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On the generalized Jacobi equation

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Abstract The standard text-book Jacobi equation (equation of geodesic deviation) arises by linearizing the geodesic equation around some chosen geodesic, where the linearization is done with respect to the coordinates and the velocities. The generalized Jacobi equation, introduced by Hodgkinson in 1972 and further developed by Mashhoon and others, arises if the linearization is done only with respect to the coordinates, but not with respect to the velocities. The resulting equation has been studied by several authors in some detail for timelike geodesics in a Lorentzian manifold. Here we begin by briefly considering the generalized Jacobi equation on affine manifolds, without a metric; then we specify to lightlike geodesics in a Lorentzian manifold. We illustrate the latter case by considering particular lightlike geodesics (a) in Schwarzschild spacetime and (b) in a plane-wave spacetime.

Keywords General relativity · Light rays · Jacobi equation

1 Introduction

The Jacobi equation, also known as the equation of geodesic deviation, describes geodesics in the neighborhood of a reference geodesic. More precisely, the Jacobi equation is the linearization of the geodesic equation around the reference geodesic, where the linearization is done with respect to the coordinates and the velocities. Thus, the Jacobi equation describes geodesics which are close to the reference geodesic and whose velocities are close to the velocity of the reference geodesic. The Jacobi equation has important applications to General Relativity: for timelike geodesics, the Jacobi equation describes the relative acceleration of freely

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falling particles around a freely falling reference particle. This relative acceleration, which is determined by the curvature tensor along the worldline of the reference particle, can be interpreted as the tidal force produced by the gravitational field. For lightlike geodesics, the Jacobi equation determines the shape of light bundles around a reference light ray. Again, the expansion and distortion of the bundle is determined by the curvature tensor along the reference light ray. Detailed discussions of the Jacobi equation (or geodesic deviation equation) can be found in almost any text-book on General Relativity, see e.g. Synge [19], Misner et al. [14] or Hawking and Ellis [8].

If a geodesic is close to a reference geodesic but its velocity is not, the Jacobi equation does not give a valid approximation for it. Such a geodesic is properly described by an equation which is linearized only with respect to the coordinates but not with respect to the velocities. Such a generalized Jacobi equation was brought forward, for timelike geodesics in a general-relativistic spacetime, by Hodgkinson [9]. This equation describes tidal forces on particles whose relative velocity is not small. Applications to astrophysics were given by Mashhoon [12; 13]. The generalized Jacobi equation was independently rediscovered by Ciufolini [6]. More recently, its implications were studied in a series of papers by Chicone and Mashhoon [3; 4; 5].

In geometric terms, the ordinary Jacobi equation approximates the geodesic flow on a tubular neighborhood of the reference geodesic in phase space (i.e., in the cotangent bundle over the base manifold). By contrast, the generalized Jacobi equation approximates the geodesic flow on a neighborhood in phase space that is unrestricted in the fiber dimension.

All the references quoted above consider the generalized Jacobi equation for timelike geodesics, which is physically of particular relevance because of its relation to tidal forces. In this article we want to discuss the generalized Jacobi equation for lightlike geodesics. Then it describes light rays around a reference light ray. The domain where the generalized Jacobi equation is applicable but the Jacobi equation is not encompasses all neighboring light rays that are close to the reference light ray but whose velocities are not.

Although we are mainly interested in lightlike geodesics, it is worthwhile to note that the geodesic equation and, hence, the Jacobi equation and the generalized Jacobi equation can be formulated on an affine manifold, without a metric. All that is needed is a connection. We begin by deriving the affine generalized Jacobi equation in Sect. 2. In Sect. 3 we specify to lightlike geodesics of a Lorentzian metric. In this case, the generalized Jacobi equation can be used to describe light bundles with arbitrarily large opening angle around a reference light ray. Of course, if the opening angle is large, the light rays will leave the neighborhood of the reference light ray soon, unless they are being refocused towards the reference light ray. So the generalized Jacobi equation can be used, in general, as a valid approximation only close to the vertex of the bundle. The general results of this section are illustrated by considering a bundle around a circular light ray at $r = 3$ m in Schwarzschild spacetime. In Sect. 4 we consider lightlike Fermi normal coordinates which allow to write the generalized Jacobi equation in terms of the curvature tensor along the reference light ray. As an example, we consider a bundle of light rays in a plane-wave spacetime.

Throughout we work with the generalized Jacobi equation in the sense of Hodgkinson, Mashhoon etc. This equation is linearized with respect to the coordinates but retains full dependence of the velocities. As an alternative, one may approximate with respect to the coordinates and the velocities up to common order N . This approach, which dates back to Bażański [1], can be used to set up an iterative scheme. One begins by solving the ordinary Jacobi equation, then calculates the corrections due to second-order terms and so on. This method was applied to particle motion in Schwarzschild and Kerr spacetimes by Kerner et al. [7; 10].

This paper is dedicated to Bahram Mashhoon on occasion of his 60th birthday. I have very much profited from discussions with him and from reading his papers. In particular, I wish to thank him for introducing me to the subject of the generalized Jacobi equation.

2 The generalized Jacobi equation in an affine manifold

An affine manifold is a manifold with an affine connection ∇ . In local coordinates, the connection is determined by its connection coefficients $\Gamma_{\nu\sigma}^\mu$,

$$\nabla_{\partial_\nu} \partial_\sigma = \Gamma_{\nu\sigma}^\mu \partial_\mu. \quad (1)$$

Here and in the following, the dimension of the manifold is n , and we use the summation convention for greek indices running from 1 to n .

Whenever we have an affine manifold, we can consider the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma_{\nu\sigma}^\mu(x) \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0. \quad (2)$$

The solution curves $x^\mu(s)$ of (2) are the geodesics (autoparallels) of the affine connection. As a geodesic remains a geodesic under affine reparametrization, $s \mapsto as + b$, it is usual to refer to s as to an “affine parameter”. Note that only the symmetric part of $\Gamma_{\nu\sigma}^\mu$ enters into the geodesic equation. The antisymmetric part

$$T_{\nu\sigma}^\mu := \Gamma_{\nu\sigma}^\mu - \Gamma_{\sigma\nu}^\mu, \quad (3)$$

which is called the “torsion”, drops out from the geodesic equation. As in the following we are interested only in the geodesic equation, and in equations derived thereof, we may replace any connection by its symmetrized version. Therefore we will assume in the following that the connection is symmetric, $\Gamma_{\nu\sigma}^\mu = \Gamma_{\sigma\nu}^\mu$ (For the Jacobi equation and some of its generalizations in terms of a non-symmetric connection see, e.g., Swaminarayan and Safko [18]).

Now fix a geodesic $X^\mu(s)$,

$$\frac{d^2X^\mu}{ds^2} + \Gamma_{\nu\sigma}^\mu(X) \frac{dX^\nu}{ds} \frac{dX^\sigma}{ds} = 0. \quad (4)$$

We will call it the “reference geodesic”, and we will assume that it is known, i.e., that we have the X^μ explicitly as functions of the curve parameter s . Then the

geodesic equation for a neighboring curve $x^\mu(s) = X^\mu(s) + \xi^\mu(s)$ results from inserting $x^\mu(s) = X^\mu(s) + \xi^\mu(s)$ into (2) and subtracting (4),

$$\begin{aligned} \frac{d^2\xi^\mu}{ds^2} + \Gamma_{v\sigma}^\mu(X + \xi) \left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds} \right) \left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds} \right) \\ - \Gamma_{v\sigma}^\mu(X) \frac{dX^\nu}{ds} \frac{dX^\sigma}{ds} = 0. \end{aligned} \quad (5)$$

With $X^\mu(s)$ known, (5) is a system of second order ordinary differential equations for the n functions $\xi^\mu(s)$. It is the exact geodesic equation, expressed in terms of the coordinate difference $\xi^\mu(s)$ with respect to the reference geodesic $X^\mu(s)$.

If we linearize (5) with respect to ξ^μ , but not with respect to $d\xi^\mu/ds$, we get

$$\begin{aligned} \frac{d^2\xi^\mu}{ds^2} + \Gamma_{v\sigma}^\mu(X) \left(2 \frac{d\xi^\nu}{ds} \frac{dX^\sigma}{ds} + \frac{d\xi^\nu}{ds} \frac{d\xi^\sigma}{ds} \right) \\ + \partial_\tau \Gamma_{v\sigma}^\mu(X) \xi^\tau \left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds} \right) \left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds} \right) = 0. \end{aligned} \quad (6)$$

This is the generalized Jacobi equation, on an affine manifold, in arbitrary coordinates.

By contrast, if we linearize (5) both with respect to ξ^μ and with respect to $d\xi^\mu/ds$, we get

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma_{v\sigma}^\mu(X) 2 \frac{d\xi^\nu}{ds} \frac{dX^\sigma}{ds} + \partial_\tau \Gamma_{v\sigma}^\mu(X) \xi^\tau \frac{dX^\nu}{ds} \frac{dX^\sigma}{ds} = 0. \quad (7)$$

This is the ordinary Jacobi equation in arbitrary coordinates. To recover the standard text-book form, we have to introduce the covariant derivative along $X(s)$, which is defined by

$$\frac{D\eta^\mu}{ds} = \frac{d\eta^\mu}{ds} + \Gamma_{\rho\tau}^\mu(X) \eta^\rho \frac{dX^\tau}{ds} \quad (8)$$

for any η^ν , and the curvature tensor

$$R_{\tau v\sigma}^\mu = \partial_v \Gamma_{\tau\sigma}^\mu - \partial_\tau \Gamma_{v\sigma}^\mu + \Gamma_{v\lambda}^\mu \Gamma_{\tau\sigma}^\lambda - \Gamma_{\tau\lambda}^\mu \Gamma_{v\sigma}^\lambda. \quad (9)$$

Then a straight-forward calculation reduces (7) to

$$\frac{D^2\xi^\mu}{ds^2} + R_{\tau v\sigma}^\mu(X) \xi^\nu \frac{dX^\tau}{ds} \frac{dX^\sigma}{ds} = 0, \quad (10)$$

which is, indeed, the standard text-book form of the ordinary Jacobi equation.

The generalized Jacobi equation (6) is a second order non-linear ordinary differential equation for $\xi^\mu(s)$. It is non-autonomous because the coefficients $\Gamma_{v\sigma}^\mu(X)$ and $\partial_\tau \Gamma_{v\sigma}^\mu(X)$ are functions of the curve parameter s . It gives a valid approximation for all those geodesics for which the $\xi^\mu(s)$ are small, whereas the $d\xi^\mu(s)/ds$ need not be small. Of course, if the $d\xi^\mu(s)/ds$ are not small, the $\xi^\mu(s)$ will in general remain small only for a small interval of the parameter s . Such geodesics will leave the neighborhood of the reference geodesic soon and the generalized Jacobi equation will, in general, give a valid approximation for them only on a small interval of the parameter s .

The generalized Jacobi equation may be modified in two ways.

- One may prefer to choose a different, non-affine, parametrization for the (approximate) geodesics.
- One may wish to choose special coordinates such that the generalized Jacobi equation takes a simpler form and that it can be easier compared for two different geodesics.

Both is possible in an affine manifold, without referring to a metric.

As to the first item, it is often convenient to use on a geodesic a (non-affine) parameter u which coincides with one of the coordinates, say $u = x^n$. This is possible for all curves which are transverse to the hypersurfaces $x^n = \text{constant}$. Then the dependence of the x^n coordinate on the parameter is known, $x^n(u) = u$, and the geodesic equation reduces to a system of differential equations for the remaining coordinate functions $(x^1(u), \dots, x^{n-1}(u))$. The condition that two curves $X(u)$ and $X(u) + \xi(u)$, both parametrized by $u = x^n$, are geometrically close is equivalent to the condition that the $\xi^i(u)$ are small for $i = 1, \dots, (n-1)$. By contrast, if an affine parameter s is used it may be that not all of the $\xi^\mu(s)$ are small even if the two geodesics are geometrically close. This happens if the affine parameter on one of the two geodesics lags behind or runs ahead of the other.

Therefore we will now rewrite the generalized Jacobi equation with respect to the new curve parameter $u = x^n$. We first observe that the $\mu = n$ component of the geodesic equation (2) can be written in the form

$$\frac{d^2u}{ds^2} + \Gamma_{v\sigma}^n(x) \frac{dx^\nu}{du} \frac{dx^\sigma}{du} \left(\frac{du}{ds} \right)^2 = 0, \quad (11)$$

and the remaining $(n-1)$ components as

$$\frac{d}{ds} \left(\frac{du}{ds} \frac{dx^i}{du} \right) + \Gamma_{v\sigma}^i(x) \frac{dx^\nu}{du} \frac{dx^\sigma}{du} \left(\frac{du}{ds} \right)^2 = 0. \quad (12)$$

Here and in the following, latin indices i, j, \dots take values 1 to $(n-1)$. Calculating the first term of (12) with the product rule and inserting (11) results in

$$\frac{d^2x^i}{du^2} - \Gamma_{v\sigma}^n(x) \frac{dx^\nu}{du} \frac{dx^\sigma}{du} \frac{dx^i}{du} + \Gamma_{v\sigma}^i(x) \frac{dx^\nu}{du} \frac{dx^\sigma}{du} = 0. \quad (13)$$

Together with the equation $x^n(u) = u$, (13) determines the geodesics (or, more precisely, those geodesics that are transverse to the surfaces $x^n = \text{constant}$) with the non-affine parametrization by $u = x^n$. Equation (13) is a system of second order ordinary differential equations for the $(n-1)$ functions $x^i(u)$. Clearly, these differential equations are non-autonomous because the coefficients depend on $u = x^n$. Also, as a result of the reparametrization we have now got a system of equations that is cubic, rather than quadratic, in the velocities.

In analogy to our earlier procedure we search for solutions of (13) in the form $x^\mu(u) = X^\mu(u) + \xi^\mu(u)$, with X^μ a known geodesic, now reparametrized by $u = x^n$. Of course, as $x^n(u) = X^n(u) = u$, we have $\xi^n(u) = 0$. Inserting $x^i(u) = X^i(u) + \xi^i(u)$ into (13), and linearizing with respect to the ξ^i , but not with respect to the

$d\xi^i/du$, gives us the reparametrized version of the generalized Jacobi equation,

$$\begin{aligned}
& \frac{d^2\xi^i}{du^2} + \Gamma_{v\sigma}^i(X) \left(2 \frac{dX^v}{du} \frac{d\xi^\sigma}{du} + \frac{d\xi^v}{du} \frac{d\xi^\sigma}{du} \right) \\
& - \Gamma_{v\sigma}^n(X) \left(2 \frac{dX^v}{du} \frac{d\xi^\sigma}{du} + \frac{d\xi^v}{du} \frac{d\xi^\sigma}{du} \right) \frac{dX^i}{du} \\
& - \Gamma_{v\sigma}^n(X) \left(\frac{dX^v}{du} + \frac{d\xi^v}{du} \right) \left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du} \right) \frac{d\xi^i}{du} \\
& + \partial_\tau \Gamma_{v\sigma}^i(X) \xi^\tau \left(\frac{dX^v}{du} + \frac{d\xi^v}{du} \right) \left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du} \right) \\
& - \partial_\tau \Gamma_{v\sigma}^n(X) \xi^\tau \left(\frac{dX^v}{du} + \frac{d\xi^v}{du} \right) \left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du} \right) \left(\frac{dX^i}{du} + \frac{d\xi^i}{du} \right) = 0.
\end{aligned} \tag{14}$$

We now turn to the question of whether the generalized Jacobi equation can be simplified by choosing special coordinates. To that end we will use Fermi coordinates. Text-books on general relativity treat Fermi coordinates near a time-like reference geodesic (or, more generally, near a timelike reference curve) in a Lorentzian manifold, see e.g. Synge [19]. However, it is fairly obvious that, by an analogous procedure, one can introduce Fermi coordinates near an arbitrary reference geodesic in an affine manifold, without a metric. The construction is as follows.

Let an affinely parametrized geodesic $X(s)$ be given. Choose an n -bein (i.e., n linearly independent vectors) at one point of the geodesic, such that the n th vector coincides with the tangent vector of the geodesic, and transport this n -bein parallelly along $X(s)$. Denote the n resulting vector fields along the geodesic by $E_1(s), \dots, E_n(s)$. As $X(s)$ is an affinely parametrized geodesic, $E_n(s)$ coincides with the tangent vector $dX(s)/ds$ for all s . Let \exp denote the exponential map of the given connection which, as indicated above, is assumed to be torsion-free. Then every point in a sufficiently small tubular neighborhood of the given geodesic can be written uniquely as $\exp_{X(u)}(x^i E_i(u))$. The n numbers (x^1, \dots, x^{n-1}, u) are the Fermi coordinates of this point. By definition of the exponential map, this means that the Fermi coordinates are determined by the following property: The point with Fermi coordinates (x^1, \dots, x^{n-1}, u) can be reached by following the geodesic with initial point $X(u)$ and initial tangent vector $x^i E_i(u)$ up to the affine parameter 1. By construction, the Fermi coordinate u coincides along the geodesic $X(s)$ with the affine parameter s . Along neighboring geodesics, however, the Fermi coordinate u will give a non-affine parametrization in general.

Note that Fermi coordinates are well-defined only on a (possibly small) tubular neighborhood of a chosen geodesic $X(s)$. Farther away from $X(s)$ they need not exist, because there may be points that cannot be reached by a geodesic starting on $X(s)$, and they need not be unique, because geodesics issuing from a point on $X(s)$ may intersect.

The crucial property of Fermi coordinates is that, in such coordinates, the connection coefficients $\Gamma_{v\sigma}^\mu$ vanish on $X(s)$. To prove this, we first observe that

$\nabla_{\partial_n} \partial_\mu = 0$ along $X(s)$ because the E_μ are parallelly transported, hence $\Gamma_{n\sigma}^\mu(X) = 0$. As the connection is symmetric, this implies that $\Gamma_{\sigma n}^\mu(X) = 0$. What remains to be proven is that $\Gamma_{ij}^\mu(X) = 0$. We use the fact that

$$\nabla_{(\partial_i + \partial_j)}(\partial_i + \partial_j) = \nabla_{\partial_i}\partial_i + \nabla_{\partial_j}\partial_j + \nabla_{\partial_i}\partial_j + \nabla_{\partial_j}\partial_i. \quad (15)$$

As for any $(n-1)$ -tuple (x^1, \dots, x^{n-1}) the integral curves of $x^i \partial_i$ that start on $X(s)$ are geodesics, the left-hand side and the first two terms on the right-hand side of (15) vanish on $X(s)$. Thus, (15) tells us that $\Gamma_{ij}^\mu(X) + \Gamma_{ji}^\mu(X) = 0$. As our connection is symmetric, this implies that, indeed, $\Gamma_{ij}^\mu(X) = 0$.

If written in Fermi coordinates, the generalized Jacobi equation (14) simplifies considerably because of $\Gamma_{v\sigma}^\mu(X) = 0$. Moreover, the use of Fermi coordinates reduces the generalized Jacobi equation to a standard form which facilitates comparison of this equation for two different geodesics. In Sect. 4 below we will make this explicit for lightlike geodesics in a Lorentzian manifold, where the Fermi coordinates can be further specified.

3 The generalized Jacobi equation for lightlike geodesics in arbitrary coordinates

We now specify the results of the preceding section to the case that ∇ is the Levi-Civita connection of a pseudo-Riemannian metric $g = g_{\mu\nu}dx^\mu dx^\nu$,

$$\Gamma_{v\sigma}^\mu = \frac{1}{2}g^{\mu\rho}(\partial_v g_{\sigma\rho} + \partial_\sigma g_{v\rho} - \partial_\rho g_{v\sigma}), \quad (16)$$

where, as usual,

$$g^{\mu\nu}g_{v\sigma} = \delta_\sigma^\mu. \quad (17)$$

We assume that the metric has Lorentzian signature $(+, \dots, +, -)$, and we want to discuss the generalized Jacobi equation for the case of lightlike geodesics,

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \quad (18)$$

If the dimension n of the manifold is equal to 4, lightlike geodesics can be interpreted as light rays in a general-relativistic spacetime. The mathematical results, however, hold for any n .

Let us briefly recall the well-known fact that, if the connection coefficients $\Gamma_{v\sigma}^\mu$ are given in terms of a metric via (16), the geodesic equation (2) can be derived from a Hamiltonian. As a matter of fact, Hamilton's equations for the Hamiltonian

$$H(x, p) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu \quad (19)$$

take the form

$$\frac{dx^\mu}{ds} = p_\mu, \quad (20)$$

$$\frac{dp_\mu}{ds} = \frac{1}{2}(\partial_\mu g^{\rho\sigma})(x)p_\rho p_\sigma. \quad (21)$$

The derivative with respect to s of (21), together with (20) and (16), indeed reproduces the geodesic equation (2). By (20), the side-condition (18) takes the form

$$H(x, p) = 0. \quad (22)$$

A reparametrization of the solution curves can be achieved for lightlike geodesics in a particularly convenient way: We just have to multiply the Hamiltonian with an appropriate function of x and p . If we want to change from the affine parameter s to the parameter $u = x^n$, we can proceed in the following way. We first recall that such a reparametrization is possible for all geodesics that are transverse to the hypersurfaces $x^n = \text{constant}$ (This condition is true for *all* lightlike geodesics if and only if the hypersurfaces $x^n = \text{constant}$ are spacelike. However, we may also consider the case that these hypersurfaces are lightlike or timelike; then the transversality condition only holds for *some* lightlike geodesics). By (20) and (18), the transversality condition is true if $g^{n\sigma}(x)p_\sigma \neq 0$. This condition determines an open subset of the cotangent bundle (phase space). On this subset we can switch from the Hamiltonian (19) to the modified Hamiltonian

$$\tilde{H}(x, p) = \frac{g^{\mu\nu}(x)p_\mu p_\nu}{2g^{n\sigma}(x)p_\sigma}. \quad (23)$$

Hamilton's equation with the Hamiltonian \tilde{H} read

$$\frac{dx^\mu}{du} = -\frac{\tilde{H}(x, p)g^{n\mu}(x)}{g^{n\sigma}(x)p_\sigma} + \frac{g^{\mu\nu}(x)p_\nu}{g^{n\sigma}(x)p_\sigma}, \quad (24)$$

$$\frac{dp_\mu}{du} = -\frac{\tilde{H}(x, p)(\partial_\mu g^{n\rho})(x)p_\rho}{g^{n\sigma}(x)p_\sigma} + \frac{(\partial_\mu g^{\rho\tau})(x)p_\rho p_\tau}{2g^{n\sigma}(x)p_\sigma}, \quad (25)$$

and the side-condition (22) is equivalent to

$$\tilde{H}(x, p) = 0. \quad (26)$$

With (26), the $\mu = n$ component of (24) reduces to

$$\frac{dx^n}{du} = 1, \quad (27)$$

so the new curve parameter u coincides, indeed, with the coordinate x^n (up to an additive constant that can be chosen at will). The other components of (24) take the form

$$\frac{dx^i}{du} = \frac{g^{i\nu}(x)p_\nu}{g^{n\sigma}(x)p_\sigma}, \quad (28)$$

where, as before, latin indices range from 1 to $(n-1)$, and (25) reduces to

$$\frac{dp_\mu}{du} = \frac{(\partial_\mu g^{\rho\tau})(x)p_\rho p_\tau}{2g^{n\sigma}(x)p_\sigma}. \quad (29)$$

Applying the derivative operator d/du to (28), and using (29), reproduces the equation (13) for u -parametrized geodesics. We have thus proven that the lightlike

geodesics of the metric $g = g_{\mu\nu}(x)dx^\mu dx^\nu$, if reparametrized by $u = x^n$, can be derived from the Hamiltonian (23) together with the side-condition (26).

This gives us a convenient method of how to derive the generalized Jacobi equation for lightlike geodesics parametrized by $u = x^n$: Start out from the Hamiltonian (23). Write Hamilton's equations with the side-condition (26). Rewrite this as second-order equations for the x^μ . Fix a solution $X^\mu(u)$ to these equations, substitute $x^\mu(u) = X^\mu(u) + \xi^\mu(u)$, and linearize with respect to the ξ^μ but not with respect to the $d\xi^\mu/du$.

The resulting set of equations describes lightlike geodesics $X(u) + \xi(u)$ which are close to the lightlike geodesic $X(u)$ but whose tangent vectors need not be close to the tangent vector of $X(u)$. For $n = 4$ these equations can be used, by choosing initial conditions appropriately, for describing a homocentric light bundle around a central light ray $X(u)$ in a general-relativistic spacetime. If the vertex of the bundle is at $u = u_0$, say, we have to choose the initial conditions as $\xi^i(u_0) = 0$. There are only $(n-2) = 2$ independent solutions because the side-condition (18) fixes one of the $d\xi^i/du$ in terms of the others. This is in agreement with the intuitively obvious fact that a homocentric light bundle has $(n-2) = 2$ dimensions transverse to the propagation direction. In contrast to the ordinary Jacobi equation, the generalized Jacobi equation can be used to describe homocentric bundles whose opening angle is arbitrarily large but the approximation is valid, in general, only for small values of $u - u_0$, i.e., close to the vertex of the bundle. For larger values of $u - u_0$, the $\xi^i(u)$ will, in general, not be small, so the fact that we linearized with respect to these quantities may produce large errors.

To be sure, there are special examples where the generalized Jacobi equation holds for a large parameter interval. An example of this kind will be given in Sect. 4; in this example even the Jacobi equation, the generalized Jacobi equation and the exact geodesic equation coincide. In general, however, the generalized Jacobi equation for lightlike geodesics is a short-time equation, describing the temporal evolution of light bundles with arbitrarily large opening angles near their vertex.

We illustrate the general results of this section with an example.

Example 1 We want to calculate, with the help of the generalized Jacobi equation, the evolution of a light bundle around a circular geodesic at $r = 3\text{ m}$ in Schwarzschild spacetime. The Schwarzschild metric is

$$g_{\mu\nu} dx^\mu dx^\nu = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2. \quad (30)$$

It is well-known that a lightlike geodesic that starts tangentially to the circle $r = 3\text{ m}$, $\vartheta = \pi/2$, will stay on this circle. We want to write the generalized Jacobi equation for lightlike geodesics near this circular geodesic. We will use the azimuthal coordinate φ for the parameter, $u = x^n = \varphi$. This excludes geodesics tangent to a half-space $\varphi = \text{constant}$. Therefore, our parametrization allows us to treat bundles with any opening angle smaller than $\pi/2$ around the circular geodesic, but not the limiting case that the opening angle is equal to $\pi/2$. Using φ for the parameter gives us directly the intersection of the bundle with any half-plane $\varphi = \text{constant}$.

After calculating the contravariant components $g^{\mu\nu}$ of the Schwarzschild metric, we write the Hamiltonian (23) with $x^n = \varphi$,

$$\tilde{H}(x, p) = \frac{1}{2} \left(p_\varphi + \frac{\sin^2 \vartheta}{p_\varphi} \left(p_\vartheta^2 + r(r-2m)p_r^2 - \frac{r^3 p_t^2}{r-2m} \right) \right). \quad (31)$$

Using the side-condition $\tilde{H}(x, p) = 0$, Hamilton's equations for the Hamiltonian \tilde{H} take the form

$$\begin{aligned} \frac{d\varphi}{du} &= 1, \quad \frac{dt}{du} = -\frac{r^3 \sin^2 \vartheta p_t}{(r-2m)p_\varphi}, \quad \frac{d\vartheta}{du} = \frac{\sin^2 \vartheta p_\vartheta}{p_\varphi}, \\ \frac{dr}{du} &= \frac{r(r-2m) \sin^2 \vartheta p_r}{p_\varphi}, \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{dp_\varphi}{du} &= 0, \quad \frac{dp_t}{du} = 0, \quad \frac{dp_\vartheta}{du} = \frac{\cos \vartheta p_\varphi}{\sin \vartheta}, \\ \frac{dp_r}{du} &= -\frac{(r-m) \sin^2 \vartheta p_r^2}{p_\varphi} + \frac{r^2(r-3m) \sin^2 \vartheta p_t^2}{(r-2m)^2 p_\varphi}. \end{aligned} \quad (33)$$

If we apply the derivative d/du to the expressions for $d\vartheta/du$ and dr/du from (32), use (33) and the side-condition $\tilde{H}(x, p) = 0$, we arrive at the following second order system for $\vartheta(u)$ and $r(u)$.

$$\frac{d^2 \vartheta}{du^2} = \frac{2 \cos \vartheta}{\sin \vartheta} \left(\frac{d\vartheta}{du} \right)^2 + \sin \vartheta \cos \vartheta. \quad (34)$$

$$\frac{d^2 r}{du^2} = \frac{2 \cos \vartheta}{\sin \vartheta} \frac{d\vartheta}{du} \frac{dr}{du} + \frac{2}{r} \left(\frac{dr}{du} \right)^2 + (r-3m) \left(\left(\frac{d\vartheta}{du} \right)^2 + \sin^2 \vartheta \right). \quad (35)$$

Obviously, this system of equations admits the solution

$$r(u) = 3m, \quad \vartheta(u) = \frac{\pi}{2}. \quad (36)$$

In order to linearize around this circular lightlike geodesic, we write

$$r(u) = 3m + \frac{1}{\sqrt{3}} \xi^r(u), \quad \vartheta(u) = \frac{\pi}{2} + \frac{1}{3m} \xi^\vartheta(u). \quad (37)$$

The numerical factors are chosen such that

$$g(\partial_{\xi^r}, \partial_{\xi^r}) = g(\partial_{\xi^\vartheta}, \partial_{\xi^\vartheta}) = 1 \quad (38)$$

at $r = 3m$, $\vartheta = \pi/2$. Inserting (37) into (34) and (35), and linearizing with respect to ξ^ϑ and ξ^r , but not with respect to $d\xi^\vartheta/du$ and $d\xi^r/du$, gives us the generalized

Fig. 1 Cross section of initially circular light bundle around a circular lightlike geodesic in Schwarzschild spacetime, calculated with the generalized Jacobi equation. The point $(\xi^r, \xi^\vartheta) = (0, 0)$ represents the circular geodesic at $r = 3m$ and $\vartheta = \pi/2$. The ξ^r axis is in the equatorial plane, pointing outwards. The ξ^ϑ axis is perpendicular to the equatorial plane, pointing downwards. The picture shows the intersection with a half-plane $\varphi = \text{constant}$ close to the vertex of the bundle, for four different opening angles

Jacobi equation:

$$\frac{d^2 \xi^\vartheta}{du^2} = -\xi^\vartheta - \frac{2\xi^\vartheta}{9m^2} \left(\frac{d\xi^\vartheta}{du} \right)^2, \quad (39)$$

$$\begin{aligned} \frac{d^2 \xi^r}{du^2} = & \xi^r + \frac{\xi^r}{9m^2} \left(\left(\frac{d\xi^\vartheta}{du} \right)^2 - \frac{2}{3} \left(\frac{d\xi^r}{du} \right)^2 \right) \\ & - \frac{2\xi^\vartheta}{9m^2} \frac{d\xi^\vartheta}{du} \frac{d\xi^r}{du} + \frac{2}{3\sqrt{3}m} \left(\frac{d\xi^r}{du} \right)^2. \end{aligned} \quad (40)$$

If we linearize also with respect to $d\xi^\vartheta/du$ and $d\xi^r/du$, only the first term on the right-hand side of (39) and of (40) survives,

$$\frac{d^2 \xi^\vartheta}{du^2} = -\xi^\vartheta, \quad (41)$$

$$\frac{d^2 \xi^r}{du^2} = \xi^r. \quad (42)$$

This is the ordinary Jacobi equation.

If we solve (41) and (42) with initial conditions

$$\xi^\vartheta(0) = 0, \quad \xi^r(0) = 0, \quad (43)$$

$$\frac{d\xi^\vartheta}{d\varphi}(0) = -\varepsilon \sin \chi, \quad \frac{d\xi^r}{d\varphi}(0) = \varepsilon \cos \chi, \quad (44)$$

with χ running from 0 to 2π and ε fixed, it gives us the cross-section of an initially circular bundle with opening angle proportional to ε . Owing to the linearity of the ordinary Jacobi equation, the cross-section of such a bundle will be elliptic for all values of $u = \varphi$. As a consequence of the plus sign on the right-hand side of (42), in contrast to the minus sign on the right-hand side of (41), the expansion of the bundle will increase in the ξ^r direction and decrease in the ξ^ϑ direction; hence the major axis of the ellipse is in the ξ^r direction. This reflects the fact that the circular geodesic at $r = 3m$ is unstable with respect to perturbations in the ξ^r direction but stable with respect to perturbations in the ξ^ϑ direction.

By contrast, if we solve the non-linear equations (39) and (40) with initial conditions (43) and (44), the cross-section of the resulting bundle will not be elliptic. The larger the opening angle ε , the stronger the deviation from the elliptic shape due to the non-linearities, see Fig. 1.

This example demonstrates how the generalized Jacobi equation can be used for calculating the shapes of light bundles, beyond the small-angle approximation that is inherent in the standard treatment based on the ordinary Jacobi equation. Of

course, one has to keep in mind that the generalized Jacobi equation is, in general, a valid approximation only close to the vertex of the bundle.

4 The generalized Jacobi equation for lightlike geodesics in Fermi coordinates

We have already discussed how, on an affine manifold, Fermi coordinates can be introduced near a reference geodesic $X(s)$. The construction involved the choice of a parallelly transported n -bein $E_1(s), \dots, E_n(s)$ along the chosen geodesic which was arbitrary apart from the fact that $E_n(s)$ should coincide with the tangent vector $dX(s)/ds$ of the reference geodesic. If our connection is the Levi-Civita connection of a Lorentzian metric, we can further specify this n -bein. For a timelike reference geodesic, it is usual to require that the n -bein be orthonormal. Then $E_1(s), \dots, E_{n-1}(s)$ span the spacelike orthocomplement of $dX(s)/ds$ at each value of s . Thus, in the Fermi coordinates (x^1, \dots, x^{n-1}, u) the hypersurfaces $u = \text{constant}$ intersect the timelike reference geodesic $X(s)$ orthogonally and are, therefore, spacelike near $X(s)$ (Farther away from $X(s)$ they need not be spacelike). These are the Fermi normal coordinates treated in standard text-books on general relativity. In these coordinates the metric attains a standard form if written up to second order in the transverse coordinates. This standard form was derived by Manasse and Misner [11] and can be found, e.g., in Misner et al. [14], p.332.

Whereas this standard text-book treatment of Fermi normal coordinates assumes a timelike geodesic, here we are interested in Fermi coordinates near a lightlike reference geodesic $X(s)$. Then the n -bein cannot be chosen orthonormal because of the requirement that $E_n(s) = dX(s)/ds$. The best choice is to have $E_{n-1}(s)$ lightlike with

$$g_{\mu\nu}(X(s)) E_{n-1}^\mu(s) E_n^\nu(s) = -1, \quad (45)$$

and the remaining vectors $E_1(s), \dots, E_{n-2}(s)$ orthonormal and perpendicular to both E_{n-1} and E_n . (Condition (45) assures that E_{n-1} is future-pointing if E_n is future-pointing.) With this choice, the resulting Fermi coordinates $(x^1, \dots, x^{n-2}, x^{n-1} = v, u)$ yield hypersurfaces $u = \text{constant}$ that are lightlike where they meet the reference geodesic $X(s)$. The hypersurface $v = 0$ is lightlike along the reference geodesic $X(s)$ which is completely contained in this hypersurface. These lightlike Fermi normal coordinates were discussed in some detail in a recent article by Blau et al. [2]. In that article, the authors derive the general expression for the metric in lightlike Fermi normal coordinates, up to second order in the coordinates away from the reference geodesic. This result is the lightlike analogue of the above-mentioned Manasse–Misner representation of the metric near a timelike geodesic. If adapted to our sign and index conventions, it reads

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu = & -2dudv + \delta_{AB} dx^A dx^B - R_{injn}(u) x^i x^j du^2 \\ & - \frac{4}{3} R_{ikjn}(u) x^i x^j dx^k du - \frac{1}{3} R_{ikjl}(u) x^i x^j dx^k dx^l \dots \end{aligned} \quad (46)$$

Here the Fermi coordinates are denoted $(x^1, \dots, x^{n-2}, x^{n-1} = v, u)$, as outlined above. As before, greek indices run from 1 to n and lower case latin indices

i, j, k, l, \dots run from 1 to $(n - 1)$. In addition, upper case latin indices A, B, \dots run from 1 to $(n - 2)$. As usual, δ_{AB} denotes the Kronecker delta. $R_{\rho\nu\sigma\tau}(u) = g_{\rho\mu}(u)R_{\nu\sigma\tau}^{\mu}(u)$ is the purely covariant version of the curvature tensor (9), evaluated at $(x^1 = 0, \dots, x^{n-2} = 0, x^{n-1} = v = 0, u)$, i.e., along the reference geodesic. The ellipses in (46) indicate terms of third and higher order with respect to the x^i .

With the contravariant metric components $g^{\mu\nu}$ calculated, up to second order, from (46), we can write the Hamiltonian (23) up to second order. This is enough to write the pertaining Hamilton equations up to first order, which together with the constraint (26) will give us the generalized Jacobi equation in lightlike Fermi normal coordinates near an arbitrary lightlike reference geodesic.

We will illustrate this with an example.

Example 2 Consider a plane-wave spacetime in Brinkmann coordinates $(x^1, \dots, x^{n-2}, v, u)$,

$$g_{\mu\nu} dx^\mu dx^\nu = -2dudv - h_{AB}(u)x^A x^B du^2 + \delta_{AB} dx^A dx^B, \quad (47)$$

where $h_{AB}(u) = h_{BA}(u)$ has a non-negative trace,

$$\delta^{AB} h_{AB}(u) \geq 0, \quad (48)$$

but is arbitrary otherwise. For $n = 4$, any such metric can be interpreted as a combined gravitational and electromagnetic plane wave. If equality holds in (48), the spacetime is Ricci-flat and, thus, a pure gravitational wave. For any choice of $h_{AB}(u)$, the vector field ∂_v is lightlike and absolutely parallel. For a discussion of the geometry of plane-wave spacetimes the reader is referred to Penrose [15].

Plane-wave spacetimes have the following interesting property, first discovered by Penrose [16]. Near *any* lightlike geodesic in *any* spacetime, the metric takes the form of a plane wave in a well-defined limit, called the Penrose limit. Thereby the original geodesic is represented as the curve $(x^1 = 0, \dots, x^{n-2} = 0, v = 0, u)$ in the limiting plane-wave spacetime (47). The Penrose limit can be conveniently written in terms of lightlike Fermi normal coordinate, as recently demonstrated by Blau, Frank and Weiss [2]. In particular, their analysis showed that Brinkmann coordinates for plane waves are lightlike Fermi normal coordinates; in this case all the higher-order terms, which are indicated in (46) by ellipses, vanish exactly. The curvature tensor is given along the geodesic $(x^1 = 0, \dots, x^{n-2} = 0, v = 0, u)$ by

$$R_{AnBn}(u) = h_{AB}(u) \quad (49)$$

and $R_{\mu\nu\sigma\tau}(u) = 0$ for all other index combinations, compare (46) with (47).

We want to write the generalized Jacobi equation near the lightlike geodesic $(x^1 = 0, \dots, x^{n-2} = 0, v = 0, u)$. We will use $u = x^n$ for the curve parameter. After calculating the contravariant metric components from (47), we can write the Hamiltonian (23):

$$\tilde{H}(x, p) = p_u - \frac{1}{2} h_{AB}(u) x^A x^B p_v - \frac{\delta^{AB} p_A p_B}{2 p_v}. \quad (50)$$

Note that this Hamiltonian is of second order with respect to the coordinates x^A and independent of the coordinate v . Hamilton's equations with the Hamiltonian (50) take the form

$$\frac{du}{du} = 1, \quad \frac{dv}{du} = -\frac{1}{2} h_{AB} x^A x^B + \frac{\delta^{AB} p_A p_B}{2 p_v^2}, \quad \frac{dx^A}{du} = -\frac{\delta^{AB} p_B}{p_v}, \quad (51)$$

$$\frac{dp_u}{du} = \frac{1}{2} \frac{dh_{AB}}{du} x^A x^B p_v, \quad \frac{dp_v}{du} = 0, \quad \frac{dp_A}{du} = h_{AB} x^B p_v. \quad (52)$$

Fig. 2 Past light cone of an event in a plane-wave spacetime (47). Three dimensions (x^1, v, u) are shown, with u future-pointing lightlike (from *lower right* to *upper left*), v future-pointing lightlike (from *lower left* to *upper right*), and x^1 orthogonal to both. The picture is valid for the case that x^1 is an eigendirection of $h_{AB}(u)$ with positive eigenvalue. In this case in the three-dimensional picture all the light rays issuing from an event into the past are refocused into another event, with the exception of one single light ray that stays on an integral curve of the absolutely parallel lightlike vector field ∂_v . A colour version of this picture can be found online in [17]. For a similar picture, hand-drawn by Roger Penrose, see [15]

Applying d/du to (51), and using (52), gives us the lightlike geodesic equation in second-order form:

$$\frac{d^2x^A}{du^2} = -\delta^{AB}h_{BC}x^C, \quad (53)$$

$$\frac{d^2v}{du^2} = -\frac{1}{2}\frac{dh_{AB}}{du}x^Ax^B - 2h_{AB}x^A\frac{dx^B}{du}. \quad (54)$$

If we write

$$x^A(u) = 0 + \xi^A(u), \quad v(u) = 0 + \xi^v(u), \quad (55)$$

and linearize with respect to ξ^A and ξ^v , we get the generalized Jacobi equation

$$\frac{d^2\xi^A}{du^2} = -\delta^{AB}h_{BC}\xi^C, \quad \frac{d^2\xi^v}{du^2} = -2h_{AB}\xi^A\frac{d\xi^B}{du}. \quad (56)$$

For the transverse coordinates $x^A = \xi^A$, these equations are independent of the velocities; hence, the generalized Jacobi equation coincides with the ordinary geodesic equation. Moreover, the (generalized) Jacobi equation even coincides with the exact geodesic equation (53). If we solve these equations with initial conditions

$$x^A(u_0) = 0, \quad \delta_{AB}\frac{dx^A}{du}(u_0)\frac{dx^B}{du}(u_0) = \varepsilon^2, \quad (57)$$

it gives us the cross-section of an initially circular light bundle around the geodesic $(x^1 = 0, \dots, x^{n-2} = 0, v = 0, u)$ with vertex at $u = u_0$. Such a bundle will have an elliptic cross-section, for arbitrarily large opening angle. The rate of expansion is positive in eigen-directions of $h_{AB}(u)$ with negative eigenvalues and negative in eigen-directions of $h_{AB}(u)$ with positive eigenvalues. Condition (48) makes sure that at least one eigenvalue is positive (unless all are zero in which case (47) is just the Minkowski metric). In directions with positive eigenvalue the light bundle will be refocussed until a conjugate point is reached. This focusing property is illustrated by Fig. 2 which displays the past-light cone of an event in a plane-wave spacetime, with three dimensions (x^1, v, u) shown.

The fact that, for the lightlike geodesic $(x^1 = 0, \dots, x^{n-2} = 0, v = 0, u)$ in a plane-wave spacetime, the generalized Jacobi equation coincides with the ordinary Jacobi equation has the following interesting consequence: in the Penrose limit the difference between the generalized Jacobi equation and the Jacobi equation vanishes.

5 Concluding remarks

In this article we have discussed a generalized Jacobi equation for lightlike geodesics, following as closely as possible the approach that was brought forward by Hodgkinson, Mashhoon and others for timelike geodesics. As an alternative, one could also apply the approach of Bażański [1], which was mentioned in the introduction, to lightlike geodesics. In that case one would consider lightlike geodesics of the form $x(s) = X(s) + \varepsilon \xi(s)$, where $X(s)$ is a lightlike reference geodesic, and solve the geodesic equation for $x(s)$ iteratively up to some order N with respect to ε . For any finite order N , the resulting equation would not be valid for light bundles of arbitrarily large opening angle, in contrast to the generalized Jacobi equation treated here. On the other hand, for ε sufficiently small it would be valid for arbitrarily large parameter intervals. An interesting application of this Bażański-type approach to lightlike geodesics, which apparently has not been considered in the literature so far, could be to study the caustics of light bundles up to some order N with respect to ε .

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