SLAC-PUB-1611
July 1975

A SEMICLASSICAL MODEL FOR THE MAGNETIC MOMENT OF AN ELECTRON*

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ABSTRACT

The magnetic moment of an electron is (semiclassically) modeled by a loop of current threaded by a flux quantum $\phi_0 = \pi \frac{\hbar c}{e}$. The integral of the charge density involved in the current is taken to equal the electronic charge. It is shown that the assumption that the moment's flux is quantized to a value ϕ_0 implies that the size of the (bare) electronic charge must be extremely small ($< 10^{-57}$ cm) but the most appropriate loop radius is on the order of the electron Compton radius $\lambda\!\!\!^{-}_e$. Using these results, a classical calculation of the static electric and magnetic energies associated with the loop is shown to be in reasonable agreement with the physically observed electron mass.

(Submitted to Am. J. Phys.)

* Work supported by the U. S. Energy Research and Development Administration.

It is often useful to construct classical or semiclassical models to assist in a conceptual understanding of various aspects of quantum mechanical systems. One such model for the magnetic moment of an elementary particle is a rotating sphere with a uniform ratio of charge to mass density. This simple model[1] gives the relationship between magnetic moment μ and angular momentum ℓ

$$\mu = \frac{e\ell}{2mc} \tag{1}$$

which is only a factor of two different from the analogous quantum mechanical Bohr magnetic moment

$$\mu_B = \frac{e\hbar}{2mc} \; . \tag{2}$$

(Recall that for the electron the projection of the spin angular momentum is equal to $\hbar/2$.) Gaussian units are used throughout this paper; e is the charge of the positron; the symbols $\hbar$, m, and c have their usual meanings.

Another semiclassical model of long standing[2] describing the spin and magnetic moment of an electron considers the electron to move with the velocity c in a circle of radius $\lambda\!\!\!^{-}_e/2$. This model gives the correct (spin) angular momentum $\hbar/2$, but gives $\mu_B/2$ for the magnetic moment, again off by (only) a factor of two. A detailed quantum mechanical analysis of a free Dirac electron has been made by Huang,[3] leading to a remarkably similar picture. In this analysis it was shown that Zitterbewegung, a phenomenon first studied by Schrödinger,[4] causes the electron to circulate in a kind of orbital motion of radius $\lambda\!\!\!^{-}_e$. This circulation is due to the components of the vacuum fluctuations of momentum less than mc, the total energy of which is essentially determined by the electron mass. Further, Huang has shown that the correct angular momentum $\hbar/2$ and magnetic moment μ_B are associated with this motion. The success of the Fermi formula[5] in describing the hyperfine splitting of atoms with zero orbital angular momentum[6] is strong experimental evidence that the magnetic moment of the electron is due to electric currents.[7]

From the above results it appears reasonable to (semiclassically) model the magnetic moment of an electron by a loop of conductor containing a current which generates the moment's flux. We shall assume that the conductor contains a charge density, the integral of which is equal to the electronic charge. The motion of this charge gives the loop current. Conceptual contact between this model and quantum mechanics may be made by viewing the loop as the result of an extended time average of the quantized circulatory motion of the electronic charge. The analysis in this paper, however, starts by considering the amount of flux associated with the moment rather than the current. Since superconductivity theory has predicted[8] and experiment has established[9] that magnetic flux is quantized (in units of ϕ_0), it is assumed here that the magnetic flux associated with the magnetic moment of the electron is similarly quantized. Of course, if the total flux of the electron is not quantized, or the quantum of flux differs significantly from ϕ_0, then this semiclassical model does not approximate the quantum mechanical system and hence loses its logical basis.

For superconducting materials the flux quantum is given by

$$\phi_0 = 2\pi \frac{\hbar c}{q} \tag{3}$$

where q is the charge of the current carrier; q = -2e and $\phi_0 \approx 2 \times 10^{-7}$ G cm^2, indicating that the superconducting current is carried by Cooper pairs[10] of electrons. This is considered a triumph in the theory of superconductivity and Eq. (3) applied to the ac Josephson effect has enabled a most precise determination[11] of the fine structure constant $\alpha = e^2/\hbar c$.

The loop of conductor of this model is depicted in Fig. 1a. A current I, due to a charge -e moving with a velocity v, is assumed to flow around the loop, creating and linking with a single flux quantum ϕ_0. There are obvious uncertainties of order two associated with this assumption, since we consider singly charged particles rather than Cooper pairs and the electron spin has a projection of $\hbar/2$ rather than $\hbar$. However, these two factors are in opposite sense. Moreover, uncertainties of this magnitude are inherent in any semi-classical model of a quantum system (in this analysis we shall consider agreement within a factor of two as satisfactory), and in any case are not large enough to materially alter the ensuing semiclassical results and their implications.

The magnetic moment of this configuration is given by[12]

$$\mu = IA/c \tag{4}$$

where A is the effective area of the loop. Classical electromagnetic calculation[13] shows the inductance L of such a loop to be

$$L = 4\pi \frac{b}{c^2} \left(\ln \frac{8b}{a} - \frac{7}{4}\right) \cong 4\pi \frac{b}{c^2} \ln\left(1.4 \frac{b}{a}\right) \tag{5}$$

where a and b are the loop radii as shown in Fig. 1b. The relative permeability has been assumed to be unity; the 4π and factors of c are added to maintain appropriate Gaussian units.[14] Using the relationship

$$\phi = LIc \tag{6}$$

to eliminate the current in favor of its associated flux, one obtains

$$\mu = \frac{A\phi_0}{Lc^2} . \tag{7}$$

Using $A = \pi b^2$ and substituting Eq. (5) into (7) yields

$$\mu = K b \phi_0 \tag{8}$$

where

$$K = \frac{1}{4 \ln (1.4 \frac{b}{a})} . \tag{9}$$

One notes that the magnetic moment of the loop is proportional to its "size" (i.e., b) while its "shape" (i.e., the ratio b/a) enters only in the argument of a logarithm, resulting in a very weak shape dependence--extremely weak for large b/a.

To show that relativistic considerations restrict the permissible range of the ratio a/b, we solve for the circulating charge velocity v required to maintain one flux quantum ϕ_0 through the loop. Noting that $I = ev/2\pi b$, one sees from Eqs. (5) and (6) that

$$v = \frac{\pi c}{2\alpha \ln (1.4\, b/a)} . \tag{10}$$

If one assumes that the speed of light is the maximum realistically allowable velocity, then Eq. (10) immediately rules out a fat doughnut shape ($a \sim b$) for the loop configuration. For $a = b$, $v = 640\, c$. Solving Eq. (10) for a/b and setting $v = c$ yields

$$\frac{a}{b} = 1.4 \exp(-\pi c/2\alpha v) = 4.6 \times 10^{-94} . \tag{11}$$

Allowing a factor two for semiclassical modeling errors (i.e., permitting $v = 2c,\ c/2$) yields

$$\frac{a}{b} = 2.5 \times 10^{-47},\ 1.5 \times 10^{-187} \tag{12}$$

respectively. It is seen from Eq. (11) that, as v is reduced yet further, the ratio a/b decreases exponentially, being infinitesimal for any $v \lesssim c$.

Eq. (8) is plotted in Fig. 2 for the above ratios of a/b; the values of v given by Eq. (10) are also indicated. To aid in orientation, the electron Compton wavelength $\lambdabar_e = 386$ F, the classical electron radius $r_e = 2.82$ F, the muon Compton wavelength $\lambdabar_\mu = 1.87$ F, the nucleon Compton wavelength $\lambdabar_N = 0.21$ F, and the nucleon radius $R_N = 0.7$ F are indicated on the abscissa, while the Bohr moment μ_B, the muon moment μ_μ, and the nuclear moment μ_N are indicated on the ordinate.

Setting $\mu = \mu_B$ and using Eqs. (8), (9), and (10) yields

$$\frac{v}{c} = \frac{\lambdabar}{b} \tag{13}$$

where $\lambdabar = \hbar/mc$. Then taking 2c as a reasonable upper limit imposed by relativity on the velocity (the factor 2 is to allow for modeling errors) will give b its smallest value, $\lambdabar_e/2$. At this limit a has its largest value, 0.5×10^{-57} cm. The analysis of this loop model gives no indication of a maximum value for b. However, quantum mechanical considerations, specifically the Heisenberg uncertainty principle and also the detailed analysis of Huang, leave little motivation for considering b to significantly exceed $\lambdabar_e$. If, motivated by quantum mechanical considerations, one assumes $b \sim \lambdabar_e$, then $v \simeq c$ and the dimension a must be infinitesimal, going like

$$b \exp(-\eta/\alpha) \tag{14}$$

where η is a number on the order of unity. Expression (14) is consistent with the premise of QED that the bare electron may be represented by a point charge.[15] To date experiments at either high energy[16] or low energy[17] reveal no violation of QED theory.

Having a feel for the dimensions of the (semiclassical) loop which are appropriate to generate the electronic magnetic moment ($b \sim \lambdabar_e$, a infinitesimal), it is now of interest to calculate the static energies associated with its electric and magnetic fields. When the flux is quantized at ϕ_0 and $\mu = \mu_B$, the magnetic energy

$$W_m = \tfrac{1}{2}LI^2 = \frac{\phi_0^2}{2Lc^2} = \frac{\phi_0^2}{8\pi\ell \ln(1.4\frac{\ell}{a})} \tag{15}$$

can, using Eqs. (2), (3), (8), and (9), be shown to be given by

$$W_m = \frac{e^2}{2\pi\ell}\left(\frac{v}{c}\right)^2 \ln\left(1.4\,\frac{\ell}{a}\right) = \frac{mc^2}{4}\left(\frac{\lambda\!\!\!^{-}}{\ell}\right)^2 . \tag{16}$$

The electrostatic energy associated with this loop is shown in Appendix A to be

$$W_e = \frac{e^2}{2\pi\ell} \ln\left(\frac{8\ell}{a}\right) . \tag{17}$$

Using Eqs. (8) and (9) to eliminate the ratio a/ℓ yields

$$W_e = \frac{mc^2}{4} + \frac{e^2}{2\pi\ell} \ln\left(\frac{8}{1.4}\right) \tag{18}$$

or approximately one-quarter of the electron mass. (The second term is $< 1\%$ of the electron mass for $\ell \sim \lambda\!\!\!^{-}_e$.)

If one were now to view the electron as a small circulating sphere of radius a, rather than spread out over the loop, the energy would, of course, diverge like a^{-1}. As an aside we remark that in this case, by taking motivation from the quantum mechanical results mentioned below, one may still retrieve the functional dependence of the energy upon a and ℓ exhibited in Eqs. (16) and (17). This can be done by taking the existence of the vacuum pairs into account (semi-classically) by means of the constitutive relations.[18]

The interesting features of Eqs. (16) and (17) are that they show that the electric and magnetic energies of the loop are comparable, can easily be made to total the electron mass for a value ℓ near $\lambda\!\!\!^{-}_e$, and diverge in the same way as a function of the parameters ℓ and a: linearly divergent in ℓ, but only logarithmically divergent in a. It is well known that the classical electron self-mass diverges linearly,[19] while the quantum mechanical self-mass was shown by Weisskopf[20] to diverge logarithmically. This reduction of the divergence from linear to logarithmic comes about through the proper consideration of vacuum polarization due to electron-positron pairs which reside in the vacuum according to Dirac's positron theory. Weisskopf also noted that, if one were to ascribe the entire electron mass to the logarithmically divergent quantum mechanical self-mass, one would obtain the quantity $\bar{\lambda}_e \exp(-1/\alpha)$ as the relevant "critical length" of the theory--a kind of quantum mechanical radius of the electron. It is evident that the dimension a is the analogous quantity for this loop model.

Referring again to Fig. 2, it is interesting to observe that any force which tends to increase ℓ will likewise tend to increase μ. Now both the self-electric and -magnetic forces of the loop can be seen to operate in this sense, implying that these forces will tend to increase the static magnetic moment (presumably after the renormalization process has, through the electron mass, determined the loop size). A semiclassical study of this model has, in fact, already been made by Koba,[21] who showed that the loop expansion due to vacuum fluctuations gives an effect larger than the reduction (due to wobbling) found by Welton,[22] yielding an overall increase in the magnetic moment equal to $\frac{\alpha}{2\pi}\mu_B$, a result first derived by Schwinger.[23] These considerations may offer at least a partial answer to the general problem pointed out by Feynman,[24] i.e., there is very little physical intuition associated with the calculations of QED - e.g., even the reason for the positive sign of the anomalous magnetic moment is not known.

This model may be applied to the muon as well as to the electron; the major difference is the reduction of the dimension b by the factor m_e/m_μ. The appropriate change in the ratio a/b is uncertain because of the insensitivity of this model to this ratio. However, because both muon and electron obey the same (Dirac) equation, appropriately modified by QED, one would expect the two ratios a/b (or at least their logarithms) to be nearly equal.[25] Due to the difficulties and uncertainties introduced by strong interactions, no attempt is made to apply this model to the nucleon.

To summarize, this semiclassical analysis of a current loop, threaded by a quantum of flux ϕ_0, and having the magnetic moment of the electron, yields loop dimensions which are compatible with quantum mechanical ideas about electron structure. That is, the loop size should be no smaller than the Compton wavelength and the circulating charge must be of extremely small dimension. Further, a calculation of the electric and magnetic energies associated with such a loop shows that they are comparable and their sum is equal to the electron mass for a loop radius on the order of the Compton wavelength. The model indicates, as does quantum mechanics, that there may be a critical length $\sim \lambda\!\!\!^{-} \exp(-\eta/\alpha)$, where $\eta \sim 1$, pertinent to charged lepton structure.

In conclusion, without an underlying quantum mechanical theory describing the structure of the electron, and giving meaning to the expression for its self-mass, the significance of these results is rather difficult to assess. However, the semiclassical loop model is simple and easy to visualize and it exhibits a mathematical behavior very similar to that of the quantum mechanical system it is to represent. Hopefully, then, through these features it can help furnish useful insights into the structure of the electron.

The author thanks S. J. Brodsky for important discussions about this model and its possible interpretations and implications.

APPENDIX A
THE ENERGY TO CHARGE A RING IN FREE SPACE

The charge Q on a conductor of (free space) capacitance C is given by

$$Q = VC \tag{A-1}$$

where V is the potential. The energy W to charge the capacitor is

$$W = \frac{Q^2}{2C} . \tag{A-2}$$

One may now use Eqs. (A-1), (A-2), and the expansion for the potential of a ring[26] to obtain the expression for the electrostatic energy,

$$W = \frac{Q^2}{2b} \sum_{n=0}^{\infty} \left(\frac{b-a}{b}\right)^n \left[P_n(0)\right]^2, \tag{A-3}$$

where the origin has been placed at the center of the ring and the point at which the potential is calculated is that part nearest the origin (i.e., $r = b - a$; cf. Fig. 1b). Using the values for the Legendre polynomial for zero argument,[27]

$$\begin{aligned} n \text{ odd:} \quad & P_n(0) = 0 \\ n \text{ even:} \quad & P_n(0) = (-1)^{\frac{n}{2}} \frac{n!}{2^n \left[\left(\frac{n}{2}\right)!\right]^2} \end{aligned} \tag{A-4}$$

and setting n = 2m yields

$$W = \frac{Q^2}{2b} \sum_{m=0}^{\infty} \left(\frac{b-a}{b}\right)^{2m} \left[\frac{(2m)!}{2^{2m}(m!)^2}\right]^2 . \tag{A-5}$$

Eq. (A-5) may be made tractable by using the Sterling formula,[28]

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \ldots\right) \tag{A-6}$$

Substituting Eq. (A-6) in (A-5) and explicitly writing the m = 0 term yields

$$W = \frac{Q^2}{2\pi b} \left\{\pi + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{y-1}{y}\right)^m \left(1 - \frac{1}{4m} + \frac{1}{32m^2} - \ldots\right)\right\} \tag{A-7}$$

where $y \equiv b/2a >> 1$. Using Peirce[29] 766 for the first expansion and Jolley[30] 1133 for the second two (for which the factor $\left(\frac{y-1}{y}\right)^m$ may be ignored) yields

$$W = \frac{Q^2}{2\pi b}\left\{\pi + \ln\frac{b}{2a} - \frac{\pi^2}{24} + \frac{1.20}{32}\right\}$$
$$\cong \frac{Q^2}{2\pi b}\ln\left(\frac{8b}{a}\right) , \qquad \text{(A-8)}$$

which is the desired result.

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$$\mu^{-1} = \epsilon = 1 \quad , \quad r > b$$

$$\mu^{-1} = \epsilon = b/r, \quad r < b,$$

it is easy to show (nonrelativistically) that

$$W_m = \frac{e^2 v^2}{3bc^2} \ln(2.72\tfrac{b}{a})$$

and

$$W_e = \frac{e^2}{2b} \ln(2.72 \frac{b}{a}) .$$

It is interesting to note that the assumption that $\mu \to 0$ (Meissner effect) and $\epsilon \to \infty$ (much like conductivity $\sigma \to \infty$) as $r \to 0$ implies that in some sense near the bare charge the vacuum resembles the superconducting state. Cf., for example, F. London, *op. cit.*, p. 14 for a good qualitative discussion of these features of superconductors. It is also relevant to note here that it has been suggested by G. Parisi, Phys. Rev. D 11, 970 (1975) that the possibility that the vacuum may behave like a superconductor could offer an explanation of the quark confinement problem.

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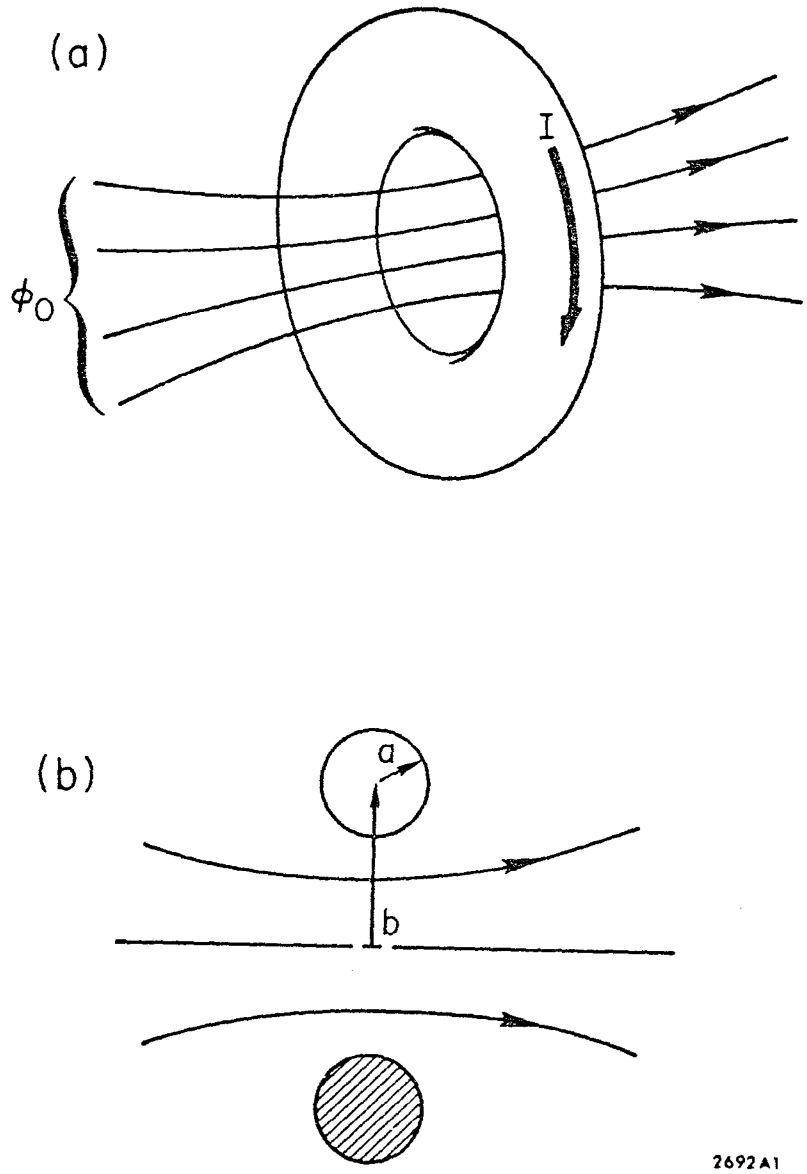
FIGURE CAPTIONS

1. Semiclassical loop model of the magnetic moment of an electron.

   (a) A current I flows around the loop linking a flux quantum ϕ_0.

   (b) The two radii of the loop are shown in a cross-sectional view.

2. A plot of magnetic moment versus the radius b of a current loop which links a flux quantum ϕ_0 for several values of the ratio a/b.



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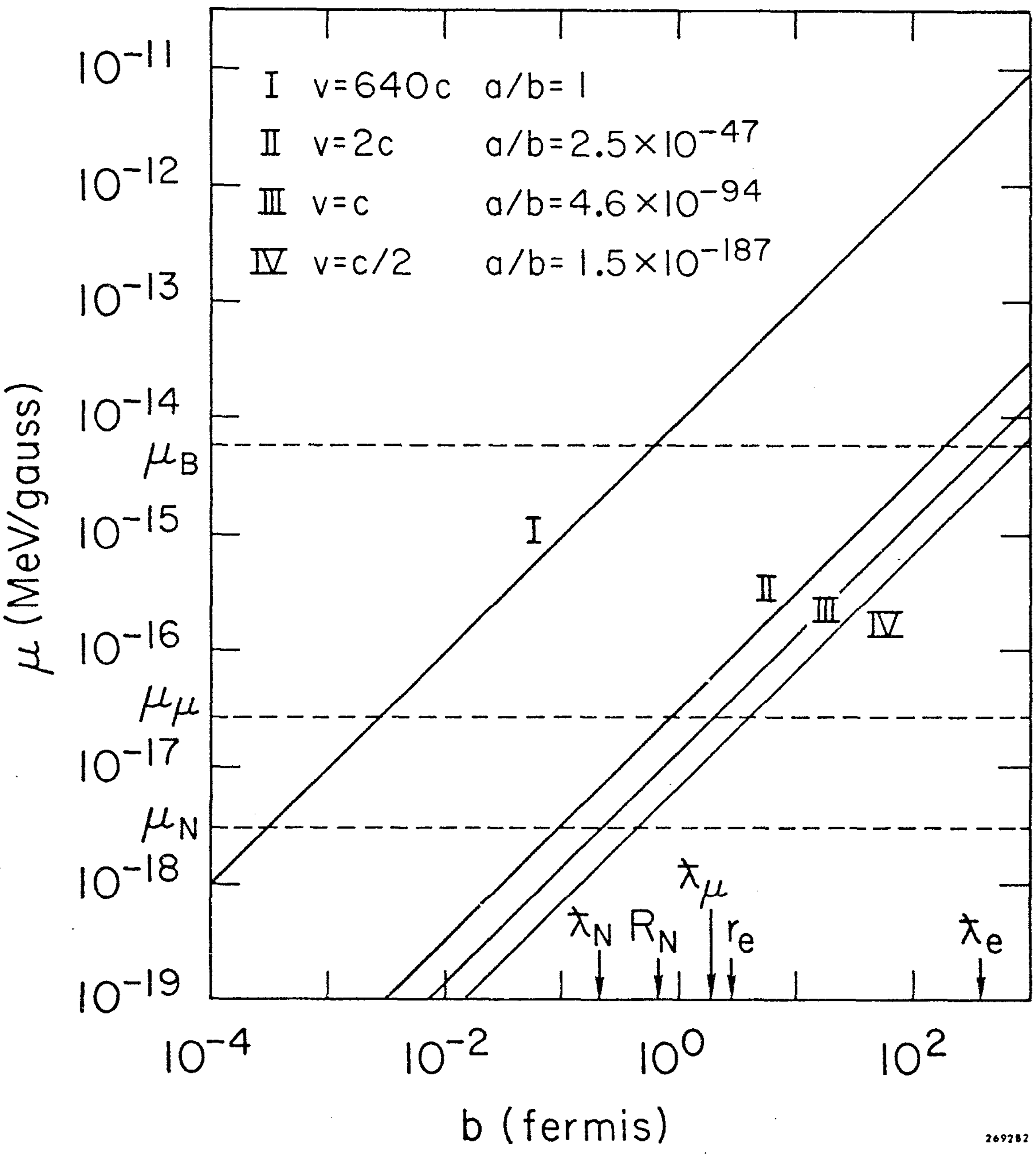

I v=640c a/b=1
II v=2c a/b=2.5×10^-47
III v=c a/b=4.6×10^-94
IV v=c/2 a/b=1.5×10^-187
μ (MeV/gauss)
b (fermis)
μB
μμ
μN
λN RN λμ re λe
10^-11
10^-12
10^-13
10^-14
10^-15
10^-16
10^-17
10^-18
10^-19
10^-4
10^-2
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I
II
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