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The Algebra and Calculus of Stochastically Perturbed Spacetime with Classical and Quantum Applications

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Abstract: We consider an alternative to dark matter as a potential solution to various remaining problems in physics: the addition of stochastic perturbations to spacetime to effectively enforce a minimum length and establish a fundamental uncertainty at minimum length (ML) scale. To explore the symmetry of spacetime to such perturbations both in classical and quantum theories, we develop some new tools of stochastic calculus. We derive the generators of rotations and boosts, along with the connection, for stochastically perturbed, minimum length spacetime ("ML spacetime"). We find the metric, the directional derivative, and the canonical commutator preserved. ML spacetime follows the Lie algebra of the Poincare group, now expressed in terms of the two-point functions of the stochastic fields (per Ito's lemma). With the fundamental uncertainty at ML scale a symmetry of spacetime, we require the translational invariance of any classical theory in classical spacetime to also include the stochastic spacetime perturbations. As an application of these ideas, we consider galaxy rotation curves for massive bodies to find that—under the Robertson–Walker minimum length theory—rotational velocity becomes constant as the distance to the center of the galaxy becomes very large. The new tools of stochastic calculus also set the stage to explore new frontiers at the quantum level. We consider a massless scalar field to derive the Ward-like identity for ML currents.

Keywords: stochastic spacetime; stochastic calculus; dark energy; classical and quantum gravity

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1. Introduction

The presence of dark matter and dark energy has been suggested to resolve various remaining problems in physics. In cosmology, examples of such problems include the observed galaxy rotation curves with velocities contrary to known laws of physics and Hubble tension. Theories such as quantum gravity and non-commutative geometry have been the subject of much additional research in attempts to quantize gravity and unify the treatment of classical cosmological problems with quantum aspects of black holes and high-energy particle physics ([1–9]).

We consider a somewhat different approach here: the addition of stochastic perturbations of spacetime effectively enforcing some minimum length (ML) scale and establishing a fundamental uncertainty at this scale. We allow for the possibility that stochastic fields can be both classical and quantum by nature to further consider the transition point between the classical and quantum regimes. Exploring the symmetry of spacetime to such classical-to-quantum perturbations both in classical and quantum theories requires some new tools but introduces the possibility of new solutions.

Promoting the stochastic perturbations to quantum operators in the quantum regime requires the formulation of canonical variables of the ML spacetime, demanding a better understanding of partial derivatives with respect to stochastic variables. We expand stochastic calculus by formulating the partial derivative relative to a stochastic spacetime at minimum length scale ("ML spacetime"). We derive the connection and the generators of translations, rotations, and boosts for ML spacetime to find that ML spacetime preserves the metric, the directional derivative, and the canonical commutator. Finally, we find that ML spacetime follows the Lie algebra of the Poincare group, now expressed in terms of the two-point functions of the stochastic fields corresponding to the fundamental uncertainty of ML spacetime.

"Planck length", historically synonymous with the notion of minimum length, originated via dimensional arguments linking the gravitational constant, the speed of light, and Planck's constant. As such, Planck length appears abstractly removed from any physical reality. Frank Wilczek, in his reply to C. Alden Mead's letter *"Walking the Planck Length through History"* [10–12], commented,

> *It is true and significant that the Planck length arises naturally when one considers the ultimate limits to measurement...*

The *"limits to measurement"* have two possible sources. The first is a measurement error due to the technological limitations of our "meter stick", such as an interferometer. The second possible source of such limits is a fundamental uncertainty in spacetime, a diffusive nature of spacetime to complement the geometry of spacetime. In this paper, we consider only the latter.

Quantum mechanics treats spacetime as fixed. General relativity warps spacetime. Here, we treat spacetime as uncertain at some small scale. By introducing stochastic fields in addition to regular fields, we formulate an effective theory of a fundamental uncertainty in spacetime.

Minimum length scale is intimately tied with the quest for a unified theory capable of explaining the vast divergence of scales between the extremely small strength of gravity compared to that of the strong force ([13], pg. 225). The most popular attempts at such unification—supersymmetry and string theory—remain experimentally elusive despite the many years dedicated to searches at particle accelerators.

Much research has focused on theories of quantum gravity at Planck scale, resulting in an emergent spacetime. Often considering various non-commutative geometries, these theories assume a quantized regime, with canonical commutation relationships defined by the generalized uncertainty principle (GUP) in place of the Heisenberg uncertainty principle (HUP). In this paper, we instead focus on introducing and developing the mathematical fundamentals of a theory where spacetime fields are naturally classical in nature but can be quantized. Future research of the concepts presented here once applied to problems with strong gravitational sources, will help us answer the question of whether such classical-to-quantum fundamental uncertainty in spacetime can lead to, or possibly relate to, emergent gravitational fields.

We focus on situations devoid of or under extremely weak sources of gravity. This has the additional benefit of allowing us to treat the metric as separate and distinct from the diffusive effects of the stochastic spacetime. Potential research focusing on the effects of a stochastic spacetime in the presence of a large source and in the region of the Schwarzschild radius may shed some light on the complex question of how the metric may be influenced by a stochastic spacetime. But for now, with the narrower focus of this paper, we are free to assume very small to non-existent effects of gravitational fields with no impact on the metric.

Stochastic fields in Langevin equations are commonly used for analyzing classical problems in physics. In the classical regime, stochastic spacetime fields enforce an effective ML scale often defined by the scale of the problem. Generally, these classical ML scales can be much greater than Planck length. In the classical cosmological problems, for example, given the present-day estimate for Hubble parameter H, its diffusive counterpart D [14], and the vacuum temperature, ML scale can be as large as 10^{-1} m.

While HUP applies only in the quantum regime, in its saturated form, HUP provides a transition point where classical stochastic fields may be quantized and where we can thus promote the stochastic canonical variables to quantum operators. Understanding how this quantization may come about physically—and to what extent a mix of such quantum and classical representations of the canonical variables is possible around the transition point—remains for further research and consideration.

We will see that, in the quantum regime, canonical variables preserve the non-stochastic commutation relationship, thus not requiring the extension of HUP to GUP. In contrast, modified canonical commutation relations of various quantum gravity theories rely on a GUP to enforce ML scale (See [8] for an excellent overview). Finally, the classical regime lends itself to the application of the equipartition theorem, and we take advantage of this in the case of the classic harmonic oscillator.

Over the years, various suggestions have been made about what happens at ML scale, including the possibility that the laws of physics break down. In [14], we instead made the assumption that the fundamental uncertainty at ML scale must be a symmetry of spacetime: we required an invariance of the proper time functional in general relativity to the introduction of stochastic perturbations. This allowed us to expand the Christoffel connection in general relativity to include the diffusive effects of the fundamental ML scale uncertainty and to ultimately suggest a means of resolving Hubble tension.

We continue to apply this approach here. Consistent with the steps taken in [14], for any classical theory in classical spacetime, the requirement of invariance due to infinitesimal translations must also include a stochastic element at ML scale. As shown in [14], adding an infinitesimal stochastic translation to the regular infinitesimal translation leads to an extended Christoffel connection, now also a function of the two-point functions of the stochastic spacetime fields. The application of the extended connection in the Robertson–Walker (RW) metric provides for an energy momentum tensor and Friedmann equations expanded to include the effects of stochastic diffusion in addition to geometry. Working in a spatially local frame, the RW theory in the absence of stochastic perturbations and the ML theory in the Minkowski metric are both conformally Minkowski. However, together, they scale at different powers of the RW scale factor and, thus, are no longer conformally Minkowski. We estimate the diffusive counterpart to H, D, by analyzing available redshift data to find that, at the present time, $D \sim \frac{1}{70}H$ [14]. In doing so, we resolve the Hubble tension.

Here, we take advantage of the results in [14] to consider the galaxy rotation curves of a massive body far from the center of the galaxy ([15–21]). Under the Friedmann–Lemaitre–Robertson–Walker metric in the limit of a very weak gravitational field, together with the stochastic spacetime perturbations of the Robertson–Walker minimum length theory (RWML) [14], we discover rotational velocity approaching a constant value at large distances from the center of the galaxy, consistent with many cosmological observations.

In contrast to classical theories, quantum field theories (QFTs) must be applied in the quantum regime of ML spacetime. We set the stage for QFTs in ML spacetime and consider an example of a massless scalar field to derive a Ward-like identity, suggesting ML currents (defined in (54)) decouple from massless scalar fields.

The paper is organized as follows. We begin by defining a stochastic spacetime, thus introducing a fundamental uncertainty in spacetime as a mathematical reality. We then consider classical problems formulated via Langevin equations in the classical regime. We next derive the stochastic partial derivative and consider the resulting algebra of stochastic spacetime. This allows us to promote the stochastic fields to the quantum regime. Equipped with these derivations, we formulate maps and a connection to further allow us to consider quantum field theories (QFTs) in ML spacetime. We consider an application of these ideas in the classical regime to the galaxy rotation curves and take a look at the data provided in Spitzer Photometry and Accurate Rotation Curves (SPARC) [22]. In the quantum regime, we consider massless scalar fields. We conclude with a brief discussion of possible critiques and a few words on future research.

Throughout the paper, we thus build on the existing, experimentally proven theories, tools, and approaches in stochastic calculus, QFT, and general relativity (GR) by expanding on the notion of uncertainty at ML scale and observing what the resulting math tells us.

Unless stated otherwise, we work in natural units, with a mostly positive Minkowski metric, using Ito formalism.

2. ML Spacetime, $\tilde{X}$

We begin with a simple definition of $\delta\xi_x$ as a stochastic contravariant vector field,

$$\delta\xi^{\mu}_{x} \sim \mathcal{N}(0, \sigma_{\xi}\sqrt{\delta t}), \quad \langle \delta\xi^{\mu}_{x} \rangle = 0 \tag{1}$$

Equation (1) characterizes the stochastic spacetime vector fields as normally distributed, with a mean of zero and a standard deviation that scales with $\sigma_{\xi}\sqrt{\delta t}$. We take δt to represent the ML scale and σ_{ξ} as the magnitude of perturbations.

Most readers are probably familiar with the concept of a random walk, where the position of an object has a standard deviation defined by $\Delta r = \sqrt{D_r \Delta t}$. In (1), it is σ^2_{ξ} that plays the role of D_r and thus represents the diffusive parameter for the stochastic spacetime. Also, note that, consistent with the typical literature on random walks, the diffusive parameter has the dimension $[\sigma^2_{\xi}] = L$, and therefore, $[\sigma_{\xi}] = L^{\frac{1}{2}}$.

We require $\delta\xi_x$ to be unique to a point in spacetime, x, and we employ a less simplistic approach to (1) to formally define the magnitude of the perturbations via a two-point function,

$$\langle \delta\xi^{\mu}_{x} \delta\xi^{\nu}_{y} \rangle = \delta^4(x-y)\, h^{\mu\nu}_{\rho} \delta x^{\rho} \tag{2}$$

Above, we take the most general form for the two-point function of the random terms consistent with [14] and proportional to the ML scale captured by the magnitude of $\delta x \sim \delta t$.

The simplest application of (2) is in an isotropic spacetime, where we might define $h^{\mu\nu}_{\rho} \delta x^{\rho} \equiv g^{\mu\nu}\sigma^2_{\xi}\delta t$, with g being the metric. Figure 1a provides a simplified view of such a stochastic vector field at some point A in two-dimensional space. In this simple case, the stochastic vector field is a two-dimensional vector in space with random elements.

We define ML spacetime, $\tilde{X}$, to exhibit a fundamental uncertainty corresponding to this stochastic vector field,

$$\tilde{x} = x + \delta\xi_x \tag{3}$$

In our analysis, we also need additional two-point functions for the kinetic stochastic terms, which we define most generally as follows,

$$\langle \partial_{\alpha}\delta\xi^{\mu}_{x} \rangle = 0, \quad \langle \partial_{\alpha}\delta\xi^{\mu}_{x} \partial_{\beta}\delta\xi^{\nu}_{x} \rangle \equiv k^{\mu\nu}_{\rho\alpha\beta}\delta x^{\rho} \tag{4}$$

$$\langle \delta\xi^{\mu}_{x} \partial_{\alpha}\delta\xi^{\nu}_{x} \rangle = -\langle \partial_{\alpha}\delta\xi^{\mu}_{x} \delta\xi^{\nu}_{x} \rangle \equiv a^{\mu\nu}_{\rho\alpha}\delta x^{\rho} \tag{5}$$

It is important to recognize that Equation (4) defines the behavior of kinetic stochastic fields rather than spacetime stochastic fields. The resulting diffusion corresponding to these kinetic fields has a different dimension than what the reader may be used to in dealing with random walks in spacetime. Specifically, from (4), we see that the required dimension for $k^{\mu\nu}_{\rho\alpha\beta}$ must be $[k^{\mu\nu}_{\rho\alpha\beta}] = L^{-1}$.

For a better insight into the diffusion of kinetic stochastic fields, consider an isotropic spacetime. In this case, we may, for example, simplify (4) with $k^{\mu\nu}_{\rho\alpha\beta}\delta x^{\rho} \equiv 2g^{\mu\nu}\delta_{\alpha\beta}c_{\xi}\sigma^2_{\xi}\delta t$. This choice of parametrization gives us a diffusive parameter $D \sim c_{\xi}\sigma^2_{\xi}$, where c_{ξ} is some new parameter of dimension $[c_{\xi}] = L^{-2}$. Hence, the diffusive parameter corresponding to kinetic stochastic fields is of dimension $[D] = L^{-1}$. Note that D is of the same dimension as the Hubble parameter H. As shown in [14], D is the diffusive equivalent to H and plays an important role in potentially explaining the Hubble tension (see Equations (16)–(20) in [14] for the role that diffusion parameter D plays in defining the redshift).

Alternatively, we may consider $k^{\mu\nu}_{\rho\alpha\beta}\delta x^{\rho} \equiv 2g^{\mu\nu}g_{\alpha\beta}c_{\xi}\sigma^2_{\xi}\delta t$. Either choice does not impact the considerations in this paper. To gain some intuition on this topic, consider that $\partial_{\mu}\delta\xi^{\alpha}$ is also a normally distributed stochastic variable (a normally distributed variable is characterized by a kurtosis equal to three times the second moment squared) in the α

direction, regardless of the choice of μ (as illustrated in Figure 1b), albeit with a different standard deviation. This suggests δ is the more appropriate choice, further supported by results in [14]. A more general choice may be to take $k^{\mu\nu}_{\rho\alpha\beta}\delta x^{\rho} \equiv 2g^{\mu\nu}\rho_{\alpha\beta}c_{\xi}\sigma_{\xi}^{2}\delta t$, with ρ some positive definite dimensionless matrix, $[\rho] = L^{0}$.

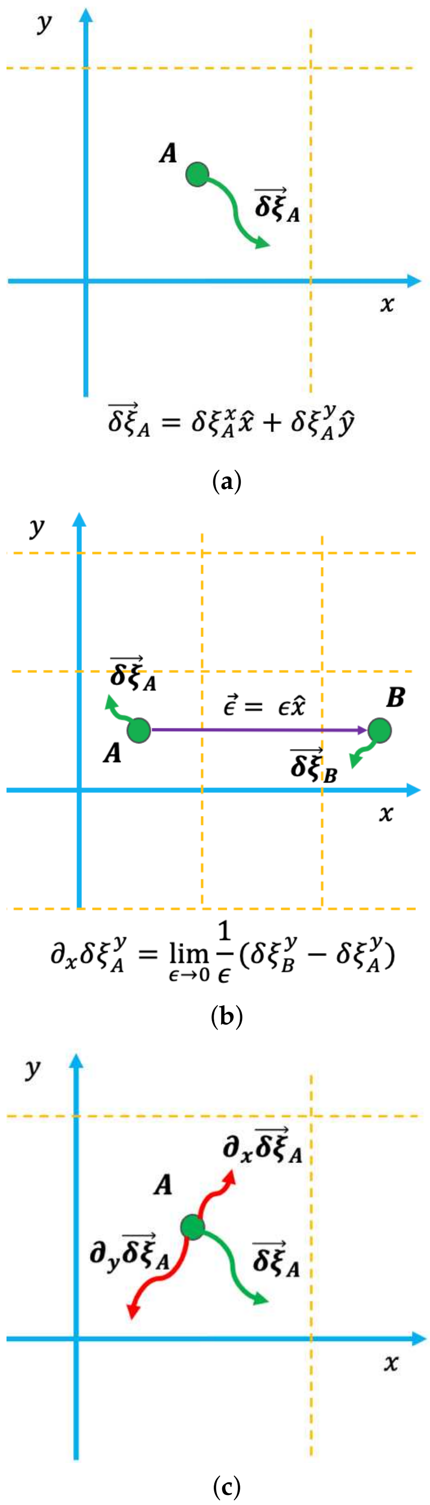


Figure 1. (**a**) Stochastic variable in two-dimensional space at point A. (**b**) Partial derivative of a stochastic variable in two-dimensional space. (**c**) All the stochastic vector fields at point A in two-dimensional space.

We make no assumptions as to whether partial derivatives of stochastic fields may or may not be well-defined. However, we do take the approach that the point functions involving such partial derivatives are indeed well-defined. Additionally, a reader may recognize that the parametrization of $\rho_{\alpha\beta}$ in the classical regime is a function of the choice of the Ito vs. Stratonovich approach and, more specifically, the choice of phase in the definition of the derivative.

In general, for four-dimensional spacetime, each point x is thus associated with a total of five contravariant stochastic vector fields—one for the spacetime stochastic vector and an additional four kinetic vector fields corresponding to each partial derivative. Figure 1c shows all the stochastic fields associated with point A for the case of two spatial dimensions. In addition to the two-dimensional original vector field $\delta\vec{\xi}_A$, we have the two additional kinetic stochastic vector fields corresponding to the two partial derivatives in this example, $\partial_x\delta\vec{\xi}_A$ and $\partial_y\delta\vec{\xi}_A$.

A more realistic and yet simplified approach to the assumption of an isotropic spacetime involves separating the diffusive effects in space vs. time, as performed in [14] and described in Appendix A.

For spherical potentials, it is also necessary to formulate the spherical stochastic terms, derived from the stochastic terms in Cartesian coordinates and described in detail in Appendix A. It is interesting to note that in ML spacetime, $\tilde{r}$ obtains a diffusive term with a singularity at $r = 0$, $\langle\delta\tilde{r}\rangle = \frac{\sigma_\xi^2\delta t}{2r}$.

3. Langevin Equations in Classical Regime

Langevin equations allow us to consider what happens under infinitesimal changes that include some form of uncertainty. They allow us to gain an understanding of the impact that the uncertainty at ML scale has on the motion of a classical particle.

The uncertainty in position results in an uncertainty in momentum, and vice versa, albeit at potentially very different scales. Figure 2 shows a particle of mass m on a world path line parametrized by λ. This parametrization allows us to treat the classic particle's position and momentum as functions of time along the world line.

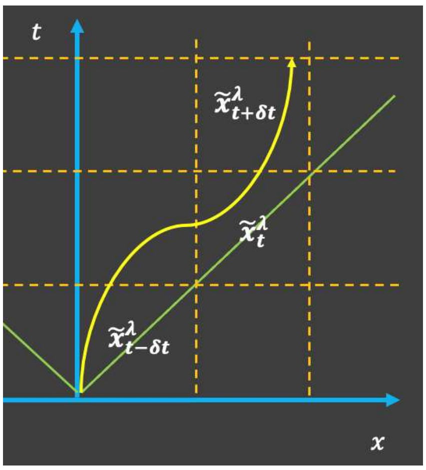


Figure 2. Particle on a world line.

Langevin equations for position and momentum at some time t are then given by

$$\delta\tilde{x}^\lambda_t = \frac{1}{m}\tilde{p}^\lambda_t\delta t + \delta\xi^\lambda_t \quad (6)$$

$$\begin{aligned}\delta\tilde{p}^\lambda_t &= -\nabla V\delta t + m\,\partial_t\delta\xi^\lambda_t \\ &= -\nabla V\delta t + m\sqrt{c_\xi}(\delta\xi^\lambda_t - \delta\xi^\lambda_{t-\delta t}),\end{aligned} \quad (7)$$

where $\delta\xi_t^\lambda$ is the stochastic term, where we have defined

$$\partial_t \delta\xi_t^\lambda \equiv \sqrt{c_\xi}(\delta\xi_t^\lambda - \delta\xi_{t-\delta t}^\lambda), \quad c_\xi \equiv \frac{1}{\delta t^2} \tag{8}$$

(Note, with δt the smallest time interval, we can approximate the two-point function of the kinetic term,

$$\langle(\partial_t \delta\xi_t^\lambda)^2\rangle = \frac{1}{\delta t^2}\langle(\delta\xi_t^\lambda - \delta\xi_{t-\delta t}^\lambda)^2\rangle = \frac{2\sigma_\xi^2}{\delta t}, \tag{9}$$

Parametrizing the kinetic two-point function with $\langle(\partial_t \delta\xi_t^\lambda)^2\rangle = 2c_\xi\sigma_\xi^2\delta t$, as in (A2), we have $c_\xi = \frac{1}{\delta t^2}$ in this case.) We also assume that the stochastic variables exhibit no autocorrelation:

$$\langle(\delta\xi_t^\lambda)^2\rangle = \sigma_\xi^2\delta t \tag{10}$$

$$\langle\delta\xi_t^\lambda \delta\xi_{t-\delta t}^\lambda\rangle = 0 \tag{11}$$

$$\langle(\partial_t \delta\xi_t^\lambda)^2\rangle = 2c_\xi\sigma_\xi^2\delta t \tag{12}$$

In the simple case of a free particle, with the potential $V = 0$, stepping through the process defined by (6) and (7), we obtain the following for position and momentum at some time t relative to some earlier time $t = 0$ to order δt,

$$\begin{aligned} \tilde{x}_t^\lambda &= \frac{1}{m}\sum_n \tilde{p}_{n\delta t}^\lambda \delta t + \sum_n \delta\xi_{n\delta t}^\lambda \\ &\approx \frac{1}{m} p_0^\lambda n\delta t + \sum_n \delta\xi_{n\delta t}^\lambda \end{aligned} \tag{13}$$

$$\tilde{p}_t^\lambda = p_0^\lambda + m\sqrt{c_\xi}\delta\xi_t^\lambda \tag{14}$$

Equation (13) clearly shows a random walk at ML scale, where the uncertainty grows with the square root of time (see Appendix A.1 of [14] for further discussion on the fundamental aspects of stochastic variables). However, (14) is not a random walk at all! Instead, the momentum of a free particle obtains a simple uncertainty given by $\Delta p = m\sqrt{c_\xi\sigma_\xi^2\delta t} = m\sqrt{\frac{\sigma_\xi^2}{\delta t}}$, which is constant through time.

While all stochastic processes are diffusive, not all stochastic processes follow a random walk. The resulting simulated phase space diagram in Figure 3a also shows this clearly: the uncertainty in the momentum remains bounded. Figure 3b shows the path of the particle, with a non-discernible effect of the random walk on the position of the particle given the scale of the momentum.

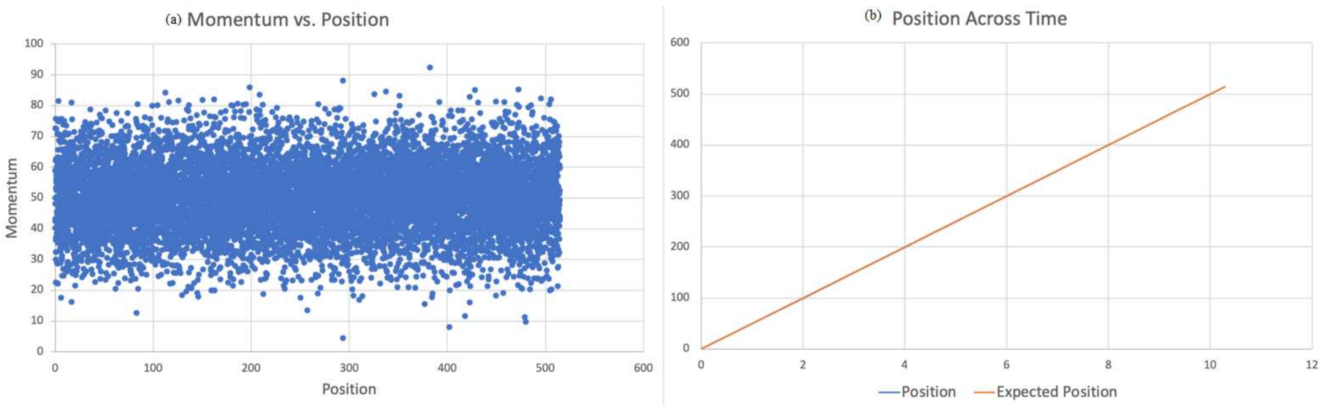
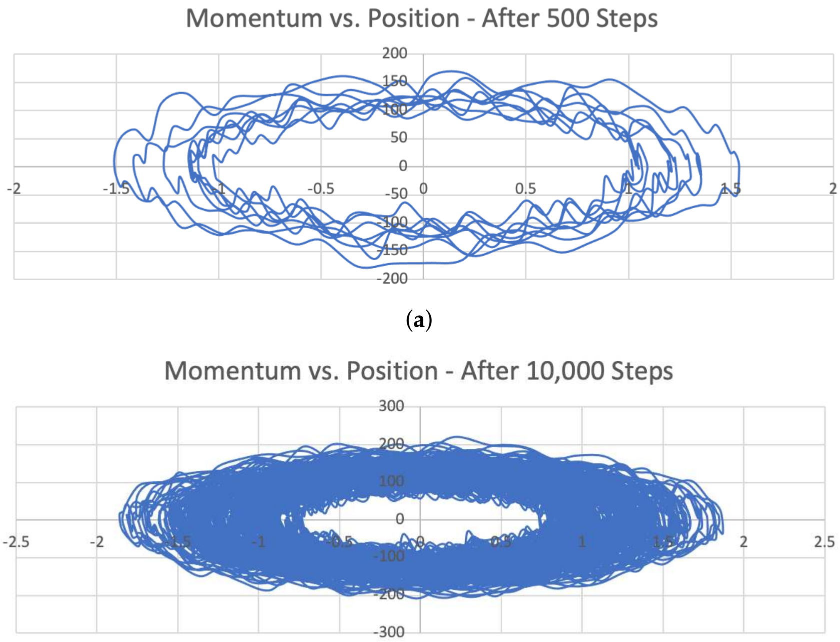


Figure 3. Simulated phase space and position through time for a free particle.

In Appendix B, we approximate the path of a free particle of very small mass with a null path and require the invariance of its Lagrangian to a translation in space with a corresponding stochastic element. We find that the particle gains an acceleration proportional to $c_\xi \sigma_\xi^2$. ML spacetime acts as a diffusive force, creating an accelerating universe.

Similarly, we can simulate the phase space of two particles coupled in a harmonic oscillator, a toy model for a Cooper pair. Figure 4a,b show the simulated phase space after a smaller and then much greater number of steps, respectively. The uncertainty at ML scale gives the two coupled particles a minimum expected positive separation distance.



(**a**)

(**b**)

Figure 4. Simulated phase space for a coupled harmonic oscillator after (**a**) 500 and (**b**) 10,000 steps.

Simulations of the harmonic oscillator may help us better understand what it means for a harmonic oscillator to be quantized as a function of its frequency ω relative to $\sqrt{2c_\xi}$. There are two possible regimes, one where ω is far smaller than $\sqrt{2c_\xi}$ and the other where the two terms scale similarly, resulting in position and momentum taking on discrete values. Figure 5 shows an example of such a simulation.

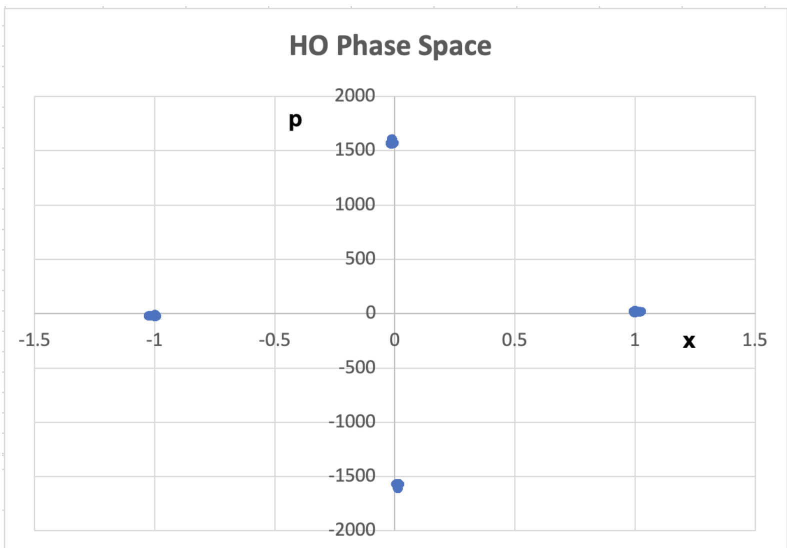


Figure 5. Quantizing the coupled harmonic oscillator.

In cosmology, we can take advantage of various astronomical observations to observe the effect of very small things on very large physical phenomena, such as in the case of Hubble tension and galaxy rotation curves. But, how do we test for other types of classical effects of ML spacetime? An experiment analyzing the Cooper pair copper oxide may provide the answer [23].

The addition of ML perturbations to the classical harmonic oscillator also adds energy to the system. Applying the equipartition theorem to the classical harmonic oscillator approaching a squeezed state at the critical temperature gives us a means of roughly estimating the velocity of such perturbations at the point of classical-to-quantum regime transition.

The harmonic oscillator toy model roughly predicts the speed of the resulting ML scale perturbations for a Cooper pair copper oxide corresponding to the critical temperature and length described in [23] at $1.8 \cdot 10^5 \frac{\text{m}}{\text{s}}$ (Appendix B provides the details of this derivation), which favorably compares to the experimental result of $1.5 \cdot 10^5$ m/s [23].

Additional experiments, such as those described in [23], can be helpful in testing this relationship at the regime transition point between the velocity of the perturbations, critical temperature, and the length scale of the system, as well as the very existence of ML spacetime perturbations.

But, perhaps the most definitive experiments to determine the possibility of ML effects may be in the realm of high-energy particle physics. Answering questions such as how the Higgs might relate to ML perturbations or whether ML perturbations play a role in vector-like quark theories of composite Higgs models requires further theoretical research with consequent experimental particle searches.

However, before we can even begin discussing the possible impact of ML spacetime in high-energy particle physics, we must first develop the necessary mathematical tools required to promote canonical stochastic variables to the quantum regime. This means, first and foremost, gaining an additional understanding of stochastic calculus.

4. Stochastic Calculus of ML Spacetime in Quantum Regime

Stochastic calculus ([24,25]) provides us with well-understood integral relationships (alternatively, summation relationships). For example,

$$\int_{t_1}^{t_3} d\xi_t = \int_{t_1}^{t_2} d\xi_t + \int_{t_2}^{t_3} d\xi_t, \quad t_1 < t_2 < t_3 \tag{15}$$

or the two-point function relationship,

$$\begin{aligned} \langle d\xi_t d\xi_{t'} \rangle = \delta(t-t')\sigma_\xi^2 dt \quad \Rightarrow \\ \langle \int_{t_1}^{t_2} d\xi_t \int_{t_1}^{t_2} d\xi_{t'} \rangle = \sigma_\xi^2 (t_2 - t_1) \end{aligned} \tag{16}$$

However, the derivatives of stochastic variables, as well as derivatives with respect to stochastic variables, are generally considered ill-defined. Instead, we explicitly define the directional derivative of a stochastic variable as well as derive the partial derivative in stochastic ML spacetime by taking advantage of the extremely small magnitude of stochastic perturbations.

Classical theories in classical spacetime must be invariant under infinitesimal translations. However, even in classical spacetime, we must now require that the infinitesimal translations also include ML uncertainty. For example, requiring invariance of the proper time functional in general relativity to Planck scale uncertainty allows ML diffusive effects to play an important role in addition to that of the metric. Together, geometry and diffusion offer a potential solution to Hubble tension [14].

In requiring an invariance of a classical theory to an infinitesimal translation with ML uncertainty, we are readily equipped to apply the available tools of stochastic calculus. In contrast, to apply QFTs in ML spacetime, we are forced to introduce new extensions to stochastic calculus. We ultimately see that ML spacetime preserves the metric, the directional derivative, and the canonical commutator.

In classical spacetime, the directional derivative of some function $\phi(x)$ in ϵ direction is defined as

$$\epsilon^{\mu}\partial_{\mu}\phi(x) = \phi(x+\epsilon) - \phi(x) \tag{17}$$

This definition applies to any function of x and thus also applies to a stochastic variable,

$$\epsilon^{\mu}\partial_{\mu}\delta\xi^{\alpha}_{x} = \delta\xi^{\alpha}_{x+\epsilon} - \delta\xi^{\alpha}_{x} \tag{18}$$

In classical spacetime, we can Taylor expand any function of x around a particular value. This is equally applicable to a function of $\tilde{x}$,

$$\begin{aligned}\phi(\tilde{x}) = \phi(x+\delta\xi_x) = \phi(x) + \delta\xi^{\alpha}_{x}\,\partial_{\alpha}\phi(x)\\ + \frac{1}{2}\delta\xi^{\alpha}_{x}\delta\xi^{\beta}_{x}\,\partial_{\alpha}\partial_{\beta}\phi(x),\end{aligned} \tag{19}$$

where we go out to second order in stochastic variables, i.e., first order in δx.

While we clearly know how to take partial derivatives of a function of x, how do we take "stochastic partial derivatives" of a function of $\tilde{x}$, $\tilde{\partial}_{\alpha} \equiv \frac{\partial}{\partial\tilde{x}^{\alpha}}$? Applying the chain rule to a partial derivative of a function in $\tilde{x}$ gives us

$$\partial_{\mu}\phi(\tilde{x}) = \tilde{\partial}_{\alpha}\phi(\tilde{x})\,\partial_{\mu}\tilde{x}^{\alpha}, \tag{20}$$

where $\tilde{\partial}$ is yet to be defined. Since

$$\partial_{\mu}\tilde{x}^{\alpha} = g^{\alpha}_{\mu} + \partial_{\mu}\delta\xi^{\alpha}_{x}, \tag{21}$$

we have

$$\begin{aligned}\partial_{\mu}\phi(\tilde{x}) &= \tilde{\partial}_{\alpha}\phi(\tilde{x})\,(g^{\alpha}_{\mu} + \partial_{\mu}\delta\xi^{\alpha}_{x})\\ &= \tilde{\partial}_{\mu}\phi(\tilde{x}) + \partial_{\mu}\delta\xi^{\alpha}_{x}\,\tilde{\partial}_{\alpha}\phi(\tilde{x}),\end{aligned} \tag{22}$$

and thus,

$$\tilde{\partial}_{\mu}\phi(\tilde{x}) = \partial_{\mu}\phi(\tilde{x}) - \partial_{\mu}\delta\xi^{\alpha}_{x}\,\tilde{\partial}_{\alpha}\phi(\tilde{x}) \tag{23}$$

To solve for $\tilde{\partial}$, we apply (23) iteratively, keeping terms to order δx,

$$\begin{aligned}\tilde{\partial}_{\mu}\phi(\tilde{x}) &= \partial_{\mu}\phi(\tilde{x}) -\\ &\partial_{\mu}\delta\xi^{\alpha}_{x}\ \{\partial_{\alpha}\phi(\tilde{x}) - \partial_{\alpha}\delta\xi^{\beta}_{x}\,\tilde{\partial}_{\beta}\phi(\tilde{x})\}\\ &= \{\partial_{\mu} - \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha} + \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha}\delta\xi^{\beta}_{x}\,\partial_{\beta}\}\ \phi(\tilde{x})\end{aligned} \tag{24}$$

We thus obtain the stochastic partial derivative (the stochastic partial derivative can be generalized to any stochastic variable as long as the perturbations remain small):

$$\tilde{\partial}_{\mu} = \partial_{\mu} - \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha} + \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha}\delta\xi^{\beta}_{x}\,\partial_{\beta} \tag{25}$$

Consider two applications.

First, the stochastic partial derivative of $\tilde{x}$ is given by

$$\begin{aligned}&\tilde{\partial}_{\mu}\tilde{x}^{\rho} =\\ &\{\partial_{\mu} - \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha} + \partial_{\mu}\delta\xi^{\alpha}_{x}\,\partial_{\alpha}\delta\xi^{\beta}_{x}\,\partial_{\beta}\}(x^{\rho} + \delta\xi^{\rho})\\ &\quad = g^{\rho}_{\mu}\end{aligned} \tag{26}$$

We see that ML spacetime preserves the metric

$$\partial_\mu x^\rho = \tilde{\partial}_\mu \tilde{x}^\rho = g^\rho_\mu \tag{27}$$

Second, consider the directional derivative in ML spacetime along $\tilde{\epsilon}$, where $\tilde{\epsilon}$ connects two points in ML spacetime, as shown in Figure 6,

$$\tilde{\epsilon}^\alpha = \epsilon^\alpha + \delta\xi^\alpha_{x+\epsilon} - \delta\xi^\alpha_x = \epsilon^\alpha + \epsilon^\beta\, \partial_\beta \delta\xi^\alpha_x \tag{28}$$

$$\Rightarrow \quad \tilde{\epsilon}^\mu \tilde{\partial}_\mu = \epsilon^\mu \partial_\mu \tag{29}$$

ML spacetime also preserves the directional derivative.

Equipped with the new tools of stochastic calculus, we can consider the algebra of ML spacetime.

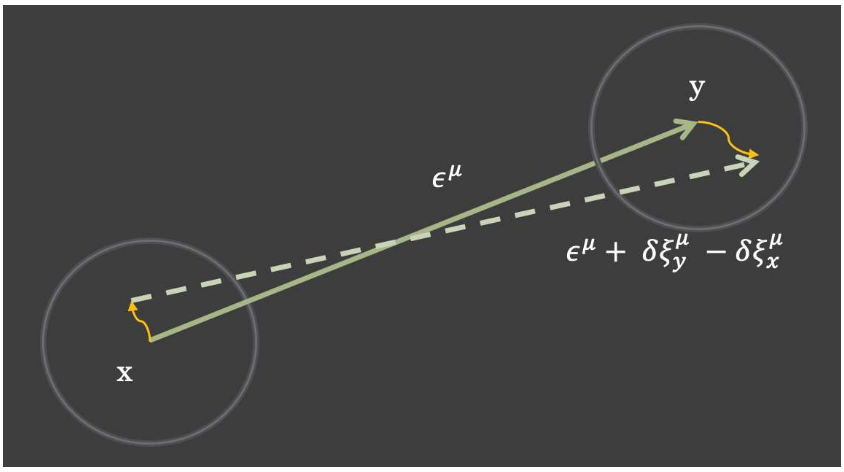


Figure 6. Classical and ML four vectors between two points, x and y.

5. ML Spacetime Algebra

We formulate the generator of rotations and boosts in ML spacetime (to order δx):

$$\begin{aligned}\tilde{J}^{\mu\nu} &= (-i)(\tilde{x}^\mu \tilde{\partial}^\nu - \tilde{x}^\nu \tilde{\partial}^\mu) \\ &= (-i)\{(x^\mu \partial^\nu - x^\nu \partial^\mu) + (\delta\xi^\mu \partial^\nu - \delta\xi^\nu \partial^\mu) \\ &- (x^\mu\, \partial^\nu \delta\xi^\alpha \;-\; x^\nu\, \partial^\mu \delta\xi^\alpha)\partial_\alpha \\ &- (\delta\xi^\mu\, \partial^\nu \delta\xi^\alpha \;-\; \delta\xi^\nu\, \partial^\mu \delta\xi^\alpha)\partial_\alpha \\ &+ (x^\mu\, \partial^\nu \delta\xi^\alpha_x\, \partial_\alpha \delta\xi^\beta_x \;-\; x^\nu\, \partial^\mu \delta\xi^\alpha_x\, \partial_\alpha \delta\xi^\beta_x)\, \partial_\beta\}\end{aligned} \tag{30}$$

Our algebraic derivations can be simplified by the choice of frame. We choose to work in a local frame by setting $x = 0$ without any loss of generality. After a bit of algebra, we find that

$$\begin{aligned}[\tilde{J}^{\mu\nu}, \tilde{J}^{\rho\sigma}] &= (-i)(g^{\nu\rho} \tilde{J}^{\mu\sigma} - g^{\nu\sigma} \tilde{J}^{\mu\rho} \\ &\quad - g^{\mu\rho} \tilde{J}^{\nu\sigma} + g^{\mu\sigma} \tilde{J}^{\nu\rho})\end{aligned} \tag{31}$$

$$[\tilde{J}^{\mu\nu}, \tilde{p}^\rho] = i(g^{\mu\rho} \tilde{p}^\nu - g^{\nu\rho} \tilde{p}^\mu) \tag{32}$$

$$[\tilde{p}^\mu, \tilde{p}^\nu] = 0, \tag{33}$$

where $\tilde{p} = i\tilde{\partial}$.

ML spacetime follows the Lie algebra of the Poincare group free of any conditions on the values for the two-point functions. It can be useful to take expectation values and express the generators in terms of two-point functions of ML spacetime. When we do so in a local frame, $x = 0$, we obtain

$$\langle\tilde{\partial}_\mu\rangle = \partial_\mu + k^{\alpha\beta}_{\sigma\mu\alpha}\delta x^\sigma \partial_\beta \tag{34}$$

$$\langle\tilde{J}^{\mu\nu}\rangle = J^{\mu\nu} + (-i)(g^{\mu\rho}a^{\nu\alpha}_{\sigma\rho} - g^{\nu\rho}a^{\mu\alpha}_{\sigma\rho})\delta x^\sigma \partial_\alpha \tag{35}$$

Finally, we find that the canonical commutator is conserved:

$$[\tilde{x}^\mu, \tilde{p}^\nu] = [x^\mu, p^\nu] \tag{36}$$

The generators of boosts and rotations in ML spacetime follow the commutation rules of Lorentz algebra. While this is encouraging, requiring relativistic invariance of ML spacetime in QFT is not enough. We must also understand how to formulate connections and covariant derivatives in ML spacetime.

6. Quantum Field Theory (QFT) in ML Spacetime

In order to consider QFT in ML spacetime, we must first construct the maps between classical and ML spacetime. These maps allow for the construction of the connection in ML spacetime, both with and without additional background fields. In the absence of any additional background fields, we show that the covariant derivative in ML spacetime equals the stochastic partial derivative in ML spacetime, $\tilde{D}_\mu \to \tilde{\partial}_\mu$, just as in the case of classical spacetime, $D_\mu \to \partial_\mu$.

The map from classical to ML spacetime, $T_{\delta\xi_x} : X \longrightarrow \tilde{X}$, is readily obtained from (19):

$$T_{\delta\xi_x}\phi(x) = \phi(\tilde{x}) \tag{37}$$

$$T_{\delta\xi_x} = 1 + \delta\xi^\alpha_x\, \partial_\alpha + \frac{1}{2}\delta\xi^\alpha_x\delta\xi^\beta_x\, \partial_\alpha\partial_\beta \tag{38}$$

However, the map from ML to classical spacetime is clearly not given by $T_{-\delta\xi_x}$ as $T_{-\delta\xi_x}T_{\delta\xi_x} \neq 1$. Instead, we can construct the map $\tilde{T}_{-\delta\xi_x} : \tilde{X} \longrightarrow X$ by requiring that $\tilde{T}_{-\delta\xi_x}T_{\delta\xi_x} = T_{\delta\xi_x}\tilde{T}_{-\delta\xi_x} = 1$:

$$\tilde{T}_{-\delta\xi_x}\phi(\tilde{x}) = \phi(x) \tag{39}$$

$$\begin{aligned}\tilde{T}_{-\delta\xi_x} &= 1 - \delta\xi^\alpha_x\, \partial_\alpha + \delta\xi^\alpha_x\, \partial_\alpha\delta\xi^\beta\, \partial_\beta + \frac{1}{2}\delta\xi^\alpha_x\delta\xi^\beta_x\, \partial_\alpha\partial_\beta \\ &= 1 - \delta\xi^\alpha_x\, \tilde{\partial}_\alpha + \frac{1}{2}\delta\xi^\alpha_x\delta\xi^\beta_x\, \tilde{\partial}_\alpha\tilde{\partial}_\beta\end{aligned} \tag{40}$$

Equipped with the maps that allow us to move easily between the two spacetimes, we are ready to formulate the covariant derivative and the connection. The connection in ML spacetime must follow the regular transformation laws ([26–28]),

$$U(\tilde{x} + \tilde{\epsilon}, \tilde{x}) = T_{\delta\xi_{x+\epsilon}}U(x+\epsilon, x)\tilde{T}_{-\delta\xi_x} \tag{41}$$

In the limit $\epsilon \longrightarrow 0$, we have

$$U(\tilde{x} + \tilde{\epsilon}, \tilde{x}) \longrightarrow T_{\delta\xi_x}U(x, x)\tilde{T}_{-\delta\xi_x} = T_{\delta\xi_x}\tilde{T}_{-\delta\xi_x} = 1, \tag{42}$$

as required. In the absence of any additional fields, $U(x+\epsilon, x) = 1$, we are left with

$$\begin{aligned}U(\tilde{x} + \tilde{\epsilon}, \tilde{x}) &= T_{\delta\xi_{x+\epsilon}}\tilde{T}_{-\delta\xi_x} \\ &= 1 + \epsilon^\alpha\, \{\partial_\alpha\delta\xi^\beta\, \partial_\beta - \partial_\alpha\delta\xi^\beta\, \partial_\beta\delta\xi^\rho\, \partial_\rho\},\end{aligned} \tag{43}$$

where we have taken advantage of the directional derivative of a stochastic variable (18) and the formulations of map operators.

The covariant derivative in ML spacetime can now be written as

$$\begin{aligned}&\epsilon^\mu\tilde{D}_\mu\phi(x + \delta\xi_x) = \\ &\quad \phi(x + \epsilon + \delta\xi_{x+\epsilon}) - U(\tilde{x} + \tilde{\epsilon}, \tilde{x})\phi(x + \delta\xi_x),\end{aligned} \tag{44}$$

and with

$$x + \epsilon + \delta\xi_{x+\epsilon} = \tilde{x} + \tilde{\epsilon} \tag{45}$$

$$\phi(\tilde{x} + \tilde{\epsilon}) = (1 + \tilde{\epsilon}^{\alpha}\tilde{\partial}_{\alpha})\phi(\tilde{x}) = (1 + \epsilon^{\alpha}\partial_{\alpha})\phi(\tilde{x}) \tag{46}$$

and (43), we obtain $\tilde{D} = \tilde{\partial}$, as expected in the absence of additional fields.

With the above, we set the stage for applications of QFTs in ML spacetime.

We next consider two applications: galaxy rotation curves in the classical regime and a massless scalar field in the quantum regime.

7. Galaxy Rotation Curves

For any classical theory in classical spacetime, the requirement of invariance under infinitesimal translations must now also include a stochastic element at ML scale. In general relativity, the one universal law, $u^2 = -1$, gives us the proper time functional. Under invariance due to infinitesimal translations—including ML scale uncertainty—it provides us with the extended Christoffel connection and a redshift defined not just by the Hubble parameter H but also its diffusive counterpart D originating from the fundamental uncertainty at ML scale [14].

In [14], we developed the RWML theory working in a spatially local frame and in the absence of any gravitational sources. In order to better understand galaxy rotation curves, we reconsider RWML theory, now in the limit of a very large radius and thus also in the weak gravity limit.

Also, in [14], we found that the redshift under RWML is a function of not only H but also D. In contrast, a massive body in the presence of a weak gravitational field has a radial velocity and angular momentum that are entirely functions of H. We find that the radial velocity is extremely small and approximately given by $\dot{R} = HR$, while its angular momentum grows slowly with time according to $\dot{L} = HL$. The presence of the diffusive effect ($D \neq 0$) makes these new dynamics possible. However, D plays no role in defining the radial velocity and the growth of the angular momentum of a massive body at the periphery of a galaxy. Furthermore, the rotation velocity V of a massive body, in the limit of very large R, becomes constant.

To see the above, we consider the RWML universe, just as in [14] but now in spherical coordinates, with the choice of a frame at $\theta = \frac{\pi}{2}$. The extended Christoffel connection terms provided in [14] (see Equations (A51–55) with $\kappa \approx 0$ in [14]) can be written in spherical terms, where we assume a weak gravity limit. From [14], we also know that diffusion in time is far smaller in magnitude than diffusion in space (in terms of the diffusion parameters defined in [14], this means that we take $\sigma_{\epsilon}^2 \approx 0$). Given these assumptions, we now have

$$\tilde{\Gamma}^t_{rr} = \frac{1}{r^2}\tilde{\Gamma}^t_{\phi\phi} = a\dot{a} + 3c_{\xi}\sigma^2_{\xi} \tag{47}$$

where a is the RW metric scale factor. To obtain the propagation velocity of a massive particle, we use the geodesic equation expressed in terms of dt rather than the proper time, as explained in Equation (9) in [29], which we repeat here:

$$\frac{d^2x^{\mu}}{dt^2} = -\tilde{\Gamma}^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} + \tilde{\Gamma}^{0}_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}\frac{dx^{\mu}}{dt} \tag{48}$$

The above Equations (47) and (48) provide us with the equation of motion for the radius R, given in [29], (see [29], Appendix A) but now expanded to also include the effects of diffusion, $D = \frac{c_{\xi}\sigma^2_{\xi}}{a^2}$:

$$\begin{aligned}\ddot{R} = &-\frac{GM}{R^2} + R\dot{\phi}^2 + H^2R + 3D(\dot{R} - HR)^3\\ &+ 3D(\dot{R} - HR)R^2\dot{\phi}^2\end{aligned} \tag{49}$$

where the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ and where we have made the assumption that $\dot{H} = \frac{\ddot{a}}{a} - H^2 \approx 0$. If we further take $H^2 \approx 0$ and retain only terms dominant in R as R gets extremely large, we obtain

$$\ddot{R} \approx 3D(\dot{R} - HR)R^2\dot{\phi}^2 \tag{50}$$

Assuming that the radial acceleration is approximately zero, the above gives us $\dot{R} = HR$ and, thus, the solution for R over time: $R = aR_0$, where R_0 represents the effective radius of the galaxy. Note that this solution also gives us $\ddot{R} = H^2R \approx 0$, thus justifying our assumption of near zero radial acceleration. Finally, without ML effects, i.e., with $D = 0$, the above term disappears, and we are back to the classical means of obtaining the radial velocity. But, $D > 0$ provides new dominant terms in R and, thus, another set of dynamics altogether.

The second equation of motion is for ϕ:

$$R\ddot{\phi} = -2(\dot{R} - HR)\dot{\phi} - HR\dot{\phi}, , \tag{51}$$

which reduces to the following upon taking advantage of the solution for R:

$$\ddot{\phi} = -H\dot{\phi} \tag{52}$$

We thus have $\dot{\phi} = \frac{\dot{\phi}_0}{a}$. Together, the two solutions give us a rotational velocity, $V = R\dot{\phi} = R_0\dot{\phi}_0$, which is constant both in time and distance to the center of the galaxy. Finally, we also find that angular momentum grows with time at a rate of the Hubble constant: $\dot{L} = 2R\dot{R}\dot{\phi} + R^2\ddot{\phi} = HL$.

We take a look at 175 disk galaxies and perform a simple analysis using SPARC [22] rotation curve data. The data sample provides the stellar mass estimates based on an analysis of the atomic hydrogen (H1) gas, the H1 radius R for each galaxy at each point of rotational velocity observation, and the corresponding velocity estimate along the flat part of the rotation curve (see Table 1 of [22] and the description on page 2). Figure 7 is a plot of all the velocity observations for each galaxy across R. It is clear that there are some galaxies where the radius is quite large. We select such galaxies and show the plot in Figure 8. As our theory suggests that the rotational velocity is constant for each galaxy far away from the center, we estimate this asymptotic velocity corresponding to each galaxy by averaging galaxy velocities for $R > 40$ kpc.

We find that a simple average of the asymptotic velocity outperforms both Newtonian and MOND regressions. Figure 9 shows a plot of velocity across the chosen galaxies in terms of the maximum radius of observation for each galaxy, which we use as a proxy for the present-day radius of the galaxy. Figure 10 shows the velocity with respect to the galaxy stellar mass, along with the average and the MOND fitted curve. We also perform a log regression of the velocity to stellar mass to find that the p-value is zero, just as in the case of Newtonian and MOND regressions. Granted, this is a very simplistic and selective analysis, but it does appear to suggest that the galaxy's asymptotic velocity, when analyzed across galaxies, is not a function of stellar mass.

Equation (49) is extremely interesting and may make us wonder about scenarios in which $-3DHV^2R$ becomes dominant, corresponding to an equation of motion for a harmonic oscillator. But, to properly treat problems in QFT, we must use appropriate covariant derivatives derived via (41) and (44) in a corresponding algebra to ensure gauge invariance. We next consider the simplest possible application in QFT, the case of a massless scalar particle.

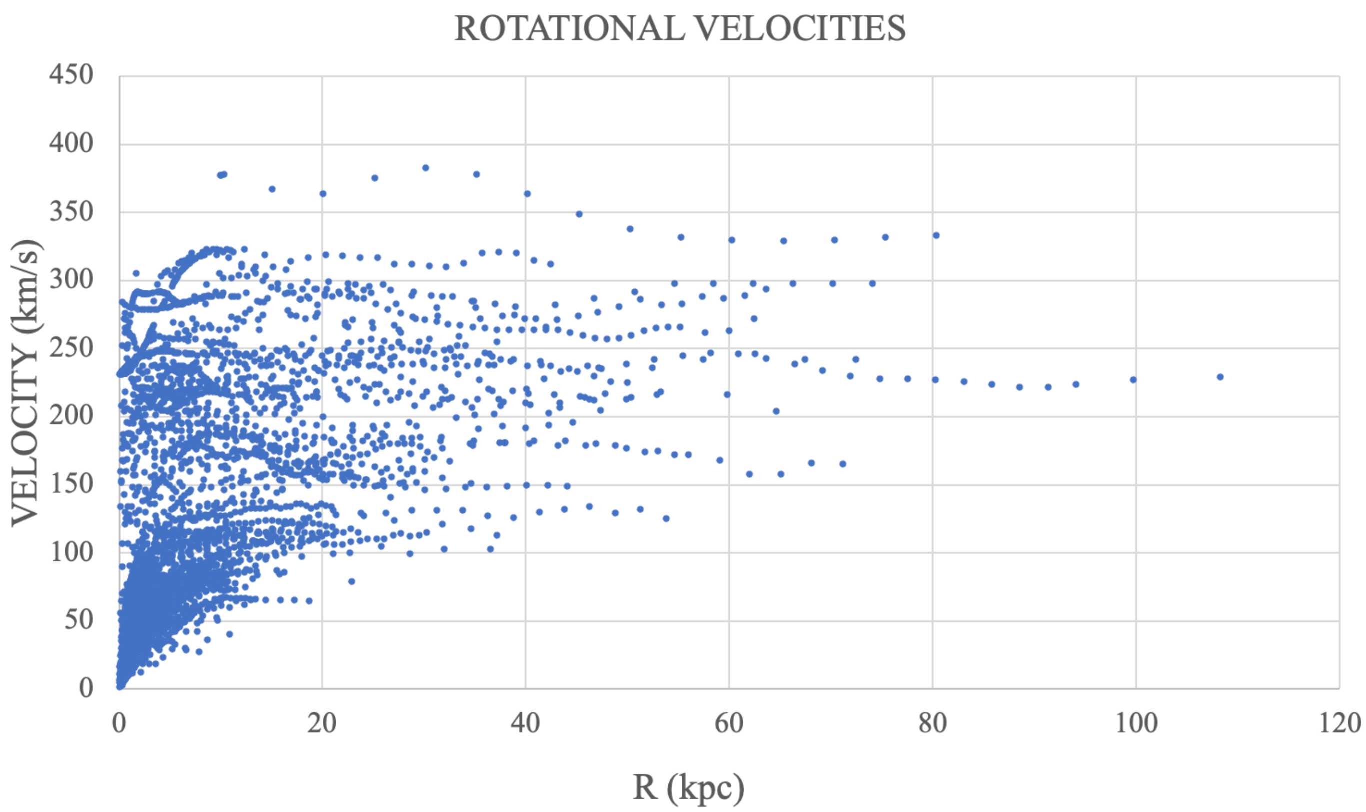


Figure 7. Rotational velocity for all 175 galaxies in the SPARC [22] database.

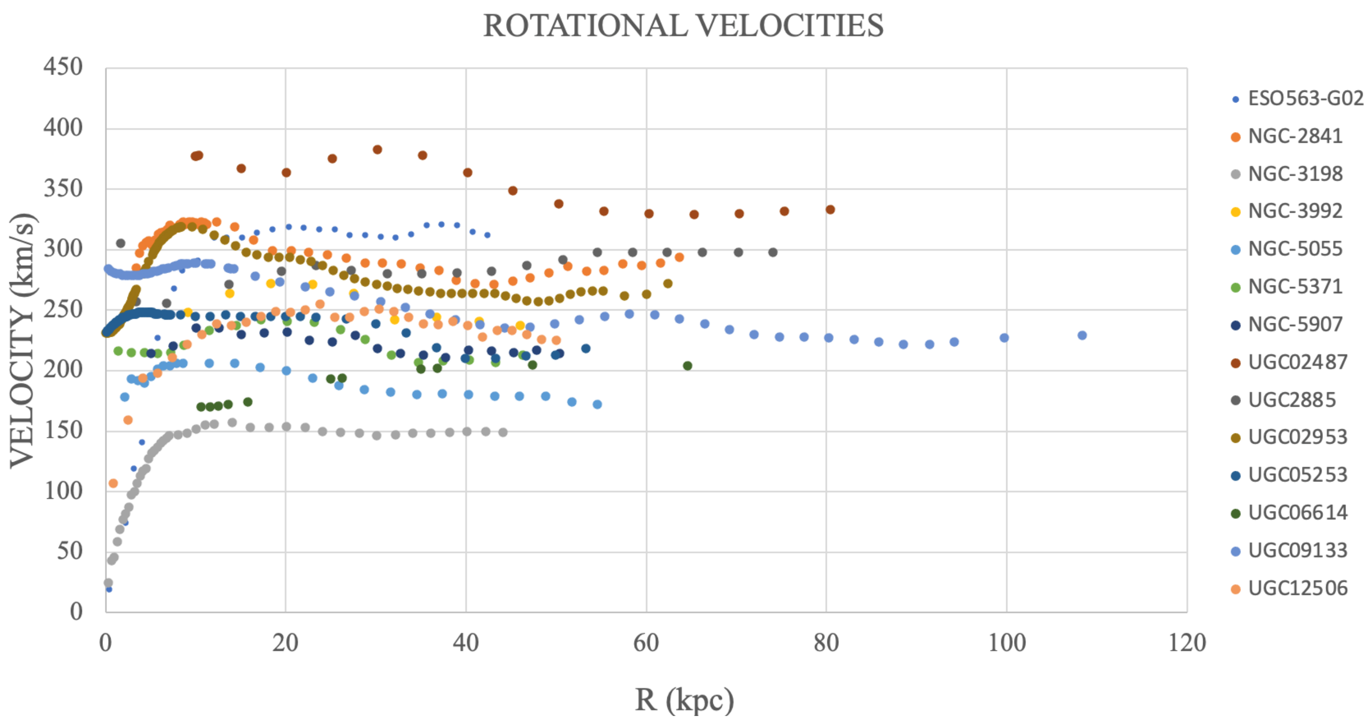


Figure 8. Rotational velocity for the selected galaxies from the SPARC [22] database.

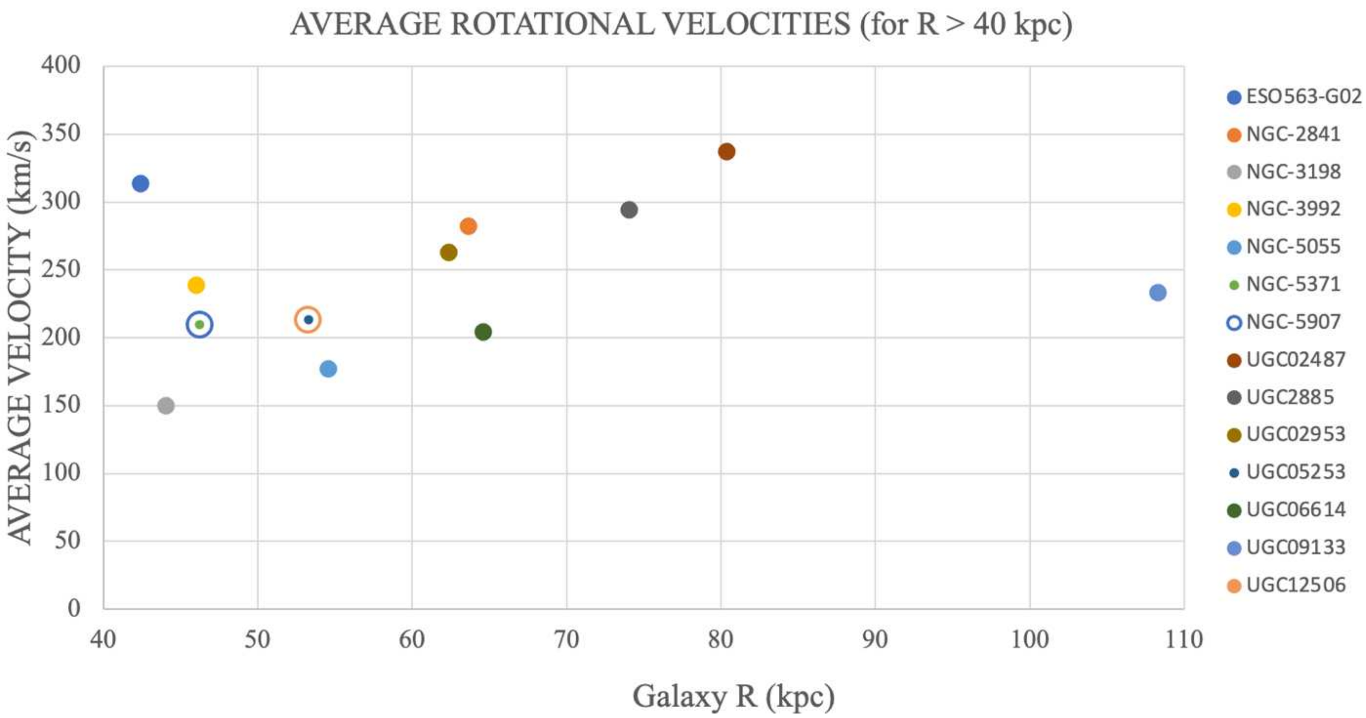


Figure 9. Average rotational velocity (R > 40 kpc) across galaxy radius (max R used as a proxy).

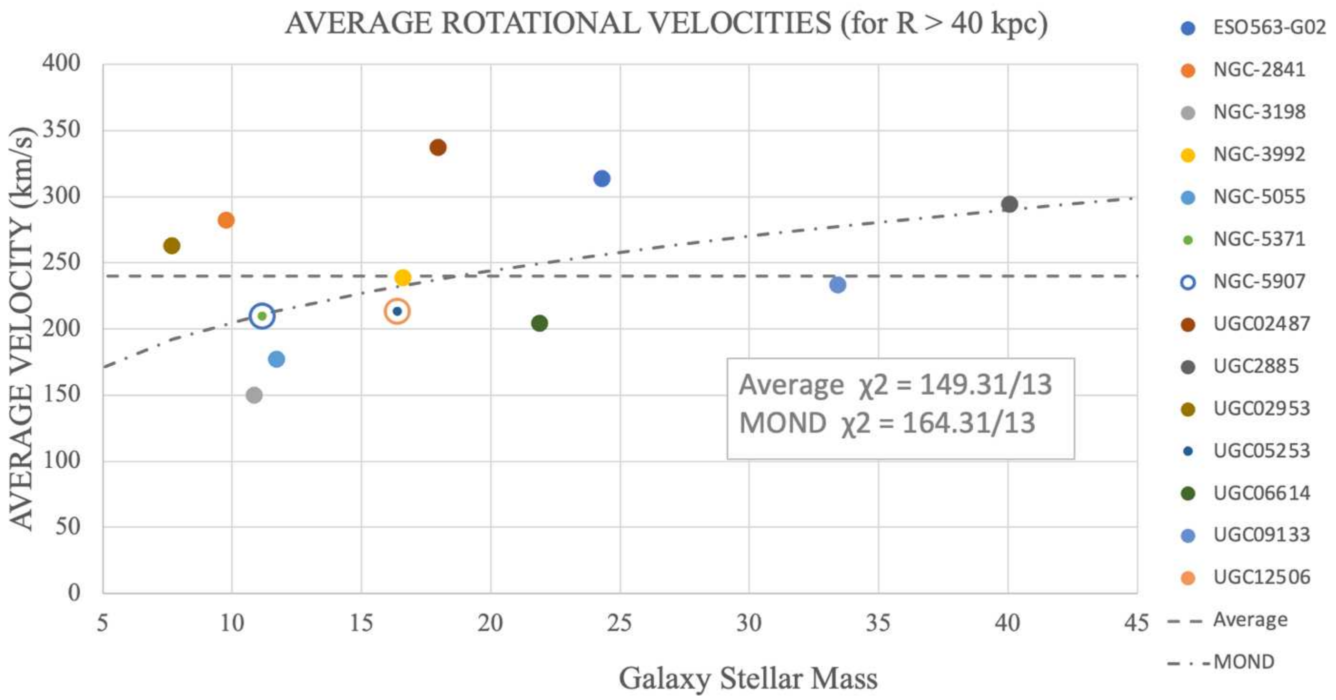


Figure 10. Average rotational velocity ($R > 40$ kpc) across galaxy stellar mass.

8. Ward-like Identity for ML Currents

The action for a free massless scalar particle in ML spacetime is given by $\mathcal{L}(\tilde{x}) = (\tilde{\partial}\phi(\tilde{x}))^2$. Requiring the uncertainty of the Planck spacetime as a symmetry of the system gives us $\langle \delta\mathcal{L} \rangle = \langle \mathcal{L}(\tilde{x}) - \mathcal{L}(x) \rangle = 0$. After a bit of algebra, we find that

$$\begin{aligned} \langle (\tilde{\partial}\phi(\tilde{x}))^2 \rangle = (\partial\phi(x))^2 + \\ a^{[\alpha\beta]}_{\rho\mu}\delta x^{\rho}\, \partial^{\mu}\phi(x)\, \partial_{\alpha}\partial_{\beta}\phi(x) \end{aligned} \tag{53}$$

The diffusive term in (53) is the ML spacetime current,

$$a^{[\alpha\beta]}_{\rho\mu}\delta x^{\rho}|_{\alpha=\beta} = \langle \delta\xi^{\alpha}\, \partial_{\mu}\delta\xi^{\beta}\rangle - \langle \partial_{\mu}\delta\xi^{\alpha}\, \delta\xi^{\beta}\rangle$$
$$\equiv i(j_{\delta\xi})_{\mu}, \tag{54}$$

which is non-zero only for $\alpha = \beta$. In momentum space, (54) gives us

$$k \cdot j_{\delta\xi} = 0, \tag{55}$$

where k is the four-momentum of the scalar field passing through the ML spacetime. Note that on-shell $\partial^2\phi$ in (53) equals zero for a zero mass field. Ward-like identity suggests that off-shell a massless scalar field decouples from ML currents.

9. Potential Critiques

Perhaps nature is entirely deterministic and there is no fundamental uncertainty, even at Planck scale? To continue with Frank Wilczek's comments in his reply to C. Alden Mead's letter *"Walking the Planck Length through History"* [10–12],

> *... refined length measurement requires large momentum, according to Heisenberg's uncertainty principle, but when the momentum becomes too large, its gravitational effect becomes strong, curving spacetime and distorting the interval one seeks to measure. Thus a fundamental difficulty arises in resolving lengths below the Planck scale.*

By treating the ML scale in the quantum regime with a stochastic approach, we may be obscuring a deterministic reality and missing out on some important relationships we have yet to understand. On the other hand, the "failure of our measuring tools" at Planck scale might hide a deeper (perhaps stochastic) reality at the smallest of scales.

10. Conclusions

Some quantum gravity theories allow for an uncertainty in the energy-momentum tensor with an emergent ML spacetime. In contrast, we focus not on gravity but instead on a stochastic spacetime. We formulate the emergence of a diffusive universe in the classical regime, which can be tested. Initial results warrant additional deeper analysis and research. In the quantum regime, we set the stage for QFT applications.

It is worth noting that the Stratonovich scheme is typically the choice for interval parametrization in physics applications. For example, path integral formulation in quantum mechanics and the QFT Wilson loop construction in the infinitesimal form [27] both use the Stratonovich approach. The choice of Ito vs. Stratonovich formalism does not impact the ideas presented in this paper but does impact the parametrization of two-point functions for classical Langevin processes. The contingency of parametrization on the choice of approach is itself an interesting topic not covered in this discussion but worth further consideration.

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Appendix A. ML Spacetime 2-Point Functions

The two-point functions in (2), (4), and (5) can be greatly simplified by separating out the uncertainty in time vs. space and applying the results of [14], which show the magnitude of uncertainty in time, σ_ϵ, as far smaller than the magnitude in spatial coordinates. With the approximation $\sigma_\epsilon^2 \approx 0$, under the Minkowski metric, we can write

$$\langle \delta\xi^0 \delta\xi^0 \rangle \approx 0, \qquad \langle \delta\xi^i \delta\xi^j \rangle = g^{ij}\sigma_\xi^2 \delta t \tag{A1}$$

We can also greatly simplify the two-point functions for the kinetic stochastic terms:

$$\langle \partial_\alpha \delta\xi^0 \partial_\beta \delta\xi^0 \rangle \approx 0, \quad \langle \partial_\alpha \delta\xi^i \partial_\beta \delta\xi^j \rangle = g^{ij} c\sigma_\xi^2 \delta t \tag{A2}$$

In dealing with spherical potentials, we must derive the corresponding spherical ML spacetime. In this case, with the choice of $\theta = \frac{\pi}{2}$ and with the above two-point function simplifications, we have

$$\tilde{r} = r + \frac{\sigma_\xi^2 \delta t}{2r} + \delta\xi^r_{r,\phi}, \quad \langle \delta\xi^r_{r,\phi} \rangle = 0, \quad \langle (\delta\xi^r_{r,\phi})^2 \rangle = \sigma_\xi^2 \delta t \tag{A3}$$

$$\tilde{\phi} = \phi + \delta\xi^\phi_{r,\phi}, \quad \langle \delta\xi^\phi_{r,\phi} \rangle = 0, \quad r^2 \langle (\delta\xi^\phi_{r,\phi})^2 \rangle = \sigma_\xi^2 \delta t, \tag{A4}$$

where

$$\delta\xi^r = cos\phi \; \delta\xi^x + sin\phi \; \delta\xi^y \tag{A5}$$

$$r\delta\xi^\phi = -sin\phi \; \delta\xi^x + cos\phi \; \delta\xi^y \tag{A6}$$

Note that, in spherical coordinates, the ML radius obtains a singularity at $r = 0$ proportional to the diffusive parameter σ_ξ^2.

Appendix B. Free Classical Particle and Coupled Harmonic Oscillator

The Lagrangian for a free classical particle,

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2, \tag{A7}$$

under the translation $x \to x + \delta x + \delta\xi$ gives us

$$\langle \delta\mathcal{L} \rangle = m(-\ddot{x}\delta x + c_\xi \sigma_\xi^2 \delta t) \tag{A8}$$

Under a Minkowski metric for a null path of a massless particle, $\delta t \sim \delta x$. If we here assume a very small mass to approximate a null path, we obtain $\ddot{x} \sim c_\xi \sigma_\xi^2$. Planck scale uncertainty gives the particle an acceleration: the diffusive effect of uncertainty has the effect of a force (note, to simulate Langevin Equations (6) and (7) under the constraint of an invariant Lagrangian, we must set $-\nabla V = 2c_\xi \sigma_\xi^2$).

We use SI units for the remainder of this section.

In the case of a coupled harmonic oscillator as a toy model for a Cooper pair, the Lagrangian under infinitesimal translations with ML uncertainty gives a new contribution to the energy of the classic coupled harmonic oscillator with frequency ω. The Lagrangian in one spatial dimension, under spacetime perturbations, acquires an additional harmonic oscillator at ML scale:

$$\langle \tilde{\mathcal{L}} \rangle = \mathcal{L} + \frac{1}{2} m \langle (\partial_t \delta\xi_t)^2 - \omega^2 (\delta\xi_t)^2 \rangle \tag{A9}$$

Energy obtains a new contribution from the ML perturbations:

$$\Delta E = \frac{1}{2} m (2 c_\xi \sigma_\xi^2 + \omega^2 \sigma_\xi^2) \delta t = \frac{m\sigma_\xi^2}{\delta t} (1 + \frac{(\omega_\xi \delta t)^2}{2}) \tag{A10}$$

If we assume $\omega\delta t << 1$, we are left only with the kinetic diffusive term. In this case, applying the equipartition theorem for a single spatial dimension, $\Delta E = \frac{1}{2} k_B T$, at the critical temperature corresponding to the point of classical-to-quantum transition, we have

$$\Delta x \sim \sqrt{\langle(\delta\xi_t)^2\rangle} = \sqrt{\sigma_\xi^2 \delta t} \tag{A11}$$

$$\Delta p \sim m\sqrt{\langle(\partial_t \delta\xi_t)^2\rangle} = m\sqrt{2c_\xi \sigma_\xi^2 \delta t} = m\sqrt{\frac{2\sigma_\xi^2}{\delta t}} \tag{A12}$$

With HUP approximately saturated, $\Delta x \Delta p = \frac{\hbar}{2}$, we obtain

$$m\sigma_\xi^2 \sim \frac{\hbar}{2\sqrt{2}} \tag{A13}$$

$$T_c \sim \frac{\sqrt{2} v\hbar}{k_B \delta l}, \tag{A14}$$

where $v = \frac{\delta l}{\delta t}$ is the excitation velocity and δl the length scale defined by the problem. For the Cooper pair experiment in [23], we have $\delta l \sim 30, \quad T_c = 328$ K, obtaining $v = 1.8 \cdot 10^5 \frac{\mathrm{m}}{\mathrm{s}}$. This rough estimate can be compared to the experimental observation of $v = 1.51 \cdot 10^5 \frac{\mathrm{m}}{\mathrm{s}}$ [23].

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