XXIXth Spanish Relativity Meeting (ERE 2006) IOP Publishing
Journal of Physics: Conference Series **66** (2007) 012033 doi:10.1088/1742-6596/66/1/012033

Exact Wave Propagation in a Spacetime with a Cosmic String

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Abstract. We present exact solutions of the massless Klein-Gordon equation in a spacetime in which an infinite straight cosmic string resides. The first solution represents a plane wave entering perpendicular to the string direction. We also present and analyze a solution with a static point-like source. In the short wavelength limit these solutions approach the results obtained by using the geometrical optics approximation: magnification occurs if the observer lies in front of the string within a strip of angular width $8\pi G\mu$, where μ is the string tension. We find that when the distance from the observer to the string is less than $10^{-3}(G\mu)^{-2}\lambda \sim 150\mathrm{Mpc}(\lambda/\mathrm{AU})(G\mu/10^{-8})^{-2}$, where λ is the wave length, the magnification is significantly reduced compared with the estimate based on the geometrical optics due to the diffraction effect.

1. Introduction

Typical wavelength of gravitational waves from astrophysical compact objects such as BH(black hole)-BH binaries is in some cases very long so that wave optics must be used instead of geometrical optics when we discuss gravitational lensing. More precisely, if the wavelength becomes comparable or longer than the Schwarzschild radius of the lens object, the diffraction effect becomes important and as a result the magnification factor approaches unity [1, 2, 3, 4, 5]. Mainly due to the possibility that the wave effects could be observed by future gravitational wave observations, several authors [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] have studied wave effects in gravitational lensing in recent years.

In most of the works which studied gravitational lensing phenomenon in the framework of wave optics, isolated and normal astronomical objects such as galaxies are concerned as lens objects. Recently Yamamoto and Tsunoda[12] studied wave effects in gravitational lensing by an infinite straight cosmic string. The metric around a cosmic string is completely different from that around a usual massive object.

Cosmic strings generically arise as solitons in a grand unified theory and could be produced in the early universe as a result of symmetry breaking phase transition[16, 17]. If symmetry breaking occurred after inflation, the strings might survive until the present universe. Recently, cosmic strings attract a renewed interest partly because a variant of their formation mechanism was proposed in the context of the brane inflation scenario[18, 19, 20, 21, 22, 23, 24]. In this scenario inflation is driven by the attractive force between parallel D-branes and parallel anti D-branes in a higher dimensional spacetime. When those brane-anti-brane pairs collide and annihilate at the end of inflation, lower-dimensional D-branes, which behave like monopoles, cosmic strings or domain walls from the view point of four-dimensional observers, are formed generically [25, 26, 27, 28, 29].

For some time, cosmic string was a candidate for the seed of structure formation of our universe, but this possibility was ruled out by the measurements of the spectrum of cosmic microwave background (CMB) anisotropies[30, 31]. The current upper bound on the dimensionless string tension $G\mu$ is around $10^{-7} \sim 10^{-6}$, which comes from the observations of CMB[32, 33, 34, 35] and/or the pulsar timing [36, 37, 38, 39]. Although cosmic string cannot occupy dominant fraction of the energy density of the universe, its non-negligible population is still allowed observationally[40, 41]. In fact, Sazhin et al.[42, 43] reported that CSL-1, which is a double image of elliptical galaxies with angular separation 1.9 arcsec, could be the first case of the gravitational lensing by a cosmic string with $G\mu \approx 4 \times 10^{-7}$.

We study in detail wave effects in the gravitational lensing by an infinite straight cosmic string. In Ref. [12], wave propagation around a cosmic string was studied but they put the waveform around the string by hand. Their prescription is correct only in the limit of geometrical optics, which breaks down when the wavelength becomes longer than a certain characteristic length. In this paper, we present exact solutions of the (scalar) wave equation in a spacetime with a cosmic string. We analytically show that our solutions reduce to the results of the geometrical optics in the short wavelength limit. We derive a simple analytic formula of the leading order corrections to the geometrical optics due to the finite wavelength effects and also an expression for the long wavelength limit. Interference caused by the lensing remains due to the diffraction effects even when only a single image can be seen in the geometrical optics. This fact increases the lensing probability by cosmic strings.

2. A solution of the wave equation around an infinite straight cosmic string

A solution of Einstein equations around an infinite straight cosmic string to first order in $G\mu$ is given by [44]

$$d^2s = -dt^2 + dr^2 + (1-\Delta)^2 r^2 d\theta^2 + dz^2, \tag{1}$$

where (r, z, θ) is a cylindrical coordinate$(0 \leq \theta < 2\pi)$ and $2\pi\Delta \approx 8\pi G\mu$ is the deficit angle around the cosmic string. Spatial part of the above metric describes the Euclidean space with a wedge of angular size $2\pi\Delta$ removed. Due to the deficit angle around a string, double images of the source are observed with an angular separation $2\pi\Delta$ when a source is located behind the string in the limit of geometrical optics. In general for a wave with a finite wavelength, some interference pattern appears. An exact solution of Einstein equations around a finite thickness string has been already obtained [45], but we use the metric (1) as a background since the string thickness is negligibly small compared with the Einstein radius, $\approx \pi D\Delta$, where D is the distance from the observer to the string.

Throughout the paper, we consider waves of a massless scalar field instead of gravitational waves for simplicity, but the wave equations are essentially the same in these two cases. An extension to the cosmological setup is straightforwardly done by adding an overall scale factor. In that case the time coordinate t is to be understood as the conformal time. The wave equation remains unchanged if we consider a conformally coupled field, but it is modified for the other cases due to curvature scattering. The correction due to curvature scattering of the Friedmann universe is suppressed by the square of the ratio between the wavelength and the Hubble length, which can be neglected in any situations of our interest.

Our goal of this section is to construct a solution of the wave equation which corresponds to a plane wave injected perpendicularly to and scattered by the cosmic string. This situation occurs if the distance between the source and the string is infinitely large. In order to construct such a solution, we introduce a monochromatic source uniformly extended in the z-direction and localized in $r-\theta$ plane,

$$S=\frac{B}{(1-\Delta)}\delta(r-r_o)\delta(\theta-\pi)e^{-i\omega t}, \tag{2}$$

where ω is the frequency and we have introduced B, a constant independent of Δ, to adjust the overall normalization when we later take the limit $r_o\to\infty$. The factor $(1-\Delta)^{-1}$ appears because θ-coordinate used in the metric (1) differs from the usual angle

$$\varphi\equiv(1-\Delta)\theta. \tag{3}$$

Here we consider a uniformly extended source instead of a point source since the former is easier to hundle. When the limit $r_o\to\infty$ is taken, the answers are identical in these two cases. The case with a point-like source at a finite distance can be treated in a similar way.

The solution of the wave equation for $\phi(r,\theta)$ when $r_o\to\infty$ can be written as

$$\phi(r,\theta)=\frac{-iB}{2\sqrt{2}(1-\Delta)}\sqrt{\frac{r_o}{\pi\omega}}e^{i\omega r_o-i\frac{\pi}{4}}\sum_{m=0}^{\infty}\epsilon_m i^m e^{-\frac{im\pi\Delta}{2(1-\Delta)}}J_{\nu_m}(\omega r)\cos m\theta. \tag{4}$$

We determine the overall normalization of the source amplitude B, independently of $G\mu$, so that Eq. (4) becomes a plane wave $e^{i\omega r\cos\theta}$ when $G\mu=0$. This condition leads to $B=-2\sqrt{\frac{2\pi\omega}{r_o}}e^{-i\omega r_o-i\pi/4}$. Then, finally ϕ becomes

$$\phi(r,\theta)=\frac{1}{1-\Delta}\sum_{m=0}^{\infty}\epsilon_m i^m e^{-\frac{im\Delta\pi}{2(1-\Delta)}}J_{\nu_m}(\omega r)\cos m\theta. \tag{5}$$

3. Limiting behaviors of the solution

3.1. Approximate waveform in the wave zone

The solution (5) describes the waveform propagating around a cosmic string. But it is not easy to understand the behavior of the solution because it is given by a series. In fact, it takes much time to perform the summation in Eq. (5) numerically for a realistic value of tension of the string, say, $G\mu\approx10^{-6}$ because of slow convergence of the series. In particular it is not manifest whether the amplification of the solution in the short wavelength limit coincides with the one which is obtained by the geometrical optics approximation. Therefore it will be quite useful if one can derive a simpler analytic expression. Here we only quote the final result which keeps terms up to $O(1/\sqrt{\xi})$,

$$\phi(\xi,\theta)\approx\exp\left(i\xi\cos\frac{\alpha(\theta)}{\sqrt{\xi}}\right)\Theta(\alpha(\theta))-\frac{\sigma(\theta)}{\sqrt{\pi}}e^{i\xi-\frac{i}{2}\alpha^2(\theta)}\mathrm{Erfc}\left(\frac{|\alpha(\theta)|}{\sqrt{2}}e^{-i\pi/4}\right)+(\theta\to-\theta). \tag{6}$$

where

$$\alpha(\theta) := (\pi\Delta-(1-\Delta)\theta)\sqrt{\xi}, \tag{7}$$

$$\sigma(\theta) := \mathrm{sign}(\alpha(\theta)), \tag{8}$$

and

$$\mathrm{Erfc}(x):=\int_x^{+\infty}dt\,e^{-t^2}. \tag{9}$$

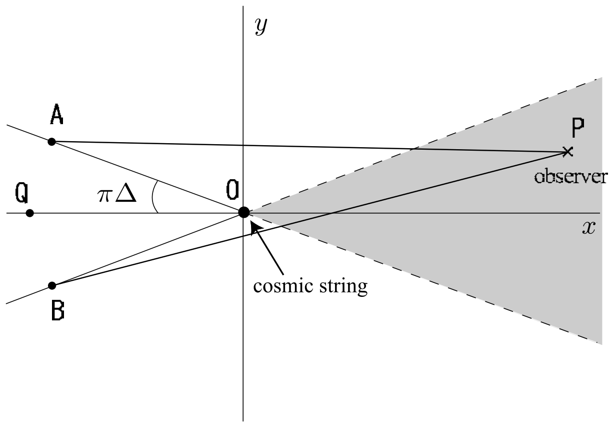

Figure 1. Configuration of the source, the cosmic string and the observer. A and B are the positions of a source. O and P are the positions of the cosmic string and the observer, respectively. In this figure, the wedge AOB is removed and thus A and B must be identified.

3.2. Geometrical optics limit

Geometrical optics limit corresponds to the limit $\xi \to \infty$ with Δ and θ fixed. In this limit $\alpha(\theta)$ also goes to $+\infty$, and hence the error function in Eq. (6) vanishes. Hence the waveform in the geometrical optics limit, which we denote as $\phi_{\rm go}$, becomes

$$\phi_{\rm go}(\xi,\theta) = e^{i\xi\cos(\pi\Delta+\varphi)}\Theta(\pi\Delta+\varphi) + e^{i\xi\cos(\pi\Delta-\varphi)}\Theta(\pi\Delta-\varphi), \tag{10}$$

where φ is defined by Eq. (3).

Since ϕ and hence $\phi_{\rm go}$ are even in θ, it is sufficient to consider the case with $\theta > 0$. In Fig. 1, the configuration of the source, the lense and the observer is drawn in the coordinates in which the deficit angle $2\pi\Delta$ is manifest, i.e., the wedge AOB is removed from the spacetime. Both points A and B indicate the location of the source. The lines OA and OB are to be identified. The angle made by these two lines is the deficit angle. The locations of the string and the observer are represented by O and P, respectively. In our current setup the distance between O and A ($= r_o$) is taken to be infinite. When $\varphi > \pi\Delta$, only the source A can be seen from the observer. This corresponds to the fact that only the first term remains for $\varphi > \pi\Delta$ in Eq. (10). For $\varphi > \pi\Delta$, we have

$$\phi_{\rm go}(\xi,\theta) = e^{i\xi\cos(\varphi+\pi\Delta)}. \tag{11}$$

This is a plane wave whose traveling direction is $\varphi = -\pi\Delta$, which is the direction of $\overrightarrow{\rm AP}$ in Fig. 1 in the limit $r_o = |\overrightarrow{\rm AO}| \to \infty$.

For $|\varphi| < \pi\Delta$, $\phi_{\rm go}$ is

$$\phi_{\rm go}(\xi,\theta) = e^{i\xi\cos(\varphi-\pi\Delta)} + e^{i\xi\cos(\varphi+\pi\Delta)}. \tag{12}$$

This is the superposition of two plane waves whose traveling directions are different by the deficit angle $2\pi\Delta$. Hence amplification of the images and interference occur for $|\varphi| < \pi\Delta$ as expected.

As we shall explain below, Eq. (10) coincides with the one derived under the geometrical optics. In geometrical optics, wave form is given by [11]

$$\phi_{go} = \sum_j |u(\vec{x}_j)|^{1/2} \exp[i\omega T(\vec{x}_j) - i\pi n_j], \tag{13}$$

where $\vec{x}$ represents a two-dimensional vector on the lens plane and $T(\vec{x})$ represents the summation of time of flight of the light ray from the source to the point $\vec{x}$ on the lens plane and that from the point $\vec{x}$ to the observer. $\vec{x}_j$ is a stationary point of $T(\vec{x})$, and $n_j = 0, 1/2, 1$ when $\vec{x}_j$ is a minimum, saddle and maximum point of $T(\vec{x})$, respectively. The amplitude ratio $|u(\vec{x})|^{1/2}$ is written as

$$u(\vec{x}) = 1/\det[\delta_{ab} - \partial_a\partial_b\psi(\vec{x})], \tag{14}$$

where $\psi(\vec{x})$ in Eq. (14) is the deflection potential [46] which is the integral of the gravitational potential of the lens along the trajectory between the source and the observer. Eq. (13) represents that the wave form is obtained by taking the sum of the amplitude ratio $|u(\vec{x}_j)|^{1/2}$ of each images with the phase factor $e^{i\omega T(\vec{x}_j)-i\pi n_j}$. If the lens is the straight string, the spacetime is locally flat everywhere except for right on the string. This means that the deflection potential $\psi(\vec{x})$ is zero and hence the amplitude ratio is unity for all images [46] and the trajectory where the time of flight $T(\vec{x})$ takes the extremal value is a geodesic in the conical space, and $T(\vec{x})$ of any geodesic takes minimum, which means $n_j = 0$. There are two geodesics if the observer is in the shaded region in Fig. 1. The time of flight along the trajectory AP is

$$T_{\mathtt{A}} = \lim_{r_o\to\infty} |\overrightarrow{\mathtt{AP}}| \approx r_o + r\cos(\pi\Delta + \varphi), \tag{15}$$

where $r \equiv |\overrightarrow{\mathtt{OP}}|$. The time of flight along the trajectory BP is obtained by just replacing φ with $-\varphi$. Hence, substituting (15) into (13), we find that the waveform in the geometrical optics is the same as Eq. (12) except for an overall phase $e^{ir_o\xi}$. This factor has been already absorbed in the choice of the normalization factor B in our formula (5).

We define the amplification factor

$$F(\xi,\theta) = \frac{\phi(\xi,\theta)}{\phi_{\rm UL}(\xi,\theta)}, \tag{16}$$

where $\phi_{\rm UL}$ is the unlensed waveform. Using Eq. (12), the amplification factor of $\phi_{\rm go}$ for $|\varphi| < \pi\Delta$ is given by

$$F_{\rm go}(\xi,\theta) \approx 2e^{-i\frac{\xi}{2}(\pi\Delta)^2}\cos(\pi\Delta\xi\varphi), \tag{17}$$

where we have assumed φ and Δ are small and dropped terms higher than quadratic order. It might be more suggestive to rewrite the above formula into

$$|F_{\rm go}(\xi,\theta)| \approx 2\cos(\pi\Delta\omega y), \tag{18}$$

where $y = r\sin\varphi$. The distance from a node to the next of when the observer is moved in y-direction is $\lambda/\pi\Delta$, where λ is a wavelength.

3.3. Quasi-geometical optics approximation

In the previous subsection, we have derived the waveform in the limit $\xi, |\alpha(\pm\theta)| \to \infty$ which corresponds to the geometrical optics approximation. Here we expand the waveform (6) to the lowest order in $1/\alpha(\pm\theta)$. This includes the leading order corrections to the geometrical optics approximation due to the finite wavelength effects.

Using the asymptotic formula for the error function, the leading order correction due to the finite wavelength, which we denote as $\delta\phi_{\rm qgo}$, is obtained as

$$\delta\phi_{\rm qgo}(\xi,\theta) = -\frac{e^{i\xi+i\pi/4}}{\sqrt{2\pi}}\left(\frac{1}{\alpha(\theta)} + \frac{1}{\alpha(-\theta)}\right) = -\frac{e^{i\xi+i\pi/4}}{\sqrt{2\pi\xi}}\frac{2\pi\Delta}{(\pi\Delta)^2 - \varphi^2}, \tag{19}$$

As is expected, the correction blows up for $|\varphi| \approx \pi\Delta$, where $\alpha(\theta)$ or $\alpha(-\theta)$ vanishes, irrespectively of the value of ξ. In such cases, we have to evaluate the error function directly, going back to Eq. (6).

The expression on the first line in Eq.(19) manifestly depends only on $\alpha(\pm\theta)$ aside from the common phase factor $e^{i\xi}$. This feature remains true even if we consider a small value of $\alpha(\pm\theta)$. This can be seen by rewriting Eq. (6) as

$$\phi(\xi,\theta) \approx \frac{e^{i\xi-\frac{i}{2}\alpha^2(\theta)}}{\sqrt{\pi}}\mathrm{Erfc}\left(\frac{-\alpha(\theta)}{\sqrt{2i}}\right)+(\theta\to-\theta). \tag{20}$$

The common phase $e^{i\xi}$ does not affect the absolute magnitude of the wave. Except for this unimportant overall phase, the waveform is completely determined by $\alpha(\pm\theta)$.

The geometrical meaning of these parameters $\alpha(\pm\theta)$ is the ratio of two length scales defined on the lens plane. To explain this, let us take the picture that a wave is composed of a superposition of waves which go through various points on the lens plane. In the geometrical optics limit the pathes passing through stationary points of $T(\vec{x})$, which we call the image points, contribute to the waveform. The first length scale is $r_s = |\alpha(\pm\theta)|/\sqrt{\xi}\times r$ which is defined as the separation between an image point and the string on the lens plane. In this picture we expect that pathes whose pathlength is longer or shorter than the value at an image point by about one wavelength will not give a significant contribution because of the phase cancellation. Namely, only the pathes which pass within a certain radius from an image point need to be taken into account. Then such a radius will be given by $r_F = \sqrt{\lambda r}$, which we call Fresnel radius. Namely, a wave with a finite wavelength can be recognized as an extended beam whose transverse size is given by r_F. The ratio of these two scales gives $\alpha(\pm\theta)$:

$$|\alpha(\pm\theta)| = \frac{\sqrt{2\pi}r_s}{r_F}.$$

When $r_s \gg r_F$, i.e., $\alpha(\pm\theta)\gg 1$, the beam width is smaller than the separation. In this case the beam image is not shadowed by the string, and therefore the geometrical optics becomes a good approximation. When $r_s < r_F$, i.e.,

$$\alpha(\pm\theta) < 1, \tag{21}$$

we cannot see the whole image of the beam, truncated at the location of the string. Then the diffraction effect becomes important. The ratio of the beam image eclipsed by the string determines the phase shift and the amplification of the wave coming from each image. If we substitute $|\varphi|\approx 0$ as a typical value, we obtain a rough criterion that the diffraction effect becomes important when

$$\lambda > 2\pi(\pi\Delta)^2 r, \tag{22}$$

or $\xi < (\pi\Delta)^{-2}$ in terms of ξ.

The same logic applies for a usual compact lens object. In this case the Fresnel radius does not change but the typical separation of the image from the lens is given by the Einstein radius $r_E \approx \sqrt{4GMr}$, where M is the mass of the lens. Then the ratio between r_E and r_F is given by $r_E/r_F = \sqrt{GM/\lambda}$, which leads to the usual criterion that the diffraction effect becomes important when $\lambda > GM$[1, 2, 3, 4, 5].

From the above formula (19), we can read that the leading order corrections scales like $\propto \sqrt{\lambda/r}$. This dependences on λ and r differ from the cases that the lens is composed of a normal localized object, in which the leading order correction due to the finite wavelength is $\mathcal{O}(\lambda/M)$ [15].

The condition for the diffraction effect to be important (21) can be also derived directly from Eq. (19). In order that the current expansion is a good approximation, $\phi_{\rm qgo}$ must be smaller than $\phi_{\rm go}$. This requires that $1/\alpha(\pm\theta)\gg 1$, which is identical to (21).

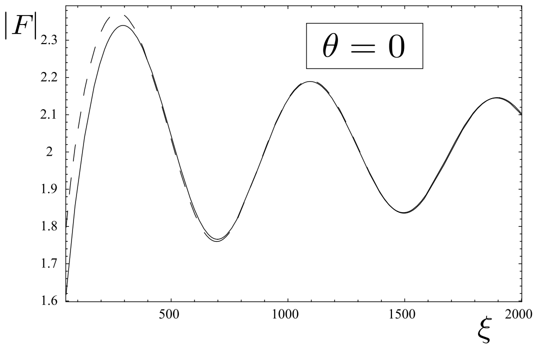

Figure 2. The absolute value of the amplification factor as a function of ξ for $\theta = 0$. Black line and dashed one correspond to Eq. (6) and the quasi-geometrical optics approximation, respectively. The string tension is chosen to be $G\mu = 10^{-2}$.

We plot the absolute value of the amplification factor under the quasi-geometrical optics approximation as dashed line in Fig. 2. We find that the quasi-geometrical optics approximation is a good approximation for $\xi > \Delta^{-2}$. For $\xi < \Delta^{-2}$, the quasi-geometrical optics approximation gives a larger amplification factor than the exact one.

In the quasi-geometrical optics approximation, we find from Eqs. (12) and (19) the absolute value of the amplification factor for $\varphi = 0$ is

$$|F(\xi, 0)| \approx 2 \left[1 - \sqrt{\frac{2}{\pi\xi(\pi\Delta^2}} \cos\left(\frac{\xi}{2}(\pi\Delta)^2 + \frac{\pi}{4} \right) \right]^{1/2} . \tag{23}$$

From this expression, we find that the position of the first peak of the amplification factor lies at $\xi \approx 4.25 \times (\pi\Delta)^{-2}$, which can be also verified from Fig. 2. For $\xi < \Delta^{-2}$ the present approximation is not valid, but we know that the amplification factor should converge to unity in the limit $\xi \to 0$, where r_F is much larger than r_s.

We show in Fig. 3 the absolute value of the amplification factor as a function of φ for four cases of ξ around Δ^{-2}. Top left, top right, bottom left and bottom right pannels correspond to $\xi(\pi\Delta)^2 = 0.5, 1, 2$ and 4, respectively. Black curves are plots for Eq. (6) and the dotted ones are plots for the quasi-geometrical optics approximation. As is expected, the error of the quasi-geometrical optics approximation becomes very large near $\varphi = \pi\Delta$, where $\alpha(\theta)$ vanishes. As the value of ξ increases, the angular region in which the quasi-geometrical optics breaks down is reduced.

Interestingly, the absolute value of the amplification factor deviates from unity even for $\varphi > \pi\Delta$ which is not observed in the geometrical optics limit. This is a consequence of diffraction of waves, the amplitude of oscillation of the interference pattern becomes smaller as θ becomes larger, which is a typical diffraction pattern formed when a wave passes through a single slit. The broadening of the interference pattern due to the diffraction effect means that the observers even in the region $|\varphi| > \pi\Delta$ can detect signatures of the presense of a cosmic string.

But the deviation of the amplification from unity outside the wedge $\varphi > \pi\Delta$ is rather small except for the special case $\xi(\pi\Delta)^2 \approx 1$: for $\xi(\pi\Delta)^2 \ll 1$ the magnification is inefficient and for $\xi(\pi\Delta)^2 \gg 1$ the magnification itself does not occur. Hence the increase of the event rates of lensing by cosmic strings compared with the estimate under the geometrical optics approximation could be important only when the relation $\xi(\pi\Delta)^2 \approx 1$ is satisfied. If we take $D = 10^{28}$cm and $\omega = 10^{-3}$Hz which is in the frequency band of LISA(Laser Interferometer Space Antenna)[47], we find that the typical value of $G\mu$ is $\approx 2 \times 10^{-9}$.

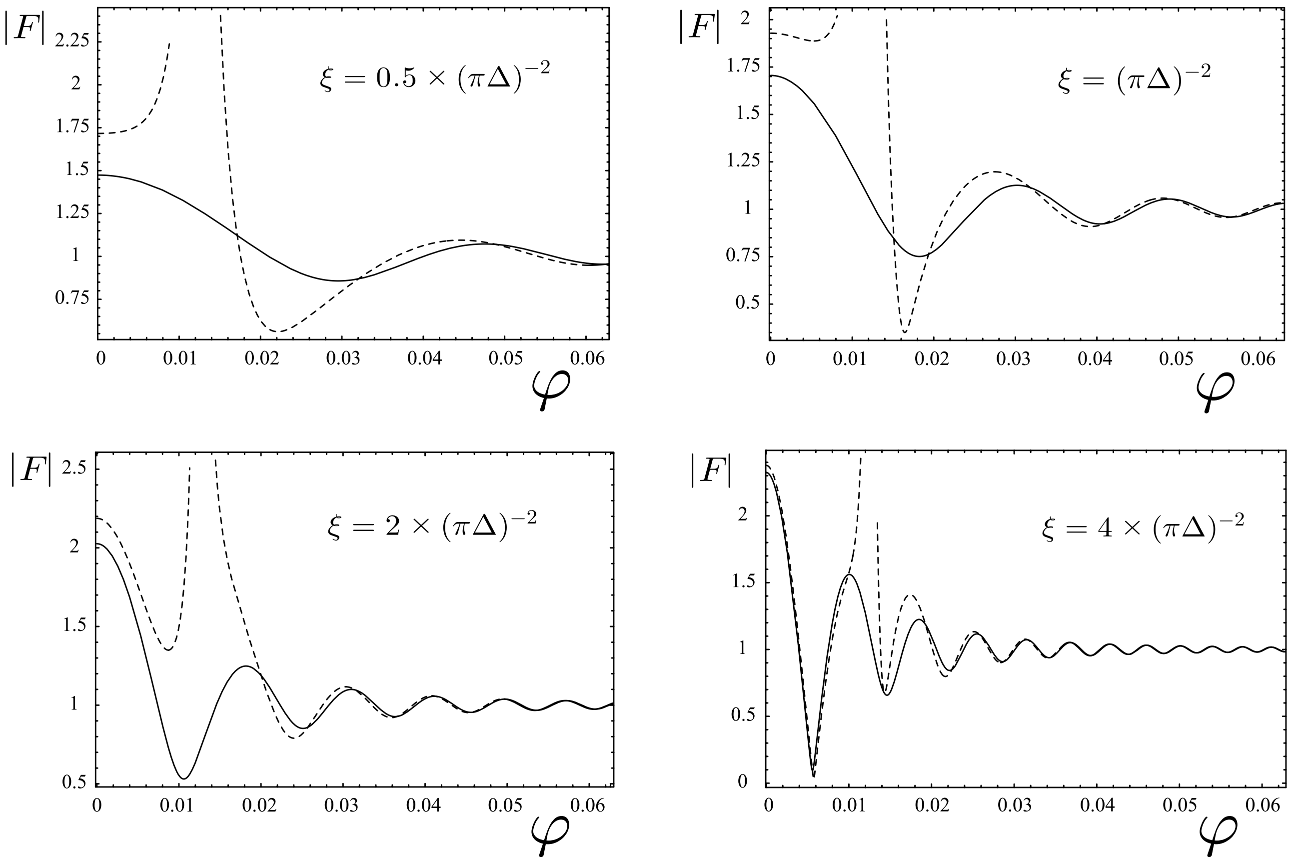

Figure 3. Black line and dotted one correspond to Eq. (6) and the quasi-geometrical optics approximation, respectively. The string tension is chosen to be $G\mu = 10^{-3}$.

So far, we have considered the stringy source rather than a point source. Extension to a point source can be done in a similar manner to the case of the stringy source. The result is

$$\phi(r, \theta, z) \approx -\frac{1}{4\pi D} e^{i\omega D} \mathcal{F}\left(\frac{\omega r r_o}{D}, \theta\right), \tag{24}$$

where $D = \sqrt{(r + r_o)^2 + z^2}$ is the distance between the source and the observer and

$$\mathcal{F}(x, \theta) := \frac{1}{2i\pi(1 - \Delta)} \int_{\infty - i\pi}^{\infty + i\pi} dt' \, e^{\frac{ix}{2}(t' - i\frac{\pi}{2})^2} \left(\frac{1}{1 - e^{-\frac{1}{1-\Delta}(t' + i\pi/2) + i\theta - \epsilon}} + (\theta \to -\theta) \right). \tag{25}$$

In particular, assuming that $\Delta, \varphi \ll 1$, and keeping terms which could remain for $\omega r, \omega r_o \gg 1$, we have

$$F\!\left(\frac{\omega r r_o}{D}, \theta\right) \approx e^{-\frac{i}{2}\frac{\omega r r_o}{D}(\pi\Delta - \varphi)^2} \frac{1}{\sqrt{\pi}} \mathrm{Erfc}\left(\frac{\varphi - \pi\Delta}{\sqrt{2i}} \sqrt{\frac{\omega r r_o}{D}} \right) + (\theta \to -\theta). \tag{26}$$

3.4. Simpler derivation of Eq. (20).

We have derived an approximate waveform (20) which is valid in the wave zone from the exact solution of the wave equation Eq. (5). Here we show that Eq. (20) can be obtained by a more intuitive and simpler method. In the path integral formalism [11], the wave form is given by the sum of the amplitude exp $(i\omega T(s))$ for all posibble pathes which connect the source and the observer. Here $T(s)$ is the time of flight along the path s. If the cosmic string resides between the source and the observer, the wave form will be given by the sum of two terms one of which is obtained by the path integral over the pathes which pass through the upper side of the string $(y > 0)$ in Fig. 1, and the other through the lower side of it $(y < 0)$. The waveform coming from the formar contribution will be given by

$$A \int_{-\infty}^{\infty} dz_{\mathtt{Q}} \int_{0}^{\infty} dy_{\mathtt{Q}}\, e^{i\omega(|\overrightarrow{\mathtt{AQ}}|+|\overrightarrow{\mathtt{QP}}|)}, \tag{27}$$

where $\mathtt{Q} = (0, y_{\mathtt{Q}}, z_{\mathtt{Q}})$ is a point on the lens plane specified by $x = 0$. One can determine the normalization constant A by a little more detailed analysis, but we do not persue it further here. By integrating Eq. (27), we recover the first term in Eq. (20).

4. Summary

We have constructed a solution of the Klein-Gordon equation for a massless scalar field in the flat spacetime with a deficit angle $2\pi\Delta \approx 8\pi G\mu$ caused by an infinite straight cosmic string. We showed analytically that the solution in the short wavelength limit reduces to the geometrical optics limit. We have also derived the correction to the amplification factor obtained in the geometrical optics approximation due to the finite wavelength effect and the expression in the long wavelength limit.

The waveform is characterized by a ratio of two different length scales. One length scale r_s is defined as the separation between the image position on the lens plane in the geometrical optics and the string. We have two r_s since there are two images corresponding to which side of the string the ray travels. (When the image cannot be seen directly, we asign a negative number to r_s.) The other length scale r_F, which is called Fresnel radius, is the geometrical mean of the wavelength and the typical separation among the source, the lens and the observer. The waveform is characterized by the ratios between r_s and r_F. If $r_F > r_s$, the diffraction effect becomes important and the interference patterns are formed. Even when the image in the geometrical optics is not directly seen by the observer, the interference patterns remain. In contrast, in the geometrical optics magnification and interference occur only when the observer can see two images which travel both sides of the string. Namely, the angular range where lensing signals exist is broadened by the diffraction effect.

Acknowledgments

T.S. thanks Kunihito Ioka, Takashi Nakamura and Hiroyuki Tashiro for useful comments. This work is supported in part by Grant-in-Aid for Scientific Research, Nos. 14047212 and 16740141, and by that for the 21st Century COE "Center for Diversity and Universality in Physics" at Kyoto university, both from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] Ohanian H C 1973 *Phys. Rev. D* **8** 2734, Ohanian H C 1974 *Int. J. Theor. Phys.* **9** 425, Ohanian H C 1983 *Astrophys. J.* **271** 551
[2] Bliokh P V and Minakov A A 1975 *Ap& SS.* **34** L7
[3] Bontz R J and Haugan M P 1981 *Ap& SS.* **78** 199
[4] Thorne K S 1983 in Gravitational Radiation, ed N Deruell and T Piran (Amsterdam: North-Holland) p 28
[5] Deguchi S and Watson W D 1986 *Astrophys. J.* **307** 30
[6] Takahashi R and Nakamura T 2003 *Astrophys. J.* **595** 1039 (*Preprint* astro-ph/0305055)
[7] Seto N 2004 *Phys. Rev.* D **69** 022002 (*Preprint* astro-ph/0305605)
[8] Nakamura T T 1998 *Phys. Rev. Lett.* **80** 1138

[9] Yamamoto K Path Integral Formulation for Wave Effect in Multi-lens System 2003 *Preprint* astro-ph/0309696
[10] Baraldo C, Hosoya A and Nakamura T T 1999 *Phys. Rev.* D **59** 083001
[11] Nakamura T T and Deguchi S 1999 *Prog. Theor. Phys. Suppl.* **133** 137
[12] Yamamoto K and Tsunoda K 2003 *Phys. Rev.* D **68** 041302 (*Preprint* astro-ph/0309694)
[13] Suyama T, Takahashi R and Michikoshi S 2005 Wave propagation in a weak gravitational field and the validity of the thin *Preprint* astro-ph/0505023
[14] Takahashi R, Suyama T and Michikoshi S, *Astron. Astrophys.* **438** L5 (*Preprint* astro-ph/0503343)
[15] Takahashi R 2004 *Astron. Astrophys.* **423** 787 (*Preprint* astro-ph/0402165)
[16] Hindmarsh M B and Kibble T W B 1994 *Rept. Prog. Phys.* **58** 477 (*Preprint* hep-ph/9411342)
[17] Vilenkin A and Shellard E P S 2000 *Cosmic Strings and Other Topological Defects* (Cambridge: Cambridge University Press)
[18] Dvali G R and Tye S H H 1998 Brane inflation *Preprint* hep-ph/9812483
[19] Dvali G R 1999 *Phys. Lett.* B **459** 489 (*Preprint* hep-ph/9905204)
[20] Burgess C P, Majumdar M, Nolte D, Quevedo F, Rajesh G and Zhang R J 2001 *JHEP* **0107** 047 (*Preprint* hep-th/0105204)
[21] Alexander S H S 2002 *Phys. Rev.* D **65** 023507 (*Preprint* hep-th/0105032)
[22] Dvali G R, Shafi Q and Solganik S 2001 D-brane inflation *Preprint* hep-th/0105203
[23] Jones N, Stoica H and Tye S H H 2002 *JHEP* **0207** 051 (*Preprint* hep-th/0203163)
[24] Shiu G and Tye S H H 2001 *Phys. Lett.* B **516** 421 (*Preprint* hep-th/0106274)
[25] Majumdar M and Christine-Davis A 2002 *JHEP* **0203** 056 (*Preprint* hep-th/0202148)
[26] Dvali G and Vilenkin A 2003 *Phys. Rev.* D **67** 046002 (*Preprint* hep-th/0209217)
[27] Jones N T, Stoica H and Tye S H H 2003 *Phys. Lett.* B **563** 6 (*Preprint* hep-th/0303269)
[28] Dvali G and Vilenkin A 2004 *JCAP* **0403** 010 (*Preprint* hep-th/0312007)
[29] Copeland E J, Myers R C and Polchinski J 2004 *JHEP* **0406** 013 (*Preprint* hep-th/0312067)
[30] Spergel D N *et al.* [WMAP Collaboration] 2003 *Astrophys. J. Suppl.* **148** 175 (*Preprint* astro-ph/0302209)
[31] Percival W J *et al.* [The 2dFGRS Team Collaboration] 2002 *Mon. Not. Roy. Astron. Soc.* **337** 1068 (*Preprint* astro-ph/0206256)
[32] Jeong E and Smoot G F 2005 *Astrophys. J.* **624** 21 (*Preprint* astro-ph/0406432)
[33] Pogosian L, Tye S H H, Wasserman I and Wyman M 2003 *Phys. Rev.* D **68** 023506 (*Preprint* hep-th/0304188)
[34] Pogosian L, Wyman M C and Wasserman I 2004 Observational constraints on cosmic strings: Bayesian analysis in a three dimensional parameter space *Preprint* astro-ph/0403268
[35] Wyman M, Pogosian L and Wasserman I 2005 *Phys. Rev.* D **72** 023513 (*Preprint* astro-ph/0503364)
[36] Kaspi V M, Taylor J H and Ryba M F 1994 *Astrophys. J.* **428** 713
[37] Thorsett S E and Dewey R J 1996 *Phys. Rev.* D **53** 3468
[38] McHugh M P, Zalamansky G, Vernotte F and Lantz E 1996 *Phys. Rev.* D **54** 5993
[39] Lommen A N 2002 New limits on gravitational radiation using pulsars *Preprint* astro-ph/0208572
[40] Bouchet F R, Peter P, Riazuelo A and Sakellariadou M 2002 *Phys. Rev.* D **65** 021301 (*Preprint* astro-ph/0005022)
[41] Rocher J and Sakellariadou M 2005 *Phys. Rev. Lett.* **94** 011303 (*Preprint* hep-ph/0412143)
[42] Sazhin M, Capaccioli M, Longo G, Paolillo M and Kovanskaya O 2005 Further spectroscopic observations of the CSL-1 object *Preprint* astro-ph/0506400
[43] Sazhin M *et al.* 2003 *Mon. Not. Roy. Astron. Soc.* **343** 353 (*Preprint* astro-ph/0302547)
[44] Vilenkin A 1981 *Phys. Rev.* D **23** 852
[45] Gott J R I 1985 *Astrophys. J.* **288** 422
[46] Schneider P, Ehlers J and Falco E E 1992 *Gravitational Lenses* (New York: Springer)
[47] See http://lisa.jpl.nasa.gov/