further in terms of the ansatz scalars G^{ij}, χ^{ij}, ξ, ω^{ij} gives

$$\Box\chi^{ij} = M_{21}G^{ij} + M_{22}\chi^{ij} + M_{23}\xi^{ij} + M_{24}\omega^{ij} \tag{2.130}$$

$$\Box\xi = M_{31}\,\mathrm{Tr\,G} + \mathrm{M_{32}Tr}\,\chi + \mathrm{M_{33}}\xi + \mathrm{M_{34}Tr}\,\omega \tag{2.131}$$

$$\Box\omega = M_{41}G^{ij} + M_{42}\chi^{ij} + M_{43}\xi^{ij} + M_{44}\omega^{ij}. \tag{2.132}$$

The M_{ij} in the above equations, combined with the M_{1j} in the linearized Einstein equation give the 4×4 mass matrix $\mathbf{M}$.

2.8 Eigenmodes

Plugging our background and fluctuation ansatze into the linearized Einstein and Maxwell equations yielded four equations for the four AdS scalars that were part of the fluctuation ansatz. These four equations can be conveniently written as

$$(\mathbf{1}_4\Box - \mathbf{M})\begin{pmatrix} G^{ij} \\ \chi^{ij} \\ \xi \\ \omega^{ij} \end{pmatrix} = 0. \tag{2.133}$$

The matrix $\mathbf{M}$ is called the 'mass matrix'.

If $\mathbf{S}$ is the matrix that diagonalizes $\mathbf{M}$ then we have

$$(\mathbf{1}_4\Box - \mathbf{D})\begin{pmatrix} \phi_1^{ij} \\ \phi_2^{ij} \\ \phi_3 \\ \phi_4^{ij} \end{pmatrix} = 0, \tag{2.134}$$

where

$$\mathbf{D} = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}, \tag{2.135}$$

and

$$\begin{pmatrix} \phi_1^{ij} \\ \phi_2^{ij} \\ \phi_3 \\ \phi_4^{ij} \end{pmatrix} = \mathbf{S} \begin{pmatrix} G^{ij} \\ \chi^{ij} \\ \xi \\ \omega^{ij} \end{pmatrix}. \tag{2.136}$$

The matrix $\mathbf{D}$ is diagonal and its entries are the eigenvalues of $\mathbf{M}$, which are the squared masses of the scalar fields ϕ_1^{ij}, ϕ_2^{ij}, ϕ_3, and ϕ_4^{ij}.

Explicitly, the mass matrix is

$$\mathbf{M} = \begin{pmatrix} 24 & 32\sqrt{2} & 0 & -16\sqrt{2} \\ 0 & 24-16\sqrt{2} & 0 & 8\sqrt{2} \\ -8\sqrt{2}\delta^{ij} & 16(-1+\sqrt{2})\delta^{ij} & 16 & 8\delta^{ij} \\ -24\sqrt{2} & 16(-2+\sqrt{2}) & 0 & 8(5+2\sqrt{2}) \end{pmatrix}. \tag{2.137}$$

By diagonalizing $\mathbf{M}$ one can find the ϕ_a^{ij}, ϕ_3, and the squared masses.

The ϕ_a^{ij} and ϕ_3 are found to be

$$\phi_1^{ij} = -\frac{1}{10}(2+3\sqrt{2})G^{ij} + \frac{1}{5}(-4+\sqrt{2})\chi^{ij} + \frac{1}{5}(3+\sqrt{2})\omega^{ij} \tag{2.138}$$

$$\phi_2^{ij} = \frac{1}{2}G^{ij} + \chi^{ij} \tag{2.139}$$

$$\phi_3 = \frac{2}{3}\mathrm{Tr}\,\chi + \xi - \frac{1}{3}\mathrm{Tr}\,\omega \tag{2.140}$$

$$\phi_4^{ij} = \frac{3}{10}(-1+\sqrt{2})G^{ij} - \frac{1}{5}(1+\sqrt{2})\chi^{ij} + \frac{1}{5}(2-\sqrt{2})\omega^{ij}. \tag{2.141}$$

The squared mass values are 72, 24, 16, and -8, respectively. Multiplying the squared mass values by the AdS radius squared gives the dimensionless squared mass values. The AdS radius squared was found to be $l^2 = \frac{3}{8}$ in section 4, so the dimensionless squared mass values are 27, 9, 6, and -3.

The Breitenlohner-Freedman bound is [32] $m^2 l^2 = -\frac{9}{4}$, so the eigenmode ϕ_4^{ij}, which has dimensionless squared mass value -3, is unstable.

Peeling off the different mass modes

Inverting the above equations gives

$$G^{ij} = \frac{1}{7}(2 - 3\sqrt{2})\phi_1^{ij} + \phi_2^{ij} + (1 + \sqrt{2})\phi_4^{ij} \tag{2.142}$$

$$\chi^{ij} = \frac{1}{14}(3\sqrt{2} - 2)\phi_1^{ij} + \frac{1}{2}\phi_2^{ij} - \frac{1}{2}(1 + \sqrt{2})\phi_4^{ij} \tag{2.143}$$

$$\xi = \frac{1}{7}(3 - \sqrt{2})\mathrm{Tr}\,\phi_1 + \phi_3 + \frac{1}{3}(2 + \sqrt{2})\mathrm{Tr}\,\phi_4 \tag{2.144}$$

$$\omega^{ij} = \phi_1^{ij} + \phi_2^{ij} + \phi_4^{ij}. \tag{2.145}$$

These expressions can be plugged into the perturbation ansatz of section 5, and the different mass modes can be 'peeled off'.

The metric parts of the 'peeled-off' mass modes are

$$h_{a,mn} = 2c_{a,1}\phi_a^{ij}(x)W_{mn}^{ij},\ a = 1, 2, 4 \tag{2.146}$$

$$h_{3,mn} = 0, \tag{2.147}$$

and the 3-form potential parts of the 'peeled-off' mass modes are

$$a_{a,mnp} = \phi_a^{ij}(x)\mathcal{A}_{a,mnp}^{ij},\ a = 1, 2, 4 \tag{2.148}$$

$$a_{3,mnp} = \phi_3(x)Y_{mnp}, \tag{2.149}$$

where

$$\mathcal{A}_{a,mnp}^{ij} = c_{a,2}X_{mnp}^{ij} + c_{a,3}\delta^{ij}Y_{mnp} + 2c_{a,4}Z_{mnp}^{ij}. \tag{2.150}$$

The $c_{a,i}$ are the coefficients in equations (141)-(144), e.g.

$$c_{4,1} = 1 + \sqrt{2} \tag{2.151}$$

$$c_{4,2} = -\frac{1}{2}(1 + \sqrt{2}) \tag{2.152}$$

$$c_{4,3} = \frac{1}{3}(2 + \sqrt{2}) \tag{2.153}$$

$$c_{4,4} = 1. \tag{2.154}$$

Degeneracies of the masses

The spinors η^i, $i = 1, \ldots, 6$, are used to construct the tensors W^{ij}, X^{ij}, and Z^{ij} that are defined in section 5. W^{ij} is symmetric in i and j, so there are at most 21 of them that are linearly independent. In fact, $\mathrm{Tr\,W} = \sum_{\mathrm{i=1}}^{6} \mathrm{W^{ii}} = 0$, so there are actually at most 20 linearly independent W^{ij}. Likewise, the $\mathcal{A}_a^{ij}$ are symmetric in i and j, and $\mathrm{Tr}\,\mathcal{A}_\mathrm{a} = 0$, so there are at most 20 linearly independent $\mathcal{A}_1^{ij}$, $\mathcal{A}_2^{ij}$, and $\mathcal{A}_4^{ij}$.

The η^i realize the $\mathbf{6}$ of SU(4), so W^{ij} and $\mathcal{A}_a^{ij}$ each realize the symmetric product $\mathbf{6} \times_s \mathbf{6}$. This symmetric product breaks as [80]:

$$\mathbf{6} \times_s \mathbf{6} \to \mathbf{20'} + \mathbf{1}, \tag{2.155}$$

where $\mathbf{1}$ is the trace part of $\mathbf{6} \times_s \mathbf{6}$ and $\mathbf{20'}$ is the symmetric traceless part. Since $\mathrm{Tr\,W} = \mathrm{Tr}\,\mathcal{A}_\mathrm{a} = 0$, it follows that W^{ij} and $\mathcal{A}_a^{ij}$ each realize the $\mathbf{20'}$. So there are exactly 20 linearly independent W^{ij} and $\mathcal{A}_a^{ij}$.

Therefore, the squared mass values 27, 9, and -3 each have degeneracy 20, and the squared mass value 6 has degeneracy 1.

2.9 Comparison to known results

The instability of the Pope-Warner solution on S^7 was first demonstrated in [32]. There it was found that there are unstable modes with dimensionless squared mass -3 that realize the $\mathbf{20'}$ of $SU(4)$. These modes are recovered here.

The result of [32] was extended to tri-Sasakian manifolds in [34]. There a consistent truncation of $d = 11$ supergravity was carried out on a 7-dimensional tri-Sasakian manifold to give a $d = 4$ supergravity theory. In carrying out the truncation, a scalar potential for the $d = 4$ theory was extracted. This potential has the Pope-Warner solution as a fixed point. By computing the second derivatives of this potential at the Pope-Warner fixed point, the squared masses of the scalars at the Pope-Warner fixed point can be found. These masses are not given in [34], but we found them to be $(27, 18^{\mathbf{2}}, 9^{\mathbf{2}}, 6, 2.25^{\mathbf{2}}, -3, 0^{\mathbf{7}})$, where the superscripts denote the degeneracies. Therefore, the mass values found here are a subset of those found in [34].

In the special case where i and j are set equal to a fixed value, the perturbation ansatz given in section 5 is contained in the consistent truncation ansatz of [34]. To go between the ansatz of [34] and the one here it suffices to express the canonical 1-forms and 2-forms of the tri-Sasakian structure used in [34] in terms of the spinors used here. In terms of the spinors used here, the 1-forms of [34] are

$$\vartheta^1_a = \bar{\zeta}\Gamma_a\eta \tag{2.156}$$

$$\vartheta^2 = \bar{\psi}\Gamma_a\eta \tag{2.157}$$

$$\vartheta^3 = -\bar{\zeta}\Gamma_a\psi, \tag{2.158}$$

and the 2-forms of [34] are

$$J^1_{ab} = -\bar{\zeta}\Gamma_{ab}\eta - 2\bar{\psi}\Gamma_{[a}\eta\bar{\zeta}\Gamma_{b]}\psi \tag{2.159}$$

$$J^2_{ab} = -\bar{\psi}\Gamma_{ab}\eta - 2\bar{\zeta}\Gamma_{[a}\psi\bar{\zeta}\Gamma_{b]}\eta \tag{2.160}$$

$$J^3_{ab} = \bar{\zeta}\Gamma_{ab}\psi + 2\bar{\zeta}\Gamma_{[a}\eta\bar{\psi}\Gamma_{b]}\eta, \tag{2.161}$$

where η is an arbitrary linear combination of the η^i with real coefficients and unit norm.

In [34] a single unstable mode with squared mass -3 was found. There it was supposed that on S^7 this single mode was one of the 20 unstable modes found in [32]. Here we have explicitly shown that this is indeed the case.

3 The solutions

In this chapter we obtain the FR, skew-whiffed FR, and PW solutions of eleven-dimensional supergravity. Even though we obtained the PW solution in chapter two, we obtain it again here because the conventions used in chapter 2 are different from those used in the rest of this dissertation.

The FR and PW solutions of eleven-dimensional supergravity on a SE manifold, M_7, can be derived from the following general Ansatz,

$$ds^2_{11} = ds^2_{AdS_4(L)} + a^2\, ds^2_7\,, \tag{3.1}$$

$$\mathcal{F}_{(4)} = f_0\, \text{vol}_{AdS_4(L)} + f_i\, \Phi_{(4)}\,. \tag{3.2}$$

A regular SE manifold, M_7, is the total space of a U(1) fibration over a Kähler-Einstein (KE) base, B_6, and the internal metric in (3.1) can be written locally as

$$ds_7^2 = ds_{B_6}^2 + c^2\big(d\psi + A\big)^2\,, \tag{3.3}$$

were A is the Kähler potential on B_6, ψ is the angle along the fiber and c is the squashing parameter. The potential for the internal flux in (3.2) is given by the real part of a canonical complex three-form, Ω, on M_7, such that

$$\Phi_{(4)} = d(\Omega + \overline{\Omega})\,. \tag{3.4}$$

The constants, a, c, f_0 and f_i in (3.1)-(3.2) are fixed by the equations of motion in terms of the AdS_4 radius, L, which sets the overall scale of the solution:

- Supersymmetric and skew-whiffed FR solutions

$$a = 2L\,, \qquad c = 1\,, \qquad f_0 = \kappa\,\frac{3}{2L}\,, \qquad f_i = 0\,, \tag{3.5}$$

  where $\kappa = -1$ and $+1$, respectively.

- PW solution

$$a = 2\sqrt{\frac{2}{3}}\,L\,, \qquad c = \sqrt{2}\,, \qquad f_0 = \frac{\sqrt{3}}{2L}\,, \qquad f_i = \frac{4}{3}\sqrt{\frac{2}{3}}\,L^3\,. \tag{3.6}$$

In (3.1), we have factored out the overall scale, a^2, of the internal metric so the KE metric, g_{B_6}, and the SE metric, g_{M_7}, obtained by setting $c = 1$ in (3.3), are canonically normalized with

$$Ric_{B_6} = 8\,g_{B_6}\,, \qquad Ric_{M_7} = 6\,g_{M_7}\,. \tag{3.7}$$

In the following, we will refer to the SE metric on M_7 as the "round" metric.

The one form $\vartheta = d\psi + A$, called the contact form, is globally defined on M_7, and is dual to the Reeb vector field, $\xi = \partial_\psi$, which is nowhere vanishing and has length one. The other two globally defined forms of the SE geometry are the real two form, J, and a complex three form, Ω, with its complex conjugate, $\overline{\Omega}$. They satisfy

$$d\vartheta = 2J\,, \qquad d\Omega = 4i\,\vartheta \wedge \Omega\,. \tag{3.8}$$

Note that the ansatz (3.1)-(3.4) is in fact written in terms of globally defined objects of the SE geometry.

It is convenient to choose special frames, $\mathring{e}^a$, $a = 1, \ldots, 7$, on M_7, that are orthonormal with respect to the round metric and such that

$$J = \frac{i}{2}\left(\mathring{e}^{z_1} \wedge \mathring{e}^{\bar{z}_1} + \mathring{e}^{z_2} \wedge \mathring{e}^{\bar{z}_2} + \mathring{e}^{z_3} \wedge \mathring{e}^{\bar{z}_3}\right)\,, \qquad \Omega = e^{4i\psi}\,\mathring{e}^{z_1} \wedge \mathring{e}^{z_2} \wedge \mathring{e}^{z_3}\,, \qquad \vartheta = \mathring{e}^7\,, \tag{3.9}$$

where

$$\mathring{e}^{z_1} = \mathring{e}^1 + i\,\mathring{e}^2\,, \qquad \mathring{e}^{z_2} = \mathring{e}^3 + i\,\mathring{e}^4\,, \qquad \mathring{e}^{z_3} = \mathring{e}^5 + i\,\mathring{e}^6\,, \tag{3.10}$$

is a local holomorphic frame on the KE base. This shows that J is the pull-back of the Kahler form, while Ω is, up to a phase along the fiber, the pull-back of the holomorphic $(3,0)$-form on B_6. We will denote the components of the round metric by $\mathring{g}_{ab} = \delta_{ab}$ and of the squashed metric (3.3) by g_{ab}. Then the components of the eleven-dimensional metric (3.1) along the internal manifold are $\mathfrak{g}_{ab} = a^2\, g_{ab}$.

One can also express ϑ, J and Ω as bilinears in Killing spinors, η^α,

$$\mathring{D}_a \eta^\alpha = \frac{i}{2}\Gamma_a \eta^\alpha\,, \qquad \bar{\eta}^\alpha \eta^\beta = \delta^{\alpha\beta}\,, \qquad \alpha, \beta = 1, 2\,, \tag{3.11}$$

that are globally defined on M_7 and whose existence is equivalent to M_7 being a SE manifold. In terms of η^α's we have (see, e.g., [61])

$$\vartheta_a = i\,\bar\eta^1\Gamma_a\eta^2\,, \qquad J_{ab} = \bar\eta^1\Gamma_{ab}\eta^2\,, \qquad \Omega_{abc} = -\frac{1}{2}(\bar\eta^1 + i\,\bar\eta^2)\Gamma_{abc}(\eta^1 + i\,\eta^2)\,. \tag{3.12}$$

Using this realization together with Fierz identities, it is straightforward to prove a number of useful identities summarized in appendix B.

To verify the solutions (3.5) and (3.6), we note that the covariant derivatives for the squashed and round metric are related by

$$D_a V_b = \mathring{D}_a V_b - 2\,(c^2-1)\vartheta_{(a}J_{b)}{}^c\,V_c\,, \tag{3.13}$$

where we have adopted a convention to raise and lower indices with the round metric, $\mathring{g}_{ab}$. For the Ricci tensors, using identities in appendix B, we have

$$R_{ab} = \mathring{R}_{ab} + 2(1-c^2)\mathring{g}_{ab} + 2(3c^4+c^2-4)\vartheta_a\vartheta_b\,. \tag{3.14}$$

These are also the components of the Ricci tensor, $\mathcal{R}_{ab}$, along the internal manifold. The Ricci tensor for AdS_4 of radius, L, is

$$Ric_{AdS_4} = -\frac{3}{L^2}g_{AdS_4}\,. \tag{3.15}$$

so that the eleven-dimensional Ricci scalar is

$$\mathcal{R} = -\frac{12}{L^2} + \frac{6}{a^2}(8-c^2)\,. \tag{3.16}$$

The energy momentum tensor in (1.1) has only diagonal contributions from the flux along AdS_4 and M_7 that are straightforward to evaluate. Then the Einstein equations (1.1) reduce to three algebraic equations:

$$a^2 = \frac{4}{3}(c^2-4)L^2\,, \qquad f_0^2 = \frac{3(7c^2-16)}{4L^2(c^2-4)}\,, \qquad f_i^2 = \frac{2}{27}c^2(c^2-4)^3(c^2-1)L^6\,, \tag{3.17}$$

for the size of the internal part of the metric and the parameters of the flux.

We now turn to the Maxwell equations (1.2). Let us denote by $*$ the Hodge dual on M_7 with respect to the round metric with the volume form

$$\mathrm{vol}_{M_7} = \frac{1}{6}\,J\wedge J\wedge J\wedge\vartheta = \frac{3}{8}\,i\,\Omega\wedge\overline{\Omega}\wedge\vartheta\,, \tag{3.18}$$

The volume form for the squashed metric is then $c\,\mathrm{vol}_{M_7}$, while $ca^7\mathrm{vol}_{AdS_4}\wedge\mathrm{vol}_{M_7}$ is the volume form in eleven-dimensions.

It follows from (3.8) and (3.18) that

$$*\,d\Omega = 4\,\Omega\,, \qquad *\,\Omega = \frac{1}{4}\,d\Omega\,. \tag{3.19}$$

Then for the flux, $\mathcal{F}_{(4)}$, in (3.2) and (3.4), we have

$$\star\,\mathcal{F}_{(4)} = f_0\star\mathrm{vol}_{AdS_4} - \frac{4f_i}{ac}\,\mathrm{vol}_{AdS_4}\wedge(\Omega+\overline{\Omega})\,, \tag{3.20}$$

so that

$$d\star\mathcal{F}_{(4)} = -\frac{4f_i}{ac}\,\mathrm{vol}_{AdS_4}\wedge\Phi_{(4)}\,. \tag{3.21}$$

The second term in (1.2) yields

$$\mathcal{F}_{(4)}\wedge\mathcal{F}_{(4)} = 2f_0f_i\,\mathrm{vol}_{AdS_4}\wedge\Phi_{(4)}\,, \tag{3.22}$$

which shows that the Maxwell equations reduce to a single equation

$$f_i\left(f_0-\frac{2}{ac}\right) = 0\,. \tag{3.23}$$

Assuming that both a and c are positive, one verifies that (3.5) and (3.6) exhaust all solutions to (3.17) and (3.23). Note that the only difference between the supersymmetric and skew-whiffed FR solutions is the sign, κ, of the flux along AdS_4. Equivalently, one could reverse the orientation of the internal manifold, which changes the sign of the Hodge dual in (1.2). Here, we will keep the orientation of M_7 fixed as in (3.18).

4 The linearized analysis

We will not attempt here a complete analysis of the Kaluza-Klein spectrum around the PW solution, but instead will identify a small set of harmonics for the low lying scalar modes on which the scalar mass operator in the linearized expansions around both the FR and PW backgrounds can be diagonalized. In doing that, we will be guided both by the explicit structure of the linearized equations of motion and by the properties of unstable modes on S^7 that were identified in [32].

The scalar modes we want to consider correspond to fluctuations of the internal metric and the internal three-form potential,

$$\delta\mathfrak{g}_{ab} = \varphi(x) h_{ab}\,, \qquad \delta\mathcal{A}_{(3)} = \varphi(x)\,\alpha_{(3)}\,, \tag{4.1}$$

where $\varphi(x)$ is a scalar field on AdS_4, while h_{ab} and $\alpha_{(3)}$ are, respectively, a symmetric tensor and a three form harmonic on M_7.

4.1 Linearized Einstein equations

We begin with the metric harmonic and the linearization of the Einstein equations (1.1). Following a crucial obervation in [32] for the unstable modes on S^7, we will assume that metric harmonic, h_{ab}, corresponds to a deformation of the internal metric along the $(2,0)$ and $(0,2)$ components on the KE base. Specifically, h_{ab} is horizontal, that is $\vartheta^a h_{ab} = 0$, and its only nonvanishing components in the basis (3.10) are $h_{z_i z_j}$ and $h_{\bar{z}_i \bar{z}_j}$. It is then automaticaly traceless. Finally, we will assume that it is transverse with respect to the round metric, $\mathring{D}^a h_{ab} = 0$. It follows then from (3.13) that it is also transverse with respect to the internal metric with any value of the squashing parameter, c.

With those assumptions, the metric fluctuation (4.1) is both transverse and traceless in eleven dimensions, so that the expansion of the Ricci tensor in (1.1) yields only one term with the Lichnerowicz operator (see, e.g., [62]), and there are no terms from the Ricci scalar to linear order. The eleven-dimensional Lichnerowicz operator becomes then a sum,[3]

$$\Box_{AdS_4} - \frac{1}{a^2}\Delta_L^c\,, \tag{4.2}$$

where Δ_L^c is the Lichnerowicz operator on M_7 with respect to the squashed metric. Then on the metric harmonics, h_{ab}, as above,

$$\Delta_L^c h_{ab} = \left[\Delta_L + 4(1-c^2) + \Big(1 - \frac{1}{c^2}\Big)\mathcal{L}_\xi^2\right] h_{ab}\,, \tag{4.3}$$

where Δ_L is the Lichnerowicz operator for the round metric, and $\mathcal{L}_\xi$ is the Lie derivative along the Reeb vector.

[3] We use $\Box_{AdS_4} = g^{\mu\nu}\nabla_\mu\nabla_\mu$, but $\Delta_L = -g^{ab}D_aD_b + \dots$

Let's denote the combination on the left hand side in (1.1) by $\mathcal{E}_{MN}$. After collecting all the terms in the expansion and using (3.16), we obtain

$$\delta\mathcal{E}_{ab} = -\frac{1}{2}\Big(\Box_{AdS_4} - \nu^2\Big)\,\varphi(x)\,h_{ab}\,, \tag{4.4}$$

where

$$\nu^2 = \frac{1}{a^2}\Delta_L + \Big(\frac{84}{a^2} - \frac{24}{L^2}\Big) - \frac{1}{a^2}\Big(1 - \frac{1}{c^2}\Big)(16c^2 - \mathcal{L}_\xi^2)\,. \tag{4.5}$$

Let us now turn to the expansion of the energy momentum tensor, $\mathcal{T}_{MN}$, on the right hand side in (1.1). For the metric variation as above, the only terms that contribute to the linear expansion of the energy momentum tensor come from the flux,

$$\delta\mathcal{T}_{ab} = \frac{f_i}{3a^6}\,g^{cf}g^{dg}g^{dh}(d\alpha_{acde}\Phi_{bfgh} + d\alpha_{bcde}\Phi_{afgh})\varphi(x)\,. \tag{4.6}$$

Assuming that metric harmonic is an eigentensor of the Lichnerowicz operator in (4.5), we see that in order to diagonalize the linearized Einstein equations we must find a flux harmonic such that the symmetric tensor in (4.6) is proportional to h_{ab}.

4.2 Linearized Maxwell equations

The expansion of the Maxwell equations is

$$d\star\delta\mathcal{F}_{(4)} + 2\,\mathcal{F}_{(4)}\wedge\delta\mathcal{F}_{(4)} + d\,\delta(\star)\mathcal{F}_{(4)} = 0\,, \tag{4.7}$$

where

$$\delta\mathcal{F}_{(4)} = d\varphi\wedge\alpha_{(3)} + \varphi\,d\alpha_{(3)}\,, \tag{4.8}$$

and $\delta(\star)$ is the variation of the Hodge dual due to fluctuation of the metric. Define a four form

$$(\delta\mathfrak{g}\cdot\mathcal{F})_{MNPQ} = \mathfrak{g}^{M'M''}\delta\mathfrak{g}_{MM'}\mathcal{F}_{M''NPQ} + \ldots + \mathfrak{g}^{Q'Q''}\delta\mathfrak{g}_{QQ'}\mathcal{F}_{MNPQ''}\,. \tag{4.9}$$

Then for a traceless fluctuation of the metric,

$$\delta(\star)\mathcal{F}_{(4)} = -\star(\delta\mathfrak{g}\cdot\mathcal{F}_{(4)})\,. \tag{4.10}$$

Specializing to the background flux (3.2) and the fluctuations (4.1), the linearization (4.7) splits into terms that are one, three, and four forms along AdS_4, respectively. They yield the following equations

$$d*_c\alpha = 0\,,\qquad \alpha\wedge\Phi_{(4)} = 0\,, \tag{4.11}$$

and

$$(\Box_{AdS_4}\varphi)*_c\alpha_{(3)} + \varphi\left[\frac{2f_0}{a}d\alpha_{(3)} - \frac{1}{a^2}d*_c d\alpha_{(3)} + \frac{f_i}{a^4}d*_c(h\cdot\Phi_{(4)})\right] = 0\,, \tag{4.12}$$

where $*_c$ denotes the dual with respect to the squashed metric and $h\cdot\Phi_{(4)}$ is defined as in (4.9) using the round metric. The various factors of the internal radius, a, in (4.12) are consistent with the overall $1/L^2$ dependence of the mass terms on the AdS_4 radius.

4.3 The master harmonic

Our task now is to identify the smallest set of harmonics on which we can diagonalize the Maxwell equation (4.12). The first step will be to streamline the evaluation of the Hodge duals.

Any k-form, Ξ_k, on M_7 can be uniquely decomposed into the sum,

$$\Xi_k = \omega_k + \vartheta\wedge\omega_{k-1}\,, \tag{4.13}$$

where ω_k and ω_{k-1} are horizontal forms, that is, $\imath_\xi\omega_k = \imath_\xi\omega_{k-1} = 0$. Then

$$*_c\,\Xi_k = c*\omega_k + \frac{1}{c}*(\vartheta\wedge\omega_{k-1})\,, \tag{4.14}$$

where $*$ is the Hodge dual with respect to the round metric. We may simplify this further by introducing another Hodge dual, $\bullet$, in the space perpendicular to the fiber, or, equivalently on the KE base, B_6. Then for a horizontal form, ω, using $\mathrm{vol}_{M_7} = \mathrm{vol}_{B_6} \wedge \vartheta$, we have[4]

$$* \, \omega = \vartheta \wedge \bullet \, \omega \,, \qquad * \, (\omega \wedge \vartheta) = \bullet \, \omega \,, \tag{4.15}$$

and hence

$$*_c \, \Xi_k = c \, \vartheta \wedge \bullet \, \omega_k + \frac{1}{c} \, (-1)^{k-1} \bullet \, \omega_{k-1} \,. \tag{4.16}$$

To further restrict the Ansatz for the flux harmonic, let us look at the last term in (4.12), which is already constrained by the conditions we have imposed in section 4.1 on the metric harmonic, h_{ab}. Since h_{ab} has nonvanishing components only along the KE base, we have

$$h \cdot \Phi_{(4)} = 4i \, \vartheta \wedge h \cdot (\Omega - \overline{\Omega}) \,, \tag{4.17}$$

where all contractions between the metric harmonic and the background flux form are with the round metric.

We can now evaluate the forms (4.17) on S^7 using the metric harmonics given in [32]. It turns out that $h \cdot (\Omega - \overline{\Omega})$ is closed (!) and it is both horizontal and invariant along the fiber. This means that it is a closed basic three-form with nonvanishing $(2,1)$ and $(1,2)$ components. On S^7, it is then a pull-back of the corresponding closed form on $\mathbb{CP}^3$ and thus is exact. Indeed, we find that

$$h \cdot (\Omega - \overline{\Omega}) = -64 \, d\omega \,, \tag{4.18}$$

where ω is a basic, primitive, transverse (1,1)-form, and an eigenform of the Laplacian, with

[4] Note that on a k-form, $*^2 = *_c^2 = 1$, while $\bullet^2 = (-1)^k$.

the eigenvalue 24. In the following we will show that a similar construction can be carried out on a general SE manifold, M_7.

We start with a primitive, $(1,1)$-form, ω on the KE which is a transverse eigenform of the Hodge-de Rham Laplacian with the eigenvalue λ_ω. Its pull-back to M_7 is then a basic form, satisfying

$$\imath_\xi \omega = 0\,, \qquad \mathcal{L}_\xi \omega = 0\,, \tag{4.19}$$

which we also denote by ω. We will now discuss the conditions on ω and derive some identities that are used later.

(i) The condition that ω is a primitive $(1,1)$-form means that

$$J^{ab}\omega_{ab} = 0\,, \qquad J_a{}^c J_b{}^d \omega_{cd} = \omega_{cd}\,, \tag{4.20}$$

where the first condition can be equivalently written as

$$J\wedge \bullet\, \omega = 0 \qquad \text{or} \qquad J\wedge J\wedge \omega = 0\,. \tag{4.21}$$

It follows from (4.20) that on B_6 and M_7, respectively,[5]

$$J\wedge\omega = -\bullet\,\omega \qquad \text{and} \qquad *(J\wedge\omega) = -\vartheta\wedge\omega\,. \tag{4.22}$$

(ii) Transversality on the KE base

$$d\bullet\omega = 0\,, \tag{4.23}$$

[5]The operator $\bullet\, J\wedge$ on a six-dimensional Kahler manifold maps two-forms into two-forms. It has eigenvalues -1, 1 and 2 with degeneracies 8, 6 and 1, respectively, corresponding to the primitive $(1,1)$-forms, $(2,0)+(0,2)$-forms and $(1,1)$-forms proportional to J.

implies transversality on the SE manifold,

$$
\begin{aligned}
d * \omega &= d(\vartheta \wedge \bullet\, \omega) \\
&= 2J \wedge \bullet\, \omega - \vartheta \wedge d \bullet \omega \\
&= 0\,,
\end{aligned}
\tag{4.24}
$$

where the last step follows from (4.22) and (4.23).

By taking the exterior derivative of (4.22), we get

$$
J \wedge d\omega = 0\,, \tag{4.25}
$$

and a somewhat less obvious

$$
J \wedge \bullet\, d\omega = 0\,. \tag{4.26}
$$

Since the last identity is on the KE base, upon taking a dual we obtain a 1-form with components proportional to

$$
2J^{\alpha\beta}\nabla_\alpha \omega_{\beta\gamma} + J^{\alpha\beta}\nabla_\gamma \omega_{\alpha\beta}\,. \tag{4.27}
$$

On a Kähler manifold, J is covariantly constant and the first term can be written as

$$
\begin{aligned}
J^{\alpha\beta}\nabla_\alpha \omega_{\beta\gamma} &= \nabla^\alpha (J_\alpha{}^\beta \omega_{\beta\gamma}) \\
&= -\nabla^\alpha (J_\gamma{}^\beta \omega_{\alpha\beta}) \\
&= 0\,,
\end{aligned}
\tag{4.28}
$$

where we used that ω is a transverse $(1,1)$-form. The vanishing of the second term in (4.27) is shown similarly.

(iii) Finally, ω is an eigenfunction of the Hodge-de Rham Laplacian operator, $\Delta_{(1,1)}$ on B_6, which for a transverse form is simply,

$$
\Delta_{(1,1)}\omega \equiv \bullet\, d \bullet d\omega = \lambda_\omega\, \omega\,. \tag{4.29}
$$

Then the Laplacian on M_7, after using (4.15) and (4.26) is

$$\begin{aligned}\Delta\,\omega &\equiv -*d*d\omega \\ &= -*d(\vartheta\wedge\bullet\, d\omega) \\ &= -*(2J\wedge\bullet\, d\omega-\vartheta\wedge d\bullet d\omega) \\ &= \lambda_\omega\,\omega\,.\end{aligned} \tag{4.30}$$

Hence ω is also an eigenfunction of the Laplacian on M_7 with the same eigenvalue, λ_ω.

4.4 The metric harmonic

We now take the following Ansatz for the metric harmonic in terms of a pure imaginary (1,1)-form, ω,

$$h_{ab}=(d\omega)_{acd}(\Omega_b{}^{cd}-\overline{\Omega}_b{}^{cd})+(a\leftrightarrow b)\,. \tag{4.31}$$

This tensor is manifestly horizontal and has only $(2,0)$ and $(0,2)$ components as we have required in section 4.1. It also satisfies (4.18), as one can verify using identities in section 4.3 and appendix A. We will now show that h_{ab} is a transverse eigentensor of the Lichnerowicz operator on M_7,

$$\Delta_L h_{ab}=\lambda_h h_{ab}\,,\qquad \lambda_h=\lambda_\omega+4\,, \tag{4.32}$$

with the eigenvalue, λ_h, fixed by λ_ω.

Before we present a somewhat lengthy proof, let us note that the same relation between the eigenvalues of the Hodge-de Rham Laplacian and the Lichnerowicz operator has been derived in [56] through a general analysis of the fermion/boson mass relations on manifolds with Killing spinors, see Appendix G. In particular, it was shown that if a two-form and a symmetric tensor harmonics arise from the same spin-3/2 harmonic by a supersymmetry transformation generated by Killing spinors, the resulting shift of the eigenvalues is precisely the one given in (4.32). While we have not derived the intermediate spin-3/2 harmonic in general, some explicit checks on S^7 (or, more generally on tri-Saskian manifolds), where all the forms in (4.31) can be realized in terms of Killing spinors,[6] have convinced us that our construction here and in the following sections yields a subset of harmonics in a single $\mathcal{N}=2$ supermultiplet as in [56]. We will discuss it further in chapter 5, where we identify this supermultiplet as the long Z-vector multiplet [57].

We also note that a similar construction for tensor harmonics on a five-dimensional SE manifolds has been recently carried out in [63] and it follows a much earlier construction for four-dimensional Kahler manifolds in [64].

4.4.1 Proof of transversality

There are four types of terms in the transversality condition,[7] $D^a h_{ab}=0$. First, we have

$$D^a(d\omega)_{acd}\Omega_b{}^{cd} = -\lambda_\omega \omega_{cd}\Omega_b{}^{cd} = 0\,, \tag{4.33}$$

since ω is a $(1,1)$-form. Secondly,

$$(d\omega)_{acd}D^a\Omega_b{}^{cd} = 4i\,(d\omega)^{acd}\vartheta_{[a}\Omega_{bcd]} = 0\,, \tag{4.34}$$

since $d\omega$ is horizontal, and hence $d\omega^{abc}\vartheta_a=0$. Similarly, the full contraction between $d\omega$, which is a sum of a $(2,1)$ and a $(1,2)$ form, and the $(3,0)$ form, Ω, must vanish. The third

[6]See, section 5.1.

[7]Throughout this section, D_a is the covariant derivative with respect to the round metric.

type of terms are

$$D^a(d\omega)_{bcd}\Omega_a{}^{cd} = D_a(d\omega)_{bcd}\Omega^{acd}\,. \tag{4.35}$$

Since $D_{[a}(d\omega)_{bcd]} = 0$, we have

$$\begin{aligned}
3D_a(d\omega)_{bcd}\Omega^{acd} &= D_b(d\omega)_{acd}\Omega^{acd} \\
&= D_b\left[(d\omega)_{acd}\Omega^{acd}\right] - (d\omega)_{acd}D_b\Omega^{acd} \\
&= -4i\,(d\omega)^{acd}\vartheta_{[b}\Omega_{acd]} \\
&= 0\,,
\end{aligned} \tag{4.36}$$

as $d\omega$ is either contracted with ϑ or fully contracted with Ω. Finally, the last type of terms are

$$(d\omega)_{bcd}D^a\Omega_a{}^{cd} = 0\,, \tag{4.37}$$

since Ω is itself transverse, see, e.g., (C.12). Transversality of the terms with $\overline{\Omega}$ is verified similarly.

4.4.2 Proof of (4.32)

The Lichnerowicz operator,[8] Δ_L, on k-forms coincides with the Hodge-de Rham Laplacian,

$$\Delta = d\delta + \delta d\,, \qquad \delta = (-1)^k * d * \,. \tag{4.38}$$

We have assumed that $\Delta\omega = \lambda_\omega\omega$. Using (3.19) we also find

$$\Delta\Omega = 16\,\Omega\,. \tag{4.39}$$

For an arbitrary tensor, the Lichnerowicz operator is defined by

$$\Delta_L T_{a_1\ldots a_k} = -\Box T_{a_1\ldots a_k} + (R^a{}_{a_1}T_{a\,a_2\ldots a_k} + \ldots) - 2(R^a{}_{a_1}{}^b{}_{a_2}T_{a\,b\,a_3\ldots a_k} + \ldots)\,, \tag{4.40}$$

[8] For a list of properties of the Lichnerowicz operator, see, e.g., [62].

where there are k-terms in the first bracket and $\frac{1}{2}k(k-1)$ in the second. An important property, which we are going to exploit in the following, is that Δ_L commutes with the contraction.

Consider the tensor

$$t_{acdbef} = (d\omega)_{acd}\Omega_{bef}\,, \tag{4.41}$$

from which the $(2,0)$-part of h_{ab} is obtained by contracting over the pairs ce and df and then symmetrizing over ab. It follows from the definition (4.40) that

$$(\Delta_L t)_{acdbef} = (\Delta_L d\omega)_{acd}\Omega_{bef} + (d\omega)_{acd}(\Delta_L\Omega)_{bef} - 2D^g(d\omega)_{acd}D_g\Omega_{bef} + R\text{-terms}\,, \tag{4.42}$$

where the R-terms involve split contractions with both $d\omega$ and Ω,

$$R\text{-terms} = -2\left[(d\omega)_{gcd}\Omega_{hef}R^g{}_a{}^h{}_b + 8\text{-terms}\right] \tag{4.43}$$

We will now show that all terms in (4.42) give contributions to $\Delta_L h_{ab}$ that are proportional to h_{ab} and evaluate the proportionality constants.

From the first two terms we get $(\lambda_\omega + 16)h_{ab}$. Next, we consider the R-terms, which can be traded for covariant derivatives acting on Ω using

$$[D_a, D_b]\Omega_{cde} = -\Omega_{fde}R^f{}_{cab} - \dots\,. \tag{4.44}$$

This gives

$$R\text{-terms} = 2\big((d\omega)_{gcd}[D^g, D_a] + (d\omega)_{agd}[D^g, D_c] + (d\omega)_{acg}[D^g, D_d]\big)\Omega_{bef}\,. \tag{4.45}$$

The covariant derivatives acting on Ω can be evaluated using (C.12). This yields terms that are products of the form

$$d\omega_{\times\times\times}J_{\times\times}\Omega_{\times\times\times} \qquad \text{or} \qquad d\omega_{\times\times\times}\vartheta_\times\vartheta_\times\Omega_{\times\times\times}\,. \tag{4.46}$$

Performing the contractions as in the definition of h_{ab}, see (4.31), we are left with two free indices with all other ones contracted. Because of the symmetrization, the free indices in the terms of the first type in (4.46) must be on two different tensors. In particular, this implies that J is always contracted with either Ω or $d\omega$ or both. All terms in which J is contracted with Ω are simplified using (C.7) and yield terms proportional to h_{ab}. This leaves terms in which J is contracted with $d\omega$. By inspection, in all those terms $d\omega$ is doubly contracted with Ω, which means that the contraction with J is once more a multiplication by i. The second type terms in (4.46) all vanish except when the two ϑ's are contracted. Collecting all the terms we find that the total contribution from the R-terms to $\Delta_L h_{ab}$ is $-10h_{ab}$.

Finally, we consider the third term in (4.42). Since $d\omega$ is closed, we rewrite this term as

$$-2D_g(d\omega)_{acd}D_g\Omega_b{}^{cd}+(a\leftrightarrow b) = -2D_a(d\omega)_{gcd}D_g\Omega_b{}^{cd}-4D_c(d\omega)_{agd}D^g\Omega_b{}^{cd}+(a\leftrightarrow b)\,. \quad (4.47)$$

Let's start with the first term in (4.47). Since

$$(d\omega)_{gcd}D^g\Omega_b{}^{cd} = 4i\,(d\omega)^g{}_{cd}\vartheta_{[g}\Omega_{bcd]} = 0\,, \quad (4.48)$$

we have

$$D_a(d\omega)_{gcd}D^g\Omega_b{}^{cd} = -(d\omega)_{gcd}D_aD^g\Omega_b{}^{cd}\,. \quad (4.49)$$

Expanding the covariant derivatives using (C.12), we find that all terms involving ϑ vanish. The remaining terms have $d\omega$ contracted with J and twice contracted with Ω, which reduces the contraction with J to the multiplication by i. Then the net contribution from this term to $\Delta_L h_{ab}$ is $-6h_{ab}$.

This leaves us with the second term in (4.47), which we once more rewrite using the Leibnitz rule. However, now the total derivative term does not vanish, but yields the derivative

D^c of the following terms,

$$(d\omega)_{agd} D^g \Omega_{bc}{}^d = i(d\omega)_{agd}(2\,\vartheta^g \Omega_{bc}{}^d + \vartheta_b \Omega_c{}^{gd} - \vartheta_c \Omega_b{}^{gd})\,. \tag{4.50}$$

The first term on the rhs vanishes as $d\omega$ is horizontal. The second term can be rewritten as

$$i(d\omega)_{agd}\vartheta_b \Omega_c{}^{gd} = i\,h_{ac}\vartheta_b - i(d\omega)_{cgd}\Omega_a^{gd}\vartheta_b\,. \tag{4.51}$$

Acting with D^c and using

$$3\,D^a D_{[a}\omega_{bc]} = D^a(d\omega)_{abc} = -\lambda_\omega\,\omega_{bc}\,, \tag{4.52}$$

and the transversality of h_{ab}, we get

$$ih_{ac}J_c{}^b - i(d\omega)_{cgd} D_c \Omega_a{}^{gd}\vartheta_b - i(d\omega)_{cgd}\Omega_a{}^{gd}J^c{}_b = h_{ab} + 0 - (d\omega)_{bgd}\Omega_a{}^{gd}\,, \tag{4.53}$$

which gives $-4h_{ab}$ contribution in $\Delta_L h_{ab}$. The last term in (4.50) is

$$-\,iD^c(\vartheta_c(d\omega)_{agd}\Omega_b{}^{gd}) = -i\vartheta_c\,D^c(d\omega)_{agd}\Omega_b{}^{gd} - i\vartheta_c(d\omega)_{agd}D^c\Omega_b{}^{gd}\,. \tag{4.54}$$

Using $d^2\omega = 0$, the first term on the right hand side above can be simplified using

$$\begin{aligned}
-i\vartheta_c D^c(d\omega)_{agd} &= -i\vartheta_c D_a(d\omega)^c{}_{gd} - i\vartheta_c D_g(d\omega)_a{}^c{}_d - i\vartheta_c D_d(d\omega)_{ag}{}^c \\
&= iJ_{ac}(d\omega)^c{}_{gd} + iJ_{gc}(d\omega)_a{}^c{}_d + iJ_{dc}(d\omega)_{ag}{}^c \\
&= (d\omega)_{agd}\,,
\end{aligned} \tag{4.55}$$

where the second line follows using the Leibnitz rule, horizontality of $d\omega$ and (C.12). The second in term (4.54), using (C.12), is

$$-\,i\vartheta_c(d\omega)_{agd}D^c\Omega_b{}^{gd} = (d\omega)_{agd}\Omega_b{}^{gd}\,. \tag{4.56}$$

Hence the last term in (4.50) is $-2(d\omega)_{agd}\Omega_b{}^{gd}$, and by (4.47) it contributes $-8h_{ab}$ to $\Delta_L h_{ab}$.

Finally, using the Leibnitz rule, we are left with

$$4(d\omega)_{agd}D_c D_g \Omega_{bcd} = 16\, h_{ab}\,. \tag{4.57}$$

Hence all terms in (4.42) are indeed proportional to h_{ab}, with the net result

$$\lambda_h = \lambda_\omega + 16 - 10 - 6 - 4 - 8 + 16 = \lambda_\omega + 4\,. \tag{4.58}$$

This concludes the proof of (4.32).

4.5 The flux harmonics

We take as internal flux harmonic the linear combination

$$\alpha_{(3)} = t_1\, \vartheta \wedge \omega + t_2\, * d\,(\vartheta \wedge \omega)\,, \tag{4.59}$$

where t_1 and t_2 are arbitrary pure imaginary parameters.[9]

The harmonics that arise in the expansion of the Maxwell equation (4.12) are: $d\alpha$, $*_c\alpha$, and $d *_c d\alpha$. We will now show that for α given by (4.59), each of those terms is a linear combination of the following two linearly independent harmonics:

$$\Lambda_1 = *(\vartheta \wedge \omega) \qquad \text{and} \qquad \Lambda_2 = d(\vartheta \wedge \omega)\,. \tag{4.60}$$

Specifically, we find

$$d\alpha = \lambda_\omega\, t_2\, \Lambda_1 + (t_1 - 2t_2)\, \Lambda_2\,, \tag{4.61}$$

$$*_c\alpha = \frac{1}{c}\big(t_1 + 2\, t_2\, (c^2 - 1)\big)\, \Lambda_1 + c\, t_2\, \Lambda_2\,, \tag{4.62}$$

$$d *_c d\alpha = \frac{\lambda_\omega}{c}(t_1 - 2t_2)\, \Lambda_1 + c\,(\lambda_\omega t_2 - 2(t_1 - 2t_2))\, \Lambda_2\,. \tag{4.63}$$

[9] In the following, we denote this harmonic simply by α.

The first identity follows from

$$
\begin{aligned}
d * d(\vartheta \wedge \omega) &= d * (2J \wedge \omega) - d * (\vartheta \wedge d\omega) \\
&= -2\, d(\vartheta \wedge \omega) + d \bullet d\omega \\
&= -2\, d(\vartheta \wedge \omega) + \lambda_\omega \bullet \omega \\
&= -2\, d(\vartheta \wedge \omega) + \lambda_\omega *(\vartheta \wedge \omega)\,.
\end{aligned}
\tag{4.64}
$$

where we used (3.8), (4.15), (4.22) and (4.29). The second one is an immediate consequence of (4.14), (3.8) and (4.22). For the third one, we have

$$
\begin{aligned}
d *_c d\, \alpha &= t_2 \lambda_\omega d \;*_c *(\vartheta \wedge \omega) + (t_1 - 2t_2)\, d *_c d(\vartheta \wedge \omega) \\
&= c\, \lambda_\omega\, t_2\, d(\vartheta \wedge \omega) + (t_1 - 2t_2) \frac{1}{c} \left[d * d(\vartheta \wedge \omega) + 2(c^2 - 1)\, d(\vartheta \wedge \omega) \right] \\
&= \frac{\lambda_\omega}{c} (t_1 - 2t_2) *(\vartheta \wedge \omega) - c\, \left[2(t_1 - 2t_2) - \lambda_\omega t_2 \right]\, d(\vartheta \wedge \omega)\,.
\end{aligned}
\tag{4.65}
$$

In evaluating the contribution from the metric fluctuation to the linearized Maxwell equations (4.12) we also need the indentity

$$
h \cdot \Phi_{(4)} = -128\, i\, \vartheta \wedge d\omega\,. \tag{4.66}
$$

To prove it, we note that by the second identity in (3.8),

$$
\Phi_{(4)} = 4i\vartheta \wedge (\Omega - \overline{\Omega})\,. \tag{4.67}
$$

Since h_{ab} is horizontal,

$$
h \cdot \Phi_{(4)} = -i(h \cdot \Omega) \wedge \vartheta + i(h \cdot \overline{\Omega}) \wedge \vartheta\,, \tag{4.68}
$$

where $(h \cdot \Omega)_{abc} = 3 h_{d[a} \Omega_{bc]}{}^d$. Using the definition (4.31) and the identity (C.8), we find that only $\overline{\Omega}$ terms in h_{ab} contribute to the contraction $h \cdot \Omega$. The three terms in that contraction are then evaluated using (C.9) and (C.11). The result is given in (4.18), but now we have shown that it holds on any SE manifold. Including the conjugate terms yields (4.66).

Finally,

$$d *_c (\vartheta \wedge d\omega) = -\frac{\lambda_\omega}{c} *(\vartheta \wedge \omega) = -\frac{\lambda_\omega}{c} \Lambda_1 \,. \tag{4.69}$$

This proves that all terms in (4.12) are linear combinations of the two basis harmonics (4.60).

It also follows from (4.69) that $d\Lambda_1 = 0$. Since $d\Lambda_2 = 0$ as well, we have $d *_c \alpha = 0$ as required by (4.11). The other equation in (4.11) is satisfied automatically.

We must also evaluate the linearized energy momentum tensor (4.6). To this end we note that the two basis harmonics (4.60), using (3.8) and (4.22), can be written as

$$\Lambda_1 = -J \wedge \omega \,, \qquad \Lambda_2 = -2\Lambda_1 - \vartheta \wedge d\omega \,. \tag{4.70}$$

Hence $d\alpha$ in (4.61) is a linear combination of a horizontal $(2,2)$-form $J \wedge \omega$ and a mixed form $\vartheta \wedge d\omega$. Given (4.67), the contraction in (4.6) with $J \wedge \omega$ must vanish. Similarly, the only nonvanishing terms in the contraction with the second form are those in which the free indices are along the base and the two ϑ's are contracted. This gives

$$g^{cf} g^{dg} g^{eh} (\vartheta \wedge \omega)_{acde} \Phi_{fgh} = \frac{12\, i}{c^2} (d\omega)_{ade} (\Omega b^{de} - \overline{\Omega}_b{}^{de}) \,, \tag{4.71}$$

where the indices on the right hand side are raised with the round metric. The full expansion of (4.6) is then

$$\delta \mathcal{T}_{ab} = -\frac{4\, i}{a^6 c^2} f_i \, (t_1 - 2t_2) \varphi(x) \, h_{ab} \,, \tag{4.72}$$

and is indeed proportional to the metric harmonic.

4.6 The masses

For a scalar field, $\varphi(x)$, satisfying (1.4) with mass, m, and the metric and flux harmonics as above, the linearized Einstein equations (4.4)-(4.6) become diagonal,

$$-\frac{1}{2}\left[m^2-\frac{1}{a^2}(\lambda_\omega+4)+\frac{24}{L^2}+\frac{4}{a^2}\left(4c^2-\frac{4}{c^2}-21\right)\right]h_{ab}=-4i\,\frac{f_i}{a^6\,c^2}(t_1-2\,t_2)\,h_{ab}\,. \quad (4.73)$$

To evaluate the left hand side, we have used (4.32) and $\mathcal{L}_\xi^2 h_{ab}=-16h_{ab}$. The latter follows from the observation that the R-charge of the metric harmonic is $q=4$ and is the same as of the background flux. The contraction in the fluctuation of the energy momentum tensor on the right hand side has been evaluated in (4.72).

The linearized Maxwell equation (4.12) can be simplified using (4.61)-(4.63). After projecting onto the basis harmonics, Λ_1 and Λ_2, it yields two equations

$$\begin{aligned}\frac{1}{c}\left(m^2-\frac{\lambda_\omega}{a^2}\right)t_1+\left[\frac{2}{c}(c^2-1)m^2+2\,\lambda_\omega\left(\frac{1}{a^2c}+\frac{f_0}{a}\right)\right]\,t_2&=-128\,i\,\frac{f_i}{a^4}\,\frac{\lambda_\omega}{c}\,,\\ 2\left(\frac{c}{a^2}+\frac{f_0}{a}\right)t_1+\left[c\left(m^2-\frac{\lambda_\omega}{a^2}-\frac{4}{a^2}\right)-4\,\frac{f_0}{a}\right]\,t_2&=0\,.\end{aligned} \quad (4.74)$$

For the FR solutions (3.5) there is no internal flux, $f_i=0$, and the Einstein and Maxwell equations decouple. From the first one we get the same mass,

$$m_1^2L^2=\frac{\lambda_\omega}{4}-2, \quad (4.75)$$

for both the supersymmetric and skew-whiffed solution. The other two masses in (1.8) and (1.9) are then obtained by setting the determinant of the homogeneous system of equations (4.74) for t_1 and t_2 to zero. This yields a quadratic equation for m^2, whose solutions are either

$$m_2^2L^2=\frac{\lambda_\omega}{4}+\sqrt{\lambda_\omega+1}-1\,,\qquad m_3^2L^2=\frac{\lambda_\omega}{4}-\sqrt{\lambda_\omega+1}-1\,, \quad (4.76)$$

for the supersymmetric or

$$m_2^2 L^2 = \frac{\lambda_\omega}{4} + 2\sqrt{\lambda_\omega + 1} + 2\,, \qquad m_3^2 L^2 = \frac{\lambda_\omega}{4} - 2\sqrt{\lambda_\omega + 1} + 2\,, \tag{4.77}$$

for the skew-whiffed solutions, respectively.

For the PW solution, all three equations are coupled by the non-vanishing internal flux. Solving (4.74) for t_1 and t_2 and plaguing into (4.73) yields a cubic equation for m^2, whose solutions are

$$m_1^2 L^2 = \frac{3}{8}\lambda_\omega\,, \qquad m_2^2 L^2 = \frac{3}{8}\lambda_\omega + 3\sqrt{1 + \lambda_\omega} + 3\,, \qquad m_3^2 L^2 = \frac{3}{8}\lambda_\omega - 3\sqrt{1 + \lambda_\omega} + 3\,. \tag{4.78}$$

For each of the masses there is a fluctuation of the metric and the flux that together diagonalize the linearized equations of motion around the PW solution. As we have already discussed in section 1.3, the last mass will violate the BF bound when λ_ω lies in the range (1.11). One may note that the masses m_2^2 and m_3^2 for the PW solution are simply 3/2 of the masses for the flux modes in the skew-whiffed FR solution.

4.7 Additional bosonic modes in the Z multiplet

In addition to the three scalar fields in the Z-vector multiplet, there are two additional scalar fields that are associated with symmetric tensor harmonics, as seen in Table 2 below. It is reasonable to ask what happens to these two scalar fields. In particular it would be nice to know what the masses of these fields are at the FR and PW points.

In order to determine the masses of these two scalars, one needs to know the symmetric tensor harmonics associated with them, and their eigenvalues under the Lichnerowitz operator. Since these two fields lie in the same supermultiplet as fields whose associated harmonics we know, it is in fact possible to construct their associated harmonics.

Key results useful for carrying out such a construction are provided in the paper [56]. In [56] the authors provide a formula that gives a spinor-vector harmonic in terms of a 3-form harmonic and a Killing spinor. In table I of [56] they give

$$\Xi_\alpha = a\tau_{\alpha\mu\nu\rho}\eta Y_{\mu\nu\rho} + b\tau_{\mu\nu}\eta Y_{\alpha\mu\nu} + c\tau_{\mu\nu\rho}\eta\mathcal{D}_\alpha Y_{\mu\nu\rho}, \tag{4.79}$$

where Ξ_α is a spinor-vector harmonic, the τ are Dirac matrices, η is a Killing spinor, $Y_{\mu\nu\rho}$ is a 3-form harmonic, and a, b, and c are given constants. The relation between the eigenvalue of the spinor-vector under the Rarita-Schwinger operator and the eigenvalue of the 3-form under the 'square root of the Hodge-de Rham operator' is given to be

$$M_{(3/2)(1/2)^2} = -4(M_{(1)^3} + 1), \tag{4.80}$$

where $M_{(3/2)(1/2)^2}$ is the eigenvalue of the spinor-vector, and $M_{(1)^3}$ is the eigenvalue of the 3-form.

In the same table, the authors provide a formula that gives a symmetric tensor harmonic in terms of a spinor-vector harmonic. It is given by

$$Y_{(\alpha\beta)} = a\bar{\eta}\tau_{\{\alpha}\Xi_{\beta\}} + b\bar{\eta}\mathcal{D}_{\{\alpha}\Xi_{\beta\}}, \tag{4.81}$$

where $Y_{(\alpha\beta)}$ is a symmetric tensor harmonic. The relation between the eigenvalue of the symmetric tensor under a Lichnerowicz-like operator and the eigenvalue of the spinor-vector harmonic is given by

$$M_{(2)(0)^2} = (M_{(3/2)(1/2)^2} + 4)(M_{(3/2)(1/2)^2} + 8). \tag{4.82}$$

In general, one can see that using these formulas to obtain harmonics from known harmonics will give objects that will be unwieldy to deal with. However, in the special case where the internal manifold is tri-Sasaki and the known harmonics are, as in Chapter 2, constructed only in terms of Killing spinors and Dirac matrices, it is expected that the resulting objects will be easier to deal with.

In particular, in Chapter 2 we constructed two 3-form harmonics at the FR point, $H_1 = X - \frac{2}{3}Y$ and $H_2 = X + Y + 5Z$. Let H denote either of these 3-form harmonics. Then using the formula from [56], one obtains the spinor-vector harmonics

$$\begin{aligned} \Xi_r &= \Gamma_{rmnp}\zeta H_{mnp} \\ \Pi_r &= \Gamma_{rmnp}\psi H_{mnp}. \end{aligned} \tag{4.83}$$

One finds, at least for the tri-Sasaki case, that in the formula provided in [56], all three terms are proportional to each other, and so it is sufficient to keep only the first term.

In turn, one can use the formula given by [56] to obtain symmetric tensor harmonics from these spinor-vectors. One obtains the symmetric tensor harmonics

$$\begin{aligned} h^{(1)}_{(mn)} &= \bar{\zeta}\Gamma_{\{m}\Xi_{n\}} - \bar{\psi}\Gamma_{\{m}\Pi_{n\}} \\ h^{(2)}_{(mn)} &= \bar{\zeta}\Gamma_{\{m}\Pi_{n\}} \\ h^{(3)}_{(mn)} &= \bar{\zeta}\Gamma_{\{m}\Xi_{n\}} + \bar{\psi}\Gamma_{\{m}\Pi_{n\}} \end{aligned} \tag{4.84}$$

As in the formula for the spinor-vectors, one finds that the derivative terms are proportional to the non-derivative terms, and so can be dropped.

The constructed symmetric tensor $h^{(1)}$ is actually proportional to the symmetric tensor harmonic that we have already constructed in terms of the $(1,1)$-form ω and the 3-form Ω.

In fact, the other two symmetric tensors $h^{(2)}$ and $h^{(3)}$ are actually proportional to symmetric tensor harmonics that are simply constructed from ω and J or Ω. It is found that, up to constants,

$$\begin{aligned} h^{(1)}_{(ab)} &= (d\omega)_a{}^{cd}(\Omega_{bcd} - \bar{\Omega}_{bcd}) + (a \leftrightarrow b) \\ h^{(2)}_{(ab)} &= (d\omega)_a{}^{cd}(\Omega_{bcd} + \bar{\Omega}_{bcd}) + (a \leftrightarrow b) \\ h^{(3)}_{(ab)} &= \omega_{c(a} J^c_{b)}. \end{aligned} \tag{4.85}$$

Even though these relations were found in the particular case where the internal manifold is tri-Sasaki, it would not be surprising if these relations were true in general. That is, it should be expected that, in general, the symmetric tensors $h^{(2)}$ and $h^{(3)}$ given above are indeed the symmetric tensor harmonics that lie in the same supermultiplet as $h^{(1)}$.

To verify this expectation, one can first note that the $U(1)_R$ charges are in agreement with what is given in table 2. In addition, one can show that the tensors are transverse, and that they are eigenfunctions of the Lichnerowicz operator, with the correct eigenvalues, i.e. $\lambda_\omega + 4$. Since $h^{(2)}$ is the same as $h^{(1)}$ except for a sign, the calculations for $h^{(2)}$ are the same as those for $h^{(1)}$, which have already been done. Hence, we need only do the calculations for $h^{(3)}$.

First, we show $h^{(3)}$ is transverse.

$$D^a W_{ab} = \frac{1}{2} D^a(\omega_{ca} J_b{}^c + \omega_{cb} J_a{}^c) \tag{4.86}$$

The first term on the right-hand-side is

$$D^a(\omega_{ca} J_b{}^c) = \omega_{ca} D^a J_b{}^c = \omega_{ca}(-\delta^a{}_b \vartheta^c + \mathring{g}^{ac}\vartheta_b) = -\omega_{cb}\vartheta^c + \omega_{ca}\mathring{g}^{ac}\vartheta_b = 0. \tag{4.87}$$

The first equality follows from transversality of ω, the second equality follows from (B.12), and the third equality follows from the fact that ω is horizontal and traceless.

The second term on the right-hand-side is

$$D^a(\omega_{cb}J_a{}^c) = D^a(J_c{}^e J_b{}^f \omega_{ef} J_a{}^c) = -D^a(\delta_a{}^e J_b{}^f \omega_{ef}) = D^a(\omega_{fa} J_b{}^f) = 0. \tag{4.88}$$

The last equality follows from the calculation for the first term on the right-hand-side. Therefore, $h^{(3)}$ is indeed transverse.

Now we want to obtain the eigenvalue of $h^{(3)}$ under the Lichnerowicz operator. This calculation follows in similar fashion to the same calculation for $h^{(2)}$.

Letting $t_{abcd} = \omega_{ab}J_{cd}$, so that $h^{(3)}$ is obtained by the appropriate contraction and symmetrization, the action of the Lichnerowitz operator on this tensor is given by

$$(\Delta_L t)_{abcd} = (\Delta\omega)J_{cd} + \omega_{ab}(\Delta J)_{cd} - 2(D^e\omega_{ab})(D_e J_{cd}) + \text{R-terms}. \tag{4.89}$$

One finds that the R-terms are given by

$$\text{R-terms} = 2(\omega_{eb}[D^e, D_a] + \omega_{ae}[D^e, D_b])J_{cd}. \tag{4.90}$$

Computing the second derivatives of J, one finds

$$[D^e, D_a]J_{cd} = 2J^e{}_{[c}\mathring{g}_{d]a} - 2J_{a[c}\delta_{d]}{}^e. \tag{4.91}$$

Plugging this into the expression for the R-terms gives

$$\text{R-terms} = 4(\omega_{eb}J^e{}_{[c}\mathring{g}_{d]a} - J_{a[c}\omega_{d]b}) - (a \leftrightarrow b). \tag{4.92}$$

Carrying out the appropriate contraction and symmetrization to obtain the symmetric tensor $h^{(3)}$ from t gives that the contribution to $\Delta_L h^{(3)}$ from the R-terms is $-6h^{(3)}$.

Next, we want to get the contribution to $\Delta_L h^{(3)}$ from the term $-2(D^e\omega_{ab})(D_e J_{cd})$. Using (B.12) gives

$$(D^e\omega_{ab})(D_e J_{cd}) = -(D_c\omega_{ab})\vartheta_d + (D_d\omega_{ab})\vartheta_c \tag{4.93}$$

In contracting the indices a and d, one uses the fact that ω is transverse and horizontal, metric compatibilty, and (B.12) to find that

$$(D^e\omega_{ab})(D_e J_c{}^a) = \omega_{ab} J_c{}^a. \tag{4.94}$$

So symmetrizing the indices to obtain $h^{(3)}$ from t gives that the contribution to $\Delta_L h^{(3)}$ from the term $-2(D^e\omega_{ab})(D_e J_{cd})$ is $-2h^{(3)}$.

To obtain the contribution to $\Delta_L h^{(3)}$ from the term $\omega_{ab}(\Delta J)_{cd}$, one simply needs to know the eigenvalue of J under the Hodge-de Rham Laplacian. The Hodge-de Rham Laplacian on J is given by $\Delta J = d\delta J$. One uses (B.12) to find that

$$(\delta J)_m = -D^l J_{lm} = 6\vartheta_m, \tag{4.95}$$

so that $\Delta J = 6d\vartheta = 12J$. Hence, the contribution to $\Delta_L h^{(3)}$ from the term $\omega_{ab}(\Delta J)_{cd}$ is $12h^{(3)}$.

Adding up the contributions from all the terms gives

$$\Delta_L h^{(3)} = (\lambda_\omega + 12 - 2 - 6)h^{(3)} = (\lambda_\omega + 4)h^{(3)}, \tag{4.96}$$

as expected.

The symmetric tensor harmonics $h^{(2)}$ and $h^{(3)}$ are associated with AdS_4 scalars $\phi^{(2)}$ and $\phi^{(3)}$, so that the eleven-dimensional metric fluctuations are $\phi^{(2)}h^{(2)}_{ab}$ and $\phi^{(3)}h^{(3)}_{ab}$. The

2-form harmonic ω is associated with an AdS_4 vector field Z, so that the appropriate eleven-dimensional object is the 3-form fluctuation $Z_\mu \omega_{mn}$. We would like to plug the metric fluctuations $\phi^{(2)} h^{(2)}_{ab}$ and $\phi^{(3)} h^{(3)}_{ab}$ and the 3-form fluctuation $Z_\mu \omega_{mn}$ into the linearized field equations to obtain the masses of the scalars $\phi^{(2)}$ and $\phi^{(3)}$ at the PW point.

Plugging the fluctuations into the linearized Einstein equation

$$\begin{aligned}\frac{1}{2}\hat{\Delta} h_{AB} + \breve{\nabla}_{(A}\breve{\nabla}^C h_{B)C} - \frac{1}{2}\breve{\nabla}_A \breve{\nabla}_B h^C_C &= -F_A{}^{CNP} F_B{}^M{}_{NP} h_{MC} - \frac{1}{36} h_{AB} F_{CDEF} F^{CDEF} \\ &\quad + \frac{1}{9} g_{AB} h^{CM} F_{CDEF} F_M^{DEF} + \frac{2}{3} F_{(A}{}^{MNP} f_{B)MNP} \\ &\quad - \frac{1}{18} g_{AB} F^{MNPQ} f_{MNPQ} \qquad (4.97)\end{aligned}$$

gives the two equations

$$\begin{aligned}\frac{1}{2}\hat{\Delta} h^{(2)}_{AB} &= -\frac{1}{36} h^{(2)}_{AB} F_{CDEF} F^{CDEF} \\ \frac{1}{2}\hat{\Delta} h^{(3)}_{AB} &= -F_A{}^{CNP} F_B{}^M{}_{NP} h^{(3)}_{MC} - \frac{1}{36} h^{(3)}_{AB} F_{CDEF} F^{CDEF}, \qquad (4.98)\end{aligned}$$

one for each of the metric fluctuations. The first term on the right-hand-side of the equation for $h^{(3)}$ is 0 for $h^{(2)}$ because $h^{(2)}$ has terms of type $(2,0)$ and $(0,2)$, whereas $h^{(3)}$ is of type $(1,1)$.

From the first equation one finds that at the FR point the scalar $\phi^{(2)}$ has mass $m^2 L^2 = \frac{1}{4}(\lambda_\omega - 8)$, as expected, and at the PW point it has mass $\frac{3}{8}\lambda_\omega$. From the second equation one finds that at the FR point the scalar $\phi^{(3)}$ has mass $m^2 L^2 = \frac{1}{4}(\lambda_\omega - 8)$, as expected, and at the PW point it has mass $\frac{3}{8}(\lambda_\omega - 8)$. Note that the masses of the scalars are larger at the PW point than at the FR point, so their values are stable.

The linearized Maxwell equation is

$$\breve{\nabla}_A f^{ABCD} + 4\breve{\nabla}_A(F^{M[ABC}h^{D]}{}_M) - \frac{1}{2}F^{BCDR}\breve{\nabla}_R h_A{}^A = -\frac{1}{288}\epsilon^{BCDEFGHIJKL}F_{EFGH}f_{IJKL}$$
$$-\frac{1}{1152}Tr(g^{-1}h)\epsilon^{BCDEFGHIJKL}F_{EFGH}F_{IJKL} \quad (4.99)$$

Plugging the fluctuations into the linearized Maxwell equations, one finds that only the first two terms on the left-hand-side are non-zero. The first term on the left-hand-side yields the free massive vector field equation for Z. As expected, it gives that the mass of Z is given by the eigenvalue of ω under the Hodge-de Rham operator. The first and second terms together give that the divergence of Z, $D_\mu Z^\mu$ is 0 at the FR point but proportional to the scalar $\phi^{(2)}$ away from it.

5 Examples

In this chapter we will construct explicitly the $(1,1)$-form(s), ω, leading to an instability of the PW solutions for two classes of SE manifolds: the tri-Sasakian manifolds and the homogeneous manifolds (1.3). Throughout this section we take ω to be real. The unstable modes in chapter 4 are then constructed using the form $i\,\omega$.

5.1 Tri-Sasakian manifolds

The eleven-dimensional supergravity admits a consistent truncation on an arbitrary tri-Sasakian manifold to a $\mathcal{N} = 3$, $d = 4$ gauged supergravity [34]. As shown in [34], the instability of the PW solution follows then from the existence of a single scalar mode with the mass $m^2 = -3$ in the spectrum of fluctuations around the corresponding critical point of the scalar potential.

Starting with that unstable scalar mode in the four-dimensional theory, one can follow the truncation and reconstruct the unstable mode in eleven-dimensions. However, it is simpler to look directly for a $(1,1)$-form, ω, in terms of the geometric data on a tri-Sasakian manifold.

A tri-Sasakian manifold admits three globally defined orthonormal Killing spinors, η^i, in terms of which the three one-forms, K^i, dual to the SU(2) Killing vectors, are given by

$$K^i_a = \frac{i}{2}\epsilon^{ijk}\,\bar{\eta}^j\Gamma_a\eta^k\,. \tag{5.1}$$

Define

$$M^i = -\frac{1}{2}dK^i\,, \qquad M^i_{ab} = -\frac{1}{2}\epsilon^{ijk}\,\bar{\eta}^j\Gamma_{ab}\eta^k\,. \tag{5.2}$$

The forms K^i and M^i satisfy [65]

$$\mathring{D}_a K^i_b = -M^i_{ab}\,, \tag{5.3}$$

$$\mathring{D}_a M^i_{bc} = 2\mathring{g}_{a[b}K^i_{c]}\,, \tag{5.4}$$

$$K^i_a K^{ja} = \delta^{ij}\,, \tag{5.5}$$

$$M^i_{ab}K^{jb} = \epsilon^{ijk}K^k_a\,, \tag{5.6}$$

$$M^i_{ac}M^{jc}{}_b = K^i_a K^j_b - \delta^{ij}\mathring{g}_{ab} + \epsilon^{ijk}M^k_{ab}\,. \tag{5.7}$$

Using those identities we show that the two-forms

$$\omega^i = \frac{1}{2}\,\epsilon^{ijk}\,K^j\wedge K^k + \frac{1}{3}\,M^i\,, \tag{5.8}$$

are transverse eigenforms of the Hodge-de Rham Laplacian,

$$\Delta\omega^i = 24\,\omega^i\,. \tag{5.9}$$

Indeed, the transversality follows directly from (5.3)-(5.6), which imply that

$$\mathring{D}^a M^i_{ab} = 6K^i_b\,, \qquad \mathring{D}^a (K^j_{[a} K^k_{b]}\,) = -\epsilon^{jki} K^i_b\,. \tag{5.10}$$

To prove (5.9), we note that on a transverse form, ω^i,

$$\Delta \omega^i_{ab} = -\mathring{D}^c (d\omega^i)_{abc}\,, \tag{5.11}$$

where

$$d\omega^i = -2\epsilon^{ijk} M^j \wedge K^k\,. \tag{5.12}$$

The divergence in (5.11) is then evaluated by first using (5.3) and (5.4) and then simplifying the resulting contractions using (5.5)-(5.7).

The PW solution is now obtained by choosing any two orthonormal Killing spinors that fix a particular SE structure. Given the SU(2) isometry, we may simply take $(\eta^\alpha) = (\eta^1, \eta^2)$ and set $\chi = \eta^3$ to be the additional Killing spinor. Then $\vartheta = K^3$ and $J = -M^3$, see (3.12). Consider the two form

$$\omega = K^1 \wedge K^2 - \frac{1}{3}\, J\,, \tag{5.13}$$

with components

$$\omega_{ab} = -2(\bar\eta^1 \Gamma_{[a}\chi)(\bar\eta^2 \Gamma_{b]}\chi) - \frac{1}{3}\bar\eta^1 \Gamma_{ab}\eta^2\,. \tag{5.14}$$

It follows from (5.5) and (5.6) that ω is horizontal. Similarly, the form

$$d(K^1 \wedge K^2) = -2(M^1 \wedge K^2 - K^1 \wedge M^2)\,, \tag{5.15}$$

is horizontal, so that ω is in fact basic. Finally, by contracting with J, we check that ω is a primitive (1,1)-form.

We have checked that the unstable mode arising from ω in (5.13) reproduces precisely the unstable mode in the truncation in [34]. We also note that a more complete construction and classification of harmonics on $N^{1,1}$ in terms of Killing spinors, including the forms above, can be found in [66].

While the construction above gives an unstable mode on any tri-Sasakian manifold, there will be additional modes if the manifold admits more than three Killing spinors.[10] In particular, to construct the unstable modes on S^7 found in [32], we can generalize the foregoing as follows. Let χ^j be the additional six Killing spinors and let

$$K_a^{\alpha j} = i\,\bar{\eta}^\alpha \Gamma_a \chi^j\,, \qquad \alpha = 1, 2\,, \quad j = 1, \ldots, 6\,. \tag{5.16}$$

Then

$$\omega^{ij} = \frac{1}{2}(K^{1i} \wedge K^{2j} + K^{1j} \wedge K^{2i}) - \frac{1}{3} J\, \delta^{ij}\,, \tag{5.17}$$

are symmetric, $\omega^{ij} = \omega^{ji}$, and traceless, $\omega^{ij}\delta_{ij} = 0$, and transform in $\mathbf{20'}$ of SU(4), which is the isometry of the KE base, $\mathbb{CP}^3$. In the same way as above, one checks that ω^{ij} are basic (1,1)-forms and that the diagonal forms, ω^{jj}, are transverse and $\Delta\omega^{jj} = 24\omega^{jj}$. By the **SU(4)** symmetry, the same holds for the remaining forms.

5.2 Homogeneous Sasaki-Einstein manifolds

The homogeneous SE manifolds (1.3) are given by G/H coset spaces, which is a convenient realization for a calculation of the KK spectrum of the corresponding $AdS_4 \times M_7$ compactification of the eleven dimensional supergravity. However, one can also realize any

[10] In fact, the only regular manifold with more than three Kiling spinors is S^7 [23].

Spin	Field	Energy	$\mathrm{U}(1)_R$	m^2L^2
1	Z	E_0+1	q	$4E_0(E_0-1)$
0	π	E_0+2	q	$(E_0+2)(E_0-1)$
0	ϕ	E_0+1	$q+4$	$(E_0+1)(E_0-2)$
0	ϕ	E_0+1	q	$(E_0+1)(E_0-2)$
0	ϕ	E_0+1	$q-4$	$(E_0+1)(E_0-2)$
0	π	E_0	q	$E_0(E_0-3)$

Table 2: The bosonic sector of a Z-vector multiplet.

homogeneous SE manifold as a hypersurface in some $\mathbb{C}^N$, in some cases modded out by a continous Abelian symmetry. This has been discussed in detail in [53, 54, 55, 46, 52, 67]. In this section we use the latter construction to find explicitly stability violating $(1,1)$-forms, ω, on each of the spaces (1.3). An advantage of this method is that the required properties of ω are either manifest or easy to verify.

In principle, one could try to identify $(1,1)$-forms leading to instabilities of PW solutions by examining the KK spectra that have been studied for all homogeneous SE manifolds in references in Table 1. Indeed, in the KK reduction of the three-form potential, $\mathcal{A}_{(3)}$, a transverse two-form harmonic gives rise to a vector field whose mass is given by the eigenvalue of the Hodge-de Rham Laplacian [68, 39]. In the terminology of [56], the vector field is called the Z-vector field and it is present in the KK towers of the following $\mathcal{N}=2$ supermultiplets [57, 50]: the long and/or semi-long graviton multiplet, the two long and/or semi-long gravitino multiplets and the Z-vector multiplet.

However, the mere presence in the KK spectrum of a two-form harmonic, ω, whose mass, λ_ω, lies in the instability range (1.11), is not yet sufficient to conclude that the PW solution is unstable. One must also show that ω is a transverse, primitive, basic, $(1,1)$-form, which is by no means obvious. For that reason, we first construct explicitly stability violating $(1,1)$-forms, ω, and then check whether both λ_ω and the supersymmetric FR scalar masses (4.75) and (4.76) agree with the known KK spectra, in particular, whether the corresponding fields: the Z-vector field, the scalar and the two pseudo-scalar fields lie in a long Z-vector multiplet. The comparison works perfectly for S^7, $N^{1,1}$ and $M^{3,2}$, but reveals missing multiplets in the published KK spectra for $Q^{1,1,1}$ and $V^{5,2}$.

The bosonic fields of a long Z-multiplet are listed in Table 2 , with the R-charge in the second column and the masses in the last column given in the conventions used in this paper. Specifically, the R-charge is twice the charge in the original tables in the KK literature (see, e.g., Table 3 in [50]). We define the mass of a Z-vector as the eigenvalue of the corresponding Hodge-de Rham operator. This agrees with the usual definition used in the references in Table 1, except that our normalization of the metric for the FR solution introduces a factor of four difference,

$$m_Z^2 L^2 = \frac{1}{4}\frac{M_Z^2}{e^2}\,. \tag{5.18}$$

The masses of the scalar fields are related by

$$m_{\phi,\pi}^2 L^2 = \frac{1}{16}\Big(\frac{M_{\phi,\pi}^2}{e^2} - 32\Big)\,, \tag{5.19}$$

where $e^2 = 1/(16L^2)$ is usually set to one.

5.2.1 S^7

We represent S^7 as the unit sphere in $\mathbb{C}^4$,

$$|u^1|^2 + \ldots + |u^4|^2 = 1\,. \tag{5.20}$$

The $\mathrm{U}(1)_R$ symmetry is the rotation by the phase. Let $\Phi_{ij\bar{k}\bar{l}}$ be a constant complex tensor in $\mathbb{C}^4$ that is antisymmetric in $[ij]$ and $[\bar{k}\bar{l}]$, primitive with respect to the canonical complex structure in $\mathbb{C}^4$, and satisfies the reality condition $\Phi_{ij\bar{k}\bar{l}} = -(\Phi_{kl\bar{i}\bar{j}})^*$. Then the pull-back onto S^7 of

$$\omega = \Phi_{ij\bar{k}\bar{l}}\, u^i \bar{u}^{\bar{k}}\, du^j \wedge d\bar{u}^{\bar{l}}\,, \tag{5.21}$$

yields 20 basic $(1,1)$-forms with $\lambda_\omega = 24$, which give rise to the unstable modes obtained in [32].

Our calculation agrees with the general result for the spectrum of the Hodge-de Rham Laplacian on $\mathbb{CP}^3$ [43], conveniently summarized in Table 2 in [58]. There we find that there is a single tower of $(1,1)$-forms in $[k,2,k]$ irrep of $\mathrm{SU}(4)$ with the eigenvalues

$$\lambda_{(1,1)} = 4(k+2)(k+3)\,, \qquad k = 0,1,2,\ldots\,. \tag{5.22}$$

The forms (5.21) lie at the bottom of the tower with $k = 0$. The higher level forms with $k \geq 1$ have $\lambda_{(1,1)} \geq 48$ and thus lie outside the instability bound (1.11).

One may note that those forms are not the lowest lying transverse two-forms on S^7. Indeed, the spectrum of the Laplacian on two-forms on S^7 is [44]

$$\lambda_{(2)} = (p+2)(p+4)\,, \qquad p = 1,2,3,\ldots\,, \tag{5.23}$$

of which (5.22) is a subset. For $p = 1$ and 2, the eigenvalues are 15 and 24, respectively, and satisfy (1.11). However, the two-forms with $\lambda_{(2)} = 15$ are trilinear in u^i and $\bar{u}^i$, hence have a nonzero R-charge and are not basic.

5.2.2 $N^{1,1}$

The (hyper-)Kähler quotient construction for $N^{1,1}$ [46, 67] starts with $\mathbb{C}^3 \oplus \overline{\mathbb{C}}^3$ with coordinates (u^j, v_j), $j = 1, 2, 3$, that transform as **3** and $\bar{\mathbf{3}}$ under SU(3), respectively, and with $(u^j, -\bar{v}^j)$ transforming as doublets under SU(2). The $N^{1,1}$ manifold is then the surface

$$|u^j|^2 = |v_j|^2 = 1\,, \qquad u^j v_j = 0\,, \tag{5.24}$$

modded by the U(1) action $(u^i, v_i) \sim (e^{i\delta} u^i, e^{-i\delta} v_i)$. The standard metric [69, 70] is obtained by a reduction from the flat metric in $\mathbb{C}^6$. We refer the reader to [67] for a detailed discussion of the metrics and for explicit angular coordinates.

The three Killing forms in section 5.1 can be taken as

$$K^1 = \frac{1}{2}(u^j dv_j + \bar{u}_j d\bar{v}^j)\,, \qquad K^2 = -\frac{i}{2}(u^j dv_j - \bar{u}_j d\bar{v}^j)\,, \qquad K^3 = \frac{i}{2}(u^j d\bar{u}_j + v_j d\bar{v}^j)\,, \tag{5.25}$$

in terms of which the form, ω, is given by (5.13). It is now manifest that ω is a $(1,1)$-form, which is invariant under the U(1) action of the K"ahler quotient, and hence a well-defined form on $N^{1,1}$. It is also a singlet of SU(3) and is invariant under the $\mathrm{U}(1)_R \subset \mathrm{SU}(2)$ isometry, $(u^j, v_j) \to (e^{i\psi} u^j, e^{i\psi} v_j)$, along the SE fiber. Evaluating it in angular coordinates, we verify that it is basic and primitive.

The complete KK spectrum on this space was obtained in [45] (see, also [71, 46, 47, 66]), where one finds 21 towers of two-form harmonics. Specifying to the **(1,3)** irreducible representation of SU(3) × SU(2), $M_1 = M_2 = 0$ and $J = 1$ in the notation in [45], leaves two possible eigenvalues $\lambda^{(2)}_{12} = 96$ and $\lambda^{(2)}_{21} = 48$ lying in the series E_8 with $j = 0$. The first three forms are the ones constructed above in (5.8), one of which is the sought after (1, 1)-form, ω, with $\lambda_\omega = 24$. The remaining three are the three canonical two-forms, M^i, on the tri-Sasakian manifold, one of which is the complex structure, and hence is not primitive, while the other two are not basic.

In this example the $\mathcal{N} = 2$ long Z-vector multiplet is a part of a long $\mathcal{N} = 3$ gravitino supermultiplet, see Table 4 in [66]. Following [56], all harmonics in this multiplet can be constructed in terms of the three Killing spinors on $N^{1,1}$ [66].

5.2.3 $M^{3,2}$

The $\mathcal{N} = 2$ supersymmetry of the FR solution on $M^{3,2}$ was proved in [72]. The complete Kaluza-Klein spectrum was obtained in [48] (see also [49]) and further analyzed more recently in [50]. The KE base of $M^{3,2}$ is $\mathbb{CP}^2 \times \mathbb{CP}^1$ and the SE metric in the form (2.3) is given by [73, 74]

$$ds^2 = \frac{3}{4} ds^2_{\mathbb{CP}^2} + \frac{1}{2} ds^2_{\mathbb{CP}^1} + (d\psi + \frac{3}{4} A_{\mathbb{CP}^2} + \frac{1}{2} A_{\mathbb{CP}^1})^2, \tag{5.26}$$

where the $ds^2_{\mathbb{CP}^k}$ is the Fubini-Study metric and $A_{\mathbb{CP}^k}$ is the Kähler potential with $dA_{\mathbb{CP}^k} = 2J_{\mathbb{CP}^k}$.

The Kähler quotient construction for this SE manifold is explained in Appendix H. See also the references [54, 55]. The construction starts with $\mathbb{C}^3 \oplus \mathbb{C}^2$ with coordinates, u^i and v^α. In terms of these coordinates, $M^{3,2}$ is the surface defined by the equations

$$2u^j \bar{u}_j = 3v^\alpha \bar{v}_\alpha = 1, \tag{5.27}$$

modded by the $U(1)$ symmetry, $(u^i, v^\alpha) \sim (e^{2i\delta}u^i, e^{-3i\delta}v_\alpha)$. Once more the $\mathrm{U}(1)_\mathrm{R}$ symmetry is $(u^i, v^\alpha) \to (e^{i\psi}u^i, e^{i\psi}v^\alpha)$.

In light of what was discussed in Chapter 4, one would like to find or construct (1,1)-forms on the space that are basic, primitive, and transverse, and that are eigenfunctions of the Hodge-de Rham Laplacian with eigenvalues in the range between $2(9-4\sqrt{3})$ and $2(9+4\sqrt{3})$.

The condition that the $(1,1)$-forms be basic amounts to imposing that they are invariant under the $U(1)_R$ symmetry, or in other words, that they live on the KE base, and the condition that the eigenvalues lie in the given range means that one should look for modes of the Laplacian that are low-lying. As discussed, the existence of such a primitive and transverse $(1,1)$-form means that there is a scalar that causes instability.

Modes that satisfy the desired conditions were in fact constructed in [58], and the construction of [58] relies upon an important result that was found in [75]. In [75] it is shown that if there exists a Killing vector K^a on a KE space, then there exists a scalar ψ on the space such that $K^a = J^{ab}\partial_b\psi$. The scalar ψ is shown to be an eigenfunction of the Laplacian with eigenvalue 2Λ, i.e. $\Box\psi + 2\Lambda\psi = 0$, where Λ is such that $R_{ab} = \Lambda g_{ab}$. The converse of this statement is also shown to be true. That is, if a KE space has a scalar ψ that is an eigenfunction of the Laplacian, with eigenvalue 2Λ, then the object $K^a = J^{ab}\partial_b\psi$ is a Killing vector on the space.

Therefore, if a KE space has a Lie group symmetry, then it is guaranteed to possess as many scalar harmonics with eigenvalue 2Λ as there are dimensions in the group. For example, since $\mathbb{CP}^2$ has symmetry group SU(3), it has eight scalar harmonics with eigenvalue 2Λ. Each of the eight generators of SU(3) corresponds to a Killing vector on $\mathbb{CP}^2$, and each of these

Killing vectors can be expressed in terms of a scalar harmonic in the way explained above.

For the KE space $\mathbb{CP}^n$ it is shown in [75] that the scalar harmonics with eigenvalue 2Λ that generate the Killing vectors are

$$\psi = \frac{1}{Z^A \bar{Z}_A} T_A{}^B Z^A \bar{Z}_B, \tag{5.28}$$

where the Z^A are the homogeneous coordinates on $\mathbb{CP}^n$ and $T_A{}^B$ is an arbitrary Hermitian traceless tensor.

In [58] the authors study the stability of AdS_5 solutions of M-theory compactified on six-dimensional KE spaces. Among other spaces, they look at the KE space $\mathbb{CP}^2 \times \mathbb{CP}^1$, which is the KE base of the SE manifold $M^{3,2}$. On this space, they note that given a scalar harmonic Y on $\mathbb{CP}^2$, one can construct a primitive transverse $(1,1)$-form ω from it that is an eigenfunction of the Hodge-de Rham Laplacian with the same eigenvalue as the scalar Y.

This $(1,1)$-form ω is a linear combination of the forms $\partial_B \bar{\partial}_B Y$, $YJ^{(4)}$, and $YJ^{(2)}$. It is straightforward to see that each of these forms is an eigenfunction of the Hodge-de Rham Laplacian $\Delta = d\delta + \delta d$ on the KE space, with the same eigenvalue as Y. For the first form, the Dolbeault operators commute with Δ, so clearly it is an eigenfunction with the same eigenvalue as Y. For the second and third forms, one notes that d and δ of the complex structures are zero because their covariant derivatives are 0, and so Δ only acts on Y. So it is clear that the second and third forms are also eigenfunctions of Δ with the same eigenvalue as Y. The coefficients in ω are fixed by imposing that it be primitive and transverse.

In the special case where Y is taken to be a scalar harmonic that generates a Killing vector on $\mathbb{CP}^2$, which is given explicitly above, and where $\Lambda = 8$, one finds ω to be

$$\omega = 2i\partial_B\bar{\partial}_B Y + 16YJ^{(4)} - 16YJ^{(2)}, \tag{5.29}$$

where the SE two form is $J = J^{(4)} + J^{(2)}$ with $J^{(4)} = \frac{3}{4}J_{\mathbb{CP}^2}$ and $J^{(2)} = \frac{1}{2}J_{\mathbb{CP}^1}$, and ∂_B and $\bar{\partial}_B$ are the Dolbeault operators (see, e.g., [6]).

In the notation and conventions used in this work,

$$Y = t_i{}^j u^i \bar{u}_j, \tag{5.30}$$

where t_{ij} is a constant hermitian, traceless matrix and

$$\Delta Y = 16Y. \tag{5.31}$$

As discussed, this $(1,1)$-form is primitive, transverse, and basic, and it is an eigenfunction of Δ with eigenvalue 16. It is thus associated with an unstable scalar at the Pope-Warner point. Furthermore, there are eight such $(1,1)$-forms because there are eight scalar harmonics with eigenvalue 16, and thus there are eight unstable scalar modes at the Pope-Warner point transforming in $(\mathbf{8},\mathbf{1})$ of $\mathrm{SU}(3)\times\mathrm{SU}(2)$.

These eight unstable scalar modes have the mass

$$m_3^2L^2 = 9 - 3\sqrt{17} \approx -3.3693. \tag{5.32}$$

In the KK spectrum for the supersymmetric solution, we should find a Z-vector multiplet with the masses

$$m_Z^2L^2 = 16, \qquad m_\phi^2L^2 = 2\,, \qquad m_\pi^2L^2 = 3 \pm \sqrt{17}\,. \tag{5.33}$$

Indeed, there is such a multiplet given by eqs. (3.23) and (3.24) in [55]. In [55] an irrep of $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is specified by (M_1, M_2, J, Y), where M_1 is the number of columns in the $SU(3)$ tableau with one box, M_2 is the number of columns in the $SU(3)$ tableau with two boxes, J is the $SU(2)$-irrep 'quantum' number, and Y is the $U(1)$ charge. In (3.23) and (3.24) we must set $M_1 = M_2 = 1$ and $J = 0$. Doing so gives

$$E_0 = \frac{1}{2}(1 + \sqrt{17}), \tag{5.34}$$

which reproduces the masses (5.33) using formulae in table 2.

The space $M^{3,2}$ and its complete KK spectrum were treated in much detail in [55]. It is therefore appropriate to present the harmonics associated with the unstable scalars in the language and conventions used in that paper.

Let

$$\begin{aligned}
\Phi &= \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \\
\Phi_i &= \begin{array}{|c|c|}\hline 1 & i \\ \hline 2 \\ \cline{1-1}\end{array} \\
\Phi_i^* &= \epsilon^{ij} \begin{array}{|c|c|}\hline j & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \ \left(\epsilon^{12} = 1\right) \\
\Phi_{ij} &= \frac{1}{2}\left(\begin{array}{|c|c|}\hline i & j \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline j & i \\ \hline 3 \\ \cline{1-1}\end{array} \right).
\end{aligned} \tag{5.35}$$

The two 3-form harmonics and the metric harmonic associated with the unstable scalars will be constructed from these tensors.

The two 3-form harmonics are constructed from the five 3-forms A^j given below. These

3-forms are pieces of the 3-form H decomposition given in [55].

$$\begin{aligned}
A^1_{ABC} = & \ \epsilon_{ABCD}\left(\lambda^D_{3i}\Phi_i - \lambda^D_{i3}\Phi^*_i\right) \\
A^2_{Amn} = & \ \epsilon_{mn}\left(\lambda^A_{3i}\Phi_i + \lambda^A_{i3}\Phi^*_i\right) \\
A^3_{AB3} = & \ \lambda^{[A}_{i3}\lambda^{B]}_{3j}\epsilon^{ik}\Phi_{kj} \\
A^4_{AB3} = & \ \lambda^{[A}_{i3}\lambda^{B]}_{3i}\Phi \\
A^5_{mn3} = & \ \epsilon_{mn}\Phi.
\end{aligned} \tag{5.36}$$

The remaining components of the A^j are 0.

The A^j close under the action of $Q = \star d$. (Note that d in [55] is defined differently than usual, so that on 3-forms it is $\frac{1}{4}$ of the usual one.)

$$\begin{aligned}
QA^1 = & \ -i\frac{4}{\sqrt{3}}A^5 \\
QA^2 = & \ i\frac{4}{\sqrt{3}}A^3 + \frac{2}{\sqrt{3}}A^4 \\
QA^3 = & \ -i\frac{\sqrt{3}}{2}A^2 + A^3 \\
QA^4 = & \ \frac{1}{\sqrt{3}}A^2 - A^4 + 2A^5 \\
QA^5 = & \ A^4 + i\frac{1}{\sqrt{3}}A^1.
\end{aligned} \tag{5.37}$$

For $A = c_jA^j$ we can use the above to determine the c_j and μ such that

$$QA = \mu A. \tag{5.38}$$

Solving these equations gives the eigenvalues $\mu = \frac{1}{2}(1 \pm \sqrt{17})$, 2, 0, and -3. These eigenvalues agree with those listed in equation (6.37) of [55].

In particular the modes corresponding to $\mu = \frac{1}{2}(1 \pm \sqrt{17})$ are the ones associated with unstable scalars. For these the constants are found to be (fixing $c_1 = 1$)

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -i \\ c_3 &= \frac{1}{6}\left(\sqrt{3} \pm \sqrt{51}\right) \\ c_4 &= -i\frac{\sqrt{3}}{4}\left(1 \pm \sqrt{17}\right) \\ c_5 &= -i\frac{1}{2}\left(\sqrt{3} \pm \sqrt{51}\right). \end{aligned} \tag{5.39}$$

The corresponding masses, in the conventions of [37], of these two modes are obtained using the 3-form mass formula (equation B.3 of [37])

$$m_f^2 = 16(Q-2)(Q-1). \tag{5.40}$$

They are $m_f^2 = 16\left(5 \mp \sqrt{17}\right)$. Note that these are the masses in the supersymmetric, non-skew-whiffed case.

In going from the skew-whiffed Freund-Rubin solution to the Pope-Warner solution, the two 3-form modes 'mix' with a metric mode to form a new mode. We obtain this metric mode by contracting the 4-form $f = dA = \star QA$ with the background internal flux over three indices and symmetrizing over the remaining two indices.

$$Y_{\alpha\beta} = F_{(\alpha}{}^{\gamma\delta\epsilon} f_{\beta)\gamma\delta\epsilon}, \tag{5.41}$$

giving, without worrying about the overall constant,

$$\begin{aligned} Y_{1A} &= \epsilon_{ij}(\lambda^A_{j3}\Phi_i + \lambda^A_{3j}\Phi^*_i) \\ Y_{2A} &= i\epsilon_{ij}(\lambda^A_{j3}\Phi_i - \lambda^A_{3j}\Phi^*_i), \end{aligned} \tag{5.42}$$

with all other components equal to 0.

5.2.4 $Q^{1,1,1}$

Recall that $Q^{1,1,1}$ is a U(1) bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, with the metric (see, e.g., [74])

$$ds^2 = \frac{1}{2}(ds^2_{\mathbb{CP}^1_{(1)}} + ds^2_{\mathbb{CP}^1_{(2)}} + ds^2_{\mathbb{CP}^1_{(3)}}) + \left[d\psi + \frac{1}{2}\left(A_{\mathbb{CP}^1_{(1)}} + A_{\mathbb{CP}^1_{(2)}} + A_{\mathbb{CP}^1_{(3)}}\right)\right]^2 . \tag{5.43}$$

The Kahler quotient construction for this manifold [54, 55], has three $\mathbb{C}^2$'s, with coordinates u^α, v^α and w^α, respectively, one for each $\mathbb{CP}^1$ factor in the KE base. Then $Q^{1,1,1}$ is the surface in $\mathbb{C}^6$,

$$u^\alpha \bar{u}_\alpha = v^\alpha \bar{v}_\alpha = w^\alpha \bar{w}_\alpha = 1 , \tag{5.44}$$

modded by two U(1) symmetries, $(u^\alpha, v^\alpha, w^\alpha) \sim (e^{i\delta} u^\alpha, e^{i\theta} v^\alpha, e^{-i\delta - i\theta} w^\alpha)$. In terms of the projective coordinates, z_i, on $\mathbb{CP}^1_{(i)}$, and the fiber angle, ψ, we have

$$\begin{aligned} u^1 &= \frac{z_1 e^{2i\psi/3}}{(1-|z_1|)^{1/2}} , \qquad & v^1 &= \frac{z_2 e^{2i\psi/3}}{(1-|z_2|)^{1/2}} , \qquad & w^1 &= \frac{z_3 e^{2i\psi/3}}{(1-|z_3|)^{1/2}} , \\ u^2 &= \frac{e^{2i\psi/3}}{(1-|z_1|)^{1/2}} , \qquad & v^2 &= \frac{e^{2i\psi/3}}{(1-|z_2|)^{1/2}} , \qquad & w^2 &= \frac{e^{2i\psi/3}}{(1-|z_3|)^{1/2}} . \end{aligned} \tag{5.45}$$

The SU(2) Killing vectors on each $\mathbb{CP}^1$ yield triplets of scalar harmonics,

$$Y_{(1)} = t^{(1)}{}_\alpha{}^\beta u^\alpha \bar{u}_\beta , \qquad Y_{(2)} = t^{(2)}{}_\alpha{}^\beta v^\alpha \bar{v}_\beta , \qquad Y_{(3)} = t^{(3)}{}_\alpha{}^\beta w^\alpha \bar{w}_\beta , \tag{5.46}$$

which are eigenfunctions of the Laplacian with the eigenvalue 16 [74]. The two forms

$$\omega_{(1)} = Y_{(1)}(J_{\mathbb{CP}^1_{(2)}} - J_{\mathbb{CP}^1_{(3)}}) , \qquad \omega_{(2)} = Y_{(2)}(J_{\mathbb{CP}^1_{(3)}} - J_{\mathbb{CP}^1_{(1)}}) , \qquad \omega_{(3)} = Y_{(3)}(J_{\mathbb{CP}^1_{(1)}} - J_{\mathbb{CP}^1_{(2)}}) , \tag{5.47}$$

are primitive, transverse $(1,1)$-eigenforms of the Hodge-de Rham operator with the same eigenvalue [58]. This gives nine unstable modes for the PW solution on $Q^{1,1,1}$ in the adjoint representation of $SU(2) \times SU(2) \times SU(2)$.

Clearly, numerical values of all the masses of the Z-vector field and the scalar and pseudoscalar fields at the FR solution are the same as for $M^{3,2}$, and one expects to find a similar structure of $\mathcal{N} = 2$ supermultiplets as well. Hence it is surprising that the KK spectrum in section 4 in [51] does not contain a long Z-vector multiplet in the adjoint of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ with the energy (5.34). In fact, there is also no graviton multiplet corresponding to the scalar harmonics (5.46). However, a closer examination of the allowed harmonics on $Q^{1,1,1}$ and their masses, which are listed in section 3 of the same paper, shows that the Z-multiplet we are looking for should have been included in the final "complete classification."

5.2.5 $V^{5,2}$

As discussed in [53] (see, also [52, 76, 77]), the Stiefel manifold, $V^{5,2}$, is the intersection of the Kahler cone in $\mathbb{C}^5$,

$$(u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (u^5)^2 = 0\,, \tag{5.48}$$

with the unit sphere,

$$|u^1|^2 + |u^2|^2 + |u^3|^2 + |u^4|^2 + |u^5|^2 = 1\,. \tag{5.49}$$

Writing $u^j = x^j + iy^j$, the real and imaginary part vectors (x^j) and (y^j) in $\mathbb{R}^5$ can be parametrized by the Euler angles of the coset space SO(5)/SO(3), [11]

$$\begin{pmatrix} x^1 & y^1 \\ x^2 & y^2 \\ x^3 & y^3 \\ x^4 & y^4 \\ x^5 & y^5 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_3(\alpha_1,\alpha_2,\alpha_3) & 0 \\ 0 & \mathcal{R}_2(\phi) \end{pmatrix} \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\mu \\ 0 & 0 \\ \sin\theta & 0 \\ 0 & \sin\mu \end{pmatrix} \mathcal{R}_2\left(\frac{4}{3}\psi\right) , \tag{5.50}$$

where

$$0 \le \alpha_1, \alpha_3 < 2\pi\,, \qquad 0 \le \alpha_2\,, \phi < \pi\,, \qquad -\frac{\pi}{2} \le \mu\,, \theta < \frac{\pi}{2}\,, \qquad 0 \le \psi < \frac{3\pi}{8}\,, \tag{5.51}$$

and $\mathcal{R}_2$ and $\mathcal{R}_3$ are rotation matrices. In terms of the coordinates on the cone and the angles, the SE metric on $V^{5,2}$ is

$$\begin{aligned} ds^2 &= \frac{3}{2} du^j d\bar{u}^j - \frac{3}{16}|u^j d\bar{u}^j|^2 \\ &= \frac{3}{8}\Big[d\mu^2 + \cos^2\mu\,\sigma_1^2 + d\theta^2 + \cos^2\theta\,\sigma_2^2 \\ &\qquad + \frac{1}{2}\sin^2(\mu-\theta)(\sigma_3+d\phi)^2 + \frac{1}{2}\sin^2(\mu+\theta)(\sigma_3-d\phi)^2\Big] \\ &\qquad\quad + \Big[d\psi + \frac{3}{8}\cos(\mu-\theta)(\sigma_3+d\phi) + \frac{3}{8}\cos(\mu+\theta)(\sigma_3-d\phi)\Big]^2 , \end{aligned} \tag{5.52}$$

where σ_i are the SO(3)-invariant forms, $d\sigma_i = \sigma_j \wedge \sigma_k$. The metric (5.52) is the canonical SE form of the U(1) fibration over the KE base, which is the Grassmannian, $Gr_2(\mathbb{R}^5)$. The volume of the space is computed from this metric in Appendix E.

The harmonics on $V^{5,2}$ are obtained by the pullback of tensors in $\mathbb{C}^5$ and decompose into $\mathrm{SO}(5) \times \mathrm{U}(1)_R$. Here SO(5) acts on u^j in the real vector representation, while $\mathrm{U}(1)_R$ is the

[11] A somewhat different explicit parametrization of $V^{5,2}$ is given in [76, 77].

phase rotation, $u^j \to e^{i\psi} u^j$.

The lowest lying scalar harmonic that is invariant under $\mathrm{U}(1)_R$ is $\Phi^{ij} = u^i \bar{u}^j - u^j \bar{u}^i$. It is an eigenfunction of the Laplacian with the eigenvalue 16 [52]. Similarly, the lowest lying $(1,1)$-forms that are not proportional to the Kahler form are

$$\omega^i = \epsilon^{ijklm} u^j \bar{u}^k du^l d\bar{u}^m \,. \tag{5.53}$$

They transform as **5** of SO(5) and are invariant under $\mathrm{U}(1)_R$. Expanding those forms using (5.50) confirms that they are basic. They satisfy

$$\Delta\omega^i = \frac{32}{3}\,\omega^i \,, \tag{5.54}$$

and hence give rise to five unstable modes of the PW solution with the mass

$$m_3^2 L^2 = 7 - \sqrt{105} \approx -3.2469 \,. \tag{5.55}$$

The masses for the supersymmetric solution are

$$m_Z^2 L^2 = \frac{32}{3}\,, \qquad m_\phi^2 L^2 = \frac{2}{3}\,, \qquad m_\pi^2 L^2 = \frac{5}{3} \pm \sqrt{\frac{35}{3}}\,. \tag{5.56}$$

The Z-vector multiplet has then

$$E_0 = \frac{1}{6}(3 + \sqrt{105})\,. \tag{5.57}$$

While such a multiplet is not listed in the tables in [52], the authors note at the end of section 2 that there might be an additional vector supermultiplet with this energy.[12] In appendix C, we list all bosonic harmonics on $V^{5,2}$ that transform in **5** of SO(5) and show that they decompose unambigously into $\mathcal{N} = 2$ supermultiplets including a long Z-vector multiplet in agreement with our construction.

[12]We thank A. Ceresole and G. Dall'Agata for correspondence, which clarified this point.

5.3 Orbifolds

Homogeneous SE manifolds also admit discrete symmetries such that the quotient maniofld, M_7/Γ is still SE. The natural question is what happens to the master (1,1)-forms in this projection and whether the PW solution for the quotient SE manifold is stable. We will now examine this for some examples of SE discrete quotients that were considered in the literature.

For S^7, it has been shown in [32] that if the discrete symmetry group Γ is a subgroup of SU(4), it will preserve some of the unstable modes. The same reasoning applies to the (1,1)-forms (5.21) and shows that some of them will be well-defined on the quotient.

Orbifolds of $M^{3,2}$, $Q^{1,1,1}$, and S^7, can be obtained as limits of the $Y^{p,k}$ Sasaki-Einstein manifolds [78]. Specifically, when $2k = 3p$ and $p = 2r$, one has that $Y^{2r,3r}(\mathbb{CP}^2) = M^{3,2}/\mathbb{Z}_r$, where $\mathbb{Z}_r$ is a finite subgroup of SU(2) acting on $\mathbb{CP}^1$. Since the master 2-forms for $M^{3,2}$ are constructed from scalar harmonics on the $\mathbb{CP}^2$, see (5.30) and (5.29), they are preserved under the orbifolding. Hence the instability persists for these orbifolds of $M^{3,2}$.

Similarly, when $k = p$, one has $Y^{p,p}(\mathbb{CP}^1 \times \mathbb{CP}^1) = Q^{1,1,1}/\mathbb{Z}_p$, where $\mathbb{Z}_p$ is a finite subgroup of SU(2) acting on one of the three $\mathbb{CP}_1$'s. Each independent master (1,1)-form on $Q^{1,1,1}$, see (5.46) and (5.47), is constructed from a scalar harmonic on one of the $\mathbb{CP}^1$ factors. For the SU(2) acting on $\mathbb{CP}^1_{(i)}$, the forms $\omega_{(j)}$, $j \neq i$, are invariant under $\mathbb{Z}_p$ and hence are well defined on the quotient $Q^{1,1,1}/\mathbb{Z}_p$.

For $k = 3p$, one has that $Y^{p,3p} = S^7/\mathbb{Z}_{3p}$, where $\mathbb{Z}_{3p} \subset$ SU(4) acts by

$$(u^1, u^2, u^3, u^4) \quad \longrightarrow \quad (e^{2\pi i/3p}u^1, e^{2\pi i/3p}u^2, e^{2\pi i/3p}u^3, e^{-2\pi i/p}u^4). \tag{5.58}$$

The six master $(1,1)$-forms on S^7 that contain precisely one u^4 or $\bar{u}^4$ are not invariant under (5.58). This yields fourteen unstable modes on that space.

The orbifolds $V^{5,2}/\mathbb{Z}_k$ have been discussed in [77]. The finite group here is $\mathbb{Z}_k \subset \mathrm{U}(1)_b$, where $\mathrm{U}(1)_b$ is a diagonal subgroup of the $\mathrm{SO}(2)\times\mathrm{SO}(2)$ rotation in the (12) and (34) planes in $\mathbb{C}^5$. Clearly, the master 2-form ω^5, see (5.53), is invariant under this action and yields one unstable mode on $V^{5,2}/\mathbb{Z}_k$.

6 Conclusion

In the research presented in this dissertation we have analyzed a subset of scalar modes in the linearized spectrum of eleven-dimensional supergravity around the Pope-Warner solution on an arbitrary SE manifold and derived a condition under which the solution becomes perturbatively unstable. Specifically, we have shown that when the manifold admits a basic, transverse, primitive $(1,1)$-form within a certain range of eigenvalues of the Hodge-de Rham Laplacian, then there are scalar modes violating the BF bound. We have also constructed such destabilizing $(1,1)$-forms on all homogenous SE manifolds, and on their orbifolds, and found that when viewed as harmonics for fluctuations around the supersymmetric solution, those forms give rise to a long Z-long vector supermultiplet in the KK spectrum.

Using this fact it would be straightforward to rephrase the stability condition in terms of spinor-vector harmonics on the SE manifold. To do so one could use the formulae given in, e.g., [56], since by the construction in [56], spinor-vector harmonics give rise to long Z-vector supermultiplets.

Throughout this work we have assumed that the SE manifold was quasi-regular, i.e. regular or non-regular, and the quasi-regularity was used explicitly in some of the proofs. In particular quasi-regularity was used in establishing the shift between the eigenvalues of the symmetric tensor and 2-form harmonics under their respective mass operators. However, since this proof is local, one would expect that our construction should hold for an arbitrary SE manifold. Indeed, the fact that this same shift was proven in [56] (see Appendix G) for any internal manifold with a Killing spinor indicates that this expectation does actually hold.

It remains an open problem to see whether stability violating 2-forms exist on any SE manifold. In other words, even though the PW solution turned out to be unstable for all the concrete SE manifolds we looked at, it is not yet known whether there exists an SE manifold for which the PW solution is stable. It would be notable to find such a SE manifold. As discussed, if the manifold is quasi-regular, the question of stability reduces to the problem of determining the low lying spectrum of the Hodge-de Rham Laplacian on a six-dimensional KE manifold, which in itself is a difficult problem with rather few explicit results (see, e.g., [81]). If the manifold is irregular, perhaps the results of [91] may be of use. In this paper the author presents a generalization of the identity $\Delta = 2\Delta_{\bar{\partial}}$ to SE manifolds.

There is also an analogue of the PW solution in type IIB supergravity [82], which is known to be unstable within the $\mathcal{N} = 8$ $d = 5$ supergravity [83, 84] obtained by compactification on S^5. It would be interesting, and perhaps simpler, to examine the stability of this type of solutions for the new class of five-dimensional SE manifolds [27, 28] for which the spectra of the scalar Laplacian were obtained in [85, 86].

As discussed in the introduction, the main motivation for recent interest in PW solutions came from the "top-down" construction of holographic models of superconductors in [10, 12, 11]. The PW solutions are then dual to zero entropy states with emergent conformal invariance at $T = 0$. In light of this duality to a conformal theory, it would be nice to see what the PW instability discussed here looks like in the dual CFT under the AdS/CFT correspondence. A possible starting point for such an endeavor is provided in the papers [66] and [47]. In these papers the authors discuss a special $N = 3$ long gravitino multiplet that, in fact, contains the tri-Sasaki modes discussed here that lead to PW instability. In particular, the authors give a composite CFT operator that they claim corresponds to this long gravitino multiplet.

A Conventions

We use the same conventions as in [79] and [32], with the mostly plus space-time metric and the bosonic field equations of eleven-dimensional supergravity given in (1.1) and (1.2), and the gravitino supersymmetry transformations

$$\delta\psi_M = D_M\epsilon + \frac{1}{144}\left(\Gamma_M{}^{NPQR} - 8\,\delta_M{}^N\Gamma^{PQR}\right)\mathcal{F}_{NPQR}\,\epsilon\,. \tag{A.1}$$

On a manifold with a Minkowski signature metric, $\mathfrak{g}$, we define the Hodge dual, $\star$, by

$$\star\Lambda\wedge\Lambda = -|\Lambda|\mathrm{vol}_{\mathfrak{g}}\,. \tag{A.2}$$

The Hodge dual, $*$, for a riemannian metric, g, is then defined without the minus sign.

The eleven-dimensional Dirac matrices in the $4+7$ decomposition are

$$\begin{aligned} \mathbb{\Gamma}^{\mu} &= \gamma^{\mu-1} \otimes \mathbf{1}\,, \qquad \mu = 1,\ldots,4\,, \\ \mathbb{\Gamma}^{a+4} &= -\gamma^{5} \otimes \Gamma^{a}\,, \quad a = 1,\ldots,7\,, \end{aligned} \tag{A.3}$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \Gamma^7 = i\,\Gamma^1\ldots\Gamma^6\,. \tag{A.4}$$

Then

$$\mathbb{\Gamma}^1\mathbb{\Gamma}^2\ldots\mathbb{\Gamma}^{11} = \epsilon^{12\ldots 11}\,\mathbf{1} = \mathbf{1}\,. \tag{A.5}$$

We use the representation in which the four-dimensional γ-matrices are real, while the seven-dimensional Γ-matrices are pure imaginary and antisymmetric. For a real spinor, η, on the internal manifold, we then have $\bar{\eta} = \eta^T$.

B Sasaki-Einstein manifolds: definitions and relevant information

Due to their prominence in string theory and M-theory, it is worthwhile to properly define what a Sasaki-Einstein manifold is.

In this appendix we define what a Sasaki-Einstein manifold is, and provide information about them that is relevant in subsequent sections. The content of this section follow principally from the contents of [23] and [26]. A thorough treatment of the subject is given in [24].

Contact manifolds

In defining what a Sasaki-Einstein manifold is, it is natural to start by first defining what a contact manifold is. A **contact manifold** is a $(2n-1)$-dimensional manifold M such that there exists a 1-form η on it, with the property that

$$\eta \wedge (d\eta)^n \neq 0 \tag{B.1}$$

at each point of M. Such a 1-form η is called a **contact 1-form**.

Given a contact manifold M with contact 1-form η, there is a unique vector field ξ called the **Reeb vector field**. The Reeb vector field ξ is defined to be the unique vector field satisfying the conditions

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0. \tag{B.2}$$

At each point p on the manifold M one can consider the hyperplane $\ker \eta(p)$. This hyperplane is a (2n-2)-dimensional subspace of the tangent space TM_p, and the bundle D of all such hyperplanes, $D = \ker \eta$, is a sub-bundle of the tangent bundle TM. In this way the contact 1-form η specifies a distribution D of (2n-2)-dimensional hyperplanes on the manifold M. D is called the *contact distribution*.

The contact distribution D is maximally non-integrable, which translates into the fact that $d\eta$ is nondegenerate, i.e. for every vector X on a hyperplane there exists a Y on the hyperplane such that $d\eta(X,Y) \neq 0$. Intuitively, the 2-form $d\eta$ is a way to measure the failure of the parallelogram, formed by the vectors X and Y in the hyperplane, to close in the Reeb vector direction. For more details on contact manifolds see [87].

Since it is nondegenerate, the 2-form $d\eta$ can be regarded as a symplectic form ω on D.

In addition to a symplectic form on D one would also like to have an almost complex structure J, which is a type (1,1) tensor, on D. Furthermore, one would like for this J to be compatible with ω. Compatibility with ω means the relations $d(JX, JY) = \omega(X, Y)$ and $\omega(JX, X) > 0$ are satisfied. In terms of indices, the first relation is equivalent to $J^i{}_m J^j{}_n \omega_{ij} = \omega_{mn}$.

This J can be used to get a Riemannian metric on D, namely $g_D(X, Y) = \omega(JX, Y)$, where X and Y are smooth sections of D. This metric is compatible with the almost complex structure J, which means that $g_D(JX, JY) = g_D(X, Y)$. In terms of indices, this expression is equivalent to $J^i{}_m J^j{}_n g_{ij} = g_{mn}$.

Also, in terms of indices, the expression $g_D(JX, Y) = -\omega(X, Y)$ is equivalent to $g_{ik} J^k{}_j = \omega_{ij}$. So ω can be thought of as J with its index lowered with the metric.

J on D can be extended to a tensor Φ of type (1,1) on TM by letting $\Phi = J$ on D and $\Phi\xi = 0$. Also, the metric g_D on D can be extended to a metric g on TM by letting $g(X, Y) = g_D(X, Y) + \eta(X)\eta(Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

It is clear that under this metric the Reeb vector ξ is orthogonal to the vectors in D, i.e. $g(X, \xi) = 0$ for any section X of D. The orthogonality follows from the definition of D, i.e. $D = \ker \eta$, and from the definition of the Reeb vector, which requires that $i_\xi d\eta = 0$. Furthermore, this metric satisfies the compatibility condition $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$.

The above construction motivates the definition of a metric contact structure: If in addition to the contact 1-form η and its associated Reeb vector field ξ, there is a tensor field Φ of type (1,1) and a Riemannian metric g that satisfy

$$\Phi^2 = -\mathbb{1} + \xi \otimes \eta, \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{B.3}$$

then the contact manifold M is said to have a **metric contact structure**.

Sasakian and Sasaki-Einstein manifolds

Given a compact manifold M with Riemannian metric g, the **metric cone** over M is defined to be the space $\mathbb{R}_+ \times M$ with metric $ds^2 = dr^2 + r^2 g$. If the metric cone over M is Kähler, then M is defined to be a **Sasakian manifold**.

A Sasakian manifold is automatically a contact manifold with a metric contact structure, and its type (1,1) tensor Φ and metric g are as in the construction above, i.e. $\Phi = J$ on $D = \ker \eta$, $\Phi\xi = 0$, and $g(X,Y) = g_D(X,Y) + \eta(X)\eta(Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

Finally, a **Sasaki-Einstein manifold** is defined to be a Sasakian manifold with $Ric_g = 2(n-1)g$. The metric cone over a Sasaki-Einstein manifold is Ricci-flat Kähler, hence Calabi-Yau. The converse of this statement is true, i.e. a manifold whose cone is Kähler Ricci-flat is a Sasaki-Einstein manifold.

An interesting special type of Sasaki-Einstein manifold is a 3-Sasakian manifold. A **3-Sasakian manifold** is a Sasakian manifold whose metric cone is hyper-Kähler. This means the holonomy of the cone metric is contained in $Sp(p)$. $Sp(p) \subset SU(2p)$, so a hyper-Kähler manifold is a Calabi-Yau manifold, and a 3-Sasakian manifold is a Sasaki-Einstein manifold.

Reeb foliation

The Reeb vector field ξ was defined to be the unique vector field satisfying the conditions

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0. \tag{B.4}$$

From the first condition it is clear that ξ is nowhere vanishing. Since it is nowhere vanishing, it can be used to generate a 1-parameter family of diffeomorphisms of the space on which it is defined.

Therefore, given a Sasakian manifold M with Sasakian structure (ξ,η,Φ,g), one can partition M into disjoint orbits of the diffeomorphism generated by the Reeb vector ξ. Each orbit is a 1-dimensional space. Partitioning M in this way is called the **Reeb foliation**, and the orbits are the **leaves** of the Reeb foliation.

Sasakian manifolds split into three different classes, depending on the nature of the Reeb foliation. If the leaves of the foliation close, so that they are circles, then the Sasakian manifold is said to be **quasi-regular**. For a quasi-regular manifold, the Reeb vector generates a U(1) action. This U(1) action is always locally free. If in addition the U(1) action is free overall, then there is no point on the manifold that is fixed by a nontrivial element of the U(1) action. In this case the Sasakian manifold is said to be **regular**.

If the $U(1)$ action is not free overall, then it must 'wrap around' an orbit an integer number of times, so that the orbit is fixed by a discrete subgroup of the $U(1)$ action. In this case the manifold is said to be **non-regular**.

If the leaves of the foliation do not close, then they are noncompact. In this case the manifold is said to be **irregular**.

Transverse Kahler structure

Motivated by the definition of the distribution $D = \ker\eta$ and the form of the metric g, the tangent bundle can be split into the direct sum $TM = D \oplus L_\xi$, so that at a point p on M, $TM_p = D_p \oplus L_{\xi,p}$, where D_p is a (2n-2)-dimensional hyperplane, and $L_{\xi,p}$ is the 1-dimensional line that is tangent to the Reeb vector at p. The spaces D_p and $L_{\xi,p}$ are orthogonal with respect to the metric g.

As discussed previously D naturally has a (almost) complex structure $J = \Phi|_D$, a symplectic structure $d\eta$, and a metric $g_D(X,Y) = d\eta(JX,Y)$. $(D, J, d\eta)$ gives the Sasakian manifold M what is referred to in [23] as a *transverse Kähler* structure. It is important to note that in general this (2n-2)-dimensional Kähler structure holds only locally. One would like to know, however, when this Kähler structure holds globally.

For a Sasakian manifold M, let $\mathcal{Z}$ be the space of leaves of its Reeb foliation. Then if the Reeb foliation is quasi-regular then the (2n-2)-dimensional Kähler structure holds globally. In particular, if the Reeb foliation is regular then $\mathcal{Z}$ has the structure of Kahler manifold, and if the Reeb foliation is non-regular then $\mathcal{Z}$ has the structure of an orbifolded Kahler manifold. For necessary details about the orbifold structure in the non-regular case see [23] and [26].

The converse of this statement holds true, i.e. given that $\mathcal{Z}$ is a Kahler manifold or a proper orbifold of one, a principal $U(1)$ bundle M over $\mathcal{Z}$ is a Sasakian orbifold with metric $\pi^*h + \eta \otimes \eta$, where π is the projection from M to $\mathcal{Z}$, h is the metric on $\mathcal{Z}$, and η is a 1-form on M such that $d\eta = 2\pi^*\omega$, where ω is the symplectic structure of $\mathcal{Z}$. For necessary details see [23] and [26].

If the Reeb foliation of a Sasakian manifold is irregular, then the situtation is more complicated. However, it is known that in this case the closure of the group action generated by the Reeb vector is isomorphic to a torus $\mathbb{T}^k$, with $k \geq 2$.

Finally, if M is 3-Sasakian, then it is an $SU(2)$ bundle over a 4-dimensional quaternionic Kähler manifold or orbifold. Accordingly, its metric can be written as $g = g_{\mathcal{O}} + \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3$, where $g_{\mathcal{O}}$ is the metric of the 4-dimensional quaternionic Kahler manifold or orbifold, and the η^i are 1-forms that are dual to a triplet of Reeb vectors ξ^i that form an

$su2$ lie algebra. For more details see [25] and [26]. A nice fact about 3-Sasakian manifolds is that they are automatically Einstein.

Homogeneous Sasaki-Einstein manifolds

There is a special type of Sasaki-Einstein manifold that has been well-known to supergravity theorists since the 1980s, namely homogeneous Sasaki-Einstein manifolds.

A Sasaki-Einstein manifold is **homogeneous** if there is a group of isometries G that acts transitively on it and preserves the Sasakian structure. A group action is transitive if there is a point in the space such that every other point in the space can be obtained via a group action on that point; so there is only one orbit of the group action. Hence, a homogeneous Sasaki-Einstein manifold can be expressed as a coset space.

There are in fact only five seven-dimensional homogeneous Sasaki-Einstein manifolds: S^7, N^{010}, $V^{5,2}$, M^{32}, and Q^{111} [CRW]. They are principal U(1) bundles over the Kähler-Einstein spaces $\mathbb{CP}^3$, $\mathrm{SU}(3)/T^2$, $\mathrm{Gr}_2(\mathbb{R}^5)$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, and $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, respectively.

The seven-dimensional homogeneous Sasaki-Einstein spaces have all been used to compactify eleven dimensional supergravity to AdS_4, and the complete Kaluza-Klein spectra of these compactifications have been determined in [44], [45], [46], [52], [55], and [51].

Killing spinors

A Killing spinor is a spinor that satisfies the relation

$$\nabla_Y \psi = \alpha Y \cdot \psi \tag{B.5}$$

for any vector field Y, where $Y \cdot \psi = Y^m \Gamma_m$ and α is a constant. For applications in supergravity, string theory, and M-theory, it is important to know when a Sasaki-Einstein manifold admits Killing spinors and how many of them it possesses.

If a seven-dimensional Sasaki-Einstein manifold is simply connected it admits at least two Killing spinors, and both of them satisfy the defining relation with the same constant α, with $\alpha > 0$ [23, 26]. If a seven-dimensional 3-Sasakian manifold is simply connected it admits at least three Killing spinors, and all of them of them satisfy the defining relation with the same constant α, with $\alpha > 0$ [23].

C Sasaki-Einstein identities

In the local frame

$$e^{1,2,3} = e^{r/L} dx^{0,1,2}\,, \qquad e^4 = dr\,, \qquad e^{a+4} = 2L\,\mathring{e}^a\,, \quad a = 1, \ldots, 7\,, \tag{C.1}$$

on $AdS_4 \times M_7$, cf. (3.9), the unbroken supersymmetries are given by $\epsilon = \varepsilon \otimes \eta$,

$$\varepsilon = e^{r/L} \varepsilon_0\,, \qquad \gamma^{012} \varepsilon_0 = \varepsilon_0\,, \tag{C.2}$$

and

$$\eta = (\cos(2\psi) + \sin(2\psi)\Gamma^{12})\eta_0\,, \qquad \Gamma^{12}\eta_0 = \Gamma^{34}\eta_0 = \Gamma^{56}\eta_0\,, \tag{C.3}$$

where ε_0 and η_0 are constant spinors. We choose the two independent solutions, η^1 and η^2, of (C.3) such that the components of the two SE tensors in (3.9) and (3.12) are the same.

Given the Reeb vector field of unit length, [13]

$$\xi^a \xi_a = \vartheta_a \vartheta^a = 1\,, \tag{C.4}$$

the projection operator

$$\pi^a{}_b = \delta^a{}_b - \vartheta^a \vartheta_b\,, \tag{C.5}$$

[13] All indices are raised and lowered with the SE metric, $\mathring{g}_{ab}$.

is a map onto the subspace perpendicular to the Reeb vector. Any tensor $H_{ab\ldots c}$ satisfying

$$\vartheta^a H_{ab\ldots c} = \vartheta^b H_{ab\ldots c} = \ldots = \vartheta^c H_{ab\ldots c} = 0\,, \tag{C.6}$$

will be invariant under the projection, and, modulo its dependence on the fiber coordinate, ψ, can be thought of as a tensor on the Kahler-Einstein base. We refer to such tensors as horizontal.

For complex horizontal tensors of rank n there is a further decomposition into (p,q)-type tensors, where p and q, $p+q=n$, refer to the number of holomorphic and anti-holomorphic indices according to the corresponding decomposition along the Kahler-Einstein base. In particular, J_{ab} and Ω_{abc}, are horizontal tensors of type $(1,1)$ and $(3,0)$, respectively. A contraction of J with a $(p,0)$-type and $(0,p)$-type horizontal tensor is a multiplication by $+i$ and $-i$, respectively. For example,

$$J_a{}^d\,\Omega_{bcd} = i\,\Omega_{abc}\,, \qquad J_a{}^d\,\overline{\Omega}_{bcd} = -i\,\overline{\Omega}_{abc}\,. \tag{C.7}$$

Horizontal tensors (forms) that are in addition invariant along the Reeb vector field are called basic.

Using the explicit realization of the Sasaki-Einstein forms in terms of Killing spinors (3.12), one can prove additional identities, which we use frequently. First, we have the following "single contraction" identities

$$J^{ac}J_{bc} = \pi^a{}_b\,, \qquad \Omega^{abe}\Omega_{cde} = 0\,, \tag{C.8}$$

$$\Omega^{abe}\,\overline{\Omega}_{cde} = 4\,\pi^{[a}{}_{[c}\pi^{b]}{}_{d]} - 4\,J^{[a}{}_{[c}J^{b]}{}_{d]} - 8i\,\pi^{[a}{}_{[c}J^{b]}{}_{d]}\,, \tag{C.9}$$

from which the higher contractions follow,

$$J_{ab}J^{ab} = 6\,, \qquad \Omega^{acd}\,\overline{\Omega}_{bcd} = 8\pi^a{}_b - 8iJ^a{}_b\,, \qquad \Omega^{abc}\overline{\Omega}_{abc} = 48\,. \tag{C.10}$$

We also need the following uncontracted identity

$$\Omega^{abc}\,\overline{\Omega}_{def} = 6\,\pi^{[a}{}_{[d}\pi^{b}{}_{e}\pi^{c]}{}_{f]} - 18i\,\pi^{[a}{}_{[d}\pi^{b}{}_{e}J^{c]}{}_{f]} - 18\,\pi^{[a}{}_{[d}J^{b}{}_{e}J^{c]}{}_{f]} + 6i\,J^{[a}{}_{[d}J^{b}{}_{e}J^{c]}{}_{f]}\,. \qquad \text{(C.11)}$$

and covariant derivatives of the Sasaki-Einstein forms that are given by

$$\mathring{D}_a\vartheta_b = J_{ab}\,, \qquad \mathring{D}_aJ_{bc} = -2\,\mathring{g}_{a[b}\vartheta_{c]}\,, \qquad \mathring{D}_a\Omega_{bcd} = 4i\,\vartheta_{[a}\Omega_{bcd]}\,. \qquad \text{(C.12)}$$

Identities (3.8) follow from (C.12) by antisymmetrization.

D Some harmonics on $V^{5,2}$

The classification of supermultiplets in the KK spectrum on $V^{5,2}$ given in Tables 2-6 in [52] does not include any long Z-vector supermultiplet. However, the discussion in section 2 in that paper suggests that some vector multiplets might be missing from the classification. In this appendix, we use standard group theory methods (see, e.g., [49]) to list all harmonics on $V^{5,2}$ that transform in **5** of SO(5). This allows us to determine unambigously that there must be a long Z-vector supermultiplet in the KK spectrum consistent with the explicit construction in section 5.2.5. We refer the reader to [52] and the references therein for the group theoretic set-up of the harmonic analysis on this space.

The $V^{5,2}$ manifold is a G/H coset space,

$$V^{5,2} = \frac{\mathrm{SO}(5)\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}\,, \qquad \text{(D.1)}$$

where the embeding of H in G is defined by the branching rule

$$\mathbf{5}_Q \quad\longrightarrow\quad \mathbf{3}_Q + \mathbf{1}_{Q+1} + \mathbf{1}_{Q-1}\,. \qquad \text{(D.2)}$$

It then follows that the embedding of H into the tangent SO(7) group is given by

$$\mathbf{1} \longrightarrow \mathbf{1}_0\,, \tag{D.3}$$

$$\mathbf{7} \longrightarrow \mathbf{3}_1 + \mathbf{3}_{-1} + \mathbf{1}_0\,, \tag{D.4}$$

$$\mathbf{8} \longrightarrow \mathbf{3}_{1/2} + \mathbf{3}_{-1/2} + \mathbf{1}_{3/2} + \mathbf{1}_{-3/2}\,. \tag{D.5}$$

This shows that the embedding is through the chain

$$\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(3) \times \mathrm{U}(1) \subset \mathrm{SU}(4) \subset \mathrm{SO}(7)\,, \tag{D.6}$$

where SU(2) ⊂ SU(3) is the maximal embedding. The other two embeddings are regular, except that the normalization of the U(1) charge is half the conventional one [80].

In addition to (D.4), we also need the branchings of **21**, **35** and **27** of SO(7), which determine the two-form, the three-form and the symmetric tensor harmonics, respectively,

$$\begin{aligned}
\mathbf{21} &\longrightarrow \mathbf{1}_0 + \mathbf{3}_2 + \mathbf{3}_1 + \mathbf{3}_0 + \mathbf{3}_{-1} + \mathbf{3}_{-2} + \mathbf{5}_0\,, \\
\mathbf{35} &\longrightarrow \mathbf{1}_3 + \mathbf{1}_1 + \mathbf{1}_0 + \mathbf{1}_{-1} + \mathbf{1}_{-3} + \mathbf{3}_2 + \mathbf{3}_1 + \mathbf{3}_0 + \mathbf{3}_{-1} + \mathbf{3}_{-2} + \mathbf{5}_1 + \mathbf{5}_0 + \mathbf{5}_{-1}\,, \\
\mathbf{27} &\longrightarrow \mathbf{1}_2 + \mathbf{1}_0 + \mathbf{1}_{-2} + \mathbf{3}_1 + \mathbf{3}_0 + \mathbf{3}_{-1} + \mathbf{5}_2 + \mathbf{5}_0 + \mathbf{5}_{-2}\,.
\end{aligned} \tag{D.7}$$

We recall that each independent harmonic is completely specified by its $G \times H$ representation. It follows from (D.2) that only representations $\mathbf{3}_q$ and $\mathbf{1}_q$ in the branchings (D.3), (D.4) and (D.7) give rise to harmonics in $\mathbf{5}_Q$ of $\mathrm{SO}(5) \times \mathrm{U}(1)_R$. Specifically, each $\mathbf{3}_q$ yields a single harmonic, $(\mathbf{5}_q, \mathbf{3}_q)$, while each $\mathbf{1}_q$ yields two harmonics, $(\mathbf{5}_{q-1}, \mathbf{1}_q)$ and $(\mathbf{5}_{q+1}, \mathbf{1}_q)$.

After compiling the list of all harmonics, one must identify the longitudinal ones, which do not give rise to four-dimensional fields in the KK expansion. This can be done by looking at the representation labels of the harmonics. For example, there are two scalar

$Q=$	4	3	2	1	0	-1	-2	-3	-4
h				sg_+		sg_-			
Z			sg_-	sg_+	Z	sg_-	sg_+		
A				sg_+		sg_-			
W				W_+		W_-			
π	W_+		W_-, H	W_+	Z, Z	W_-	W_+, H		W_-
ϕ		Z		W_+	Z	W_-		Z	
Σ				W_+		W_-			
S				H		H			

Table 3: The $\mathcal{N}=2$ supermultiplets on $V^{5,2}$ in $\mathbf{5}$ of SO(5).

harmonics in $(\mathbf{5}_1, \mathbf{1}_0)$ and $(\mathbf{5}_{-1}, \mathbf{1}_0)$, and four vector harmonics in $(\mathbf{5}_1, \mathbf{3}_1)$, $(\mathbf{5}_{-1}, \mathbf{3}_{-1})$, $(\mathbf{5}_1, \mathbf{1}_0)$ and $(\mathbf{5}_{-1}, \mathbf{1}_0)$. The last two are in the same representations as the scalar harmonics and are longitudinal. Indeed, the scalar harmonics are the functions z^i and $\bar{z}^i$, respectively, and the corresponding longitudinal vector harmonics are dz^i and $d\bar{z}^i$. The remaining two transverse vector harmonics are obtained from $z^i z^j d\bar{z}^j$ and $\bar{z}^i \bar{z}^j dz^j$. The same procedure is used to count the two-form, the three-form, and the symmetric tensor longitudinal harmonics.

Using KK expansions in [56] (see also [66] for a succinct summary), it is then straightforward to identify the four dimensional fields corresponding to the transverse harmonics and arrange them into $\mathcal{N}=2$ supermultiplets, whose field content is given, e.g., in Tables 1-9 in

[50]. The result is summarized in Table 3, where the first column lists the four-dimensional fields. The remaining columns are labelled by the $U(1)$ charges of the R-symmetry subgroup of G. The R-charge in [50] is $y_0 = 2Q/3$. Each entry in those columns corresponds to a transverse harmonic in the $\mathbf{5}_Q$ representation of $\mathrm{SO}(5) \times \mathrm{U}(1)_\mathrm{R}$, with the symbol indicating the $\mathcal{N} = 2$ supermultiplet that the corresponding four-dimensional field belongs to: $sg_\pm$ – short graviton multiplets, Z – a long Z-vector multiplet, $W_\pm$ – long W-vector multiplets, and H – a hypermultiplet.

E Volume of $V^{5,2}$

The metric is

$$ds^2 = ds^2(\mathrm{KE}) + [\mathrm{d}\psi + \frac{3}{8}\cos(\mu-\theta)(\sigma_3 + \mathrm{d}\phi) + \frac{3}{8}\cos(\mu+\theta)(\sigma_3 - \mathrm{d}\phi)]^2, \tag{E.1}$$

where

$$\begin{aligned} ds^2(\mathrm{KE}) = \frac{3}{8}[&\mathrm{d}\mu^2 + \cos^2\mu\,\sigma_1^2 + \mathrm{d}\theta^2 + \cos^2\theta\,\sigma_2^2 \\ &+ \frac{1}{2}\sin^2(\mu-\theta)(\sigma_3 + d\phi)^2 + \frac{1}{2}\sin^2(\mu+\theta)(\sigma_3 - d\phi)^2]. \end{aligned} \tag{E.2}$$

The σ_j are SO(3) left-invariant forms.

$$\begin{aligned} \sigma_1 &= \cos\gamma\,d\alpha + \sin\gamma\,\sin\alpha\,d\beta \\ \sigma_2 &= \sin\gamma\,d\alpha - \cos\gamma\,\sin\alpha\,d\beta \\ \sigma_3 &= d\gamma + \cos\alpha\,d\beta, \end{aligned} \tag{E.3}$$

where

$$0 \le \alpha \le \pi, \qquad 0 \le \beta \le 2\pi, \qquad 0 \le \gamma \le 2\pi. \tag{E.4}$$

The ranges of the other angles in the metric are

$$0 \le \phi < \pi, \qquad -\frac{\pi}{2} \le \mu, \theta < \frac{\pi}{2}, \qquad 0 \le \psi < \frac{3\pi}{8}. \tag{E.5}$$

We find that the determinant of this metric is

$$\mathrm{Det}\, g = \left(\frac{27}{512}\right)^2 \sin^2\alpha\ \cos^2\theta\ \cos^2\mu\ \sin^2(\mu-\theta)\ \sin^2(\mu+\theta), \tag{E.6}$$

so that

$$(\mathrm{Det}\, g)^{1/2} = \frac{27}{512} \sin\alpha\ \cos\theta\ \cos\mu\ \sin(\mu-\theta)\ \sin(\mu+\theta). \tag{E.7}$$

Note that the quantity above will have both positive and negative (and 0) values in the coordinate patch. So in computing the volume of the space, the absolute value of it must be used.

The volume of the space is

$$\begin{aligned}
\mathrm{Vol} =& \int |(\mathrm{Det}\, g)^{1/2}| d\beta\, d\gamma\, d\phi\, d\psi\, d\alpha\, d\theta\, d\mu \\
=& \int d\beta\, d\gamma\, d\phi\, d\psi \int (\mathrm{Det}\, g)^{1/2}\, d\alpha\, d\theta\, d\mu \\
=& \frac{3}{2}\pi^4 \int |(\mathrm{Det}\, g)^{1/2}|\, d\alpha\, d\theta\, d\mu \\
=& \frac{81}{1024}\pi^4 \int \sin\alpha d\alpha \int \cos\theta\ \cos\mu\, |\sin(\mu-\theta)\ \sin(\mu+\theta)| d\theta\, d\mu \\
=& \frac{81}{512}\pi^4 \int \cos\theta\ \cos\mu\, |\sin(\mu-\theta)\ \sin(\mu+\theta)| d\theta\, d\mu.
\end{aligned} \tag{E.8}$$

The integrals are over the ranges of the coordinates given in (E.4) and (E.5).

To do the last integral it is convenient to re-write the expression inside the absolute value as

$$\sin(\mu-\theta)\ \sin(\mu+\theta) = (\cos\theta + \cos\mu)(\cos\theta - \cos\mu), \tag{E.9}$$

and let

$$f(\theta,\mu) = \cos\theta\ \cos\mu(\cos\theta + \cos\mu)(\cos\theta - \cos\mu). \tag{E.10}$$

Then the volume is given by

$$\mathrm{Vol} = \frac{81}{512}\pi^4 \int |\mathrm{f}(\theta,\mu)|\mathrm{d}\theta\mathrm{d}\mu. \tag{E.11}$$

Computing this integral is a little tedious but straightforward. One must determine the values of θ and μ for which f is positive and for which it is negative.

Since θ and μ are both in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have

$$\begin{aligned} f(\theta,\mu) > &\ \ 0,\ \ -|\mu| < \theta < |\mu| \\ < &\ \ 0,\ \ -\frac{\pi}{2} < \theta < -|\mu|,\ \ |\mu| < \theta < \frac{\pi}{2}, \end{aligned} \tag{E.12}$$

so that

$$\begin{aligned} |f(\theta,\mu)| = &\ \ f(\theta,\mu),\ \ -|\mu| < \theta < |\mu| \\ = &\ \ -f(\theta,\mu),\ \ -\frac{\pi}{2} < \theta < -|\mu|,\ \ |\mu| < \theta < \frac{\pi}{2}. \end{aligned} \tag{E.13}$$

Therefore,

$$\int |f(\theta,\mu)| d\theta\, d\mu = \int_+ f - \int_- f, \tag{E.14}$$

where the first integral is over the region where f is positive and the second integral is over the region where f is negative. One can see that

$$\int_+ f = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\mu \int_{-|\mu|}^{|\mu|} f(\theta,\mu) d\theta, \tag{E.15}$$

and that

$$\int_- f = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\mu \left(\int_{-\frac{\pi}{2}}^{-|\mu|} + \int_{|\mu|}^{\frac{\pi}{2}} \right) f(\theta,\mu) d\theta. \tag{E.16}$$

These double integrals are readily computed by Mathematica. We find them to be

$$\begin{aligned}\int_{+} f &= \frac{2}{3} \\ \int_{-} f &= -\frac{2}{3}.\end{aligned} \tag{E.17}$$

So the volume is found to be

$$\mathrm{Vol} = \frac{27}{128}\pi^4. \tag{E.18}$$

This value for the volume is in agreement with what is calculated in [76].

F Linearized bosonic field equations of $D = 11$ supergravity

(I) The bosonic field equations of d=11 supergravity

The bosonic sector of $d = 11$ supergravity consists of a metric g_{AB} and a 3-form potential A_{ABC}. The exterior derivative of the 3-form potential gives a 4-form flux F_{ABCD}. Classically these fields must satisfy the $d = 11$ supergravity bosonic field equations. These field equations consist of an Einstein equation, a Maxwell equation, and the Bianchi identity for F_{ABCD}.

The Einstein equation is

$$R_{AB} = \frac{1}{3}F_{ACDE}F_B{}^{CDE} - \frac{1}{36}g_{AB}F_{CDEF}F^{CDEF}, \tag{F.1}$$

the Maxwell equation is

$$\nabla_A F^{ABCD} = -\frac{1}{576}\epsilon^{BCDEFGHIJKL}F_{EFGH}F_{IJKL}, \tag{F.2}$$

and the Bianchi identity is

$$\nabla_{[A}F_{BCDE]} = 0. \tag{F.3}$$

Let g_{AB} and F_{ABCD} be a solution to the field equations. We take this solution to be the 'background solution'. We would like to perturb the background fields, g_{AB} and F_{ABCD}, in such a way that the perturbed fields still satisfy the equations of motion.

Let h_{AB} and f_{MNPQ} be the perturbations to the metric and flux, respectively. The perturbed fields are then

$$\begin{aligned} \mathsf{g}_{AB} &= \quad g_{AB} + h_{AB} \\ \mathcal{F}_{ABCD} &= \quad F_{ABCD} + f_{MNPQ}. \end{aligned} \tag{F.4}$$

We would like to put these perturbed fields into the equations of motion and determine the equations the perturbations h_{AB} and f_{ABCD} must satisfy in order for g_{AB} and $\mathcal{F}_{ABCD}$ to be solutions. The equations h_{AB} and f_{ABCD} must satisfy to first order are the 'linearized field equations'.

(II) Linearizing the Einstein equation

Let $\mathcal{R}_{AB}$ be the Ricci tensor obtained from the perturbed metric g_{AB}. Then the Einstein equation is

$$\mathcal{R}_{AB} = \frac{1}{3}\mathcal{F}_{ACDE}\mathcal{F}_B^{\ CDE} - \frac{1}{36}\mathsf{g}_{AB}\mathcal{F}_{CDEF}\mathcal{F}^{CDEF}. \tag{F.5}$$

We want to expand each of the terms to first order in the perturbations and obtain the linearized Einstein equation.

(II.1) Linearizing $\mathcal{R}_{AB}$

Expanding $\mathcal{R}_{AB}$ to first order gives

$$\mathcal{R}_{AB} = R_{AB} + \delta R_{AB}, \tag{F.6}$$

where

$$\delta R_{AB} = \frac{1}{2}\hat{\Delta} h_{AB} + \breve{\nabla}_{(A}\breve{\nabla}^{C} h_{B)C} - \frac{1}{2}\breve{\nabla}_{A}\breve{\nabla}_{B} h^{C}_{C}. \tag{F.7}$$

$\breve{\nabla}_A$ is the covariant derivative for the background metric. $\hat{\Delta} h_{AB}$ is called the 'Lichnerowicz operator', and its action on h_{AB} is

$$\hat{\Delta} h_{AB} = -\breve{\nabla}_{C}\breve{\nabla}^{C} h_{AB} - 2R_{ACBD}h^{CD} + 2R_{(A}^{\ C} h_{B)C}. \tag{F.8}$$

(II.2) Linearizing $\mathcal{F}_{ACDE}\mathcal{F}_{B}^{CDE}$

Expanding $\mathcal{F}_{ACDE}\mathcal{F}_{B}^{CDE}$ gives

$$\begin{aligned}\mathcal{F}_{ACDE}\mathcal{F}_{B}{}^{CDE} &= \mathsf{g}^{MC}\mathsf{g}^{ND}\mathsf{g}^{PE}\mathcal{F}_{ACDE}\mathcal{F}_{BMNP} \\ &= (g^{MC} - h^{MC})\ldots(F_{BMNP} + f_{BMNP}) \\ &= F_{ACDE}F_{B}{}^{CDE} + O(h) + O(f),\end{aligned} \tag{F.9}$$

where

$$\begin{aligned}O(h) &= -(h^{MC}g^{ND}g^{PE} + g^{MC}h^{ND}g^{PE} + g^{MC}g^{ND}h^{PE})F_{ACDE}F_{BMNP} \\ O(f) &= g^{MC}g^{ND}g^{PE}F_{ACDE}f_{BMNP} + g^{MC}g^{ND}g^{PE}f_{ACDE}F_{BMNP}.\end{aligned} \tag{F.10}$$

After some straightforward manipulations

$$\begin{aligned}O(h) &= -3F_{A}{}^{CNP}F_{B}{}^{M}{}_{NP}h_{MC} \\ O(f) &= 2F_{(A}{}^{MNP}f_{B)MNP}.\end{aligned} \tag{F.11}$$

So

$$\mathcal{F}_{ACDE}\mathcal{F}_{B}{}^{CDE} = F_{ACDE}F_{B}{}^{CDE} - 3F_{A}{}^{CNP}F_{B}{}^{M}{}_{NP}h_{MC} + 2F_{(A}{}^{MNP}f_{B)MNP}. \tag{F.12}$$

(II.3) Linearizing $\mathsf{g}_{AB}\mathcal{F}_{CDEF}\mathcal{F}^{CDEF}$

Expanding $\mathsf{g}_{AB}\mathcal{F}_{CDEF}\mathcal{F}^{CDEF}$ gives

$$\begin{aligned}\mathsf{g}_{AB}\mathcal{F}_{CDEF}\mathcal{F}^{CDEF} &= \mathsf{g}_{AB}\mathsf{g}^{CM}\mathsf{g}^{DN}\mathsf{g}^{EP}\mathsf{g}^{FQ} \qquad \mathcal{F}_{CDEF}\mathcal{F}_{MNPQ} \\ &= (g_{AB}+h_{AB})(g^{CM}-h^{CM})\cdots \\ &\quad (F_{CDEF}+f_{CDEF})(F_{MNPQ}+f_{MNPQ}) \\ &= g_{AB}F_{CDEF}F^{CDEF}+O(h)+O(f).\end{aligned} \tag{F.13}$$

$O(h)$ is the part that is first order in the h_{AB}. It is

$$\begin{aligned}O(h) = &\ h_{AB}F_{CDEF}F^{CDEF} \\ &-g_{AB}(h^{CM}g^{DN}g^{EP}g^{FQ}+g^{CM}h^{DN}g^{EP}g^{FQ})F_{CDEF}F_{MNPQ} \\ &-g_{AB}(g^{CM}g^{DN}h^{EP}g^{FQ}+g^{CM}g^{DN}g^{EP}h^{FQ})F_{CDEF}F_{MNPQ}.\end{aligned} \tag{F.14}$$

After some straightforward manipulation

$$O(h) = h_{AB}F_{CDEF}F^{CDEF} - 4g_{AB}h^{CM}F_{CDEF}F_M^{\ DEF}. \tag{F.15}$$

$O(f)$ is the part that is first order in the f_{ABCD}. It is straighforward to obtain that

$$O(f) = 2g_{AB}F^{MNPQ}f_{MNPQ}. \tag{F.16}$$

So

$$\begin{aligned}\mathsf{g}_{AB}\mathcal{F}_{CDEF}\mathcal{F}^{CDEF} = &\ g_{AB}F_{CDEF}F^{CDEF}+h_{AB}F_{CDEF}F^{CDEF} \\ &-4g_{AB}h^{CM}F_{CDEF}F_M^{\ DEF}+2g_{AB}F^{MNPQ}f_{MNPQ}.\end{aligned} \tag{F.17}$$

(II.4) The linearized Einstein equation

Putting together equations (F.5), (F.6), (F.7), (F.12), and (F.17) gives the linearlized

Einstein equation. It is

$$\frac{1}{2}\hat{\Delta}h_{AB} + \breve{\nabla}_{(A}\breve{\nabla}^C h_{B)C} - \frac{1}{2}\breve{\nabla}_A\breve{\nabla}_B h^C_C = -F_A{}^{CNP}F_B{}^M{}_{NP}h_{MC} - \frac{1}{36}h_{AB}F_{CDEF}F^{CDEF} + \frac{1}{9}g_{AB}h^{CM}F_{CDEF}F_M^{DEF} + \frac{2}{3}F_{(A}{}^{MNP}f_{B)MNP} - \frac{1}{18}g_{AB}F^{MNPQ}f_{MNPQ} \tag{F.18}$$

(III) Linearizing the Maxwell equation

The Maxwell equation is

$$\nabla_A\mathcal{F}^{ABCD} = -\frac{1}{576}\epsilon^{BCDEFGHIJKL}\mathcal{F}_{EFGH}\mathcal{F}_{IJKL}. \tag{F.19}$$

We want to expand each of the terms to first order in the perturbations and obtain the linearized Maxwell equation.

(III.1) Linearizing $\nabla_A\mathcal{F}^{ABCD}$

Expanding $\nabla_A\mathcal{F}^{ABCD}$ gives

$$\begin{aligned}\nabla_A\mathcal{F}^{ABCD} &= \mathsf{g}^{AM}\mathsf{g}^{BN}\mathsf{g}^{CP}\mathsf{g}^{DQ}\nabla_A\mathcal{F}_{MNPQ} \\ &= (g^{AM}-h^{AM})\dots(g^{DQ}-h^{DQ})\nabla_A(F_{MNPQ}+f_{MNPQ}) \\ &= \breve{\nabla}_AF^{ABCD} + O(f) + O(h) + O(\partial h).\end{aligned} \tag{F.20}$$

$O(f)$ is the term that results from varying $\mathcal{F}$, $O(h)$ is the term that results from varying the 4 inverse metrics g^{AM}, and $O(\partial h)$ is the term that results from varying the Christoffel symbols in the covariant derivative ∇_A.

First, we obtain $O(f)$. It is straightforward to see that

$$O(f) = \breve{\nabla}_Af^{ABCD}. \tag{F.21}$$

Next, we obtain O(h). After shuffling terms around it is possible to obtain $O(h)$ in a nice, compact form.

$$
\begin{aligned}
O(h) = & -(h^{AM}g^{BN}g^{CP}g^{DQ} + g^{AM}h^{BN}g^{CP}g^{DQ} \\
& \qquad + g^{AM}g^{BN}h^{CP}g^{DQ} + g^{AM}g^{BN}g^{CP}h^{DQ})\breve{\nabla}_A F_{MNPQ} \\
= & \; -h^{AM}\breve{\nabla}_A F_M{}^{BCD} - h^{BN}\breve{\nabla}_A F^A{}_N{}^{CD} - h^{CP}\breve{\nabla}_A F^{AB}{}_P{}^D - h^{DQ}\breve{\nabla}_A F^{ABC}{}_Q \\
= & \; -4h^{M[A}\breve{\nabla}_A F_M^{BCD]} \\
= & \; 4\breve{\nabla}_A F^{M[ABC}h^{D]}{}_M + 4(\breve{\nabla}_A h^{M[A})F_M{}^{BCD]} \\
= & \; 4\breve{\nabla}_A F^{M[ABC}h^{D]}{}_M - F^{BCDM}\breve{\nabla}_A h^A{}_M + 3F^{AM[BC}\breve{\nabla}_A h^{D]}{}_M
\end{aligned}
\tag{F.22}
$$

So

$$
O(h) = 4\breve{\nabla}_A F^{M[ABC}h^{D]}{}_M - F^{BCDM}\breve{\nabla}_A h^A{}_M + 3F^{AM[BC}\breve{\nabla}_A h^{D]}{}_M. \tag{F.23}
$$

Now, we want to obtain $O(\partial h)$. This term arises from varying the Christoffel symbols.

$$
\begin{aligned}
\Gamma^R_{AM} = & \; \frac{1}{2}\mathsf{g}^{RS}(\partial_A \mathsf{g}_{MS} + \partial_M \mathsf{g}_{AS} - \partial_S \mathsf{g}_{AM}) \\
= & \; \frac{1}{2}(g^{RS} - h^{RS})[\partial_A(g_{MS} + h_{MS}) + \partial_M(g_{AS} + h_{AS}) - \partial_S(g_{AM} + h_{AM})] \\
= & \; \breve{\Gamma}^R_{AM} + \gamma^R_{AM},
\end{aligned}
\tag{F.24}
$$

where $\breve{\Gamma}^R_{AM}$ is the Christoffel symbol for the background metric and

$$
\gamma^R_{AM} = -\frac{1}{2}h^{RS}(\partial_A g_{MS} + \partial_M g_{AS} - \partial_S g_{AM}) + \frac{1}{2}g^{RS}(\partial_A h_{MS} + \partial_M h_{AS} - \partial_S h_{AM}). \tag{F.25}
$$

It is possible to express the first term of γ^R_{AM} in terms of $\breve{\Gamma}^K_{AM}$. Doing so gives

$$
\gamma^R_{AM} = -h^{RS}g_{SK}\breve{\Gamma}^K_{AM} + \frac{1}{2}g^{RS}(\partial_A h_{MS} + \partial_M h_{AS} - \partial_S h_{AM}). \tag{F.26}
$$

Furthermore, using the fact that

$$
\breve{\nabla}_A h_{MS} = \partial_A h_{MS} - \breve{\Gamma}^K_{AM}h_{KS} - \breve{\Gamma}^K_{AS}h_{MK}, \tag{F.27}
$$

it is possible to show that

$$\gamma^R_{AM} = \frac{1}{2}(\breve{\nabla}_A h_M{}^R + \breve{\nabla}_M h_A{}^R - \breve{\nabla}^R h_{AM}). \tag{F.28}$$

The covariant derivative of F is

$$\begin{aligned} \nabla_A F_{MNPQ} = & \ \partial_A F_{MNPQ} + 4\Gamma^R_{A[M} F_{NPQ]R} \\ = & \ \breve{\nabla}_A F_{MNPQ} + 4\gamma^R_{A[M} F_{NPQ]R}, \end{aligned} \tag{F.29}$$

so

$$\begin{aligned} O(\partial h) = & \ 4g^{AM} g^{BN} g^{CP} g^{DQ} \gamma^R_{A[M} F_{NPQ]R} \\ = & \ 2\left[(\breve{\nabla}_A h_R{}^{[A}) F^{BCD]R} + (\breve{\nabla}^{[A} h_{AR}) F^{BCD]R} - (\breve{\nabla}_R h_A{}^{[A}) F^{BCD]R}\right]. \end{aligned} \tag{F.30}$$

Expanding the antisymmetrizations and simplifying gives

$$O(\partial h) = F^{BCDR} \breve{\nabla}_A h^A{}_R - 3F^{AR[BC} \breve{\nabla}_A h_R{}^{D]} - \frac{1}{2} F^{BCDR} \breve{\nabla}_R h_A{}^A. \tag{F.31}$$

Finally, we put the parts together to obtain

$$\breve{\nabla}_A \mathcal{F}^{ABCD} = \breve{\nabla}_A F^{ABCD} + \breve{\nabla}_A f^{ABCD} + 4\breve{\nabla}_A (F^{M[ABC} h^{D]}{}_M) - \frac{1}{2} F^{BCDR} \breve{\nabla}_R h_A{}^A \tag{F.32}$$

(III.2) Linearizing $\epsilon^{BCDEFGHIJKL} \mathcal{F}_{EFGH} \mathcal{F}_{IJKL}$

Expanding the right hand side of the Maxwell equation gives

$$\begin{aligned} \epsilon^{BCDEFGHIJKL} \mathcal{F}_{EFGH} \mathcal{F}_{IJKL} = & \ (-\mathsf{g})^{-1/2} \tilde{\epsilon}^{BCDEFGHIJKL} \\ & \quad (F_{EFGH} + f_{EFGH})(F_{IJKL} + f_{IJKL}) \\ = & \ \epsilon^{BCDEFGHIJKL} F_{EFGH} F_{IJKL} + O(f) + O(h), \end{aligned} \tag{F.33}$$

where g is the determinant of the metric, $O(f)$ is the term that arises from varying $\mathcal{F}$, and $O(h)$ is the term that arises from varying the determinant of the metric g.

It is straightforward to see that

$$O(f) = 2\epsilon^{BCDEFGHIJKL} F_{EFGH} f_{IJKL}. \tag{F.34}$$

To get $O(h)$ one needs to use the fact that, to first order,

$$det(\mathsf{g}_{AB}) = det(g_{AB}) + det(g_{AB}) Tr(g^{-1}h), \tag{F.35}$$

where $\mathsf{g}^{-1}h$ is the matrix multiplication of the inverse metric and the metric perturbation.

So

$$\begin{aligned}
(-\mathsf{g})^{-1/2} &= (-g - gTr(g^{-1}h))^{-1/2} \\
&= (-g)^{-1/2}(1 + Tr(g^{-1}h))^{-1/2} \\
&= (-g)^{-1/2}(1 - \frac{1}{2}Tr(g^{-1}h)) \\
&= (-g)^{-1/2} - \frac{1}{2}(-g)^{-1/2} Tr(g^{-1}h).
\end{aligned} \tag{F.36}$$

This gives

$$O(h) = -\frac{1}{2} Tr(g^{-1}h)\epsilon^{BCDEFGHIJKL} F_{EFGH} F_{IJKL}. \tag{F.37}$$

(III.3) The linearized Maxwell equation

Putting together equations (F.19), (F.32), (F.33), (F.34), and (F.37) gives the linearized Maxwell equation. It is

$$\begin{aligned}
\breve{\nabla}_A f^{ABCD} + 4\breve{\nabla}_A (F^{M[ABC} h^{D]}{}_M) - \frac{1}{2} F^{BCDR} \breve{\nabla}_R h_A{}^A = -\frac{1}{288}\epsilon^{BCDEFGHIJKL} F_{EFGH} f_{IJKL} \\
-\frac{1}{1152} Tr(g^{-1}h)\epsilon^{BCDEFGHIJKL} F_{EFGH} F_{IJKL}
\end{aligned} \tag{F.38}$$

(IV) The linearized Bianchi identity

Expanding the Bianchi identity gives

$$\nabla_{[A}\mathcal{F}_{BCDE]} = \nabla_{[A} F_{BCDE]} + \nabla_{[A} f_{BCDE]} = 0. \tag{F.39}$$

So the linearized Bianchi identity is

$$\breve{\nabla}_{[A}f_{BCDE]} = 0. \tag{F.40}$$

Note that there is no need to consider the variation of the covariant derivative because the Christoffel symbols vanish when taking an exterior derivative.

(V) Summary

So to summarize, the linearized field equations are the linearized Einstein equation

$$\begin{aligned} \frac{1}{2}\hat{\Delta}h_{AB} + \breve{\nabla}_{(A}\breve{\nabla}^{C}h_{B)C} - \frac{1}{2}\breve{\nabla}_{A}\breve{\nabla}_{B}h^{C}_{C} &= -F_{A}{}^{CNP}F_{B}{}^{M}{}_{NP}h_{MC} - \frac{1}{36}h_{AB}F_{CDEF}F^{CDEF} \\ &\quad + \frac{1}{9}g_{AB}h^{CM}F_{CDEF}F_{M}^{DEF} + \frac{2}{3}F_{(A}{}^{MNP}f_{B)MNP} \\ &\quad - \frac{1}{18}g_{AB}F^{MNPQ}f_{MNPQ} \end{aligned} \tag{F.41}$$

the linearized Maxwell equation

$$\begin{aligned} \breve{\nabla}_{A}f^{ABCD} + 4\breve{\nabla}_{A}(F^{M[ABC}h^{D]}{}_{M}) - \frac{1}{2}F^{BCDR}\breve{\nabla}_{R}h_{A}{}^{A} &= -\frac{1}{288}\epsilon^{BCDEFGHIJKL}F_{EFGH}f_{IJKL} \\ &\quad - \frac{1}{1152}Tr(g^{-1}h)\epsilon^{BCDEFGHIJKL}F_{EFGH}F_{IJKL} \end{aligned} \tag{F.42}$$

and the linearized Bianchi identity

$$\breve{\nabla}_{[A}f_{BCDE]} = 0. \tag{F.43}$$

G Conventions of [56] and [50]

In this appendix we translate the definition of the scalar mass used in [56] and [50] into the definition of it used here. We also clarify the 'Lichnerowicz-like' operator used in [56] and demonstrate that this paper agrees with our result that $\Delta_L = \Delta_2 + 4$.

AdS Klein-Gordan equation and scalar field mass used here

Here the Klein-Gordan equation for a scalar field in AdS space is taken to be

$$\Box\phi = \frac{\nu}{L^2}\phi. \tag{G.1}$$

$\Box$ is the Laplacian in AdS with the metric sign convention $(-+++)$, and L^2 is the AdS radius squared. $\nu = m^2L^2$ is regarded as the **dimensionless mass**, and $\frac{\nu}{L^2}$ is regarded as the **mass**. So the mass is the eigenvalue of the scalar field ϕ under the Laplacian, and the dimensionless mass is obtained from the mass by multiplying by L^2.

AdS Klein-Gordan equation and scalar field mass in [56] and [50]

In [56] and [50] the Klein-Gordan equation for a scalar field in AdS space is taken to be (equations (3.22a) and (3.22b) of [56])

$$(\Box_f - 32)\phi = -m_f^2\phi. \tag{G.2}$$

$\Box_f$ is the Laplacian in AdS with the metric sign convention $(+---)$.

In [56] and [50] m_f^2 is regarded as the **mass**. This Klein-Gordan equation is obtained from the one derived in [49], which is

$$(\Box_f + \frac{1}{3}R)\phi = -m_f^2\phi, \tag{G.3}$$

where R is the Ricci scalar of AdS.

It is important to note that some authors have a denominator of 6 instead of 3 in the Ricci scalar term in the Klein-Gordan equation, see e.g. [6]. In the Klein-Gordan equation above the denominator is 3 because the authors define their Riemann tensor so that it is $\frac{1}{2}$ of what it is traditionally ((A.1.28) of [7]).

Using this convention, the Ricci tensor for AdS in [72] is given to be

$$R^a{}_c = \frac{3}{2}\lambda\delta^a_c. \tag{G.4}$$

Comparing with (G.2) we see that in [56] and [50] the size of AdS is fixed so that $\lambda = -16$.

The mass here in terms of the mass of [56] and [50]

Here the traditional definition of the Riemann tensor is used, and the Ricci tensor is

$$R^a{}_c = -\frac{3}{L^2}\delta^a_c. \tag{G.5}$$

Setting the the right-hand-side of (G.4) to $\frac{1}{2}$ the right-hand-side of (G.5) gives

$$\lambda = -\frac{1}{L^2}, \tag{G.6}$$

so that the square of the AdS radius in [56] and [50] is

$$L^2 = \frac{1}{16}. \tag{G.7}$$

We would like to have the mass used here in terms the mass of [56] and [50]. To do this we note that for a given AdS radius $\Box_f = -\Box_p$ because [56] and [50] uses the opposite metric sign convention used here.

So the Klein-Gordan equation of [56] and [50] becomes

$$(-\Box - 32)\phi = -m_f^2\phi, \tag{G.8}$$

which with further massaging becomes

$$\Box\phi = (m_f^2 - 32)\phi. \tag{G.9}$$

Comparing with (G.1), and setting $L^2 = \frac{1}{16}$, gives

$$\nu = \frac{1}{16}(m_f^2 - 32). \tag{G.10}$$

D'Auria and Fre's 'Lichnerowicz-like' operator

In [56] the authors use what they call the 'Lichnerowicz-like' operator on symmetric tensors. In equation (2.11e) of that reference they give it to be

$$M_{(2)(0)^2}Y_{(\alpha\beta)} = \left[(\Box + 40)\delta^{(\alpha\beta)}_{(\lambda\mu)} - 4C^{\alpha\lambda\beta\mu}\right] Y_{(\lambda\mu)}. \tag{G.11}$$

The tensor $C^{\alpha\beta\mu\nu}$ is the Weyl tensor on the internal space, which is Einstein. In equation (2.9a) of [56] give it to be

$$C^{\alpha\beta\mu\nu} = R^{\alpha\beta}{}_{\mu\nu} - 4e^2\delta^{\alpha\beta}_{\mu\nu}, \tag{G.12}$$

where the first term is the Riemann tensor and the second term is the antisymmetrized product of δ's, i.e.

$$\delta^{\alpha\beta}_{\mu\nu} = \delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}. \tag{G.13}$$

Acting with the antisymmetrized δ's on the symmetric tensor Y gives

$$\begin{aligned} \delta^{\alpha\lambda}_{\beta\mu}Y_{(\lambda\mu)} &= \frac{1}{4}(\delta^{\alpha}_{\beta}\delta^{\lambda}_{\mu} - \delta^{\lambda}_{\beta}\delta^{\alpha}_{\mu})(Y_{\lambda\mu} + Y_{\mu\lambda}) \\ &= -\frac{1}{4}(Y_{\beta\alpha} + Y_{\alpha\beta}) \\ &= -\frac{1}{2}Y_{(\alpha\beta)}. \end{aligned} \tag{G.14}$$

To get the second equality it is assumed that the symmetric tensor Y is traceless.

Therefore acting with the Weyl tensor on Y gives

$$C^{\alpha\lambda\beta\mu}Y_{(\lambda\mu)} = R^{\alpha\lambda}{}_{\beta\mu}Y_{(\lambda\mu)} + 2e^2Y_{(\alpha\beta)}, \tag{G.15}$$

and putting this, with $e = 1$, into the action of the Lichnerowicz-like operator given in equation (G.11) gives

$$M_{(2)(0)^2}Y_{(\alpha\beta)} = \Box Y_{(\alpha\beta)} - 4R^{\alpha\lambda}_{\cdot\beta\mu}Y_{(\lambda\mu)} + 32Y_{(\alpha\beta)}. \tag{G.16}$$

It is important to note that the above is not the usual Lichnerowicz operator, which is given in equation (V.4.111e) of [49], but in the case of an Einstein space differs from it by a constant.

The usual Lichnerowicz operator does not appear to be given in [56], but it is given in equation (V.4.111e) of [49] to be

$$-\Delta_L Y_{(ab)} = \left[(\Box + 48)\delta_{(ab)}^{(de)} - 4R_{a\cdot b\cdot}^{\cdot d\cdot e}\right] Y_{(de)}. \tag{G.17}$$

(N.B. In equation (V.4.109) of the same publication, i.e. reference [49], the authors give the same operator as above, but with the opposite sign in front of the Riemann tensor. The operator above seems to be the correct one.)

Comparing equations (G.16) and (G.17) gives that

$$-\Delta_L = M_{(2)(0)^2} + 16, \tag{G.18}$$

where the operator on the left-hand-side is the usual Lichnerowicz operator and the operator on the right-hand-side is what the authors of [56] call the 'Lichnerowicz-like' operator.

From equations (4.27), (4.71), (3.23b), and (3.23g) of [56] one obtains

$$-M_{(2)(0)^2} = M_{(1)^2(0)}, \tag{G.19}$$

where the operator on the right-hand-side is the Hodge-de-Rham operator on 2-forms.

This relation together with equation (G.18) gives

$$-\Delta_L = -M_{(1)^2(0)} + 16, \tag{G.20}$$

where the operator on the left-hand-side is the usual Lichnerowitz operator given in [49] and the operator on the right-hand-side is the Hodge-de-Rham operator on 2-forms.

Equations (2.11c) of [56] and (V.4.111c) of [49] both give

$$M_{(1)^2(0)} Y_{[\alpha\beta]} = 3D^\mu D_{[\mu} Y_{\alpha\beta]}. \tag{G.21}$$

Equation (G.20) is for the case when the size of the space is such that $R_{\alpha\beta} = 24g_{\alpha\beta}$, for the usual definition of the Riemann tensor. In the case when the size of the space is such that $R_{\alpha\beta} = 6g_{\alpha\beta}$ the differential operators get scaled down by $\frac{1}{4}$.

Hence, in the case that size of the space is such that $R_{\alpha\beta} = 6g_{\alpha\beta}$, the relation that is equation (G.20) becomes

$$-\Delta_L = -M_{(1)^2(0)} + 4, \tag{G.22}$$

where the operator on the left-hand-side is the usual Lichnerowitz operator given in [56] and the operator on the right-hand-side is the Hodge-de-Rham operator given in [56] and [49].

H Toric homogeneous Sasaki-Einstein manifolds via Kähler quotient

The Kähler quotient provides a straightforward way to construct a Kähler manifold from a higher-dimensional one. The higher-dimensional Kähler manifold is taken to be a simpler space, e.g. typically $\mathbb{C}^n$. As a result, objects in the constructed lower-dimensional space, e.g. the metric and Kähler 2-form, can be more easily described in terms of those in this simpler higher-dimensional space.

In this appendix, we explain the Kähler and hyper-Kähler quotients, and obtain the toric homogeneous Sasaki-Einstein manifolds, $M^{3,2}$, $Q^{1,1,1}$, and $N^{1,1}$ in terms of them.

For more details on the Kähler and hyper-Kähler quotients, see [87], [88], [89], and [90].

Kähler quotient

Suppose a Lie group G acts on a symplectic manifold M. The basis elements of the Lie algebra $\mathfrak{g}$ are vector fields V^a on M, where the index a runs from 1 to $\mathrm{Dim}(\mathfrak{g})$. Each V^a is a vector field that can in turn be written in terms of the $\partial/\partial x^i$, where the x^i are local coordinates on M. These vector fields V^a can be regarded as Hamiltonian vector fields that generate Hamiltonian phase flows on the manifold M. In other words, a vector field V^a on M, which is a basis element of $\mathfrak{g}$, gives rise to a Hamiltonian μ^a on M given by

$$d\mu^a = i_{V^a}\omega, \tag{H.1}$$

where ω is the symplectic form on M. The μ^a are components of an object μ, which is called a **moment map**, and is also known as a **momentum map**.

The moment map μ is to be regarded as a map from M to the dual of the Lie algebra, i.e. $\mu : M \to \mathfrak{g}^*$. In other words, μ is to be regarded as a 1-form on M. μ can be written as $\mu = \mu^a V_a$, where V_a is the 1-form dual to the vector field V^a, so that $V_a(V^b) = \delta^b_a$. Given any element $\xi = \xi_a V^a$ of $\mathfrak{g}$ one has that

$$d\langle \mu, \xi \rangle = d(\mu^a \xi_a) = i_\xi \omega, \tag{H.2}$$

and so $\langle \mu, \xi \rangle$ can be seen as a Hamiltonian on M with corresponding Hamiltonian vector field ξ.

In the case of Euclidean three-dimensional configuration space, the phase space, i.e. the space of positions together with momenta, is six-dimensional. In this case, $M = \mathbb{C}^3$. When the group of symmetries is taken to be the group of spatial rotations about the origin, the components of the moment map are the values of the angular momentum, and when the group of symmetries is taken to be spatial translations, the components of the moment map are the values of the momentum [87]. Hence the name 'moment map' or 'momentum map'.

Given an element in the dual of the Lie algebra $p \in \mathfrak{g}^*$, one can consider the set of points in M defined by $M_p = \mu^{-1}(p)$, which is the set of all the points in M that map under the moment map to the dual Lie algebra element p. Such a set of points is called a **level set**.

In general, a group action will move a point in a level set into another level set, but it is shown in [87] that M_p is fixed under the action of the subgroup G_p of G consisting of those elements $g \in G$ such that $Ad_g^* p = p$. In the case that $p = 0$ one of course has that G_p is the entire group G.

Since M_p is fixed under G_p one can mod out the action of G_p on M_p and consider the space of G_p-orbits of M_p. This quotient space is called, e.g. in [87], a **reduced phase space**. In the case that M is a Kähler manifold, the Kähler 2-form is the symplectic form and the quotient is called a **Kähler quotient**.

The simplest example of the Kähler quotient construction is when the starting manifold is $M = \mathbb{C}^n$, the symmetry group is $G = \mathrm{U}(1)^{\mathrm{r}}$, and the level set is taken for the dual Lie algebra element $p = 0$. The subgroup that fixes the level set is of course $G_p = G$. The action of the group G is given by

$$z_i \to e^{i\xi_a Q_i^a} z_i, \tag{H.3}$$

where $\xi = (\xi_1, \ldots, \xi_r)$ is an element of the Lie algebra $u(1)^r \cong \mathbb{R}^r$. To obtain the Lie algebra vector fields, consider a scalar function $F(z_i, \bar{z}_i)$ on M. Under the group action, one has

$$\begin{aligned} F(z_i, \bar{z}_i) &\rightarrow F(e^{i\xi_a Q_i^a} z_i, e^{-i\xi_a Q_i^a} \bar{z}_i) \\ &\simeq F((1 + i\xi_a Q_i^a) z_i, (1 - i\xi_a Q_i^a) \bar{z}_i) \\ &\simeq F(z_i, \bar{z}_i) + \sum_{i=1}^{n} (i\xi_a Q_i^a z_i \frac{\partial F}{\partial z_i} - i\xi_a Q_i^a \bar{z}_i \frac{\partial F}{\partial \bar{z}_i}). \end{aligned} \tag{H.4}$$

From this one can see that the Lie algebra vectors are

$$V^a = \frac{\partial}{\partial \xi_a} = i \sum_{i=1}^{n} Q_i^a (z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}). \tag{H.5}$$

Inserting this vector into the equation

$$d\mu^a = i_{V^a} \omega,$$

with Kähler form

$$\omega = -i \sum_i dz_i \wedge d\bar{z}_i, \tag{H.6}$$

one can solve for the moment map. The right hand side of the equation gives

$$\begin{aligned} i_{V^a} \omega &= -i \sum_i dz_i(V^a) d\bar{z}_i - dz_i d\bar{z}_i(V^a) \\ &= \sum_i Q_i^a (z_i d\bar{z}_i + \bar{z}_i dz_i) \\ &= \sum_i Q_i^a d(z_i \bar{z}_i). \end{aligned} \tag{H.7}$$

One can then see that the moment map is given by

$$\mu^a = \sum_i Q_i^a |z_i|^2 - t^a, \tag{H.8}$$

where the t^a are integration constants. Setting the integration constants equal to 0 and restricting to the level set corresponding to the 0 element of $\mathfrak{g}^*$ gives the set of points defined by the r algebraic equations ($a = 1, \ldots, r$)

$$\sum_i Q_i^a |z_i|^2 = 0. \tag{H.9}$$

The space of orbits obtained by further quotienting out by the group action $z_i \to e^{i\xi_a Q_i^a} z_i$ is the Kähler quotient. An important fact is that if the charges satisfy the condition $\sum_i Q_i^a = 0$ for each a, then the resulting space is a toric Calabi-Yau manifold.

$M^{3,2}$ and $Q^{1,1,1}$ via Kähler quotients

If one starts with the space $M = \mathbb{C}^5$ parameterized by the complex coordinates $(u^1, u^2, u^3, v^1, v^2)$, and takes the Kähler quotient by U(1), with charge 2 for the u^i and charge -3 for the v^i,

	u^i	v^i
U(1) charges	2	-3

then one obtains the space defined by the equation

$$2(|u^1|^2 + |u^2|^2 + |u^3|^2) = 3(|v^1|^2 + |v^1|^2), \tag{H.10}$$

with the coordinates identified according to the U(1) action $(u^i, v^i) \to (e^{2i\xi} u^i, e^{-3i\xi} v^i)$.

This space is in fact the Calabi-Yau cone over the homogeneous Sasaki-Einstein manifold $M^{3,2}$. $M^{3,2}$ is obtained by further restricting to a fixed radius in the cone, which is achieved by setting

$$2(|u^1|^2 + |u^2|^2 + |u^3|^2) = 3(|v^1|^2 + |v^1|^2) = 1. \tag{H.11}$$

The homogeneous Sasaki-Einstein manifold $Q^{1,1,1}$ can also be obtained as a Kähler quotient. If one starts with the space $M = \mathbb{C}^6$ parameterized by the complex coordinates $(a^1, a^2, b^1, b^2, c^1, c^2)$, and takes the Kähler quotient by $\mathrm{U}(1)^2$ with charges as given in the table,

	a^i	b^i	c^i
U(1)$_1$ charges	1	0	-1
U(1)$_2$ charges	0	1	-1

then one obtains the space defined by the equations

$$|a^1|^2 + |a^2|^2 = |b^1|^2 + |b^2|^2 = |c^1|^2 + |c^2|^2, \tag{H.12}$$

with the coordinates identified according to the U(1)^2 action $(a^i, b^i, c^i) \to (e^{i\xi_1}a^i, e^{i\xi_2}b^i, e^{-i\xi_1-i\xi_2}c^i)$.

This space is in fact the cone over the homogeneous Sasaki-Einstein manifold $Q^{1,1,1}$. $Q^{1,1,1}$ is obtained by further restricting to a fixed radius in the cone, which is achieved by setting

$$|a^1|^2 + |a^2|^2 = |b^1|^2 + |b^2|^2 = |c^1|^2 + |c^2|^2 = 1. \tag{H.13}$$

Hyper-Kähler quotient

Whereas a Kähler manifold looks locally like $\mathbb{C}^n$, a hyper-Kähler manifold looks locally like $\mathbb{H}^n$, where $\mathbb{H}$ is the space of quaternions $q = a+ib+jc+kd$, $a, b, c, d \in \mathbb{R}$. The imaginary unit i in $\mathbb{C}$ gives rise in Kähler manifolds to the complex structure J, and analogously, in hyper-Kähler manifolds the units i, j, and k in $\mathbb{H}$ give rise to three complex structures, I, J, and K. Practically speaking, Kähler 2-forms are obtained by lowering the upper indices on the complex structures with the metric. So a hyper-Kähler manifold has three Kähler 2-forms as well.

The space of quaternions $\mathbb{H}$ can be seen as $\mathbb{R}^4$, so the unit quaternions can be seen as the 3-sphere S^3. S^3 is the same as the Lie group SU(2) when considered as a manifold, and when considering the multiplication of quaternions, the unit quaternions can be identified as SU(2).

More concretely, a quaternion can be represented in terms of the Pauli matrices, and in terms of two complex numbers as

$$q = q^4 \mathbb{1}_2 + i\vec{\sigma}\cdot\vec{q} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}, \tag{H.14}$$

where $u = q^4 + iq^3$ and $v = q^2 + iq^1$. The units i, j, and k are represented as the Pauli matrices, $i\sigma^1$, $i\sigma^2$, and $i\sigma^3$, respectively. This representation of the quaternions makes it clear that the units i, j, and k transform as a triplet, i.e. in the adjoint representation, under SU(2).

Since the unit quaternions i, j, and k transform as a triplet under SU(2), the three complex structures and the three Kähler 2-forms do as well. The triplet of Kähler 2-forms for $\mathbb{H}$, which transform in the adjoint representation of SU(2), is given by the relation

$$i\vec{\omega}\cdot\vec{\sigma} = \frac{1}{2} dq \wedge d\bar{q}, \tag{H.15}$$

which more explicitly is

$$i \begin{pmatrix} \omega^3 & \omega^1 - i\omega^2 \\ \omega^1 + i\omega^2 & -\omega^3 \end{pmatrix} = \begin{pmatrix} du & dv \\ -d\bar{v} & d\bar{u} \end{pmatrix} \wedge \begin{pmatrix} d\bar{u} & -dv \\ d\bar{v} & du \end{pmatrix}. \tag{H.16}$$

(Note that the conjugate of a quaterion is $\bar{q} = a - ib - jc - kd$.) This relation gives the Kähler 2-forms to be

$$\begin{aligned} \omega^3 &= -\frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v}) \\ \omega^1 - i\omega^2 &= i(du \wedge dv). \end{aligned} \tag{H.17}$$

If there is a Lie group G that acts on a hyper-Kähler manifold M, then there is a construction, called the **hyper-Kähler quotient**, that gives a hyper-Kähler manifold of lower dimension. In what follows, we assume that the starting hyper-Kähler manifold is $M = \mathbb{H}^n$, and that the Lie group is of the form $G = \mathrm{U}(1)^{\mathrm{r}}$.

In particular, G is taken to act as

$$q_i \to q_i e^{iQ_i^a \sigma^3 \xi_a}, \tag{H.18}$$

which in terms of the u_i and v_i is

$$\begin{aligned} u_i &\to u_i e^{iQ_i^a \xi_a} \\ v_i &\to v_i e^{-iQ_i^a \xi_a}. \end{aligned} \tag{H.19}$$

In the same way that they were derived in the Kähler quotient case, i.e. by Taylor expanding a scalar function to first order, one can derive the Lie algebra vector fields in the hyper-Kähler case. They are found to be

$$V^a = \frac{\partial}{\partial \xi_a} = i \sum_i Q_i^a (u_i \frac{\partial}{\partial u_i} - \bar{u}_i \frac{\partial}{\partial \bar{u}_i} - v_i \frac{\partial}{\partial v_i} + \bar{v}_i \frac{\partial}{\partial \bar{v}_i}). \tag{H.20}$$

In the Kähler case, each component of the moment map was a scalar. However, in the hyper-Kähler case there is a triplet of Kähler 2-forms, so each component of the moment map will be a triplet. In particular, the moment map is given by

$$d\vec{\mu}^a = i_{V^a} \vec{\omega}. \tag{H.21}$$

Plugging the Lie algebra vector fields and the Kähler 2-forms into this equation, one can obtain the moment map in the same way it was derived in the Kähler quotient case. One finds

$$\begin{aligned} \mu_3^a &= \frac{1}{2} \sum_i Q_i^a (|u_i|^2 - |v_i|^2) \\ \mu_1^a - i\mu_2^a &= -\sum_i Q_i^a u_i v_i. \end{aligned} \tag{H.22}$$

As in the Kähler case, we care about the level set corresponding to the 0 element of the dual Lie algebra. This set of points is the solution to the equations ($a = 1, \ldots, r$)

$$\sum_i Q_i^a(|u_i|^2 - |v_i|^2) = 0$$
$$\sum_i Q_i^a u_i v_i = 0. \tag{H.23}$$

The space obtained by further quotienting out by the group action $q_i \to q_i e^{iQ_i^a \sigma^3 \xi_a}$ is the hyper-Kähler quotient.

$N^{1,1}$ as a hyper-Kähler quotient

If one starts with the space $M = \mathbb{H}^3$ (or $\mathbb{C}^6$) parameterized by the coordinates $(u^1, v^1, u^2, v^2, u^3, v^3)$, and takes the hyper-Kähler quotient by U(1), with charge 1 for each $q^i = (u^i, v^i)$,

	q^1	q^2	q^3
$U(1)$ charges	1	1	1

then one obtains the space defined by the equations

$$\sum_i |u_i|^2 - |v_i|^2 = 0$$
$$\sum_i u_i v_i = 0, \tag{H.24}$$

with the coordinates identified according to the U(1) action $(u^i, v^i) \to (e^{i\xi} u^i, e^{-i\xi} v^i)$.

This space is in fact the Calabi-Yau cone over the homogeneous Sasaki-Einstein manifold $N^{1,1}$. $N^{1,1}$ is obtained by further restricting to a fixed radius in the cone, which is achieved by setting

$$\sum_i |u_i|^2 = \sum_i |v_i|^2 = 1. \tag{H.25}$$

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