UNIVERSITY OF SOUTHAMPTON

FACULTY OF PHYSICAL SCIENCES AND ENGINEERING

School of Physics and Astronomy

Leptons and Higgs with Discrete Flavour and Charge-Parity Symmetries

by

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ABSTRACT

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This thesis concerns itself with two seemingly disjoint topics that are interesting on their own, but that come to their full bloom when combined. These two topics are the violation of CP and the so-called flavour problem. CP violation is simply necessary for all our existence, however currently no strong enough source of it is known to ensure successful baryogenesis. The flavour problem on the other hand is a loose collection of questions concerning fermions in the standard model, especially why several flavours exist at all, and why their properties appear to be so chaotic. The overlap between the two topics happens, as in the SM CP is violated in the flavour sector. After an introduction, so-called residual flavour and CP symmetries are explored as possible explanations of the parameter structure of the lepton Yukawa sector. Such residual symmetries are embedded into larger groups at high energy and from the breaking patterns constraints on observables are derived. There it was found that an important class of subgroups of $U(3)$, namely $\Delta(6n^2)$ groups, can indeed explain the observed lepton mixing. Several variations of this approach, combining residual flavour and CP symmetries, are explored. This was the first time that such an infinite series of finite groups was analysed in this way. After this, motivated by the need for breaking of flavour and CP symmetries and the search for additional sources of CP violation, a large number of candidate scalar potentials are explored, especially for their CP properties. A necessary tool for this are CP-odd Higgs basis invariants, the theory of which was further developed to enable such analyses. Using this approach, many very complicated potentials were tested for their CP properties for the first time and new sources of CP violation were found in new and known potentials.

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Declaration of Authorship

I, Thomas Neder, declare that the thesis entitled *Leptons and Higgs with Discrete Flavour and Charge-Parity Symmetries* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
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- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
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Signed: Thomas Neder

Date: Southampton, October 7, 2016

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1

Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.
There is another theory which states that this has already happened.

– Douglas Adams, *The Restaurant at the End of the Universe*

This thesis concerns itself with two topics that seem quite unconnected at first and are indeed interesting on their own, but in addition, when looked at closely, have interesting and intricate connections. These topics are the violation of combined conjugation of charge and parity, and the so-called fermion flavour problem. The central unanswered questions behind these topics could be (slightly polemically) stated as "Why does anything exist?" and "Now that something exists, why are there unnecessary copies of it?", respectively.

Both topics will be introduced properly at a non-technical level in the next two sections. Following these sections, what was until the discovery of neutrino oscillations [7] the unbeaten[1] champion of particle physics is introduced, namely the standard model of particle physics. The conceptionally simplest explanation of neutrino oscillations is that neutrino have small but finite masses. This is in contradiction to the standard model in its original formulation, in which neutrinos were massless by construction [8, 9]. There are various extensions of the standard model that can explain these oscillations by the introduction of additional fields such that the model allows for neutrino masses. All this is discussed after a review of the remainder of the standard model, followed by sections containing technical

[1]At least concerning the 5% of the universe that are described by it.

discussions of fermion flavour and CP, and an outline of the main chapters of the thesis, which concludes the introduction.

1.1 The flavour problem

The standard model of particle physics is a relativistic quantum field theory. It is probably safe to say that the latter is still the best known unification of special relativity with quantum mechanics and at the same time the best description of elementary particles available. The fact that it is a relativistic theory means that the action is invariant under Poincare transformations whereas the fact that it is a quantum theory means that symmetry transformations act on the quantum states of the theory via unitary (or antiunitary) operators[10].

How quantum states transform under operators representing the Poincare group induces transformations on operators generating the various quantum states. These generators are then combined into fields that again form representations under the Poincare group. These are in turn are combined into a Lagrangian covariant under Poincare transformations which when integrated over spacetime yields an invariant action.

The Poincare group has two parts, spacetime translations, and boosts and rotations. Space-time translations are generated by the 4-momentum operator whose square commutes with the other operators of the Poincare group. This causes single-particle states to be characterized by their rest mass. The second part, boosts and rotation, causes the representations to be characterized by two spin quantum numbers. The reason for this is that the group formed by boosts and rotations, the Lorentz group, is locally isomorphic to $SU(2) \times SU(2)$ and each representation is labelled by the spin quantum number under each of the $SU(2)$ factors. There are two representations that correspond to (constituents of) all known fermions, $(1/2, 0)$ and $(0, 1/2)$, where this notation means that these representations are two-dimensional and transform either with a matrix from the first or the second $SU(2)$ factor, respectively. These 2-dimensional representations are often called Weyl spinors and play an important role in the construction of the standard model. Massive fermions are represented by Dirac or Majorana spinors, both of which correspond to the direct sum of two Weyl spinors, $(1/2, 0) \oplus (0, 1/2)$, with Majorana spinors fulfilling an additional reality condition. Note that this short exposition has completely ignored the gauge structure of the standard model, which can easily be considered its most successful part and the same for any quantum field theory, but is entirely irrelevant to the essence of the flavour problem. The gauge structure of the standard model will be summarized in subsection 1.3.2.

Fermions are distinguished by their quantum numbers under the various symmetries of the standard model, except for the fact that of each type, (at least) three copies seem to exist that merely differ by their Yukawa couplings and thus, at our low energies, their rest mass. These copies, one set of which is for example electron, muon, and tauon, are called flavours.[2]

The flavour problem (or puzzle) is a loose collection of unsolved questions related to the properties of fermions in the standard model among which the most fundamental ones are, Why are there even different flavours of fermions, and Why is flavour even necessary?[3] From the point of view of the author, the essence of the flavour problem can in the context of the above exposition be formulated as follows: *There is no such thing as flavour — in the Poincare group*, by which is meant that to describe several flavours, the particles of each flavour are just additional copies of the representations of the Poincare group. (As are the fermions within each generation, however, these are at least distinguished by gauge interactions.) The question now is, could there be any symmetry principle that explains why different flavours exist and why their properties and interactions are what they are? Ideally, also the seemingly chaotic structure of the flavour sector of the standard model would be explained too by this symmetry. Such a symmetry, that extends the symmetry of the model in consideration and under which the generations form a representation, is called a *flavour symmetry*.

The various no-go theorems about symmetries of the S-matrix [11, 12] are often interpreted as stating that the only symmetries that act on spacetime of a quantum field theory in flat space can be the Poincare group (or supersymmetry), and that all other symmetries have to be internal, which means that while they may depend on spacetime, spacetime itself is not transformed by these symmetries. Mathematically this means that the possible symmetries of a quantum field theory are a direct product of Poincare and internal symmetries, and that the internal symmetries act trivially on spacetime. This is for example the case for the standard model, where with $\mathcal{P}$ the Poincare group, (and the role of the remaining factors explained in the following section,)

$$G_{SM} = \mathcal{P} \times SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{1.1}$$

[2]Equivalently flavours are sometimes called families or generations.
[3]And other question concerning their precise properties and interactions.

These arguments have some loop-holes, as (among other things) the theorems only concern themselves with symmetries of the S-matrix and a larger group containing both Poincare and other symmetries might be broken spontaneously, such that at the level of the S-matrix the only surviving symmetries are indeed a direct product of the Poincare group and internal symmetries.

Nevertheless, apart from this loop-hole, any symmetries that could provide information about the properties and origin of flavour have to be internal symmetries that extend the usual standard model symmetries by relating different flavours with each other. Technically this often means that the three generations transform as some representation of a new group, G_{Flavour} that extends the symmetry of the theory:

$$G = G_{SM} \times G_{\text{Flavour}}. \tag{1.2}$$

This new group will have to be broken, because, as will be shown in section 1.4.1, the symmetries under transformations of fermion flavours that are present in the standard model are fairly small and by themselves do not contain much information about the flavour problem.

1.2 Why we all should not exist

The second topic of this thesis is a symmetry, that, if it was realized in the universe, would make it impossible to distinguish between matter and antimatter. The existence of a sufficient number of particles over antiparticles, and thus all our existence, would be impossible. This symmetry is called charge-parity conjugation, or in short CP, and its precise definition will have to wait until a later section.

The process by which matter is selected over antimatter during the evolution of the universe is generically called *baryogenesis*. There are two main quantitative observables that measure the effect of baryogenesis: the density of baryons minus the density of antibaryons over the density of photons [13]

$$n_B/n_\gamma = (6.10 \pm 0.04) \times 10^{-10} \tag{1.3}$$

and the fraction of antibaryons over baryons. Concerning the latter, the fraction of antiprotons in cosmic rays has been measured to [14]

$$n_{\bar{p}}/n_p \approx 10^{-4}. \tag{1.4}$$

Note that the surplus of baryon over antibaryons is rather small.[4] It was realized very early that four ingredients are needed for successful baryogenesis [15]: departure from thermodynamic equilibrium, baryon number violation, violation of charge conjugation, and violation of CP. The departure from thermodynamic equilibrium is supplied by the expansion of the universe, out-of-equilibrium-decays of heavy particles, first-order phase transitions or by other things, while the charge conjugation symmetry is violated in the standard model by construction. Baryon number is conserved by the renormalisable Lagrangian of the standard model but violated at the perturbative level by an anomaly, the effect of which however is almost vanishingly small, such that on one hand baryon number could safely be considered a perturbative symmetry of the standard model. On the other hand, there is a non-perturbative effect which is effected by certain non-local field configurations, called Sphalerons, that may be of considerable size. A reasonably large baryon-number violating effect may occur if the phase transition from unbroken to broken electroweak symmetry was of first order, which unfortunately is not the case in the standard model with a single Higgs doublet because the mass of the recently discovered Higgs boson is too large. And even if the phase transition was of first order, it was shown that the amount of CP violation in the standard model that is confirmed to this date is not sufficient to explain the baryon asymmetry.

A measure of the strength of CP violation at the phase transition is given by the so-called Jarlskog invariant J divided by the Higgs vacuum expectation value v squared. What both of these are will be explained later in the introduction. Nevertheless, one obtains that

$$\frac{J}{v^2} = \frac{\mathrm{Im}[V_{ij}V_{kl}V_{il}^*V_{kj}^*]}{v^2} \approx \frac{3\times 10^{-5}}{(246\ \mathrm{GeV})^2} \approx 5\times 10^{-10}\ \mathrm{GeV}^{-2} \tag{1.5}$$

and all CP-violating observables have to be proportional to this number, which will force the rates to be of a similar order of magnitude.

While it remains an interesting question what the precise origin of the violation of baryon number is, this question will not be considered further in this work, but the focus will lie on CP violation as such. An additional reason for this approach is that while CP violation is necessary for dynamically generating the observed baryon asymmetry, it might still even be the case that the universe was just started (if it was started), or just exists with a positive baryon number. CP violation as such would still be interesting because CP is violated in the standard model and a question that remains is, Was CP ever a good symmetry of the universe? Via

[4] Although there is of course no second universe to compare with.

the CPT theorem [16], CP violation is directly related to the violation of time inversion invariance, such that the question becomes, Is the direction of increasing time built into the universe, or the result of some dynamics?[5]

The last remark highlights another reason to study the violation of discrete symmetries generally. P and T are parts of the Poincare group and the question if any of them had ever been conserved is actually a question about the fundamental symmetries of nature. A similar argument can be found for C, as while it is not included in the Poincare group, it arises as a symmetry of the theory when two copies of a field with otherwise identical quantum numbers under the Poincare group are combined to a complex field.

1.3 The standard model of particle physics

In this and the following sections concepts that were merely mentioned in the previous sections will be made precise, starting with relativistic invariance, followed by gauge invariance, the breaking of the latter, the resulting masses of particles, and eventually flavour and CP symmetries of the standard model and as an extension of the standard model.

1.3.1 Relativity

The starting point is the relativistic invariance of the theory. Technically this means that the theory is to be invariant under Poincare transformations, often also called inhomogeneous Lorentz transformations. In quantum mechanics, symmetry transformations acting on states[6] have to be linear and unitary or antilinear and antiunitary operators [10]. For this reason, in the following, first, the construction of (anti-)unitary irreducible representations of the Poincare group will be outlined and which transformation properties are induced onto field operators.

The defining representation of the Poincare group acts on flat spacetime in the following way,

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1.6}$$

[5] In addition, one could wonder if the arrow of time that is generated by the violation of CP in the standard model is caused by the same dynamics as the macroscopic arrow of time, or if they are different effects.

[6] It is a postulate of quantum mechanics that any complex multiple of a state in Hilbert space will represent the same physical state. This has to be taken into account when discussing symmetries on the state space. Sets of states that only differ by arbitrary phases are called rays.

where a is a constant, real 4-vector and Λ is a constant, real 4×4 matrix that leaves the flat Minkowski metric η invariant:

$$\Lambda^T \eta \Lambda = \eta. \tag{1.7}$$

The Poincare group decays into four parts, namely those transformations that can be reached from the identity transformations by continuously changing the parameters in Λ and a, denoted by $\mathcal{P}_+^\uparrow$, and those that are only connected with the identity via mirror operations acting on time, denoted T, and acting on space, denoted P, with the matrix Λ appearing in Eq. (1.6) taking the forms of

$$\Lambda_T = \mathrm{diag}(-1, 1, 1, 1) \text{ and } \Lambda_P = \mathrm{diag}(1, -1, -1, -1). \tag{1.8}$$

P is also called parity and T time-inversion. In the following, only $\mathcal{P}_+^\uparrow$ will be considered and often just denoted by $\mathcal{P}$.

To arrive at other representations than Eq. (1.6) of the Poincare group, one can start with infinitesimal transformations,

$$x^\mu \mapsto (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu + \epsilon^\mu. \tag{1.9}$$

As ω is an antisymmetric[7] 4×4 matrix, an arbitrary infinitesimal Poincare transformation has 10 free real parameters. In any other representation $U(\Lambda)$ of the group, an infinitesimal element can only depend on these parameters, and one can expand this element in these parameters

$$U(\Lambda) = 1 + i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu + \ldots \tag{1.10}$$

If $U(\Lambda)$ is a unitary representation, then $J^{\mu\nu}$ and P^μ are hermitian operators.

From the multiplication rules of Λ and the fact that $U(\Lambda)$ is a representation, one obtains the Lie algebra of the Poincare group [17],

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \tag{1.11}$$

$$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu rho} P^\sigma - \eta^{\mu\sigma} P^\rho \tag{1.12}$$

$$[P^\mu, P^\rho] = 0. \tag{1.13}$$

Because $P^2 = P^\mu P_\mu$ commutes with all other generators, states can be classified by their quantum number under it. For what will be single particle states, this

[7]From Eq. (1.7) follows $\omega^T \eta + \eta \omega = 0$.

will correspond to the square of the rest mass of the particle. To obtain the other quantum numbers of states first define generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \text{ and } K^i = J^{0i} \tag{1.14}$$

which can be combined to [18]

$$\mathbf{J}_\pm = \frac{1}{2}(\mathbf{L} \pm i\mathbf{K}) \tag{1.15}$$

One can show now that $\mathbf{J}_\pm$ fulfil separate spin algebras $SU(2)_\pm$ and commute with each other. Up to complications when considering states with $p^2 = 0$, this shows that single-particle states are given by

$$\left|m^2, \vec{p}, j_1, j_2\right\rangle \tag{1.16}$$

where j_1 and j_2 are the spins under $SU(2)_\pm$. A scalar has simply $j_1 = j_2 = 0$, while a left-handed Weyl fermion has $(1/2, 0)$ and a right-handed Weyl fermion has $(0, 1/2)$. Two Weyl fermions can be combined to a Dirac fermion, $(1/2, 0) \oplus (0, 1/2)$.[8] The states transform with unitary operators acting on them,

$$\left|m^2, \vec{p}, j_1, j_2\right\rangle \mapsto U(\Lambda, a)\left|m^2, \vec{p}, j_1, j_2\right\rangle \tag{1.17}$$

and this induces transformations on the operators generating the states. These generators are combined into fields and at the end of the day one obtains the following transformation properties of scalars

$$\phi(x) \mapsto \phi(\Lambda^{-1}x), \tag{1.18}$$

Dirac fermions,

$$\psi(x) \mapsto \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \tag{1.19}$$

where $\Lambda_{\frac{1}{2}}$ fulfils

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu \tag{1.20}$$

Gamma matrix calculation rules can be found in most textbooks, but [19] and [20] are particularly complete.

[8] Because T and P act as automorphisms on $\mathcal{P}_+^\uparrow$, irreps of the full Poincare group are direct sums of the irreps of $\mathcal{P}_+^\uparrow$. E.g. as $P(1/2, 0) \propto (0, 1/2)$, Weyl fermions would be forced to combine to a Dirac fermion.

The fundamental fermionic irreducible representations are Weyl spinors, often denoted as ψ_L or ψ_R. This notation is safe in the chiral basis of gamma matrices. Dirac spinors are the direct sum of Weyl spinors,

$$\psi_{\text{Dirac}} = \psi_D = \psi_L + \psi_R. \tag{1.21}$$

Vice versa, Weyl spinors can be extracted from a Dirac spinor using projectors $\gamma_{L/R} = (1 \pm \gamma_5)$,

$$\psi_{L/R} = \gamma_{L/R}\psi_D \tag{1.22}$$

and Majorana spinors fulfil an additional reality condition, ,

$$\psi^c = e^{i\zeta}\psi \tag{1.23}$$

with some real phase ζ; additionally, in above equation, the charge conjugated spinor was used which is defined via

$$\psi^c = C\overline{\psi}^T \tag{1.24}$$

with the charge conjugation operator or matrix, which is defined via

$$\gamma_\mu C = -C\gamma_\mu^T. \tag{1.25}$$

For now it is sufficient to have this definition of charge conjugation to be able to mention what Majorana spinors are. Charge conjugation will be discussed in greater detail in section 1.5.

1.3.2 The gauge structure of the standard model

The standard model gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ factor accounts for the strong nuclear interaction, while the $SU(2)_L \times U(1)$ factor is responsible for the electroweak interactions. The $SU(2)_L$ factor is sometimes called weak isospin, and the $U(1)$ factor (weak) hypercharge. First, the Lagrangian will be stated and after that the properties of the various components under symmetry transformations will be discussed.

The three ingredients in terms of physical fields are gauge fields for the three types of interactions, fermions, and the Higgs fields. Consequently, the Lagrangian consists of kinetic terms and self-interactions of gauge fields $\mathcal{L}_{gauge}$, kinetic terms, which include the gauge interactions, of fermions $\mathcal{L}_{fermion}$, kinetic terms (again with built-in gauge interactions) of scalar fields $\mathcal{L}_{\text{Higgs}}$, self-interactions of the Higgs fields $\mathcal{L}_V$, and of fermions with the Higgs field, $\mathcal{L}_{ffH}$ (these are often called Yukawa interactions),

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_V + \mathcal{L}_{ffH}. \tag{1.26}$$

In the first three parts, the elegance of gauge, or local, invariance is expressed in its fullest. For a single gauge group G in some representation with generators T^a_G, the covariant derivative is given by

$$D_\mu = \partial_\mu - igA^a_\mu T^a_G. \tag{1.27}$$

Define the field strength $F^a_{\mu\nu}$ tensor via

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T^a_G \tag{1.28}$$

which with the structure constant[9] f^{abc}_G expands to

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}_G A^b_\mu A^c_\nu. \tag{1.30}$$

From this field strength tensor, a gauge invariant and Lorentz covariant quantity can be built, namely $F^a_{\mu\nu}F^{a\mu\nu}$. This quantity contains kinetic terms and self-interactions of the gauge fields. In the standard model, the gauge group is a direct product of three groups and the Lagrangian that accounts for the corresponding gauge fields is a sum of three versions of the aforementioned invariant, one for each multiplet of gauge fields. The number of gauge fields for each group equals the number of generators, and apart from that the only difference lies in the structure constant of the groups. With that the gauge Lagrangian becomes, with $G^c_{\mu\nu}$ the field strength of 8 $SU(3)$ gauge bosons, called gluons, $W^a_{\mu\nu}$ 3 $SU(2)$ gauge bosons, and 1 $U(1)$ gauge boson $B_{\mu\nu}$,

$$\mathcal{L}_{gauge} = -\frac{1}{4}G^c_{\mu\nu}G^{c\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1.31}$$

The fields strength tensors appearing in the above Lagrangian are all of the form as in Eq. (1.30). Note that gauge invariance of the theory does not allow for mass

[9] The structure constant specifies the algebraic relations between group generators,

$$[T^a, T^b] = if^{abc}T^c. \tag{1.29}$$

Inversely, if the group generators are known, the structure constants can be obtained using this relation.

terms of the gauge bosons.

Gluons are mentioned in this section for completeness only and will not play a role in the remainder of this thesis. The kinetic terms and gauge interactions of fermions are all given through the covariant derivative, for a gauge group with $G = \prod G_i$ and the standard model respectively

$$D_\mu = \partial_\mu + \sum_{G_i} i g_i A^a_\mu T^a_{G_i} \tag{1.32}$$

$$= \partial_\mu + i g_{strong} G^c_\mu T^c_{SU(3)} + i g W^a_\mu T^a_{SU(2)} + i g' \frac{Y}{2} B_\mu \tag{1.33}$$

where A^a_μ denotes a generic gauge field, associated with a generator $T^a_{G_i}$ of the group G_i, and for the standard model, G^c_μ are the 8 gluon fields that come with the 8 Gell-Mann matrices $T^c_{SU(3)}$, W^a_μ are the $SU(2)$ fields, associated with the $SU(2)$ generators $T^a_{SU(2)} =: T^a$, and $U(1)_Y$ gauge field B_μ with the generator $Y/2$. The values of the generators of the three subgroups for the various kinds of fermion fields in the standard model are summarised in Table 1.1. The kinetic terms and gauge interactions of all standard model fermions can now be written using this covariant derivative,

$$\mathcal{L}_{fermion} = \sum_{flavours} \sum_{\psi = Q_{iL}, u_{iR}, d_{iR}, E_{iL}, l_{iR}} i \bar{\psi} \gamma^\mu D_\mu \psi + h.c. \tag{1.34}$$

The fermion fields of the standard model are listed in Table 1.1, together with the values of the generators for each type of fermion.

The kinetic terms and gauge interactions of the Higgs fields are given by

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) \tag{1.35}$$

with the values of the different generators given in Table 1.2 where for later use also the charge-conjugate of the Higgs doublet is given.

There are further parts in the Lagrangian, which arise during the quantization procedure but are nevertheless an essential part of the model, namely terms for the so-called ghost fields, and gauge fix terms. They will not play a role in this thesis and information about them can be found in [18, 17].

Name	Notation	$T^c_{SU(3)}$	$T^a_{SU(2)}$	Y
left-handed quarks	$Q_{iL} = \begin{pmatrix} u_{iL} \\ d_{iL} \end{pmatrix}$	λ^c	$\frac{\tau^a}{2}$	$1/3$
right-handed up-quarks	u_{iR}	λ^c	0	$4/3$
right-handed down-quarks	d_{iR}	λ^c	0	$-2/3$
left-handed leptons	$E_{iL} = \begin{pmatrix} \nu_{iL} \\ l_{iL} \end{pmatrix}$	0	$\frac{\tau^a}{2}$	-1
right-handed charged leptons	l_{iR}	0	0	-2

Table 1.1: The fermions in the standard model of particle physics together with their transformation properties under the factors of the symmetry group. Here, the lower index $i = 1, 2, 3$ indicates flavour, and again one can see that the different flavours are just copies of each other, at least in terms of their gauge properties. λ^c denotes the Gell-Mann matrices, which will not be needed again in this thesis are just mentioned for completeness. Their values can be found e.g. in [18]. τ^a are Pauli matrices.

Name	Notation	$T^c_{SU(3)}$	$T^a_{SU(2)}$	Y
Higgs Doublet	$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	0	$\frac{\tau^a}{2}$	1
charge-conjugated Higgs Doublet	$\tilde{\phi} = i\tau_2\phi^* = \begin{pmatrix} \phi^{0*} \\ \phi^- \end{pmatrix}$	0	$\frac{\tau^a}{2}$	-1

Table 1.2: The Higgs doublet of the standard model of particle physics together with its transformation properties under the factors of the symmetry group. τ_2 is the second Pauli matrix.

1.3.3 Symmetry breaking

The self-interactions of the Higgs doublet are

$$\mathcal{L}_V = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2. \tag{1.36}$$

When the parameters in the above part of the Lagrangian fulfil $\mu^2 < 0$ and $\lambda > 0$, then classically the value of ϕ which minimizes the potential energy is not $\phi = (0, 0)^T$ as this would be unstable and would (now quantum-mechanically) decay into a configuration that is more stable. The minimum of the potential Eq. 1.36 lies at $\phi^\dagger\phi = -\mu^2/(2\lambda)$ (μ^2 was negative). The basis of the Higgs field can be chosen such that the vacuum expectation value always lies in the lower component of the doublet,

$$\langle 0|\,\phi\,|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{1.37}$$

with $v = \sqrt{-\mu^2/\lambda}$. Note that only in this basis the notation assigned to the components of the fermion doublet in Table 1.1 makes sense, as in any other basis they would be mixed accordingly. Before the phase transition, the components of the fermion doublets are indistinguishable anyway. It was in a way already assumed that the Higgs vev would appear in the lower component of ϕ.

Note that the vacuum alignment in Eq. (1.37) does not break $U(1)_Q$. However, as always some $U(1)$ subgroup of $SU(2)_L \times U(1)$ would remain unbroken, this subgroup would define what electric charge is and could always be chosen physically. Only when additional Higgs doublets are considered, one has to make sure that all their VEVs point in the same direction such that they leave the same $U(1)$ subgroup invariant. In this case it is simply practical to chose all VEVs to lie in the lower components of their Higgs doublets.

1.3.4 Gauge boson masses

As discussed, after EWSB[10], the Higgs field acquires a vacuum expectation value and one can expand the Higgs doublet around its vacuum expectation value,

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(v + h^0 + i\phi^0) \end{pmatrix} \tag{1.38}$$

The field components ϕ^+ and ϕ^0 will become the longitudinal components of charged and neutral massive gauge bosons. Their appearance in the above expansion is gauge dependent and the gauge in which they disappear and only h^0 remains is called the unitary gauge. If one inserts the expansion of the field around its vev into $\mathcal{L}_V$, one obtains a mass term for h^0 with mass

$$M_{h^0}^2 = -2\mu^2. \tag{1.39}$$

[10]Normally, a phase transition requires first of all a macroscopic system that can undergo phase changes when macroscopic observables change. The Lagrangian of the standard model as formulated in Eq. (1.26) only considers the microscopic degrees of freedom at zero temperature.

Note that the factor of 2 in above equation arises not from $\mathcal{L}_V$, but from the fact that the normalisation of h^0 when expanding ϕ in $\mathcal{L}_{Higgs}$, Eq. (1.35), produces a factor of 1/2 multiplying $(\partial_\mu h^0)(\partial^\mu h^0)$.

Next, via the Higgs-gauge interactions, $\mathcal{L}_{Higgs}$, the vacuum expectation value of the Higgs fields leads to the following mass terms for gauge bosons,

$$g^2 W^a_\mu W^{a\mu} + g g' B_\mu W^{3\mu} + \frac{(g')^2}{4} B_\mu B^\mu \tag{1.40}$$

Diagonalising these mass terms in W^a and B leads to the physical fields

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm W^2_\mu) \text{ and} \tag{1.41}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & s_W \\ -s_W & c_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{1.42}$$

with

$$c_W = \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \quad s_W = \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{1.43}$$

and θ_W the weak mixing angle.[11] The masses of the physical physical gauge bosons then become

$$M_W = \frac{gv}{2}, \; M_Z = \frac{v}{2}\sqrt{g^2 + g'^2}, \; M_A = 0. \tag{1.44}$$

With these, the weak mixing angle can be expressed as

$$c_W = \frac{M_W}{M_Z} \tag{1.45}$$

And the elementary electric charge becomes

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}}. \tag{1.46}$$

With the Higgs field expanded in this way, the Lagrangian is now only invariant under a subgroup of the original gauge group, namely $SU(3)_C \times U(1)_{em}$, where the generator of the remaining $U(1)$ group corresponds to electric charge and can be written as

$$Q = T^3_{SU(2)} + \frac{Y}{2}. \tag{1.47}$$

[11]Sometimes θ_W is wrongly called the Weinberg angle, despite the fact that it was first introduced by Glashow [8].

The weak hypercharge of the various particles was chosen exactly such that their electromagnetic charges are reproduced via above charge operator. The electroweak part of the covariant derivative becomes

$$D_\mu = \partial_\mu - \frac{1}{\sqrt{2}}(W_\mu^+(T^1+T^2) + W_\mu^-(T^1-T^2)) - \frac{ig}{\sqrt{2}c_W}Z_\mu(T^3 - s_W^2 Q) - iA_\mu eQ. \tag{1.48}$$

1.3.5 Charged fermion masses

The weak interaction within the standard model is chiral, which means that left- and right-handed fields transform differently under gauge interactions. This is only possible if the fermions are Weyl spinors before EWSB. The Weyl nature of fermions before EWSB taken together with their quantum numbers forbids mass terms for all fermions in the standard model. Only after symmetry breaking, they can acquire mass terms and after diagonalising their mass terms become of definite Dirac or Majorana nature[12], as will be discussed a little later. In the renormalisable standard model, no mass term is allowed for neutrinos, as they have no partner to couple to the Higgs boson with and a Majorana mass term is forbidden because of their hypercharge. This means that before EWSB, the only parameter whose value has to be given in terms of a physical unit of energy, e.g. GeV, is the Higgs mass, m_H. Experimentally it is known that at least without an extended gauge or scalar sector, only 3 fermion flavours can exist [21].

The renormalisable interactions of the Higgs boson with fermions are

$$\mathcal{L}_{ffH} = -(\overline{Q_L}\phi Y^d d_R + \overline{Q_L}\tilde{\phi} Y^u u_R + \overline{E_L}\phi Y^l l_R + h.c.) \tag{1.49}$$

with the charge conjugated Higgs field, $\tilde{\phi} = i\tau_2\phi^* = (\phi^{0*}, \phi^-)^T$. Expanding ϕ around its vev, they lead to the following mass terms of the standard model fermions,

$$\mathcal{L}_{ffH} \to -(\overline{d_L}M^d d_R + \overline{u_L}M^u u_R + \overline{l_L}M^l l_R) \tag{1.50}$$

with

$$M^d = Y^d v,\ M^u = Y^u v,\ M^l = Y^l v. \tag{1.51}$$

[12]This is the normal explanation, however, the breaking is not really dynamically performed, but the Higgs parameters are already chosen such that fermions are massive. As this also means that external fermions are massive, the fields that generate them transform under a massive irrep of the Poincare group.

At this point, neutrinos are still forced to remain massless. Neutrino masses will be discussed in the next subsection. The M^ψ, $(\psi = d, u, l)$, are 3×3 matrices and can be diagonalised via biunitary transformations in flavour space,

$$d_{L/R} \mapsto U_{d_{L/R}} d_{L/R},\ u_{L/R} \mapsto U_{u_{L/R}} u_{L/R},\ d_{l/R} \mapsto U_{l_{L/R}} l_{L/R} \tag{1.52}$$

and the only other place where these matrices show up in the standard model as discussed so far is in the interactions of physical W boson with quarks,

$$\overline{u_L} d_L W \mapsto \overline{u_L} U_u^\dagger U_d d_L W \tag{1.53}$$

where one can drop the index indicating whether the field is left or right-handed here because right-handed fields do not take part in gauge interactions. The matrix appearing in the interaction of left-handed quarks and W boson,

$$U_{\text{CKM}} := U_u^\dagger U_d \tag{1.54}$$

is called the CKM matrix after [22, 23] and is often also simply called the quark mixing matrix. Because unitary matrices have fewer real degrees of freedom than they have real entries (i.e. double the number of complex entries), in practice, parametrizations of the CKM matrix are used that only depend on the relevant degrees of freedom. The mostly commonly used such parametrization is the one prescribed by the Particle Data Group (PDG). The quark mixing matrix will not appear in the remainder, but only the equivalent matrices in the neutrino sector and this parametrization will discussed in the next subsection together with the complications that can arise from the fact that Neutrinos may be Majorana fermions.

Finally, looking at Eq. 1.50, as $(\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$, one can see that the Weyl fermions $\psi_{L/R}$ have combined to Dirac fermions $\psi_L + \psi_R$. [13]

1.3.6 Neutrino masses

One can see now that in the standard model left-handed neutrinos, ν_L, have no mass term, essentially because of their hypercharge. However, when loosening

[13] This is not completely correct to say, because the external states of the SM after EWSB are massive states and Weyl fields cannot act on these. The fields in the effective theory below EWSB are massive irreps and chiral fields are projections of massive fields which are now different from Weyl fields.

the restriction of renormalisability, there is only one unique operator with mass dimension 5, namely the so-called Weinberg operator,

$$\mathcal{L}_{d=5} = -\overline{E_L}\tilde{\phi}Y^M\tilde{\phi}^T E_L^c + h.c. \tag{1.55}$$

where with $E_L^c = C\overline{E_L}^T$ the charge-conjugated lepton doublet has been introduced. Generally, Lorentz-invariant quantities can be constructed not only using γ_0, which is the Clebsch-Gordan coefficient for constructing a Lorentz-singlet out of fermion and hermitian conjugate, but with the charge conjugation operator, singlets can be constructed from fermion and transposed fermion field. Charge conjugation will be discussed in detail in a later section, for now it is sufficient to know that after EWSB this operator results among other things in a so-called Majorana mass term for left-handed neutrinos,

$$\mathcal{L}_{d=5} = \overline{\nu_L}v^{*2}Y^M\nu_L^c + h.c. \tag{1.56}$$

Note that this operator has no UV completion within the standard model without any other particles added onto it. The various see-saw models that are being discussed in the literature are essentially about obtaining this operator via processes in renormalisable models. The Majorana mass matrix $M^M := v^{*2}Y^M$ in the mass term in Eq. (1.56) is symmetric by construction and can thus be made diagonal and real by a single unitary transformation of left-handed neutrinos in flavour space,

$$\nu_L \mapsto U_{\nu_L}\nu_L. \tag{1.57}$$

Similar to quarks, this matrix only shows up in the interaction term of W boson, neutrinos and charged leptons,

$$\overline{\nu_L}l_LW \mapsto \overline{\nu_L}U_\nu^\dagger U_l l_L W \tag{1.58}$$

with the matrix $U_l = U_{l_L}$ from Eq. (1.52). The matrix $U_{PMNS} = U_\nu^\dagger U_l$ is called the PMNS matrix after their discoverers, or simply the lepton mixing matrix. Note that while the matrix U_ν was introduced here for a Majorana mass term, Eqs. (1.57) and (1.58) will be identical for Dirac neutrinos, as well as the definition of the PMNS matrix. As mentioned before for the CKM matrix, the PMNS matrix, being unitary, has fewer real degrees of freedom than it has real entries and parametrizations in terms of mixing angles and relevant complex phases have been introduced. The so-called PDG (Particle Data Group) parametrization has 4 degrees of freedom for Dirac fermions and 6 for Majorana fermions, out of which

3 are mixing angles, and the remainder complex phases:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} K \quad (1.59)$$

where $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$. K is just the identity matrix for Dirac fermions and $K = (1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$ for Majorana fermions. The phase δ is called CP or Dirac phase, and the phases α_{21}, α_{31} Majorana phases. This is the parametrization that will be used throughout this thesis.

From the existence of a Majorana mass term $\overline{\nu} m \nu + h.c.$ for some field ν follows that those fields indeed have to be Majorana fermions, as for real mass m the mass eigenstates become $\nu + \nu^c$.

The maybe most minimal extension of the standard model is the addition of a number n_{ν_R} of gauge singlets to the fermion content of the model.[14] These fermionic gauge singlets are often called right-handed neutrinos, ν_R, as they have the right quantum numbers to appear in Yukawa couplings, analogously to right-handed up-quarks,

$$\overline{E_L}\tilde{\phi} Y^\nu \nu_R + h.c.. \quad (1.60)$$

After EWSB this coupling results in a Dirac mass term for neutrinos,

$$\overline{\nu_L} v^* Y^\nu \nu_R + h.c. \quad (1.61)$$

Such a mass term, if appearing on its own could, analogously to quarks, be made diagonal and real by a biunitary transformation, which would result in a mixing matrix in the $Wl\nu$ interaction of the same form as in Eq. (1.58). In addition to this, right-handed neutrinos, as they are gauge-singlets can have a Majorana mass term by themselves, before, or rather, completely independently of EWSB,

$$\mathcal{L}_{M^R} = \overline{\nu_R} M^R \nu_R^c + h.c. \quad (1.62)$$

This mass term gives rise to the so-called see-saw mechanism of type 1 . Before explaining what is meant by that, note that even with the addition of gauge singlets, left-handed neutrinos can still not have a Majorana mass term in a renormalisable model. Now, the mass matrix M^R of right-handed neutrinos can be chosen

[14] Such singlets are in particular predicted by unified theories, where the standard model gauge group is embedded into a larger gauge group, e.g. $SO(10)$. They are also predicted in left-right-symmetric models.

diagonal and real. Then for each right-handed neutrino that is heavy enough such that it is sufficient to take its effect on theory into account by adding effective operators to the Lagrangian that arise when integrating this heavy neutrino out, a contribution to a Majorana mass term arises,

$$\mathcal{L}_{see-saw} = \sum_{heavy\ \nu} \overline{\nu_{iL}}(Y^{\nu})_{ij}\frac{1}{m_{\nu_{jR}}}(Y^{\nu T})_{jk}\nu^{c}_{jL} + h.c. \tag{1.63}$$

where the generated Majorana mass matrix of left-handed neutrinos has the form

$$M^{M} = \sum_{j}(Y^{\nu})_{ij}\frac{1}{m_{\nu_{jR}}}(Y^{\nu T})_{jk} \tag{1.64}$$

where the index j runs over all integrated-out neutrinos. In the following M^M will be used to denote a generic left-handed Majorana mass matrix, not necessarily generated by this see-saw mechanism. For this expansion to be valid, the mass of right-handed neutrinos that are integrated out must be much larger than masses or momenta that appear in the standard model, the mass terms generated for left-handed neutrinos are very small and this is often explored as a possible explanation of the smallness of the mass of the observed neutrinos. The mass of lightest neutrino is not known experimentally yet, and only an upper limit is known. If the lightest neutrino was massless, the minimal number of particles that is needed as heavy partners in the above see-saw mechanism is thus 2, to account for the masses of the two light non-massless neutrino.

Note that above procedure is equivalent to approximately diagonalising a combined Majorana-Dirac mass term. Generally, not only in above approximation this also shows that fermions that have both a Majorana and a Dirac mass term are of Majorana nature. Furthermore, note that in a sufficiently complicated model, both Dirac and Majorana neutrinos may appear.

In the standard model without additional particles that would allow Majorana neutrino mass terms, there are two accidental global $U(1)$ symmetries, namely Lepton number, under which all leptons transform with a common phase, and Baryon number, under which all Baryons transform with a common phase. Majorana mass terms violate Lepton number and vice versa, by enforcing Lepton number and extending it to right-handed neutrinos, one can forbid Majorana mass terms both for left- and right-handed neutrinos, even in models with appropriate additional fields.

Attempts to measure neutrino mass and mixing parameters are an active experimental field and the status will be reviewed in the following chapters where appropriate.

1.4 Flavour symmetries

As already hinted at earlier, a flavour symmetry is a symmetry under which the fermion flavours form a representation. In the standard model, the kinetic terms and gauge interactions of fermions are invariant under arbitrary transformations in flavour space. However, this large symmetry is broken explicitly by the Yukawa interactions. Nevertheless, even the Yukawa terms have small unbroken flavour symmetries in each sector. These are the topic of subsection 1.4.1. Eventually, relations between the different accidental flavour symmetries that stem from an embedding into a larger flavour group determine the structure of the Yukawa sector and some of the main results of this thesis (which will be discussed in later chapters) are concerned with this.

1.4.1 Residual flavour symmetries of fermions

Previously, basis transformations in flavour space were used to diagonalise mass matrices. The purpose of this subsection is to analyse for the different parts of the Lagrangian which basis transformations in flavour space will actually leave those parts unchanged. In other words, one would like to know what the accidental flavour symmetries of the different fermionic parts of the Lagrangian are. First, mass terms of several fermions will be analysed for both the Majorana and the Dirac case and relations between these accidental flavour symmetries and the matrices that diagonalise the mass matrices will be found. After that, accidental flavour symmetries will be discussed for all parts of the standard model Lagrangian that involve fermions and special emphasis will be put on the Yukawa sector.

To start with, consider n Majorana fermions $\nu_i = (\nu_1, \ldots, \nu_n)$ with a Majorana mass term,

$$\mathcal{L} \supset \nu_i^T M_{ij}^M \nu_j + h.c. = \begin{pmatrix} \nu_1^T & \ldots & \nu_n^T \end{pmatrix} M^M \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix} + h.c.. \tag{1.65}$$

In a basis where M^M is diagonal, and assuming that all mass eigenvalues are different, this term is invariant under individual sign changes of all fields, e.g. $\nu_1 \mapsto -\nu_1$. All of these sign changes can be written as diagonal matrices g^ν_{ij} multiplying the vector ν_i:

$$\nu_i \mapsto g^\nu_{ij}\nu_j \tag{1.66}$$

with e.g. for $n = 3$,

$$g^\nu = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{1.67}$$

or all possible products of the matrices on the rhs of Eq. (1.67). These matrices form a representation of a $(Z_2)^n$ group. (So for $n = 3$ a $(Z_2)^3$ group.) The group formed by all of these 2^n matrices shall be called the *maximally allowed residual symmetry* for Majorana fermions. One of these Z_2 groups is generated just by an overall sign change of all ν_i fields. The remaining $(Z_2)^{n-1}$ group is then generated by all diagonal matrices that have exactly one +1 on the diagonal and otherwise only −1s, e.g. for $n = 3$:

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}. \tag{1.68}$$

The point of this division will be explained in the following. A non-diagonal Majorana mass matrix M^M can be diagonalised by a unitary transformation of the Majorana fermion fields, cf. Eq. (1.57). In this diagonal basis, the mass term has the above-mentioned symmetry. In the non-diagonal basis, this symmetry exists as well but with the group elements, in particular the generators, transformed into the new basis. Call U the matrix that diagonalises M^M, and u_i the three columns of U, $U = (u_1 u_2 u_3)$. If the basis of the fields is changed actively with U,

$$\nu_i \mapsto U_{ij}\nu_j, \tag{1.69}$$

then the generators g_k (where k enumerates the generators in some way) of the $(Z_2)^n$ group transform in the following way:

$$(g_k) \mapsto U(g_k)U^\dagger. \tag{1.70}$$

For those g_k that are proportional to the identity matrix, this does not change anything, however, for those particular generators with only one +1 on the diagonal and otherwise only −1, one can expand U in its columns and obtains e.g. for $n = 3$,

$$U \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} U^\dagger = +u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger \tag{1.71}$$

and similarly for all matrices of this form, again for $n = 3$:

$$G_1 = +u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger \tag{1.72}$$
$$G_2 = -u_1 u_1^\dagger + u_2 u_2^\dagger - u_3 u_3^\dagger \tag{1.73}$$
$$G_3 = -u_1 u_1^\dagger - u_2 u_2^\dagger + u_3 u_3^\dagger . \tag{1.74}$$

Note that because of the order in which a column u_i is multiplied with its hermitian conjugate, the G_i of Eqs. (1.72)-(1.74) are matrices.[15] The matrices G_i form a $Z_2 \times Z_2$ group for $n = 3$ and a $(Z_2)^{n-1}$ group for arbitrary n. Their most important property is that the i-th column of the matrix U, u_i is the eigenvector with eigenvalue +1 of the matrix G_i:

$$G_i u_i = +u_i . \tag{1.75}$$

Thus, knowing the form of all G_i is equivalent to knowing U, except for the phases of the columns u_i. This $(Z_2)^{n-1}$ group shall be called the *minimally necessary residual symmetry* of Majorana fermions, simply because it is the smallest symmetry that if it is known completely, determines the form of the matrix U, up to the ordering of the columns, and up to the phase of each column. Equivalently, for arbitrary n, U is determined by the $(Z_2)^{n-1}$ group of matrices that have only one eigenvalue +1 and otherwise −1. [16] If one was to force the mass matrix to be symmetric under a symmetry that is larger than the maximally allowed symmetry, this would force some states to be massless or degenerate in mass, depending on the nature of this symmetry. Note that vice versa, if the lightest state of a flavour multiplet was massless, then the residual symmetry would be enhanced to a $U(1)$ factor for this field (keeping the Z_2 factors for the other fields). This was considered in [24].[17]

[15] These matrices are often in the literature called S, U, SU.

[16] Note that this is true if those n Majorana fermions have identical gauge quantum numbers under all other unbroken symmetries at low energies. For the SM that means that after electroweak breaking, all Majorana fermions mix as the only left-over quantum number is electric charge (and, technically, colour), which for Majorana fermions has to be zero.

[17] This is particularly interesting in light of the result of chapter 2.

Often in the literature, when residual flavour symmetries are discussed, the minimally necessary residual symmetry or a subgroup of it is just called the residual symmetry. Furthermore, often the residual symmetry is only considered when it is actually embedded into a larger flavour group in the model in consideration.[18] The philosophy here however is that these symmetries are more of an accidental nature and exist whether they are embedded into a larger group or not.[19] In particular, the relation between residual symmetries always exists, such that through knowledge of the residual symmetry information can be gained about the mixing matrices.

Next, for n Dirac fermions that share a mass term [20], where one now has to distinguish left-handed and right-handed fields,

$$\mathcal{L} \supset \nu^\dagger_{i,R} M^D_{ij} \nu_{j,L} + h.c. = \begin{pmatrix} \nu^\dagger_{1,R} & \dots & \nu^\dagger_{n,R} \end{pmatrix} M^D \begin{pmatrix} \nu_{1,L} \\ \vdots \\ \nu_{n,L} \end{pmatrix} + h.c., \tag{1.76}$$

in a basis where M^D is diagonal, the mass term is invariant under a change of the phase of each individual field, $\nu_{i,L/R} \mapsto e^{i\alpha_i}\nu_{i,L/R}$, simultaneously for left- and right-handed fields. These transformations form a $U(1)^n$ group which constitutes the maximally allowed residual symmetry for Dirac fermions. As an arbitrary Dirac mass matrix is diagonalised by a bi-unitary transformation, $\nu_{i,L} \mapsto U^L_{ij}\nu_{j,L}$ and $\nu_{i,R} \mapsto U^R_{ij}\nu_{j,R}$, two seemingly different but isomorphic residual symmetry groups exist for left- and right-handed Dirac fermions, namely those given by

$$G^L = \{U^L \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_n}) U^{L^\dagger}\} \text{ and } G^R = \{U^R \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_n}) U^{R^\dagger}\} \tag{1.77}$$

for arbitrary α_i. If the basis of right-handed fermions is not fixed by other considerations, one can change the basis of right-handed fermions such that $U^R = U^L$.

Again, there are subgroups of these $U(1)^n$ groups G^L and G^R that already completely determine the matrices U^L and U^R. For Dirac fermions, the smallest groups that completely determine the diagonalisation matrices are Z_3 and again $Z_2 \times Z_2$.

[18] Often such symmetries are also called remnant symmetries.

[19] One could thus say that when for example no embedding of the residual symmetry is discussed that in a way existing accidental symmetry (even if only in separate sectors of the Lagrangian) is unaccounted for.

[20] Which again means that they have equal gauge quantum numbers after all symmetry breaking.

In the basis where M^D is diagonal, the relevant generator of Z_3 has the form

$$g_{Z_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{1.78}$$

with $\omega = e^{2\pi i/3}$ the third root of unity.[21] The generators of the minimal $Z_2 \times Z_2$ group have the same form in the basis where M^D is diagonal as for Majorana fermions, cf. Eq. (1.68). For a residual symmetry Z_3 or any other group of the form Z_k with $k \geq 3$, the columns of the corresponding mixing matrix are the eigenvectors of an element g of the residual symmetry group:

$$g u_i = e^{i\alpha_i} u_i \tag{1.79}$$

with u_i a column of $U^{L/R}$. Again, forcing the mass matrix to be invariant under a symmetry larger than the maximally allowed symmetry would force the fermions to be degenerate in mass or to be massless. On the other hand, because the maximally allowed residual symmetry is larger for Dirac fermions than for Majorana fermions, any discrete subgroup of $U(1)^n$ could serve as a subgroup that determines the matrices $U^{L/R}$.

Next, one can analyse what kind of residual flavour symmetries exist in the SM, also allowing for Higgs transformations. While in the literature, as in chapters 2–4 of this thesis only residual symmetries of mass matrices are analysed, this is in contrast to the also usual assumption that the flavour symmetry is broken above the electroweak scale. This would mean that the flavour symmetry may be broken to a much larger residual symmetry, enabled by Higgs transformations under the flavour group, which is then only later broken to the residual symmetry of mass matrices by the standard model Higgs.

Before EWSB, the fermionic degrees of freedom are, cf. Table 1.1 , $SU(2)_L$ doublets $Q_L = (u_L, d_L)$ of quarks and $E_L = (\nu_L, l_L)$ of leptons as well as singlets u_R, d_R of quarks and l_R of charged leptons. One can add additional singlets ν_R and omit the discussion of quarks, which will be analogous to the Dirac case in the following. First of all, the pure kinetic terms without gauge interactions are invariant under the largest possible unitary transformation of all fermions, namely $U(48)$ if one counts components of doublets separately (and even more, if one counts quark colours). However, of course, fermions are partly distinguished from each other

[21] In the literature, such a generator is often denoted by T.

by their gauge interactions, and the respective flavour symmetries of the gauge interaction terms are separate $U(3)$s for each of Q_L, u_R, d_R, E_L, l_R, ν_R.

The remaining terms of the Lagrangian up to mass dimension 5 that involve fermions are the Yukawa interactions and the Weinberg operator, cf. Eqs. (1.49) and (1.55), where now also the notation omits things that only play a role within the spinor space of fermions, but not in flavour space,

$$\mathcal{L} \supset l_R^\dagger Y^l E_L \phi^\dagger + \nu_R^\dagger Y^\nu E_L \tilde{\phi}^\dagger + \nu_R^T M^R \nu_R + E_L^T \tilde{\phi} Y^M \tilde{\phi}^T E_L + h.c. \tag{1.80}$$

Additionally, the Higgs field can contribute to transformations of the Yukawa matrices. With the transformations

$$E_L \mapsto g_L E_L,\ l_R \mapsto g_{l_R} l_R,\ \nu_R \mapsto g_{\nu_R} \nu_R,\ \phi \mapsto e^{i\alpha_\phi}\phi,\ \tilde{\phi} \mapsto e^{-i\alpha_\phi}\tilde{\phi}, \tag{1.81}$$

the following transformations of Majorana-type terms

$$Y^M \mapsto g_L^T Y^M g_L e^{-i2\alpha_\phi}, \tag{1.82}$$

$$M^R \mapsto g_{\nu_R}^\dagger M^R g_{\nu_R} \tag{1.83}$$

and of Dirac-type terms

$$Y^l \mapsto g_{l_R}^\dagger Y^l g_L e^{-i\alpha_\phi}, \tag{1.84}$$

$$Y^\nu \mapsto g_{\nu_R}^\dagger Y^\nu g_L e^{i\alpha_\phi} \tag{1.85}$$

are induced. More formally, above equations express the fact that the Yukawa and Majorana-type couplings are tensors under the various residual symmetries under which the fermions and the Higgs doublet transform. This formulation may also be extended to allow for additional Higgs doublets and scalar singlets. One should also mention that a potential of several scalars can have its own large residual symmetries which are generally unrelated to the residual symmetries in the Yukawa sector.

Taking into account the fact that an additional phase can be absorbed in the Higgs field, but otherwise following the discussion at the beginning of this section, one finds that the maximally allowed symmetries of the different terms are as follows:

The Y^M term is in the basis where it is diagonal invariant under

$$\{g_L\} = \{e^{i\alpha_\phi} \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}\} \simeq U(1) \times (Z_2)^2. \tag{1.86}$$

and the M^R term under a $(Z_2)^{n_{\nu_R}}$, where n_{ν_R} is the number of ν_R fields. The Y^l and Y^ν terms have each a symmetry under simultaneous $U(1)^3$ of e_R and E_L, and ν_R and E_L, respectively.[22] After EWSB, as the Higgs field acquires a vacuum expectation value, $\langle 0|\,\phi\,|0\rangle = (v, 0)$, the components of doublets become separate degrees of freedom. The fermionic degrees of freedom are now $u_L, u_R, d_L, d_R, l_L, l_R, \nu_L, \nu_R$. The residual symmetries of the kinetic terms remain unchanged and correspond to the largest possible unitary change of basis, while for gauge interactions, no longer an $U(3)$ symmetry for each of the degrees of freedom exists as was the case before EWSB. In particular, the $\bar{l}\nu W$ interaction does not allow for arbitrary individual basis transformations of l and ν.[23]

The following mass terms arise from the interaction terms of Higgs field and fermions, again only showing leptonic terms,

$$\mathcal{L} \supset l_R^\dagger Y^l l_L v^* + \nu_R^\dagger Y^\nu \nu_L v^* + \nu_R^T M^R \nu_R + \nu_L^T v^2 Y^M \nu_L + h.c. \tag{1.87}$$

With the transformations

$$\nu_L \mapsto g_{\nu_L}\nu_L,\ \nu_R \mapsto g_{\nu_R}\nu_R,\ l_L \mapsto g_{l_L} l_L,\ l_R \mapsto g_{l_R} l_R \tag{1.88}$$

one can see firstly, that the Majorana mass terms have $(Z_2)^{n_{\nu_R}}$ and $(Z_2)^3$ symmetries, respectively. Secondly, the Dirac mass terms only have symmetries under simultaneous transformations of left- and right-handed fields, which then again correspond to arbitrary phase transformations, $\{g_L^l = g_R\} \simeq U(1)^3$.[24]

[22] If there were, hypothetically, three Higgs doublets that in the basis where Y^M is diagonal were to generate one dimension 5 operator for each generation, then the residual fermionic symmetry of this Majorana mass term would become a $U(1)^3$ too, interestingly.

[23] However, allowing for a phase to be absorbed in the W field (which would need to correctly be accounted for in other places), one could imagine transformations where $g_l^\dagger g_\nu$ is a phase times the identity.

[24] In principle one could allow for basis transformations of the Higgs vev $v \mapsto e^{i\alpha_v} v$ to extend the accidental symmetry to the same groups as before EWSB. Such transformations would need to be accompanied with transformations of h^0 and ϕ^0 and thus also Z^0. So what happens to the accidental flavour symmetries in EWSB in a way is that they survive but get intertwined with phase transformations of massive gauge fields.

Note that all of the accidental flavour symmetry groups in the Yukawa sector mentioned in this section so far are not really symmetries of the whole theory but only remain unbroken in a small sector of the Lagrangian. The interaction term of W boson, left-handed electron and neutrino, in the basis where both Y^l and Y^M have been diagonalised, transforms as

$$W\bar{l}_L U_l^\dagger U_\nu \nu_L + h.c. \mapsto W\bar{l}_L g_L^{l\dagger} U_l^\dagger U_\nu g_L^\nu \nu_L + h.c.. \tag{1.89}$$

At this stage this just means that the mixing matrices $U_l^\dagger U_\nu$ and $g_L^{l\dagger} U_l^\dagger U_\nu g_L^\nu$ are physically equivalent, as the transformations g_L^l and g_L^ν leave the remaining Lagrangian invariant.

1.4.2 Breaking of larger flavour symmetries

As already hinted at, a flavour symmetry relates some fermionic degrees of freedom with each other (maybe even all of them). In practice however, flavour symmetries are often subgroups of $U(3)$. The reason for this is that as mentioned above, before EWSB, the residual flavour symmetry of the kinetic and gauge terms is an individual $U(3)$ for every flavour multiplet of each fermion type, except for ν_R, for which a Majorana mass term is allowed, which reduces the residual flavour symmetry in ν_R to $(Z_2)^{n_{\nu_R}}$. If this mass term for ν_R is set to zero, then the n_{ν_R} singlets have an $U(n_{\nu_R})$ residual flavour symmetry.

Consider thus that at some higher energy the symmetry of the standard model was extended by a group $G_F \subset U(3)$ to $G_{SM} \times G_F$, such that the various fermion multiplets ψ transform under G_F with some representation ρ_ψ of G_F (that acts in flavour space),

$$\psi \mapsto \rho_\psi(g)\psi \text{ with } \psi = Q_L, u_R, d_R, E_L, l_R, \nu_R, \tag{1.90}$$

and $g \in G_F$. In addition, as this symmetry may be spontaneously broken, also Higgs fields ϕ_i (of which now several might exist[25]) are to transform under G_F,

$$\phi_i \mapsto \rho_{\phi_i}(g)\phi_i. \tag{1.91}$$

[25] If additional scalar fields exist, one has to be careful to avoid flavour-changing neutral currents, either by symmetry arguments, maybe contained in G_F or by suppressing the corresponding parameters below the currently experimentally detectable level.

Furthermore, there may be scalars which are singlets under the standard model group but who still transform under G_F and may contribute to the spontaneous breaking of G_F.[26] As the scalar fields can be allowed to transform non-trivially under G_F, also the scalar potential is constrained by G_F.

G_F is then broken, for example spontaneously or radiatively, to subgroups in the different sectors of the Lagrangian. Note that in the literature these unbroken subgroups of G_F are often called residual symmetries but do not have to be identical to the residual flavour symmetries discussed in the previous subsection. In particular, G_F might be broken completely in parts of or even the whole Lagrangian.

In fact, in the literature (e.g. [25]), models with flavour symmetries are often classified by the subgroup of G_F that remains unbroken in the Majorana mass term of left-handed neutrinos. If the whole minimally necessary $Z_2 \times Z_2$ subgroup is contained in G_F, models would be called *direct*[27]. If only one of the Z_2 factors is considered to be part of G_F, a model would be called *semi-direct*[28], and *indirect* if no part of the residual symmetry is part of G_F. Later in this thesis in chapter 4, a distinction between neutrino-semidirect, where $G_\nu = Z_2$, and charged-lepton-semidirect, where $G_l = Z_2$ is introduced.

In any case, if the symmetry G_F was restored at some point, then whatever was in the place of the Yukawa matrices Y^l, Y^ν and the Majorana type couplings Y^M, M^R had to transform accordingly under G_F (maybe by simply being zero). Through some mechanism (e.g. by spontaneously or radiatively breaking G_F), these couplings then acquired constant values that are no longer invariant under the whole of G_F but only under the respective residual symmetries of each coupling.

The situation could arise that only parts of the maximally allowed residual symmetries at low energies are embedded as subgroups into G_F, as would be the case for G_F a discrete group as at least Y^l has a continuous maximally allowed residual symmetry. At higher energies, when G_F is unbroken, whatever is to become Y^l at low energies would in this case have to be invariant under G_F times the remainder of the maximally allowed residual symmetry of Y^l.

If $G_F \neq U(3)$ but is a proper subgroup of it, then flavour basis transformations $\in U(3)$ may change the form of the generators of G_F. However, the residual

[26] Note that to avoid additional massless Goldstone bosons, for which no experimental evidence or hints exist, the scalar potential should after the breaking of G_F not be invariant under any continuous symmetries, so also in particular no continuous subgroups of G_F. Furthermore one should note that additional scalars could also be triplets, etc.

[27] A $Z_2 \times Z_2$ group is also called a Klein group and sometimes denoted by K_4.

[28] Not to be confused with the semidirect product between groups.

symmetries to which G_F is broken in the different parts of the Lagrangian are embedded in some fixed way in G_F and their relative orientation will not change through flavour basis changes. If one basis exists in which G_F holds, then it is a true symmetry of the Lagrangian.

The open question is now, which groups could be candidates for G_F, and as a consequence of this, which representations of G_F the known and possible new fields transform under. In this respect, already a lot of information can be gained by specifying G_F and the subgroups it is broken to in the different parts of the Lagrangian without specifying a breaking mechanism or model, just by analysing and exploiting the relations imposed onto the residual symmetries via their embedding in G_F.

While all subgroups of $SU(3)$ are known, the subgroups of $U(3)$ are not yet known systematically. However, there has both been progress towards a classification [26, 27, 28], as well as searches using the group theory software GAP [29]. In later chapters, more references concerning both approaches are given. As the maximally allowed residual symmetry of a Majorana mass term is discrete, quite a bit of attention has been given to discrete subgroups of $U(3)$. Choosing a discrete group for G_F has also additionally the advantage that no Goldstone bosons appear, except when choosing the discrete group too restrictive.[29]

In chapters 2 and 4, analyses will be presented where all allowed subgroups that could be preserved in the neutrino sector as they would appear in direct or semi-direct model were systematically scanned for all possible choices of $G_F = \Delta(6n^2)$ for arbitrary n. It will be seen there that these groups are not as obscure as the name suggests and that there are good reasons to believe that it is worth analysing them thoroughly.

To summarize this section, first, in the previous subsection, residual residual symmetries of Dirac and Majorana mass terms have been discussed. These symmetries are of residual nature and exist always, however only in parts of the Lagrangian, namely the mass terms. After that, still in the previous subsection, such residual residual symmetries of the various sectors of the Lagrangian were analysed, with a particular focus on the Yukawa couplings and Majorana type couplings like the Weinberg operator. Next, flavour symmetries that are symmetries of the whole Lagrangian at some higher energy were discussed. These extend the symmetry

[29]A possible problem that might however arise are so-called domain walls, where different parts of the universe end up in separate vacua which are not connected by a flat direction and thus can coexist in a stable way.

group to $G_{SM} \times G_F$. The breaking of G_F has been sketched and the relation between subgroups of G_F that remain unbroken in various sectors of the Lagrangian with the residual flavour symmetries from the previous subsection was discussed, which happens to not always be one-to-one.

The results of this section will come in useful again in the next section, when residual and larger CP symmetries will be analysed in a similar way after a thorough introduction into the general properties of and puzzles associated with CP.

1.5 CP

CP conjugation, which is short for charge-parity conjugation, is a discrete symmetry which is a combination of space parity and charge conjugation. In this section, first, some general properties of CP will be analysed, starting from considering C and P separately and combining them afterwards. After this, accidental residual CP symmetries of the different sectors of the standard model will be discussed in analogy to the discussion of accidental flavour symmetries in the previous section. There, also relations between accidental residual flavour and accidental residual CP symmetries are discussed. In the subsection following that one, CP as a symmetry of the theory at higher energies and its breaking will be discussed. There, also the famous issue of consistency will be introduced. In later chapters, especially chapters 3-5, this will be discussed in greater detail, also concerning open questions. In those chapters, also plenty of additional references can be found.

To start with, in the following the discrete symmetries space parity P, charge conjugation C, and their combination CP will be introduced and the transformation properties of the different kinds of fields are listed by spin without considering their gauge properties yet [20]. When fields appear in multiplets of symmetries, slight additional complications arise which will be discussed after this first part.

Space parity, or often simply called parity and abbreviated as P, is a discrete transformation of spacetime defined by changing the sign of all space components, but not the time component,

$$P: \; x^\mu \mapsto x_\mu = (t, -\vec{x}). \tag{1.92}$$

This connects single-particle states with momentum $\vec{p}$ with such with momentum $-\vec{p}$. Charge conjugation on the other hand does not arise as a symmetry of spacetime but is defined to connect single-particle states with opposite $U(1)$ charges.

The unitary and linear operator which implements the operation of space parity on the Fock space of states is called $\mathcal{P}$. (This is standard notation, cf. [20], but should not be confused with the full Poincare group, which is often also denoted by $\mathcal{P}$.) Similarly, the linear and unitary operator which implements charge conjugation on the Fock space is called $\mathcal{C}$. The properties of the above operators $\mathcal{P}$ and $\mathcal{C}$ induce the transformation properties of field operators under parity and charge conjugation, which are listed in the following.

A scalar singlet transforms under parity as

$$\mathcal{P}\phi(x^\mu)\mathcal{P}^\dagger = e^{i\alpha_P}\phi(x_\mu) \tag{1.93}$$

where $e^{i\alpha_P}$ is an arbitrary phase factor in between the definitions of $\mathcal{P}$ and how it acts on field operators and under charge conjugation as

$$\mathcal{C}\phi(x^\mu)\mathcal{C}^\dagger = e^{i\alpha_C}\phi^\dagger(x^\mu), \tag{1.94}$$

again with an arbitrary phase factor $e^{i\alpha_C}$. A CP transformation is the combination of both of these operations.

For chiral spinors (so especially in the chiral basis also Weyl spinors) $\psi_{L/R}$, parity transforms each of them as

$$\mathcal{P}\psi_{L/R}(x^\mu)\mathcal{P}^\dagger = e^{i\beta_P}\gamma^0\psi_{L/R}(x_\mu). \tag{1.95}$$

Now, defining

$$[\psi(x^\mu)]^P := \gamma^0\psi(x_\mu), \tag{1.96}$$

then with $\gamma^0\gamma_{L/R} = \gamma_{R/L}\gamma^0$, it follows that parity turns a left-handed Weyl spinor into a right-handed spinor and vice-versa,

$$(\psi_{L/R})^P = (\psi^P)_{R/L}. \tag{1.97}$$

As a Dirac spinor is the direct sum of two chiral spinors with opposite chiralities,

$$\psi_D = \psi_L + \psi_R, \tag{1.98}$$

its transformation properties under parity follow to have the same form as for chiral spinors

$$\mathcal{P}\psi_D(x^\mu)\mathcal{P}^\dagger = e^{i\beta_P}\gamma^0\psi_D(x_\mu). \tag{1.99}$$

For a spinor with Majorana property, $\psi^c = e^{i\zeta}\psi$, cf. Eq. (1.23), from which follows that from any Dirac or Weyl spinor $\psi_{L/R/D}$, a Majorana spinor can be constructed via

$$\psi_M = \psi_{L/R/D} + e^{-i\zeta}(\psi_{L/R/D})^c, \tag{1.100}$$

and it obeys the normal transformation law under parity transformations, too:

$$\mathcal{P}\psi_M(x_\mu)\mathcal{P}^\dagger = e^{i\beta_P}\gamma^0\psi_M(x_\mu). \tag{1.101}$$

Under charge conjugation, chiral, Dirac and Majorana spinors behave the same,

$$\mathcal{C}\psi_{L/R/D/M}\mathcal{C}^\dagger = e^{i\beta_C}(\psi_{L/R/D/M})^c \tag{1.102}$$

where for Dirac and Majorana spinors,

$$(\psi_{D/M})^c = C\overline{\psi_{D/M}}^T. \tag{1.103}$$

The charge conjugation matrix C is defined via

$$\gamma_\mu C = -C\gamma_\mu^T. \tag{1.104}$$

For chiral spinors, $(\psi_{L/R})$, the same transformation properties hold and the transformation properties of Weyl spinors can be extracted.

Finally, for a vector field A^μ, as it is in particular a four-vector, parity acts on it as on the spacetime vector x^μ,

$$\mathcal{P}A^\mu(x^\mu)\mathcal{P}^\dagger = e^{i\xi_P}A_\mu(x_\mu), \tag{1.105}$$

while charge conjugation just adds a phase

$$\mathcal{C}A^\mu(x^\mu)\mathcal{C}^\dagger = e^{i\xi_C}A^\mu(x^\mu) \tag{1.106}$$

CP transformations combine the above transformations and one obtains

$$\mathcal{CP}\phi(x^\mu)\mathcal{CP}^\dagger = e^{i\alpha}\phi^\dagger(x_\mu) \tag{1.107}$$

$$\mathcal{CP}\psi(x^\mu)\mathcal{CP}^\dagger = e^{i\beta}\gamma^0 C\gamma^{0T}\psi^{\dagger T}(x_\mu) \tag{1.108}$$

$$\mathcal{CP}\bar{\psi}(x^\mu)\mathcal{CP}^\dagger = -e^{-i\beta}\psi^T(x_\mu)C^{-1}\gamma^0 \tag{1.109}$$

$$\mathcal{CP}A^\mu(x^\mu)\mathcal{CP}^\dagger = e^{i\xi_A}A_\mu(x_\mu) \tag{1.110}$$

The above transformations hold for chiral spinors, Dirac and Majorana spinors, where additional constraints arise from the chirality or Majorana conditions, $\psi = \gamma_{L/R}\psi$, and $\psi^c = e^{i\zeta}\psi$, respectively. In the chiral basis of gamma matrices also the two-component conditions for Weyl spinors can be obtained immediately.[30]

For a theory with many different fields, like especially the standard model, in a basis where the fermion mass matrices are diagonal and real (and if there is only one Higgs doublet), for each field a separate arbitrary phase appears in the P, C, or CP transformation. A theory now conserves P, C, or CP, if at least one combination of values of all these phases exists such that with these phases appearing in the transformation, the Lagrangian is invariant under it. In the basis where the mass matrices are diagonal and real the phases on CP transformations of $W^\pm$, $e^{\pm i\xi_W}$, (which are complex conjugated because $W^\pm$ are related by C conjugation), i-th up-type quark, $e^{i\xi_{u_i}}$, and j-th down-type quark, $e^{i\xi_{d_j}}$ appear in the $W\overline{u}d$ coupling. From this follow conditions on the elements of the CKM matrix that needs to be fulfilled for the theory to conserve CP,

$$V_{ij}^* = e^{i(\xi_W+\xi_j-\xi_i)}V_{ij}. \tag{1.111}$$

In the standard model it so happens that all CP violation appears in the quark mixing matrix and it is not necessary to take into account the possibility of flavour basis transformations in CP transformations. However, the most general CP transformations have to take these into account as CP conservation/violation does not depend on these internal basis transformations that are possible in flavour space. Furthermore, if several copies of Higgs fields with the same quantum numbers are part of the model, then basis transformations that act on the copies of Higgs fields are possible and in that case it is not a priory clear what a Higgs basis is in which a reasonably simple condition like Eq. (1.111) holds. The most general CP transformations in the standard model, now extended by n_{ν_R} fermionic

[30]For spinors, note that the hermitian conjugation acts the whole of the field, in particular the generators and annihilators in the field, while the transposition acts only within spinor space but does not act on the generators.

singlets and allowing for several Higgs doublets to exist, that take internal basis transformations into account, are,

$$\mathcal{CP}\phi_i(x^\mu)\mathcal{CP}^\dagger = (X_\phi)_{ij}\phi_j^\dagger(x_\mu) \tag{1.112}$$

$$\mathcal{CP}\psi_i(x^\mu)\mathcal{CP}^\dagger = (X_\psi)_{ij}\gamma^0 C\gamma^{0T}\psi_j^{\dagger T}(x_\mu) \tag{1.113}$$

where before EWSB $\psi = Q_L, u_R, d_R, E_L, l_R, \nu_R$ and similarly after EWSB $\psi = u_L, d_L, u_R, d_R, l_L, \nu_L, l_R, \nu_R$.[31] Furthermore, after EWSB, the Higgs field is expanded around its vev and the different components $h^0, \phi^0, \phi^\pm$ can have separate transformations. While more complicated models, where it is entirely necessary to use general CP transformations, do often not allow for simple relations indicating CP violation, as Eq. (1.111), this role is not taken by CP-odd basis invariants. The most famous of such invariants is the Jarlskog invariant. Similarly, invariants exist for models with Majorana neutrinos. CP-odd invariants, in particular such involving scalar parameters will be discussed in great detail in chapter 5, where also plenty of references will be given.

Pure gauge theories can never violate CP. In the standard model, QED and QCD are already CP-invariant (except for the strong CP problem, which unfortunately will not be discussed here). The weak interactions violate both invariance under space parity and charge conjugation. However, in a model with the standard model gauge group but only a single generation of fermions and no additional neutrino, CP is not violated. The only CP violation that is currently experimentally confirmed arises via quark mixing and is only possible if at least three generations of fermions exist.

In the next subsection, it will be analysed in analogy to subsection 1.4.1 which sets of matrices X appearing in general CP transformation still leave the various part of the Lagrangian invariant, starting with Dirac and Majorana mass matrices and after that focussing on the Yukawa couplings.

1.5.1 Residual CP symmetries of fermions

In the same way as the various terms of the fermion Lagrangian have different residual flavour symmetries, parts of a Lagrangian of a model also have different residual (general) CP symmetries. First, as earlier for flavour symmetries, the

[31] As ghost fields are an essential part of the theory, they also have definite transformation properties.

residual CP symmetries of fermion mass terms will be discussed as these will correspond to the smallest residual symmetries, at least in the standard model. Next, the residual CP symmetries of fermions in the standard model before and after EWSB will be analysed.

To start with, consider again a mass term of several Majorana fermions, cf. Eq. (1.65),

$$\mathcal{L} \supset \nu_i^T M_{ij}^M \nu_j + h.c. \tag{1.114}$$

$$= \nu_i^T M_{ij}^M \nu_j - \nu_i^\dagger M_{ij}^{M*} \nu_j^* \tag{1.115}$$

where the minus sign in the second row results from the minus sign between the two terms results from the fact that the spinors anticommute. (Everything not necessary for the discussion which will essentially happen in flavour space has been ignored.)

With transformations of the kind

$$\nu_i \mapsto X_{ij} \nu_j^*, \tag{1.116}$$

from which follows

$$\nu_i^T \mapsto \nu_j^\dagger (X^T)_{ji}, \tag{1.117}$$

in a basis where M^M is diagonal and real, the X matrices corresponding to the maximally allowed residual CP symmetries are given by i times independent sign changes of each field ,

$$X = \begin{pmatrix} \pm i & & \\ & \ddots & \\ & & \pm i \end{pmatrix}. \tag{1.118}$$

Recall that the maximally allowed flavour symmetries for such a Majorana mass term just corresponds to arbitrary sign changes of each field. Call these diagonal flavour and CP transformation matrices $\tilde{G}$ and $\tilde{X}$, respectively. In some other, arbitrary, basis, $\nu \mapsto U\nu$ with a unitary matrix U, these transformations then become

$$U^\dagger \tilde{G} U =: G \tag{1.119}$$

and

$$U^\dagger \tilde{X} U^* =: X. \tag{1.120}$$

These residual flavour and CP symmetries fulfil the following consistency conditions on flavour and CP transformations (as do the diagonal transformations):

$$XX^* = 1 \tag{1.121}$$

and

$$XG^*X^\dagger = G. \tag{1.122}$$

The first of above conditions is equivalent to X being symmetric and was found already in [30].

Above consistency conditions are stricter than a similar set of conditions found in the literature. Without wanting to anticipate the next subsection, here is a good place to discuss this issue. In the literature, often some fixed general CP transformation that is obtained in some way is imposed onto the mass matrix to generate constrains that in contrast to normal residual flavour symmetries also extend to the phases of the mass matrix. To not to overconstrain the mass matrix, often it is demanded that X matrices only fulfil the following more loose conditions:

$$XX^* = G \tag{1.123}$$

and

$$XG^*X^\dagger = G' \tag{1.124}$$

with G and G' elements of the residual flavour symmetry. The precise statement is that if for every G of the residual symmetry group a G' is contained in the group such that above equations hold, then X can be used in a residual CP symmetry such that it does not enlarge the residual flavour symmetry. The origin of these consistency conditions will be discussed in section 1.5.2. If the residual flavour symmetry that is imposed onto the mass matrix gets larger than the maximally allowed residual symmetry, then mass eigenvalues will be forced to vanish or to be degenerate. However for every possible subgroup of the residual $(Z_2)^{n_\nu}$ that is identified with a subgroup of a larger flavour symmetry G_F there are examples of X matrices which enlarge the residual flavour symmetry beyond the maximally allowed one. First of all, for $\{G\} = \{\pm 1\}$, all (anti-)orthogonal matrices X would be candidates for residual CP transformations, which clearly enlarge the group of residual flavour transformations. For $\{\pm 1\} \neq \{G\} \simeq Z_2$ one can see in a diagonal basis that the looser consistency conditions allow permutation matrices interchanging two equal eigenvalues. For larger groups $\{G\}$ the situation only gets worse. To summarize, the correct consistency conditions that residual CP and flavour symmetries on Majorana mass matrices have to fulfil are Eqs. (1.121) and (1.122). Unfortunately, also in some of the papers of the author which are based on later chapters, the looser consistency conditions were used. Where this happens, it will be clearly remarked and if possible corrected.

Now, for Dirac fermions with a mass term as in Eq. (1.76),

$$\mathcal{L} \supset \nu_{i,R}^\dagger M_{ij}^D \nu_{j,L} - \nu_{i,L}^T M_{ij}^{D*} \nu_{j,R}^* \tag{1.125}$$

under transformations of the form

$$\nu_{i,L/R} \mapsto (X_{L/R})_{ij} \nu_{j,L/R}^*, \tag{1.126}$$

in a basis where M^D is diagonal and real, the mass term has a symmetry again under simultaneous phase changes of each field,

$$\nu_{i,L} \mapsto e^{i\alpha_i} \nu_{i,L}^* \tag{1.127}$$

and

$$\nu_{i,R} \mapsto -e^{i\alpha_i} \nu_{i,R}^*. \tag{1.128}$$

The matrices $X_{L/R}$ then contain these phases on their diagonals. In an arbitrary basis, the form of these CP transformations is obtained using similar transformations to Eqs. (1.119),(1.120), except that the basis of left- and right-handed fermions may be chosen differently, $\nu_{L/R} \mapsto U_{\nu_{L/R}} \nu_{L/R}$. In a basis where M^D is diagonal and real, CP transformation matrices $X_{L/R}$ and residual symmetries $G_{L/R}$ fulfil the following consistency conditions,

$$X_L X_L^* = X_R X_R^* = 1,\ X_L X_R^* = X_R X_L^* = -1 \tag{1.129}$$

and a relation relating flavour and CP, with a minus sign on the rhs if one X_L and one X_R appear on the lhs.

$$X G_{L/R} X^\dagger = G_{L/R}. \tag{1.130}$$

In an arbitrary basis, transformation matrices $U_{\nu_{L/R}}$ need to be inserted in the correct places. In those relations in which both left- and right-handed CP or flavour matrices appear now products of basis transformation matrices can appear. However, this is of little significance in the standard model as in this case, the right-handed basis is arbitrary and can be chosen identical to the left-handed basis.

Next, the residual CP symmetries of different terms involving fermions in the standard model (and a little beyond) will be analysed. Again, similarly to the discussion of residual flavour symmetries, CP could be violated at a scale above the electroweak scale. Thus the residual CP symmetries of the Yukawa sector before EWSB may be embedded into a larger CP symmetry. These larger residual CP symmetries would then be broken to the residual symmetries of mass matrices by the standard model Higgs doublet. To make the discussion slightly more general, a number n_{ν_R} of fermionic standard model singlets are allowed as well as a number n_ϕ of Higgs doublets instead of only one. Before EWSB, the fermionic degrees of freedom relevant for the discussion here are $E_L = (e_L, \nu_L)$, e_R, and ν_R, as the quark sector will be analogous to Dirac leptons. The kinetic terms of fermions are invariant under the most general CP transformation that mixes all fermionic fields, but as fermions are distinguished by their gauge interactions, the relevant transformations only transform fermions in identical gauge transformations into fermions of the same type. Ignoring spinor indices, gamma matrices and charge conjugation matrices, the Yukawa terms plus the Weinberg operator in such a model can be written as

$$\begin{aligned}\mathcal{L} \supset\ & l_R^\dagger Y_a^l E_L \phi_a^\dagger + \nu_R^\dagger Y_a^\nu E_L \tilde{\phi}_a^\dagger + \nu_R^T M^R \nu_R + E_L^T \tilde{\phi}_a Y_{ab}^M \tilde{\phi}_b^T E_L \\ & -(l_R^T Y_a^{l*} E_L^* \phi_a^T + \nu_R^T Y_a^{\nu *} E_L^* \tilde{\phi}_a^T + \nu_R^\dagger M^{R*} \nu_R^* + E_L^\dagger \tilde{\phi}_a^* Y_{ab}^{M*} \tilde{\phi}_b^\dagger E_L^*)\end{aligned} \tag{1.131}$$

As the index denoting different Higgs fields has been made explicit, possible transposition of the Higgs fields in only in the $SU(2)$ space of E_L and ϕ_i relevant. The question is now again, which transformations can be applied to the various terms that leave them invariant for fixed but arbitrary Yukawa and Majorana type coupling matrices. With the transformations

$$E_L \mapsto X_L E_L^*,\ l_R \mapsto X_{l_R} e_R^*,\ \nu_R \mapsto X_{\nu_R} \nu_R^*,\ \phi \mapsto X_\phi \phi^\dagger,\ \tilde{\phi} \mapsto X_\phi^* \tilde{\phi}^* \tag{1.132}$$

where the transformation of $\tilde{\phi}$ follows from the one of ϕ, the following conditions arise, for Majorana-type terms

$$Y_{cd}^{M*} = -X_L^\dagger Y_{ab}^M X_L (X_\phi^*)_{ac} (X_\phi^*)_{bd} \tag{1.133}$$

$$M^{R*} = -X_{\nu_R}^\dagger M^R X_{\nu_R} \tag{1.134}$$

and of Dirac-type terms

$$Y_b^{l*} = -X_{l_R}^\dagger Y_a^l X_{E_L} (X_\phi^\dagger)_{ab}, \tag{1.135}$$

$$Y_b^{\nu *} = -X_{\nu_R}^\dagger Y_a^\nu X_{E_L} (X_\phi^T)_{ab}. \tag{1.136}$$

The sets of maximally allowed symmetries CP of each term consist of those transformation that fulfil above conditions for each of the involved fields in that term. For M^R one obtains the same result as previously.

To avoid flavour-changing neutral currents (FCNC), all of the Yukawa matrices of each fermion pair need to be diagonal in the same basis (or at least off-diagonal elements need to be undetectably small).

After EWSB, the fermion and Higgs doublets are split into their components, and similarly to flavour symmetries, the larger CP symmetries above EWSB are broken to residual CP symmetries of mass terms

1.5.2 CP as a symmetry

Again similar to flavour symmetries, an unbroken CP symmetry at high energies may relate some (or even all) fermionic degrees of freedom with each other, where one is normally restricted to transforming fields with opposite $U(1)$ charges into each other and otherwise equivalent gauge quantum numbers.

This larger CP symmetry would then be broken in some way to the respective residual CP symmetries of the various sectors of the standard model. By embedding the residual CP symmetries into this larger CP symmetry, relations between the various residual CP symmetries arise. If the CP symmetry is broken, then also scalars transform under the CP symmetry and can obtain residual CP symmetries, as mentioned in the previous section.

And again similarly to flavour symmetries, parts or all of the residual CP symmetries can be embedded into the larger CP symmetry.

Whatever is in place of the Yukawas at high energies must transform as tensors under the unbroken transformations of fermions and scalars. If at the same time at high energies an unbroken flavour symmetry is present, then whatever was in place of the Yukawas must transform simultaneously as tensors under flavour and CP transformations. From this normally the consistency condition between unbroken flavour and CP transformations is derived, as will be done in chapters 3 and 4. However, as the only condition on flavour and CP transformations at high energies is that the Yukawas or generally all parameters of the whole Lagrangian transform as simultaneous tensors under flavour and CP, in principle arbitrary CP transformations could be imposed as long as the residual flavour and CP symmetries fulfil the consistency conditions, Eqs. (1.121) and (1.122) at low energies. Of course when enforcing arbitrary flavour and CP symmetries to simultaneously hold at high energies, one can end up forcing the Yukawa sector to vanish completely and the normal consistency condition ensures that this does not happen.

To summarize, mass terms have not only residual flavour symmetries but also residual general CP symmetries. These need to fulfil a rather strict consistency condition. The residual flavour and CP symmetries of mass terms actually arise from the spontaneous breaking via the standard model Higgs of larger residual symmetries of the Yukawa sector that hold above EWSB. As the standard model Higgs may not play a role in the breaking of flavour or CP symmetry (except this last step), it is at least worth mentioning, even if in the remainder of this thesis only residual symmetries of mass terms will be considered.

1.6 Outline of the remainder of the thesis

The following chapter analyses the mixing predictions in direct models with $G_F = \Delta(6n^2)$. This analysis is extended with consistent CP in chapter 3. In chapter 4, the symmetry is weakened and the mixing predictions in semidirect models, again with consistent CP are studied. Chapter 5 breaks away a little bit from the pure study of fermion residual symmetries and the focus is put on CP odd invariants for multi-Higgs models. Chapter 6 concludes the thesis.

2

Lepton mixing predictions from direct models with a $\Delta(6n^2)$ flavour symmetry

This chapter presents results that had been partly published in [1] and [3]. The results are significant, as for the first time, predictions of lepton mixing parameters for Majorana neutrinos were obtained for direct models based on $\Delta(6n^2)$ flavour symmetry groups for arbitrarily large n. The contribution of the author of this thesis to the research presented here lies in performing all necessary calculations, and writing the majority of [1] and the entirety of [3].

Before the measurement of a rather large reactor mixing angle by the Daya Bay [31], RENO [32], and Double Chooz [33] collaborations, all measurements of this parameter were compatible with it being zero. In this case, the smallest flavour symmetry that could explain the structure of the mixing matrix via residual symmetries of mass matrices in a direct model is S_4 [34, 35]. This and all other models that predicted $\theta_{13} = 0$ were ruled out by the clear evidence of the contrary. Nevertheless, as has been emphasized in previous sections, a residual symmetry of the neutrino mass matrix that can completely determine the diagonalisation matrix V^ν always exists. One can now ask which more complicated flavour symmetry groups these residual symmetries could be part of and what the predictions of such an embedding would be. It will be found then that many members of the $\Delta(6n^2)$ group series are good candidates for flavour symmetry groups and especially that the middle column of the mixing matrix is trimaximal, the mixing angle θ_{13} is fixed up to a discrete choice, no CP violation would be allowed in the lepton sector (which is a great prediction as it is in conflict with the emerging hints at non-zero CP leptonic violation), and that a sum rule holds, namely $\theta_{23} = 45^\circ \mp \theta_{13}/\sqrt{2}$.

As discussed in section 1.4.2, to obtain these predictions, this larger flavour symmetry will then be spontaneously broken in order to generate the observed fermionic masses and mixings [36]. However before even considering the construction of a model, it may be insightful to know some of the possible candidate symmetries for $\mathcal{G}_f$ and the goal of [1], was then to shed light on a particular class of candidates for $\mathcal{G}_f$, namely the $\Delta(6n^2)$ groups. These groups are really not as obscure as the name suggests, as will become clear when the group theory of $\Delta(6n^2)$ groups will be reviewed and developed later this sections. Oversimplifying a little, these groups can be thought of as a way of combining discrete phases multiplying flavours and permutations acting on flavours.

These groups were chosen, partly due to the past and current popularity of $S_4 \cong \Delta(24)$ ($n = 2$) in flavour model building (see [25] and references contained therein) but in particular publications that had appeared recently at the time, demonstrating that $\Delta(96)$ ($n = 4$)[37, 38, 39, 40], $\Delta(150)$ ($n = 5$)[41, 42], $\Delta(600)$ ($n = 10$)[42, 43] and $\Delta(1536)$ ($n = 16$)[43] generate phenomenologically viable predictions for the lepton mixing angles, where [41, 42, 43] are numerical searches using the program GAP [44, 45, 46, 47], that indicated that out of discrete groups up to a certain size, only members of the $\Delta(6n^2)$ series were able in direct models to provide mixing matrices that were compatible with global fits taking the θ_{13} measurement into account.[1] In contrast to the above computational studies or studies of single groups, here, the whole of the infinite group series of $\Delta(6n^2)$ will be tested as flavour group candidates using analytical methods.

In the following, first, another derivation of the relation between diagonalisation matrices and residual symmetries will be given. It will be found that only $\Delta(6n^2)$ groups with n even contain a full $Z_2 \times Z_2$ subgroup, which, again, is the smallest residual symmetry which for 3 Majorana neutrinos completely determines the mixing matrix (up to Majorana phases). After that, the groups $\Delta(6n^2)$ will be introduced, where they will also be analysed for subgroups that may be identified with residual symmetries, followed by an analysis of the representations of $\Delta(6n^2)$. Next, the lepton mixing results will be given, before the chapter concludes with a summary and an outlook.

[1] $\Delta(6n^2)$ flavour symmetry models in which not the complete Klein symmetry was identified as a subgroup of the flavour symmetry had also been studied (see e.g. [48]).

2.1 From $\mathcal{G}_f$ to lepton mixing: A shortcut

As previously mentioned, a discrete flavour symmetry will be introduced which is spontaneously broken to different subgroups in the charged lepton and neutrino sectors, thereby generating the observed lepton masses and mixings. The flavour group is broken to some abelian subgroup Z_m^T (m an integer) in the charged lepton sector and to the $Z_2^S \times Z_2^U$ Klein Symmetry Group in the neutrino sector.

The superscripts denote that S, T and U are the generators of their corresponding Z_m group in the diagonal charged lepton basis. Hence, the $Z_2^S \times Z_2^U$ transformations on ν_L and the Z_m^T transformations on $e_{L,R}$ leave the Lagrangian invariant. This implies that

$$[S, M^M M^{M\dagger}] = [U, M^M M^{M\dagger}] = 0 \text{ and } [T, M^l M^{l\dagger}] = 0, \tag{2.1}$$

where M^M and M^l represent the mass matrices of left-handed Majorana neutrinos and charged leptons, cf. section 1.3.5. Since S and U commute with $M^M M^{M\dagger}$ (and with each other), all three are diagonalised by the same matrix V_L^ν. Similarly T and M^l are diagonalised by the same matrix V_L^l. The PMNS matrix is then given by

$$V = V_L^{l\dagger} V_L^\nu. \tag{2.2}$$

To obtain the matrices V^ν and V^l, and hence the PMNS matrix, one only needs to diagonalise the generators S, T, U. In practice, this amounts to finding the eigenvectors of S, U and T which form the columns of V^ν and V^l. This is straightforward for T since the eigenvalues are non-degenerate due to the fact that T must be an element of $\mathcal{G}_f$ of order 3 or greater. However for the S and U generators the situation is slightly different because they are 3×3 matrices of order 2. Thus, each eigenvalue of S or U can only be ± 1. Without loss of generality, one can choose $\det(S) = \det(U) = +1$, so that each generator has two -1 eigenvalues, rendering the corresponding eigenvectors non-unique. Since the three matrices S, U and SU each have one (unique) $+1$ eigenvalue this allows for the calculation of three unique eigenvectors (one for each non-trivial Klein group generator), each providing an ith column of the matrix V^ν:

$$G_i V_i^\nu = +V_i^\nu, \text{ for } G_i \in \{S, U, SU\}. \tag{2.3}$$

In this way all three columns of V^ν can be obtained.

Again, this method enables the calculation of the lepton mixing matrix by only considering the flavour group's representation matrices [41, 43]. However, this requires explicit representation matrices for the $\Delta(6n^2)$ group's representations. These are given in the following.

2.2 The group theory of $\Delta(6n^2)$

The $\Delta(6n^2)$ groups are finite non-Abelian subgroups of $SU(3)$ (and thus also of $U(3)$) of order $6n^2$. They are isomorphic to the semidirect product [49],

$$\Delta(6n^2) \cong (Z_n \times Z_n) \rtimes S_3. \tag{2.4}$$

The Klein group $Z_2^S \times Z_2^U$ (in direct models) can either originate purely from the $Z_n \times Z_n$ or it will involve the S_3 generators as well, both possibilities requiring even n. The S_3 subgroup can be expanded in its factors to obtain

$$\Delta(6n^2) \cong (Z_n^c \times Z_n^d) \rtimes (Z_3^a \rtimes Z_2^b). \tag{2.5}$$

Notice that in Eq. (2.5), $(Z_n^c \times Z_n^d)$ forms a normal, abelian subgroup of $\Delta(6n^2)$, generated by the elements c and d, and $(Z_3^a \rtimes Z_2^b)$ is nothing more than S_3 rewritten in terms of its generators a and b. From Eq. (2.5) follows that a presentation of $\Delta(6n^2)$ is [49]:

$$\begin{aligned} a^3 = b^2 = (ab)^2 = c^n = d^n = 1, \quad cd = dc, \\ aca^{-1} = c^{-1}d^{-1}, \ ada^{-1} = c, \\ bcb^{-1} = d^{-1}, \ bdb^{-1} = c^{-1}. \end{aligned} \tag{2.6}$$

An advantage of the presentation in Eqs. (2.4)-(2.5) is that every group element can be written as

$$g = a^\alpha b^\beta c^\gamma d^\delta, \tag{2.7}$$

with $\alpha = 0, 1, 2$, $\beta = 0, 1$ and $\gamma, \delta = 0, \ldots, n-1$, making the computation of all group elements for a certain representation/basis computationally simple. All that needs to be known next is the explicit forms of generators.

In order to find the explicit forms for the generators, one can restrict oneself to 3-dimensional irreducible representations of $\Delta(6n^2)$. Then, it can be shown that $\Delta(6n^2)$ has $2(n-1)$ 3-dimensional irreducible representations denoted by $\mathbf{3}_k^l$ and explicitly generated by [49]:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = (-1)^{k+1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}, \tag{2.8}$$

where $\eta = e^{2\pi i/n}$; $k = 1, 2$; and $l = 1, \ldots, n-1$.

One can restrict the analysis to faithful irreducible representations of $\Delta(6n^2)$. Thus, all representations in Eq. (2.8) where l divides n can be excluded, as they are unfaithful. Of the remaining representations, $\mathbf{3}_k^l$ and $\mathbf{3}_k^{l'}$ are complex conjugates of each other if $l + l' = n$. Therefore, they will provide complex conjugated mixing matrices. The remaining representations provide the same sets of mixing matrices because the generators a and b are the same for all l and

$$c(\mathbf{3}_k^l) = c(\mathbf{3}_k^1)^l \text{ and } d(\mathbf{3}_k^l) = d(\mathbf{3}_k^1)^l. \tag{2.9}$$

Then, from Eq. (2.7) and Eq. (2.9) follows that each power of the c and d generators will appear in every 3-dimensional irreducible representation. For these reasons, it suffices if one only considers S, T, and U as representation matrices from $\mathbf{3}_2^1$. Notice that $k = 2$ has been chosen because in this case the determinant of the elements of order 2 is $+1$.

Having reduced the possible cases needed for consideration, the next step is to calculate all Klein subgroups of $\Delta(6n^2)$. This is accomplished by first calculating all order two elements. From the generators and rules given in Eq. (2.6) it follows that all order 2 elements in $\Delta(6n^2)$ are given by:

$$c^{n/2}, \; d^{n/2}, \; c^{n/2}d^{n/2}, \; bc^{\epsilon}d^{\epsilon}, \; abc^{\gamma}, \text{ and } a^2bd^{\delta}, \tag{2.10}$$

where $\epsilon, \gamma, \delta = 0, \ldots, n-1$.

The order 2 elements found in Eq. (2.10) serve as a starting point for calculating Klein Symmetry groups of $\Delta(6n^2)$. Using Eq. (2.6) and Eq. (2.10), the Klein subgroups of $\Delta(6n^2)$ for even n are:

$$\{1, c^{n/2}, d^{n/2}, c^{n/2}d^{n/2}\}, \tag{2.11}$$

$$\{1, c^{n/2}, abc^{\gamma'}, abc^{\gamma'+n/2}\}, \tag{2.12}$$

$$\{1, d^{n/2}, a^2bd^{\delta'}, a^2bd^{\delta'+n/2}\}, \tag{2.13}$$

$$\{1, c^{n/2}d^{n/2}, bc^{\epsilon'}d^{\epsilon'}, bc^{\epsilon'-n/2}d^{\epsilon'-n/2}\}, \tag{2.14}$$

where $\gamma', \delta', \epsilon' = 1, \ldots, n/2$. Notice that Eq. (2.11) corresponds to the Klein symmetry originating completely from $Z_n \times Z_n$ whereas Eqs. (2.12)-(2.14) involve also S_3. In the basis of Eq. (2.8), one of the Klein generators (taken to be S) is diagonal for all cases, while in the case of Eq. (2.11) both Klein generators S, U are diagonal [2].

The T generator which controls the charged lepton sector must be at least of order 3 and only the minimal order 3 case is phenomenologically viable, thus only this possibility is considered.[3] In $\Delta(6n^2)$ groups where 3 does not divide n, all elements of order 3 are expressible as [49]:

[2] As an example of the Klein subgroups in Eqs. (2.11)-(2.14), in $\Delta(96)$(n=4)[37, 38, 39, 40], it was found that for the bi-trimaximal mixing example $S = d^2$ and $U = a^2bd^3$, implying that these generators are contained in the Klein subgroups defined in Eq. (2.13).

[3] T generators of order greater than 3 are not viable: Consider the order of T to be even. Then, $T^m = 1$ with $m = 2q$ where q is an integer. Note that diagonal T candidates in the basis of Eq. (2.8) will not lead to acceptable mixing. After removing unphysical phases, all non-diagonal T candidates of even order $m = 2q$ can be written without loss of generality as,

$$T = bc^{\xi n/q}, \xi = 1, \ldots, q-1. \tag{2.15}$$

The matrices of Eq. (2.15) are diagonalised by

$$V^l = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-i\pi\xi/q} & -e^{-i\pi\xi/q} \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2.16}$$

Applying the above matrix to $c^{n/2}$ results in:

$$U \to V^{l\dagger} c^{n/2} V^l = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.17}$$

The unique eigenvector of this generator is given by $(0, 1, 1)/\sqrt{2}$. Picking the smallest element of the mixing matrix as V_{13} gives $V_{13} = 0$. For $n = 2$ this results in a completely bimaximal mixing matrix [50, 51]. If the order of T is not even but can be divided by 3, application of a unitary transformation $R = c^x d^y$ can remove all phases implying only $T = a$ remains, yielding the previously discussed predictions for $T = a$. Continuing the systematic consideration of candidate T generators leads to the case of a T generator in which the order is odd, not divisible by 3 but larger than 3. A $\Delta(6n^2)$ group can only contain such an element if m divides n. Then, for this case the possible T generators are given by

$$T = c^{\mu n/m} d^{\rho n/m} \tag{2.18}$$

$$ac^\gamma d^\delta, a^2 c^\gamma d^\delta \tag{2.19}$$

where $\delta, \gamma = 0 \dots n-1$ [4].

In [1], the order three generator was chosen to be

$$T = a, \tag{2.20}$$

since a and a^2 only differ by a permutation of rows and columns and it was assumed[5] there that in the basis of Eq. (2.8), multiplication by $c^\gamma d^\delta$ only yields phases which may be absorbed into the charged lepton fields.

Notice that the T of Eq. (2.20) can be diagonalised by the matrix,

$$V^l = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^2 & \omega & 1 \\ \omega & \omega^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{2.21}$$

where $\omega = e^{2\pi i/3}$. The ordering of the columns and rows in the above V^l determines the ordering of the eigenvalues in T:

$$T \to V^{l\dagger} a V^l = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.22}$$

For example, changing the order of the eigenvalues of T by applying a^α to T by $a^{\alpha\dagger} T a^\alpha$ $(\alpha = 1, 2)$ changes V^l to $a^\alpha V^l$ which just permutes the rows of V in Eq. (2.2).

Note that it is not always the case that the generators S, T, U above generate the full $\Delta(6n^2)$ group. It turns out that the Klein subgroup in Eq. (2.12), in combination with the residual Z_3^T in Eq. (2.20), will only generate the full $\Delta(6n^2)$ symmetry group if and only if γ' does not divide n. From the top down point of view of choosing G_F this is acceptable since one is only interested in the possible predictions that can arise from $\Delta(6n^2)$.

where $\mu, \rho = 0, \dots, m-1$ and μ, ρ are not simultaneously zero. These yield no phenomenologically viable predictions. Therefore, only candidate T generators from Z_3 subgroups of $\Delta(6n^2)$ are phenomenologically viable.

[4]When n is divisible by 3, there exist more order three elements given by $c^{n/3}$, $c^{2n/3}$, $d^{n/3}$, $d^{2n/3}$, $c^{n/3}d^{n/3}$, $c^{2n/3}d^{n/3}$, $c^{n/3}d^{2n/3}$, $c^{2n/3}d^{2n/3}$. In the basis of Eq. (2.8), these are diagonal matrices of phases. Since S is also diagonal in this basis, this would result in phenomenologically unacceptable predictions for leptonic mixing.

[5]This second assumption was not correct, as was later seen in [4].

2.3 Results

Using the results of the previous section one can compute the columns of the lepton mixing matrix which correspond to each possible Klein subgroup of a certain $\Delta(6n^2)$ group where n is even with $T = a$. The steps for this procedure are summarised as follows.

All Klein group elements in Eqs. (2.11)-(2.14) in the explicit $\mathbf{3}_2^1$ representation matrices given in Eq. (2.8) are generated, then each Klein group's elements are transformed to the basis where T is diagonal via V^l, cf. Eq. (2.22). Here, the eigenvectors with eigenvalue $+1$ correspond to the columns of possible mixing matrices as in Eq. (2.3). Since the ordering of the columns and rows of the mixing matrix calculated this way is arbitrary, without loss of generality the smallest absolute value from each mixing matrix is taken and assigned as V_{13} with its corresponding column being the third column of V. This completed procedure is unique up to interchanging the second and third rows of V, corresponding to two predictions for the atmospheric angle.

Implementing the preceding procedure for calculating the mixing matrix resulting from the Klein group in Eq. (2.11) with $T = a$ yields the *old* trimaximal mixing matrix [52, 53] which is given by the V^l in Eq. (2.21) up to permutation of its rows and columns. Clearly, this is not a phenomenologically viable mixing matrix, so this possibility is discarded.

One does not have to consider all the Klein groups in Eqs. (2.12)-(2.14) since they all result in identical PMNS matrices up to permutations of rows and columns. This is because the Klein group elements in Eq. (2.13) and Eq. (2.14) are related to G_i in Eq. (2.12) by $a^2 G_i a$ and $a G_i a^2$ respectively, where a and a^2 from Eq. (2.8) interchange rows and columns.

Thus, it is sufficient to consider the Klein subgroup given in Eq. (2.12), where the element $c^{n/2}$ becomes the "traditional" S generator in the basis in which T is diagonal,

$$S \to V^{l\dagger} c^{n/2} V^l = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{2.23}$$

This predicts one trimaximal middle column (TM_2), i.e. $(1,1,1)^T/\sqrt{3}$ [54, 55, 56], in lepton mixing [6]. This was also assumed in [43]. The other elements of the same

[6]Note that a Klein symmetry corresponding to V with a fixed column of $1/\sqrt{6}(2,-1,-1)^T$ (TM_1 mixing) cannot be identified as a subgroup of $\Delta(6n^2)$.

Klein subgroup also provide columns of V which is then up to the order of rows and columns given by

$$V = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos(\vartheta) & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin(\vartheta) \\ -\sqrt{\frac{2}{3}}\sin\left(\frac{\pi}{6}+\vartheta\right) & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\cos\left(\frac{\pi}{6}+\vartheta\right) \\ \sqrt{\frac{2}{3}}\sin\left(\frac{\pi}{6}-\vartheta\right) & -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\cos\left(\frac{\pi}{6}-\vartheta\right) \end{pmatrix}, \tag{2.24}$$

where $\vartheta = \pi\gamma'/n$ (cf. [43]). Since $\gamma' = 1, \ldots, n/2$, discrete predictions for the mixing angles corresponding to $\vartheta = \pi/n, \ldots, \pi/2$ are obtained. In general one cannot predict the order of the rows and columns with this method, so the entry with the smallest absolute value is picked and assigned to be $|V_{13}|$. Notice that for the different values of ϑ, different elements of Eq. (2.24) play the role of V_{13}. After V_{13} has been fixed, the second and third row can still be interchanged, leading to two different predictions for the atmospheric angle, corresponding to $\delta_{CP} = 0$ and $\delta_{CP} = \pi$, leading to the testable sum rules, $\theta_{23} = 45^\circ \mp \theta_{13}/\sqrt{2}$, respectively [25]. (Note that Klein subgroups do not predict Majorana phases which would correspond to a matrix $K = \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$ multiplied onto Eq. 2.24 from the right.) These sum rule relations follow from considering the atmospheric angle sum rule given in [57] for the cases $\delta_{CP} = 0, \pi$. The sum rule $\theta_{23} = 45^\circ - \theta_{13}/\sqrt{2}$ was also proposed in [58] in a different context.

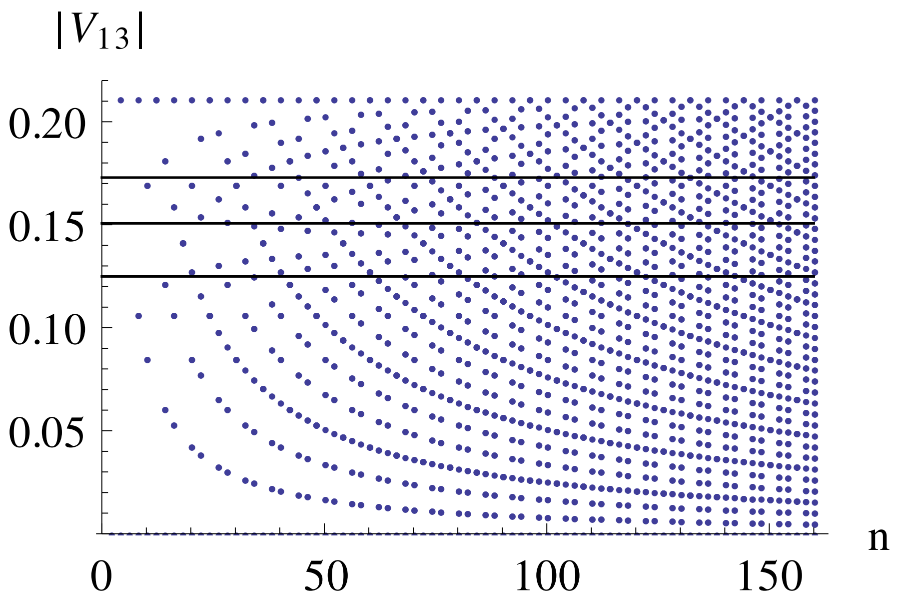


Figure 2.1: The possible values that $|V_{13}|$ can take in $\Delta(6n^2)$ flavour symmetry groups with even n. Examples include $|V_{13}| = 0.211, 0.170, 0.160, 0.154$ for $n = 4, 10, 16, 22$, respectively. The lines denote the present approximate 3σ range of $|V_{13}|$.

Fig. 2.1 shows all possible predictions for $|V_{13}|$ corresponding to the different Klein subgroups for each $\Delta(6n^2)$ of even n one obtains using the method previously discussed.[7] As n increases, the number of possible values of $|V_{13}|$ predicted by $\Delta(6n^2)$ also increases according to the above discussion.

2.4 Conclusions

In this chapter, predictions of lepton mixing parameters for direct models based on $\Delta(6n^2)$ flavour symmetry groups for arbitrarily large n in which the full Klein symmetry is identified as a subgroup of the flavour symmetry were obtained. After reviewing and developing the group theory associated with $\Delta(6n^2)$, some known results of the at the time recent numerical searches are reviewed here and many new possible mixing patterns for large n able to yield lepton mixing angle predictions within 3σ of recent global fits were found. Previously, $\Delta(6n^2)$ had only been analysed within particular scans up to a much lower order than considered here. All the examples predict exact TM_2 mixing with oscillation phase zero or π corresponding to two possible predictions for the atmospheric angle but differ in the prediction of $|V_{13}|$ as shown in FIG. 2.1.

For large n, it is clear that the predictions for $|V_{13}|$ densely fill the allowed range. Nevertheless, this general method of analysing $\Delta(6n^2)$ flavour symmetry groups is of interest since it represents for the first time a model independent treatment of an infinite class of theories. The general predictions for the considered class of theories based on $\Delta(6n^2)$ are Majorana neutrinos, trimaximal lepton mixing with reactor angle fixed up to a discrete choice, an oscillation phase of either zero or π and sum rules $\theta_{23} = 45° \mp \theta_{13}/\sqrt{2}$, respectively, which are consistent with the recent global fits and will be tested in the near future.

Some time after [1] was published, it was shown, [60], using (even) more mathematical methods, that for 3 Majorana neutrinos, in direct models indeed the only remaining flavour symmetry groups in which the residual symmetries may be embedded into are $\Delta(6n^2)$ and $(Z_m \times Z_{m/3}) \rtimes S_3$, depending on the choices of n and m. For further details see [60].

[7] The group with $n = 42$ produces no predictions within the three sigma range, contrasting well-regarded hints in the literature [59].

3

Lepton mixing predictions including Majorana phases from $\Delta(6n^2)$ flavour symmetry and general CP

The work presented in this chapter has been partially published in [2]. The contribution of the author to the research presented here consisted in performing all calculations and writing the majority of [2]. In the following, the results presented in the previous chapter are extended by also considering the effect of residual CP symmetries. For this, first an argument will be given concerning which general CP transformations one should consider.[1] After that, the mixing matrices that are allowed by the different cases of residual symmetries are presented.

This work is mostly motivated by the fact that general CP transformations are the only known framework which allows to predict Majorana phases in a flavour model purely from symmetry. Furthermore, it is the first time that general CP transformations are investigated for an infinite series of finite groups, namely again $\Delta(6n^2) = (Z_n \times Z_n) \rtimes S_3$. While in direct models the mixing angles and Dirac CP phase are solely predicted from symmetry and $\Delta(6n^2)$ flavour symmetry provides many examples of viable predictions for mixing angles, the Majorana phases remain entirely unconstrained by the pure flavour symmetry.

[1]This argument as published in [2] turned out to be incomplete and it will be remarked where necessary in the following, where gaps turned up. However, the mixing prediction remain valid.

As in the previous chapter, for all groups the predicted mixing matrix has a trimaximal middle column and the Dirac CP phase is 0 or π. The Majorana phases are predicted from residual flavour and CP symmetries where α_{21} can take several discrete values for each n and the Majorana phase α_{31} is a multiple of π. In the second half of this chapter, first, constraints constraints on the groups and CP transformations from measurements of the neutrino mixing angles were discussed. After that, as it is the most accessible observable for Majorana phases, also the constraints from neutrinoless double-beta decay were analysed.

3.1 Introduction

The question of the origin of neutrino masses and mixing parameters is of fundamental importance. One approach are so-called direct models of neutrino masses [25] where a discrete non-Abelian flavour symmetry group is broken to a $Z_2 \times Z_2$ group in the Neutrino sector, and a Z_3 subgroup in the charged lepton sector. In such a model the lepton mixing angles and the lepton Dirac CP phase are completely fixed by symmetry.

Recently such direct models have been analysed with the help of the group database GAP [41, 43]. The only flavour groups that can produce viable mixing parameters in a direct model belong to the group series $\Delta(6n^2)$ or are subgroups of such groups. The group theory of $\Delta(6n^2)$ groups has been analysed in [49]. The consequences for neutrino mixing from a $\Delta(6n^2)$ flavour symmetry in direct models have been studied in detail in [1] for arbitrary even n. Some examples of $\Delta(6n^2)$ groups or subgroups have previously been studied in [37, 38, 39, 40, 48, 61, 62, 63].

In the Standard Model, violation of CP occurs in the flavour sector. Promoting CP to a symmetry at high energies which is then broken allows to impose further constraints on mass matrices of charged leptons and Majorana neutrinos. In this case the interplay between CP and flavour symmetries has to be carefully discussed[64, 65, 66, 30, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77]. For direct models, especially with a flavour group from $\Delta(6n^2)$, CP symmetries have not been studied in detail yet.

In the following, first, a class of general CP (gCP) transformations consistent with $\Delta(6n^2)$ groups for arbitrary n will be examined. After reviewing flavour and general CP transformations and stating their effect on mass matrices, in the following section the general theory of gCP transformations in the presence of flavour symmetries in a general context is reviewed and developed. Afterwards direct models with $\Delta(6n^2)$ as a flavour group are considered, where the lepton mixing matrix including Majorana phases for arbitrary even n for all possible breaking patterns of the flavour group and of the considered class of gCP transformations is computed. Here also the constraints from measurements of the mixing angles and from neutrinoless double-beta-decay on these models were analysed. The last section concludes this chapter.

3.2 General CP transformations, flavour symmetries, automorphisms and the character table

In this section the interplay between flavour symmetries and CP symmetries is reviewed, which has especially been discussed in [64, 30, 68, 70, 73] and general arguments are used to show that for a class of groups G, of which $G = \Delta(6n^2)$ is an example, physical CP transformations correspond to $X_r \in e^{i\alpha}G$ with α a real number.[2]

3.2.1 General CP transformations and flavour symmetries

Consider a theory where generations of fermions are assigned to multiplets of representations r of a flavour group G and that is invariant under transformations of the multiplets φ_r under the group G

$$\varphi_r \mapsto \rho_r(g)\varphi_r \tag{3.1}$$

where $\rho_r(g)$ is the representation matrix for $g \in G$ in the representation r.

Further consider the group G being broken to a Klein subgroup $G_\nu \simeq Z_2 \times Z_2$ subgroup in the neutrino sector and an abelian subgroup $G_e \simeq Z_m$ with $m > 2$ in the charged lepton sector. If these subgroups remain unbroken at all energies, in the

[2]The more correct way of stating this, is that $X_r \in e^{i\alpha}G$ are definitely consistent CP transformations. However, the argument is not complete in showing that these are in fact all consistent CP transformations.

low-energy-limit constraints on the mass matrices of charged leptons and neutrinos are imposed. Left-handed doublets transform under the same representation r. The charged lepton mass matrix M^l has to fulfil

$$\rho_r(g)^\dagger M^l(M^l)^\dagger \rho_r(g) = M^l(M^l)^\dagger \tag{3.2}$$

with $\rho_r(g)$ being the representation matrix of $g \in G_e$ in the representation r. The Majorana neutrino mass matrix is constrained by

$$\rho_r(g)^T M^M \rho_r(g) = M^M \tag{3.3}$$

with $g \in G_\nu$.

Define general CP (gCP) by

$$\varphi_r \mapsto X_r(\varphi_r^*(x^P)) \tag{3.4}$$

where r is the representation of G according to which φ_r transforms. [3] X_r is a unitary matrix. One needs to find all matrices X_r that are "allowed" in coexistence with a flavour group G. The aforesaid will be made a more precise statement in the following section, where the conditions for the existence of gCP transformations as well as their properties will be discussed.

If the theory at the low-energy end is invariant under residual gCP transformations with matrices X_r^l for charged leptons and X_r^ν for neutrinos then the mass matrices will be constrained by

$$X_r^{l\dagger} M^l(M^l)^\dagger X_r^l = (M^l)^*(M^l)^T \tag{3.5}$$

for charged leptons and by

$$X_r^{\nu T} M^M X_r^\nu = (M^M)^* \tag{3.6}$$

for Majorana neutrinos.

If $X_r^\nu \in G_\nu$ ($X_r^l \in G_e$), no new constraints on the neutrino (charged lepton) mass matrix follow but it being real. With $g, h \in (Z_2 \times Z_2)$ from $\rho_r(g)X_r\rho_r(h)$ only the same constraints as for X_r follow for the mass matrix. This means only X_r that

[3] Other Authors consider transformations of the type $\varphi_r \mapsto \varphi_{r'}^*$, where r, r' can be different. In [64] has been shown that only gCP transformations where $r = r'$ actually make observables (e.g. particle decays) conserve CP.

are not in $(Z_2 \times Z_2)$ allow for a mass matrix that is not real and at the same time impose new constraints on it.

3.2.2 The consistency equation

One would like to know which transformations of the type

$$\varphi_r \mapsto X_r \varphi_r^*(x^P) \tag{3.7}$$

can be applied to the theory without destroying the invariance under G, i.e. which matrices X_r can appear in Eq. (3.7) that preserve symmetry under G? Consider performing a gCP transformation followed by a flavour transformation followed by the inverse gCP transformation. From invariance of the theory under G follows that the matrix X_r is allowed in a gCP transformation if for every $g \in G$ there is a $g' \in G$ such that

$$X_r \rho_r^*(g) X_r^\dagger = \rho_r(g'). \tag{3.8}$$

Eq. (3.8) is called the consistency equation and an X_r that fulfils it is called consistent with G.

If r is a faithful representation, which is equivalent to saying that ρ_r is injective, one can define a bijective mapping $u_X : G \to G$ between the elements of the group:

$$u_X(g) := \rho_r^{-1}(X_r \rho_r^*(g) X_r^\dagger). \tag{3.9}$$

(One can drop the index r on u_{X_r} because for all faithful irreps the mapping generated by Eq. (3.9) will be the same). For faithful representations r, $u_X(g)$ is an automorphism of the group G.

3.2.3 Inner and outer automorphisms

Group automorphisms come in two kinds: Inner and outer automorphisms. Inner automorphisms $Inn(G)$ are such automorphisms $u : G \to G$ where for all $g \in G$ one single group element h_u exists such that

$$u(g) = h_u^{-1} g h_u. \tag{3.10}$$

All inner automorphisms are given by $Inn(G) = G/Z(G)$, where $Z(G)$ is the center of G, i.e. all elements of G that commute with every other group element. Outer automorphisms $Out(G)$ are all automorphisms that are not inner.

An inner automorphism will map each element into its original conjugacy class. An outer automorphism however is not inner which means that there is at least one $g' \in G$ for which with all $h \in G$ $u(g) \neq h^{-1}g'h$ (compare with the definition of inner automorphisms before Eq. (3.10)), i.e. there is at least one $g' \in G$ which is not mapped back into its original conjugacy class.[4] Also if g is in the class C_k and it is mapped onto $u(g)$ which is in the class C_l, every element in C_k is mapped on an element in C_l by u.

This proves also that an automorphism that maps each element back into its original conjugacy class is inner, as well that an automorphism that maps elements from at least two conjugacy classes on each other is outer. [5]

Now return to the automorphism u_X (3.9) that is induced by the consistency equation (3.8). If $\rho_r(g)$ is real and $X_r \in G$ then u_X will be an inner automorphism. This is also true if $X_r \in e^{i\alpha}G$.

[4]This argument neglects that in fact outer automorphisms can exist that map elements back into the same class.

[5] An outer automorphism u also generates mappings between different representations of G. For two representations ρ_r and ρ_s define

$$u_{sr} = \rho_s \circ u \circ \rho_r^{-1} \tag{3.11}$$

with which follows

$$(u_{sr} \circ \rho_r)(g) = \rho_s(u(g)). \tag{3.12}$$

The outer automorphism u acting inside the group thus interchanges columns of the character table while when acting between representations via u_{sr} interchanges rows of the character table. We call a symmetry of the character table

$$\chi_{jk} = \mathrm{tr}\rho_j(g_k), \; g_k \in C_k \tag{3.13}$$

any transformation of the type

$$\chi_{jk} \mapsto P_{ij}\chi_{kl}Q_{kl} \tag{3.14}$$

with permutation matrices P and Q that leaves χ invariant, i.e.

$$P_{ij}\chi_{kl}Q_{kl} = \chi_{ij} \tag{3.15}$$

and where only classes of the same size and element-order are interchanged, i.e $|C_l| = |C_j|$ and $\mathrm{ord}g_l = \mathrm{ord}g_j$ for $g_l \in C_l$ and $g_j \in C_j$. An outer automorphism will always generate a non-trivial symmetry of the character table, just as a symmetry of the character table always gives rise to an outer automorphism: Define the automorphism by the action on the conjugacy classes, a corresponding permutation of the representations is always given by any outer automorphism via u_{sr}.

If on the other hand u is an outer automorphism it follows that a matrix X_r that could mediate u á la Eq. (3.9) is not in $e^{i\alpha}G$ (if it exists).

One could ask now if there can be a matrix $\tilde{X}_r$ that is not in $e^{i\alpha}G$ for that $u_{\tilde{X}}$ only connects elements within the same conjugacy class, i.e. that generates an inner automorphism? As for an inner automorphism u there always is a single $h_u \in G$ such that the automorphism is given by $u(g) = h_u^{-1} g h_u$ it follows that

$$\tilde{X}_r \rho_r^*(g_k) \tilde{X}_r^\dagger = \rho_r(h_u)\rho_r(g_k)\rho_r(h_u^{-1}). \tag{3.16}$$

For a real matrix $\rho_r(g)$ multiplying by $\tilde{X}_r$ from the right and by $\rho_r(h_u^{-1})$ from the left yields

$$\rho_r(h_u^{-1})\tilde{X}_r\rho_r(g_k) = \rho_r(g_k)\rho(h_u^{-1})\tilde{X}_r. \tag{3.17}$$

As g_k can be every element of G, $\rho_r(h_u^{-1})\tilde{X}_r$ commutes with every group element. One can now apply Schur's Lemma [6] to find that

$$\tilde{X}_r = \lambda \rho_r(h_u) \tag{3.18}$$

where $|\lambda| = 1$ to keep $\tilde{X}_r$ unitary. As $\tilde{X}_r$ was supposed to not be in $e^{i\alpha}G$ this is in contradiction to the assumptions.[7] For real $\rho_r(g)$ this proves that inner automorphisms correspond to $X \in e^{i\alpha}G$. For real representations, there is always a basis where this is the case, i.e where $\rho_r(g)$ is real for every $g \in G$.

If $\rho_r(g)$ is complex one has to deal with complex conjugation: Assume there is a matrix w_r such that by applying complex conjugation and this matrix on an element of G, the element is mapped into the class of its inverse, $C(g^{-1})$:

$$\rho_r(g) \mapsto w_r^\dagger \rho_r(g)^* w_r \in C(g^{-1}). \tag{3.19}$$

This can be thought of as an automorphism mapping $g \mapsto g^{-1}$ followed by an automorphism that maps g^{-1} onto another element in the same class. As in the second step every element is sent into the original class, this second mapping is an inner automorphism[8] and therefore by definition a single group element h exists

[6] To be precise one uses the second part of Schur's Lemma which states that an operator that in some representation commutes with every group element is proportional to the identity.

[7] Again, here it was neglected that outer automorphisms could exist that send every element back into its class and the matrix $\tilde{X}_r$ could precisely correspond to such a class-preserving outer automorphism. However, if one happens to know that the group G has actually no class-preserving outer automorphisms, then the argument holds.

[8] Again, also this mapping could be a class-preserving outer automorphism, in which case a matrix facilitating the mapping (if it even exists) would not be in G.

which inverts this step such that

$$\rho_r(h)^\dagger(w_r^\dagger \rho_r(g)^* w_r)\rho_r(h) = g^{-1}. \tag{3.20}$$

For this reason it is assumed in the following that the matrix w_r maps elements directly onto their inverses. Using this, the general mapping induced by the consistency equation is given by:

$$u_X(g) = \rho_r^{-1}(X_r w_r \rho_r(g^{-1}) w_r^\dagger X_r^\dagger) \tag{3.21}$$

This mapping can be seen as an automorphism mapping g on g^{-1} followed by an automorphism given by $X_r w_r$:

$$u_X(g) = u_{Xw}(g^{-1}). \tag{3.22}$$

If both w_r and X_r are contained in $e^{i\alpha}G$, u_X will map g in the same conjugacy class as g^{-1}. For $\Delta(6n^2)$, $w_r = \rho_r(b)$ maps elements into the class of the inverse[9] and is contained in the group. $w_r \notin G$ was thus not considered further.

Analogous to real irreps above one can now ask if there can be matrices $\tilde{X}_r$ that are not in $e^{i\alpha}G$ but that with $w_r \in e^{i\alpha}G$ will map g in the conjugacy class of g^{-1}? This would be equivalent to $u_{\tilde{X}w}$ being an inner automorphism[10] which would mean that for each group element $g \in G$ there is a single $h_u \in G$ such that

$$\rho_r(h_u)\rho_r(g^{-1})\rho_r(h_u^{-1}) = X_r w_r \rho_r(g^{-1}) w_r^\dagger X_r^\dagger. \tag{3.23}$$

Again one can use Schur's Lemma and finds there is $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$X_r = \lambda \rho_r(h_u) w_r^\dagger \tag{3.24}$$

with $|\lambda| = 1$ to make X_r unitary. This contradicts $X_r \notin e^{i\alpha}G$. We have proved now that if $w_r \in e^{i\alpha}G$ then if and only if $X \in e^{i\alpha}G$ $u_X(g)$ will be in the conjugacy class of g^{-1}.[11] In [64] it was shown that only gCP transformations that map elements into the class of its inverse element make observables conserve CP.[12] It

[9] This was later found to only hold for $3 \nmid n$, as will be discussed in section 4.2. However by excluding unfaithful 2-dimensional representations of $\Delta(6n^2)$ from the discussion, the automorphism generated by b becomes class-inverting again.

[10] Again, $\tilde{X}_r$ could be a class-preserving outer automorphism.

[11] If G has no class-preserving outer automorphisms.

[12] This holds only if one considers all irreps of G simultaneously. For a specific model with a limited representation content, further CP transformations can be consistent with G, especially if the model is renormalisable and not even in Kronecker products all irreps have to appear.

was thus proved here that such transformations are given by $X_r \in e^{i\alpha}G$.[13] [14] In the following G is specialised to be $\Delta(6n^2)$.

3.3 gCP Symmetries and $\Delta(6n^2)$ groups

In this section the effect of gCP transformations where $X \in e^{i\alpha}G$ for $G = \Delta(6n^2)$ on mass and mixing matrices is considered. First the gCP transformations that are consistent with $G_\nu = Z_2 \times Z_2$ and $G_2 = Z_3$ are derived. Afterwards the constrained mass matrices and the lepton mixing matrix are stated. After this constraints from measurements of lepton mixing angles and from neutrinoless double-beta decay for arbitrary n are discussed.

If one wants to break the flavour symmetry to $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$ subgroups, the residual flavour and residual gCP transformations are not independent, as they still have to fulfil the consistency equation.[15] If e.g. in one sector $\rho_r(g)$ and X_r are unbroken, then also $X_r \rho_r(g)^* X_r^\dagger$ must be unbroken. Thus the allowed residual gCP transformations have to map elements from the Klein group in consideration into said Klein group.

The Klein subgroups of $\Delta(6n^2)$ are given by [1]

$$\{1, c^{n/2}, d^{n/2}, c^{n/2}d^{n/2}\}, \tag{3.25}$$

$$\{1, c^{n/2}, abc^{\gamma}, abc^{\gamma+n/2}\}, \tag{3.26}$$

$$\{1, d^{n/2}, a^2bd^{\delta}, a^2bd^{\delta+n/2}\}, \tag{3.27}$$

$$\{1, c^{n/2}d^{n/2}, bc^{\epsilon}d^{\epsilon}, bc^{\epsilon-n/2}d^{\epsilon-n/2}\}, \tag{3.28}$$

where $\gamma, \delta, \epsilon = 1, \ldots, n/2$. The group Eq. (3.25) will produce a mixing matrix with $|V_{ij}| = 1/\sqrt{3}$, and it will not be considered further. The bottom three Klein subgroups will generate the same mixing matrix, thus it is sufficient to only consider the mixing matrices generated by group Eq. (3.26). The allowed matrices X_r in the low-energy-limit have to be contained in $e^{i\alpha}G_\varphi$. A matrix X_r is allowed if for a Klein subgroup K holds that for each $g \in K$ also $u(g) \in K$. For said

[13] Under the assumptions mentioned in the various remarks concerning things that were only understood after [2] was published.

[14] One would now be able to find all $X_r \notin e^{i\alpha}G$ by reading off all automorphisms from the symmetries of the character table that do not map the class of g on the class of g^{-1}. (This would often contain the identity transformation on the character table.)

[15] As was shown in section 1.5.1, this condition is not strong enough and will still allow for residual CP transformations that force mass eigenvalues to be zero or degenerate. As this is unphysical

Klein subgroup $K = \{1, c^{n/2}, abc^{\gamma}, abc^{\gamma+n/2}\}$ one finds that the allowed matrices $X \in e^{i\alpha}G$ are given by the representation matrices for

$$X_r = \rho_r(e^{i\alpha}c^x d^{2x+2\gamma}), \rho_r(e^{i\alpha}c^x d^{2\gamma+2x+n/2}), \rho_r(e^{i\alpha}abc^x d^{2x}), \rho_r(e^{i\alpha}abc^x d^{2x+n/2}) \tag{3.29}$$

with $\alpha \in \mathbb{R}$ and $x = 0, \ldots, n-1$.

Without loss of generality, left-handed doublets $(\nu_L, e_L)^T$ are assigned to the representation 3_2^1 (cf.[1]). Invariance of the mass matrix under the Klein subgroup in consideration plus invariance under one of the transformations from Eq. (3.29) constrains the Majorana neutrino mass matrix to

$$M_\nu = \begin{pmatrix} |m_{22}|e^{2i\pi\frac{\gamma}{n}}e^{i\varphi_1} & |m_{21}|e^{i\varphi_1} & 0 \\ |m_{21}|e^{i\varphi_1} & |m_{22}|e^{-2i\pi\frac{\gamma}{n}}e^{i\varphi_1} & 0 \\ 0 & 0 & |m_{33}|e^{i\varphi_3} \end{pmatrix} \tag{3.30}$$

where the values of φ_1 and φ_3 can be found in table (3.1). In principle, several gCP transformations can remain unbroken. However, the phases φ_1, φ_3 are already fixed by one single unbroken transformation. Leaving a second gCP transformation unbroken with incompatible constraints on the phase φ_i will force the corresponding mass parameters $|m_{..}|$ to be zero. The masses of neutrinos are $|m_{33}|$ and $||m_{21}| \pm |m_{22}||$. Thus $|m_{21}| = 0$ or $|m_{22}| = 0$ will result in a pair of degenerate neutrino states. It is not possible to have $|m_{33}| = 0$ without $|m_{21}| = 0$ or $|m_{22}| = 0$. Leaving a second gCP transformation unbroken is never physically viable.

X_r	φ_1	φ_3
$\rho_r(e^{i\alpha}c^x d^{2x+2\gamma})$	$-\alpha - 2\pi(\gamma + x)/n$	$-\alpha + 4\pi(\gamma + x)/n$
$\rho_r(e^{i\alpha}c^x d^{2\gamma+2x+n/2})$	$-\alpha - \pi/2 - 2\pi(\gamma + x)/n$	$-\alpha + \pi + 4\pi(\gamma + x)/n$
$\rho_r(e^{i\alpha}abc^x d^{2x})$	$-\alpha - 2\pi x/n$	$-\alpha + 4\pi x/n$
$\rho_r(e^{i\alpha}abc^x d^{2x+n/2})$	$-\alpha - \pi/2 - 2\pi x/n$	$-\alpha + \pi + 4\pi x/n$

Table 3.1: Values of φ_1 and φ_3 for gCP transformations consistent with the residual Klein symmetry

The neutrino mass matrix Eq. (3.30) will be diagonalised by a unitary matrix U_ν via $U_\nu^T M_\nu U_\nu$. A matrix U_ν such that the diagonalised mass matrix is real and positive is given by

$$U_\nu^{(+)} = \begin{pmatrix} -\frac{e^{i\left(-\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & \frac{e^{i\left(-\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & 0 \\ \frac{e^{i\left(\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & \frac{e^{i\left(\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & 0 \\ 0 & 0 & e^{-\frac{i\varphi_3}{2}} \end{pmatrix} \tag{3.31}$$

for $|m_{21}| > |m_{22}|$ and by

$$U_\nu^{(-)} = \begin{pmatrix} -\frac{e^{i\left(\frac{-\pi\gamma}{n}-\frac{\varphi_1}{2}+\frac{\pi}{2}\right)}}{\sqrt{2}} & \frac{e^{i\left(-\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & 0 \\ \frac{e^{i\left(\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}+\frac{\pi}{2}\right)}}{\sqrt{2}} & \frac{e^{i\left(\frac{\pi\gamma}{n}-\frac{\varphi_1}{2}\right)}}{\sqrt{2}} & 0 \\ 0 & 0 & e^{-\frac{i\varphi_3}{2}} \end{pmatrix} \tag{3.32}$$

for $|m_{21}| < |m_{22}|$.

For charged leptons, the allowed gCP transformations with $X_r \in e^{i\alpha}G$ have to be consistent with $G_e = \{1, a, a^2\}$ and are given by

$$X_r = c^y d^{-y}, ac^y d^{-y}, a^2 c^y d^{-y}, bc^y d^{-y}, abc^y d^{-y}, a^2 c^y d^{-y} \tag{3.33}$$

where $3y = 0 \bmod n$. Especially when 3 divides n there is a huge number of allowed X matrices. But, as the charged lepton mass matrix is already invariant under transformations with a and transformations with $c^y d^{-y}$ force it to be zero (for $3y \neq 0 \bmod n$) or produce no new constraint (for $3y = 0 \bmod n$), the only transformations that produce physical constraints are given by

$$X_r = \rho_r(1), \rho_r(b). \tag{3.34}$$

For $X_r = \rho_r(1)$ the mass matrix of charged leptons is restrained to

$$M_{l1} M_{l1}^\dagger = \begin{pmatrix} m_3^l & m_1^l & m_2^l \\ m_2^l & m_3^l & m_1^l \\ m_1^l & m_2^l & m_3^l \end{pmatrix} \tag{3.35}$$

with all parameters being real or for $X_r = \rho_r(b)$ to

$$M_{lb} M_{lb}^\dagger = \begin{pmatrix} m_3^l & m_1^l & (m_1^l)^* \\ (m_1^l)^* & m_3^l & m_1^l \\ m_1^l & (m_1^l)^* & m_3^l \end{pmatrix} \tag{3.36}$$

with m_1^l complex and m_3^l real. Both charged lepton mass matrices can be diagonalised by

$$U^l = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix} . \tag{3.37}$$

Above charged lepton mass matrices only differ by unphysical phases which can be absorbed into the charged lepton fields.

After removing an overall phase $e^{-i\varphi_1/2}$ to render the top left entry real, the physical mixing matrix is given by $U_{\text{PMNS}}^{(+)/(-)} = (U_e)^\dagger U_\nu^{(+)/(-)}$ (For $U_\nu^{(+)}$ and $U_\nu^{(-)}$ cf. Eq. (3.31) and Eq. (3.32)):

$$U_{\text{PMNS}}^{(+)/(-)} = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\left(\frac{\pi\gamma}{n}\right) & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin\left(\frac{\pi\gamma}{n}\right) \\ -\sqrt{\frac{2}{3}}\sin\left(\pi\left(\frac{\gamma}{n}+\frac{1}{6}\right)\right) & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\cos\left(\pi\left(\frac{\gamma}{n}+\frac{1}{6}\right)\right) \\ \sqrt{\frac{2}{3}}\sin\left(\pi\left(\frac{1}{6}-\frac{\gamma}{n}\right)\right) & -\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\cos\left(\pi\left(\frac{1}{6}-\frac{\gamma}{n}\right)\right) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & [i]ie^{-i6\pi(\gamma+x)/n} & 0 \\ 0 & 0 & [i]i \end{pmatrix} \tag{3.38}$$

where the additional factors of i in square brackets only appear in $U_{\text{PMNS}}^{(-)}$. As the ordering of the mixing matrix is arbitrary at this point, one would like to fix it by requiring that the smallest entry of the matrix has to be the top-right entry, i.e. U_{13}. For small γ/n the first row and third column are in the right place in the above matrix.

As this matrix is now in the PDG convention, the values of Majorana phases α_{21} and α_{31} as well as the Dirac CP phase δ_{CP} for this ordering of the mixing matrix can be read off the matrix. Recall that the PDG convention is $U_{\text{PMNS}} = R_{23}U_{13}R_{12}P$ in terms of $s_{ij} = \sin(\theta_{ij})$, $c_{ij} = \cos(\theta_{ij})$, the Dirac CP violating phase δ_{CP} and further Majorana phases contained in $P = \text{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})$.

The Majorana phase α_{21} is then given by

$$\alpha_{21} = \varphi_1 - \varphi_3 \tag{3.39}$$

With table [3.1] follows that

$$\varphi_1 - \varphi_3 = -\frac{6\pi(\gamma+x)}{n} \text{ for } X = c^x d^{2x+2\gamma}, abc^x d^{2x} \tag{3.40}$$

or

$$\varphi_1 - \varphi_3 = -\frac{3\pi}{2} - \frac{6\pi(\gamma+x)}{n} \text{ for } X = c^x d^{2x+2\gamma+n/2}, abc^x d^{2x+n/2}. \tag{3.41}$$

The values of all CP phases depend on the ordering of Eq. (3.38) which needs to be changed for higher values of γ/n. The possible values of the CP phases can be found in table (3.2). There, U' denotes the mixing matrix after reordering such that the entry with the smallest absolute value is in the top right corner. As for every γ/n the second and third row can be interchanged, which results in changing δ_{CP} by π while changing the prediction for U_{23} and U_{33} and thus the prediction for θ_{23}. The Dirac CP phase is hence predicted to be 0 or π, and since the lepton mixing matrix has the tri-maximal form for the second column, referred to as TM2, this leads to the mixing sum rules $\theta_{23} = 45° \mp \theta_{13}/\sqrt{2}$ for $\delta_{CP} = 0, \pi$, respectively, as previously noted in [1] (for a review of sum rules see [25]).

The prediction of α_{31} also depends on the order of these rows. In the table (3.2) the second row of the mixing matrix after reordering it is indicated in the column U'_{23}. Improved measurements of θ_{23} will constrain this freedom of interchanging the second and third row.

γ/n	U'_{13}	U'_{23}	$\delta_{CP}^{(-)/(+)}$	$\alpha_{21}^{(-)}$	$\alpha_{21}^{(+)}$	$\alpha_{31}^{(-)}$	$\alpha_{31}^{(+)}$
$0/12\ldots1/12$	U_{13}	U_{23}	0	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3$	2π	$-\pi$
	U_{13}	U_{33}	$-\pi$	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3$	0	π
$1/12\ldots2/12$	U_{31}	U_{21}	0	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3-\pi$	2π	π
	U_{31}	U_{11}	$-\pi$	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3-\pi$	0	$-\pi$
$2/12\ldots3/12$	U_{31}	U_{11}	0	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3-\pi$	0	$-\pi$
	U_{31}	U_{21}	$-\pi$	$\varphi_1-\varphi_3$	$\varphi_1-\varphi_3-\pi$	2π	π
$3/12\ldots4/12$	U_{23}	U_{13}	0	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+2\pi$	0	π
	U_{23}	U_{33}	$-\pi$	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+2\pi$	2π	$-\pi$
$4/12\ldots5/12$	U_{23}	U_{33}	0	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+2\pi$	2π	$-\pi$
	U_{23}	U_{13}	$-\pi$	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+2\pi$	0	π
$5/12\ldots6/12$	U_{11}	U_{31}	0	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+\pi$	2π	π
	U_{11}	U_{21}	$-\pi$	$\varphi_1-\varphi_3+2\pi$	$\varphi_1-\varphi_3+\pi$	0	$-\pi$

Table 3.2: Values of CP phases after reordering for different values of γ/n in $U_{\text{PMNS}}^{(-)/[(+)]}$. In each row, γ/n can take arbitrary values in the interval indicated. U' denotes the matrix after reordering.

The key observable for Majorana phases is neutrino-less double beta decay ($0\nu\beta\beta$). The effective mass of neutrinoless double-beta decay is given by

$$|m_{ee}| = |\frac{2}{3}m_1\cos^2(\frac{\pi\gamma'}{n}) + \frac{1}{3}m_2 e^{i\alpha_{21}} + \frac{2}{3}m_3\sin^2(\frac{\pi\gamma'}{n})e^{i(\alpha_{31}-2\delta)}| \tag{3.42}$$

with

$$m_1 = m_l \ , \ m_2 = \sqrt{m_l^2 + \Delta m_{21}^2} \ , \ m_3 = \sqrt{m_l^2 + \Delta m_{31}^2} \tag{3.43}$$

for normal ordering and

$$m_1 = \sqrt{m_l^2 + \Delta m_{31}^2} \ , \ m_2 = \sqrt{m_l^2 + \Delta m_{21}^2 + \Delta m_{31}^2} \ , \ m_3 = m_l \tag{3.44}$$

for inverted ordering, where m_l is the mass of the lightest neutrino and

$$\gamma' = \gamma \bmod \frac{1}{6}. \tag{3.45}$$

The absolute values of the entries of the mixing matrix after reordering are periodic in γ/n which is why one can simplify the analysis by defining γ' in this way.

There are 8 cases to distinguish for combinations of phases. Adding a multiple of 2π will not change the effect of α_{21} or $\alpha_{31} - 2\delta$. For this reason, for both Eq. (3.40) and Eq. (3.41) the 12 cases in table (3.2) reduce to 8 cases of values for

$$\bar{\alpha}_{21} = \alpha_{21} + 6\pi\frac{\gamma + x}{n} \ , \ \bar{\alpha}_{31} = \alpha_{31} - 2\delta \tag{3.46}$$

that are given by

$$(\bar{\alpha}_{21}, \bar{\alpha}_{31}) = (0,0), (\pi/2,0), (\pi,0), (3\pi/2,0), (0,\pi), (\pi/2,\pi), (\pi,\pi), (3\pi/2,\pi). \tag{3.47}$$

The by far most stringent constraint on γ/n comes from the measurement of θ_{13}. The current 3 sigma range for θ_{13} from [78] yields values of γ'/n in the range $0.0460 \ldots 0.0627$.

It is generally fine to not only consider γ'/n in this range but even γ/n because changing γ by $1/6$ only changes α_{21} by π, which is included in the four cases discussed above.

In order to understand predictions of $\Delta(6n^2)$ groups for $0\nu\beta\beta$ decay on a general level, in Figure 3.1, the effective mass $|m_{ee}|$ of $0\nu\beta\beta$ is plotted against the mass of the lightest neutrino m_l for all combinations of $\bar{\alpha}_{21}$ and $\bar{\alpha}_{31}$. In these plots, models defined by some values of γ/n and x/n correspond to single fine lines. γ/n takes 11 values, starting with the 3 sigma lower bound and increases in 10 equal steps until it reaches the 3 sigma upper bound. x/n takes values $0, 0.1, 0.2, \ldots, 1$.

Δm^2_{21} and Δm^2_{31} are not varied, as doing so only would almost unnoticeably broaden each single line. Instead the best fit value from [78] was used:

$$\Delta m^2_{21} = 7.54 \times 10^{-5}\ \mathrm{eV}^2, \tag{3.48}$$

$$\Delta m^2_{31} = 2.41 \times 10^{-3}\ \mathrm{eV}^2. \tag{3.49}$$

In Figure 3.1, magenta lines correspond to predictions assuming inverted hierarchy, red lines to normal hierarchy. Dashed blue and yellow lines indicate the currently allowed three sigma region for normal and inverted hierarchy, respectively. The three sigma ranges for mixing angles are taken from [78]. The upper bound $|m_{ee}| < 0.140$ eV is given from measurements by the EXO-200 experiment [79]. Planck data in combination with other CMB and BAO measurements [80] provides a limit on the sum of neutrino masses of $m_1 + m_2 + m_3 < 0.230$ eV from which the upper limit on the mass of the lightest neutrino can be derived.

The main features of the results from Figure 3.1 are as follows:

- For inverted hierarchy there is no particular structure visible. Additionally, the predicted values for $|m_{ee}|$ are well within the reach of e.g. phase III of the GERDA experiment of $|m_{ee}^{\mathrm{exp}}| \sim 0.02 \ldots 0.03$ eV [81].
- For normal ordering, it follows from Figure 3.1 that for the values of γ/n and x/n considered is always a lower limit on $|m_{ee}|$ which means that these parameters are accessible to future experiments.[16]
- Further for normal ordering, in the very low m_{lightest} region, predicted values of $|m_{ee}|$ are closer to the upper end of the blue three sigma range.
- With the current data, no combination of $\bar{\alpha}_{21}$ and $\bar{\alpha}_{31}$ is favoured. Only for values of $|m_{ee}| \lesssim 0.0001$ eV and $m_{\mathrm{lightest}} \lesssim 0.01 \ldots 0.001$ eV it would be possible to distinguish different values of $\bar{\alpha}_{21}$ and $\bar{\alpha}_{31}$.

The necessary precisions on $|m_{ee}|$ and m_{lightest} are unfortunately outside of the range of any projected experiments known to the author. Nevertheless, the red curves corresponding to fixed values of γ/n and x/n are often close to the blue dashed three sigma range. With increasingly precise knowledge of the values of the mixing angles, especially θ_{13}, the three sigma ranges will shrink, perhaps

[16] However, one can solve Eq. (3.42) either analytically or numerically to obtain solutions for n, γ, and x such that $|m_{ee}| = 0$.

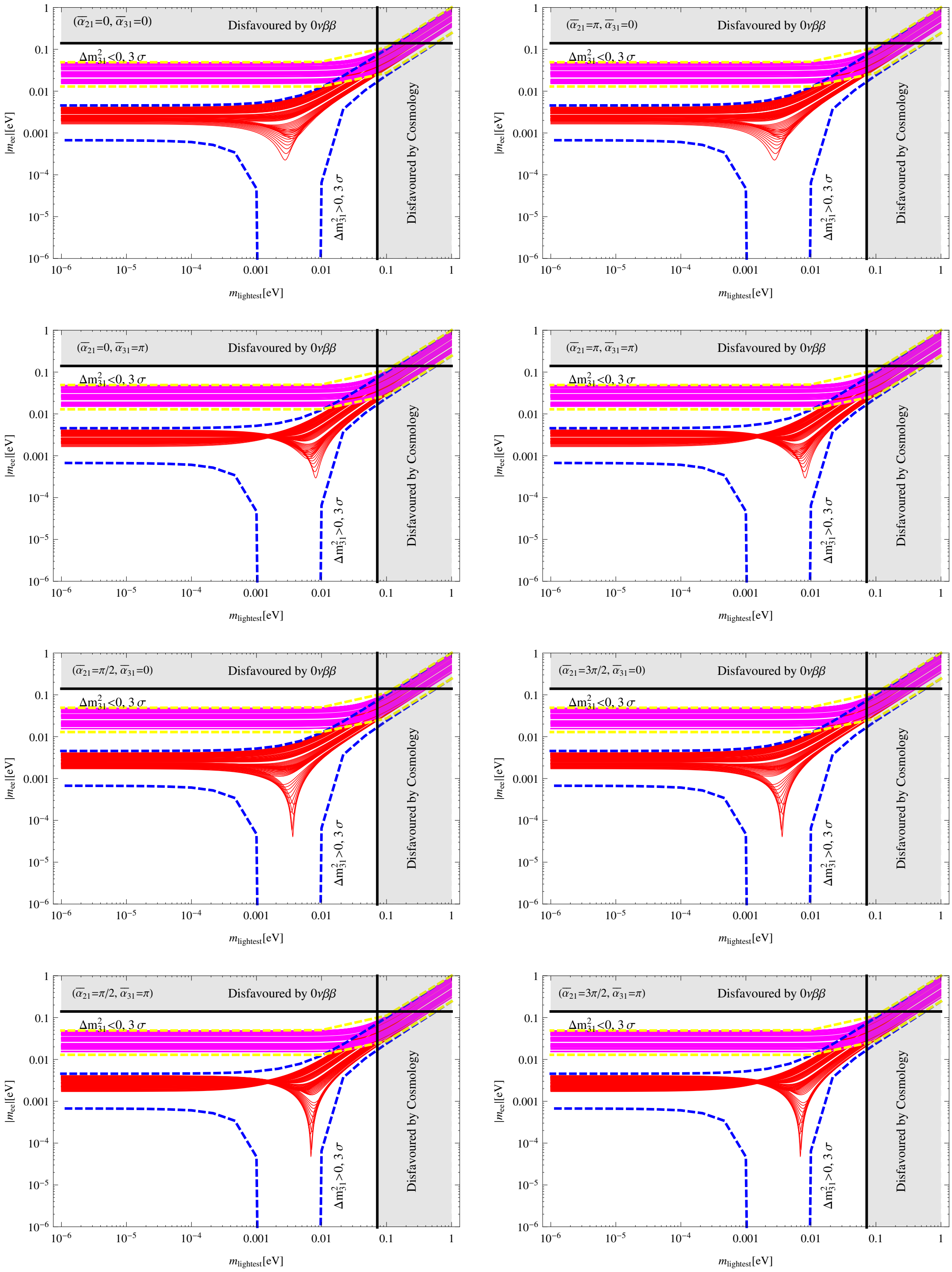


Figure 3.1: Effective Mass of $0\nu\beta\beta$ decay. γ/n is varied between the lower and upper 3 sigma bound, $x/n = 0, 0.1, 0.2, \ldots, 1$. For the definition of $\bar{\alpha}_{21}$ and $\bar{\alpha}_{31}$ cf. Eqs. (3.40), (3.41).

making it possible to draw conclusions about γ/n and x/n without an overly precise measurement of $|m_{ee}|$ or of the mass of the lightest neutrino.

To recapitulate, the following assumptions went into producing these results: There are 3 left-handed doublets of leptons, which in turn transform as a triplet under a $\Delta(6n^2)$ group. The neutrinos are Majorana fermions and $\Delta(6n^2)$ is broken to a $Z_2 \times Z_2$ subgroup in the neutrino sector and to Z_3 in the charged lepton sector. The mixing angles are solely predicted from the aforementioned assumptions. There is a general CP symmetry consistent with $\Delta(6n^2)$ which is broken to one element in each sector. From this gCP symmetry the Majorana phases are predicted.

If one of the mixing angles would be found to be incompatible with any of the predictions this would mean that either $\Delta(6n^2)$ is not broken to $Z_2 \times Z_2$ or that the flavour group is not $\Delta(6n^2)$ or that one of the more fundamental assumptions is wrong. The neutrinos could still be Majorana fermions as $\Delta(6n^2)$ could still be broken to a single Z_2, as discussed in the next chapter, or broken completely.

3.4 Conclusions

In this chapter the interplay of $\Delta(6n^2)$ groups and general CP transformations (gCP) in a direct model for three generations of Dirac charged leptons and Majorana neutrinos was examined. One finds that gCP transformations with $X_r \in e^{i\alpha}\Delta(6n^2)$ are physical CP transformations. Leaving a single gCP transformation unbroken will constrain the mixing matrix such that all phases, Dirac and Majorana are predicted and depend only on the $\Delta(6n^2)$ group, the residual $Z_2 \times Z_2$ group (parametrised by γ) and the residual gCP transformation (parametrised by x) in the neutrino sector. Leaving two or more gCP transformations unbroken was found not to be physically viable.[17]

Comparing the predictions for the mixing angles with experimental data one finds that the strongest constraint on γ/n is imposed by the relatively precise measurement of θ_{13}. The smallest group where θ_{13} lies within three sigma of the central value has $n = 14$. Furthermore, as the Majorana CP violating phases had been predicted, predictions for neutrinoless double-beta decay were studied. One finds

[17]This was only found because the residual general CP matrices that were analysed in this paper did not fulfil the correct consistency condition, but one that is too loose and allowed for some CP transformations to force mass eigenvalues to be zero or degenerate.

that for inverted ordering, the predicted $|m_{ee}|$ is within the reach of upcoming experiments like GERDA III. (As is always the case in this scenario.) For normal ordering, measuring $|m_{ee}|$ down to 10^{-4}eV could exclude large regions of γ/n and x/n, depending on the value of δ_{CP}.

In conclusion, these results represent the first time that an infinite series of finite groups has been examined for general CP transformations that are consistent with it. The important role of $\Delta(6n^2)$ among the subgroups of $SU(3)$ with triplet irreducible representations needs to be emphasized, especially in light of the results which had appeared shortly after parts of the results in this chapter were published in [2] that confirmed that $\Delta(6n^2)$ groups are indeed among the last viable flavour symmetries in direct models [60] . Moreover, as [60] does not consider CP symmetries which in the opinion of the author of this thesis are entirely on the same footing as pure flavour symmetries, the study in this chapter is not only the first, but still remains the most complete study of direct models with general CP of groups that are still experimentally viable.

If the Dirac CP phase is measured to differ from 0 or π, or the mixing angles deviate from the sum rules $\theta_{23} = 45° \mp \theta_{13}/\sqrt{2}$, respectively, then this would mean that in general a potential flavour group $\Delta(6n^2)$ cannot be broken to $Z_2 \times Z_2$, as in the case of the direct approach assumed here. However the semi-direct approach, in which a Z_2 subgroup is preserved, would remain a possibility for theories based on $\Delta(6n^2)$ and this is precisely the topic of the next chapter.

4

General CP and $\Delta(6n^2)$ flavour symmetry in semi-direct models of leptons

In this chapter, a detailed analysis of $\Delta(6n^2)$ flavour symmetry combined with a general CP symmetry in the lepton sector in semi-direct models will be performed. These results were previously published in [4] and discussed in [5]. The author was sharing the computaional load with collaborator G.J. Ding and writing the corresponding parts of [4]. In addition, [5] was completely written by the author.

This chapter is rather similar in methodology to the previous one and the difference lies in the different choice of residual flavour symmetries that are embedded into G_F. Again, flavour and CP symmetry are broken to different residual symmetries G_ν in the neutrino and G_l in the charged lepton sector, together with residual CP symmetries in each sector. The mixing prediction for all possible breakings of $\Delta(6n^2)$ to $G_\nu = Z_2$ with $G_l = K_4, Z_p, p > 2$ and to $G_\nu = K_4$ with $G_l = Z_2$ are analysed. Because of the large number of individual results, the more tedious results will be exiled to an appendix, while this chapter only summarizes the most important findings and small differences to the previous chapter. As the above-mentioned breakings have a smaller preserved symmetry than the full Klein group, predictions depend on additional undetermined parameters, which in most cases depend on the reactor angle and the Majorana phases. Out of the large number of combinations of charged lepton and neutrino residual symmetries, only five are phenomenologically allowed and are studied in slightly greater detail.

4.1 Introduction

The work in this chapter was mostly motivated by the fact that global fits of lepton mixing parameters started showing a slight tendency towards a non-trivial value of δ_{CP} [82, 78, 83]. While this could be caused by a statistical fluctuation [84], a definite measurement confirming a non-trivial value of δ_{CP} would immediately rule out all direct models with finite flavour groups, so in particular also all direct models with $\Delta(6n^2)$, which were the topic of the two previous chapters. Measuring the Dirac CP phase is definitely one of the primary goals of the next generation neutrino oscillation experiments. Furthermore, CP violation has been firmly established in the quark sector and it would be surprising if no CP violation was discovered in the lepton sector.[1]

As mentioned before, in semidirect models, not the entire minimally necessary residual flavour symmetry is embedded into a larger flavour symmetry. Instead, either only a Z_2 factor of the flavour group is preserved in the neutrino sector, while the charged lepton sector is completely fixed by symmetry, or vice versa, a Z_2 factor is preserved in the lepton sector, while the neutrino sector is completely fixed. [2]

One could argue that the most minimal extension of direct models is actually given by semidirect models with residual CP symmetries, because in this way fewer additional parameters that are not related to the breaking of the group are introduced. In addition, residual CP symmetries also constrain the Majorana phases, which currently may be far from being measured, however there is certain hope that with $0\nu\beta\beta$ and $\nu \to \bar{\nu}$ oscillation at least within the lifetime of the author of this thesis these parameters will become accessible. Moreover, as fermion mass terms always have some residual CP symmetry, one could adopt the viewpoint that CP symmetries are on the same footing as flavour symmetries and that always some CP symmetry might exist at high energies into which residual CP symmetries are embedded.

In this chapter thus, after quickly recapitulating some notation, the analysis of mappings facilitated by CP transformations within the flavour group will be extended, confirming a conjecture from the last chapter. The program GAP was used

[1] Although human intuition has not the best record concerning such things.

[2] There are also what could be called double-semidirect models, where in both sectors only a Z_2 factor is preserved [85].

to computationally generate the automorphisms of $\Delta(6n^2)$ to obtain CP transformations consistent with the group. After this, residual CP transformations that are consistent with residual flavour symmetries are given. [3]

Concrete semi-direct S_4 flavour models with a general CP symmetry had been constructed in Refs. [30, 72, 86, 87, 67, 88, 71] where the spontaneous breaking of the $S_4 \rtimes H_{\mathrm{CP}}$ down to $Z_2 \times$ CP in the neutrino sector was implemented. Other models with a flavour symmetry and a general CP symmetry can also be found in Refs. [89, 90, 68, 66]. The interplay between flavour symmetries and CP symmetries has been generally discussed in [73, 64]. In addition, there are other theoretical approaches involving both flavour symmetry and CP violation [91, 92, 93, 65]. The work presented here follows on from a similar analysis of semi-direct models based on the group $\Delta(96)$ [94]. While this paper was being prepared, a study of general CP within the semi-direct approach appeared based on the infinite series of finite groups $\Delta(6n^2)$ and $\Delta(3n^2)$ [95]. Where the results overlap for $\Delta(6n^2)$ they appear to be broadly in agreement, although the case that the residual symmetry $Z_2 \times CP$ is preserved by the charged lepton sector was not considered in [95]. This work focuses exclusively on $\Delta(6n^2)$, and, apart from considering extra cases not previously considered, presents the numerical results in a quite different and complementary way. Many of the numerical results contained here, for example, the predictions for neutrinoless double beta decay, were not previously considered at all.

The remainder of this chapter is organised as follows. In Section 4.2 the interplay of general CP transformations with $\Delta(6n^2)$ is analysed. In Section 4.3 lepton mixing predictions in neutrino-semidirect models with residual symmetry $Z_2 \times CP$ in the neutrino sector are given. In Section 4.4 lepton mixing predictions in charged-lepton-semidirect models with residual symmetry $Z_2 \times CP$ in the charged lepton sector are analysed. The phenomenological predictions of the neutrinoless double beta decay for all the viable cases are presented in Section 4.5, Finally Section 4.6 concludes this chapter.

[3] Where unfortunately, again not the consistency conditions as discussed in section 1.5.1 were used, resulting in too many CP candidates. This, however, does not invalidate the mixing results obtained, as inconsistent residual CP symmetries can never produce a physically correct mass spectrum.

4.2 General CP with $\Delta(6n^2)$

As usual by now, consider a theory with both flavour symmetry G_F and general CP symmetry at high energy scale. A field multiplet $\varphi_{\mathbf{r}}$ transforms under the action of the flavour symmetry group G_F as

$$\varphi_{\mathbf{r}} \xmapsto{g} \rho_{\mathbf{r}}(g)\varphi_{\mathbf{r}}, \quad g \in G_F\,, \tag{4.1}$$

where $\rho_{\mathbf{r}}(g)$ is the representation matrix of g in the representation $\mathbf{r}$ and a general CP transformation acts on the field as:

$$\varphi_{\mathbf{r}} \xmapsto{CP} X_{\mathbf{r}}\varphi^*_{\mathbf{r}}(x_\mu). \tag{4.2}$$

The general CP symmetry has to be consistent with the flavour symmetry. In [96, 70, 73, 30] it has been argued that the general CP symmetry can only be compatible with the flavour symmetry if the following consistency equation is satisfied:

$$X_{\mathbf{r}}\rho^*_{\mathbf{r}}(g)X^\dagger_{\mathbf{r}} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f\,. \tag{4.3}$$

Hence a general CP transformation is related to an automorphism which maps g into g' as before. Furthermore, it was recently shown under some (fairly limiting) assumptions that physical CP transformations have to be given by class-inverting automorphism of G_F [64]. In this chapter the flavour symmetry is given by a $\Delta(6n^2)$ group for some n The group theory of $\Delta(6n^2)$ is presented in Appendix 7.4. With the help of the computer algebra program system `GAP` [44, 45, 46, 47] the automorphism group of the $\Delta(6n^2)$ until $n = 19$ was studied.[4]. The results are collected in Table 4.1. One finds that the outer automorphism groups of members of the $\Delta(6n^2)$ series are generally non-trivial except for $\Delta(6) \cong S_3$ and $\Delta(24) \cong S_4$. However, there is only one class-inverting outer automorphism for $n \neq 3\mathbb{Z}$ while no class-inverting automorphism exists for $n = 3\mathbb{Z}$. In fact, one finds a class-inverting automorphism u acting on the generators as:

$$a \xmapsto{u} a^2, \quad b \xmapsto{u} b, \quad c \xmapsto{u} d, \quad d \xmapsto{u} c\,. \tag{4.4}$$

which confirms an assumption made in the previous chapter. It can be checked that for $n \neq 3\mathbb{Z}$ this automorphism u maps each conjugacy class onto the class of

[4]The $\Delta(6n^2)$ group with $n > 19$ are not available in `GAP` so far.

inverse elements. In the case of $n = 3\mathbb{Z}$,

$$\frac{2n^2}{3}C_2^{(\tau)} \stackrel{u}{\longmapsto} \frac{2n^2}{3}C_2^{(-\tau)}, \qquad \left(\frac{2n^2}{3}C_2^{(\tau)}\right)^{-1} = \frac{2n^2}{3}C_2^{(\tau)}, \quad \tau = 0, 1, 2\,. \tag{4.5}$$

Hence both $\frac{2n^2}{3}C_2^{(1)}$ and $\frac{2n^2}{3}C_2^{(2)}$ are not mapped into their inverse classes although the latter is still true for the remaining classes. As a result, one can conjecture that the $\Delta(6n^2)$ group with $n \neq 3\mathbb{Z}$ admits a unique class-inverting automorphism given by Eq. (4.4). One can nevertheless apply CP transformations obtained for $n \neq 3Z$ to examples with $n = 3Z$ to see what kind of constraints are obtained, as, although it is unclear what their precise relation to this particular group is at high energies, they are definitely CP transformations. When it doubt, assume $n \neq 3Z$. The general CP transformation corresponding to u, which is denoted by $X_{\mathbf{r}}(u)$, would be physically well-defined, as suggested in Ref. [64]. Its concrete form is fixed by the consistency equations as follows:

$$\begin{aligned}
X_{\mathbf{r}}\left(u\right)\rho^{*}_{\mathbf{r}}(a)X^{\dagger}_{\mathbf{r}}\left(u\right) &= \rho_{\mathbf{r}}\left(u\left(a\right)\right) = \rho_{\mathbf{r}}(a^2)\,,\\
X_{\mathbf{r}}\left(u\right)\rho^{*}_{\mathbf{r}}(b)X^{\dagger}_{\mathbf{r}}\left(u\right) &= \rho_{\mathbf{r}}\left(u\left(b\right)\right) = \rho_{\mathbf{r}}\left(b\right)\,,\\
X_{\mathbf{r}}\left(u\right)\rho^{*}_{\mathbf{r}}(c)X^{\dagger}_{\mathbf{r}}\left(u\right) &= \rho_{\mathbf{r}}\left(u\left(c\right)\right) = \rho_{\mathbf{r}}\left(d\right)\,,\\
X_{\mathbf{r}}\left(u\right)\rho^{*}_{\mathbf{r}}(d)X^{\dagger}_{\mathbf{r}}\left(u\right) &= \rho_{\mathbf{r}}\left(u\left(d\right)\right) = \rho_{\mathbf{r}}\left(c\right)\,.
\end{aligned} \tag{4.6}$$

In our basis, presented in section 7.4, we can determine that

$$X_{\mathbf{r}}\left(u\right) = \rho_{\mathbf{r}}(b)\,. \tag{4.7}$$

Furthermore, including inner automorphisms, the full[5] set of general CP transformations compatible with $\Delta(6n^2)$ flavour symmetry is

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(g), \quad g \in \Delta(6n^2)\,. \tag{4.8}$$

Consequently the general CP transformations are of the same form as the flavour symmetry transformations in the chosen basis. In particular, we see that the conventional CP transformation with $\rho_{\mathbf{r}}(1)$=1 is allowed. As a consequence, all coupling constants would be real in a $\Delta(6n^2)$ model with imposed CP symmetry since all the CG coefficients are real, as shown in Appendix 7.4.1. In the case of $n = 3\mathbb{Z}$, the consistency equations of Eq. (4.6) are also satisfied except when $\mathbf{r}$ is the doublet representations $\mathbf{2_2}$, $\mathbf{2_3}$ or $\mathbf{2_4}$. Hence the general CP transformations

[5] class-inverting

n	G_F	GAP-Id	Inn(G_F)	Out(G_F)	Num.
1	$\Delta(6) \equiv S_3$	[6,1]	S_3	Z_1	1
2	$\Delta(24) \equiv S_4$	[24,12]	S_4	Z_1	1
3	$\Delta(54)$	[54,8]	$(Z_3 \times Z_3) \rtimes Z_2$	S_4	0
4	$\Delta(96)$	[96,64]	$\Delta(96)$	Z_2	1
5	$\Delta(150)$	[150,5]	$\Delta(150)$	Z_4	1
6	$\Delta(216)$	[216,95]	$(Z_3 \times A_4) \rtimes Z_2$	S_3	0
7	$\Delta(294)$	[294,7]	$\Delta(294)$	Z_6	1
8	$\Delta(384)$	[384,568]	$\Delta(384)$	K_4	1
9	$\Delta(486)$	[486,61]	$((Z_9 \times Z_3) \rtimes Z_3) \rtimes Z_2$	$Z_3 \times S_3$	0
10	$\Delta(600)$	[600,179]	$\Delta(600)$	Z_4	1
11	$\Delta(726)$	[726,5]	$\Delta(726)$	Z_{10}	1
12	$\Delta(864)$	[864,701]	$(Z_3 \times ((Z_4 \times Z_4) \rtimes Z_3)) \rtimes Z_2$	D_{12}	0
13	$\Delta(1014)$	[1014,7]	$\Delta(1014)$	Z_{12}	1
14	$\Delta(1176)$	[1176,243]	$\Delta(1176)$	Z_6	1
15	$\Delta(1350)$	[1350,46]	$(Z_3 \times ((Z_5 \times Z_5) \rtimes Z_3)) \rtimes Z_2$	$Z_4 \times S_3$	0
16	$\Delta(1536)$	[1536,408544632]	$\Delta(1536)$	$Z_4 \times Z_2$	1
17	$\Delta(1734)$	[1734,5]	$\Delta(1734)$	Z_{16}	1
18	$\Delta(1944)$	[1944,849]	$((Z_{18} \times Z_6) \rtimes Z_3) \rtimes Z_2$	$Z_3 \times S_3$	0
19	$\Delta(2166)$	[2166,15]	$\Delta(2166)$	Z_{18}	1

Table 4.1: The automorphism groups of the $\Delta(6n^2)$ group series, where Inn(G_F) and Out(G_F) denote inner automorphism group and outer automorphism group of the flavour symmetry group G_F respectively. The last column gives the number of class-inverting outer automorphisms. Note that the inner automorphism group of $\Delta(6n^2)$ with $n = 3\mathbb{Z}$ is isomorphic to $\Delta(6n^2)/Z_3$ since its center is the Z_3 subgroup generated by $c^{\frac{n}{3}}d^{\frac{2n}{3}}$.

in Eq. (4.7) can also be imposed on a model with $n = 3\mathbb{Z}$ if the fields transforming as $\mathbf{2_2}$, $\mathbf{2_3}$ or $\mathbf{2_4}$ are absent.

4.3 Lepton mixing with residual symmetry $Z_2 \times CP$ in the neutrino sector

In the following, all lepton mixing patterns in neutrino-semidirect models, ie. where $G_\nu \simeq Z_2$, will be listed. Examples had been considered in [25, 72, 86, 87, 71, 66, 94]. The full symmetry group is $\Delta(6n^2) \rtimes H_{CP}$, which is broken down to $G_l \rtimes H^l_{CP}$ and $Z_2 \times H^\nu_{CP}$ residual symmetries in the charged lepton and neutrino sectors respectively. G_l is usually taken to be an abelian subgroup of $\Delta(6n^2)$ of order larger than 2 to avoid degenerate charged lepton masses. The misalignment between the two residual symmetries generates the PMNS matrix. Again, only residual symmetries are considered and how the required symmetry breaking is dynamically achieved is not discussed, as there are generally more than one mechanism and many possible specific model realizations. One column of the lepton mixing matrix can be fixed and the resulting lepton mixing parameters are generally constrained to depend on only one free parameter in this approach. As usual the three generation of the left-handed lepton doublet fields are assigned to the faithful representation $\mathbf{3}_{1,1}$ which is denoted by $\mathbf{3}$ in the following.

4.3.1 Charged lepton sector

From a residual symmetry G_l follows that the charged lepton mass matrix is invariant under the transformation $\ell_L \to \rho_{\mathbf{3}}(g_l)\ell_L$, where ℓ_L stands for the three generations of left-handed lepton doublets, g_l is the generator of G_l, and $\rho_{\mathbf{3}}(g_l)$ is the representation matrix of g_l in the triplet representation $\mathbf{3}$. As a consequence, the charged lepton mass matrix satisfies

$$\rho^{\dagger}_{\mathbf{3}}(g_l)m^{\dagger}_l m_l \rho_{\mathbf{3}}(g_l) = m^{\dagger}_l m_l\,, \tag{4.9}$$

where the charged lepton mass matrix m_l is defined in the convention, $\ell^c m_l \ell_L$. Let us denote the diagonalization matrix of $m^{\dagger}_l m_l$ by U_l, i.e.

$$U^{\dagger}_l m^{\dagger}_l m_l U_l = \text{diag}\left(m^2_e, m^2_\mu, m^2_\tau\right) \equiv \widehat{m}^2_l\,. \tag{4.10}$$

where m_e, m_μ and m_τ are the electron, muon and tau masses respectively. Substituting Eq. (4.10) into Eq. (4.9), we obtain

$$\widehat{m}^2_l \left[U^{\dagger}_l \rho_{\mathbf{3}}(g_l) U_l\right] = \left[U^{\dagger}_l \rho_{\mathbf{3}}(g_l) U_l\right] \widehat{m}^2_l\,. \tag{4.11}$$

One can see that $U^{\dagger}_l \rho_{\mathbf{3}}(g_l) U_l$ has to be diagonal. Therefore U_l not only diagonalizes $m^{\dagger}_l m_l$ but also the matrix $\rho_{\mathbf{3}}(g_l)$. As a result, the unitary diagonalization matrix U_l is completely fixed by the residual flavour symmetry G_l once the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ are non-degenerate. In the present work, only the case that G_l is a cyclic subgroup of $\Delta(6n^2)$ is considered. Hence the generator g_l of G_l could be of the form $c^s d^t$, $bc^s d^t$, $ac^s d^t$, $a^2 c^s d^t$, $abc^s d^t$ or $a^2 bc^s d^t$ with $s, t = 0, 1, \ldots, n-1$. If the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ are degenerate such that its diagonalization matrix U_l can not be fixed uniquely, one could extend G_l from a single cyclic subgroup to the product of several cyclic subgroups. This scenario is beyond the scope of this work except that the simplest K_4 extension is included. (As has been before, in [97].) Given the explicit form of the representation matrices listed in Appendix 7.4, the charged lepton diagonalization matrices U_l for different cases of G_l can be calculated, and the results are summarized in Appendix 7.1.2. Since the charged lepton masses can not be constrained at all in the present approach (in other word, the order of the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ is indeterminate), U_l can undergo rephasing and permutations from the left.

4.3.2 Neutrino sector

In the present work, we assume the light neutrinos are Majorana particles. As a consequence, the residual flavour symmetry G_ν in the neutrino sector can only be a K_4 or Z_2 subgroup. The phenomenological consequence of $G_\nu = K_4$ has been studied in Refs. [2] by two of us. Here we shall concentrate on $G_\nu = Z_2$ case with general CP symmetry which allows us to predict CP phases. The Z_2 subgroups of $\Delta(6n^2)$ can be generated by

$$bc^x d^x, \quad abc^y, \quad a^2bd^z, \qquad x, y, z = 0, 1 \ldots n-1 \tag{4.12}$$

and additionally

$$c^{n/2}, \quad d^{n/2}, \quad c^{n/2}d^{n/2} \tag{4.13}$$

for $n = 2\mathbb{Z}$. It is notable that the Z_2 elements in Eq. (4.12) and Eq. (4.13) are conjugate to each other respectively:

$$\begin{aligned}
&\left(c^\gamma d^\delta\right) bc^x d^x \left(c^\gamma d^\delta\right)^{-1} = bc^{x-\delta-\gamma} d^{x-\delta-\gamma}, && \left(bc^\gamma d^\delta\right) bc^x d^x \left(bc^\gamma d^\delta\right)^{-1} = bc^{-x+\delta+\gamma} d^{-x+\delta+\gamma}, \\
&\left(ac^\gamma d^\delta\right) bc^x d^x \left(ac^\gamma d^\delta\right)^{-1} = a^2bd^{-x+\delta+\gamma}, && \left(a^2c^\gamma d^\delta\right) bc^x d^x \left(a^2c^\gamma d^\delta\right)^{-1} = abc^{-x+\delta+\gamma}, \\
&\left(abc^\gamma d^\delta\right) bc^x d^x \left(abc^\gamma d^\delta\right)^{-1} = a^2bd^{x-\delta-\gamma}, && \left(a^2bc^\gamma d^\delta\right) bc^x d^x \left(a^2bc^\gamma d^\delta\right)^{-1} = abc^{x-\delta-\gamma}.
\end{aligned} \tag{4.14a}$$

$$\begin{aligned}
&\left(c^\gamma d^\delta\right) c^{n/2} \left(c^\gamma d^\delta\right)^{-1} = c^{n/2}, && \left(bc^\gamma d^\delta\right) c^{n/2} \left(bc^\gamma d^\delta\right)^{-1} = d^{n/2}, \\
&\left(ac^\gamma d^\delta\right) c^{n/2} \left(ac^\gamma d^\delta\right)^{-1} = c^{n/2}d^{n/2}, && \left(a^2c^\gamma d^\delta\right) c^{n/2} \left(a^2c^\gamma d^\delta\right)^{-1} = d^{n/2}, \\
&\left(abc^\gamma d^\delta\right) c^{n/2} \left(abc^\gamma d^\delta\right)^{-1} = c^{n/2}, && \left(a^2bc^\gamma d^\delta\right) c^{n/2} \left(a^2bc^\gamma d^\delta\right)^{-1} = c^{n/2}d^{n/2}.
\end{aligned} \tag{4.14b}$$

The residual general CP symmetry should be compatible with the residual Z_2 symmetry in the neutrino sector, and therefore the corresponding consistency equation should be satisfied, i.e.,

$$X_{\nu\mathbf{r}}\rho^*_{\mathbf{r}}(g)X^{-1}_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(g), \quad g \in Z_2\,, \tag{4.15}$$

which means that the residual CP and residual flavour transformations are commutable with each other [30, 72] in the neutrino sector. For a given solution $X_{\nu\mathbf{r}}$ of Eq. (4.15), one can check that $\rho_{\mathbf{r}}(g)X_{\nu\mathbf{r}}$ is also a solution. The residual CP symmetries consistent with the Z_2 elements in Eq. (4.12) and Eq. (4.13) are summarized as follows. Note that residual CP symmetries with permutations are those that fulfil constency condition Eq. (4.3), but not the stricter actual condition Eq. (1.122).

- $g = bc^x d^x, \quad x = 0, 1, 2 \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \quad \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma}), \quad \gamma = 0, 1, 2 \ldots n-1\,. \tag{4.16}$$

- $g = abc^y, \quad y = 0, 1, 2 \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \quad \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma}), \quad \gamma = 0, 1, 2 \ldots n-1\,. \tag{4.17}$$

- $g = a^2bd^z, \quad z = 0, 1, 2 \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^{2z+2\delta} d^\delta), \quad \rho_{\mathbf{r}}(a^2bc^{2\delta} d^\delta), \quad \delta = 0, 1, 2 \ldots n-1\,. \tag{4.18}$$

- $g = c^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(abc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \ldots n-1\,. \tag{4.19}$$

- $g = d^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(a^2bc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \ldots n-1\,. \tag{4.20}$$

- $g = c^{n/2}d^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(bc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \ldots n-1\,. \tag{4.21}$$

As we shall demonstrate in the following, the residual CP symmetry should be symmetric to avoid degenerate lepton masses. Then the viable CP transformations would be constrained to be $\rho_{\mathbf{r}}(abc^\gamma d^{2\gamma})$, $\rho_{\mathbf{r}}(a^2bc^{2\delta}d^\delta)$ and $\rho_{\mathbf{r}}(bc^\gamma d^{-\gamma})$ together with

$\rho_{\mathbf{r}}(c^\gamma d^\delta)$ for $g = c^{n/2}$, $d^{n/2}$ and $c^{n/2}d^{n/2}$ respectively. The full symmetry $\Delta(6n^2) \rtimes H_{CP}$ is broken down to $Z_2 \times H^\nu_{CP}$ in the neutrino sector. The invariance of the light neutrino mass matrix m_ν under the residual flavour symmetry $G_\nu = Z_2$ and the residual CP symmetry H^ν_{CP} leads to

$$\begin{aligned} &\rho^T_{\mathbf{3}}(g_\nu) m_\nu \rho_{\mathbf{3}}(g_\nu) = m_\nu, \quad g_\nu \in Z^\nu_2\,, \\ &X^T_{\nu\mathbf{3}} m_\nu X_{\nu\mathbf{3}} = m^*_\nu, \quad X_\nu \in H^\nu_{CP}\,, \end{aligned} \tag{4.22}$$

from which we can construct the explicit form of m_ν and then diagonalize it. The Majorana mass matrices that fulfil these constraints and the diagonalisation matrices U_ν that arise are listed in Appendix 7.1.1.

4.3.3 Predictions for lepton flavour mixing

With the possible forms of the neutrino and charged lepton mass matrices and their diagonalization matrices worked out in previous sections, the lepton flavour mixing matrix candidates are of course given by

$$U_{PMNS} = U^\dagger_l U_\nu\,. \tag{4.23}$$

Because the ordering of the charged-lepton and neutrino masses is not fixed by the residual symmetries, the PMNS matrix U_{PMNS} is only determined up to independent permutations of rows and columns. From Eqs. (4.14a,4.14b), follows that the residual Z_2 symmetries generated by bc^xd^x, abc^y, a^2bd^z are conjugate to each other, and the same is true for the Z_2 symmetry generated by $c^{n/2}$, $d^{n/2}$ and $c^{n/2}d^{n/2}$. If a pair of residual flavour symmetries (G'_ν, G'_l) is conjugated to the pair of groups (G_ν, G_l) under the group element $g \in \Delta(6n^2)$, then both pairs lead to the same result for U_{PMNS} even after the general CP symmetry is included [71]. As a consequence, one only needs to needs to consider the representative residual symmetries $G_\nu = Z_2^{bc^xd^x}$, $Z_2^{c^{n/2}}$ and $G_l = \langle c^sd^t\rangle$, $\langle bc^sd^t\rangle$, $\langle ac^sd^t\rangle$, $\langle abc^sd^t\rangle$ and $\langle a^2bc^sd^t\rangle$. Because the residual flavour symmetry in the neutrino sector is taken to be a Z_2 instead of a K_4 subgroup, only one column of the PMNS matrix can be fixed up to permutation and rephasing of the elements in this scenario. The form of the fixed columns for different choices of the residual flavour symmetry is summarized in Table 4.2. The present 3σ confidence level ranges for the magnitude of

	$G_\nu = Z_2^{bc^x d^x}$	$G_\nu = Z_2^{c^{n/2}}$
$G_l = \langle c^s d^t \rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ ✗	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ✗
$G_l = \langle bc^s d^t \rangle$	$\begin{pmatrix} 0 \\ \cos\left(\frac{s+t-2x}{2n}\pi\right) \\ \sin\left(\frac{s+t-2x}{2n}\pi\right) \end{pmatrix}$ ✗	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ ✗
$G_l = \langle ac^s d^t \rangle$	$\sqrt{\frac{2}{3}}\begin{pmatrix} \sin\left(\frac{s-x}{n}\pi\right) \\ \cos\left(\frac{\pi}{6} - \frac{s-x}{n}\pi\right) \\ \cos\left(\frac{\pi}{6} + \frac{s-x}{n}\pi\right) \end{pmatrix}$ ✓	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓
$G_l = \langle abc^s d^t \rangle$	$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}$ ✓	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ✗
$G_l = \langle a^2bc^s d^t \rangle$	$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}$ ✓	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ ✗

Table 4.2: The form of the column of the PMNS matrix which is fixed for different residual symmetries G_ν and G_l. The symbol "✗" denotes that the resulting lepton mixing is ruled out since there is at least one zero element in the fixed column, and the symbol "✓" denotes that the resulting mixing is experimentally still allowed. Note that for $G_\nu = Z_2^{bc^x d^x}$, the cases of $G_l = \langle abc^s d^t \rangle$ and $G_l = \langle a^2bc^s d^t \rangle$ are not independent as $b\,(abc^s d^t)\,b = a^2bc^{-t}d^{-s}$ and $b\,(bc^x d^x)\,b = bc^{-x}d^{-x}$.

the elements of the leptonic mixing matrix are given by a global fit [78]:

$$||U_{PMNS}||_{3\sigma} = \begin{pmatrix} 0.789 \to 0.853 & 0.501 \to 0.594 & 0.133 \to 0.172 \\ 0.194 \to 0.558 & 0.408 \to 0.735 & 0.602 \to 0.784 \\ 0.194 \to 0.558 & 0.408 \to 0.735 & 0.602 \to 0.784 \end{pmatrix}, \qquad (4.24)$$

for normal ordering of the neutrino mass spectrum, and a very similar result is obtained for inverted ordering. No entry of the PMNS matrix can be zero. As a result, the mixing patterns with a zero element have been ruled out by experimental data of neutrino mixing. In the following, the viable cases in which no element of the fixed column is zero are presented, and the predictions for the lepton flavour mixing parameters will be investigated for the various residual CP symmetries compatible with residual flavour symmetry. Only the mixing matrices as such will be given with some discussion and with more detailed results listed in Appendix 7.1.3.

(I) $G_l = \langle ac^s d^t \rangle$, $G_\nu = Z_2^{bc^x d^x}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma})\}$

The PMNS matrix is found to be

$$U^I_{PMNS} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2}\sin\varphi_1 & e^{i\varphi_2}\cos\theta - \sqrt{2}\sin\theta\cos\varphi_1 & e^{i\varphi_2}\sin\theta + \sqrt{2}\cos\theta\cos\varphi_1 \\ \sqrt{2}\cos\left(\frac{\pi}{6}-\varphi_1\right) & -e^{i\varphi_2}\cos\theta - \sqrt{2}\sin\theta\sin\left(\frac{\pi}{6}-\varphi_1\right) & -e^{i\varphi_2}\sin\theta + \sqrt{2}\cos\theta\sin\left(\frac{\pi}{6}-\varphi_1\right) \\ \sqrt{2}\cos\left(\frac{\pi}{6}+\varphi_1\right) & e^{i\varphi_2}\cos\theta + \sqrt{2}\sin\theta\sin\left(\frac{\pi}{6}+\varphi_1\right) & e^{i\varphi_2}\sin\theta - \sqrt{2}\cos\theta\sin\left(\frac{\pi}{6}+\varphi_1\right) \end{pmatrix}, \tag{4.25}$$

where

$$\varphi_1 = \frac{s-x}{n}\pi, \qquad \varphi_2 = \frac{2t-s-3(\gamma+x)}{n}\pi\,. \tag{4.26}$$

These two parameters φ_1 and φ_2 are interdependent of each other, and they can take the discrete values

$$\begin{aligned} &\varphi_1 = 0, \pm\frac{1}{n}\pi, \pm\frac{2}{n}\pi, \ldots \pm \frac{n-1}{n}\pi\,, \\ &\varphi_2 \bmod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \ldots \frac{2n-1}{n}\pi\,. \end{aligned} \tag{4.27}$$

Now concerning the permutations of the rows and the columns, the PMNS matrix can be multiplied by a 3×3 permutation matrix from both the left- and the right-hand side. There are six permutation matrices corresponding to six possible orderings of rows (or columns):

$$\begin{aligned} P_{123} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ P_{231} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.28}$$

The atmospheric mixing angle θ_{23} becomes $\pi/2 - \theta_{23}$, the Dirac CP phases δ_{CP} becomes $\pi + \delta_{CP}$ and the other mixing parameters are unchanged if the second and third row of the PMNS matrix are exchanged. The permutation of the second and third row will not be given explicitly in the following. The PMNS matrix can be rearranged as follows:

$$U^{I,1st}_{PMNS} = U^I_{PMNS}, \qquad U^{I,2nd}_{PMNS} = P_{231}U^I_{PMNS}, \qquad U^{I,3rd}_{PMNS} = P_{312}U^I_{PMNS}\,. \tag{4.29}$$

The above three arrangements are related:

$$\begin{aligned}
U^{I,2nd}_{PMNS}(\theta,\varphi_1,\varphi_2) &= \mathrm{diag}(1,1,-1)U^{I,1st}_{PMNS}(\pi-\theta,\frac{\pi}{3}+\varphi_1,\varphi_2)\mathrm{diag}(1,1,-1),\\
U^{I,3rd}_{PMNS}(\theta,\varphi_1,\varphi_2) &= \mathrm{diag}(-1,1,1)U^{I,1st}_{PMNS}(-\theta,-\frac{\pi}{3}+\varphi_1,\varphi_2)\mathrm{diag}(1,-1,1)\,,
\end{aligned} \tag{4.30}$$

where the phase factor diag $(\pm1,\pm1,\pm1)$ can be absorbed by the lepton fields. Hence it is sufficient to only discuss the first PMNS matrix $U^{I,1st}_{PMNS}$ in detail, the phenomenological predictions for the other two can be obtained by variable substitution.

Taking into account measured values of θ_{12} and θ_{13} [78], we obtain the constraint on φ_1 as

$$0.417\pi \leq \varphi_1 \leq 0.583\pi, \qquad \text{or} \qquad -0.583\pi \leq \varphi_1 \leq -0.417\pi\,, \tag{4.31}$$

which indicates that φ_1 is around $\pm\pi/2$. This mixing pattern can accommodate the present neutrino oscillation data very well. The 3σ allowed values of the lepton mixing parameters for $n = 2, 3, \ldots, 100$ are displayed in Fig. 4.1 and Fig. 4.2. Analytic expressions for these parameters can be found in appendix 7.1.3. In the case that n is divisible by 3, the doublet representations $\mathbf{2_2}$, $\mathbf{2_3}$ and $\mathbf{2_4}$ are assumed to be absent such that the general CP symmetry in Eq. (4.8) is consistently defined. If n is divisible by 3, the three permutations $U^{I,1st}_{PMNS}$, $U^{I,2nd}_{PMNS}$ and $U^{I,3rd}_{PMNS}$ give rise to the same predictions for the mixing parameters. The observed values of the three lepton mixing angles can not be achieved for $n = 3$. In case of $n = 2$ and $n = 4$, both the atmospheric mixing angle θ_{23} and the Dirac CP phase δ_{CP} are maximal while the Majorana phases are zero. It is remarkable that the three CP phases can take any values for sufficiently large n, while θ_{12} is always constrained to be in the range of $0.313 \leq \sin^2\theta_{12} \leq 0.344$. Hence this mixing pattern can be tested by precisely measuring the solar mixing angle θ_{12}. Notice that θ_{12} can be measured with rather good accuracy by JUNO experiment [98].

Correlations between mixing parameters for $n \to \infty$ and $n = 8$ are shown in Fig. 7.1 where only the phenomenologically viable cases are given for which the observed values of θ_{12}, θ_{13} and θ_{23} can be accommodated for at least some values of the parameter θ.

The vector $\sqrt{2/3}\,(\sin\varphi_1, \cos(\pi/6-\varphi_1)\,, \cos(\pi/6+\varphi_1))^T$ enforced by the residual Z_2 symmetry could also be the second column of the PMNS matrix. Ignoring

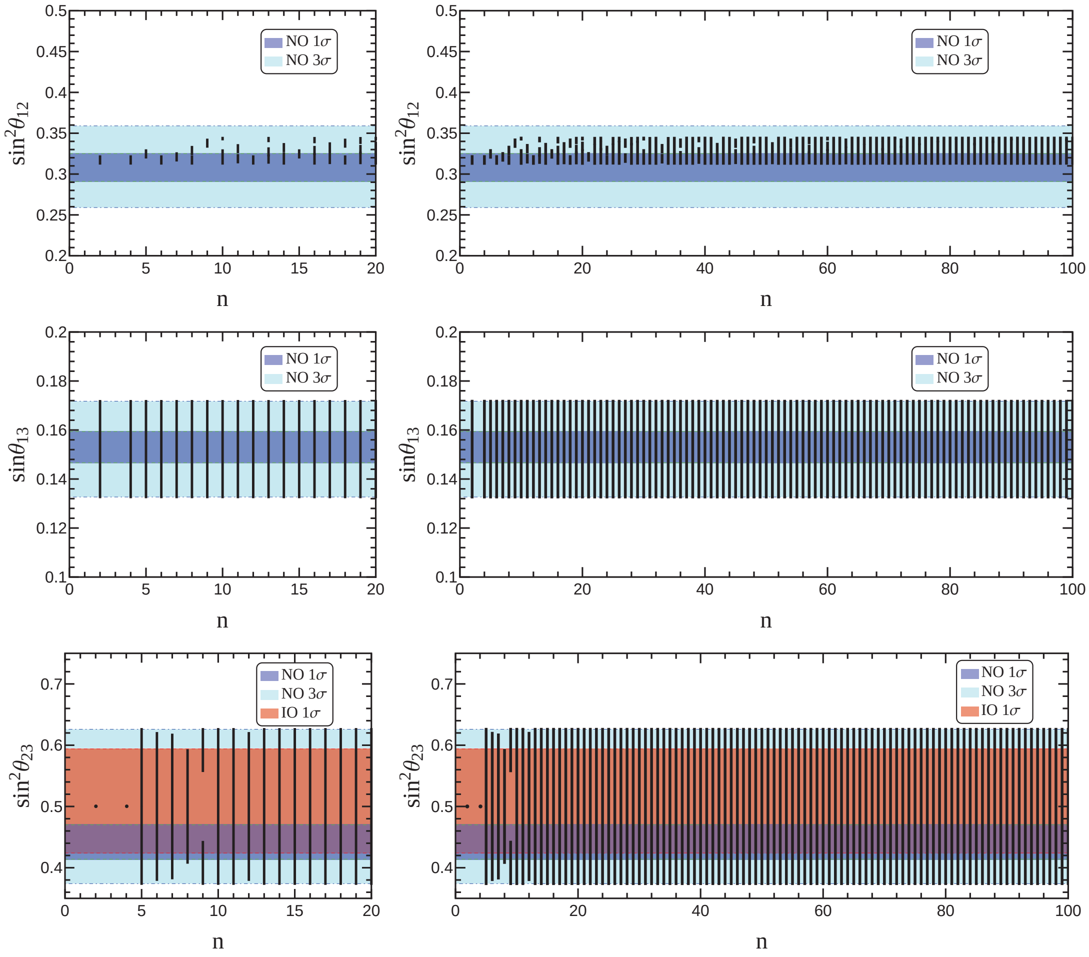


Figure 4.1: Numerical results for case I, 1st-3rd ordering with the PMNS matrices given in Eq. (4.29): allowed values of $\sin^2\theta_{12}$, $\sin\theta_{13}$ and $\sin^2\theta_{23}$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

exchanging the second and the third rows, three rearrangements are possible,

$$U^{I,4th}_{PMNS} = U^{I}_{PMNS}P_{213}, \quad U^{I,5th}_{PMNS} = P_{231}U^{I}_{PMNS}P_{213}, \quad U^{I,6th}_{PMNS} = P_{312}U^{I}_{PMNS}P_{213}\,. \tag{4.32}$$

Analogously to Eq. (4.30), these three forms of the PMNS matrix are related by parameter redefinition as follows

$$\begin{aligned} U^{I,5th}_{PMNS}(\theta,\varphi_1,\varphi_2) &= \mathrm{diag}(1,1,-1)U^{I,4th}_{PMNS}(\pi-\theta,\frac{\pi}{3}+\varphi_1,\varphi_2)\mathrm{diag}(1,1,-1),\\ U^{6th}_{PMNS}(\theta,\varphi_1,\varphi_2) &= \mathrm{diag}(-1,1,1)U^{I,4th}_{PMNS}(-\theta,-\frac{\pi}{3}+\varphi_1,\varphi_2)\mathrm{diag}(-1,1,1)\,. \end{aligned} \tag{4.33}$$

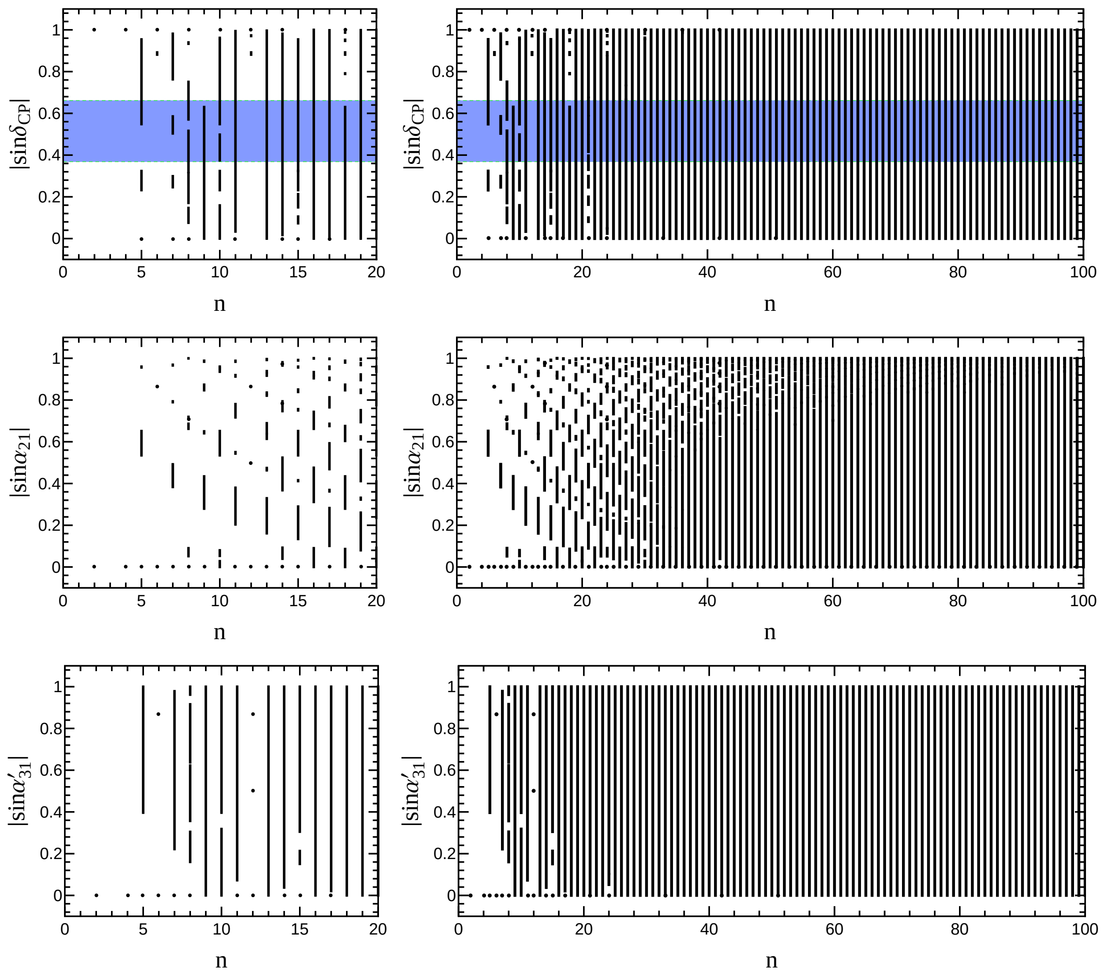


Figure 4.2: Numerical results in case I, 1st-3rd ordering with the PMNS matrices given in Eq. (4.29): the possible values of $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ for different n, where the three lepton mixing angles are required to lie in the 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

In this case, one finds the following relation,

$$3\sin^2\theta_{12}\cos^2\theta_{13} = 2\sin^2\varphi_1\,, \tag{4.34}$$

which yields $0.614 \le |\sin\varphi_1| \le 0.727$ at 3σ confidence level, and therefore the parameter φ_1 is to be in the range

$$\varphi_1 \in \pm\left([0.210\pi, 0.259\pi]\right) \cup [0.741\pi, 0.790\pi]\right)\,. \tag{4.35}$$

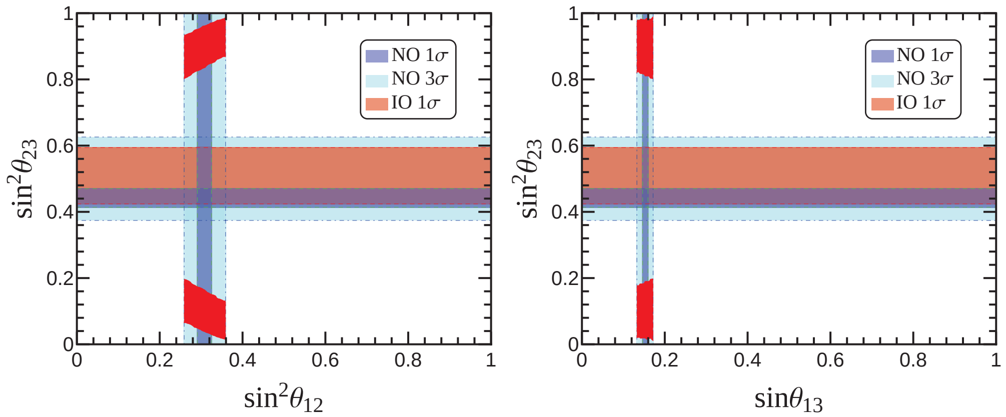


Figure 4.3: Numerical results in case I, 4th-6th ordering with the PMNS matrices given in Eq. (4.32). The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_1 and φ_2 to be continuous (which is equivalent to taking the limit $n \to \infty$), where θ_{12} and θ_{13} are required to lie in their 3σ ranges. The resulting predictions for θ_{23} are far beyond its 3σ range. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

For the representative values $\pm\pi/4$ and $\pm 3\pi/4$ of φ_1, the relatively small θ_{13} leads to

$$(\varphi_1, \varphi_2, \theta) \simeq (\pm\frac{\pi}{4}, 0, \frac{3\pi}{4}), \quad (\pm\frac{\pi}{4}, \pi, \frac{\pi}{4}), \quad (\pm\frac{3\pi}{4}, 0, \frac{\pi}{4}), \quad (\pm\frac{3\pi}{4}, \pi, \frac{3\pi}{4})\,. \tag{4.36}$$

Accordingly the atmospheric mixing angle θ_{23} would be

$$\sin^2\theta_{23} \simeq \frac{1}{4}(2-\sqrt{3}) \simeq 0.067, \qquad \text{or} \qquad \sin^2\theta_{23} \simeq \frac{1}{4}(2+\sqrt{3}) \simeq 0.933\,, \tag{4.37}$$

which is not compatible with the global analysis of neutrino oscillation data [78]. As a result, the three lepton mixing angles can not be accommodated simultaneously in this case, and this mixing pattern is not viable. The detailed numerical results are presented in Fig. 4.3. The correct values of the atmospheric mixing angle really cannot be achieved for realistic θ_{12} and θ_{13}.

Finally the fixed column $\sqrt{2/3}\,(\sin\varphi_1, \cos(\pi/6-\varphi_1), \cos(\pi/6+\varphi_1))^T$ can be placed in the third column. Using the freedom of exchanging the rows of the PMNS matrix, three equivalent configurations are found,

$$U^{I,7th}_{PMNS} = U^{I}_{PMNS} P_{321}, \quad U^{I,8th}_{PMNS} = P_{231} U^{I}_{PMNS} P_{321}, \quad U^{I,9th}_{PMNS} = P_{312} U^{I}_{PMNS} P_{321}\,, \tag{4.38}$$

which are related by

$$U^{I,8th}_{PMNS}(\theta,\varphi_1,\varphi_2)=\text{diag}(1,1,-1)U^{I,7th}_{PMNS}(\pi-\theta,\frac{\pi}{3}+\varphi_1,\varphi_2)\text{diag}(-1,1,1),$$
$$U^{I,9th}_{PMNS}(\theta,\varphi_1,\varphi_2)=\text{diag}(-1,1,1)U^{I,7th}_{PMNS}(-\theta,-\frac{\pi}{3}+\varphi_1,\varphi_2)\text{diag}(1,-1,1)\,. \tag{4.39}$$

For the 3σ interval $1.76\times10^{-2}\leq\sin^2\theta_{13}\leq2.95\times10^{-2}$ [78], one finds

$$0.378\leq\sin^2\theta_{23}\leq0.406,\qquad\text{or}\qquad0.594\leq\sin^2\theta_{23}\leq0.622\,. \tag{4.40}$$

This mixing pattern can be directly tested by future atmospheric neutrino oscillation experiments or long baseline neutrino oscillation experiments. If θ_{23} is found to be nearly maximal, this mixing would be ruled out. Furthermore, the precisely measured θ_{13} leads to $0.162\leq|\sin\varphi_1|\leq0.210$, and therefore φ_1 has to be in the following range

$$\varphi_1\in\pm\left([0.0519\pi,0.0675\pi]\cup[0.933\pi,0.948\pi]\right)\,, \tag{4.41}$$

which implies that φ_1 should be rather close to 0 or π. To reproduce the observed value of the reactor mixing angle, the two smallest values for n are 5 and 10, i.e. at least $\Delta(150)$ or $\Delta(600)$ is needed to produce viable mixing in this case. The admissible values of $\sin^2\theta_{23}$ and $\sin\theta_{13}$ for $n=5$, 10, 20 and 30 are plotted in Fig. 7.2. Furthermore, the variation of the allowed values of the lepton mixing parameters with respect to n are shown in Fig. 4.4 and Fig. 4.5. Compared with previous cases, both θ_{23} and θ_{13} are predicted to take several discrete values until $n=100$ in this case. It is interesting that the Majorana phase α'_{31} is constrained to be in the range of $0\leq|\sin\alpha'_{31}|\leq0.91$ while both δ_{CP} and α_{21} can take any values between 0 and 2π for large n.

(II) $G_l=\langle abc^sd^t\rangle$, $G_\nu=Z_2^{bc^xd^x}$, $X_{\nu\mathbf{r}}=\{\rho_\mathbf{r}(c^\gamma d^{-2x-\gamma}),\rho_\mathbf{r}(bc^{x+\gamma}d^{-x-\gamma})\}$

In this case, the PMNS matrix is determined to be

$$U^{II}_{PMNS}=\frac{1}{2}\begin{pmatrix}-\sin\theta-\sqrt{2}e^{i\varphi_3}\cos\theta & 1 & \cos\theta-\sqrt{2}e^{i\varphi_3}\sin\theta\\ -\sin\theta+\sqrt{2}e^{i\varphi_3}\cos\theta & 1 & \cos\theta+\sqrt{2}e^{i\varphi_3}\sin\theta\\ -\sqrt{2}\sin\theta & -\sqrt{2} & \sqrt{2}\cos\theta\end{pmatrix}, \tag{4.42}$$

or the one obtained by exchanging the second and the third rows, where the parameter φ_3 is

$$\varphi_3=-\frac{3\gamma+2s-t+2x}{n}\pi\,. \tag{4.43}$$

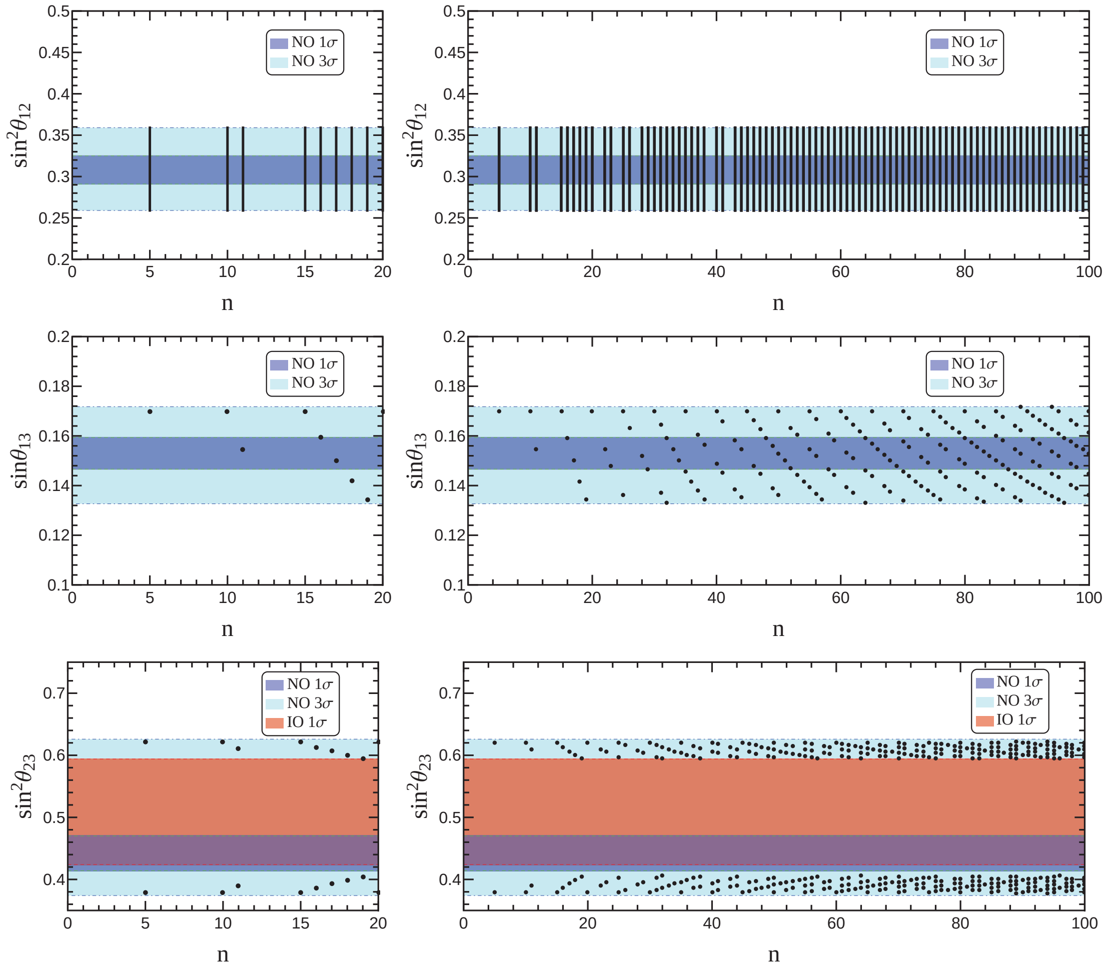


Figure 4.4: Numerical results in case I, 7th-9th ordering with the PMNS matrices given in Eq. (4.38): the allowed values of $\sin^2\theta_{12}$, $\sin\theta_{13}$ and $\sin^2\theta_{23}$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

It can take $2n$ discrete values:

$$\varphi_3 \mod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \ldots, \frac{2n-1}{n}\pi\,. \tag{4.44}$$

The eigenvalues of $abc^s d^t$ would be degenerate for $t=0$ such that the unitary transformation U_l can be made unique. If that is the case, one could choose the residual symmetry to be $G_l = K_4^{(c^{n/2},abc^s)}$ which leads to same PMNS matrix shown in Eq. (4.42) with $t=0$. This mixing pattern has one column $\left(1/2,1/2,-1/\sqrt{2}\right)^T$ which is the same as the first (second) column of the bimaximal mixing up to permutations. In order to in accordance with the experimental data, the fixed vector $\left(1/2,1/2,-1/\sqrt{2}\right)^T$ can only be the second column of the PMNS matrix.

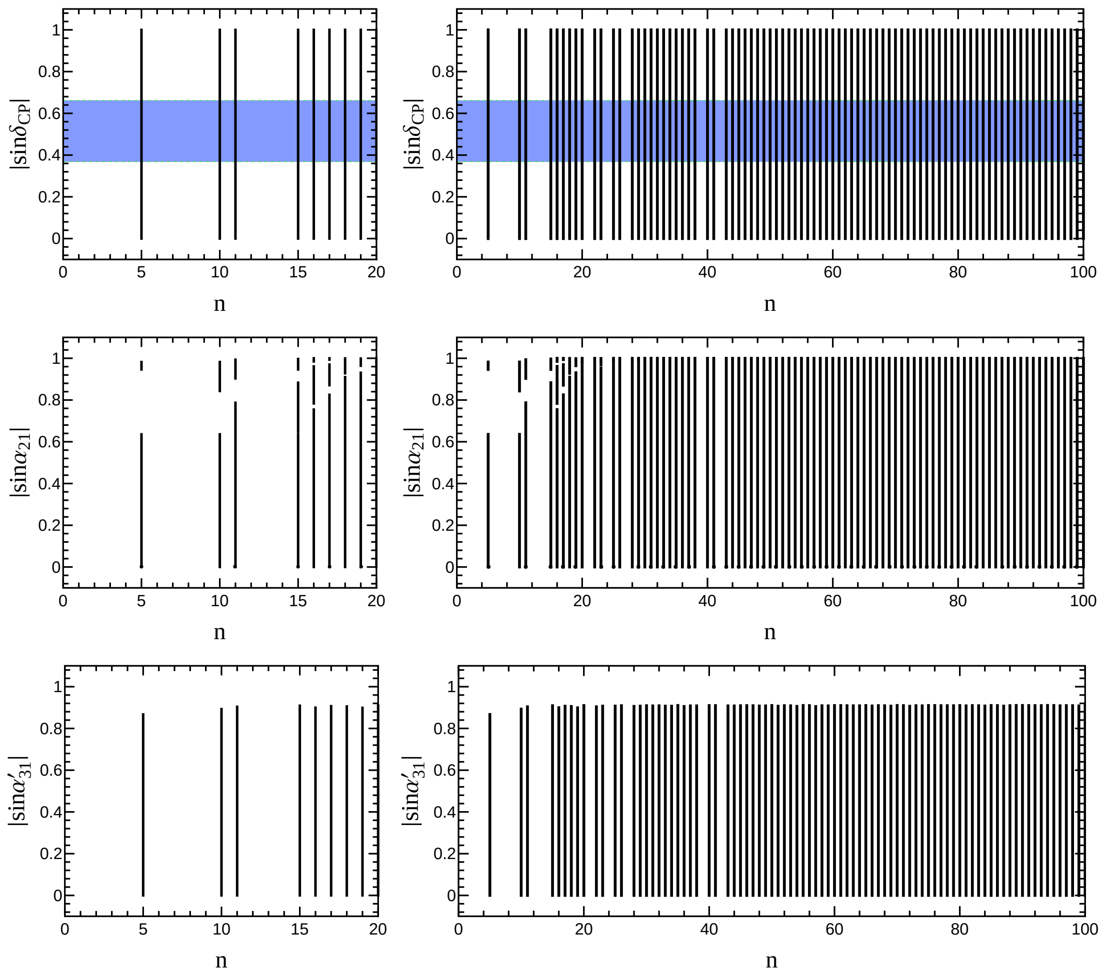


Figure 4.5: Numerical results in case I, 7th-9th ordering with the PMNS matrices given in Eq. (4.38): the allowed values of $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing angles are taken from Ref. [78].

The following correlation is found:

$$4\sin^2\theta_{12}\cos^2\theta_{13} = 1\,, \tag{4.45}$$

which leads to $0.254 \leq \sin^2\theta_{12} \leq 0.258$ for the measured value of the reactor mixing angle [78]. Therefore $\sin^2\theta_{12}$ is predicted to be very close to its 3σ lower bound 0.259 [78] in this case. Furthermore, the expression for $\sin^2\theta_{13}$ in Eq. (7.46) yields

$$\frac{1}{8}\left(3-\sqrt{1+8\cos^2\varphi_3}\right) \leq \sin^2\theta_{13} \leq \frac{1}{8}\left(3+\sqrt{1+8\cos^2\varphi_3}\right)\,. \tag{4.46}$$

In order to be in accordance with experimental data, the parameter φ_3 has to be in the range

$$\varphi_3 \in [0, 0.135\pi] \cup [0.865\pi, 1.135\pi] \cup [1.865\pi, 2\pi] \,. \tag{4.47}$$

The allowed values of the mixing parameters with respect to n are shown in Fig. 4.6 and Fig. 4.7, and the correlations between them are plotted in Fig. 7.3, where the 3σ lower bound of $\sin^2\theta_{12}$ is chosen to be 0.254 instead of 0.259 given in Ref. [78]. The values of $\varphi_3 = 0, \pi$ are always acceptable, and the corresponding Dirac and Majorana CP phases are conserved. Note that only CP conserving cases are allowed for $n = 2, 3, \ldots, 7$. Moreover, the CP violating phases δ_{CP} and α_{21} are predicted to fulfil $|\sin\delta_{CP}| \leq 0.895$ and $|\sin\alpha_{21}| \leq 0.545$ while α'_{31} is not constrained at all for large n.

(III) $G_l = \langle ac^s d^t\rangle$, $G_\nu = Z_2^{c^{n/2}}$, $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$

This case is only possible if n is divisible by 2, and the PMNS matrix takes the form

$$U^{III}_{PMNS} = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\varphi_4}\cos\theta - e^{i\varphi_5}\sin\theta & 1 & e^{i\varphi_4}\sin\theta + e^{i\varphi_5}\cos\theta \\ \omega e^{i\varphi_4}\cos\theta - \omega^2 e^{i\varphi_5}\sin\theta & 1 & \omega e^{i\varphi_4}\sin\theta + \omega^2 e^{i\varphi_5}\cos\theta \\ \omega^2 e^{i\varphi_4}\cos\theta - \omega e^{i\varphi_5}\sin\theta & 1 & \omega^2 e^{i\varphi_4}\sin\theta + \omega e^{i\varphi_5}\cos\theta \end{pmatrix}, \tag{4.48}$$

where

$$\varphi_4 = \frac{\gamma + \delta + 2s}{n}\pi, \qquad \varphi_5 = \frac{2\delta - \gamma + 2t}{n}\pi \,, \tag{4.49}$$

which can take the values

$$\varphi_4, \varphi_5 \bmod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \ldots, \frac{2n-1}{n}\pi \,. \tag{4.50}$$

Agreement with experimental data can be achieved only if the vector $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)^T$ is placed in the second column, which results in so-called TM_2 mixing [99]. There are three independent arrangements up to the exchange of the second and the third row,

$$U^{III,1st}_{PMNS} = U^{III}_{PMNS}, \qquad U^{III,2nd}_{PMNS} = P_{231}U^{III}_{PMNS}, \qquad U^{III,3rd}_{PMNS} = P_{312}U^{III}_{PMNS} \,. \tag{4.51}$$

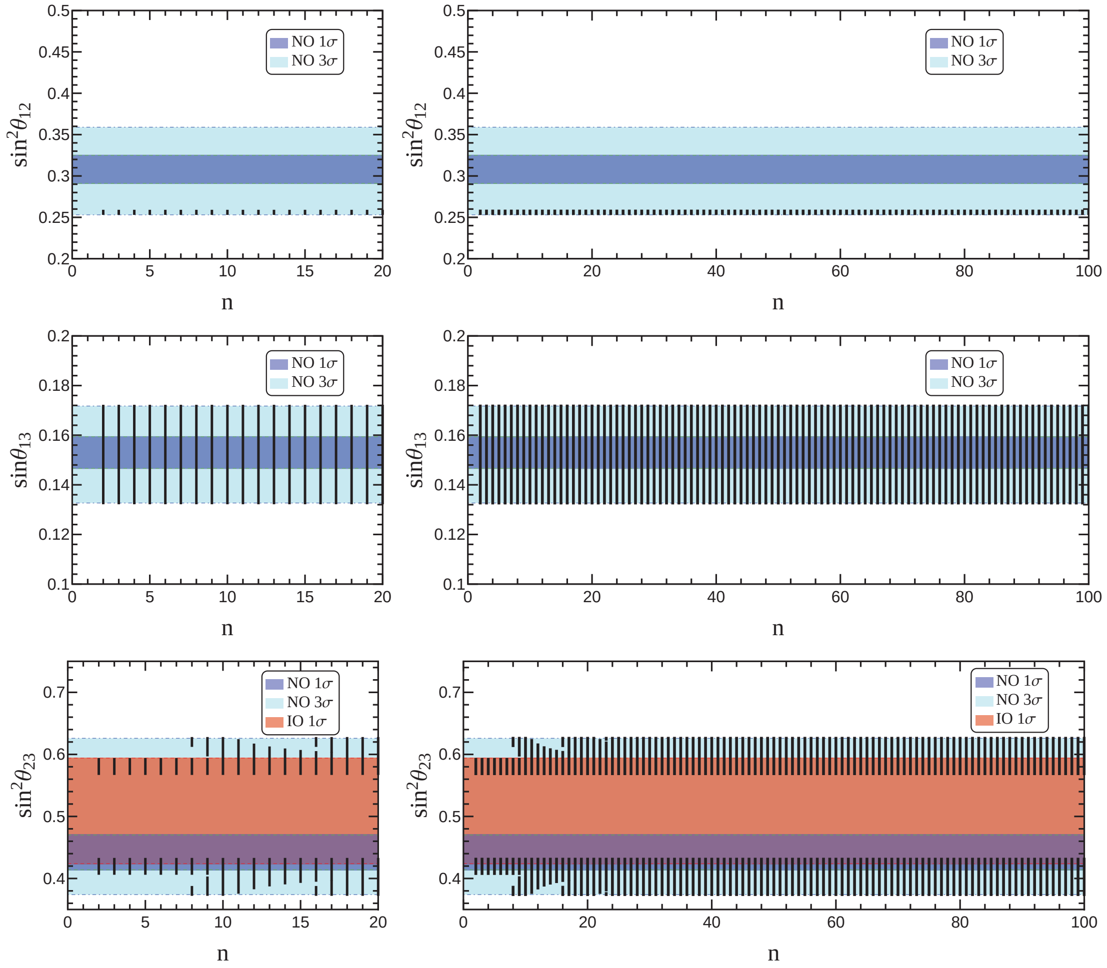


Figure 4.6: Numerical results in case II: the allowed values of $\sin^2\theta_{12}$, $\sin\theta_{13}$ and $\sin^2\theta_{23}$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2\theta_{12}$ is chosen to be 0.254 instead of 0.259 given in Ref. [78]). The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

They are related as follows,

$$U_{PMNS}^{III,2nd}(\theta,\varphi_4,\varphi_5) = U_{PMNS}^{III,1st}(\theta,\varphi_4+\frac{2\pi}{3},\varphi_5-\frac{2\pi}{3}),$$
$$U_{PMNS}^{III,3rd}(\theta,\varphi_4,\varphi_5) = U_{PMNS}^{III,1st}(\theta,\varphi_4-\frac{2\pi}{3},\varphi_5+\frac{2\pi}{3})\,. \qquad (4.52)$$

It is enough to study the phenomenological predictions of $U_{PMNS}^{III,1st}$.

All mixing parameters depend on the combination $\varphi_5-\varphi_4$ except $|\tan\alpha_{21}|$. Common to all TM_2 mixing, θ_{13} and θ_{12} are related with each other via:

$$3\cos^2\theta_{13}\sin^2\theta_{12} = 1. \qquad (4.53)$$

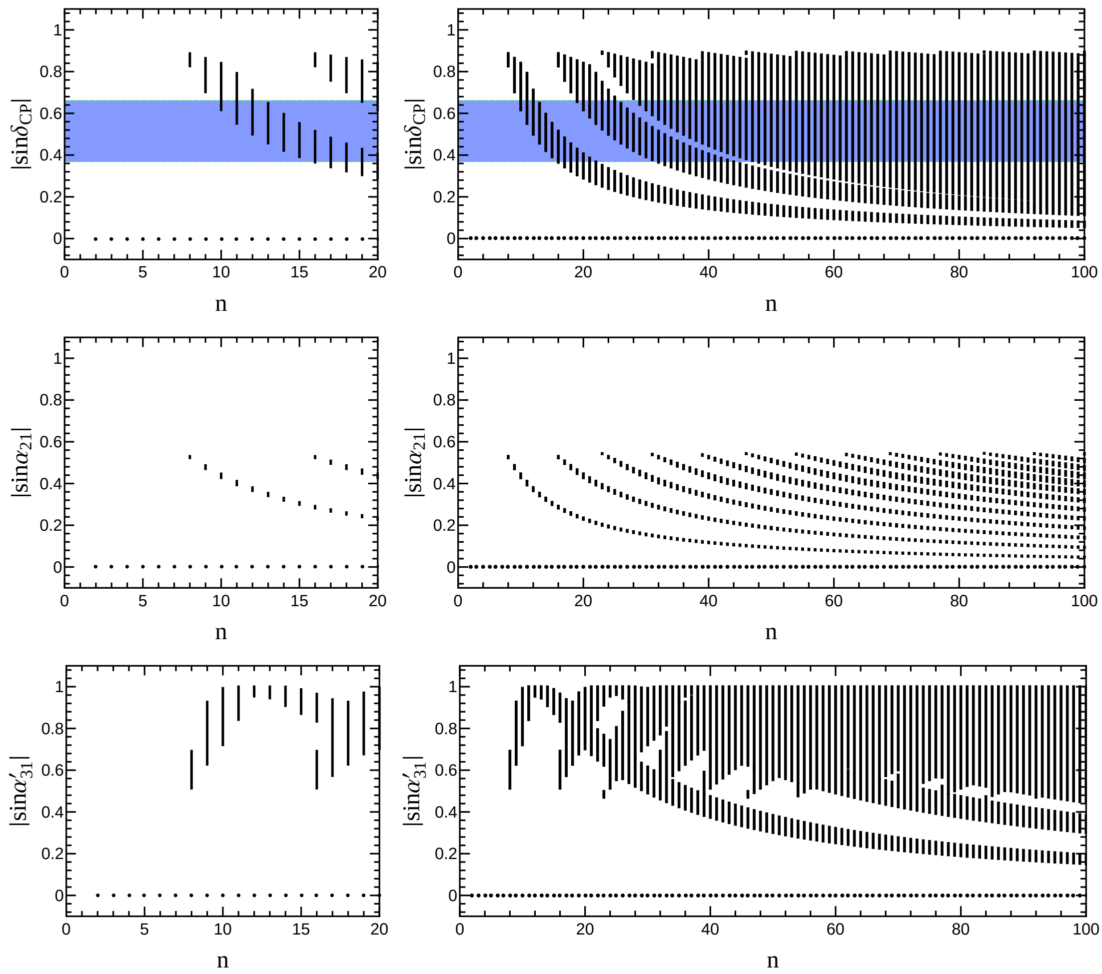


Figure 4.7: Numerical results in case II: the allowed values of $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2\theta_{12}$ is chosen to be 0.254 instead of 0.259 given in Ref. [78]). The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

Therefore θ_{12} admits a lower bound $\sin^2\theta_{12} > 1/3$. Given the 3σ interval of θ_{13} [78], we find $0.339 \leq \sin^2\theta_{12} \leq 0.343$. This prediction can be tested at JUNO in the near future. In addition, θ_{13} and θ_{23} are correlated as follows

$$\frac{3\cos^2\theta_{13}\sin^2\theta_{23}-1}{1-3\sin^2\theta_{13}} = \frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(\varphi_5-\varphi_4\right)\,. \tag{4.54}$$

The expression for θ_{13} in Eq. (7.47) implies that

$$\begin{aligned}\tfrac{1}{3}(1-|\sin 2\theta|) \leq \sin^2\theta_{13} \leq \tfrac{1}{3}(1+|\sin 2\theta|),\\ \tfrac{1}{3}(1-|\cos(\varphi_5-\varphi_4)|) \leq \sin^2\theta_{13} \leq \tfrac{1}{3}(1+|\cos(\varphi_5-\varphi_4)|)\,,\end{aligned} \tag{4.55}$$

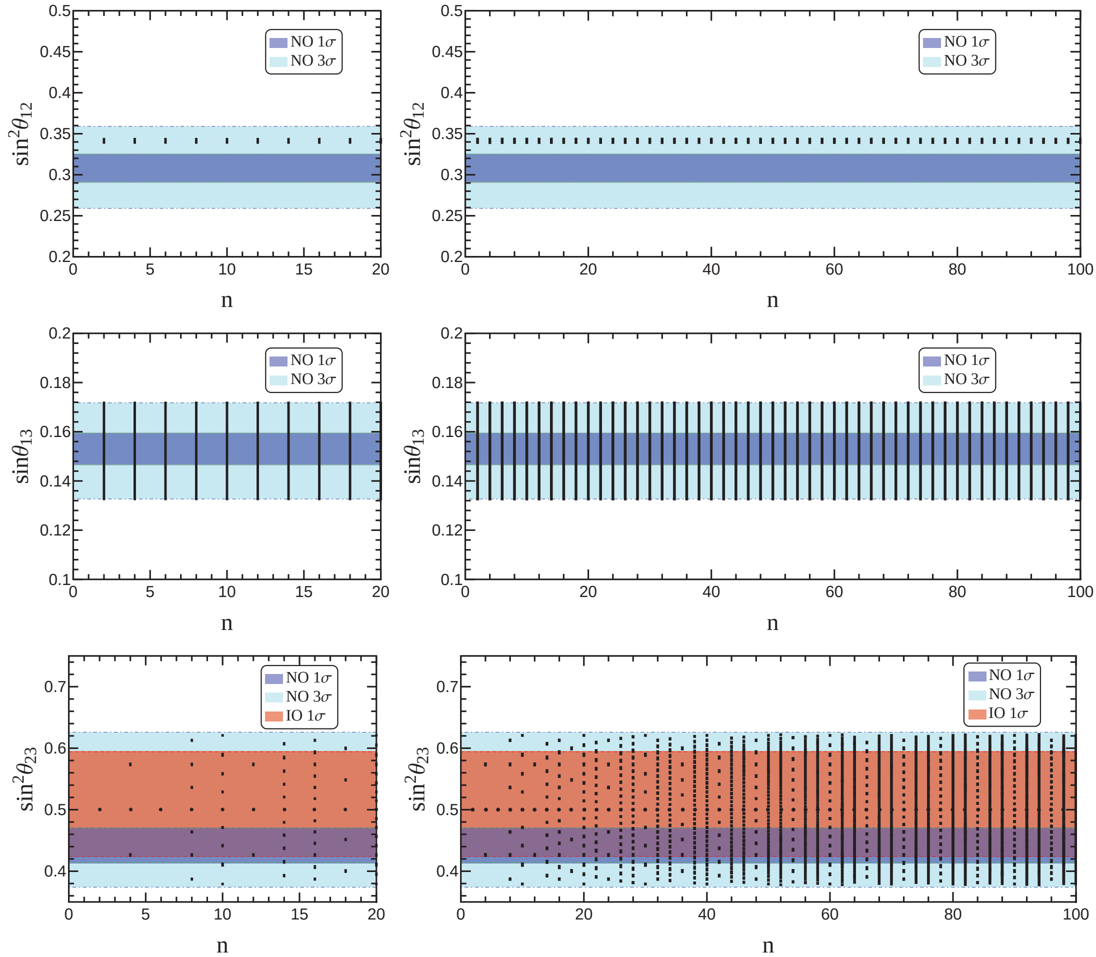


Figure 4.8: Numerical results in case III: the allowed values of $\sin^2\theta_{12}$, $\sin\theta_{13}$ and $\sin^2\theta_{23}$ for different n, where the three lepton mixing angles are required to lie in their 3σ regions. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78]. Note that n should be even in this case.

which yields

$$\theta \in [0.183\pi, 0.317\pi] \cup [0.683\pi, 0.817\pi] ,$$
$$\varphi_5 - \varphi_4 \in [-0.135\pi, 0.135\pi] \cup [0.865\pi, 1.135\pi] . \qquad (4.56)$$

The allowed values of the mixing parameters for different n are shown in Fig 4.8 and Fig. 4.9. The case of $\varphi_4 = \varphi_5$ is always viable for any n, and the resulting θ_{23} and δ_{CP} are maximal while the Majorana phase α'_{31} is trivial. Correlations among the mixing parameters are plotted in Fig. 7.4. The three CP phases can take any values for large n.

(IV) $G_l = \langle ac^s d^t \rangle$, $G_\nu = Z_2^{c^{n/2}}$, $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(abc^\gamma d^\delta)$

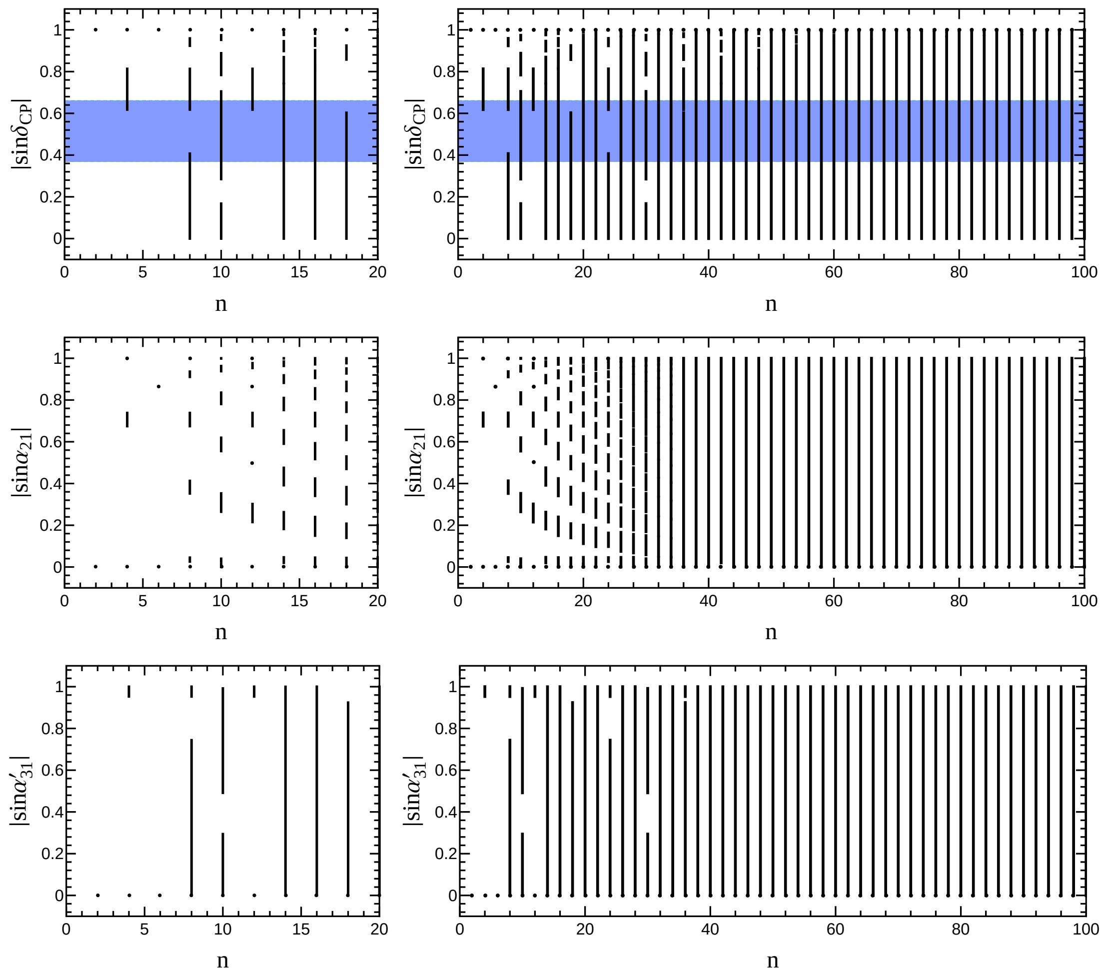


Figure 4.9: Numerical results in case III: the allowed values of $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ for different n, where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing angles are taken from Ref. [78]. Note that n needs to be even in this case.

In this case, the PMNS matrix is of the form

$$U^{IV}_{PMNS}=\frac{1}{\sqrt{3}}\begin{pmatrix} i\sqrt{2}e^{i\varphi_7}\sin\left(\varphi_6-\frac{\phi}{2}\right) & 1 & \sqrt{2}e^{i\varphi_7}\cos\left(\varphi_6-\frac{\phi}{2}\right) \\ i\sqrt{2}e^{i\varphi_7}\cos\left(\varphi_6-\frac{\phi}{2}+\frac{\pi}{6}\right) & 1 & -\sqrt{2}e^{i\varphi_7}\sin\left(\varphi_6-\frac{\phi}{2}+\frac{\pi}{6}\right) \\ -i\sqrt{2}e^{i\varphi_7}\cos\left(\varphi_6-\frac{\phi}{2}-\frac{\pi}{6}\right) & 1 & \sqrt{2}e^{i\varphi_7}\sin\left(\varphi_6-\frac{\phi}{2}-\frac{\pi}{6}\right) \end{pmatrix}, \tag{4.57}$$

with

$$\varphi_6=\frac{s-t-\gamma}{n}\pi,\qquad \varphi_7=\frac{s+t+3\gamma}{n}\pi\,. \tag{4.58}$$

The constant vector $\left(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}\right)^T$ must be the second column to account for the measured values of the lepton mixing angles. The PMNS matrices corresponding to other ordering of rows and columns are related to the above one through redefinition of the free parameter ϕ. This case differs from case III in the residual CP symmetry, and the resulting PMNS matrix in Eq. (4.57) is still of TM_2 form. The associated lepton mixing parameters read as:

$$
\begin{aligned}
&\sin^2\theta_{13}=\frac{1}{3}\left[1+\cos(\phi-2\varphi_6)\right], \quad \sin^2\theta_{12}=\frac{1}{2-\cos(\phi-2\varphi_6)}\,,\\
&\sin^2\theta_{23}=\frac{1-\sin\left(\phi-2\varphi_6+\pi/6\right)}{2-\cos(\phi-2\varphi_6)}\,,\\
&\tan\delta_{CP}=\tan\alpha'_{31}=J_{CP}=0, \qquad |\tan\alpha_{21}|=|\tan(2\varphi_7)|\,. \qquad (4.59)
\end{aligned}
$$

The contribution of φ_6 can be absorbed into the free parameter ϕ via redefinition $\phi\to\phi+2\varphi_6$, the reason for this is that the PMNS matrix in Eq. (4.57) and the resulting mixing parameters in Eq. (4.59) depend on the combination $\phi-2\varphi_6$. Regarding to the CP violating phases, both δ_{CP} and α'_{31} are always conserved while α_{21} can be any value of $0, \frac{1}{n}\pi, \frac{2}{n}\pi, \ldots, \frac{2n-1}{n}\pi$ in this scenario. Furthermore, the three mixing angles are strongly related with each other as follows:

$$
3\cos^2\theta_{13}\sin^2\theta_{12}=1, \qquad \sin^2\theta_{23}=\frac{1}{2}\pm\frac{1}{2}\tan\theta_{13}\sqrt{2-\tan^2\theta_{13}}\,. \qquad (4.60)
$$

For the best fitting value of $\sin^2\theta_{13}=0.0234$ [78], the solar and atmospheric angles are determined to be

$$
\sin^2\theta_{12}\simeq 0.341, \qquad \sin^2\theta_{23}\simeq 0.391 \quad \text{or} \quad 0.609\,, \qquad (4.61)
$$

which are compatible with the experimentally allowed regions. These correlations between the three mixing angles are shown in Fig. 7.5. We see that both θ_{12} and θ_{23} are constrained to be in a narrow range. The deviation of θ_{23} from maximal mixing is somewhat large. Hence this mixing pattern can be checked or ruled by precisely measuring θ_{12} and θ_{23} in next generation neutrino oscillation experiments.

4.4 Lepton mixing with residual symmetry $Z_2\times CP$ in the charged lepton sector

In the previous section, a $Z_2\times CP$ residual symmetry was preserved in the neutrino sector and an abelian subgroup of $\Delta(6n^2)$ in the charged lepton sector. In this section, the residual symmetry $Z_2\times CP$ is preserved in the charged lepton sector and the full symmetry $\Delta(6n^2)\rtimes H_{CP}$ is broken down to $K_4\rtimes H^{\nu}_{CP}$ in the neutrino sector. The phenomenological consequences of this scenario have been analysed for the simple flavour symmetry group $\Delta(24) = S_4$ in Ref. [87], while an extensive search in GAP was performed in [97]. All Z_2 subgroups of $\Delta(6n^2)$ had been listed in Eq. (4.12) and Eq. (4.13). The K_4 subgroups of $\Delta(6n^2)$ can be classified as follows:

$$
\begin{aligned}
K_4^{(c^{n/2},d^{n/2})} &\equiv \left\{1, c^{n/2}, d^{n/2}, c^{n/2}d^{n/2}\right\},\\
K_4^{(c^{n/2},abc^y)} &\equiv \left\{1, c^{n/2}, abc^y, abc^{y+n/2}\right\},\\
K_4^{(d^{n/2},a^2bd^z)} &\equiv \left\{1, d^{n/2}, a^2bd^z, a^2bd^{z+n/2}\right\},\\
K_4^{(c^{n/2}d^{n/2},bc^xd^x)} &\equiv \left\{1, c^{n/2}d^{n/2}, bc^xd^x, bc^{x+n/2}d^{x+n/2}\right\}, \qquad (4.62)
\end{aligned}
$$

where $K_4^{(c^{n/2},d^{n/2})}$ is a normal subgroup of $\Delta(6n^2)$, and the remaining three K_4 subgroups are conjugate to each other. This scenario is only possible if n is divisible by 2. Because of the relations relating conjugated elements, Eq. (4.14a), and Eq. (4.14b), one only needs to consider the representative cases of $G_l = Z_2^{bc^xd^x}, Z_2^{c^{n/2}}$ and $G_\nu = K_4^{(c^{n/2},d^{n/2})}$, $K_4^{(c^{n/2},abc^y)}$, $K_4^{(d^{n/2},a^2bd^z)}$ and $K_4^{(c^{n/2}d^{n/2},bc^xd^x)}$. Other possible choices of G_l and G_ν are related to these representative residual symmetry by similarity transformations, and therefore the same lepton mixing matrices are generated.

Following the same procedure as in section 4.3, the hermitian combination $m_l^\dagger m_l$ of the charged lepton mass matrix and its diagonalization matrix are calculated from the invariance under the residual symmetry. Comparing with the scenario of $Z_2 \times H^\nu_{CP}$ preserved in the neutrino sector which had been studied in section 4.3.2, we find that the unitary transformation U_l is of the same form as U_ν listed in section 4.3.2 if the both residual flavour symmetry and residual CP symmetry in the two occasions are identical and they are listed for completeness in Appendix 7.1.5.

4.4.1 Neutrino sector

In this section, the $\Delta(6n^2)$ flavour symmetry is broken down to K_4 in the neutrino sector. Hence the neutrino diagonalization matrix U_ν is entirely fixed by the residual K_4, and the residual CP symmetry allows to further determine the three leptonic CP violating phases up to π. The residual CP symmetry H^ν_{CP} in the neutrino sector must be compatible with the residual K_4 symmetry, and the consistency condition should be satisfied,[6]

$$X_{\nu\mathbf{r}}\rho^*_{\mathbf{r}}(g)X^{-1}_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(g'), \qquad g, g' \in K_4\,. \tag{4.63}$$

Solving this equation, we can find the consistent residual CP symmetries for different K_4 subgroups are as follows:

- $K_4^{(c^{n/2},d^{n/2})}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(h), \qquad h \in \Delta(6n^2)\,. \tag{4.64}$$

- $K_4^{(c^{n/2},abc^y)}$, $y = 0, 1, \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma+n/2}), \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma}), \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma+n/2})\,, \tag{4.65}$$

  with $\gamma = 0, 1, \ldots n-1$.

- $K_4^{(d^{n/2},a^2bd^z)}$, $z = 0, 1, \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \rho_{\mathbf{r}}(c^{2z+2\delta+n/2}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2\delta}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2\delta+n/2}d^\delta)\,, \tag{4.66}$$

  where $\delta = 0, 1, \ldots n-1$.

- $K_4^{(c^{n/2}d^{n/2},bc^xd^x)}$, $x = 0, 1, \ldots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma+n/2}), \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma}), \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma+n/2})\,, \tag{4.67}$$

  with $\gamma = 0, 1, \ldots n-1$.

The light neutrino mass matrix is again constrained by the residual flavour symmetry K_4 and the residual CP symmetry H^ν_{CP}:

$$\begin{aligned} &\rho^T_{\mathbf{3}}(g_\nu)m_\nu\rho_{\mathbf{3}}(g_\nu) = m_\nu, \quad g_\nu \in K_4\,, \\ &X^T_{\nu\mathbf{3}}m_\nu X_{\nu\mathbf{3}} = m^*_\nu, \quad X_\nu \in H^\nu_{CP}\,. \end{aligned} \tag{4.68}$$

The mass and diagonalisation matrices are listed in Appendix 7.1.6 and in the following directly the physical mixing results are given.

	$G_l = Z_2^{bc^{x'}d^{x'}}$	$G_l = Z_2^{c^{n/2}}$
$G_\nu = K_4^{(c^{n/2},d^{n/2})}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}^T$ ✗	$\begin{pmatrix}1\\0\\0\end{pmatrix}^T$ ✗
$G_\nu = K_4^{(c^{n/2},abc^y)}$	$\frac{1}{2}\begin{pmatrix}1\\1\\-\sqrt{2}\end{pmatrix}^T$ ✓	$\begin{pmatrix}1\\0\\0\end{pmatrix}^T$ ✗
$G_\nu = K_4^{(d^{n/2},a^2bd^z)}$	$\frac{1}{2}\begin{pmatrix}1\\1\\-\sqrt{2}\end{pmatrix}^T$ ✓	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}^T$ ✗
$G_\nu = K_4^{(c^{n/2}d^{n/2},bc^xd^x)}$	$\begin{pmatrix}\cos\left(\frac{x-x'}{n}\pi\right)\\-i\sin\left(\frac{x-x'}{n}\pi\right)\\0\end{pmatrix}^T$ ✗	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}^T$ ✗

Table 4.3: The form of the row of the PMNS matrix that is fixed for different residual symmetries G_ν and G_l which are K_4 and Z_2 subgroups of $\Delta(6n^2)$ flavour symmetry group respectively. The superscript "T" means transpose. The symbol "✗" denotes that the resulting lepton mixing is ruled out since there is at least one zero element in the fixed row, and the symbol "✓" denote that the resulting mixing is viable. Note that for $G_l = Z_2^{bc^{x'}d^{x'}}$, the cases of $G_\nu = K_4^{(c^{n/2},abc^y)}$ and $G_\nu = K_4^{(d^{n/2},a^2bd^z)}$ are equivalent because the residual symmetries are related by group conjugation as $b(bc^{x'}d^{x'})b = bc^{-x'}d^{-x'}$, $bd^{n/2}b = c^{n/2}$ and $b(a^2bd^z)b = abc^{-z}$.

4.4.2 Predictions for lepton flavour mixing

As the different residual symmetries related by group conjugation lead to the same predictions for the lepton mixing matrix, one only needs to consider the cases of $G_l = Z_2^{bc^xd^x}, Z_2^{c^{n/2}}$ and $G_\nu = K_4^{(c^{n/2},d^{n/2})}$, $K_4^{(c^{n/2},abc^y)}$, $K_4^{(d^{n/2},a^2bd^z)}$ and $K_4^{(c^{n/2}d^{n/2},bc^xd^x)}$. Compared with section 4.3, one row instead of one column of the PMNS matrix is fixed by residual flavour symmetry in this scenario.[7] The explicit form of this row vector for different residual symmetry is summarized in Table 4.3. Only one independent case is viable. Taking into account the residual CP symmetry, both mixing angles and CP phases in terms of one free parameter are predicted in terms of one free parameter that is not related to the choice of the residual symmetry itself.

[6] Again, as in the previous chapter, some of these CP symmetries are not constrained enough, because the consistency condition used is not strict enough.

[7] This leads to a new sort of sum rules.

(V) $G_l = \{1, bc^{x'}d^{x'}\}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^{\gamma'}d^{-2x'-\gamma'}), \rho_{\mathbf{r}}(bc^{x'+\gamma'}d^{-x'-\gamma'})\}$, $G_\nu = K_4^{(c^{n/2}, abc^y)}$ and $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma}d^{2y+2\gamma})\}$

Combining the unitary transformation U_l in Eq. (7.51) and U_ν in Eq. (7.84), we can pin down the lepton flavour mixing matrix as follows:

$$U_{PMNS}^{V} = \frac{1}{2}\begin{pmatrix} \sin\theta + \sqrt{2}e^{i\varphi_8}\cos\theta & \sin\theta - \sqrt{2}e^{i\varphi_8}\cos\theta & \sqrt{2}e^{i\varphi_9}\sin\theta \\ 1 & 1 & -\sqrt{2}e^{i\varphi_9} \\ \cos\theta - \sqrt{2}e^{i\varphi_8}\sin\theta & \cos\theta + \sqrt{2}e^{i\varphi_8}\sin\theta & \sqrt{2}e^{i\varphi_9}\cos\theta \end{pmatrix}, \tag{4.69}$$

with

$$\varphi_8 = \frac{3\gamma' + 2x' + 2y}{n}\pi, \qquad \varphi_9 = -\frac{3\gamma + 2x' + 2y}{n}\pi\,. \tag{4.70}$$

Here φ_8 and φ_9 are independent, they are determined by the residual symmetry, and they can take the values,

$$\varphi_8, \varphi_9 \mod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \ldots, \frac{2n-1}{n}\pi\,. \tag{4.71}$$

In order to be in accordance with the present neutrino oscillation data, the vector $\left(1/2, 1/2, -e^{i\varphi_9}/\sqrt{2}\right)$ can only be the second or the third row. Note that as usual permutation of the second and the third rows of U_{PMNS}^{V} is also viable.

The mixing angles θ_{13} and θ_{23} are related as follows

$$2\cos^2\theta_{13}\sin^2\theta_{23} = 1, \quad \text{or} \quad 2\cos^2\theta_{13}\sin^2\theta_{23} = 1 - 2\sin^2\theta_{13}\,, \tag{4.72}$$

where the second relation is for the PMNS matrix obtained by exchanging the second and the third rows of U_{PMNS}^{V}. Moreover, θ_{12} and θ_{13} are related by

$$\cos^2\theta_{13}\cos 2\theta_{12} = \pm 2\sin\theta_{13}\sqrt{\cos 2\theta_{13}}\cos\varphi_8\,, \tag{4.73}$$

which is relevant to the parameter φ_8. The 3σ bound of $\sin^2\theta_{13}$ gives the limit on θ:

$$\theta \in [0.060\pi, 0.078\pi] \cup [0.922\pi, 0.940\pi]\,. \tag{4.74}$$

The equation for $\sin^2\theta_{12}$ in Eq. (7.92) leads to

$$\frac{1}{2}\left(1 - |\cos\varphi_8|\right) \le \sin^2\theta_{12} \le \frac{1}{2}\left(1 + |\cos\varphi_8|\right)\,, \tag{4.75}$$

Hence φ_8 is constrained to lie in the region

$$\varphi_8 \in [0, 0.409\pi] \cup [0.591\pi, 1.409\pi] \cup [1.591\pi, 2\pi]\,. \tag{4.76}$$

The numerical results are displayed in Fig. 4.10 and Fig. 4.11. Note that conserved CP corresponding to $\varphi_8 = 0, \pi$ is always viable. If one requires that all three mixing angles are in their 3σ intervals, one finds that $0.141 \leq \sin\theta_{13} \leq 0.172$, $0.328 \leq \sin^2\theta_{12} \leq 0.359$, and $\sin^2\theta_{23}$ is around 0.488 and 0.512 due to the correlation shown in Eq. (4.72). Note that θ_{23} is very close to maximal mixing. Therefore precisely measuring the lepton mixing angles at JUNO or long baseline neutrino experiments can test this mixing pattern directly. For the CP phases, δ_{CP} and α_{21} are predicted to be in the intervals of $|\sin\delta_{CP}| \leq 0.586$ and $|\sin\alpha_{21}| \leq 0.396$ while α'_{31} can have any value for sufficient large n. The correlations between different mixing parameters are shown in Fig. 7.6.

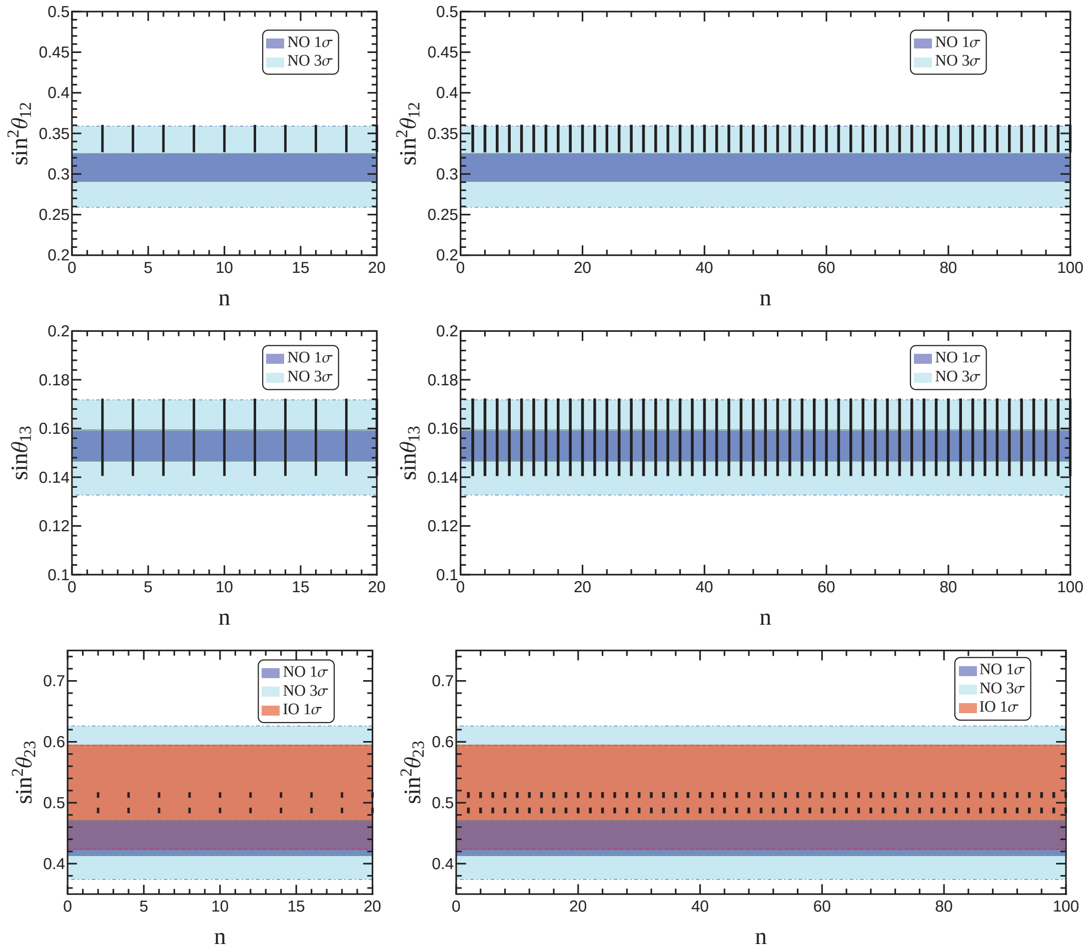


Figure 4.10: Numerical results in case V: the allowed ranges of $\sin^2\theta_{12}$, $\sin\theta_{13}$ and $\sin^2\theta_{23}$ for different n, where the three lepton mixing angles are required to lie in the 3σ regions. The 1σ and 3σ bounds of the mixing angles are taken from Ref. [78]. Note that n should be divisible by 2 in this case.

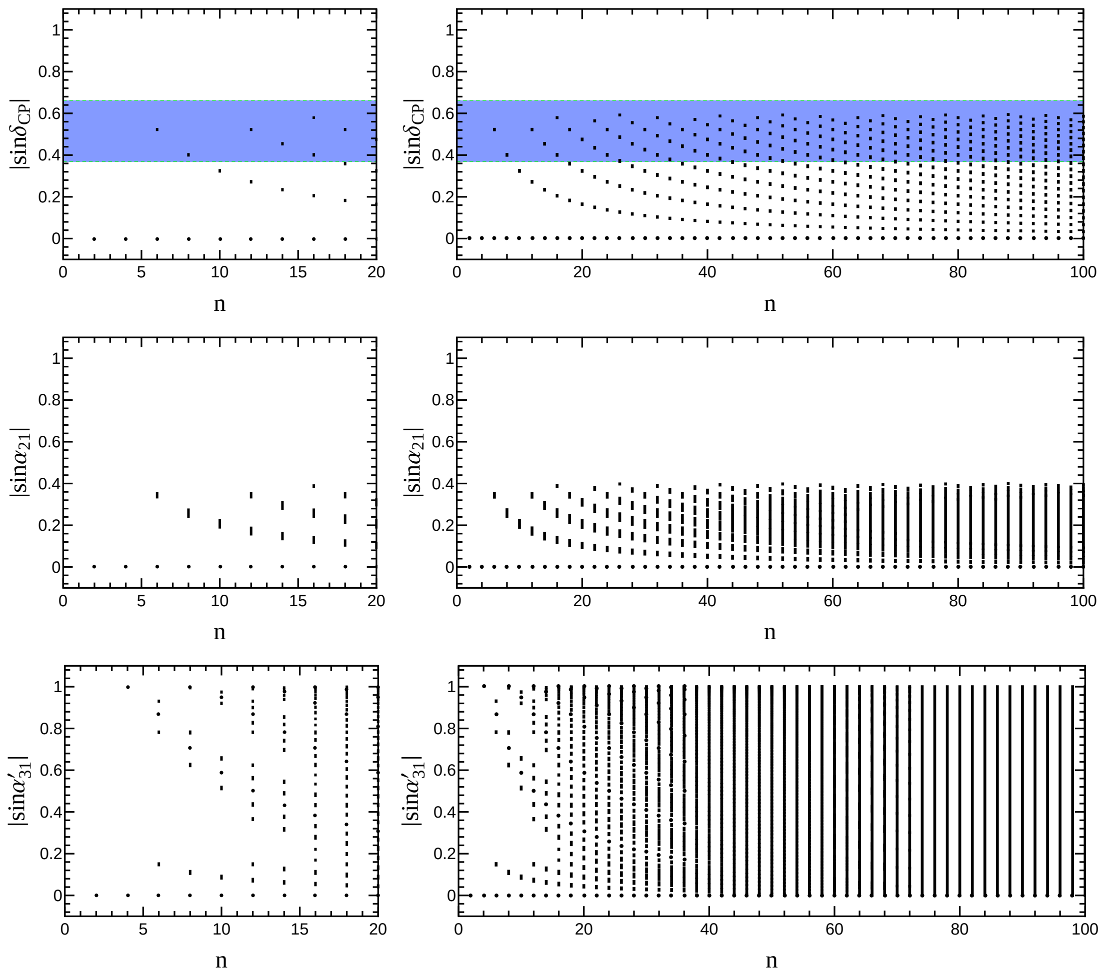


Figure 4.11: Numerical results in case V: the allowed ranges of $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ for different n, where the three lepton mixing angles are required to lie in the 3σ regions. The 1σ and 3σ bounds of the mixing angles are taken from Ref. [78]. Note that n should be divisible by 2 in this case.

4.5 Neutrinoless double-beta decay

The very rare (if possible) process of neutrinoless double-beta decay ($0\nu2\beta$), is an important probe for the Majorana nature of neutrino and lepton number violation, a sizeable number of new experiments are currently running, under construction, or in the planing phase. In models where $0\nu2\beta$ is dominated by light Majorana neutrinos, the particle physics contribution to the decay rate is parameterized by the effective mass of neutrinoless double-beta decay, which is [36]

$$|m_{ee}| = \left|(m_1 c_{12}^2 + m_2 s_{12}^2 e^{i\alpha_{21}})c_{13}^2 + m_3 s_{13}^2 e^{i\alpha'_{31}}\right| . \tag{4.77}$$

For normal hierarchy, the masses are

$$m_1 = m_l, \quad m_2 = \sqrt{m_l^2 + \delta m^2}, \quad m_3 = \sqrt{m_l^2 + \Delta m^2 + \delta m^2/2}\,, \tag{4.78}$$

and for inverted hierarchy

$$m_1 = \sqrt{m_l^2 - \Delta m^2 - \delta m^2/2}, \quad m_2 = \sqrt{m_l^2 - \Delta m^2 + \delta m^2/2}, \quad m_3 = m_l\,, \tag{4.79}$$

where m_l denotes the lightest neutrino masses, and $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ as defined in Ref. [78]. The experimental error on the neutrino mass splitting is not taken into account during the analysis, instead the best fit values from [78] are used:

$$\delta m^2 = 7.54 \times 10^{-5}\text{eV}^2, \qquad \Delta m^2 = 2.43 \times 10^{-3}(-2.38 \times 10^{-3})\text{eV}^2\,, \tag{4.80}$$

for normal (inverted) hierarchy. In the following, the properties of the effective mass are examined for all viable cases of lepton mixing discussed in this chapter. In Fig. 4.12 the allowed ranges of the effective mass are shown for each case in the limit of $n \to \infty$, where the three mixing angles are required to lie in the measured 3σ intervals [78]. (As previously mentioned, the 3σ lower bound of $\sin^2\theta_{12}$ is chosen to be 0.254 instead of 0.259 in case II.) Furthermore, the predictions for the representative value $n = 8$ ($n = 5$ in case I, 7th-9th ordering) are plotted in Fig. 4.13. The results for any finite value of n must be part of the ones shown, which correspond to $n \to \infty$. Moreover, the plot would change very little if the experimental errors on δm^2 and Δm^2 were taken into account. Note that only one distinct prediction for the effective mass arises except in case I. One reason for this is that, as discussed before, many of the possible permutations of the mixing matrix can be identified with each other. Furthermore, permuting the second and third row has no effect on the effective mass as θ_{23} does not appear in Eq. (4.77).

As shown in Fig. 4.12, for inverted hierarchy neutrino mass, almost all of the allowed 3σ range of the effective masses $|m_{ee}|$ can be reproduced in the limit $n \to \infty$ in case I, case III and case IV. However, the predictions for $|m_{ee}|$ are around the upper bound (about 0.05eV) or lower bound (about 0.013 eV) in case V. The reason is that the solar mixing angle is in a narrow region $0.328 \leq \sin^2\theta_{12} \leq 0.359$ and the Majorana phase α_{21} is constrained to be $|\sin\alpha_{21}| \leq 0.586$ in this case, as displayed in Fig. 4.10 and Fig. 4.11. Similarly $|m_{ee}|$ is near the upper bound and 0.025 eV in case II. Therefore if the effective mass is measured to be far from 0.013 eV, 0.025 eV and 0.05 eV for inverted hierarchy by future experiments, the mixing patterns in cases II and V could be ruled out.

For normal hierarchy neutrino mass, a sizeable part of the experimentally allowed 3σ region of $|m_{ee}|$ can be generated in all cases, and the effective mass could be rather small. In particular, the prediction in case I, 7th to 9th ordering approximately coincides with the present 3σ region. Unfortunately the predictions for normal hierarchy are still out of reach of projected experiments known to the author. As a result, it might turn out to be difficult to test the $\Delta(6n^2)$ flavour symmetry and general CP symmetry through neutrinoless double beta decay experiments in the case of normal mass ordering.

4.6 Conclusions

In the results presented in this chapter, a detailed analysis of $\Delta(6n^2)$ flavour symmetry combined with general CP symmetry $H_{\rm CP}$ in the lepton sector in semidirect models was performed. The lepton mixing parameters obtained from flavour symmetry $\Delta(6n^2) \rtimes H_{\rm CP}$ broken to different residual symmetries in the neutrino and charged lepton sectors were investigated.

Mass and mixing predictions were discussed for all possible cases where the $\Delta(6n^2)$ flavour symmetry with general CP is broken to $G_\nu = Z_2$ with $G_l = K_4, Z_p, p > 2$ and $G_\nu = K_4$ with $G_l = Z_2$. Five phenomenologically allowed cases survived and the resulting predictions for the PMNS parameters were presented as a function of n, as well as the predictions for neutrinoless double beta decay.

CP phases are predicted to take values different from 0, π or $\pm\pi/2$. In direct models with $\Delta(6n^2)$, $|\sin\delta_{\rm CP}| = 0$, which may contradict future measurements. In addition, both charged-lepton-semidirect and neutrino-semidirect models open up new large areas of parameter space. But still, as parts of the mixing matrix are entirely fixed and relations following from this can be tested by testing the predictions of sum rules, all semidirect models with $\Delta(6n^2)$ will eventually accessible to experiment and cannot evade exclusion forever.

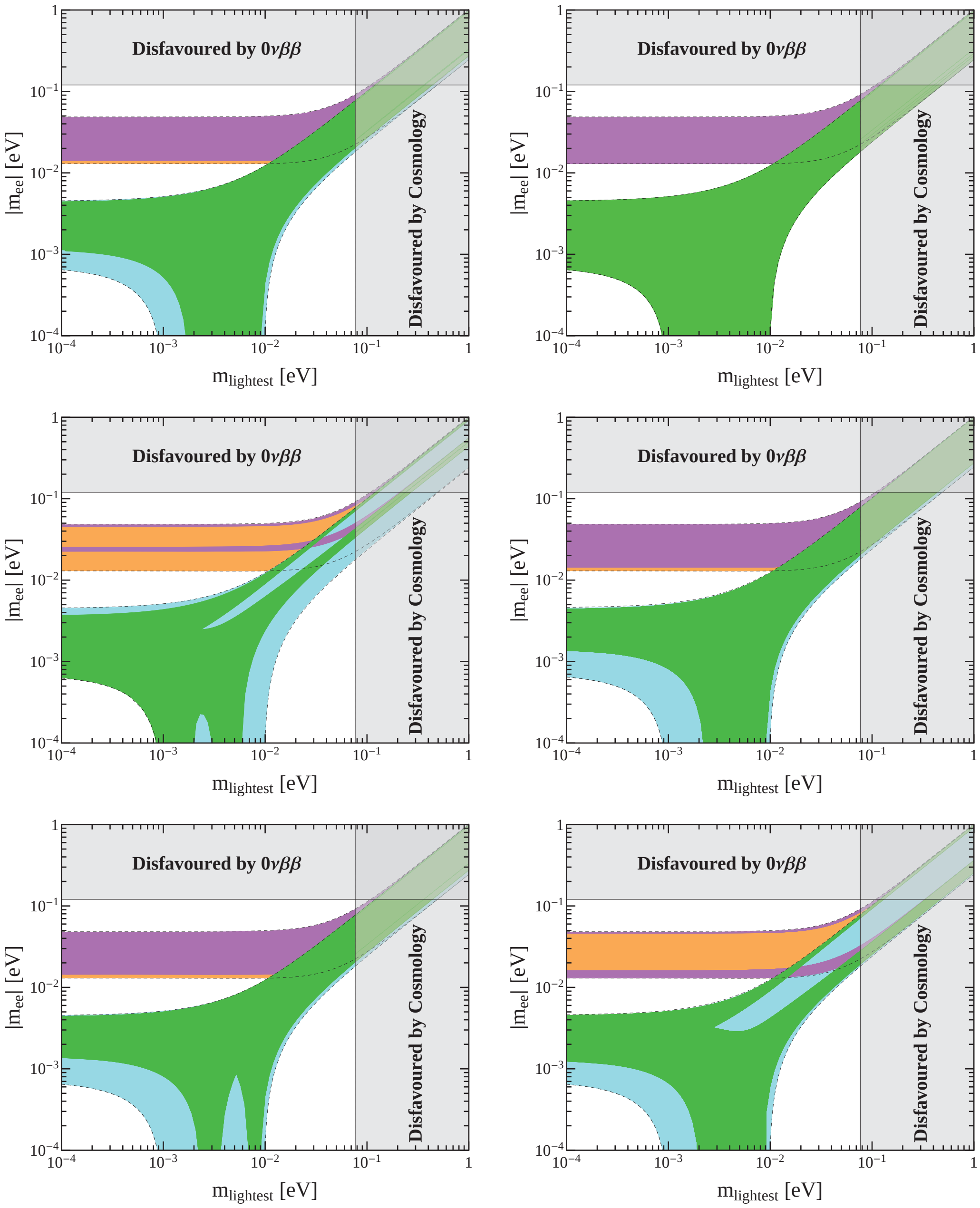


Figure 4.12: The allowed ranges of the effective mass for neutrinoless double-beta decay for all viable cases of lepton mixing in semidirect models with a $\Delta(6n^2)$ flavour group in the limit of $n \to \infty$. The top row corresponds to case I, with 1st-3rd ordering on the left and 7th to 9th ordering on the right, the middle row contains case II and III, and the bottom row case IV and V. Light blue and yellow areas indicate the currently allowed three sigma region for normal and inverted hierarchy, respectively. Purple regions correspond to predictions assuming inverted hierarchy, green regions to normal hierarchy. The upper bound $|m_{ee}| < 0.120$ eV is given by measurements by the EXO-200 [79, 100] and KamLAND-ZEN experiments [101]. Planck data in combination with other CMB and BAO measurements [80] provides a limit on the sum of neutrino masses of $m_1 + m_2 + m_3 < 0.230$ eV from which the upper limit on the mass of the lightest neutrino can be derived.

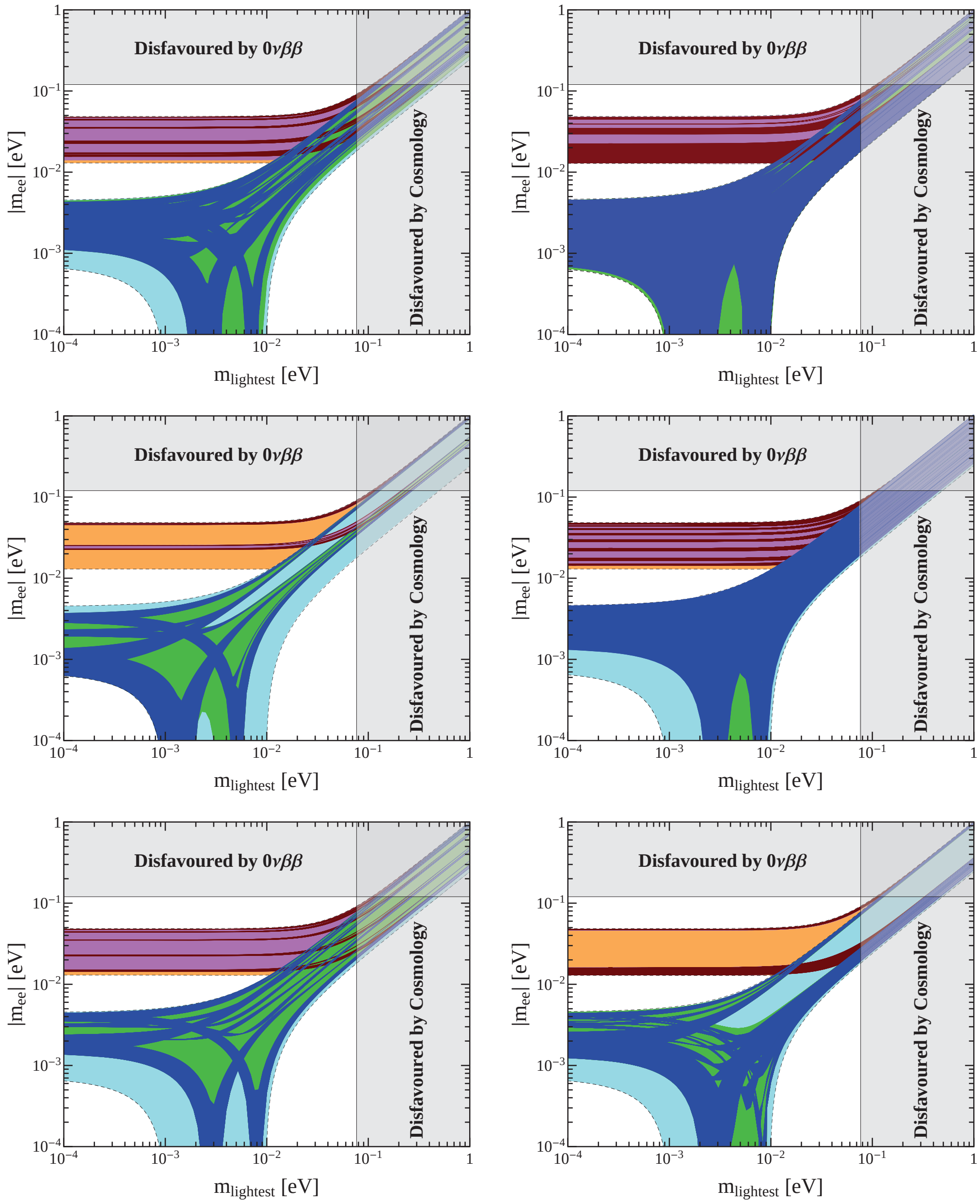


Figure 4.13: The allowed ranges of the effective mass for neutrinoless double-beta decay for all viable cases of lepton mixing in semidirect models with a $\Delta(6n^2)$ flavour group. The top row corresponds to case I, with 1st-3rd ordering on the left and 7th to 9th ordering on the right, the middle row contains case II and III, and the bottom row case IV and V. Light blue and yellow areas indicate the currently allowed three sigma region for normal and inverted hierarchy, respectively. Purple regions correspond to predictions assuming inverted hierarchy, green regions to normal hierarchy in the limit of $n \to \infty$. Blue and red regions represent predictions for normal and inverted hierarchy for the value $n = 8$ (in the top-right panel, we choose $n = 5$ which is the smallest viable value of n in that case). The upper bound $|m_{ee}| < 0.120$ eV is given by measurements by the EXO-200 [79, 100] and KamLAND-ZEN experiments [101]. Planck data in combination with other CMB and BAO measurements [80] provides a limit on the sum of neutrino masses of $m_1 + m_2 + m_3 < 0.230$ eV from which the upper limit on the mass of the lightest neutrino can be derived.

5

CP-odd invariants for multi-Higgs models and applications with discrete symmetry

This chapter presents results that were partly published previously in [6]. The contribution of the author to [6] lies in the majority of the calculations, in particular in further developing invariants methods, pioneering the contraction matrices, obtaining all invariants, evaluating invariants for example potentials, and additionally writing all relevant parts of [6], which happen to constitute the majority of the paper.

In this chapter, so-called CP-odd flavour basis invariants will be constructed and analysed. However, this was not done for fermions, where this problem is pretty much solved by the existence of the Jarlskog invariant that indicates CP violation by the Dirac CP phase and similar invariants which take into account Majorana phases. The CP-odd invariants considered in this chapter are purely constructed from the parameters of a model's scalar potential. This topic seems to lie a little out of the way of the previous development of this thesis, which it does to some extent, but it should be seen as a building block that fits well into the general effort which lies behind the previous chapters as well, namely broadly speaking the origin of CP violation, which at the end of the day seems to be necessary for all our existence.

As discussed before, only in the weak interactions of quarks in the standard model CP violation has been proven [23], however, its magnitude is not sufficient to generate the observed matter-antimatter asymmetry (among other reasons). Great hope is generally put into finding CP violation in the lepton sector. Now, the original motivation to consider residual symmetries at the beginning of this thesis was the flavour problem. With analysing direct models of $\Delta(6n^2)$ groups and finding that they can explain the currently measured lepton mixing (and later finding that they are among the last groups that do so in a direct model) progress was made. But a very important by-product was the finding that in direct models with a finite flavour group, [60], $\delta_{CP} = 0 \bmod 2\pi$. If this would turn out to be true, then this amount of CP violation in the light lepton sector, namely zero, would certainly not be enough to explain our existence. This would require an additional source of CP violation.

Approaching from a different direction: among the simplest expansions of the standard model are models with additional scalar bosons, doublets or singlets. And in fact, models with spontaneously broken flavour symmetries always require additional scalar fields to facilitate this spontaneous breaking, and as fermions are often to transform under 3-dimensional representations of the flavour group, often scalars that transform under a three-dimensional representation as well are required.[1]

In extensions with additional scalars, new sources of CP violation can arise from the scalar potential already for only one additional Higgs doublet. However, such models with two Higgs doublet are plagued with flavour-changing neutral currents and when one forbids them with additional symmetries, also the new sources of CP violation are eliminated.[2] Thus, the smallest number of Higgs doublets required for a new source of CP violation in the scalar sector that does not cause flavour-changing neutral currents is three, in line with the argument from the previous paragraph.

The questions one has ask to a model where a flavour symmetry is spontaneously broken are then, roughly in this order: Does it explain the mixing correctly? Is there sufficient CP violation in the lepton sector? Is there CP violation in the extended scalar sector? (Is there CP violation in the extended Yukawa sector?) The previous chapters concerned themselves with the first two questions, while this chapter attempts the third question, again as before not for a specific model, but developing and using more general methods to analyse a large number of models

[1] This is true for all models with spontaneously broken flavour symmetries, although in the previous paragraph only direct models had been mentioned.

[2] The parameters responsible for FNCN could be rather small, but in this case it is not clear if the CP violation in this case would be sufficient to explain the matter-antimatter asymmetry.

simultaneously. (And the fourth question is the necessary next step, but one thing at a time.)

CP-odd invariants provide a basis independent way of studying the CP properties of Lagrangians. In this chapter, the known diagrammatic method for constructing basis invariants is developed further. This method generally allows to determine whether invariants are CP-odd or CP-even and to systematically construct all of the simplest CP-odd invariants up to a given order, in the process of which many previously unknown ones are found. Additionally, such CP-odd invariants are valid for general potentials when expressed in a standard form. The diagrammatic method allows for constructing invariants that are sensitive to both explicit as well as spontaneous CP violation and can distinguish between the two kinds of CP violation. Here one should mention that while a complete so-called basis of invariants is known for models with two Higgs doublets, the invariants that constitute this basis vanish for all example potentials considered in this chapter. The newly found invariants are then used to test the CP properties of various scalar potentials involving three (or six) Higgs fields which form irreducible triplets under a discrete symmetry. The cases considered include one triplet of Standard Model (SM) gauge singlet scalars, one triplet of SM Higgs doublets, two triplets of SM singlets, and two triplets of SM Higgs doublets. For each case the potential symmetric under one of the simplest discrete symmetries with irreducible triplet representations, namely A_4, S_4, $\Delta(27)$ or $\Delta(54)$, as well as the infinite classes of discrete symmetries $\Delta(3n^2)$ or $\Delta(6n^2)$ is studied.

5.1 Introduction

The origin of the observed SM quark CP violation (CPV) is a natural consequence of three generations of quarks whose mixing is described by a complex CKM matrix. Although the CKM matrix can be parametrised in different ways, it was realised that the amount of CPV in physical processes always depends on a particular weak basis invariant which can be expressed in terms of the quark mass matrices [102]. In the SM the electroweak symmetry $SU(2)_L \times U(1)_Y$ is broken to the electromagnetic gauge group $U(1)_Q$ by a single Higgs doublet, resulting in a single physical Higgs boson which has been observed with a mass near 125 GeV [103, 104]. Although CP is automatically conserved by the Higgs potential of the SM, with more than one Higgs doublet it is possible that the Higgs potential violates CP, providing a new source of CPV [105]. This is welcome since Sakharov discovered that CPV is a necessary condition for baryon asymmetry generation [15] and CPV arising from the quark sector of the SM is insufficient [106].

It is also possible, indeed likely, that CP could be violated in the lepton sector, as is hinted at by global fits [107, 83], and such a source of CPV could also contribute to the baryon asymmetry via leptogenesis [108]. In this case one would like to construct models that explain the structure of the lepton mass matrices, through which CPV enters the processes for creating the baryon asymmetry. Typical examples of such models that use discrete symmetries to constrain the structure of mass matrices need several multiplets of scalar fields that also transform under the same symmetry (for reviews, cf. [109, 110, 27, 25, 111, 112]). Such models provide a motivation to study multiple SM Higgs singlets (sometimes called "flavons" in this context) as well as electroweak doublets. In the context of flavour models it is natural to consider Higgs doublets or singlets which play the role of "flavons" and form irreducible triplets under some spontaneously broken discrete flavour symmetry.

As already mentioned in the context of the CKM matrix, the study of CP is a subtle topic because of the basis dependent nature of the phases which control CPV. Similar considerations also apply to the phases which appear in the parameters of the potentials of multiple scalars.

An important tool to assist in determining whether CP is violated or not are basis independent CP-odd invariants (CPIs), whose usefulness has been shown in the SM in addressing CP violation arising from the CKM matrix, sourced from the Yukawa couplings. The first example of the use of such invariants was the Jarlskog invariant [102], which was reformulated in [113] in a form which is generally valid for an arbitrary number of generations. Generalising the invariant approach [113] and applying it to fermion sectors of theories with Majorana neutrinos [114] or with discrete symmetries [115, 116] leads to other relevant CPIs.

In extensions of the Higgs sector of the SM, the CP violation arising from the parameters of the scalar potential can be studied in a similar basis invariant way as for the quark sector. For example, in the general two Higgs Doublet Model (HDM) [105] (see [117] for a recent analysis) a CPI was identified in [118]. More generally, applying the invariant approach to scalar potentials has revealed relevant CPIs [119, 120, 121], including for the 2HDM [122, 123]. However, as mentioned before, while these invariants even form a basis, which means that any possible CP-odd invariant has to involve those invariant, they all vanish for more symmetric models and therefore cannot indicate whether CP is violated or not. Thus the goal is to consider yet more general Higgs potentials and improve on the methods for constructing CPIs, which subsequently are applied to potentials involving three or six Higgs fields (which can be either electroweak doublets or singlets) which form irreducible triplets under a discrete symmetry.

To begin with, the previous progress in developing a systematic approach to CPIs for arbitrary scalar potentials is reviewed, focusing on renormalisable potentials with quadratic and quartic couplings. The reader may be primarily interested in cases where the Higgs fields are electroweak $SU(2)_L$ doublets, but the formalism can also be applied to more general scalar potentials including cases where the Higgs fields are SM singlets. Methods where basis invariants [121, 122, 123] can be represented pictorially by diagrams (introduced for the 2HDM in [122]) are further developed and matrices are introduced later designated as contraction matrices, that identify how the parameters in the potential are combined to form a basis invariant. The diagrams and matrices are extremely helpful in distinguishing CPIs from basis invariants that are CP-even, as well as cataloguing each CPI uniquely in association with an element of a group of permutations. CPIs as defined via such matrices are valid for any potential, and then take specific expressions when specialising to a potential (often vanishing for cases where the potential is very symmetric, even if the potential features explicit CP violation as shown by other non-vanishing CPIs).

After that, the newly constructed CPIs are applied to physically interesting cases, beginning with the familiar example of the general 2HDM. Following this, examples of potentials which involve three or six Higgs fields which fall into irreducible triplet representations of discrete symmetries belonging to the $\Delta(3n^2)$ and $\Delta(6n^2)$ series studied extensively in the context of flavour and CP models in [30, 72, 67, 1, 69, 71, 86, 66, 2, 94, 124, 3, 87, 95, 4, 125, 126, 127, 128, 5, 85, 129, 130, 131]. Specific cases of the 3HDM [91, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147] and of the 6HDM [148, 149, 150, 151, 152, 153, 154] where the three or six Higgses are related by the discrete symmetry as one or two (flavour) triplets are considered. Although many of these cases have already been studied in the literature, systematically exploiting the formalism yields many new results. For example, although $\Delta(27)$ with a single triplet of Higgs doublets has been extensively studied in the literature [91, 134, 135, 136, 139, 140, 145], using the invariant approach and the CPIs produced several new results of interest.

Using the invariant approach, the considered cases include one triplet of SM gauge singlets, one triplet of SM Higgs doublets, two triplets of SM singlets, and two triplets of SM Higgs doublets, where for each case the potential is symmetric under one of the simplest discrete symmetries with irreducible triplet representations, namely A_4, S_4, $\Delta(27)$ or $\Delta(54)$, as well as the infinite classes of discrete symmetries $\Delta(3n^2)$ or $\Delta(6n^2)$. In each case, it is shown which potentials are in general CP conserving (all the CPIs vanish, and a CP symmetry that leaves the potential invariant proves CP invariance) or in general CP violating (in which case it is sufficient to show a single non-vanishing CPI). For the CP violating potentials imposing specific CP symmetries, lead in constraining the parameters of the potential in one way or the other to generally vanishing CPIs. As the formalism also allows for Vacuum Expectation Values (VEVs), one can obtain Spontaneous CPIs (SCPIs) that are non-vanishing if CP is spontaneously violated (as considered earlier in [119, 120]). One of these SCPIs is applied to the better studied $\Delta(27)$ potential, exploring different CP symmetries and VEVs that either conserve or spontaneously violate the imposed CP symmetry.

The layout of this chapter is as follows. Section 5.2 reviews the general formalism. In Section 5.3 the 2HDM potential is revisited where the formalism is applied and small differences to earlier developments in the literature are shown. In Section 5.4, 5.5 and 7.2 CPIs are applied to$\Delta(3n^2)$ and $\Delta(6n^2)$ groups with $n = 2$ (A_4, S_4), $n = 3$ ($\Delta(27)$, $\Delta(54)$) and $n > 3$. A summary of the results obtained for the potentials invariant under discrete symmetries is contained in Section 5.6 (including Table 5.1). Section 5.7 is dedicated to invariants that indicate spontaneous CP violation (SCPIs). Section 5.8 concludes the chapter. Further material is included in Appendix 7.3 with a complete list of the CPIs and SCPIs found and used throughout this chapter, and Appendix 7.4.2 discusses how to obtain results for $\Delta(6n^2)$ from the results of the $\Delta(3n^2)$ potentials.

5.2 CP-odd invariants for scalar potentials

5.2.1 General formalism

One important aim of this chapter is to explore the CP properties of the Higgs sector of models with several copies of SM Higgs doublets. Often, scalar potentials can be confusingly complex and it can be unclear which parameters can contribute to CP violation. This situation is made even more difficult by the possibility of choosing different bases which modify the explicit form of the potential but should not change the physics described by it. Both of these difficulties can be overcome by CPIs, in this case CP-odd (Higgs-) basis invariants, that, when non-zero, indicate CP violation. A similar CPI for the Yukawa sector of the SM is the well-known Jarlskog invariant [102].

Before defining and discussing CPIs of the scalar sector in detail, we first show how to write any possible Higgs potential in a standard form which is suitable to construct general basis invariants. This procedure has the advantage that basis invariants only have to be derived once in the standard form; their explicit form for any particular Higgs potential follows almost trivially by translating the latter into the standard parametrisation. Furthermore, invariants that are CP-odd (CPIs) for the standard form of potentials are so by construction and, if non-zero, indicate CP violation for all possible example potentials.

The relation between non-zero CPIs and CP violation can be formulated more precisely as follows. If a potential conserves CP, then all CPIs vanish automatically. Reversely, if one or several CPIs are non-zero, the potential violates CP. This statement holds for both explicit and spontaneous CP violation, and the corresponding CPIs are introduced in Sections 5.2.2 and 5.7. Note that CPIs only guarantee CP conservation if all of them vanish. This is equivalent to demanding a finite set of CPIs, the so-called basis out of which all other CPIs can be produced, to vanish. Such a basis of CPIs is known for the 2HDM [123], but, so far, not for any other more complicated scalar potentials.

In the following, we first introduce the standard form for scalar potentials. In this notation the effects of symmetry transformations, general basis transformations, complex conjugation and CP transformations on the variables and parameters of the standard form are analysed. Adopting the procedure and notation of [121, 122], any even potential of N scalar fields φ_i can, with $\phi = (\varphi_1, \ldots, \varphi_N)$ and $\phi^* = (\varphi_1^*, \ldots, \varphi_N^*)$, be written as

$$V \;=\; \phi^{*a}\, Y_a^b\, \phi_b + \phi^{*a}\phi^{*c}\, Z_{ac}^{bd}\, \phi_b\phi_d \;, \tag{5.1}$$

where the notation is such that lower indices on Y and Z are always contracted with ϕ^* and upper indices with ϕ. Y and Z are tensors that contain all possible couplings and are subject to possible symmetries acting on ϕ, as will be explained below.[3]

[3]One could also add a term such as e.g. $T_c^{ab}\phi_a\phi_b\phi^{*c}$ + h.c. to the potential to account for trilinear couplings and the discussion could be extended in this way.

Any potential of several Higgs doublets can be brought into this standard form by ϕ not containing doublets as such, but instead directly containing the components of the doublets: for n Higgs doublets $H_{i\alpha} = (h_{i,1}, h_{i,2})$, where $\alpha = 1, 2$ denotes the $SU(2)_L$ index and i goes from 1 to n,

$$\phi = (\varphi_1, \varphi_2, \ldots, \varphi_{2n-1}, \varphi_{2n}) = (h_{1,1}, h_{1,2}, \ldots, h_{n,1}, h_{n,2}) \ , \tag{5.2}$$

and the invariance of the potential under $SU(2)_L \times U(1)_Y$ will be directly reflected in the structure of Y and Z in a component-wise way. This convention, which differs from the notation of [121, 122], will be very useful later on.[4]

More explicitly, if the theory is invariant under **symmetry transformations** of a group G such that ϕ transforms in some (maybe reducible) representation $\rho(g)$ of that group, where $\rho(g)$ is the matrix that corresponds to the group element $g \in G$,

$$\phi_a \mapsto [\rho(g)]_a^{a'} \phi_{a'} \ , \tag{5.3}$$

$$\phi^{*a} \mapsto \phi^{*a'} [\rho^\dagger(g)]_{a'}^a \ , \tag{5.4}$$

then the invariance of the potential imposes the following constraints on the coupling tensors:

$$Y_a^b = \rho_a^{a'} \, Y_{a'}^{b'} \, \rho^{\dagger}{}_{b'}^{b} \ , \tag{5.5}$$

$$Z_{ac}^{bd} = \rho_a^{a'} \, \rho_c^{c'} \, Z_{a'c'}^{b'd'} \, \rho^{\dagger}{}_{b'}^{b} \, \rho^{\dagger}{}_{d'}^{d} \ , \tag{5.6}$$

where we have written $\rho_a^{a'} = [\rho(g)]_a^{a'}$ and so on. In addition to that, the quartic coupling tensor Z_{ac}^{bd} is by construction invariant under exchanging $a \leftrightarrow c$ as well as $b \leftrightarrow d$. The reason for this is that ϕ_b and ϕ_d commute so that the indices b and d can be renamed into each other to restore the original ordering of the ϕ's, and equivalently for ϕ^* with a and c.

While the theory is invariant under symmetry transformations, one also has the possibility of applying basis transformations under which the Lagrangian is not invariant. Of course, such a basis transformation should not change physics. A simple example is the transformation that diagonalises the bilinear mass terms $\phi^{*a} \, Y_a^b \, \phi_b$. As Z is generally only invariant under a smaller group than that of all

[4] In [122], for example, the $SU(2)_L$ indices are summed over outside of Z. Our definition of Z tensors can be related to [122] by explicitly highlighting the $SU(2)_L$ subindices, $\{1, 2, \ldots, 2n-1, 2n\} = \{(1,1), (1,2), \ldots, (n,1), (n,2)\}$. With this, the Z tensors in used here become $Z_{cd}^{ab} = Z_{(\tilde{c},\gamma),(\tilde{d},\delta)}^{(\tilde{a},\alpha)(\tilde{b},\beta)} = \tilde{Z}_{\tilde{c}\tilde{d}}^{\tilde{a}\tilde{b}} \delta_\gamma^\alpha \delta_\delta^\beta$, where $\tilde{Z}$ denotes the coupling tensors of [122].

basis changes, diagonalising Y would change Z.[5] Adopting our notation for the standard form of Higgs potentials, a unitary **basis transformation** in the space of the N dimensional vector ϕ, i.e. with $U \in U(N)$ a unitary $N \times N$ matrix, maps

$$\phi_a \mapsto U_a^{a'} \phi_{a'} \ , \tag{5.7}$$

$$\phi^{*a} \mapsto \phi^{*a'} U^{\dagger a}_{\ a'} \ . \tag{5.8}$$

With this definition, the kinetic terms remain unchanged while Y and Z transform to

$$Y_a^b \mapsto U_a^{a'} \, Y_{a'}^{b'} \, U^{\dagger b}_{\ b'} \ , \tag{5.9}$$

$$Z_{ac}^{bd} \mapsto U_a^{a'} \, U_c^{c'} \, Z_{a'c'}^{b'd'} \, U^{\dagger b}_{\ b'} \, U^{\dagger d}_{\ d'} \ . \tag{5.10}$$

Complex conjugation is an essential part of CP transformations and in the notation used here, changes the vertical position of the index of a field so that

$$\phi_a \mapsto (\phi_a)^* \equiv \phi^{*a} \ , \tag{5.11}$$

$$\phi^{*a} \mapsto (\phi^{*a})^* \equiv \phi_a \ . \tag{5.12}$$

Complex conjugating the Y term of the potential then results in

$$\phi^{*a} \, Y_a^b \, \phi_b \ \mapsto \ \phi_a \, (Y_a^b)^* \, \phi^{*b} = \phi^{*b} \, (Y_a^b)^* \, \phi_a = \phi^{*a} \, (Y_b^a)^* \, \phi_b \ . \tag{5.13}$$

Comparing this to the original term in the potential and demanding $V^* = V$ shows that

$$(Y_b^a)^* \ = \ Y_a^b \ . \tag{5.14}$$

A similar result is obtained for the quartic coupling, i.e.

$$(Z_{bd}^{ac})^* \ = \ Z_{ac}^{bd} \ . \tag{5.15}$$

Note that because both indices of a contracted pair interchange position under complex conjugation, no situation can arise where one would need to sum over two upper or two lower indices. However, expressions as e.g. Eqs. (5.14) and (5.15) where indices appear with exchanged vertical positions without being summed

[5]Except in the case where the components of Y conspire in such a way that the required basis transformation coincides with a symmetry transformation. Furthermore, a general basis transformation changes the form of the potential, while only transformations in the automorphism group $Aut(G)$ leave the potential form-invariant.

over need to be understood as conditions on the components of the tensors and not the tensors themselves.

Finally, all pieces are in place to define a **(general) CP transformation**[6] with a unitary matrix X on the fields as

$$\phi_a \mapsto \phi^{*a'} X^{a}_{a'} \;, \tag{5.16}$$

$$\phi^{*a} \mapsto X^{\dagger\,a'}_{\;a} \phi_{a'} \;. \tag{5.17}$$

Again, this leaves the kinetic terms invariant, while applying the CP transformation to the fields in the potential results for the Y term in

$$\begin{aligned} \phi^{*a}\, Y^{b}_{a}\, \phi_b \;\mapsto\; X^{\dagger\,a'}_{\;a} \phi_{a'}\, Y^{b}_{a}\, \phi^{*b'} X^{b}_{b'} &= X^{\dagger\,a'}_{\;a} \phi_{a'}\, (Y^{a}_{b})^{*}\, \phi^{*b'} X^{b}_{b'} \\ &= \phi^{*b'} X^{b}_{b'}\, (Y^{a}_{b})^{*}\, X^{\dagger\,a'}_{\;a} \phi_{a'} \\ &= \phi^{*a} X^{a'}_{a}\, (Y^{b'}_{a'})^{*}\, X^{\dagger\,b}_{\;b'} \phi_b \;. \end{aligned} \tag{5.18}$$

Comparing this to the original term in the potential shows that a CP transformation acting on the fields can be equally understood as the following change of Y (likewise for Z),

$$Y^{b}_{a} \mapsto X^{a'}_{a}\, (Y^{b'}_{a'})^{*}\, X^{\dagger\,b}_{\;b'} \;, \tag{5.19}$$

$$Z^{bd}_{ac} \mapsto X^{a'}_{a} X^{c'}_{c}\, (Z^{b'd'}_{a'c'})^{*}\, X^{\dagger\,b}_{\;b'} X^{\dagger\,d}_{\;d'} \;. \tag{5.20}$$

The condition for CP invariance of the standard form of the potential V in Eq. (5.1), and thus any example potential that can be brought into this standard form, can then be phrased as follows: *CP is conserved if there is an X such that the left- and right-hand sides of Eqs. (5.19) and (5.20) are identical.* As a special case of that, if the tensors Y and Z are real, the potential is invariant under a CP transformation, which we refer to as CP_0,

$$X^{a}_{a'} = \delta^{a}_{a'} \;. \tag{5.21}$$

CP_0 is often referred to as trivial or canonical CP.

In doing so, we note that a physical CP transformation will have to treat the two components of an $SU(2)_L$ doublet consistently with that symmetry, i.e. both must transform identically under the CP symmetry [70].

[6] This is often referred to as a general CP transformation.

In preparation for Section 5.7, where CPIs for spontaneous CP violation will be constructed, we define how VEVs behave under basis transformations:

$$v_a \mapsto U_a^{a'} v_{a'} \ , \tag{5.22}$$

$$v^{*a} \mapsto v^{*a'} U^{\dagger}{}_{a'}^{a} \ , \tag{5.23}$$

where $v \equiv (v_1, \ldots)$ with $v_i = \langle \varphi_i \rangle$, and U denotes the transformation matrix of the fields ϕ. Similarly, under CP transformations, they become

$$v_a \mapsto v^{*a'} X_{a'}^{a} \ , \tag{5.24}$$

$$v^{*a} \mapsto X^{\dagger}{}_{a}^{a'} v_{a'} \ . \tag{5.25}$$

5.2.2 CP-odd invariants for explicit CP violation

In the previous subsection, the standard form for even scalar potentials was introduced and the effects of symmetry transformations, basis transformations and CP transformations has been analysed. This subsection starts with a discussion of simple basis invariants constructed from Y and Z tensors. After that, the general definition of CP-odd basis invariants (CPIs) that contain Y and Z is given.

Finally, the CP properties of such invariants will be analysed. CPIs of this type, that only consist of parameters of the potential and in particular do not contain VEVs, indicate explicit violation of CP. The exact statement is that if all possible CPIs are zero, then the theory is CP conserving. Vice-versa, if at least one CPI is non-zero, the theory violates CP explicitly. Invariants including VEVs, such that they indicate spontaneous violation of CP, will be introduced in Section 5.7.

Any product of Y and Z tensors where all indices are correctly contracted forms a basis invariant. Starting with Y and considering Z a little later, the simplest invariant (that is however not CP-odd) is

$$Y_a^a . \tag{5.26}$$

For products of two Y tensors, the only possible contractions are

$$Y_a^a Y_b^b \text{ and } Y_b^a Y_a^b . \tag{5.27}$$

The above contractions correspond to the two different permutations of the two upper indices, namely firstly the identity:

$$Y_a^a Y_b^b \Leftrightarrow a \mapsto a \text{ and } b \mapsto b \ , \tag{5.28}$$

and secondly the transposition:

$$Y_b^a Y_a^b \Leftrightarrow a \mapsto b \text{ and } b \mapsto a \ . \tag{5.29}$$

More formally, one can thus also express all invariants that consist of two Y tensors by

$$Y^a_{\sigma(a)} Y^b_{\sigma(b)} \text{ with } \sigma \in S_2 \ , \tag{5.30}$$

where σ is now one of the two elements of the permutation group S_2. The invariant built from two Y tensors that corresponds to the identity of S_2 is the square of the simplest invariant. Thus, only the second invariant is irreducible, which for our purposes will be defined as not being a product or power of smaller invariants.

It is generally true that all possible invariants can be obtained through permutations of indices: all conceivable invariants built from 3 Y tensors are given by

$$Y^a_{\sigma(a)} Y^b_{\sigma(b)} Y^c_{\sigma(c)} \text{ with } \sigma \in S_3 \ , \tag{5.31}$$

or explicitly

$$Y_a^a Y_b^b Y_c^c, \ Y_a^a Y_c^b Y_b^c, \ Y_c^a Y_b^b Y_a^c, \ Y_b^a Y_a^b Y_c^c, \ Y_c^a Y_a^b Y_b^c, \ Y_b^a Y_c^b Y_a^c. \tag{5.32}$$

Here, only the last two invariants are new and irreducible, i.e. not products of smaller invariants. Additionally, they turn out to be equivalent as can be seen by renaming the indices $b \leftrightarrow c$ into each other.

The identification of invariants with elements of permutation groups will be used later to systematically identify all irreducible invariants of a given order. Beyond that, it is this formalism that is going to make it possible to determine which invariants are CP-even and which are not.

But before that, some more examples are in order, as the situation is more complicated for invariants containing Z tensors. There are already two invariants that could be built from a single Z tensor that again correspond to the two possible permutations of positions of the two upper indices:

$$Z^{ab}_{\sigma(a)\sigma(b)} \text{ with } \sigma \in S_2 \ , \tag{5.33}$$

or explicitly:

$$Z^{ab}_{ab} \text{ and } Z^{ab}_{ba} \ . \tag{5.34}$$

Because the Z tensor of potentials considered here is symmetric under exchanging both upper or both lower indices, cf. below Eq. (5.6), both invariants built from one Z tensor are equivalent. For larger numbers of tensors, the number of permutations grows quickly, however, luckily, many invariants do not need to be considered either because they are products of smaller invariants, or because they are equivalent due to the symmetry of single tensors themselves or the symmetries of the invariant. For example, for two Z tensors, generally all invariants would be given by

$$Z^{ab}_{\sigma(a)\sigma(b)} Z^{cd}_{\sigma(c)\sigma(d)} \text{ with } \sigma \in S_4 \ , \tag{5.35}$$

but the only new invariants can be chosen to be

$$Z^{ab}_{bd} Z^{cd}_{ac} \text{ and } Z^{ab}_{cd} Z^{cd}_{ab}. \tag{5.36}$$

All other 22 invariants that correspond to the remaining elements of S_4 are products of smaller invariants or equivalent to the invariants in Eq. (5.36).

Generally, a basis invariant $I^{(n_Z,m_Y)}_\sigma$ built from m_Y Y tensors and n_Z Z tensors can be written as[7]

$$I^{(n_Z,m_Y)}_\sigma \equiv Y^{a_1}_{\sigma(a_1)} \dots Y^{a_{m_Y}}_{\sigma(a_{m_Y})} Z^{b_1 b_2}_{\sigma(b_1)\sigma(b_2)} \dots Z^{b_{2n_Z-1} b_{2n_Z}}_{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \text{ with } \sigma \in S_{m_Y+2n_Z}. \tag{5.37}$$

Again, σ is a permutation of $m_Y + 2n_Z$ objects, i.e. $\sigma \in S_{m_Y+2n_Z}$. However, not all basis invariants are CP-odd, and in fact, all of the examples in Eqs. (5.26)-(5.36) turn out to be CP-even. To be able to make such statements, one needs to know how basis invariants behave under CP. Under a general CP transformation, a coupling tensor is replaced by its complex conjugate multiplied by unitary basis transformations, earlier denoted by X. But, as a basis invariant is, by definition, invariant under basis transformations, the X matrices cancel, leaving only the original product of coupling tensors with tensors replaced by their complex conjugates. The complex conjugate of a coupling tensor on the other hand can

[7] Often, not the full permutation σ will be indicated when referring to invariants, but e.g. $I^{(3,1)}_2$ would be the second invariant that was found with $n_Z = 3$ and $m_Y = 1$.

be obtained by interchanging upper with lower indices, cf. Eqs. (5.14) and (5.15). For the simplest example, Y^a_a, this works out in the following way:

$$Y^a_a \xrightarrow{CP} (Y^{a'}_{a''})^* X^{\dagger a}_{\ a'} X^{a''}_a = (Y^{a'}_{a''})^* \delta^{a''}_{a'} = (Y^a_a)^* = Y^a_a, \tag{5.38}$$

where in the last step Eq. (5.14) was used. As the right-hand side is the CP conjugate of the left-hand side and is identical to the latter, this shows that this invariant is even under CP transformations. Similarly, and using Eqs. (5.14) and (5.15), one can show that the CP conjugate of a general basis invariant can be obtained by interchanging upper and lower indices:

$$\begin{aligned} I^{(n_Z,m_Y)}_\sigma &\equiv Y^{a_1}_{\sigma(a_1)} \ldots Y^{a_{m_Y}}_{\sigma(a_{m_Y})} Z^{b_1 b_2}_{\sigma(b_1)\sigma(b_2)} \ldots Z^{b_{2n_Z-1} b_{2n_Z}}_{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \\ &\xrightarrow{CP} Y^{\sigma(a_1)}_{a_1} \ldots Y^{\sigma(a_{m_Y})}_{a_{m_Y}} Z^{\sigma(b_1)\sigma(b_2)}_{b_1 b_2} \ldots Z^{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})}_{b_{2n_Z-1} b_{2n_Z}} = [I^{(n_Z,m_Y)}_\sigma]^* \; . \end{aligned} \tag{5.39}$$

If one has found an invariant I that is not CP-even, i.e. that does not equal its CP conjugate I^*, one can extract the CP-odd part by subtracting the CP-conjugated from the original invariant:

$$\mathcal{I} = I - I^*. \tag{5.40}$$

As a CPI is already completely defined by stating half of it, I, in the following often I^* will be omitted or abbreviated. When $\mathcal{I}$ is given, it is implied that the quantity to follow is the difference between a basis invariant I and its CP conjugate.

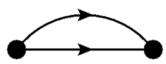
For the example invariants in Eqs. (5.26)-(5.36), interchanging upper and lower indices and possibly renaming indices shows that all of them are equal to their CP conjugate and thus CP-even. For larger invariants, this process can become quite cumbersome. Even worse, in order to show that an invariant is not CP-even, one would have to test all possible renamings of the indices, which at some point becomes too difficult. Luckily, the symmetry properties of invariants can be analysed and visualised using diagrams that encode which tensors are used and how their indices are contracted with each other. These diagrams are the topic of the next subsection. As the diagrams become more complicated, a more powerful technique relies on analysing so-called contraction matrices that also encode information about the basis invariant and reveal whether it is a CPI. We heavily rely on the contraction matrices for our systematic searches that revealed many new CPIs. These contraction matrices are introduced in section 5.2.4.

5.2.3 Diagrams for invariants

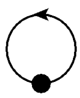
Any basis invariant consisting of contractions of Y and Z can be expressed by a diagram [122]. We use a slightly different notation from the one present in [122]. For each Y or Z draw a vertex and for any contraction of an upper index on a tensor with a lower index of a tensor draw an arrow connecting the vertices corresponding to the tensors. With $X = Y, Z$, the only rule for drawing diagrams is

$X^{a\cdot}_{\cdot\cdot} X^{\cdot\cdot}_{a\cdot} =$  (5.41)

Additionally, as Z^{ab}_{cd} is symmetric under exchange of $a \leftrightarrow b$ and/or $c \leftrightarrow d$, two lines can be attached to a vertex corresponding to a Z tensor without having to distinguish them in the diagram:

$Z^{ab}_{\cdot\cdot} Z^{\cdot\cdot}_{ab} =$  (5.42)

Contracting two indices on the same tensor with each other produces a loop:

$X^{a\cdot}_{a\cdot} =$  (5.43)

Diagrams drawn following these rules make it possible to check if an invariant is CP-even: from Eq. (5.39) follows that *the CP conjugate of an invariant produces exactly the same diagram but with inverted directions of arrows* as all upper indices have been turned into lower indices and vice versa. An invariant is identical to its CP conjugate, i.e. CP-even, if the diagrams of the invariant and its CP conjugate are identical up to the positions of the vertices. The reason for this is that in a product of Y and Z tensors, their position in the product is arbitrary and thus also the position of vertices (except for the type of tensor).[8]

A few small example diagrams for small invariants mentioned earlier in the text are shown in Figure 5.1. One can see there that for each of them, inverting the direction of the arrows produces the same diagram and thus the same invariant. This is the case for all of the small examples in Eqs. (5.26)-(5.36). The simplest invariant, Y^a_a produces the diagram in Eq. (5.43).

[8]The internal symmetry of Z^{ab}_{cd} under $a \leftrightarrow b$ and/or $c \leftrightarrow d$ is taken into account by Eq. (5.42).

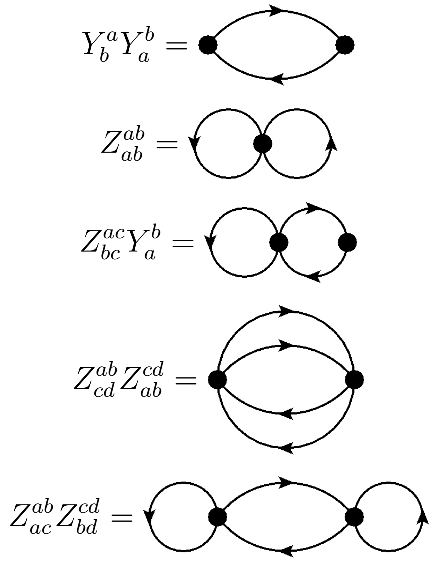
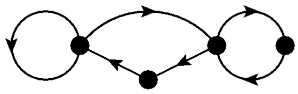
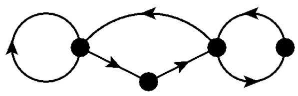


Figure 5.1: Example diagrams corresponding to small invariants.

All invariants discussed so far were CP-even. The smallest CP-odd invariant was already found in [121] and is given by the difference $\mathcal{I}_1 = I_1 - I_1^*$ of

$$I_1 \equiv Z^{ab}_{ae} Z^{cd}_{bf} Y^l_c Y^f_d =$$  (5.44)

and its CP conjugate

$$I_1^* \equiv Z^{ae}_{ab} Z^{bf}_{cd} Y^c_e Y^d_f =$$  . (5.45)

In whatever ways one tries to interchange the positions of vertices and arrows, it is impossible to make the diagrams equivalent.

Additionally, out of all possible contractions of coupling tensors, many will be related by interchanging the positions of tensors. The symmetries of the diagrams can be used to classify invariants and search for CPIs in a systematic way as will be explained in the next section. The results of this systematic search are listed in the subsequent section.

5.2.4 Symmetries of invariants

Invariants both without and with VEVs were defined via permutations of indices, cf. Eqs. (5.37) and (5.165). Firstly, it might seem as if there is a huge number of invariants, one for each possible permutation of indices, whose number grows as the factorial of the number of indices. But luckily, as already hinted at in subsection 5.2.2, invariants have symmetries which will reduce the number of inequivalent invariants. Secondly, one still has to find those index permutations which correspond to CPIs. This section concerns itself with these two issues.

Invariants were defined in the following way via index permutations $\sigma \in S_n$ where n is the total number of upper indices coming from all involved tensors and VEVs: for invariants without VEVs,

$$I_\sigma^{(m_Y n_Z)} = Y^{a_1}_{\sigma(a_1)} \dots Y^{a_{m_Y}}_{\sigma(a_{m_Y})} Z^{b_1 b_2}_{\sigma(b_1)\sigma(b_2)} \dots Z^{b_{2n_Z-1} b_{2n_Z}}_{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \text{ with } \sigma \in S_{m_Y+2n_Z}, \tag{5.46}$$

where $n = m_Y + 2n_Z$, and for invariants containing VEVs,

$$\begin{aligned} J_\sigma^{(n_v,m_Y,n_Z)} &= W^{w_1\dots w_{n_v}}_{\sigma(w_1)\dots\sigma(w_{n_v})} Y^{a_1}_{\sigma(a_1)} \dots Y^{a_{m_Y}}_{\sigma(a_{m_Y})} Z^{b_1 b_2}_{\sigma(b_1)\sigma(b_2)} \dots Z^{b_{2n_Z-1} b_{2n_Z}}_{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \\ &\mapsto W_{w_1\dots w_{n_v}}^{\sigma(w_1)\dots\sigma(w_{n_v})} Y_{a_1}^{\sigma(a_1)} \dots Y_{a_{m_Y}}^{\sigma(a_{m_Y})} Z_{b_1 b_2}^{\sigma(b_1)\sigma(b_2)} \dots Z_{b_{2n_Z-1} b_{2n_Z}}^{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \\ &= (J_\sigma^{(n_v,m_Y,n_Z)})^*, \end{aligned} \tag{5.47}$$

where now $n = n_v + m_Y + 2n_Z$ and W as defined in Eq. (5.164).

There are the following sources of symmetries of invariants: renaming of indices, permutations of tensors of the same type, and internal symmetries of tensors. Internal symmetries of tensors can refer to symmetries under exchanging indices on the tensor, and symmetries induced by the symmetry of the Lagrangian. Except for the latter, which are not discussed here, all of these symmetries exist for arbitrary invariants corresponding to arbitrary potentials. These sources of symmetries will now be discussed. To streamline notation, write all indices into a multi-index,

$$\alpha = (a_1, \dots, a_n) \ , \tag{5.48}$$

where now permutations act on α by acting on each index as usual:

$$\sigma(\alpha) = (\sigma(a_1), \dots, \sigma(a_n)). \tag{5.49}$$

Also, let $\mathcal{Z}$ stand for the product of tensors (both Y and Z) and VEVs appropriate to the invariant in discussion, then any invariant can be written as

$$I_\sigma = \mathcal{Z}^\alpha_{\sigma(\alpha)}. \tag{5.50}$$

Renaming indices into each other corresponds to another permutation of all indices. For $a_i \mapsto \pi(a_i)$ with $\pi \in S_n$, the invariant becomes

$$I_\sigma \xrightarrow{\pi} \mathcal{Z}^{\pi(\alpha)}_{\sigma(\pi(\alpha))}. \tag{5.51}$$

The original invariant and the invariant with indices renamed into each other have the same value.

Next, some elements of invariants are symmetric under *independent* permutations of upper and lower indices. For example, as discussed in section 5.2.2, the following four versions of the Z tensor are equal,

$$Z^{ab}_{cd} = Z^{ba}_{cd} = Z^{ab}_{dc} = Z^{ba}_{dc}, \tag{5.52}$$

because Z^{ab}_{cd} is symmetric under $a \leftrightarrow b$ and/or $c \leftrightarrow d$. This means that for each Z tensor in the invariant there are 4 equivalent ways of connecting it to the rest of the invariant and thus for n_Z Z tensors, there would be 4^{n_Z} σ matrices producing the same invariant and diagram. Similarly, in the tensor W that summaries the product of all VEVs and complex conjugates of VEVs, all upper and lower indices can be permuted independently of each other. Denoting any such permutation of indices that is allowed by internal symmetries of tensors by τ, then this condition becomes

$$\mathcal{Z}^{\alpha}_{\sigma(\alpha)} = \mathcal{Z}^{\tau(\alpha)}_{\sigma(\alpha)} = \mathcal{Z}^{\alpha}_{\tau(\sigma(\alpha))} = \mathcal{Z}^{\tau(\alpha)}_{\tau(\sigma(\alpha))} = I_\sigma. \tag{5.53}$$

These internal symmetries can be taken into account in the actual search for CPIs by defining a new matrix that is produced from one of the equivalent σ matrices, which maps all invariants that are related by transformations of the type τ onto a single matrix that also uniquely corresponds to the diagram corresponding to all these invariants. (In the diagram the symmetries are taken into account automatically.) This new matrix will be called contraction matrix and denoted by m. Define the following submatrices of σ and m:

$$\sigma = \begin{pmatrix} \sigma_{vv} & \sigma_{vY} & \sigma_{vZ} \\ \sigma_{Yv} & \sigma_{YY} & \sigma_{YZ} \\ \sigma_{Zv} & \sigma_{ZY} & \sigma_{ZZ} \end{pmatrix}, m = \begin{pmatrix} m_{vv} & m_{vY} & m_{vZ} \\ m_{Yv} & m_{YY} & m_{YZ} \\ m_{Zv} & m_{ZY} & m_{ZZ} \end{pmatrix}, \tag{5.54}$$

where now the vv parts correspond to contractions between VEVs, vY between VEVs and Y tensors, and so on, until ZZ, which corresponds to contractions between Z tensors. While σ is an $n \times n$ matrix with n the number of indices, m will be an $N \times N$ matrix where N is the number of tensors in the invariant. W

would only be counted once. The relations between the submatrices of σ and m are as follows:

$$\begin{aligned}
m_{vv} &= \sum_{i,j} (\sigma_{vv})_{ij} \ , \\
(m_{vY})_j &= \sum_{i} (\sigma_{vY})_{ij} \ , \\
(m_{vZ})_j &= \sum_{i} (\sigma_{vZ})_{2i-1,j} + \sum_{i} (\sigma_{vZ})_{2i,j} \ , \\
(m_{Yv})_i &= \sum_{j} (\sigma_{Yv})_{ij} \ , \\
(m_{Zv})_i &= \sum_{j} (\sigma_{Zv})_{i,2j-1} + \sum_{j} (\sigma_{Zv})_{i,2j} \ , \\
(m_{YY})_{ij} &= \sigma_{ij} \ , \\
(m_{YZ})_{ij} &= \sigma_{i,2j-1} + \sigma_{i,2j} \ , \\
(m_{ZY})_{ij} &= \sigma_{2i-1,j} + \sigma_{2i,j} \ , \\
(m_{ZZ})_{ij} &= \sigma_{2i-1,2j-1} + \sigma_{2i-1,2j} + \sigma_{2i,2j-1} + \sigma_{2i,2j} \ .
\end{aligned} \tag{5.55}$$

The element m_{ij} denotes how many arrows are pointing from the i-th tensor in the invariant to the j-th tensor. What is happening in Eq. (5.55) is that all equivalent ways of contracting the i-th and j-th tensor are summarised in m_{ij} which means that e.g. for a contraction from a Y tensor to a Z tensor, one has to add the two elements corresponding to the two possible permutations of the lower index of Z, out of which only one can be non-zero in σ. Similarly, for contractions of a Z tensor with another Z tensor (or itself), one has to add all entries in the 2×2 submatrix that corresponds to the four involved indices, out of which only two can be non-zero in σ.

For an invariant that only consists of Y tensors, the contraction matrix m is identical to σ. For invariants only consisting of Z tensors, the situation also becomes a little simpler, as the full contraction matrix is given by the last line of Eq. (5.55). As σ is a permutation matrix, in a 2×2 submatrix only either the two diagonal or the two off-diagonal elements can be non-zero at the same time and the contraction matrix decays into the sum of two smaller permutation matrices of only n_Z elements, i.e. with $\sigma_1^Z, \sigma_2^Z \in S_{n_Z}$:

$$m_{ij} = (\sigma_1^Z)_{ij} + (\sigma_2^Z)_{ij}. \tag{5.56}$$

Now, one can discuss the last source of symmetry, namely permutations of tensors of the same type. Interchanging the position of two Z tensors in an invariant,

$$Z^{a_1 a_2}_{\sigma(a_1)\sigma(a_2)} Z^{a_3 a_4}_{\sigma(a_3)\sigma(a_4)} \cdots \to Z^{a_3 a_4}_{\sigma(a_3)\sigma(a_4)} Z^{a_1 a_2}_{\sigma(a_1)\sigma(a_2)} \cdots \ , \tag{5.57}$$

induces simultaneous permutations of both upper and lower indices of the form

$$\tilde{\tau} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.58}$$

such that an invariant transforms as

$$I_\sigma \xrightarrow{\tilde{\tau}} \mathcal{Z}^{\tilde{\tau}(\alpha)}_{\tilde{\tau}(\sigma(\alpha))}. \tag{5.59}$$

For larger invariants that also contain Y tensors and VEVs the index transformation induced by permutations of tensors of equal type works similarly. Now one can rename $\tilde{\tau}(\alpha) \equiv \alpha'$ such that the invariant becomes

$$I_\sigma \to \mathcal{Z}^{\alpha'}_{\tilde{\tau}(\sigma(\tilde{\tau}^{-1}(\alpha')))} \ , \tag{5.60}$$

which shows that permutations of tensors relate different σ matrices in a way similar to conjugacy class transformations, except that the index permutations induced by tensor permutations do not generate the full permutation group S_n of the n indices. To summarise the symmetries of σ, all permutation matrices that are related to σ by conjugation with transformations of type τ, Eq. (5.53) and transformations of type $\tilde{\tau}$, Eq. (5.60),

$$\tilde{\tau} \circ \tau \circ \sigma \circ \tau' \circ \tilde{\tau}^{-1}, \tag{5.61}$$

where τ and τ' can be two different transformations, produce the same invariant as σ.

On a contraction matrix m, the permutations on tensors act in a simpler way. For all $\sigma^Y \in S_{m_Y}$ and $\sigma^Z \in S_{n_Z}$, all of the contraction matrices, first for invariants without VEVs,

$$\begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_Z \end{pmatrix} m \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_Z \end{pmatrix}^T, \tag{5.62}$$

and for invariants with VEVs,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_Y & 0 \\ 0 & 0 & \sigma_Z \end{pmatrix} m \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_Y & 0 \\ 0 & 0 & \sigma_Z \end{pmatrix}^T , \tag{5.63}$$

will produce equivalent invariants. Similarly, for invariants only involving Z tensors, all $\sigma^Z m(\sigma^Z)^T$ will produce equivalent invariants. Applying this to Eq. (5.56), this means that one of the two summands can be chosen to be a conjugacy class representative of S_{n_Z} which reduces the number of invariants that need to be considered.

Finally, all pieces are in place to discuss the CP properties first of σ and after that of m. The CP conjugate of an invariant can be obtained by interchanging upper and lower indices, or in the shorthand notation introduced in Eq. (5.50),

$$I_\sigma = \mathcal{Z}^{\alpha}_{\sigma(\alpha)} \xrightarrow{CP} \mathcal{Z}^{\sigma(\alpha)}_{\alpha}. \tag{5.64}$$

One can now rename $\sigma(\alpha) = \alpha'$ and subsequently drop the prime to obtain

$$I_\sigma \xrightarrow{CP} \mathcal{Z}^{\alpha}_{\sigma^{-1}(\alpha)}. \tag{5.65}$$

Naively, an invariant is CP-even if it equals its CP conjugate which leads to the condition

$$\sigma^2 = 1. \tag{5.66}$$

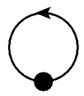
However, one has to take into account also all permutation matrices that are equivalent to σ such that the condition becomes

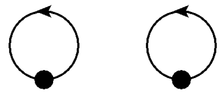
$$\sigma^{-1} = \tilde{\tau} \circ \tau \circ \sigma \circ \tau' \circ \tilde{\tau}^{-1}, \tag{5.67}$$

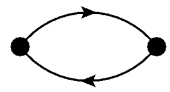
which means that as soon as any $\tau, \tau', \tilde{\tau}$ exist such that the above condition can be fulfilled, σ produces a CP-even invariant.

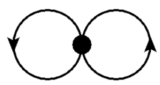
For contraction matrices, the condition testing if an invariant is CP-even simplifies. With $\sigma^{-1} = \sigma^T$, from which follows that $m \xrightarrow{CP} m^T$ and if σ_Y and σ_Z exist, such that the right-hand side is fulfilled, then the condition for invariants without VEVs to be CP-even becomes

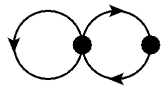
$$\text{Invariant CP-even} \Leftrightarrow m^T = \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_Z \end{pmatrix} m \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_Z \end{pmatrix}^T , \tag{5.68}$$

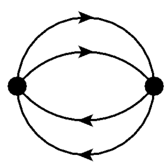
$Y^a_a =$  $= \begin{pmatrix}1\end{pmatrix}$

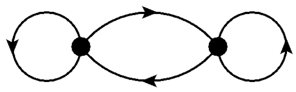
$Y^a_a Y^b_b =$  $= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$Y^a_b Y^b_a =$  $= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$Z^{ab}_{ab} =$  $= \begin{pmatrix}2\end{pmatrix}$

$Z^{ac}_{bc} Y^b_a =$  $= \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$

$Z^{ab}_{cd} Z^{cd}_{ab} =$  $= \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$

$Z^{ab}_{ac} Z^{cd}_{bd} =$  $= \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

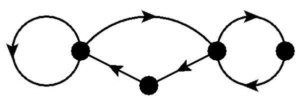
$I_1 \equiv Z^{ab}_{ae} Z^{cd}_{bf} Y^l_c Y^f_d =$  $= \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0\end{pmatrix}$

Figure 5.2: Examples of contraction matrices for small invariants. All contraction matrices are symmetric except for the CPI.

and for invariants with VEVs

$$\text{spont. Invariant CP-even} \Leftrightarrow m^T = \begin{pmatrix}1 & 0 & 0\\ 0 & \sigma_Y & 0\\ 0 & 0 & \sigma_Z\end{pmatrix} m \begin{pmatrix}1 & 0 & 0\\ 0 & \sigma_Y & 0\\ 0 & 0 & \sigma_Z\end{pmatrix}^T , \qquad (5.69)$$

where the actions of τ and τ' are absorbed in m. Figures 5.2 and 5.3 contain examples of contraction matrices for small diagrams. There, all contraction matrices happen to be trivially symmetric except for the CPIs.

It is condition Eq. (5.69) that was used to find CP-odd invariants. In the actual search, first all σ matrices for a certain number of Y and Z tensors, and

VEVs was generated. This list of σ matrices was then reduced to a list of contraction matrices, which was condensed using Eq. (5.63) to classes of equivalent contraction matrices, out of which a representative was tested for CP-oddness using Eq. (5.69). This search was performed for invariants without VEVs for $m_Y = 0$ up to $n_Z = 6$, where it was found that all invariants without Y tensors until $n_Z = 4$ are CP even. Furthermore, CP-odd invariants were found for $(m_Y, n_Z) = (1,3), (1,4), (2,2), (2,3), (3,3)$. For invariants with VEVs, only a search for invariants with $m_Y = 0$ was performed, where CP-odd invariants were found for $(n_v, n_Z) = (1,3), (2,3), (1,4)$. All inequivalent invariants from these classes are listed in section 5.2 or in appendix 7.3.

As one progresses to more complicated invariants, one has to make sure not to count invariants that are products or powers of smaller invariants. An invariant that is a product of two smaller invariants will correspond to a diagram that decays into two separate graphs. As this means that some vertices are only connected among each other while being unconnected to the rest of the diagram, such a reducible invariant will be described by a contraction matrix that can be brought to block-diagonal form only using permutation matrices. This means in particular, as for invariants with VEVs, m_{vv} denotes the number of VEVs that are only connected to other VEVs, that $m_{vv} \neq 0$ would mean that the diagram would contain graphs for $v_a v^{*a}$ that are unconnected to the rest of the diagram.

Finally, there is one last condition that relates invariants, namely the minimisation condition Eq. (5.173). In the contraction matrix for an invariant with VEVs this can be used if there is an i such that

$$m_{1i} = 1 \text{ and } m_{i1} = 2 \ , \tag{5.70}$$

or

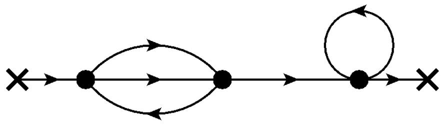
$$m_{1i} = 2 \text{ and } m_{i1} = 1. \tag{5.71}$$

In both cases, the Z tensor at position i in the invariant is connected to three VEVs.

5.2.5 CP-odd invariants only built from Z tensors

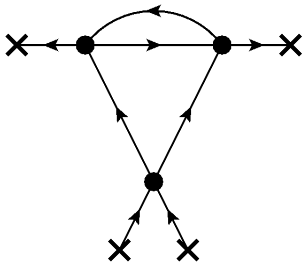
It is interesting to consider invariants that are only built from Z tensors, as these indicate CP violation that is mediated purely through the interaction of fields and does not e.g. depend on a mass splitting. One could now wonder if for a

$$J_1^{(3,1)} = Z^{a_1a_2}_{a_5a_6} Z^{a_3a_4}_{a_1a_3} Z^{a_5a_6}_{a_2a_7} v_{a_4} v^{*a_7} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$J_1^{(3,2)} \equiv Z^{a_1a_2}_{a_4a_5} Z^{a_3a_4}_{a_2a_6} Z^{a_5a_6}_{a_7a_8} v_{a_1} v_{a_3} v^{*a_7} v^{*a_8} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

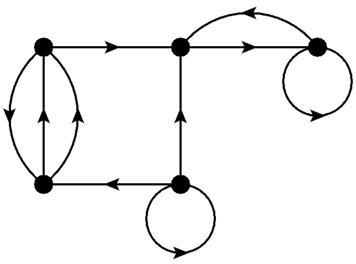
Figure 5.3: Examples for contraction matrices of CPIs for spontaneous CP violation. We draw each of the VEVs (as opposed to a single vertex for the whole W tensor).

non-diagonal Y tensor CP violating effects could be shifted between Y and Z by diagonalising Y. However, because this is just another basis change, it drops out in any basis invariants, including also CPIs.

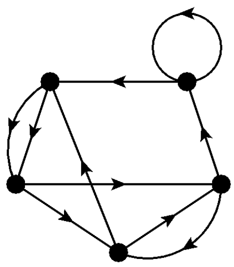
In Appendix 7.3, we list the representative CPIs with up to $n_Z = 6$ Z tensors. All other CPIs are related to these representatives by symmetries or CP conjugation.

An important first result is that *all invariants up to $n_Z = 4$ are CP-even.* For $n_Z = 5$, three different CPIs exist:

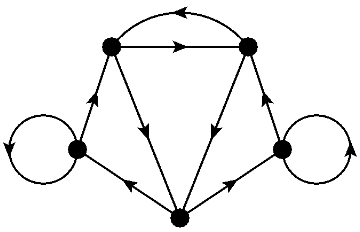
$$I_1^{(5)} \equiv Z^{a_1a_2}_{a_7a_9} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_3a_6} Z^{a_7a_8}_{a_4a_8} Z^{a_9a_{10}}_{a_1a_2} = \tag{5.72}$$



$$I_2^{(5)} \equiv Z^{a_1a_2}_{a_5a_7} Z^{a_3a_4}_{a_8a_9} Z^{a_5a_6}_{a_3a_6} Z^{a_7a_8}_{a_4a_{10}} Z^{a_9a_{10}}_{a_1a_2} = \tag{5.73}$$



$$I_3^{(5)} \equiv Z^{a_1a_2}_{a_5a_9} Z^{a_3a_4}_{a_3a_7} Z^{a_5a_6}_{a_6a_8} Z^{a_7a_8}_{a_1a_{10}} Z^{a_9a_{10}}_{a_2a_4} = \tag{5.74}$$



For $n_Z = 6$, in total 56 different invariants exist out of which three are products of the $n_Z = 5$ invariants with a completely self-contracted Z tensor, Z^{ab}_{ab}. These will not provide any new information. Next, of particular interest are those invariants that contain no self-loops, as we found that invariants with self-loops, i.e. $Z^{a\cdot}_{a\cdot}$ often vanish for the example potentials considered in this work. With $n_Z = 6$, only 5 invariants without self-loops remain:

$$I_1^{(6)} \equiv Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_7a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} , \tag{5.75}$$

$$I_2^{(6)} \equiv Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_{11}a_6} Z^{a_5a_6}_{a_9a_8} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} , \tag{5.76}$$

$$I_3^{(6)} \equiv Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_6} Z^{a_5a_6}_{a_{11}a_8} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} , \tag{5.77}$$

$$I_4^{(6)} \equiv Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_7a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_1a_4} Z^{a_{11}a_{12}}_{a_3a_2} , \tag{5.78}$$

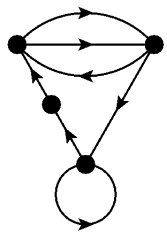
$$I_5^{(6)} \equiv Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_9a_8} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_6} Z^{a_{11}a_{12}}_{a_3a_2} . \tag{5.79}$$

The diagrams that correspond to the above invariants with $n_Z = 6$ and the remaining representative CPIs with up to $n_Z = 6$ are listed in Appendix 7.3.

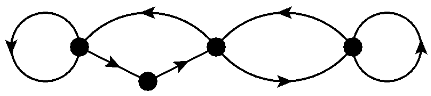
5.2.6 CP-odd invariants built from Y and Z tensors

Mixed invariants consisting of Y and Z tensors can be CP-odd at lower numbers of Z tensors than $n_Z = 5$. The reason for this is that additional asymmetries can be introduced in the diagrams. The smallest CPI found in [121], Eqs. (5.44) and (5.45), is of this type with $m_Y = 2, n_Z = 2$ and will not be repeated here. There are no other CPIs for $m_Y = 2, n_Z = 2$ that are not equivalent to the aforementioned one. The next smallest CPIs are found for $n_Z = 3$ and $m_Y = 1$. There are two different classes with the following representatives:

$$I_1^{(3,1)} \equiv Y^{a_1}_{a_4} Z^{a_2a_3}_{a_6a_7} Z^{a_4a_5}_{a_2a_5} Z^{a_6a_7}_{a_1a_3} = \tag{5.80}$$

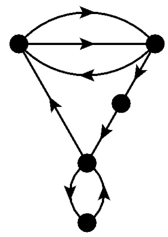


$$I_2^{(3,1)} \equiv Y^{a_1}_{a_4} Z^{a_2a_3}_{a_2a_6} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_1a_3} = \tag{5.81}$$

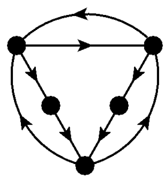


However, both invariants in Eqs. (5.80) and (5.81) contain self-loops. As these often vanish in examples, one would preferably like to find invariants without self-loops. Such invariants can already be found for $n_Z = 3, m_Y = 2$. There are in total 13 invariants with this number of Y and Z tensors, out of which 2 have no Z-self-loops. In one of them, the Y tensor is inserted in a Z loop and will only make a difference if Y is not proportional to the identity, while the other invariant has genuinely no Z-self-loops. These invariants and diagrams are

$$I_1^{(3,2)} \equiv Y^{a_1}_{a_7} Y^{a_2}_{a_5} Z^{a_3a_4}_{a_6a_7} Z^{a_5a_6}_{a_3a_4} Z^{a_7a_8}_{a_1a_2} = \tag{5.82}$$



$$I_2^{(3,2)} \equiv Y^{a_1}_{a_5} Y^{a_2}_{a_3} Z^{a_3a_4}_{a_6a_7} Z^{a_5a_6}_{a_4a_8} Z^{a_7a_8}_{a_1a_2} = \quad . \tag{5.83}$$



Naively, there are 53 classes of invariants with $m_Y = 3, n_Z = 3$. However, many of these will be products of smaller CPIs with small CP-even invariants. Eventually, there are 10 invariants without Z-self loops which are not products of smaller invariants, the representatives of which are listed in the following:

$$I_1^{(3,3)} \equiv Y^{a_1}_{a_8} Y^{a_2}_{a_6} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_7a_9} Z^{a_6a_7}_{a_3a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.84}$$

$$I_2^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_7} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_3a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.85}$$

$$I_3^{(3,3)} \equiv Y^{a_1}_{a_8} Y^{a_2}_{a_4} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_7a_9} Z^{a_6a_7}_{a_3a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.86}$$

$$I_4^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_7a_9} Z^{a_6a_7}_{a_3a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.87}$$

$$I_5^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_7} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_3a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.88}$$

$$I_6^{(3,3)} \equiv Y^{a_1}_{a_8} Y^{a_2}_{a_3} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_7a_9} Z^{a_6a_7}_{a_4a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.89}$$

$$I_7^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_7a_9} Z^{a_6a_7}_{a_4a_5} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.90}$$

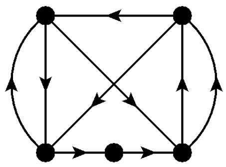
$$I_8^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_7a_8} Z^{a_6a_7}_{a_5a_9} Z^{a_8a_9}_{a_1a_2} \,, \tag{5.91}$$

$$I_9^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_7a_8} Z^{a_6a_7}_{a_1a_9} Z^{a_8a_9}_{a_2a_5} \,, \tag{5.92}$$

$$I_{10}^{(3,3)} \equiv Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4 a_5}_{a_8 a_9} Z^{a_6 a_7}_{a_1 a_5} Z^{a_8 a_9}_{a_2 a_7} \ . \tag{5.93}$$

Finally, we have also analysed invariants with $n_Z = 4$ and $m_Y = 1$. Naively, there are 18 different invariants with this number of Z and Y tensors. However, there is only one invariant with this number of coupling tensors without self-loops,

$$I_{1}^{(4,1)} \equiv Y^{a_1}_{a_6} Z^{a_2 a_3}_{a_4 a_7} Z^{a_4 a_5}_{a_8 a_9} Z^{a_6 a_7}_{a_2 a_5} Z^{a_8 a_9}_{a_1 a_3} = \tag{5.94}$$



This concludes our list of CPIs for explicit CP violation used in the main text of this chapter. Following our systematic approach, we have also calculated larger invariants and the obtained CPIs are collected in Appendix 7.3.

5.3 Two Higgs doublet model potential

As a first example for an application of CPIs that is well known in the literature we consider the most general potential of two copies of SM Higgs bosons. For this potential, a complete basis of CPIs is known [123]. All of these four CPIs have also been produced in our systematic search. Using a slightly modified version of the notation in [121], the general 2HDM potential takes the form

$$\begin{aligned} V(H_1,H_2) \;=\; & m_1^2\, H_1^\dagger H_1 + m_{12}^2\, e^{i\theta_0}\, H_1^\dagger H_2 + m_{12}^2\, e^{-i\theta_0}\, H_2^\dagger H_1 + m_2^2\, H_2^\dagger H_2 + \\ & + a_1 \left(H_1^\dagger H_1\right)^2 + a_2 \left(H_2^\dagger H_2\right)^2 \\ & + b \left(H_1^\dagger H_1\right)\left(H_2^\dagger H_2\right) + b' \left(H_1^\dagger H_2\right)\left(H_2^\dagger H_1\right) + \\ & + c_1\, e^{i\theta_1} \left(H_1^\dagger H_1\right)\left(H_2^\dagger H_1\right) + c_1\, e^{-i\theta_1} \left(H_1^\dagger H_1\right)\left(H_1^\dagger H_2\right) + \\ & + c_2\, e^{i\theta_2} \left(H_2^\dagger H_2\right)\left(H_2^\dagger H_1\right) + c_2\, e^{-i\theta_2} \left(H_2^\dagger H_2\right)\left(H_1^\dagger H_2\right) + \\ & + d\, e^{i\theta_3} \left(H_1^\dagger H_2\right)^2 + d\, e^{-i\theta_3} \left(H_2^\dagger H_1\right)^2 . \end{aligned} \tag{5.95}$$

Here $H_1 = (h_{1,1}, h_{1,2})$ and $H_2 = (h_{2,1}, h_{2,2})$ and the $SU(2)_L$ invariant contractions are indicated by the brackets e.g. $(H_1^\dagger H_1)^2 = (h_{1,1}^\dagger h_{1,1} + h_{1,2}^\dagger h_{1,2})^2$. Eq. (5.2) becomes

$$\phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}) \ , \tag{5.96}$$

such that the Z tensor corresponding to the quartic terms of the scalar potential has $4^4 = 256$ components. It is straightforward to determine these explicitly for the potential of Eq. (5.95). In the following, we display the non-vanishing ones,

$$Z^{11}_{11} = Z^{22}_{22} = 2Z^{12}_{12} = a_1 \ , \tag{5.97}$$

$$Z^{33}_{33} = Z^{44}_{44} = 2Z^{34}_{34} = a_2 \ , \tag{5.98}$$

$$4Z^{14}_{14} = 4Z^{23}_{23} = b \ , \tag{5.99}$$

$$4Z^{14}_{23} = 4Z^{23}_{14} = b' \ , \tag{5.100}$$

$$4Z^{13}_{13} = 4Z^{24}_{24} = b + b' \ , \tag{5.101}$$

$$4Z^{12}_{14} = 4Z^{12}_{23} = 2Z^{11}_{13} = 2Z^{22}_{24} = c_1 e^{i\theta_1} \ , \tag{5.102}$$

$$4Z^{14}_{34} = 4Z^{23}_{34} = 2Z^{13}_{33} = 2Z^{24}_{44} = c_2 e^{i\theta_2} \ , \tag{5.103}$$

$$2Z^{34}_{12} = Z^{33}_{11} = Z^{44}_{22} = d e^{i\theta_3} \ , \tag{5.104}$$

and remind the reader of the general relations $Z^{ab}_{cd} = Z^{ba}_{cd} = Z^{ab}_{dc} = Z^{ba}_{dc}$ and $Z^{cd}_{ab} = (Z^{ab}_{cd})^*$. Having determined the Z tensor in terms of the parameters of the potential, we can calculate CPIs explicitly.

As a first illustration, we show the results of CPIs for this potential. In our notation, the smallest one becomes

$$\begin{aligned}
\mathcal{I}_1 = & -9im_{12}^2\left(m_1^2 - m_2^2\right)\Big[c_2(2a_1 - b - b')\sin(\theta_0+\theta_2) + c_1(2a_2 - b - b')\sin(\theta_0+\theta_1) \\
& \quad + 2d(c_1\sin(\theta_0-\theta_1-\theta_3) + c_2\sin(\theta_0-\theta_2-\theta_3))\Big] \\
& - 9im_{12}^4\Big[4d(a_2-a_1)\sin(2\theta_0-\theta_3) + c_1^2\sin(2(\theta_0+\theta_1)) \\
& \quad - c_2(2c_1\sin(\theta_1-\theta_2) + c_2\sin(2(\theta_0+\theta_2)))\Big] \\
& - 9ic_1c_2\left(m_1^2-m_2^2\right)^2\sin(\theta_1-\theta_2).
\end{aligned} \tag{5.105}$$

There are many ways of setting this expression to zero, the simpler ones involve $m_{12}^2 = 0$ which leaves only the last line in the expression, which vanishes either for $m_1^2 - m_2^2$ or $\sin(\theta_1 - \theta_2)$. Alternatively, if $m_1^2 - m_2^2 = 0$ there are other combinations of constraints that make this CPI vanish, including $\sin(2\theta_0 - \theta_3) = \sin(2(\theta_0 + \theta_1)) = 0$. However, at this stage it is not clear if any of these constraints are sufficient to guarantee conservation of CP, as other CPIs could still be non-zero. The invariant $\mathcal{I}_1$ being non-zero always requires $m_{12} \neq 0$ or $m_1^2 \neq m_2^2$.

As already mentioned, a complete basis of CPIs is known for the 2HDM potential, cf. [123]. Of the four invariants given in that paper, three are equivalent to invariants given in Section 5.2 of this thesis (GH denotes the invariant in [123]):

$$I_{2Y2Z}^{(\mathrm{GH})} = I_1 = I_1^{(2,2)} \ , \tag{5.106}$$

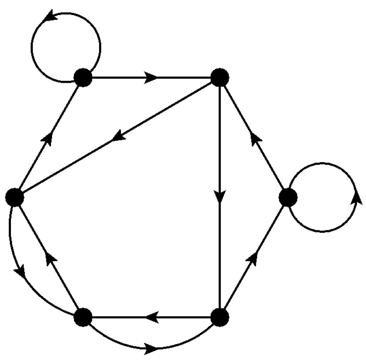
$$I_{Y3Z}^{(\mathrm{GH})} = I_2^{(3,1)} \ , \tag{5.107}$$

and

$$I_{3Y3Z}^{(\mathrm{GH})} = (I_5^{(3,3)})^* \ . \tag{5.108}$$

The fourth invariant listed in [123] has $n_Z = 6$ and has not been given here yet as it contains Z-self-loops. For completeness, we present it here following our general notation as well as diagrammatically:

$$I_{6Z}^{(\mathrm{GH})} \equiv Z_{bd}^{ac} Z_{fe}^{be} Z_{hg}^{dg} Z_{ak}^{fj} Z_{jn}^{km} Z_{mc}^{nh} = \tag{5.109}$$



All CP-odd invariants with 5 Z tensors, cf. Eqs. (5.72)-(5.74), vanish for this potential. For this potential, Z_{ac}^{ab} is non-diagonal, which is why the CPIs found in [123] produce interesting results. While the invariants from [123] form a complete basis of CPIs for the 2HDM, all of them are zero for the potentials considered in the remainder of this chapter. It is our systematic search that reveals new non-zero CPIs in those situations.

5.4 $\mathbf{A_4 = \Delta(12)}$ invariant potentials

In this section we study potentials invariant under the discrete group A_4. We start with a field content of a single triplet of SM singlets, then consider a triplet of $SU(2)_L$ doublets, two triplets of SM singlets and two triplets of $SU(2)_L$ doublets.

A_4 contains a real triplet and three one-dimensional representations. The product of two triplets decomposes as

$$\mathbf{3} \otimes \mathbf{3} \;=\; (\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 + \mathbf{3})_s + \mathbf{3}_a \;. \tag{5.110}$$

Symmetric and antisymmetric combinations are denoted by subscripts s and a, respectively. Throughout this section we work in the basis of [155] which can be easily generalised to $\Delta(27)$ and the complete $\Delta(3n^2)$ series [156, 110] studied in the Sections 5.5 and 7.2.

5.4.1 One flavour triplet

With one triplet field, only the symmetric contribution in Eq. (5.110) matters. It is convenient to define

$$V_0(\varphi) = -\, m_\varphi^2 \sum_i \varphi_i \varphi^{*i} + r \left(\sum_i \varphi_i \varphi^{*i} \right)^2 + s \sum_i (\varphi_i \varphi^{*i})^2 \;, \tag{5.111}$$

where one notes that the first two terms are $SU(3)$ invariant. We consider φ to be additionally charged under some $U(1)$ symmetry (or an appropriate discrete subgroup) such that terms of the form $\varphi_i\varphi_i$ or $\varphi_i\varphi_i\varphi_i$, for example, are not allowed. This leads to a more direct generalisation of the case where the SM gauge group applies.

The resulting renormalisable scalar potential for A_4 reads

$$\begin{aligned} V_{A_4}(\varphi) = V_0(\varphi) &+ c \left(\varphi_1 \varphi_1 \varphi^{*3} \varphi^{*3} + \varphi_2 \varphi_2 \varphi^{*1} \varphi^{*1} + \varphi_3 \varphi_3 \varphi^{*2} \varphi^{*2} \right) \\ &+ c^* \left(\varphi^{*1} \varphi^{*1} \varphi_3 \varphi_3 + \varphi^{*2} \varphi^{*2} \varphi_1 \varphi_1 + \varphi^{*3} \varphi^{*3} \varphi_2 \varphi_2 \right) \;, \end{aligned} \tag{5.112}$$

noting that this includes, as expected, four independent quartic terms. Henceforth we use the convenient abbreviations cycl. to denote the cyclic permutations, and h.c. to indicate the hermitian conjugate. We thus write the A_4 invariant potential of Eq. (5.112) in the compact form:

$$V_{A_4}(\varphi) = V_0(\varphi) + \;\; \left[c \left(\varphi_1 \varphi_1 \varphi^{*3} \varphi^{*3} + \text{cycl.} \right) \;+\; \text{h.c.} \right] \;. \tag{5.113}$$

The A_4 symmetric potential respects the general CP symmetry with a 2-3 swap, namely the CP symmetry with unitary matrix X_{23}

$$X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{5.114}$$

for arbitrary coefficients $r, s \in \mathbb{R}$ and $c \in \mathbb{C}$. Hence, despite the occurrence of the complex coupling c the A_4 symmetric potential of one triplet is invariant under this general CP symmetry. For this reason, all possible CPIs for this potential will be zero.

5.4.2 One flavour triplet of Higgs doublets

If each component of the A_4 triplet is an $SU(2)_L$ doublet,

$$H = (h_{1\alpha}, h_{2\beta}, h_{3\gamma}) , \tag{5.115}$$

the potential is rather similar to the previous case. Indeed there is one additional invariant, due to the two different ways to perform the $SU(2)_L$ contraction on the A_4 invariant $\left(\sum_i \varphi_i \varphi^{*i}\right)^2$, when the φ are replaced by Higgs doublets[9]

$$\sum_{i,j,\alpha,\beta} \left[r_1 (h_{i\alpha} h^{*i\alpha})(h_{j\beta} h^{*j\beta}) + r_2 (h_{i\alpha} h^{*i\beta})(h_{j\beta} h^{*j\alpha}) \right] . \tag{5.116}$$

Here we highlight the $SU(2)_L$ indices to clarify the distinct $SU(2)_L$ contractions. We define $V_0(H)$ in analogy with Eq. (5.111):

$$\begin{aligned} V_0(H) \;=\; & - m_h^2 \sum_{i,\alpha} h_{i\alpha} h^{*i\alpha} + \sum_{i,j,\alpha,\beta} \left[r_1 (h_{i\alpha} h^{*i\alpha})(h_{j\beta} h^{*j\beta}) + r_2 (h_{i\alpha} h^{*i\beta})(h_{j\beta} h^{*j\alpha}) \right] \\ & + s \sum_{i,\alpha,\beta} (h_{i\alpha} h^{*i\alpha})(h_{i\beta} h^{*i\beta}) , \end{aligned} \tag{5.117}$$

[9] Since the doublet **2** of $SU(2)_L$ is a pseudoreal representation, it is also possible to combine $(h_{i\alpha} h_{j\beta} \epsilon^{\alpha\beta})(h^{*i\gamma} h^{*j\delta} \epsilon_{\gamma\delta})$ using the antisymmetric ϵ tensor. However, such a term is not linearly independent of the two terms in Eq. (5.116) as can be easily seen in an explicit calculation or by noting that $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$ which entails only two independent $SU(2)_L$ invariant quartic terms.

and the A_4 potential is then

$$V_{A_4}(H) = V_0(H) + \sum_{\alpha,\beta} \left[c\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] . \tag{5.118}$$

This potential is also invariant under a CP transformation that involves swapping the second and third component in flavour space while keeping $SU(2)_L$ contractions unchanged, i.e. $h_{2\alpha} \to h^{*3\alpha}$ etc.:

$$X_{23}^H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \delta^\alpha_\beta \ . \tag{5.119}$$

Therefore, CP is conserved automatically for this potential and all possible CPIs necessarily vanish.

5.4.3 Two flavour triplets

Typically, realistic models of flavour require more than just one triplet flavon. We therefore consider the potential involving two physically different flavon fields φ and φ' which both transform in the triplet representation of A_4. In the case of two A_4 triplets distinguished by additional symmetries so that the total symmetry is $A_4 \times U(1) \times U(1)'$, the potential includes a total of seven independent mixed quartic invariants of the form $\varphi\,\varphi'\,\varphi^*\,\varphi'^*$. It is convenient to define:

$$\begin{aligned} V_1(\varphi,\varphi') = &+\ \tilde{r}_1 \left(\sum_i \varphi_i\varphi^{*i}\right)\left(\sum_j \varphi'_j\varphi'^{*j}\right) + \tilde{r}_2 \left(\sum_i \varphi_i\varphi'^{*i}\right)\left(\sum_j \varphi'_j\varphi^{*j}\right) \\ &+\ \tilde{s}_1 \sum_i \left(\varphi_i\varphi^{*i}\varphi'_i\varphi'^{*i}\right) \\ &+\ \tilde{s}_2 \left(\varphi_1\varphi^{*1}\varphi'_2\varphi'^{*2} + \varphi_2\varphi^{*2}\varphi'_3\varphi'^{*3} + \varphi_3\varphi^{*3}\varphi'_1\varphi'^{*1}\right) \\ &+\ i\,\tilde{s}_3 \Big[(\varphi_1\varphi'^{*1}\varphi'_2\varphi^{*2} + \text{cycl.}) - (\varphi^{*1}\varphi'_1\varphi'^{*2}\varphi_2 + \text{cycl.})\Big] . \end{aligned} \tag{5.120}$$

Note that in this definition, the term multiplied by $\tilde{r}_1$ contains the term multiplied by $\tilde{s}_2$ as well as the term obtained from the latter by interchanging φ with φ':

$$\left(\varphi'_1\varphi'^{*1}\varphi_2\varphi^{*2} + \varphi'_2\varphi'^{*2}\varphi_3\varphi^{*3} + \varphi'_3\varphi'^{*3}\varphi_1\varphi^{*1}\right) , \tag{5.121}$$

which is not included separately in $\tilde{s}_2$.

The A_4 symmetric renormalisable potential takes the following explicit form, with V_0 as defined in Eq. (5.111),

$$V_{A_4}(\varphi,\varphi') = V_0(\varphi) + V_0'(\varphi') + V_1(\varphi,\varphi') + \\ + \left[c\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] + \left[c'\left(\varphi_1'\varphi_1'\varphi'^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ + \left[\tilde{c}\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right], \quad (5.122)$$

where $V_0'(\varphi')$ has the same functional form as $V_0(\varphi)$ with different coefficients $m'_{\varphi'}$, r', s' and depends on φ'.

Unlike the previous A_4 invariant potentials, this potential in general violates CP, as confirmed by the non-zero CPIs listed in Table 5.1 of Section 5.6. The expressions are cumbersome and we do not reproduce them here. The non-vanishing CPIs $\mathcal{I}_2^{(6)}, \mathcal{I}_3^{(6)}, \mathcal{I}_4^{(6)}, \mathcal{I}_5^{(6)}$ (Eqs. (5.76,5.77,5.78,5.79)) all factorise as a product of $\tilde{s}_2$ with different complicated functions of the remaining parameters, for example, $\mathcal{I}_2^{(6)}$ takes the form:

$$\mathcal{I}_2^{(6)} = \tilde{s}_2 f(...)\ , \quad (5.123)$$

where f is a complicated function of the other parameters. Such a dependence on $\tilde{s}_2$ is expected because it corresponds to a CP symmetry, where one imposes X_{23} of Eq. (5.114) on both triplets, corresponding to the block matrix:

$$X_{23}^{\varphi\varphi'} = \begin{pmatrix} X_{23} & 0 \\ 0 & X_{23} \end{pmatrix}. \quad (5.124)$$

This CP symmetry constrains the potential such that $\tilde{s}_2 = 0$, which forces all CPIs to vanish as expected from the presence of a CP symmetry. Furthermore, applying instead the trivial CP symmetry CP_0 forces $\tilde{s}_3 = 0$ and all other complex parameters $(c, c', \tilde{c})$ to be real. As expected, this renders $f(...) = 0$ in Eq. (5.123), an makes all other CPIs vanish as well.

5.4.4 Two flavour triplets of Higgs doublets

Earlier, when considering a potential of an A_4 triplet of $SU(2)_L$ doublets, the only difference was that the term with coefficient r split into two different invariants corresponding to two different possible $SU(2)_L$ contractions, cf. Eq. (5.116).

Similarly, the potential of two triplets of SM doublets:

$$H = (h_{1\alpha}, h_{2\beta}, h_{3\gamma})\,, \quad H' = (h'_{1\alpha}, h'_{2\beta}, h'_{3\gamma})\ , \tag{5.125}$$

can be obtained from the corresponding potential of singlets, Eq. (5.122). In the first two parts of the potential, $V_0(\varphi)$ and $V_0(\varphi')$, as earlier, there are two different ways of $SU(2)_L$-contracting the invariants with coefficients r and r'. In the part of the potential with A_4 contractions as in $V_1(\varphi,\varphi')$, for all A_4 invariants two possible ways of $SU(2)_L$ contracting the fields exists and this part of the potential becomes

$$\begin{aligned} V_1(H,H') = &\sum_{i,j,\alpha,\beta} \left[\tilde{r}_{11} h_{i\alpha} h^{*i\alpha} h'_{j\beta} h'^{*j\beta} + \tilde{r}_{12} h_{i\alpha} h'^{*j\alpha} h'_{j\beta} h^{*i\beta}\right] \\ &+ \sum_{i,j,\alpha,\beta} \left[\tilde{r}_{21} h_{i\alpha} h'^{*i\alpha} h'_{j\beta} h^{*j\beta} + \tilde{r}_{22} h_{i\alpha} h^{*j\alpha} h'_{j\beta} h'^{*i\beta}\right] \\ &+ \sum_{i,\alpha,\beta} \left[\tilde{s}_{11} h_{i\alpha} h^{*i\alpha} h'_{i\beta} h'^{*i\beta} + \tilde{s}_{12} h_{i\alpha} h'^{*i\alpha} h'_{i\beta} h^{*i\beta}\right] \\ &+ \sum_{\alpha,\beta} \left[\tilde{s}_{21} (h_{1\alpha} h^{*1\alpha} h'_{2\beta} h'^{*2\beta} + \text{cycl.}) + \tilde{s}_{22} (h_{1\alpha} h'^{*2\alpha} h'_{2\beta} h^{*1\beta} + \text{cycl.})\right] \\ &+ i\tilde{s}_{31} \sum_{\alpha,\beta} [(h_{1\alpha} h'^{*1\alpha} h'_{2\beta} h^{*2\beta} + \text{cycl.}) - (h^{*1\alpha} h'_{1\alpha} h'^{*2\beta} h_{2\beta} + \text{cycl.})] \\ &+ i\tilde{s}_{32} \sum_{\alpha,\beta} [(h_{1\alpha} h^{*2\alpha} h'_{2\beta} h'^{*1\beta} + \text{cycl.}) - (h^{*1\alpha} h_{2\alpha} h'^{*2\beta} h'_{1\beta} + \text{cycl.})]. \end{aligned} \tag{5.126}$$

Finally, of the remainder of the potential, only the invariant with coefficient $\tilde{c}$ from Eq. (5.122) needs to be doubled:

$$\sum_{\alpha,\beta} \left[\tilde{c}_1 (h_{1\alpha} h^{*3\alpha} h'_{1\beta} h'^{*3\beta} + \text{cycl.}) + \tilde{c}_2 (h_{1\alpha} h'^{*3\alpha} h'_{1\beta} h^{*3\beta} + \text{cycl.}) + \text{h.c.}\right]. \tag{5.127}$$

We therefore write

$$\begin{aligned} V_{A_4}(H,H') = &V_0(H) + V'_0(H') + V_1(H,H') \\ &+ \sum_{\alpha,\beta} \left[c\left(h_{1\alpha} h_{1\beta} h^{*3\alpha} h^{*3\beta} + \text{cycl.}\right) + c'\left(h'_{1\alpha} h'_{1\beta} h'^{*3\alpha} h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\ &+ \sum_{\alpha,\beta} \left[\tilde{c}_1 (h_{1\alpha} h^{*3\alpha} h'_{1\beta} h'^{*3\beta} + \text{cycl.}) + \tilde{c}_2 (h_{1\alpha} h'^{*3\alpha} h'_{1\beta} h^{*3\beta} + \text{cycl.}) + \text{h.c.}\right]. \end{aligned} \tag{5.128}$$

We note that due to $SU(2)_L$ not allowing cubic invariants of H and/or H', it is sufficient to use a Z_3 symmetry to distinguish the A_4 triplets.[10]

This potential generally violates CP. This can be seen from the CP-odd invariants calculated, as $\mathcal{I}_2^{(6)}, \mathcal{I}_3^{(6)}, \mathcal{I}_4^{(6)}, \mathcal{I}_5^{(6)}$ (Eqs. (5.76,5.77,5.78,5.79)) are non-zero (see Table 5.1) but with too large expressions to display here. However, it is possible to impose a CP symmetry with

$$X_{23}^{HH'} = \begin{pmatrix} X_{23} & 0 \\ 0 & X_{23} \end{pmatrix} \otimes \delta^{\alpha}_{\beta} \ , \tag{5.129}$$

which, similarly to previous examples, restricts the coefficients in the potential, namely

$$\tilde{s}_{21} = \tilde{s}_{22} = 0 \ , \tag{5.130}$$

thereby forcing all CPIs to vanish. Imposing, alternatively, the canonical CP symmetry CP_0 leads to $\tilde{s}_{31} = \tilde{s}_{32} = 0$ as well as $c, c', \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$.

5.4.5 S_4 invariant potentials

The transition from $\Delta(3n^2)$ invariant potentials with arbitrary $n \in N$ to potentials which are symmetric under the larger group $\Delta(6n^2)$ is discussed in Appendix 7.4.2. The corresponding basis of $S_4 = \Delta(6 \times 2^2)$ can be found in [157, 49, 110]. For the A_4 potential with one triplet of singlets as well the A_4 potential with a triplet of doublets, the corresponding S_4 invariant potentials are obtained by setting

$$c^* = c \ , \tag{5.131}$$

so that

$$V_{S_4}(\varphi) = V_0(\varphi) + c\left[(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \text{cycl.}) + \text{h.c.}\right], \tag{5.132}$$

and

$$V_{S_4}(H) = V_0(H) + \sum_{\alpha,\beta} c\left[\left(h_{1\alpha}h_{1\beta}h^{*3\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right], \tag{5.133}$$

[10] The potential invariant under a Z_2 [154] would additionally allow for invariants of the form $h_{i\alpha}h'^{*i\alpha}h_{j\beta}h'^{*j\beta}$ and $h_{i\alpha}h'^{*i\beta}h_{j\beta}h'^{*j\alpha}$ where the conjugated fields are both related to H'.

where the potentials V_0 were defined in Eq. (5.111) and Eq. (5.117). For the potential of two triplets of A_4, the S_4 invariant potential arises via setting

$$\tilde{s}_2 = \tilde{s}_3 = 0 \ , \tag{5.134}$$

and additionally

$$c^* = c \ , \qquad c'^* = c' \ , \qquad \tilde{c}^* = \tilde{c} \ . \tag{5.135}$$

Defining the following abbreviation,

$$\begin{aligned} V_2(\varphi,\varphi') =& \tilde{r}_1 \left(\sum_i \varphi_i \varphi^{*i}\right)\left(\sum_j \varphi'_j \varphi'^{*j}\right) + \tilde{r}_2 \left(\sum_i \varphi_i \varphi'^{*i}\right)\left(\sum_j \varphi'_j \varphi^{*j}\right) \\ &+ \ \tilde{s}_1 \sum_i \left(\varphi_i \varphi^{*i} \varphi'_i \varphi'^{*i}\right) , \end{aligned} \tag{5.136}$$

the full potential of two S_4 triplets becomes

$$\begin{aligned} V_{S_4}(\varphi,\varphi') =& V_0(\varphi) + V'_0(\varphi') + V_2(\varphi,\varphi') + \\ &+ \ c\left[\left(\varphi_1\varphi_1\varphi^{*3}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] + c'\left[\left(\varphi'_1\varphi'_1\varphi'^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ &+ \ \tilde{c}\left[\left(\varphi_1\varphi'_1\varphi^{*3}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] . \end{aligned} \tag{5.137}$$

The S_4 potential with two triplets generally conserves CP. This can be understood from the non-vanishing CPIs obtained for A_4, which were proportional to $\tilde{s}_2$ (see Eq. (5.123)) which is zero in the case of S_4. Indeed, one CP symmetry present in $V_{S_4}(\varphi,\varphi')$ is $X_{23}^{\varphi\varphi'}$ in Eq. (5.124), because S_4 enforces $\tilde{s}_2 = 0$ and therefore the $V_{S_4}(\varphi,\varphi')$ potential is invariant under simultaneous CP transformations with 2-3-swap on φ and φ'.

Turning to the case of Higgs doublets of $SU(2)_L$, for $V_{A_4}(H,H')$, enlarging the symmetry to S_4 constrains the potential parameters as follows:

$$c^* = c, \quad c'^* = c', \quad \tilde{c}_1^* = \tilde{c}_1, \quad \tilde{c}_2^* = \tilde{c}_2, \tag{5.138}$$

and

$$\tilde{s}_{21} = \tilde{s}_{22} = \tilde{s}_{31} = \tilde{s}_{32} = 0 \ . \tag{5.139}$$

Again, introducing an abbreviation,

$$
\begin{aligned}
V_2(H,H') = \sum_{i,j,\alpha,\beta} & \left[\tilde{r}_{11} h_{i\alpha} h^{*i\alpha} h'_{j\beta} h'^{*j\beta} + \tilde{r}_{12} h_{i\alpha} h'^{*j\alpha} h'_{j\beta} h^{*i\beta}\right] \\
+ \sum_{i,j,\alpha,\beta} & \left[\tilde{r}_{21} h_{i\alpha} h'^{*i\alpha} h'_{j\beta} h^{*j\beta} + \tilde{r}_{22} h_{i\alpha} h^{*j\alpha} h'_{j\beta} h'^{*i\beta}\right] \\
+ \sum_{i,\alpha,\beta} & \left[\tilde{s}_{11} h_{i\alpha} h^{*i\alpha} h'_{i\beta} h'^{*i\beta} + \tilde{s}_{12} h_{i\alpha} h'^{*i\alpha} h'_{i\beta} h^{*i\beta}\right] , \qquad (5.140)
\end{aligned}
$$

the S_4 invariant potential of two triplets of doublets becomes

$$
\begin{aligned}
V_{S_4}(H,H') = \; & V_0(H) + V'_0(H') + V_2(H,H') \\
& + \sum_{\alpha,\beta} c \left[\left(h_{1\alpha} h_{1\beta} h^{*3\alpha} h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\
& + \sum_{\alpha,\beta} c' \left[\left(h'_{1\alpha} h'_{1\beta} h'^{*3\alpha} h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\
& + \sum_{\alpha,\beta} \tilde{c}_1 \left[\left(h_{1\alpha} h^{*3\alpha} h'_{1\beta} h'^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] \\
& + \sum_{\alpha,\beta} \tilde{c}_2 \left[\left(h_{1\alpha} h'^{*3\alpha} h'_{1\beta} h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right] . \qquad (5.141)
\end{aligned}
$$

As in Eq. (5.137), the potential $V_{S_4}(H,H')$ conserves CP. As all parameters of this potential are real, it is not surprising, that it is invariant under trivial CP, CP_0.

5.5 $\Delta(27)$ invariant potentials

In this section we concern ourselves with potentials invariant under $\Delta(27)$. As in the A_4 case, we consider the field content of a single triplet of SM singlets, then a single triplet which is also an $SU(2)_L$ doublet, then two triplets of SM singlets, and finally two $\Delta(27)$ triplets of $SU(2)_L$ doublets.

The group $\Delta(27)$ has one irreducible triplet representation $\mathbf{3}$, its conjugate $\bar{\mathbf{3}}$, and nine one-dimensional representations. The product of two triplets decomposes as

$$
\mathbf{3} \otimes \mathbf{3} \;=\; (\bar{\mathbf{3}} + \bar{\mathbf{3}})_s \;+\; \bar{\mathbf{3}}_a \;, \qquad (5.142)
$$

where the subscripts s and a denote symmetric and antisymmetric combinations, respectively. In the following we adopt the basis of [158, 156, 110].

5.5.1 One flavour triplet

Having only one triplet field, the antisymmetric contribution in Eq. (5.142) vanishes identically. As a consequence there are four independent quartic $\Delta(27)$ invariants of type $\mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$. Writing the components of the triplet field as φ_i, with $i = 1, 2, 3$, we can easily derive the renormalisable scalar potential,

$$V_{\Delta(27)}(\varphi) = V_0(\varphi) + \left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] . \tag{5.143}$$

The coefficients inside $V_0(\varphi)$ (cf. Eq. (5.111)) are real but $d \in \mathbb{C}$. The number of independent real parameters is therefore four. $V_{\Delta(27)}(\varphi)$ is accidentally also the potential for a single $\Delta(54)$ triplet [134], as discussed also in Appendix 7.4.2.

The potential of Eq. (5.143) in its most general form violates CP as can be seen from the construction of CPIs which do not vanish for general choices of the coefficients in the potential (see Table 5.1). Calculating the CPIs $\mathcal{I}_{4,5}^{(6)}$ (Eqs. (5.78,5.79)) explicitly yields the same non-zero result for this potential:

$$\mathcal{I}_{4,5}^{(6)} = -\frac{3}{32}\left(d^3 - d^{*3}\right)\left(d^3 + 6dd^*s + d^{*3} - 8s^3\right), \tag{5.144}$$

while the other explicit CPIs that are listed throughout Section 5.2 are zero for this potential. The potential in Eq. (5.143) is known to be CP conserving in the cases $\text{Arg}(d) = 0, 2\pi/3, 4\pi/3$. Indeed this is reflected in the CPIs which are proportional to

$$(d^3 - d^{*3}) . \tag{5.145}$$

This factor vanishes for $\text{Arg}(d) = 0, 2\pi/3, 4\pi/3$, where each case corresponds to a distinct CP symmetry, defined by a 3×3 matrix X. In the following, we explicitly list the CP transformations that enforce various parameter relations. The X_i-notation we use in our work matches the indices of the CP transformations listed in [68],

$$\mathrm{Arg}(d) = 0 \Longleftrightarrow X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.146}$$

$$\mathrm{Arg}(d) = 4\pi/3 \Longleftrightarrow X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \text{or} \quad X_8 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.147}$$

$$\mathrm{Arg}(d) = 2\pi/3 \Longleftrightarrow X_3 = X_2^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{or} \quad X_9 = X_8^* = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.148}$$

We recall that for each CP transformation an equivalent one can be obtained by multiplying it by an element of $\Delta(27)$. Note also that $X_1 = X_{23}$ from Eq. (5.114). Focusing on the other factor of Eq. (5.144), all CPIs we have identified vanish if we set

$$\left(d^3 + 6dd^*s + d^{*3} - 8s^3\right) = 0 \ . \tag{5.149}$$

This is a strong hint that there are CP symmetries that make the potential CP conserving, not by fixing the phase of d but by imposing specific relations between the parameters d and s. Indeed, there are three solutions to Eq. (5.149) which are listed with the corresponding CP transformations from [68],

$2s = (d + d^*) = 2\mathrm{Re}(d)$

$$\iff X_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \text{ or } X_5 = X_4 X_1 = X_4^* = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \tag{5.150}$$

$2s = -\mathrm{Re}(d) - \sqrt{3}\mathrm{Im}(d)$

$$\iff X_6 = \frac{-i}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega \\ \omega & \omega & 1 \\ \omega & 1 & \omega \end{pmatrix} \text{ or } X_{10} = X_6 X_1 = \frac{-i}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{pmatrix}, \tag{5.151}$$

$2s = -\mathrm{Re}(d) + \sqrt{3}\mathrm{Im}(d)$

$$\iff X_7 = X_6^* = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega^2 \\ \omega^2 & \omega^2 & 1 \\ \omega^2 & 1 & \omega^2 \end{pmatrix} \text{ or } X_{11} = X_7 X_1 = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \omega^2 & \omega^2 \\ \omega^2 & 1 & \omega^2 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}. \tag{5.152}$$

We conclude that there exist 12 CP symmetries, listed in [68], which correspond to two CP symmetries for each of the 6 CP conserving conditions that make either $(d^3 - d^{*3}) = 0$ or $(d^3 + 6dd^*s + d^{*3} - 8s^3) = 0$. The fact that there are two distinct classes of CP symmetries, unrelated by $\Delta(27)$ transformations, for each of the 6 CP conserving conditions is due to the $\Delta(27)$ potential being accidentally invariant under $\Delta(54)$ [134]. The two classes of CP symmetries in each case are related to each other by a $\Delta(54)$ transformation.

5.5.2 One flavour triplet of Higgs doublets

If each component of the $\Delta(27)$ triplet is an $SU(2)_L$ doublet, the potential is rather similar to the previous case, and in analogy with the A_4 potential there is one additional invariant which is contained in $V_0(H)$. The resulting potential reads

$$V_{\Delta(27)}(H) = V_0(H) \ + \ \sum_{\alpha,\beta} \left[d\left(h_{1\alpha}h_{1\beta}h^{*2\alpha}h^{*3\beta} + \text{cycl.}\right) + \text{h.c.}\right]. \tag{5.153}$$

In general the potential explicitly violates CP. It is possible to impose CP conservation as in the previous case, as follows.

Calculating the CPIs, we see that up to a prefactor, $\mathcal{I}_{4,5}^{(6)}$ have the same form as in Eq. (5.144) for the previous potential:

$$\frac{512}{315}\mathcal{I}_4^{(6)} = \frac{1024}{495}\mathcal{I}_5^{(6)} = -\left(d^3 - d^{*3}\right)\left(d^3 + 6dd^*s + d^{*3} - 8s^3\right). \tag{5.154}$$

This means that the same conditions ensure CP conservation as in the previous $\Delta(27)$ invariant potential. They are associated to CP symmetries with the X_i matrices discussed in the previous subsection, simply multiplied by δ^α_β acting on $SU(2)_L$ indices (similarly to Eq. (5.119)).

5.5.3 Two flavour triplets

As for the A_4 case, we consider the potential involving two physically different flavon fields φ and φ' which both transform in the triplet representation of $\Delta(27)$. Note that the triplet representation of $\Delta(27)$ is unique up to complex conjugation. In addition to the invariants of each field, the full potential contains also mixed terms. Confining ourselves to quartic terms of the form $\varphi\,\varphi'\,\varphi^*\,\varphi'^*$ (which can be enforced e.g. by $U(1)$ symmetries, such that the imposed symmetry is really $\Delta(27)\times U(1)\times U(1)'$), we obtain nine independent mixed invariants. The resulting renormalisable potential is then given by

$$\begin{aligned} V_{\Delta(27)}(\varphi,\varphi') =& V_0(\varphi) + V_0'(\varphi') + V_1(\varphi,\varphi') \\ &+ \left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3} + \text{cycl.}\right) + \text{h.c.}\right] + \left[d'\left(\varphi'_1\varphi'_1\varphi'^{*2}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] \\ &+ \left[\tilde{d}_1\left(\varphi_1\varphi'_1\varphi^{*2}\varphi'^{*3} + \text{cycl.}\right) + \text{h.c.}\right] + \left[\tilde{d}_2\left(\varphi_1\varphi'_1\varphi^{*3}\varphi'^{*2} + \text{cycl.}\right) + \text{h.c.}\right]. \end{aligned} \tag{5.155}$$

Here the masses as well as the coupling constants inside V_0, V_0' and V_1 are all real (note the explicit factor of i multiplying $\tilde{s}_3$), while the couplings d, d', $\tilde{d}_1$ and $\tilde{d}_2$ are generally complex.

This potential explicitly violates CP, since several of the CPIs are non-zero as can be seen in Table 5.1 of Section 5.6, but the expressions are cumbersome. However it is possible to impose CP conservation. For example, imposing trivial CP (CP_0) enforces the 4 complex coefficients $d, d', \tilde{d}_1, \tilde{d}_2$ to be real and $\tilde{s}_3 = 0$. We have verified explicitly that all CPIs vanish in this case. Alternatively, imposing $X_{23}^{\varphi\varphi'}$ Eq. (5.124) enforces $\tilde{s}_2 = 0$ and relates $\tilde{d}_1 = \tilde{d}_2^*$ as well as $d^* = d$, $d'^* = d'$, which implies that all the CPIs vanish as expected.

5.5.4 Two flavour triplets of Higgs doublets

As earlier, a potential for two triplets of $SU(2)_L$ doublets can be obtained by including all possible $SU(2)_L$ contractions of the fields in the $\Delta(27)$ invariants. The only difference of this potential to earlier Higgs potentials lies in the invariants with d-coefficients, out of which only the invariants corresponding to $\tilde{d}_1$ and $\tilde{d}_2$ in Eq. (5.155) need to be doubled. Therefore the potential is in this case:

$$\begin{aligned} V_{\Delta(27)}(H,H') =& V_0(H) + V_0'(H') + V_1(H,H') + \\ &+ \sum_{\alpha,\beta} \left[d \left(h_{1\alpha} h_{1\beta} h^{*2\alpha} h^{*3\beta} + \text{cycl.} \right) + d' \left(h'_{1\alpha} h'_{1\beta} h'^{*2\alpha} h'^{*3\beta} + \text{cycl.} \right) + \text{h.c.} \right] \\ &+ \sum_{\alpha,\beta} \left[\tilde{d}_{11} (h_{1\alpha} h^{*2\alpha} h'_{1\beta} h'^{*3\beta} + \text{cycl.}) + \tilde{d}_{12} (h_{1\alpha} h'^{*3\alpha} h'_{1\beta} h^{*2\beta} + \text{cycl.}) + \text{h.c.} \right] \\ &+ \sum_{\alpha,\beta} \left[\tilde{d}_{21} (h_{1\alpha} h^{*3\alpha} h'_{1\beta} h'^{*2\beta} + \text{cycl.}) + \tilde{d}_{22} (h_{1\alpha} h'^{*2\alpha} h'_{1\beta} h^{*3\beta} + \text{cycl.}) + \text{h.c.} \right]. \end{aligned} \tag{5.156}$$

The potential $V_{\Delta(27)}(H,H')$ is CP violating in general. Of the CPIs calculated, cf. Table 5.1, $\mathcal{I}_2^{(6)}, \mathcal{I}_3^{(6)}, \mathcal{I}_4^{(6)}, \mathcal{I}_5^{(6)}$ (Eqs. (5.76,5.77,5.78,5.79)) are non-zero, but the expressions are too large to display here.

5.5.5 Δ(54) invariant potentials

Working in the basis of [49, 159, 110], the potentials of one triplet of singlets or $SU(2)_L$ doublets are both identical for $\Delta(27)$ and $\Delta(54)$. The $\Delta(54)$ symmetric potential of two triplets of $SU(2)_L$ singlets is obtained from the corresponding $\Delta(27)$ potential by imposing the constraint of Eq. (5.134),

$$\tilde{s}_2 = \tilde{s}_3 = 0 \,, \tag{5.157}$$

as well as

$$\tilde{d}_1 = \tilde{d}_2 \,, \tag{5.158}$$

from which $V_{\Delta(27)}(\varphi,\varphi')$ becomes

$$\begin{aligned}V_{\Delta(54)}(\varphi,\varphi') =& V_0(\varphi)+V_0'(\varphi')+V_2(\varphi,\varphi')\\ &+\left[d\left(\varphi_1\varphi_1\varphi^{*2}\varphi^{*3}+\text{cycl.}\right)+\text{h.c.}\right]+\left[d'\left(\varphi_1'\varphi_1'\varphi'^{*2}\varphi'^{*3}+\text{cycl.}\right)+\text{h.c.}\right]\\ &+\tilde{d}_1\left[\left(\varphi_1\varphi_1'\varphi^{*2}\varphi'^{*3}+\text{cycl.}\right)+\left(\varphi_1\varphi_1'\varphi^{*3}\varphi'^{*2}+\text{cycl.}\right)\right]+\text{h.c.}.\end{aligned} \tag{5.159}$$

The $\Delta(54)$ potential with two triplets does not conserve CP in general, as seen in Table 5.1. The potential in Eq. (5.159) is obtained from Eq. (5.155) in the $\tilde{s}_2=\tilde{s}_3=0$, $\tilde{d}_1=\tilde{d}_2$ limit, which makes it rather similar to the CP conserving $V_{\Delta(27)}(\varphi,\varphi')$ after imposing the $X_{23}^{\varphi\varphi'}$ (cf. Eq. (5.124)), but note that $V_{\Delta(54)}(\varphi,\varphi')$ does not have $\tilde{d}_1^*=\tilde{d}_1$, $d^*=d$ nor $d'^*=d'$. Therefore, even though CPI $\mathcal{I}_2^{(6)}$ vanishes, $\mathcal{I}_3^{(6)},\mathcal{I}_4^{(6)},\mathcal{I}_5^{(6)}$ are non-zero.

For the potential of two triplets of Higgs doublets, the following conditions on the parameters arise when enlarging the symmetry to $\Delta(54)$:

$$\tilde{d}_{21}=\tilde{d}_{11}\ ,\quad \tilde{d}_{22}=\tilde{d}_{12}\ ,\quad \tilde{s}_{21}=\tilde{s}_{22}=\tilde{s}_{31}=\tilde{s}_{32}=0. \tag{5.160}$$

The potential becomes

$$\begin{aligned}V_{\Delta(54)}(H,H') =& V_0(H)+V_0'(H')+V_2(H,H')\\ &+\sum_{\alpha,\beta}\left[d\left(h_{1\alpha}h_{1\beta}h^{*2\alpha}h^{*3\beta}+\text{cycl.}\right)+d'\left(h'_{1\alpha}h'_{1\beta}h'^{*2\alpha}h'^{*3\beta}+\text{cycl.}\right)+\text{h.c.}\right]\\ &+\sum_{\alpha,\beta}\left[\tilde{d}_{11}(h_{1\alpha}h^{*2\alpha}h'_{1\beta}h'^{*3\beta}+\text{cycl.})+\tilde{d}_{12}(h_{1\alpha}h'^{*3\alpha}h'_{1\beta}h^{*2\beta}+\text{cycl.})+\text{h.c.}\right]\\ &+\sum_{\alpha,\beta}\left[\tilde{d}_{11}(h_{1\alpha}h^{*3\alpha}h'_{1\beta}h'^{*2\beta}+\text{cycl.})+\tilde{d}_{12}(h_{1\alpha}h'^{*2\alpha}h'_{1\beta}h^{*3\beta}+\text{cycl.})+\text{h.c.}\right].\end{aligned} \tag{5.161}$$

This potential is also generally CP violating and $\mathcal{I}_3^{(6)},\mathcal{I}_4^{(6)},\mathcal{I}_5^{(6)}$ are non-zero but too large to display here.

5.6 Summary of CPIs for explicit CP violation

In this section, we collect our results of Sections 5.4, 5.5 and 7.2. We have calculated CPIs for a number of different potentials which are invariant under either of the following discrete symmetries A_4, S_4, $\Delta(27)$, $\Delta(54)$, $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n>3$. All these symmetries have irreducible triplet representations. Choosing

	$\mathcal{I}_2^{(6)}$	$\mathcal{I}_3^{(6)}$	$\mathcal{I}_4^{(6)}$	$\mathcal{I}_5^{(6)}$	CP
$(\mathbf{3}_{A_4}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	Eq. (5.114)
$(\mathbf{3}_{A_4}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	Eq. (5.119)
$2 \times (\mathbf{3}_{A_4}, \mathbf{1}_{SU(2)_L})$	*	*	*	*	NA
$2 \times (\mathbf{3}_{A_4}, \mathbf{2}_{SU(2)_L})$	*	*	*	*	NA
$(\mathbf{3}_{\Delta(27)}, \mathbf{1}_{SU(2)_L})$	0	0	Eq. (5.144)	Eq. (5.144)	NA
$(\mathbf{3}_{\Delta(27)}, \mathbf{2}_{SU(2)_L})$	0	0	Eq. (5.154)	Eq. (5.154)	NA
$2 \times (\mathbf{3}_{\Delta(27)}, \mathbf{1}_{SU(2)_L})$	*	*	*	*	NA
$2 \times (\mathbf{3}_{\Delta(27)}, \mathbf{2}_{SU(2)_L})$	*	*	*	*	NA
$(\mathbf{3}_{\Delta(3n^2)}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	Eq. (5.114)
$(\mathbf{3}_{\Delta(3n^2)}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	Eq. (5.119)
$2 \times (\mathbf{3}_{\Delta(3n^2)}, \mathbf{1}_{SU(2)_L})$	Eq. (7.97)	*	*	*	NA
$2 \times (\mathbf{3}_{\Delta(3n^2)}, \mathbf{2}_{SU(2)_L})$	*	*	*	*	NA
$(\mathbf{3}_{S_4}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.114)
$(\mathbf{3}_{S_4}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.119)
$2 \times (\mathbf{3}_{S_4}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.124)
$2 \times (\mathbf{3}_{S_4}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.129)
$(\mathbf{3}_{\Delta(54)}, \mathbf{1}_{SU(2)_L})$	0	0	*	*	NA
$(\mathbf{3}_{\Delta(54)}, \mathbf{2}_{SU(2)_L})$	0	0	*	*	NA
$2 \times (\mathbf{3}_{\Delta(54)}, \mathbf{1}_{SU(2)_L})$	0	*	*	*	NA
$2 \times (\mathbf{3}_{\Delta(54)}, \mathbf{2}_{SU(2)_L})$	0	*	*	*	NA
$(\mathbf{3}_{\Delta(6n^2)}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.114)
$(\mathbf{3}_{\Delta(6n^2)}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.119)
$2 \times (\mathbf{3}_{\Delta(6n^2)}, \mathbf{1}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.124)
$2 \times (\mathbf{3}_{\Delta(6n^2)}, \mathbf{2}_{SU(2)_L})$	0	0	0	0	CP_0 & Eq. (5.129)

Table 5.1: Summary of CPIs and (if applicable) CP symmetry transformations for scalar potentials with discrete symmetry.

Higgs fields in a faithful triplet, we have determined the potential for one triplet of $SU(2)_L$ singlets, one triplet of $SU(2)_L$ doublets, two triplets of $SU(2)_L$ singlets and finally two triplets of $SU(2)_L$ doublets. The (scalar) particle content for each of these 6×4 cases is listed intuitively in the leftmost column of Table 5.1.

Many of the CPIs defined in Section 5.2 vanish for all of these 24 potentials. We have checked explicitly that $\mathcal{I}_1^{(2,2)}$, $\mathcal{I}_1^{(3,1)}$, $\mathcal{I}_2^{(3,1)}$, $\mathcal{I}_1^{(3,2)}$, $\mathcal{I}_2^{(3,2)}$, $\mathcal{I}_1^{(4,1)}$, $\mathcal{I}_1^{(5)}$, $\mathcal{I}_2^{(5)}$, $\mathcal{I}_3^{(5)}$, $\mathcal{I}_1^{(6)}$ vanish in all cases. Table 5.1 shows the relevant invariants $\mathcal{I}_2^{(6)}$, $\mathcal{I}_3^{(6)}$, $\mathcal{I}_4^{(6)}$, $\mathcal{I}_5^{(6)}$, evaluated for each potential. A 0-entry means that the corresponding CPI was found to be zero. A non-vanishing CPI is indicated by either an asterisk or an equation number, where the latter refers to the position in this thesis where the corresponding expression for the CPI is given. The asterisk is used for non-zero CPIs which we have calculated analytically but whose expressions are too large to display in the text.

We observe from Table 5.1 that 12 potentials feature explicit CP violation. On the other hand, all four CPIs shown in the table vanish for the other 12 potentials, which suggests CP is conserved in those cases. Indeed, as listed in the rightmost column, one can easily identify CP transformations which leave the potential unchanged, thereby explicitly proving that CP is conserved. We recall that trivial CP (CP_0) means complex conjugation on all scalar fields, cf. Eq. (5.21). "NA" stands for "Not Applicable" and is used for CP violating cases.

5.7 CP-odd invariants for spontaneous CP violation

So far we have discussed CPIs that signal explicit CP violation in scalar potentials. It is also useful to consider CPIs that indicate the presence of spontaneous CP violation. In order to extend our formalism (which is applicable to any potentials once translated into the standard form) we need to include also VEVs.

Recall that VEVs transform as vectors under basis transformations, cf. Eqs. (5.22) and (5.23):

$$v_a \mapsto V_a^{a'} v_{a'} \ , \tag{5.162}$$

$$v^{*a} \mapsto v^{*a'} V^{\dagger a}_{\ a'} \ . \tag{5.163}$$

When used in invariants, first, if the potential does not contain trilinear couplings, VEVs can only appear in pairs of v and corresponding v^* because otherwise indices would remain uncontracted. Furthermore, all VEVs commute and thus can be combined into one large tensor,

$$W^{w_1 \dots w_{n_v}}_{w'_1 \dots w'_{n_v}} = v_{w'_1} \dots v_{w'_{n_v}} v^{*w_1} \dots v^{*w_{n_v}} . \tag{5.164}$$

where n_v is the number of v, v^* pairs.[11] Using W, all invariants with n_v pairs of VEV and conjugated VEV can be written using

$$\begin{aligned} J_\sigma^{(n_v,m_Y,n_Z)} &\equiv W^{w_1 \dots w_{n_v}}_{\sigma(w_1) \dots \sigma(w_{n_v})} Y^{a_1}_{\sigma(a_1)} \dots Y^{a_{m_Y}}_{\sigma(a_{m_Y})} Z^{b_1 b_2}_{\sigma(b_1)\sigma(b_2)} \dots Z^{b_{2n_Z-1} b_{2n_Z}}_{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})} \\ &\mapsto W^{\sigma(w_1) \dots \sigma(w_{n_v})}_{w_1 \dots w_{n_v}} Y^{\sigma(a_1)}_{a_1} \dots Y^{\sigma(a_{m_Y})}_{a_{m_Y}} Z^{\sigma(b_1)\sigma(b_2)}_{b_1 b_2} \dots Z^{\sigma(b_{2n_Z-1})\sigma(b_{2n_Z})}_{b_{2n_Z-1} b_{2n_Z}} \\ &\equiv \left(J_\sigma^{(n_v,m_Y,n_Z)}\right)^* \ , \end{aligned} \tag{5.165}$$

[11] In [122], VEVs are always assigned in pairs to matrices $V^a_b = v^{*a} v_b$, however, since all VEVs commute, even for four or more VEVs, also all V^a_b commute and can be summarised in one large totally symmetric tensor.

with $\sigma \in S_{n_v+m_Y+2n_Z}$. When drawing diagrams, there are additional rules for contractions with VEVs, again with $X = Y, Z$:

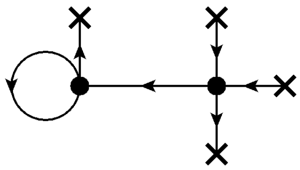
$$X^{a..}_{..} v_a = \quad (5.166)$$

and

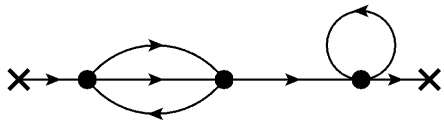
$$X^{..}_{a..} v^{*a} = \quad (5.167)$$

Invariants containing only Y tensors and VEVs are always CP-even. The smallest examples of CPIs for spontaneous CP violation built from Z tensors and VEVs are

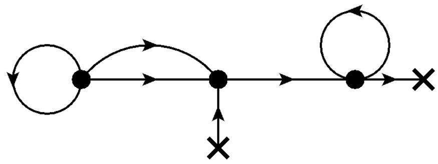
$$J^{(2,2)}_1 \equiv Z^{a_1a_2}_{a_1a_3} Z^{a_3a_4}_{a_5a_6} v_{a_2} v_{a_4} v^{*a_5} v^{*a_6} = \quad (5.168)$$



$$J^{(3,1)}_1 \equiv Z^{a_1a_2}_{a_5a_6} Z^{a_3a_4}_{a_1a_3} Z^{a_5a_6}_{a_2a_7} v_{a_4} v^{*a_7} = \quad (5.169)$$



$$J^{(3,1)}_2 \equiv Z^{a_1a_2}_{a_1a_5} Z^{a_3a_4}_{a_3a_6} Z^{a_5a_6}_{a_2a_7} v_{a_4} v^{*a_7} = \quad (5.170)$$



where the superscripts on J indicate the number of Z tensors and pairs of VEVs in the invariant. A complete search for invariants with $(n_Z, n_v) = (2,2), (3,1), (3,2), (4,1)$ was performed. The method is explained in Appendix 5.2.4 and the invariants not given in the main text are listed in Appendix 7.3.4.

5.7.1 Minimisation condition in terms of diagrams

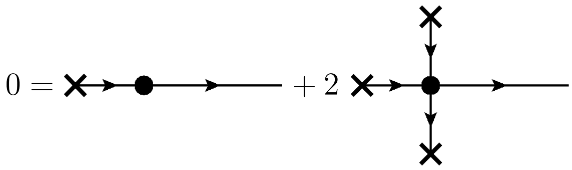
The minima of the a potential written as in Eq. (5.1) fulfil

$$0 = \frac{\partial V}{\partial \phi_e} = \phi^{*a} Y^l_a + 2\phi^{*a}\phi^{*c} Z^{ed}_{ac}\phi_d \ , \quad (5.171)$$

and

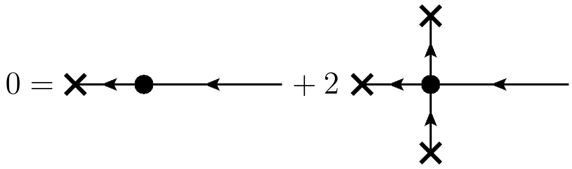
$$0 = \frac{\partial V}{\partial \phi^{*e}} = Y^b_e \phi_b + 2\phi^{*c} Z^{bd}_{ec}\phi_b\phi_d \ , \quad (5.172)$$

where the factor of 2 appears because of the symmetry of Z^{bd}_{ac} under $b \leftrightarrow d$ and $a \leftrightarrow c$. Replacing the fields by their VEVs, these minimisation conditions can be expressed in terms of diagrams:



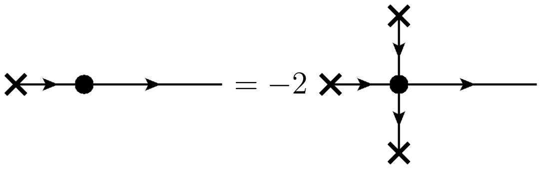
(5.173)

and



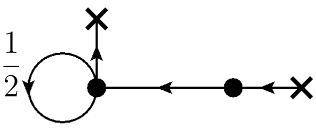
(5.174)

This can be used later to simplify CPIs, as can be seen by applying



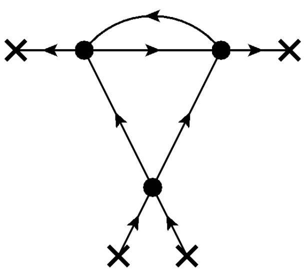
(5.175)

in Eq. (5.168). Using the minimisation condition Eq. (5.175), the invariant $J_1^{(2,2)}$ can be simplified to[12]

$$J_1^{(2,2)} \equiv -\frac{1}{2}$$ 

(5.176)

This can only be CP-odd if Y is not proportional to the identity. One can now search for more complicated invariants built from Z tensors and VEVs that will not simplify like this. The smallest CPIs for spontaneous CP violation without self-loops which also cannot be simplified using the minimisation condition for $n_Z = 3, 4$ respectively are

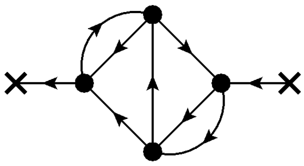
$$J_1^{(3,2)} \equiv Z^{a_1 a_2}_{a_4 a_5} Z^{a_3 a_4}_{a_2 a_6} Z^{a_5 a_6}_{a_7 a_8} v_{a_1} v_{a_3} v^{*a_7} v^{*a_8} =$$ 

(5.177)

[12]The resulting expression corresponds to the invariant J_3 in Eq. (26) of [120].

and

$$J_1^{(4,1)} \equiv Z^{a_1a_2}_{a_3a_5} Z^{a_3a_4}_{a_7a_8} Z^{a_5a_6}_{a_1a_4} Z^{a_7a_8}_{a_2a_9} v_{a_6} v^{*a_9} =$$



. (5.178)

5.7.2 Example applications

5.7.2.1 One triplet of A_4

As we have seen, the potential in Eq. (5.118) conserves CP explicitly. By an analysis of all VEVs, it has been shown [143] that CP cannot be spontaneously broken. Using our approach we have verified that the low order invariants vanish. In particular, all spontaneous invariants up to $n_Z = 3, n_v = 2$ are found to vanish for this potential.

5.7.2.2 One triplet of $\Delta(27)$

One can now calculate SCPIs for this potential for arbitrary VEVs and the smallest non-zero SCPI found is $\mathcal{J}_1^{(3,2)}$, as defined via Eq. (5.177). For the general potential $V_{\Delta(27)}(\varphi)$ (which we note is CP violating), it takes the value

$$\begin{aligned}\mathcal{J}_1^{(3,2)} =& \frac{1}{4}(d^{*3} - d^3)(|v_1|^4 + |v_2|^4 + |v_3|^4 - 2|v_1|^2|v_2|^2 - 2|v_1|^2|v_3|^2 - 2|v_2|^2|v_3|^2) \\ &+ \frac{1}{2}(dd^{*2} - 2d^*s^2 + d^2 s)(v_2v_3v_1^{*2} + v_1v_3v_2^{*2} + v_1v_2v_3^{*2}) \\ &- \frac{1}{2}(d^2d^* - 2ds^2 + d^{*2}s)(v_2^*v_3^*v_1^2 + v_1^*v_3^*v_2^2 + v_1^*v_2^*v_3^2)\ . \end{aligned} \quad (5.179)$$

In order to demonstrate the usefulness of SCPIs, let us consider the following special cases of $V_{\Delta(27)}(\varphi)$ where we impose different CP symmetries. We start by considering trivial CP (CP_0), which in this case is the X_0 matrix, forcing $\mathrm{Arg}(d) = 0$ which simplifies the SCPI expression to

$$\mathcal{J}_1^{(3,2)} = \frac{1}{2}(d^3 - 2ds^2 + d^2 s)\left[(v_2v_3v_1^{*2} + v_1v_3v_2^{*2} + v_1v_2v_3^{*2}) - (v_2^*v_3^*v_1^2 + v_1^*v_3^*v_2^2 + v_1^*v_2^*v_3^2)\right]. \quad (5.180)$$

It is known [91, 134, 135, 136] that the complex VEV $(1, \omega, \omega^2)$ is not CP violating when starting with trivial CP. This can be confirmed easily by using the SCPI above. Instead, the geometrically CP violating VEV $(\omega, 1, 1)$ does give non-zero when plugged into the SCPI. Let us consider now the CP symmetry X_3, forcing $\text{Arg}(d) = 2\pi/3$. Because d remains complex, even a real VEV like $(1,1,1)$ spontaneously violates CP [145] and this is shown by the SCPI:

$$\mathcal{J}_1^{(3,2)} = \frac{1}{2}\text{Im}(dd^{*2} - 2d^* s^2 + d^2 s)\left[(3v_1^4)\right] . \tag{5.181}$$

Another interesting case is the CP symmetry X_4, forcing $2s = (d + d^*) = 2\text{Re}(d)$. This simplifies Eq. (5.179) to

$$\mathcal{J}_1^{(3,2)} = \frac{1}{4}(d^{*3} - d^3)(|v_1|^4 + |v_2|^4 + |v_3|^4 - 2|v_1|^2|v_2|^2 - 2|v_1|^2|v_3|^2 - 2|v_2|^2|v_3|^2) . \tag{5.182}$$

It is interesting that in this case the SCPI indicates that spontaneous CP violation is independent of the phases of the VEV. Indeed, the known VEVs for the X_0 symmetric potential, such as $(0,0,1)$, $(1,1,1)$ (which are real) and $(\omega,1,1)$ are still candidate VEVs of the X_4 symmetric potential and all violate CP spontaneously, as indicated by the SCPI.

5.8 Summary of CP-odd invariants

This chapter has been concerned with CPV arising from scalar potentials which go beyond the one Higgs doublet of the SM. Powerful new tools that allow to systematically find CPIs that are valid for any scalar potential have been reviewed and further developed, and which provide a reliable indicator for whether CP is explicitly violated by the parameters of the potential. Spontaneous CPIs involving the VEVs were also considered, in order to reliably determine whether CP is spontaneously violated.

In order to illustrate the usefulness of the CPI approach, we then applied our results to multi-Higgs scalar potentials of physical interest. We first considered the general 2HDM case which was known to be CP violating, with a complete basis of CPIs known, with several small CPIs being non-zero. We then considered 3HDM and 6HDM which are symmetric under $\Delta(3n^2)$ and $\Delta(6n^2)$ groups. Many of these potentials had not been studied before and the new CPIs we found with our systematic search were needed as the previously known ones vanish even for potentials where the new CPIs reveal the presence of explicit CP violation.

For each potential, we either determined the lowest order non-zero CPIs (thereby proving that potential is CP violating) or, in cases where all the considered CPIs vanish, we derived the explicit CP symmetries that leave the potential invariant (thereby proving that potential is CP conserving). Since the potentials considered were very symmetric, we found that most of the smaller CPIs vanish. Although the CPIs apply to any potential, they take different expressions as functions of the parameters of the potential, as clearly illustrated in the 2HDM example. Furthermore, CPIs that are useful for one potential can vanish for other CP violating potentials.

We found that the A_4 potentials, although generally CP conserving for one triplet of Higgs doublets or singlets, are no longer CP conserving in general when two A_4 triplets are present (either doublets or singlets). By contrast we find that $\Delta(27)$ potentials are all CP violating in general. Although the $\Delta(27)$ potentials with a single triplet (whether the scalars are Higgs doublets or not) had previously been studied extensively, by using the calculated expression for a CPI we completely mapped specific CP symmetries to different ways to make the CPI expression vanish. For such potentials, we further analysed spontaneous CP violation when considering different CP symmetries by using a non-trivial SCPI. The potentials with $\Delta(3n^2)$ with $n > 3$ turn out to be particular cases of A_4 potentials. For such cases it is notable that the expressions for the non-zero CPI become manageable for the case with two triplets (non-Higgs), which allowed to find a CP symmetry that relates two of the real parameters of the potential. Moreover, we found that all of the $\Delta(6n^2)$ potentials are special cases of the respective $\Delta(3n^2)$ potentials. In the S_4 case, this makes even the potentials with two triplets automatically CP conserving. Although the $\Delta(54)$ potential for one triplet (whether the scalars are Higgs doublets or not) coincides with the $\Delta(27)$ potential, this is no longer the case when two triplets are present, but they still generally violate CP. $\Delta(6n^2)$ with $n > 3$ is a particular case of S_4 and therefore the potentials considered are again automatically CP conserving.

Finally, we briefly showed how our approach may also be applied to spontaneous CPV. As an illustration of this we calculated the SCPIs which are relevant for a $\Delta(27)$ potential showing how it reveals the CP properties of candidate VEVs.

In conclusion, the invariant approach to CP violation provides a reliable method for studying the CP properties of multi-Higgs potentials. We have developed a systematic formalism for determining the CPIs for multi-Higgs potentials in general, and have extensively applied this formalism to both the familiar general 2HDM as well as many examples in which the Higgs fields fall into irreducible triplet representations of a discrete symmetry. We considered not only SM Higgs doublets, but also SM singlets which play the role of flavons in flavour models. In each case we catalogued all the lowest order CPIs, many of which previously unknown, thereby elucidating the CP properties of the considered potentials and finding the relevant CP symmetry transformations where applicable.

Furthermore, invariants may provide an interesting theoretical tool. It is for example topic of debate if there is a relation between spontaneous CPV and suppressed explicit CPV, [160]. Using the equations of motion to relate spontaneous CPIs with explicit CPIs may give some insight and could be worth considering.

6

Conclusions

The purpose of this final chapter is to first succinctly summarize the contents of this thesis and after that to put them in the wider context of the current state of particle physics.

In the Standard Model, quark masses and mixing, charged-lepton masses, as well as CP violation are merely parametrised, whereas neutrino masses, mixing and dark matter are even entirely unexplained. Thus any explanation of these phenomena will have to involve physics beyond the Standard Model. Furthermore, in the SM, CP violation seems to be related to the flavour sector which is why studying one necessitates studying the other. Next, new physics can often involve new symmetries that require breaking by which additional scalars can come into play.

While typical explanation attempts point towards high energies (RHν, GUT, Planck), it is worth studying these topics now because related observable phenomena occur at low energies and corresponding experiments are on the way or will be in the near future.

Whatever may be the correct theory of flavour and CPV, two reasons make flavour and CP symmetries either as fundamental symmetries or at least as an intermediate, effective symmetry, attractive: The general success of symmetries as an organizing principle in particle physics and the fact that in the SM fermion species already have (separate) residual flavour and CP symmetries.

This thesis started by stating the flavour problem in terms of one of the most fundamental principles in physics, namely relativistic invariance. To repeat it here, from the point of view of the author, the flavour problem can be formulated in this context as: *There is no such thing as flavour — in the Poincare group*, and what was meant by that is that to describe the several flavours of the standard model, several copies of representations of the Poincare group are needed.

This already reflects the philosophy behind this thesis, namely that unsolved problems should be considered as questions about the fundamental symmetries of physics. (For lack of better guiding principles.)

After that, the problem of CP violation is introduced. And while CP violation seems to be necessary for all our existence and the standard model not providing enough of it, it was again attempted to look at it also from a fundamental symmetry perspective. It is the point of view of the author that no fundamental difference exists between flavour, general C, P, T symmetries or combinations thereof.

The remainder of the introduction then worked its way from these open questions and ideas to observable predictions, namely such as would be produced by residual symmetries in various parts of the Lagrangian, especially in the Yukawa sector and eventually lepton mixing matrices.

Three chapters then analyse different incarnations of residual symmetries in the lepton sector. The flavour group assumed was always a member of $\Delta(6n^2)$ which is an important series of subgroups of $U(3)$. These groups are similar in structure and can be analysed simultaneously for arbitrary n, which was done for the first time for such an infinite series of discrete groups.

Potentials with several Higgs doublets or singlets, that may be invariant under some discrete group can spontaneously break both flavour and CP. Furthermore, CP can be violated geometrically which means that the symmetry of the potential is so constraining that the CP phases can only take certain discrete values.

In complicated scalar potentials with additional symmetries, it is often unclear if CP is violated or not, both explicitly and spontaneously. However, CP-odd invariants can be constructed, similar to the Jarlskog invariant, that clearly indicate CP violation.

Thus, in the subsequent chapter, motivated by both the fact that flavour and CP symmetries need to broken, and the search for additional CP violation, various potentials invariant under candidates of flavour groups were studied with a focus on the possible CP violation they might introduce. The tool used for this were CP-odd Higgs basis invariants.

Among the open questions that are touched upon by the work in this thesis are the following:

- Why do fermions mix as they do? Why are the masses of fermions what they are?
- Is there CP violation in the lepton sector? In a Higgs sector? Is CP violated spontaneously? Geometrically in potentials with discrete symmetries?
- Are neutrinos Dirac or Majorana fermions?

In particular concerning the flavour symmetries examined in this thesis, experiments will soon be able to test their parameter space.

However, it is probably a good idea to look out into the future and to remind oneself of a small part of the questions that haven't been answered or addressed by the research in this thesis, starting with those that lie closer to the thesis but also slowly moving away from it:

- Could the CP violation found above explain the baryon asymmetry?
- Why do quarks and leptons mix differently? What is the difference between quarks and leptons? Are GUTs the answer?
- How is the flavour symmetry broken? Through additional scalars? Or maybe through extra dimensions?
- What are phenomenological consequences of flavour symmetries and their breaking? In rare decays of Higgs bosons or fermions? At colliders?
- Which mechanism generates neutrino masses? Is it one of the already proposed ones or something else?
- Could flavour symmetries help with explaining dark matter by explaining its stability or its weak interactions?
- If neutrinos are Dirac, could maybe continuous flavour symmetry groups help explain the flavour structure?
- What is the significance of the larger residual symmetries above EWSB?
- Could similar methods be applied to the strong CP problem? I.e. what is the origin of the CP symmetry that forbids the θ-term?
- Are there any additional light neutrinos? Could these be dark matter? Warm dark matter?

- What is the nature and origin of dark matter? Is there a warm dark matter component? Does dark matter communicate with ordinary matter (besides gravity)? What is the portal? What can neutrinos tell us about dark matter?
- Can the cosmic neutrino background be detected? The supernova neutrino background?
- What could be learned about inflation from the CνB? About baryogenesis? The statistical nature of neutrinos?
- Are there consequences for inflation in flavour models? Could a flavon be an inflaton?
- Is there a connection between gauge group (SM/GUT) and flavour group breaking?
- Is there a theoretical principle that determines what the correct description of particle physics is?
- Do flavour and gauge symmetries originate from quantum gravity?
- Is there a geometrical aspect to flavour?
- Is there a symmetry representation of quantum gravity via the representations of the diffeomorphism group?
- As quantum mechanics requires (anti)unitary representations, are symmetries a gate to testing quantum mechanics?
- Is flavour the gate to testing quantum mechanics?
- What is the nature of quantum gravity?

Clearly the research presented in this thesis only corresponds to a small technical step in the epic endeavour of mankind to uncover the laws of nature.

Particle physics as a whole is currently in a weird state: evidence of phenomena that cannot be explained by the standard model is accumulating, however when testing the standard model at the LHC, it holds up as well as always throughout its history.

However, even if the LHC doesn't find anything, there is still hope! First of all, there are other experiments, namely the various low-energy experiments attempting to detect $0\nu\beta\beta$ and dark matter. There is now such a plethora of reactor and accelerator neutrino experiments that one could easily lose track of them. These will measure the various unknown neutrino parameters, and in addition will give clues about sterile neutrinos. On the other side, cosmology has long entered the precision era, and while there are no new general purpose satellite experiments coming soon, cosmology will have no longer only to rely on light as the only messenger. A variety of experiments are being planned that may detect the cosmic neutrino background. Furthermore and more concrete, gravitational waves from colliding black holes have been detected! This proves their existence and opens the door for detecting the gravitational wave background. Beyond that, experiments are being planned to perform precision measurements of the behaviour of strong gravitational fields, e.g. the Einstein telescope [161].

On the other hand, as enough things are theoretically far from well-understood, the author is convinced that theoretical progress is necessary and possible! What seems to be needed are not only better experiments but better theoretical understanding by tying together all the bits and pieces and to find out where precisely logical gaps appear, even if this means that one has to take a step back and that one has to delve into more formal aspects of the theory. The only thing one has to fear is that no scientist can be found with the courage to do this, or even worse, that the whole of the scientific community gets distracted from having the patience for this by economical dynamics, or still worse, that mankind as a whole loses interest in the pursuit of science.

Fundamental science in particular that does not produce immediately profitable results is dependent on the goodwill of governments and eventually of the people. As science is becoming more and more expensive, it is turning from something that individuals can pursue on their own into projects that sometimes require several states to fund them. Only by abandoning the slight elitism that maybe was always part of science and returning knowledge and enthusiasm to the people that pay for it, a crisis of science can be averted.

> *And when you finish reading this book, tie a stone to it and cast it into the midst of the Euphrates.*
>
> – Jeremiah 51.63

7

Appendix

7.1 Full results for semidirect models

7.1.1 Majorana mass and diagonalisation matrices for Neutrino-semidirect models

(i) $G_\nu = Z_2^{bc^x d^x} \equiv \{1, bc^x d^x\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma})\}$

The light neutrino mass matrix satisfying Eq. (4.22) is of the following form

$$m_\nu = \begin{pmatrix} m_{11} e^{-2i\pi\frac{\gamma}{n}} & m_{12} e^{i\pi\frac{2x+\gamma}{n}} & m_{13} e^{-2i\pi\frac{x+\gamma}{n}} \\ m_{12} e^{i\pi\frac{2x+\gamma}{n}} & m_{22} e^{4i\pi\frac{x+\gamma}{n}} & m_{12} e^{i\pi\frac{\gamma}{n}} \\ m_{13} e^{-2i\pi\frac{x+\gamma}{n}} & m_{12} e^{i\pi\frac{\gamma}{n}} & m_{11} e^{-2i\pi\frac{2x+\gamma}{n}} \end{pmatrix}, \tag{7.1}$$

where m_{11}, m_{12}, m_{13} and m_{22} are real parameters. This neutrino mass matrix is diagonalized by the unitary transformation U_ν via

$$U_\nu^T m_\nu U_\nu = \text{diag}\,(m_1, m_2, m_3)\ , \tag{7.2}$$

where U_ν is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & -e^{i\pi\frac{\gamma}{n}} \sin\theta & e^{i\pi\frac{\gamma}{n}} \cos\theta \\ 0 & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2}\cos\theta & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2}\sin\theta \\ -e^{i\pi\frac{2x+\gamma}{n}} & -e^{i\pi\frac{2x+\gamma}{n}} \sin\theta & e^{i\pi\frac{2x+\gamma}{n}} \cos\theta \end{pmatrix} K_\nu\,, \tag{7.3}$$

where K_ν is a diagonal unitary matrix with entries ± 1 and $\pm i$ which encode the CP parity of the neutrino states and renders the light neutrino masses positive.

We shall omit the factor K_ν in the following cases for simplicity of notation. The angle θ is given by

$$\tan 2\theta = \frac{2\sqrt{2}m_{12}}{m_{11}+m_{13}-m_{22}}\,. \tag{7.4}$$

The light neutrino masses are

$$\begin{aligned}
m_1 &= |m_{11}-m_{13}|\,,\\
m_2 &= \frac{1}{2}\left|m_{11}+m_{13}+m_{22}-\text{sign}\left((m_{11}+m_{13}-m_{22})\cos 2\theta\right)\sqrt{(m_{11}+m_{13}-m_{22})^2+8m_{12}^2}\right|,\\
m_3 &= \frac{1}{2}\left|m_{11}+m_{13}+m_{22}+\text{sign}\left((m_{11}+m_{13}-m_{22})\cos 2\theta\right)\sqrt{(m_{11}+m_{13}-m_{22})^2+8m_{12}^2}\right|.
\end{aligned}$$

Here the order of the three eigenvalues m_1, m_2 and m_3 can not be pinned down, consequently the unitary matrix U_ν is determined up to permutations of the columns (the same turns out to be true in the following cases), and the neutrino mass spectrum can be either normal ordering or inverted ordering. Moreover, as four parameters m_{11}, m_{12}, m_{13} and m_{22} are involved in the neutrino masses, the measured mass squared splitting can be accounted for easily.

(ii) $G_\nu = Z_2^{abc^y} \equiv \{1, abc^y\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}),\ \ \rho_{\mathbf{r}}(abc^{y+\gamma}d^{2y+2\gamma})\}$

In this case, the light neutrino mass matrix takes the form:

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{13}e^{i\pi\frac{2y+\gamma}{n}} \\ m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{11}e^{-2i\pi\frac{2y+\gamma}{n}} & m_{13}e^{i\pi\frac{\gamma}{n}} \\ m_{13}e^{i\pi\frac{2y+\gamma}{n}} & m_{13}e^{i\pi\frac{\gamma}{n}} & m_{33}e^{4i\pi\frac{y+\gamma}{n}} \end{pmatrix}, \tag{7.5}$$

where m_{11}, m_{12}, m_{13} and m_{33} are real. The unitary matrix U_ν which diagonalizes the above neutrino mass matrix is given by

$$U_\nu = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}}\cos\theta & e^{i\pi\frac{2y+\gamma}{n}}\sin\theta \\ 0 & -e^{-2i\pi\frac{y+\gamma}{n}}\sqrt{2}\sin\theta & e^{-2i\pi\frac{y+\gamma}{n}}\sqrt{2}\cos\theta \end{pmatrix}, \tag{7.6}$$

with

$$\tan 2\theta = \frac{2\sqrt{2}\,m_{13}}{m_{33}-m_{11}-m_{12}}\,. \tag{7.7}$$

The light neutrino mass eigenvalues are determined to be

$$
\begin{aligned}
m_1 &= |m_{11} - m_{12}| \,, \\
m_2 &= \frac{1}{2}\left| m_{11} + m_{12} + m_{33} + \mathrm{sign}\left((m_{11} + m_{12} - m_{33})\cos 2\theta\right)\sqrt{(m_{11} + m_{12} - m_{33})^2 + 8m_{13}^2} \right| \\
m_2 &= \frac{1}{2}\left| m_{11} + m_{12} + m_{33} - \mathrm{sign}\left((m_{11} + m_{12} - m_{33})\cos 2\theta\right)\sqrt{(m_{11} + m_{12} - m_{33})^2 + 8m_{13}^2} \right|
\end{aligned}
$$

(iii) $G_\nu = Z_2^{a^2bd^z} \equiv \{1, a^2bd^z\}$, $X_{\nu\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta}d^{z+\delta})\right\}$

The light neutrino mass matrix, which is invariant under both residual flavour symmetry and residual CP symmetry, is of the form:

$$
m_\nu = \begin{pmatrix} m_{11}e^{-4i\pi\frac{z+\delta}{n}} & m_{12}e^{-i\pi\frac{\delta}{n}} & m_{12}e^{-i\pi\frac{2z+\delta}{n}} \\ m_{12}e^{-i\pi\frac{\delta}{n}} & m_{22}e^{2i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} \\ m_{12}e^{-i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} & m_{22}e^{2i\pi\frac{\delta}{n}} \end{pmatrix} , \tag{7.8}
$$

where m_{11}, m_{12}, m_{22} and m_{23} are real parameters. The neutrino diagonalization matrix U_ν is given by

$$
U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\,e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\sin\theta & e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\cos\theta \\ e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}}\cos\theta & e^{-i\pi\frac{2z+\delta}{n}}\sin\theta \\ -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}}\cos\theta & e^{-i\pi\frac{\delta}{n}}\sin\theta \end{pmatrix} , \tag{7.9}
$$

where the angle θ fulfils

$$
\tan 2\theta = \frac{2\sqrt{2}m_{12}}{m_{11} - m_{22} - m_{23}} \,. \tag{7.10}
$$

Finally the light neutrino masses are

$$
\begin{aligned}
m_1 &= |m_{22} - m_{23}| \,, \\
m_2 &= \frac{1}{2}\left| m_{11} + m_{22} + m_{23} - \mathrm{sign}\left((m_{11} - m_{22} - m_{23})\cos 2\theta\right)\sqrt{(m_{11} - m_{22} - m_{23})^2 + 8m_{12}^2} \right| \\
m_3 &= \frac{1}{2}\left| m_{11} + m_{22} + m_{23} + \mathrm{sign}\left((m_{11} - m_{22} - m_{23})\cos 2\theta\right)\sqrt{(m_{11} - m_{22} - m_{23})^2 + 8m_{12}^2} \right|
\end{aligned}
$$

(iv) $G_\nu = Z_2^{c^{n/2}} \equiv \left\{1, c^{n/2}\right\}$, $X_{\nu\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(abc^\gamma d^\delta)\right\}$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$

The light neutrino mass matrix is constrained to be of the following form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-i\pi\frac{\delta}{n}} & 0 \\ m_{12}e^{-i\pi\frac{\delta}{n}} & m_{22}e^{-2i\pi\frac{\delta-\gamma}{n}} & 0 \\ 0 & 0 & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{7.11}$$

where m_{11}, m_{12}, m_{22} and m_{33} are real. The unitary transformation U_ν is

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta & 0 \\ -e^{i\pi\frac{\delta-\gamma}{n}}\sin\theta & e^{i\pi\frac{\delta-\gamma}{n}}\cos\theta & 0 \\ 0 & 0 & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{7.12}$$

where

$$\tan 2\theta = \frac{2m_{12}}{m_{22}-m_{11}}. \tag{7.13}$$

The light neutrino masses are determined to be

$$\begin{aligned} m_1 &= \frac{1}{2}\left| m_{11}+m_{22} - \text{sign}\left((m_{22}-m_{11})\cos 2\theta\right)\sqrt{(m_{22}-m_{11})^2+4m_{12}^2}\right|, \\ m_2 &= \frac{1}{2}\left| m_{11}+m_{22} + \text{sign}\left((m_{22}-m_{11})\cos 2\theta\right)\sqrt{(m_{22}-m_{11})^2+4m_{12}^2}\right|, \\ m_3 &= |m_{33}|\,. \end{aligned} \tag{7.14}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(abc^\gamma d^\delta)$

  For the case of $\delta \neq 2\gamma \mod n$, the light neutrino masses would be partially degenerate. This is unviable. The reason is that the corresponding general CP transformation matrix is not symmetric [1]. Therefore we shall concentrate on the case of $\delta = 2\gamma \mod n$ in the following. The neutrino mass matrix is given by

$$m_\nu = \begin{pmatrix} m_{11}e^{i\phi} & m_{12}e^{-2i\pi\frac{\gamma}{n}} & 0 \\ m_{12}e^{-2i\pi\frac{\gamma}{n}} & m_{11}e^{-i(4\pi\frac{\gamma}{n}+\phi)} & 0 \\ 0 & 0 & m_{33}e^{4i\pi\frac{\gamma}{n}} \end{pmatrix}, \tag{7.15}$$

[1] In the basis in which the neutrino mass matrix is diagonal with $m_\nu = \text{diag}(m_1, m_2, m_3)$, the general CP transformation $\widehat{X}$ which leaves m_ν invariant: $\widehat{X}^T m_{mu}\widehat{X} = m_\nu^*$, should be of the form $\widehat{X} = \text{diag}(\pm1, \pm1, \pm1)$. One can go to an arbitrary basis and define the corresponding CP symmetry transformation $X = U^\dagger \widehat{X} U^*$ as a symmetry of the general neutrino mass matrix, where U is the basis transformation. As a result, the residual CP symmetry X in the neutrino sector should be symmetric. The same conclusion has been obtained in Ref. [30].

where m_{11}, m_{12}, m_{33} and ϕ are real free parameters. The neutrino diagonalization matrix is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} & e^{-i\frac{\phi}{2}} & 0 \\ -e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & 0 \\ 0 & 0 & \sqrt{2}\, e^{-2i\pi\frac{\gamma}{n}} \end{pmatrix} . \tag{7.16}$$

The light neutrino mass eigenvalues are

$$m_1 = |m_{11} - m_{12}| , \quad m_2 = |m_{11} + m_{12}| , \quad m_3 = |m_{33}| . \tag{7.17}$$

The ordering of the neutrino masses can not be determined as well.

(v) $G_\nu = Z_2^{d^{n/2}} \equiv \{1, d^{n/2}\}$, $X_{\nu\mathbf{r}} = \{\rho_\mathbf{r}(c^\gamma d^\delta), \rho_\mathbf{r}(a^2bc^\gamma d^\delta)\}$

- $X_{\nu\mathbf{r}} = \rho_\mathbf{r}(c^\gamma d^\delta)$

  The light neutrino mass matrix is constrained by residual flavour and residual CP symmetries to be

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & m_{22}e^{-2i\pi\frac{\delta-\gamma}{n}} & m_{23}e^{i\pi\frac{\gamma}{n}} \\ 0 & m_{23}e^{i\pi\frac{\gamma}{n}} & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix} , \tag{7.18}$$

  where m_{11}, m_{22}, m_{23} and m_{33} are real. The neutrino diagonalization matrix is

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}}\cos\theta & e^{i\pi\frac{\delta-\gamma}{n}}\sin\theta \\ 0 & -e^{-i\pi\frac{\delta}{n}}\sin\theta & e^{-i\pi\frac{\delta}{n}}\cos\theta \end{pmatrix} , \tag{7.19}$$

  with

$$\tan 2\theta = \frac{2m_{23}}{m_{33} - m_{22}} . \tag{7.20}$$

  The light neutrino masses take the form

$$\begin{aligned} m_1 &= |m_{11}| , \\ m_2 &= \frac{1}{2}\left| m_{22} + m_{33} - \mathrm{sign}\left((m_{33}-m_{22})\cos 2\theta\right)\sqrt{(m_{33}-m_{22})^2 + 4m_{23}^2} \right| , \\ m_3 &= \frac{1}{2}\left| m_{22} + m_{33} + \mathrm{sign}\left((m_{33}-m_{22})\cos 2\theta\right)\sqrt{(m_{33}-m_{22})^2 + 4m_{23}^2} \right| . \end{aligned} \tag{7.21}$$

- $X_{\nu\mathbf{r}} = \rho_\mathbf{r}(a^2bc^\gamma d^\delta)$

As has been shown above, $X_{\nu\mathbf{r}}$ has to be symmetric. Then the requirement $\gamma = 2\delta \mod n$ follows immediately, otherwise the light neutrino masses would be partially degenerate. In this case, the neutrino mass matrix takes the form:

$$m_\nu = \begin{pmatrix} m_{11} e^{-4i\pi\frac{\delta}{n}} & 0 & 0 \\ 0 & m_{22} e^{i\phi} & m_{23} e^{2i\pi\frac{\delta}{n}} \\ 0 & m_{23} e^{2i\pi\frac{\delta}{n}} & m_{22} e^{i(4\pi\frac{\delta}{n}-\phi)} \end{pmatrix} , \tag{7.22}$$

where m_{11}, m_{22}, m_{23} and ϕ are real. It is diagonalized by the unitary matrix

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} e^{2i\pi\frac{\delta}{n}} & 0 & 0 \\ 0 & e^{-i\frac{\phi}{2}} & e^{-i\frac{\phi}{2}} \\ 0 & -e^{i\left(\frac{\phi}{2}-2\pi\frac{\delta}{n}\right)} & e^{i\left(\frac{\phi}{2}-2\pi\frac{\delta}{n}\right)} \end{pmatrix} . \tag{7.23}$$

The light neutrino masses are

$$m_1 = |m_{11}| , \quad m_2 = |m_{22} - m_{23}| , \quad m_3 = |m_{22} + m_{23}| . \tag{7.24}$$

(vi) $G_\nu = Z_2^{c^{n/2}d^{n/2}} \equiv \left\{1, c^{n/2}d^{n/2}\right\}$, $X_{\nu\mathbf{r}} = \left\{\rho_\mathbf{r}(c^\gamma d^\delta), \rho_\mathbf{r}(bc^\gamma d^\delta)\right\}$

- $X_{\nu\mathbf{r}} = \rho_\mathbf{r}(c^\gamma d^\delta)$

  The light neutrino mass matrix invariant under both the residual flavour and residual CP symmetries is

$$m_\nu = \begin{pmatrix} m_{11} e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{13} e^{-i\pi\frac{\gamma-\delta}{n}} \\ 0 & m_{22} e^{-2i\pi\frac{\delta-\gamma}{n}} & 0 \\ m_{13} e^{-i\pi\frac{\gamma-\delta}{n}} & 0 & m_{33} e^{2i\pi\frac{\delta}{n}} \end{pmatrix} , \tag{7.25}$$

where m_{11}, m_{13}, m_{22} and m_{33} are real parameters. The unitary transformation U_ν is given by

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} \cos\theta & 0 & e^{i\pi\frac{\gamma}{n}} \sin\theta \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} & 0 \\ -e^{-i\pi\frac{\delta}{n}} \sin\theta & 0 & e^{-i\pi\frac{\delta}{n}} \cos\theta \end{pmatrix} , \tag{7.26}$$

with

$$\tan 2\theta = \frac{2m_{13}}{m_{33} - m_{11}} . \tag{7.27}$$

The light neutrino mass eigenvalues are

$$
\begin{aligned}
m_1 &= \frac{1}{2}\left|m_{11}+m_{33}-\mathrm{sign}\left((m_{33}-m_{11})\cos 2\theta\right)\sqrt{(m_{33}-m_{11})^2+4m_{13}^2}\right| ,\\
m_2 &= |m_{22}| ,\\
m_3 &= \frac{1}{2}\left|m_{11}+m_{33}+\mathrm{sign}\left((m_{33}-m_{11})\cos 2\theta\right)\sqrt{(m_{33}-m_{11})^2+4m_{13}^2}\right| .
\end{aligned}
\tag{7.28}
$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(bc^{\gamma}d^{\delta})$

  In the case of $\gamma+\delta \neq 0 \bmod n$, the general CP transformation $\rho_{\mathbf{r}}(bc^{\gamma}d^{\delta})$ is not symmetric. As a consequence, the light neutrino masses are partially degenerate. In the following, we shall focus on the case of $\gamma+\delta = 0 \bmod n$. The neutrino mass matrix is determined to be of the following form:

$$
m_\nu = \begin{pmatrix} m_{11}e^{i\phi} & 0 & m_{13}e^{-2i\pi\frac{\gamma}{n}} \\ 0 & m_{22}e^{4i\pi\frac{\gamma}{n}} & 0 \\ m_{13}e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{11}e^{-i(\phi+4\pi\frac{\gamma}{n})} \end{pmatrix} , \tag{7.29}
$$

where m_{11}, m_{13}, m_{22} and ϕ are real. The neutrino diagonalization matrix is

$$
U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 & e^{-i\frac{\phi}{2}} \\ 0 & \sqrt{2}e^{-2i\pi\frac{\gamma}{n}} & 0 \\ -e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & 0 & e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} \end{pmatrix} . \tag{7.30}
$$

Finally the light neutrino masses are given by

$$
m_1 = |m_{11}-m_{13}| , \quad m_2 = |m_{22}| , \quad m_3 = |m_{11}+m_{13}| . \tag{7.31}
$$

7.1.2 Charged lepton diagonalisation matrices in neutrino-semidirect models

- $G_l = \langle c^s d^t \rangle$

$$
U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7.32}
$$

Note that the parameters s and t should be subject to the following constraints

$$s+t \neq 0 \bmod n, \qquad s-2t \neq 0 \bmod n, \qquad t-2s \neq 0 \bmod n\,, \tag{7.33}$$

otherwise the eigenvalues of $c^s d^t$ would be degenerate and consequently U_l can not be determined uniquely. For the value of $s=t=n/2$, the residual symmetry could be chose to be $K_4^{(c^{n/2},d^{n/2})} \equiv \left\{1,\, c^{n/2},\, d^{n/2},\, c^{n/2}d^{n/2}\right\}$ instead, and then corresponding unitary transformation U_l is still a unit matrix. The constraints of Eq. (7.33) will be assumed for the subgroup $G_l = \langle c^s d^t \rangle$ in the following.

- $G_l = \langle bc^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi\frac{s+t}{2n}} & 0 & e^{-i\pi\frac{s+t}{2n}} \\ 0 & \sqrt{2} & 0 \\ -e^{i\pi\frac{s+t}{2n}} & 0 & e^{i\pi\frac{s+t}{2n}} \end{pmatrix}. \tag{7.34}$$

To avoid degenerate eigenvalues, we should exclude the values

$$s-t=0, n/3, 2n/3 \bmod n\,. \tag{7.35}$$

For the case of $s=t$, the order of the element $bc^s d^s$ is two and one could extend G_l from $\langle bc^s d^s\rangle = \{1, bc^s d^s\}$ to the Klein four subgroup $K_4^{(c^{n/2}d^{n/2}, bc^s d^s)} \equiv \left\{1,\, c^{n/2}d^{n/2},\, bc^s d^s,\, bc^{s+n/2}d^{s+n/2}\right\}$. Then the unitary transformation U_l is still of the form in Eq. (7.34) with $s=t$.

- $G_l = \langle ac^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2i\pi\frac{s}{n}} & \omega^2 e^{-2i\pi\frac{s}{n}} & \omega e^{-2i\pi\frac{s}{n}} \\ e^{-2i\pi\frac{t}{n}} & \omega e^{-2i\pi\frac{t}{n}} & \omega^2 e^{-2i\pi\frac{t}{n}} \\ 1 & 1 & 1 \end{pmatrix}, \tag{7.36}$$

where $\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is the third root of unity. Notice that the order of the element $ac^s d^t$ is three regardless of the values of s and t, and its eigenvalues are 1, ω and ω^2.

- $G_l = \langle a^2 c^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2i\pi\frac{t}{n}} & \omega^2 e^{-2i\pi\frac{t}{n}} & \omega e^{-2i\pi\frac{t}{n}} \\ e^{2i\pi\frac{s-t}{n}} & \omega e^{2i\pi\frac{s-t}{n}} & \omega^2 e^{2i\pi\frac{s-t}{n}} \\ 1 & 1 & 1 \end{pmatrix}. \tag{7.37}$$

Note that because $(ac^t d^{t-s})^2 = a^2 c^s d^t$ holds, this U_l can be obtained from the one in Eq. (7.36) by the replacement $s \to t$, $t \to t-s$.

- $G_l = \langle abc^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi \frac{t-2s}{2n}} & e^{i\pi \frac{t-2s}{2n}} & 0 \\ -e^{-i\pi \frac{t-2s}{2n}} & e^{-i\pi \frac{t-2s}{2n}} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} . \tag{7.38}$$

Non-degeneracy of the eigenvalues of $abc^s d^t$ requires $t \neq 0, n/3, 2n/3$. In the case of $t = 0$, the degeneracy can be avoided by expanding G_l to the Klein four subgroup $K_4^{(c^{n/2}, abc^s)} \equiv \{1,\ c^{n/2},\ abc^s,\ abc^{s+n/2}\}$, whose diagonalization matrix is of the same form as Eq. (7.38) with $t = 0$.

- $G_l = \langle a^2 bc^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & e^{i\pi \frac{s-2t}{2n}} & e^{i\pi \frac{s-2t}{2n}} \\ 0 & -\,e^{-i\pi \frac{s-2t}{2n}} & e^{-i\pi \frac{s-2t}{2n}} \end{pmatrix} . \tag{7.39}$$

Here the parameter s can not be equal to 0, $n/3$ or $2n/3$, otherwise two eigenvalues of $a^2 bc^s d^t$ would be identical. For the extended residual symmetry $G_l = K_4^{(d^{n/2}, a^2 bd^t)} \equiv \{1,\ d^{n/2},\ a^2 bd^t,\ a^2 bd^{t+n/2}\}$, the corresponding unitary transformation is still given by Eq. (7.39) with $s = 0$.

7.1.3 Additional mixing results for neutrino-semi-direct models

(I) In this case, the lepton mixing parameters are predicted to be

$$
\begin{aligned}
\sin^2\theta_{13} &= \frac{1}{3}\left(1+\cos^2\theta\cos 2\varphi_1+\sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1\right),\\
\sin^2\theta_{12} &= \frac{1+\sin^2\theta\cos 2\varphi_1-\sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1}{2-\cos^2\theta\cos 2\varphi_1-\sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1},\\
\sin^2\theta_{23} &= \frac{1-\cos^2\theta\sin\left(\pi/6+2\varphi_1\right)-\sqrt{2}\sin 2\theta\cos\varphi_2\sin\left(\pi/6-\varphi_1\right)}{2-\cos^2\theta\cos 2\varphi_1-\sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1},\\
|\tan\delta_{CP}| &= \Big|2\sqrt{2}\sin 2\theta\sin\varphi_2(1+2\cos 2\varphi_1)\left(2-\cos^2\theta\cos 2\varphi_1-\sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1\right)\Big/\\
&\quad \Big\{2\sin^2 2\theta\cos 2\varphi_2(\cos 3\varphi_1-2\cos\varphi_1)+\cos\varphi_1\left(9-4\cos 2\theta+3\cos 4\theta-16\cos^2\theta\cos 2\varphi_1\right)\\
&\quad -2\sqrt{2}\sin 2\theta\cos\varphi_2\left[2-\cos^2\theta(5+\cos 2\varphi_1+\cos 4\varphi_1)\right]\Big\}\Big|\\
|J_{CP}| &= \frac{1}{6\sqrt{6}}\left|\sin 2\theta\sin\varphi_2\sin 3\varphi_1\right|,\\
|\tan\alpha_{21}| &= \left|\frac{2\sin\varphi_2\left(\cos\varphi_2-\sqrt{2}\cos\varphi_1\tan\theta\right)}{\cos 2\varphi_2-2\cos\varphi_1\tan\theta\left(\sqrt{2}\cos\varphi_2-\cos\varphi_1\tan\theta\right)}\right|,\\
|\tan\alpha'_{31}| &= \left|\frac{2\sin\varphi_2\left(\cos\varphi_2+\sqrt{2}\cos\varphi_1\cot\theta\right)}{\cos 2\varphi_2+2\cos\varphi_1\cot\theta\left(\sqrt{2}\cos\varphi_2+\cos\varphi_1\cot\theta\right)}\right|,
\end{aligned}
\tag{7.40}
$$

where $\alpha'_{31}=\alpha_{31}-2\delta_{CP}$, δ_{CP} is the Dirac CP phase, α_{21} and α_{31} are the Majorana CP phases in the standard parametrisation [36]. If we embed the three generations of left-handed lepton doublets into the triplet $\mathbf{3}_{1,n-1}$ which is the complex conjugate representation of $\mathbf{3}_{1,1}$, all three CP phases δ_{CP}, α_{21} and α_{31} would become their opposite numbers modulo 2π. Furthermore, the overall sign of $\tan\alpha_{21}$ and $\tan\alpha'_{31}$ depends on the CP parity of the neutrino states which is encoded in the matrix K_ν (please see Eq. (7.3)), and the sign of the Jarlskog invariant J_{CP} depends on the ordering of rows and columns. As a result, all these quantities are presented in terms of absolute values here.

It is notable that all three CP phases depend on both the free continuous parameter θ and the discrete parameters φ_1 and φ_2 associated with flavour and CP symmetries. Both Dirac CP and Majorana CP are conserved for $\varphi_2=0$. Furthermore, the solar mixing angle θ_{12} and reactor angle θ_{13} are related by

$$3\cos^2\theta_{12}\cos^2\theta_{13}=2\sin^2\varphi_1\,, \tag{7.41}$$

which is independent of the free parameter θ.

For the lepton flavour mixing matrix U^{4th}_{PMNS}, one can extract the flavour mixing parameters:

$$
\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{3}\left(1+\cos^2\theta\cos 2\varphi_1 + \sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1\right),\\
&\sin^2\theta_{12} = \frac{2\sin^2\varphi_1}{2-\cos^2\theta\cos 2\varphi_1 - \sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1},\\
&\sin^2\theta_{23} = \frac{1-\cos^2\theta\sin\left(\pi/6+2\varphi_1\right) - \sqrt{2}\sin 2\theta\cos\varphi_2\sin\left(\pi/6-\varphi_1\right)}{2-\cos^2\theta\cos 2\varphi_1 - \sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1},\\
&|J_{CP}| = \frac{1}{6\sqrt{6}}\left|\sin 2\theta\sin\varphi_2\sin 3\varphi_1\right|,\\
&|\tan\delta_{CP}| = \Big|4\sqrt{2}\sin 2\theta\sin\varphi_2\sin 3\varphi_1\csc\varphi_1\left(2-\cos 2\varphi_1\cos^2\theta - \sqrt{2}\cos\varphi_2\cos\varphi_1\sin 2\theta\right)\Big/\\
&\quad\Big\{-16\cos 3\varphi_1\cos^2\theta + 8(1-3\cos 2\theta)\cos\varphi_1\sin^2\theta + 4\cos 2\varphi_2(\cos 3\varphi_1 - 2\cos\varphi_1)\sin^2 2\theta\\
&\quad+\sqrt{2}\cos\varphi_2\left[8(\cos 2\varphi_1+\cos 4\varphi_1)\sin\theta\cos^3\theta + 2\sin 2\theta + 5\sin 4\theta\right]\Big\}\Big|,\\
&|\tan\alpha_{21}| = \left|\frac{2\sin\varphi_2\left(\cos\varphi_2 - \sqrt{2}\cos\varphi_1\tan\theta\right)}{\cos 2\varphi_2 - 2\cos\varphi_1\tan\theta\left(\sqrt{2}\cos\varphi_2 - \cos\varphi_1\tan\theta\right)}\right|,\\
&|\tan\alpha'_{31}| = \Big|8\cos\varphi_1\left(\sqrt{2}\cos 2\varphi_1\sin 2\theta\sin\varphi_2 - 2\cos 2\theta\cos\varphi_1\sin 2\varphi_2\right)\Big/\Big\{4(3+\cos 4\theta)\\
&\times\cos 2\varphi_2\cos^2\varphi_1 - 4\sqrt{2}\cos\varphi_2\cos\varphi_1\cos 2\varphi_1\sin 4\theta - (3-\cos 4\varphi_1 + 4\cos 2\varphi_1)\sin^2 2\theta\Big\}\Big|.
\end{aligned}
\tag{7.42}
$$

The lepton mixing parameters for $U^{I,7th}_{PMNS}$ are determined to be

$$
\begin{aligned}
&\sin^2\theta_{13} = \frac{2}{3}\sin^2\varphi_1, \qquad \sin^2\theta_{12} = \frac{1+\sin^2\theta\cos 2\varphi_1 - \sqrt{2}\sin 2\theta\cos\varphi_2\cos\varphi_1}{2+\cos 2\varphi_1},\\
&\sin^2\theta_{23} = \frac{1+\sin\left(\pi/6+2\varphi_1\right)}{2+\cos 2\varphi_1}, \qquad |J_{CP}| = \frac{1}{6\sqrt{6}}\left|\sin 2\theta\sin\varphi_2\sin 3\varphi_1\right|,\\
&|\tan\delta_{CP}| = \left|\frac{\sin\varphi_2(2+\cos 2\varphi_1)}{\cos\varphi_2\cos 2\varphi_1 - 2\sqrt{2}\cot 2\theta\cos\varphi_1}\right|,\\
&|\tan\alpha_{21}| = \Big|4\sqrt{2}\cos\varphi_1\left(\cos 2\varphi_1\sin 2\theta\sin\varphi_2 - \sqrt{2}\cos 2\theta\cos\varphi_1\sin 2\varphi_2\right)\Big/\Big\{2(\cos 4\theta+3)\\
&\quad\times\cos 2\varphi_2\cos^2\varphi_1 - 2\sqrt{2}\cos\varphi_2\cos\varphi_1\cos 2\varphi_1\sin 4\theta + \left(\cos^2 2\varphi_1 - 4\cos^2\varphi_1\right)\sin^2 2\theta\Big\}\Big|,\\
&|\tan\alpha'_{31}| = \left|\frac{2\sin\varphi_2\sin\theta\left(\sqrt{2}\cos\theta\cos\varphi_1 + \cos\varphi_2\sin\theta\right)}{2\cos^2\theta\cos^2\varphi_1 + \sqrt{2}\cos\varphi_2\cos\varphi_1\sin 2\theta + \cos 2\varphi_2\sin^2\theta}\right|.
\end{aligned}
\tag{7.43}
$$

The lepton mixing parameters for $U^{I,8th}_{PMNS}$ and $U^{I,9th}_{PMNS}$ can be obtained from Eq. (7.43) by the replacement $\theta\to\pi-\theta$, $\varphi_1\to\frac{\pi}{3}+\varphi_1$ and $\theta\to-\theta$, $\varphi_1\to-\frac{\pi}{3}+\varphi_1$

respectively. We see that both θ_{13} and θ_{23} are only determined by the discrete group parameter φ_1, and they are related by

$$\sin^2\theta_{23} = \frac{1}{2} \pm \frac{1}{2}\tan\theta_{13}\sqrt{2-\tan^2\theta_{13}}\,, \tag{7.44}$$

which yields

$$\theta_{23} \simeq \frac{\pi}{4} \pm \frac{\theta_{13}}{\sqrt{2}}\,. \tag{7.45}$$

(II)

$$\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{8}\left(3-\cos 2\theta - 2\sqrt{2}\sin 2\theta\cos\varphi_3\right), \quad \sin^2\theta_{12} = \frac{2}{5+\cos 2\theta + 2\sqrt{2}\sin 2\theta\cos\varphi_3},\\
&\sin^2\theta_{23} = \frac{3-\cos 2\theta + 2\sqrt{2}\sin 2\theta\cos\varphi_3}{5+\cos 2\theta + 2\sqrt{2}\sin 2\theta\cos\varphi_3}, \qquad |J_{CP}| = \frac{1}{8\sqrt{2}}\left|\sin 2\theta\sin\varphi_3\right|,\\
&|\tan\delta_{CP}| = \left|\frac{8\cos\theta\sin^2\theta\sin 2\varphi_3 + \sqrt{2}(9\sin\theta + \sin 3\theta)\sin\varphi_3}{4\cos 3\theta + \cos\theta\left(4-8\sin^2\theta\cos 2\varphi_3\right) + \sqrt{2}(3\sin 3\theta - 5\sin\theta)\cos\varphi_3}\right|,\\
&|\tan\alpha_{21}| = \left|\frac{2\cos^2\theta\sin 2\varphi_3 + \sqrt{2}\sin 2\theta\sin\varphi_3}{\sin^2\theta + 2\cos^2\theta\cos 2\varphi_3 + \sqrt{2}\sin 2\theta\cos\varphi_3}\right|,\\
&|\tan\alpha'_{31}| = \left|\frac{16\cos 2\theta\sin 2\varphi_3 - 8\sqrt{2}\sin 2\theta\sin\varphi_3}{6\sin^2 2\theta + 4\sqrt{2}\sin 4\theta\cos\varphi_3 - 4(3+\cos 4\theta)\cos 2\varphi_3}\right|.
\end{aligned} \tag{7.46}$$

III The lepton mixing parameters are given by

$$\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{3}\left[1+\sin 2\theta\cos(\varphi_5-\varphi_4)\right], \quad \sin^2\theta_{12} = \frac{1}{2-\sin 2\theta\cos(\varphi_5-\varphi_4)},\\
&\sin^2\theta_{23} = \frac{1-\sin 2\theta\sin\left(\varphi_5-\varphi_4+\pi/6\right)}{2-\sin 2\theta\cos(\varphi_5-\varphi_4)}, \qquad |J_{CP}| = \frac{1}{6\sqrt{3}}\left|\cos 2\theta\right|,\\
&|\tan\delta_{CP}| = \left|\frac{\cot 2\theta\left[2-\sin 2\theta\cos(\varphi_5-\varphi_4)\right]}{\sin(\varphi_5-\varphi_4) - \sin 2\theta\sin(2\varphi_5-2\varphi_4)}\right|,\\
&|\tan\alpha_{21}| = \left|\frac{\cos^2\theta\sin 2\varphi_4 + \sin^2\theta\sin 2\varphi_5 - \sin 2\theta\sin(\varphi_5+\varphi_4)}{\cos^2\theta\cos 2\varphi_4 + \sin^2\theta\cos 2\varphi_5 - \sin 2\theta\cos(\varphi_5+\varphi_4)}\right|,\\
&|\tan\alpha'_{31}| = \left|\frac{4\cos 2\theta\sin(2\varphi_5-2\varphi_4)}{1-3\cos(2\varphi_5-2\varphi_4) - 2\cos 4\theta\cos^2(\varphi_5-\varphi_4)}\right|.
\end{aligned} \tag{7.47}$$

7.1.4 Correlation plots for neutrino-semidirect models

Figures 7.1–7.5 contain correlation plots between lepton mixing parameters for the various cases.

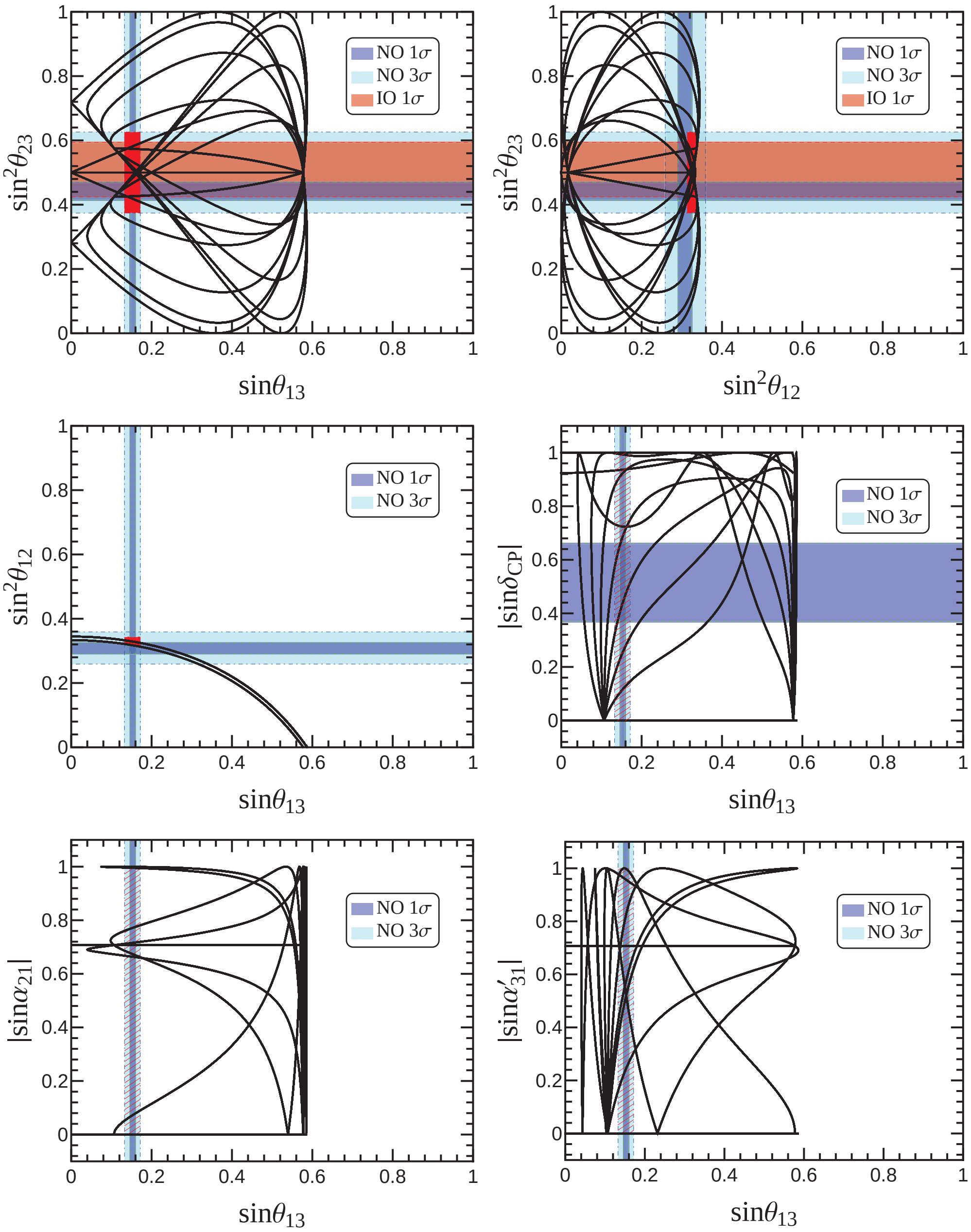


Figure 7.1: The correlations among mixing parameters in case I for the 1st-3rd ordering with the PMNS matrices given in Eq. (4.29). The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_1 and φ_2 to be continuous (which is equivalent to taking the limit $n \to \infty$) and the three mixing angles are required to lie in their 3σ regions. Note that the three CP phases δ_{CP}, α_{21} and α'_{31} are not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

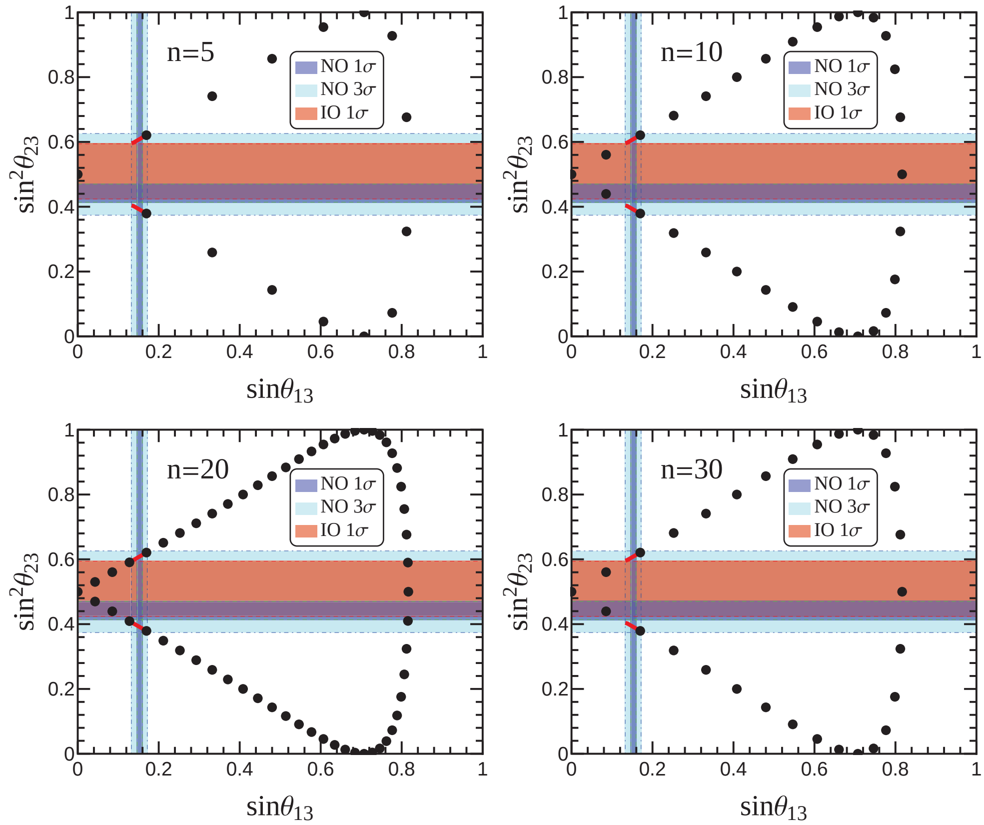


Figure 7.2: The possible values of $\sin^2\theta_{23}$ and $\sin\theta_{13}$ for the 7th-9th ordering with the PMNS matrices shown in Eq. (4.38) in case I. The 1σ and 3σ bounds of the mixing angles are taken from Ref. [78].

7.1.5 Charged lepton mass and diagonalisation matrices for charged-lepton semi-direct models

The full symmetry $\Delta(6n^2) \rtimes H_{CP}$ is broken down to $Z_2 \times H^l_{CP}$ in the charged lepton sector. Similar to section 4.3, the hermitian combination $m_l^\dagger m_l$ of the charged lepton mass matrix can be constructed from its invariance under the residual flavour symmetry Z_2 and the residual CP symmetry H^l_{CP},

$$\begin{aligned} \rho^\dagger_{\mathbf{3}}(g_l) m_l^\dagger m_l \rho_{\mathbf{3}}(g_l) &= m_l^\dagger m_l, \quad g_l \in Z_2\,, \\ X^\dagger_{l\mathbf{3}} m_l^\dagger m_l X_{l\mathbf{3}} &= \left(m_l^\dagger m_l\right)^*, \quad X_l \in H^l_{CP}\,. \end{aligned} \tag{7.48}$$

(i) $G_l = Z_2^{bc^xd^x}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma}d^{-x-\gamma})\}$

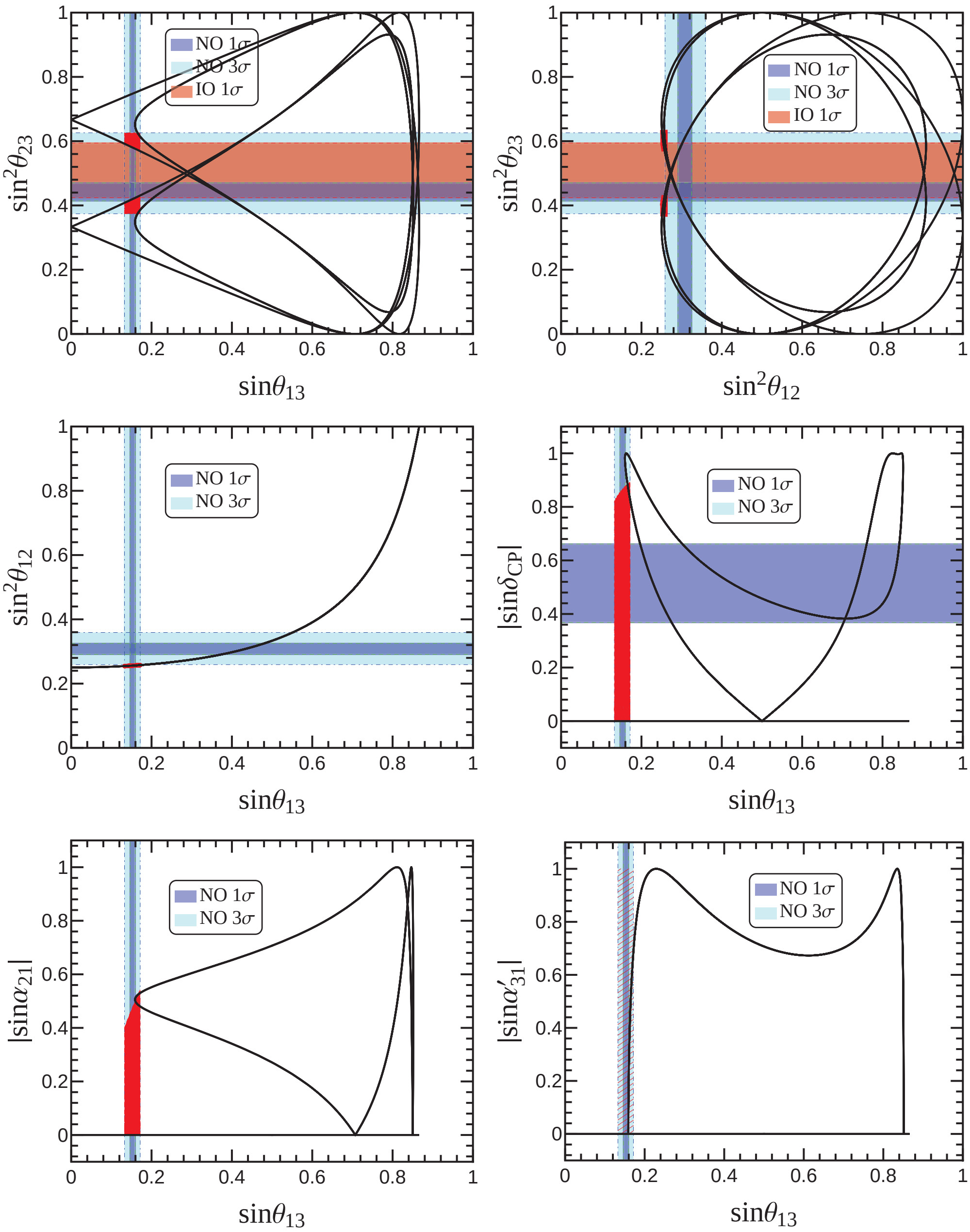


Figure 7.3: The correlations among mixing parameters in case II. The red filled regions denote the allowed values of the mixing parameters if we take the parameter φ_3 to be continuous (which is equivalent to taking the limit $n \to \infty$) and the three mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2\theta_{12}$ is chosen to be 0.254 instead of 0.259 given in Ref. [78]). Note that the Majorana phase α'_{31} is not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78]

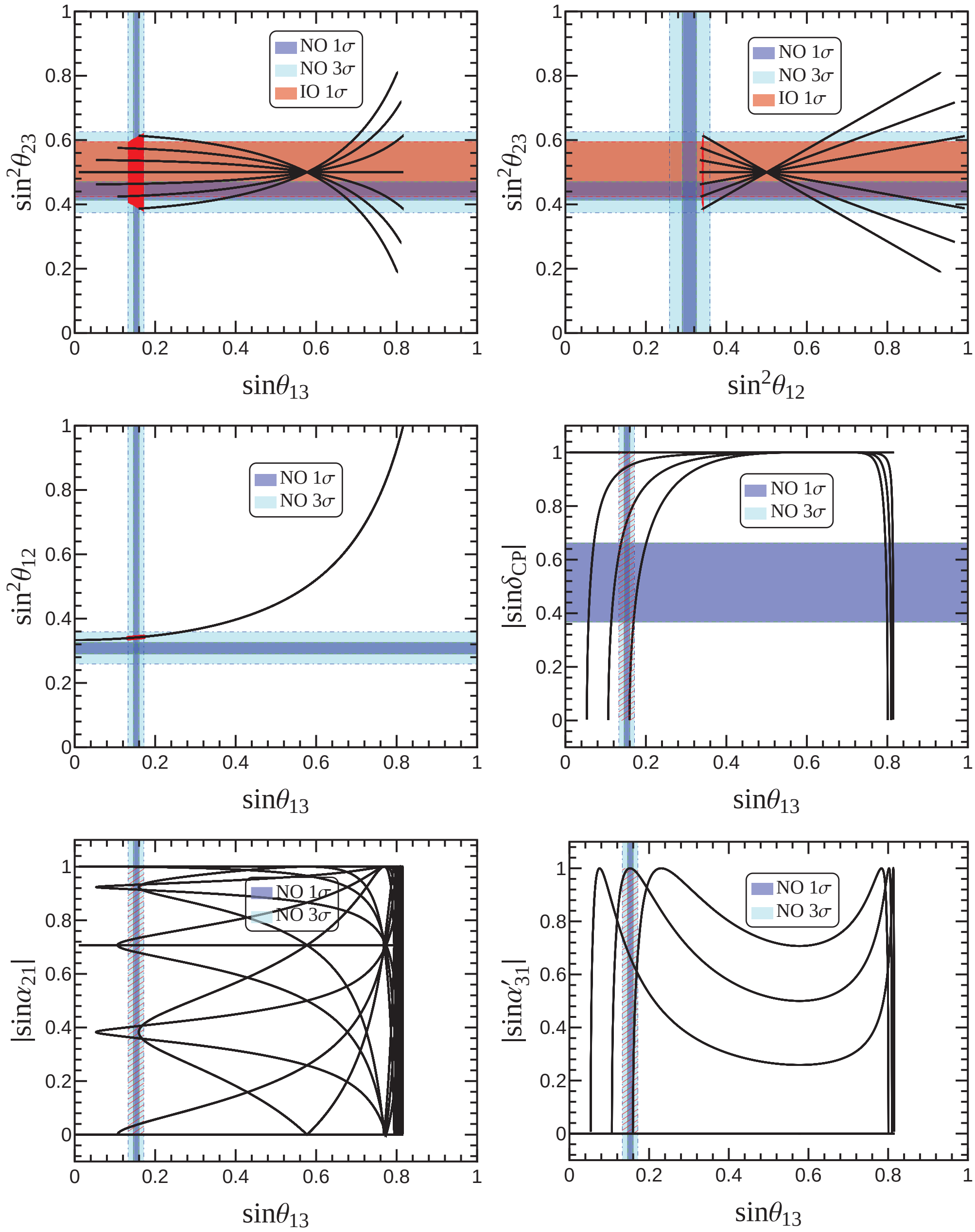


Figure 7.4: The correlations among mixing parameters in case III. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_4 and φ_5 to be continuous (which is equivalent to taking the limit $n \to \infty$) and the three mixing angles are required to lie in their 3σ ranges. Note that the three CP phases δ_{CP}, α_{21} and α'_{31} are not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

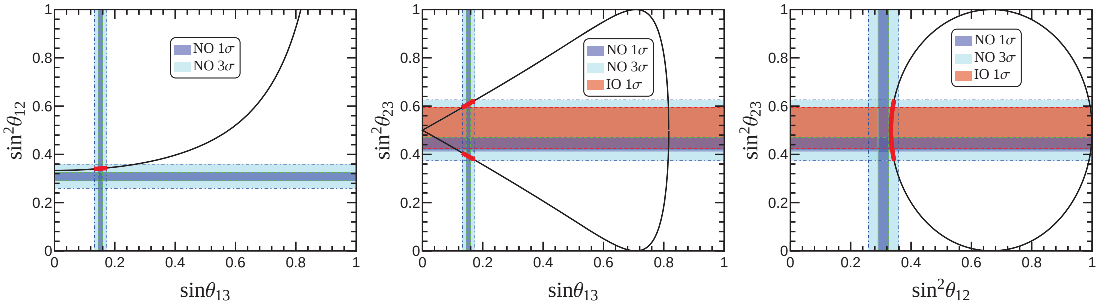


Figure 7.5: The correlations among mixing parameters in case IV. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_6 and φ_7 to be continuous (which is equivalent to taking the limit $n \to \infty$), where θ_{12} and θ_{13} are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

In this case, $m_l^\dagger m_l$ is determined to be of the form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12} e^{i\pi\frac{2x+3\gamma}{n}} & \widetilde{m}_{13} e^{-2i\pi\frac{x}{n}} \\ \widetilde{m}_{12} e^{-i\pi\frac{2x+3\gamma}{n}} & \widetilde{m}_{22} & \widetilde{m}_{12} e^{-i\pi\frac{4x+3\gamma}{n}} \\ \widetilde{m}_{13} e^{2i\pi\frac{x}{n}} & \widetilde{m}_{12} e^{i\pi\frac{4x+3\gamma}{n}} & \widetilde{m}_{11} \end{pmatrix}, \qquad (7.49)$$

where $\widetilde{m}_{11}$, $\widetilde{m}_{12}$, $\widetilde{m}_{13}$ and $\widetilde{m}_{22}$ are real parameters, and they have mass dimension of 2. This charged lepton mass matrix is diagonalized by a unitary transformation U_l via

$$U_l^\dagger m_l^\dagger m_l U_l = \text{diag}\left(m_{l_1}^2, m_{l_2}^2, m_{l_3}^2\right), \qquad (7.50)$$

with

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & -e^{i\pi\frac{\gamma}{n}} \sin\theta & e^{i\pi\frac{\gamma}{n}} \cos\theta \\ 0 & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2}\cos\theta & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2}\sin\theta \\ -e^{i\pi\frac{2x+\gamma}{n}} & -e^{i\pi\frac{2x+\gamma}{n}} \sin\theta & e^{i\pi\frac{2x+\gamma}{n}} \cos\theta \end{pmatrix}, \qquad (7.51)$$

where the angle θ is

$$\tan 2\theta = \frac{2\sqrt{2}\widetilde{m}_{12}}{\widetilde{m}_{11} + \widetilde{m}_{13} - \widetilde{m}_{22}}. \qquad (7.52)$$

It is remarkable that the unitary transformation U_l in Eq. (**??**) coincides with U_ν in Eq. (7.3). The reason is that the two cases share the same residual symmetry.

The charged lepton masses are given by

$$\begin{aligned}
m_{l_1}^2 &= \widetilde{m}_{11} - \widetilde{m}_{13}, \\
m_{l_2}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{13} + \widetilde{m}_{22} - \text{sign}\left((\widetilde{m}_{11} + \widetilde{m}_{13} - \widetilde{m}_{22})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11} + \widetilde{m}_{13} - \widetilde{m}_{22})^2 + 8\widetilde{m}_{12}^2}\right], \\
m_{l_3}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{13} + \widetilde{m}_{22} + \text{sign}\left((\widetilde{m}_{11} + \widetilde{m}_{13} - \widetilde{m}_{22})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11} + \widetilde{m}_{13} - \widetilde{m}_{22})^2 + 8\widetilde{m}_{12}^2}\right].
\end{aligned}$$

In the present framework, we can not determine the order of $m_{l_1}^2$, $m_{l_2}^2$ and $m_{l_3}^2$, i.e. we don't know which one of $m_{l_1}^2$, $m_{l_2}^2$, $m_{l_3}^2$ is electron (muon or tau) mass squared. As a result, the diagonalization matrix U_l in Eq. (**??**) is also determined up to rephasing and permutations of its column vectors. The same holds true for the following cases.

(ii) $G_l = Z_2^{abc^y}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}),\ \ \rho_{\mathbf{r}}(abc^{y+\gamma}d^{2y+2\gamma})\}$

The charged lepton mass matrix satisfying the invariant conditions of Eq. (7.48) takes the form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12}e^{-2i\pi\frac{y}{n}} & \widetilde{m}_{13}e^{i\pi\frac{2y+3\gamma}{n}} \\ \widetilde{m}_{12}e^{2i\pi\frac{y}{n}} & \widetilde{m}_{11} & \widetilde{m}_{13}e^{i\pi\frac{4y+3\gamma}{n}} \\ \widetilde{m}_{13}e^{-i\pi\frac{2y+3\gamma}{n}} & \widetilde{m}_{13}e^{-i\pi\frac{4y+3\gamma}{n}} & \widetilde{m}_{33} \end{pmatrix}, \tag{7.53}$$

where $\widetilde{m}_{11}$, $\widetilde{m}_{12}$, $\widetilde{m}_{13}$ and $\widetilde{m}_{33}$ are real. The charged lepton diagonalization matrix U_l is given by

$$U_l = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}}\cos\theta & e^{i\pi\frac{2y+\gamma}{n}}\sin\theta \\ 0 & -e^{-2i\pi\frac{y+\gamma}{n}}\sqrt{2}\sin\theta & e^{-2i\pi\frac{y+\gamma}{n}}\sqrt{2}\cos\theta \end{pmatrix}, \tag{7.54}$$

with

$$\tan 2\theta = \frac{2\sqrt{2}\,\widetilde{m}_{13}}{\widetilde{m}_{33} - \widetilde{m}_{11} - \widetilde{m}_{12}}. \tag{7.55}$$

The charged lepton masses are determined to be

$$\begin{aligned}
m_{l_1}^2 &= \widetilde{m}_{11} - \widetilde{m}_{12}, \\
m_{l_2}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{12} + \widetilde{m}_{33} + \text{sign}\left((\widetilde{m}_{11} + \widetilde{m}_{12} - \widetilde{m}_{33})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11} + \widetilde{m}_{12} - \widetilde{m}_{33})^2 + 8m_{13}^2}\right], \\
m_{l_3}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{12} + \widetilde{m}_{33} - \text{sign}\left((\widetilde{m}_{11} + \widetilde{m}_{12} - \widetilde{m}_{33})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11} + \widetilde{m}_{12} - \widetilde{m}_{33})^2 + 8m_{13}^2}\right].
\end{aligned}$$

(iii) $G_l = Z_2^{a^2bd^z}$, $X_{l\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta}d^{z+\delta})\right\}$

The charged lepton mass matrix invariant under both residual flavour and residual CP symmetries is

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12}e^{i\pi\frac{4z+3\delta}{n}} & \widetilde{m}_{12}e^{i\pi\frac{2z+3\delta}{n}} \\ \widetilde{m}_{12}e^{-i\pi\frac{4z+3\delta}{n}} & \widetilde{m}_{22} & \widetilde{m}_{23}e^{-2i\pi\frac{z}{n}} \\ \widetilde{m}_{12}e^{-i\pi\frac{2z+3\delta}{n}} & \widetilde{m}_{23}e^{2i\pi\frac{z}{n}} & \widetilde{m}_{22} \end{pmatrix}, \tag{7.56}$$

where $\widetilde{m}_{11}$, $\widetilde{m}_{12}$, $\widetilde{m}_{22}$ and $\widetilde{m}_{23}$ are real. The unitary transformation U_l follows immediately,

$$U_l = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\,e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\sin\theta & e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\cos\theta \\ e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}}\cos\theta & e^{-i\pi\frac{2z+\delta}{n}}\sin\theta \\ -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}}\cos\theta & e^{-i\pi\frac{\delta}{n}}\sin\theta \end{pmatrix}, \tag{7.57}$$

with the angle θ specified by

$$\tan 2\theta = \frac{2\sqrt{2}\widetilde{m}_{12}}{\widetilde{m}_{11}-\widetilde{m}_{22}-\widetilde{m}_{23}}\,. \tag{7.58}$$

Finally the charged lepton mass eigenvalues are

$$\begin{aligned} m^2_{l_1} &= \widetilde{m}_{22}-\widetilde{m}_{23}, \\ m^2_{l_2} &= \frac{1}{2}\left[\widetilde{m}_{11}+\widetilde{m}_{22}+\widetilde{m}_{23}-\text{sign}\left((\widetilde{m}_{11}-\widetilde{m}_{22}-\widetilde{m}_{23})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11}-\widetilde{m}_{22}-\widetilde{m}_{23})^2+8m^2_{12}}\right], \\ m^2_{l_3} &= \frac{1}{2}\left[\widetilde{m}_{11}+\widetilde{m}_{22}+\widetilde{m}_{23}+\text{sign}\left((\widetilde{m}_{11}-\widetilde{m}_{22}-\widetilde{m}_{23})\cos 2\theta\right)\sqrt{(\widetilde{m}_{11}-\widetilde{m}_{22}-\widetilde{m}_{23})^2+8m^2_{12}}\right]. \end{aligned}$$

(iv) $G_l = Z_2^{c^{n/2}} = \{1, c^{n/2}\}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(abc^\gamma d^\delta)\}$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$

  The charged lepton mass matrix is constrained to be of the following form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12}e^{i\pi\frac{2\gamma-\delta}{n}} & 0 \\ \widetilde{m}_{12}e^{-i\pi\frac{2\gamma-\delta}{n}} & \widetilde{m}_{22} & 0 \\ 0 & 0 & \widetilde{m}_{33} \end{pmatrix}, \tag{7.59}$$

  where $\widetilde{m}_{11}$, $\widetilde{m}_{12}$, $\widetilde{m}_{22}$ and $\widetilde{m}_{33}$ are real. It is diagonalized by the unitary matrix U_l with

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta & 0 \\ -e^{i\pi\frac{\delta-\gamma}{n}}\sin\theta & e^{i\pi\frac{\delta-\gamma}{n}}\cos\theta & 0 \\ 0 & 0 & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{7.60}$$

where

$$\tan 2\theta = \frac{2\widetilde{m}_{12}}{\widetilde{m}_{22} - \widetilde{m}_{11}}\,. \tag{7.61}$$

The charged lepton masses are determined to be

$$\begin{aligned}
m_{l_1}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{22} - \mathrm{sign}\left((\widetilde{m}_{22} - \widetilde{m}_{11})\cos 2\theta\right)\sqrt{(\widetilde{m}_{22} - \widetilde{m}_{11})^2 + 4m_{12}^2}\right],\\
m_{l_2}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{22} + \mathrm{sign}\left((\widetilde{m}_{22} - \widetilde{m}_{11})\cos 2\theta\right)\sqrt{(\widetilde{m}_{22} - \widetilde{m}_{11})^2 + 4m_{12}^2}\right],\\
m_{l_3}^2 &= \widetilde{m}_{33}\,.
\end{aligned} \tag{7.62}$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(abc^{\gamma}d^{\delta})$

  Similar to the discussed situation that $Z_2 \times CP$ is preserved in the neutrino sector, the CP transformation should be symmetric as well otherwise the charged lepton masses would be at least partially degenerate [2]. Therefore we shall focus on the case of $\delta = 2\gamma \mod n$ in the following. Then the charged lepton mass matrix is fixed to be

$$m_l^{\dagger} m_l = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12}e^{i\phi} & 0 \\ \widetilde{m}_{12}e^{-i\phi} & \widetilde{m}_{11} & 0 \\ 0 & 0 & \widetilde{m}_{33} \end{pmatrix}, \tag{7.63}$$

  where $\widetilde{m}_{11}$, $\widetilde{m}_{12}$, $\widetilde{m}_{33}$ and ϕ are free real parameters. Notice that $m_l^{\dagger} m_l$ is independent of the parameter γ. The unitary matrix U_l is of the form

$$U_l = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi} & e^{i\phi} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{7.64}$$

  The charged lepton masses are given by

$$\begin{aligned}
m_{l_1}^2 &= \widetilde{m}_{11} - \widetilde{m}_{12},\\
m_{l_2}^2 &= \widetilde{m}_{11} + \widetilde{m}_{12},\\
m_{l_2}^2 &= \widetilde{m}_{33}\,.
\end{aligned} \tag{7.65}$$

(v) $G_l = Z_2^{d^{n/2}} = \left\{1, d^{n/2}\right\}$, $X_{l\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^{\gamma}d^{\delta}), \rho_{\mathbf{r}}(a^2bc^{\gamma}d^{\delta})\right\}$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^{\gamma}d^{\delta})$

[2] From the residual symmetry invariant conditions in Eq. (7.48), we can derive that $U_l^{\dagger} X_{l\mathbf{3}} U_l^{*}$ should be a diagonal matrix. As a consequence, the CP transformation $X_{l\mathbf{3}}$ is symmetric.

In this case, the charged lepton mass matrix takes the form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & 0 & 0 \\ 0 & \widetilde{m}_{22} & \widetilde{m}_{23} e^{-i\pi\frac{\gamma-2\delta}{n}} \\ 0 & \widetilde{m}_{23} e^{i\pi\frac{\gamma-2\delta}{n}} & \widetilde{m}_{33} \end{pmatrix}, \tag{7.66}$$

where $\widetilde{m}_{11}$, $\widetilde{m}_{22}$, $\widetilde{m}_{23}$ and $\widetilde{m}_{33}$ are real. The charged lepton diagonalization matrix is

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} \cos\theta & e^{i\pi\frac{\delta-\gamma}{n}} \sin\theta \\ 0 & -e^{-i\pi\frac{\delta}{n}} \sin\theta & e^{-i\pi\frac{\delta}{n}} \cos\theta \end{pmatrix}, \tag{7.67}$$

with

$$\tan 2\theta = \frac{2\widetilde{m}_{23}}{\widetilde{m}_{33} - \widetilde{m}_{22}} . \tag{7.68}$$

The mass eigenvalues of the charged lepton are found to be

$$\begin{aligned} m_{l_1}^2 &= \widetilde{m}_{11}, \\ m_{l_2}^2 &= \frac{1}{2}\left[\widetilde{m}_{22} + \widetilde{m}_{33} - \text{sign}\left((\widetilde{m}_{33} - \widetilde{m}_{22})\cos 2\theta\right)\sqrt{(\widetilde{m}_{33} - \widetilde{m}_{22})^2 + 4m_{23}^2}\right], \\ m_{l_3}^2 &= \frac{1}{2}\left[\widetilde{m}_{22} + \widetilde{m}_{33} + \text{sign}\left((\widetilde{m}_{33} - \widetilde{m}_{22})\cos 2\theta\right)\sqrt{(\widetilde{m}_{33} - \widetilde{m}_{22})^2 + 4m_{23}^2}\right]. \end{aligned} \tag{7.69}$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2 b c^\gamma d^\delta)$

  This general CP transformation is symmetric only if $\gamma = 2\delta \mod n$. One can easily find that the charged lepton mass matrix is constrained to be of the form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & 0 & 0 \\ 0 & \widetilde{m}_{22} & \widetilde{m}_{23} e^{i\phi} \\ 0 & \widetilde{m}_{23} e^{-i\phi} & \widetilde{m}_{22} \end{pmatrix}, \tag{7.70}$$

  where $\widetilde{m}_{11}$, $\widetilde{m}_{22}$, $\widetilde{m}_{23}$ and ϕ are real. It is diagonalized by the unitary matrix

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & e^{i\phi} & e^{i\phi} \\ 0 & -1 & 1 \end{pmatrix} . \tag{7.71}$$

The charged lepton masses are

$$
\begin{aligned}
m_{l_1}^2 &= \widetilde{m}_{11}, \\
m_{l_2}^2 &= \widetilde{m}_{22} - \widetilde{m}_{23}, \\
m_{l_3}^2 &= \widetilde{m}_{22} + \widetilde{m}_{23}\,.
\end{aligned} \tag{7.72}
$$

(vi) $G_l = Z_2^{c^{n/2}d^{n/2}} = \left\{1, c^{n/2}d^{n/2}\right\}$, $X_{l\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(bc^\gamma d^\delta)\right\}$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$

  Remnant symmetry leads to the following charged lepton mass matrix

$$
m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & 0 & \widetilde{m}_{13}e^{i\pi\frac{\gamma+\delta}{n}} \\ 0 & \widetilde{m}_{22} & 0 \\ \widetilde{m}_{13}e^{-i\pi\frac{\gamma+\delta}{n}} & 0 & \widetilde{m}_{33} \end{pmatrix}, \tag{7.73}
$$

  where $\widetilde{m}_{11}$, $\widetilde{m}_{13}$, $\widetilde{m}_{22}$ and m_{33} are real parameters. The unitary transformation U_l is of the form

$$
U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}}\cos\theta & 0 & e^{i\pi\frac{\gamma}{n}}\sin\theta \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} & 0 \\ -e^{-i\pi\frac{\delta}{n}}\sin\theta & 0 & e^{-i\pi\frac{\delta}{n}}\cos\theta \end{pmatrix}, \tag{7.74}
$$

  with

$$
\tan 2\theta = \frac{2\widetilde{m}_{13}}{\widetilde{m}_{33} - \widetilde{m}_{11}}\,. \tag{7.75}
$$

  The charged lepton mass eigenvalues are given by

$$
\begin{aligned}
m_{l_1}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{33} - \operatorname{sign}\left((\widetilde{m}_{33} - \widetilde{m}_{11})\cos 2\theta\right)\sqrt{(\widetilde{m}_{33} - \widetilde{m}_{11})^2 + 4\widetilde{m}_{13}^2}\right], \\
m_{l_2}^2 &= \widetilde{m}_{22}, \\
m_{l_3}^2 &= \frac{1}{2}\left[\widetilde{m}_{11} + \widetilde{m}_{33} + \operatorname{sign}\left((\widetilde{m}_{33} - \widetilde{m}_{11})\cos 2\theta\right)\sqrt{(\widetilde{m}_{33} - \widetilde{m}_{11})^2 + 4\widetilde{m}_{13}^2}\right].
\end{aligned} \tag{7.76}
$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(bc^\gamma d^\delta)$

  The non-degeneracy of the charged lepton masses requires $\gamma + \delta = 0 \mod n$ for which the general CP transformation matrix $\rho_{\mathbf{r}}(bc^\gamma d^\delta)$ is symmetric. The charged lepton mass matrix fulfilling the invariant condition in Eq. (7.48) is of the form

$$m_l^\dagger m_l = \begin{pmatrix} \widetilde{m}_{11} & 0 & \widetilde{m}_{13}e^{i\phi} \\ 0 & \widetilde{m}_{22} & 0 \\ \widetilde{m}_{13}e^{-i\phi} & 0 & \widetilde{m}_{11} \end{pmatrix}, \tag{7.77}$$

where $\widetilde{m}_{11}$, $\widetilde{m}_{13}$, $\widetilde{m}_{22}$ and ϕ are real. The charged lepton diagonalization matrix is

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & 0 & e^{i\phi} \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{7.78}$$

Finally the charged lepton masses are

$$\begin{aligned} m_{l_1}^2 &= \widetilde{m}_{11} - \widetilde{m}_{13}, \\ m_{l_2}^2 &= \widetilde{m}_{22}, \\ m_{l_3}^2 &= \widetilde{m}_{11} + \widetilde{m}_{13}. \end{aligned} \tag{7.79}$$

Comparing with phenomenological predictions of $Z_2 \times CP$ in the neutrino sector analysed in section 4.3.2, we see that the diagonalization matrix U_l is of the same form as U_ν provided the residual flavour and residual CP symmetries are the same in the two occasions.

7.1.6 Neutrino mass and diagonalisation matrices for charged-lepton-semidirect models

(i) $G_\nu = K_4^{(c^{n/2}, d^{n/2})}$, $X_{\nu\mathbf{r}} = \left\{\rho_\mathbf{r}(c^\gamma d^\delta)\right\}$

Since the representation matrices of both $c^{n/2}$ and $d^{n/2}$ are diagonal, the light neutrino mass matrix is constrained to be diagonal as well. Including the residual CP symmetry, we find

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & m_{22}e^{2i\pi\frac{\gamma-\delta}{n}} & 0 \\ 0 & 0 & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{7.80}$$

where m_{11}, m_{22} and m_{33} are real parameters. The neutrino diagonalization matrix can be read out

$$U_\nu = \mathrm{diag}\left(e^{i\pi\frac{\gamma}{n}}, e^{-i\pi\frac{\gamma-\delta}{n}}, e^{-i\pi\frac{\delta}{n}}\right) K_\nu\,, \tag{7.81}$$

where K_ν is a diagonal matrix with element ± 1 or $\pm i$ to set the light neutrino masses being positive. The light neutrino masses are

$$m_1 = |m_{11}|\,, \qquad m_2 = |m_{22}|\,, \qquad m_3 = |m_{33}|\;. \tag{7.82}$$

We see that the light neutrino masses depend on only three real parameters, and we would like to stress again that the order of the light neutrino masses can not be fixed here, and therefore U_ν here and henceforth is determined up to column permutations. For other residual CP symmetries $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(bc^\gamma d^\delta)$, $\rho_{\mathbf{r}}(ac^\gamma d^\delta)$, $\rho_{\mathbf{r}}(a^2c^\gamma d^\delta)$, $\rho_{\mathbf{r}}(abc^\gamma d^\delta)$ and $\rho_{\mathbf{r}}(a^2bc^\gamma d^\delta)$ with $\gamma, \delta = 0, 1, \ldots n-1$, the light neutrino masses are partially degenerate such that they are not viable.

(ii) $G_\nu = K_4^{(c^{n/2}, abc^y)}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma}d^{2y+2\gamma})\}$

In this case, the light neutrino mass matrix takes the form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & 0 \\ m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{11}e^{-2i\pi\frac{2y+\gamma}{n}} & 0 \\ 0 & 0 & m_{33}e^{4i\pi\frac{y+\gamma}{n}} \end{pmatrix}, \tag{7.83}$$

where m_{11}, m_{12} and m_{33} are real. It is diagonalized by the unitary matrix U_ν with

$$U_\nu = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}} & 0 \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}} & 0 \\ 0 & 0 & \sqrt{2}e^{-2i\pi\frac{y+\gamma}{n}} \end{pmatrix}. \tag{7.84}$$

The light neutrino masses are given by

$$m_1 = |m_{11} - m_{12}|\,, \qquad m_2 = |m_{11} + m_{12}|\,, \qquad m_3 = |m_{33}|\;. \tag{7.85}$$

For the case of $X_{\nu\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma+n/2}), \rho_{\mathbf{r}}(abc^{y+\gamma}d^{2y+2\gamma+n/2})\right\}$, the light neutrino masses are degenerate, and therefore are not discussed here.

(iii) $G_\nu = K_4^{(d^{n/2}, a^2bd^z)}$, $X_{\nu\mathbf{r}} = \left\{\rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta}d^{z+\delta})\right\}$

The light neutrino mass matrix, which is invariant under both residual flavour and residual CP symmetry, is determined to be

$$m_\nu = \begin{pmatrix} m_{11}e^{-4i\pi\frac{z+\delta}{n}} & 0 & 0 \\ 0 & m_{22}e^{2i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} \\ 0 & m_{23}e^{2i\pi\frac{z+\delta}{n}} & m_{22}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{7.86}$$

where m_{11}, m_{22} and m_{23} are real. The unitary matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}e^{2i\pi\frac{z+\delta}{n}} & 0 & 0 \\ 0 & e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}} \\ 0 & -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}} \end{pmatrix} . \tag{7.87}$$

The light neutrino mass eigenvalues are given by

$$m_1 = |m_{11}| , \qquad m_2 = |m_{22} - m_{23}| , \qquad m_3 = |m_{22} + m_{23}| . \tag{7.88}$$

For the value of $X_{\nu\mathbf{r}} = \left\{\rho_\mathbf{r}(c^{2z+2\delta+n/2}d^\delta), \rho_\mathbf{r}(a^2bc^{2z+2\delta+n/2}d^{z+\delta})\right\}$, the neutrino masses are degenerate.

(iv) $G_\nu = K_4^{(c^{n/2}d^{n/2}, bc^x d^x)}$, $X_{\nu\mathbf{r}} = \{\rho_\mathbf{r}(c^\gamma d^{-2x-\gamma}), \rho_\mathbf{r}(bc^{x+\gamma}d^{-x-\gamma})\}$

In this case, we find the light neutrino mass matrix is of the form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{13}e^{-2i\pi\frac{x+\gamma}{n}} \\ 0 & m_{22}e^{4i\pi\frac{x+\gamma}{n}} & 0 \\ m_{13}e^{-2i\pi\frac{x+\gamma}{n}} & 0 & m_{11}e^{-2i\pi\frac{2x+\gamma}{n}} \end{pmatrix} , \tag{7.89}$$

where m_{11}, m_{13} and m_{22} are real. The unitary matrix U_ν diagonalizing this neutrino mass matrix is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & e^{i\pi\frac{\gamma}{n}} \\ 0 & \sqrt{2}e^{-2i\pi\frac{x+\gamma}{n}} & 0 \\ -e^{i\pi\frac{2x+\gamma}{n}} & 0 & e^{i\pi\frac{2x+\gamma}{n}} \end{pmatrix} . \tag{7.90}$$

Finally the neutrino masses are

$$m_1 = |m_{11} - m_{13}| , \qquad m_2 = |m_{22}| , \qquad m_3 = |m_{11} + m_{13}| . \tag{7.91}$$

For the remaining value of $X_{\nu\mathbf{r}} = \left\{\rho_\mathbf{r}(c^\gamma d^{-2x-\gamma+n/2}), \rho_\mathbf{r}(bc^{x+\gamma}d^{-x-\gamma+n/2})\right\}$, the light neutrino masses are degenerate.

7.1.7 Additional mixing results for charged-lepton-semidirect models

$$
\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{2}\sin^2\theta, \qquad \sin^2\theta_{12} = \frac{1}{2} - \frac{\sqrt{2}\sin 2\theta\cos\varphi_8}{3+\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{2}{3+\cos 2\theta}\,,\\
&|\tan\delta_{CP}| = \left|\frac{(3+\cos 2\theta)\tan\varphi_8}{1+3\cos 2\theta}\right|, \qquad |J_{CP}| = \frac{1}{8\sqrt{2}}\,|\sin 2\theta\sin\varphi_8|\,,\\
&|\tan\alpha_{21}| = \left|\frac{8\sqrt{2}(1+3\cos 2\theta)\sin 2\theta\sin\varphi_8}{7+12\cos 2\theta+13\cos 4\theta+8\sin^2 2\theta\cos 2\varphi_8}\right|,\\
&|\tan\alpha'_{31}| = \left|\frac{\sin^2\theta\sin 2\varphi_9+\sqrt{2}\sin 2\theta\sin(2\varphi_9-\varphi_8)+2\cos^2\theta\sin(2\varphi_9-2\varphi_8)}{\sin^2\theta\cos 2\varphi_9+\sqrt{2}\sin 2\theta\cos(2\varphi_9-\varphi_8)+2\cos^2\theta\cos(2\varphi_9-2\varphi_8)}\right|\,.
\end{aligned}
\tag{7.92}
$$

All mixing parameters depend on θ and φ_8 except $|\tan\alpha'_{31}|$ which involves φ_9 additionally.

7.2 Analysing $\mathbf{\Delta(3n^2)}$ invariant potentials with $\mathbf{n > 3}$ with CP-odd invariants

So far we have considered the finite groups $A_4 = \Delta(3\cdot 2^2)$ and $\Delta(27) = \Delta(3\cdot 3^2)$ which correspond to the first two non-Abelian members of the series $\Delta(3n^2)$ with $n \in \mathbb{N}$. In this section we derive renormalisable potentials which are invariant under $\Delta(3n^2)$ with $n > 3$. The field contents considered are a single triplet of SM singlets, then one triplet of $SU(2)_L$ doublets, then two triplets of SM singlets and finally two triplets of $SU(2)_L$ doublets. Following [156], a triplet of $\Delta(3n^2)$ can be written as $\mathbf{3}_{(k,l)}$, where $k, l = 0, 1, ..., n-1$. The complex conjugate of $\mathbf{3}_{(k,l)}$ is given by $\mathbf{3}_{(-k,-l)}$, which we sometimes denote as $\bar{\mathbf{3}}$, dropping the indices. The cyclic permutation symmetry included in $\Delta(3n^2)$ entails an ambiguity in labelling the *same* triplet representation such that $\mathbf{3}_{(k,l)} = \mathbf{3}_{(l,-k-l)} = \mathbf{3}_{(-k-l,k)}$. With these preliminary remarks, we can determine the product of two identical triplet representations [156]

$$\mathbf{3}_{(k,l)} \otimes \mathbf{3}_{(k,l)} \;=\; [\mathbf{3}_{(2k,2l)} + \mathbf{3}_{(-k,-l)}]_s + [\mathbf{3}_{(-k,-l)}]_a\ . \tag{7.93}$$

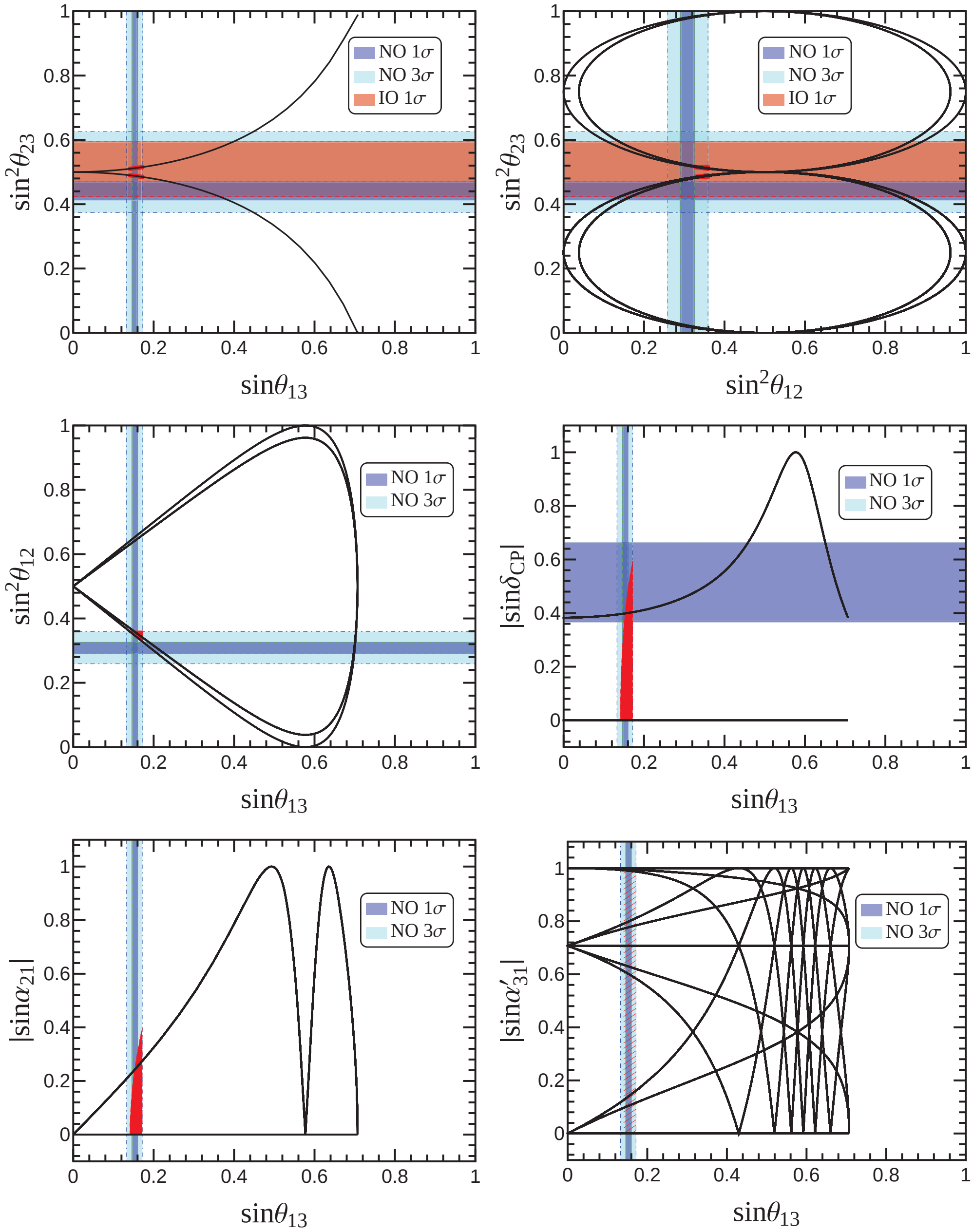


Figure 7.6: The correlations among mixing parameters in case V. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_8 and φ_9 to be continuous (which is equivalent to taking the limit $n \to \infty$) and the three mixing angles are required to lie in their 3σ ranges. Note that the Majorana phase α'_{31} is not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from Ref. [78].

Again the subscripts s and a denote symmetric and antisymmetric combinations. Assuming the original triplet $\mathbf{3}_{(k,l)}$ to be a faithful (and thus irreducible) representation of $\Delta(3n^2)$, all representations on the right-hand side are irreducible for $n \neq 2$. Excluding moreover the case with $n = 3$, the triplets $\mathbf{3}_{(2k,2l)}$ and $\mathbf{3}_{(-k,-l)}$ denote different representations. Throughout this section we adopt the basis of [156, 110].

7.2.1 One flavour triplet

With one triplet field, only the symmetric part of Eq. (7.93) is relevant for constructing quartic terms of the form $\mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$. Considering $n > 3$, the two triplets in the symmetric contraction of Eq. (7.93) are distinct, so that only two independent quartic invariants can be constructed. The renormalisable scalar potential, which is additionally invariant under a $U(1)$ symmetry, thus takes the form

$$V_{\Delta(3n^2)}(\varphi) = V_0(\varphi) \ , \tag{7.94}$$

where the explicit form of $V_0(\varphi)$ is given in Eq. (5.111). This potential always explicitly conserves CP. It is a reduced version of the A_4 symmetric potential $V_{A_4}(\varphi)$ of Eq. (5.113) which generally conserves CP. Therefore it is clear that $V_{\Delta(3n^2)}(\varphi)$ is left invariant under the same CP symmetry, i.e. the one defined with a 2-3 swap, X_{23}. In addition, $V_{\Delta(3n^2)}(\varphi)$ respects the trivial CP symmetry CP_0 (which $V_{A_4}(\varphi)$ in general does not).

7.2.2 One flavour triplet of Higgs doublets

If each component of the faithful $\Delta(3n^2)$ triplet transforms as an $SU(2)_L$ doublet, the corresponding renormalisable potential consists of four independent terms. As described in Section 5.4, the different ways of contracting the $SU(2)_L$ indices entail a doubling of the $\Delta(3n^2)$ invariant term in Eq. (7.94) which is proportional to r. The resulting Higgs potential then takes the form

$$V_{\Delta(3n^2)}(H) = V_0(H) \ , \tag{7.95}$$

with the right-hand side defined in Eq. (5.117). This potential always conserves CP explicitly (for any choice of parameters). Similar to the corresponding A_4

case, $V_{\Delta(3n^2)}(H)$ is left invariant under a CP transformation with a 2-3 swap. Additionally, it also respects the trivial CP symmetry CP_0.

7.2.3 Two flavour triplets

We now turn to the case of two flavour multiplets, φ and φ', in the same faithful triplet representation. The potential can be simplified by imposing individual $U(1)$ symmetries for each of the scalar fields, such that the actual symmetry of the potential is given by $\Delta(3n^2) \times U(1) \times U(1)'$. In addition to the potential of the individual (non-interacting) fields, only mixed terms of the form $\varphi\,\varphi'\,\varphi^*\,\varphi'^*$ are possible; in particular cubic terms are absent. In order to construct the mixed quartic terms, we consider the Kronecker product given in Eq. (7.93), now also including the antisymmetric combination. Multiplying the right-hand side with its complex conjugate, we see that there are five independent mixed quartic $\Delta(3n^2)$ invariants if $n > 3$. The renormalisable potential can be written as follows,

$$V(\varphi, \varphi')_{\Delta(3n^2)} = V_0(\varphi) + V_0'(\varphi') + V_1(\varphi, \varphi') \ , \tag{7.96}$$

where the individual contributions to the right-hand side are defined in Eqs. (5.111) and (5.120).

Unlike the previous $\Delta(3n^2)$ invariant potentials for $n > 3$, this potential generally violates CP, as confirmed by the non-zero CPI $\mathcal{I}_2^{(6)}$ (Eq. (5.76)) which for this potential becomes

$$\mathcal{I}_2^{(6)} = \frac{3}{512} i\tilde{s}_2\tilde{s}_3(-3\tilde{r}_2^2 + \tilde{s}_3^2)(-\tilde{s}_1^2 + \tilde{s}_1\tilde{s}_2 + \tilde{r}_2(-2\tilde{s}_1 + \tilde{s}_2) + \tilde{s}_3^2) \ . \tag{7.97}$$

Imposing the trivial CP symmetry CP_0 entails $\tilde{s}_3 = 0$, whereas the $U_{23}^{\varphi\varphi'}$ 2-3 swap CP symmetry constrains the potential such that $\tilde{s}_2 = 0$. As expected, both CP symmetries enforce $\mathcal{I}_2^{(6)} = 0$ (and make any other CPIs vanish), but they are distinct CP symmetries with distinct effects on the potential.

Inspection of other CPIs reveals that also the factor $(-3\tilde{r}_2^2 + \tilde{s}_3^2)$ is present in each non-vanishing CPI we found. This raises the question if there exists a CP symmetry which is associated with setting this factor to zero. Such a symmetry must relate different terms of the potential in Eq. (7.96), namely

$$\tilde{r}_2\Big(\sum_i \varphi_i\varphi'^{*i}\Big)\Big(\sum_j \varphi'_j\varphi^{*j}\Big) \ + \ i\,\tilde{s}_3\Big[(\varphi_1\varphi'^{*1}\varphi'_2\varphi^{*2} + \text{cycl.}) - (\varphi^{*1}\varphi'_1\varphi'^{*2}\varphi_2 + \text{cycl.})\Big] \ .$$

Clearly, the term proportional to $\tilde{r}_2$ is invariant under a general CP transformation where the unitary matrix X is block diagonal and the blocks are the same for both triplets φ and φ'. Hence, we are led to more general choices with different 3×3 blocks X_φ and $X_{\varphi'}$ for φ and φ', respectively. Pursuing the simple ansatz

$$X^{\varphi\varphi'} = \begin{pmatrix} X_\varphi & 0 \\ 0 & X_{\varphi'} \end{pmatrix}, \quad \text{with} \quad X_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad X_{\varphi'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{7.98}$$

we find that the potential remains invariant under the corresponding general CP transformation if and only if

$$\tilde{s}_3 \;=\; \tilde{r}_2\, i(\omega - \omega^2)\ . \tag{7.99}$$

Inserting $\omega = e^{2\pi i/3}$, we get $\tilde{s}_3 = -\sqrt{3}\tilde{r}_2$ which corresponds to one solution of the quadratic equation $3\tilde{r}_2^2 - \tilde{s}_3^2 = 0$. The other solution, $\tilde{s}_3 = \sqrt{3}\tilde{r}_2$, is related to the CP transformation where the roles of the explicit matrices in Eq. (7.98) are exchanged. Imposing either of the two CP symmetries guarantees that all CPIs vanish.

An example of a larger non-trivial CPI is provided by $\mathcal{I}_1^{(7,2)}$, defined in Eq. (7.241) of Appendix 7.3.6. Explicit evaluation in the parametrisation of Eq. (7.96) yields

$$\begin{aligned}\mathcal{I}_1^{(7,2)} =& \frac{9}{8192} i\tilde{s}_2\tilde{s}_3\left(3\tilde{r}_2^2 - \tilde{s}_3^2\right)\left(m_\varphi^2 - m_{\varphi'}^2\right)^2\left(\tilde{r}_1 + \tilde{r}_2 + \tilde{s}_1\right) \times \Big[16(s^2 + ss' + s'^2) + \\ &+ 8r(2s+s') + 8r'(2s'+s) + \tilde{s}_1^2 + \tilde{s}_2^2 - \tilde{s}_3^2 - \tilde{s}_1\tilde{s}_2 + \tilde{r}_2(2\tilde{s}_1 - \tilde{s}_2)\Big]\ .\end{aligned} \tag{7.100}$$

While this more complicated CPI vanishes for $m_\varphi = m_{\varphi'}$, we already know that such a relation is not a consequence of any CP symmetry because the simpler CPI derived above does not depend on the masses. In other words, any CP symmetry that would relate the masses by $m_\varphi = m_{\varphi'}$ would have to impose additional constraints on the other parameters of the potential.

Having identified the CP symmetries corresponding to the zeros of $\tilde{s}_2\tilde{s}_3\left(3\tilde{r}_2^2 - \tilde{s}_3^2\right)$, one may wonder about the consequences of imposing other CP symmetries on the potential of Eq. (7.96). As an example, one could for instance consider the situation where X is given by the block matrix where X_φ and X'_φ are both given by one of the matrices of Eq. (5.150). A straightforward but somewhat tedious calculation reveals that such a "general" CP symmetry would require vanishing coefficients for all non-$SU(3)$ type terms. In other words $s = s' = \tilde{s}_1 = \tilde{s}_2 = \tilde{s}_3 = 0$. The symmetry of the resulting potential would therefore be enhanced from $\Delta(3n^2)$ to $SU(3)$ in addition to preserving CP.

7.2.4 Two flavour triplets of Higgs doublets

The potential of two triplets of $SU(2)_L$ doublets can be deduced from the potential of two flavour triplets of $SU(2)_L$ singlets. It is a particular case of the corresponding A_4 potential. We therefore write the potential in terms of the expressions defined in Eqs. (5.117) and (5.126),

$$V_{\Delta(3n^2)}(H, H') = V_0(H) + V_0'(H') + V_1(H, H') \ . \tag{7.101}$$

We note again that due to the $SU(2)_L \times U(1)_Y$ gauge group, the potential cannot contain any cubic terms. In fact, each term must have an equal number of Higgs and complex conjugate Higgs fields. Hence it is sufficient to impose e.g. a Z_3 symmetry with non-trivial charge for only one of the two triplets of Higgs doublets in order to enforce the potential of Eq. (7.101). This potential in Eq. (7.101) generally violates CP explicitly. Of the CP-odd invariants calculated, cf. Table 5.1, $\mathcal{I}_2^{(6)}, \mathcal{I}_3^{(6)}, \mathcal{I}_4^{(6)}, \mathcal{I}_5^{(6)}$ (Eqs. (5.76,5.77,5.78,5.79)) are non-zero, but the expressions are too large to display here.

7.2.5 $\mathbf{\Delta(6n^2)}$ invariant potentials with $\mathbf{n > 3}$

Working in the basis of [49, 110], it is straightforward to enhance the symmetry of $\Delta(3n^2)$ invariant potentials to $\Delta(6n^2)$ by imposing extra constraints, see Appendix 7.4.2. With only one flavour triplet φ or H, the renormalisable potentials are automatically symmetric under $\Delta(6n^2)$, i.e.

$$V_{\Delta(6n^2)}(\varphi) = V_{\Delta(3n^2)}(\varphi) = V_0(\varphi) \ , \qquad V_{\Delta(6n^2)}(H) = V_{\Delta(3n^2)}(H) = V_0(H) \ , \tag{7.102}$$

where $V_0(\varphi)$ and $V_0(H)$ are defined in Eqs. (5.111) and (5.117), respectively. With two flavour triplets, it is necessary to impose $\tilde{s}_2 = \tilde{s}_3 = 0$ for $V_{\Delta(6n^2)}(\varphi, \varphi')$ and $\tilde{s}_{21} = \tilde{s}_{22} = \tilde{s}_{31} = \tilde{s}_{32} = 0$ for $V_{\Delta(6n^2)}(H, H')$. Using the definitions of Eqs. (5.136) and (5.140), we then have

$$V_{\Delta(6n^2)}(\varphi, \varphi') = V_0(\varphi) + V_0'(\varphi') + V_2(\varphi, \varphi') \ , \tag{7.103}$$

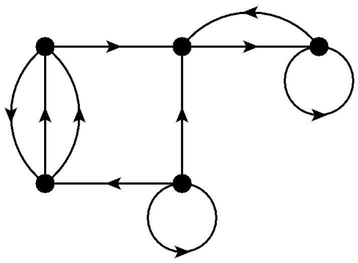
$$V_{\Delta(6n^2)}(H, H') = V_0(H) + V_0'(H') + V_2(H, H') \ . \tag{7.104}$$

All of the above $\Delta(6n^2)$ invariant potentials (with $n > 3$) conserve CP explicitly. For instance, one can easily show that the respective trivial CP transformations CP_0 as well as the respective CP transformations with a 2-3 swap (X_{23}, X_{23}^{H}, $X_{23}^{\varphi\varphi'}$, $X_{23}^{HH'}$) do not constrain the parameters of the potentials as they are all real.

7.3 List of invariants

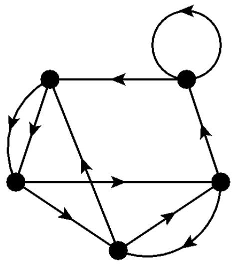
7.3.1 Contraction matrices of $\mathbf{n_Z = 5}$ invariants

$$I_1^{(5)} = Z^{a_1 a_2}_{a_7 a_9} Z^{a_3 a_4}_{a_5 a_{10}} Z^{a_5 a_6}_{a_3 a_6} Z^{a_7 a_8}_{a_4 a_8} Z^{a_9 a_{10}}_{a_1 a_2} =$$



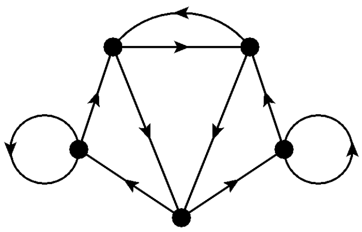
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{7.105}$$

$$I_2^{(5)} = Z^{a_1 a_2}_{a_5 a_7} Z^{a_3 a_4}_{a_8 a_9} Z^{a_5 a_6}_{a_3 a_6} Z^{a_7 a_8}_{a_4 a_{10}} Z^{a_9 a_{10}}_{a_1 a_2} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \tag{7.106}$$

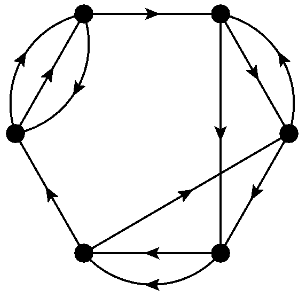
$$I_3^{(5)} = Z^{a_1 a_2}_{a_5 a_9} Z^{a_3 a_4}_{a_3 a_7} Z^{a_5 a_6}_{a_6 a_8} Z^{a_7 a_8}_{a_1 a_{10}} Z^{a_9 a_{10}}_{a_2 a_4} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{7.107}$$

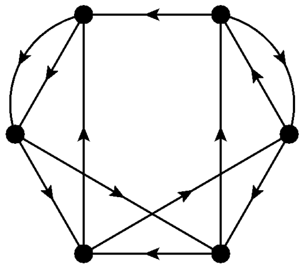
7.3.2 Contraction matrices of $\mathbf{n_Z = 6}$ invariants without Z-self-loops

$$I_1^{(6)} = Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_7a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} =$$



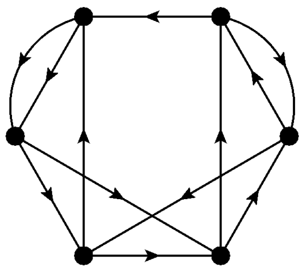
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{7.108}$$

$$I_2^{(6)} = Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_{11}a_6} Z^{a_5a_6}_{a_9a_8} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} =$$



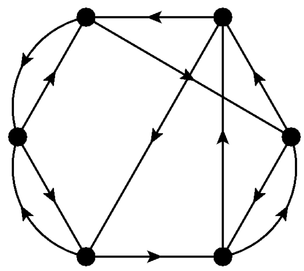
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{7.109}$$

$$I_3^{(6)} = Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_6} Z^{a_5a_6}_{a_{11}a_8} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} =$$



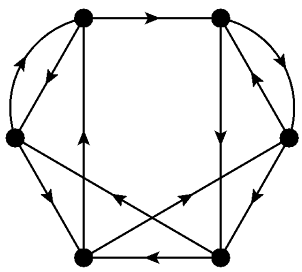
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \tag{7.110}$$

$$I_4^{(6)} = Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_7a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_1a_4} Z^{a_{11}a_{12}}_{a_3a_2} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{7.111}$$

$$I_5^{(6)} = Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_9a_8} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_6} Z^{a_{11}a_{12}}_{a_3a_2} =$$



$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{7.112}$$

7.3.3 $\mathbf{n_Z = 6}$ invariants with self-loops

$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_{10}} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.113}$$
$$Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_5a_{10}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.114}$$
$$Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_7a_8} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.115}$$
$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_7a_{10}} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.116}$$
$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_3a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.117}$$
$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_7a_{10}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.118}$$
$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_9a_{12}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{10}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.119}$$
$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{10}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.120}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_7a_{12}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_{10}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.121}$$
$$Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_9a_8} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_{10}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.122}$$
$$Z^{a_1a_2}_{a_9a_{10}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.123}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_{11}a_{10}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.124}$$
$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_9a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.125}$$
$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_8} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_5a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.126}$$
$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_7a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.127}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_{10}} Z^{a_9a_{10}}_{a_7a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.128}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_7a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.129}$$
$$Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.130}$$
$$Z^{a_1a_2}_{a_5a_{12}} Z^{a_3a_4}_{a_{11}a_{10}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.131}$$
$$Z^{a_1a_2}_{a_5a_{10}} Z^{a_3a_4}_{a_{11}a_6} Z^{a_5a_6}_{a_7a_{12}} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.132}$$
$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.133}$$
$$Z^{a_1a_2}_{a_5a_{10}} Z^{a_3a_4}_{a_7a_{12}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.134}$$
$$Z^{a_1a_2}_{a_5a_{10}} Z^{a_3a_4}_{a_{11}a_8} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_9a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.135}$$
$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_7a_6} Z^{a_5a_6}_{a_5a_{10}} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.136}$$
$$Z^{a_1a_2}_{a_9a_{10}} Z^{a_3a_4}_{a_7a_6} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.137}$$
$$Z^{a_1a_2}_{a_5a_{10}} Z^{a_3a_4}_{a_9a_{12}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.138}$$
$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_9a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.139}$$
$$Z^{a_1a_2}_{a_5a_{10}} Z^{a_3a_4}_{a_7a_{12}} Z^{a_5a_6}_{a_9a_6} Z^{a_7a_8}_{a_3a_8} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.140}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_3a_{12}} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.141}$$
$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_3a_{12}} Z^{a_5a_6}_{a_5a_{10}} Z^{a_7a_8}_{a_7a_6} Z^{a_9a_{10}}_{a_{11}a_4} Z^{a_{11}a_{12}}_{a_1a_2} \tag{7.142}$$
$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_{10}} Z^{a_9a_{10}}_{a_1a_4} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.143}$$
$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_7a_{10}} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_9a_6} Z^{a_9a_{10}}_{a_1a_4} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.144}$$

$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_7a_{10}} Z^{a_5a_6}_{a_5a_8} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_6} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.145}$$

$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_{12}} Z^{a_5a_6}_{a_5a_8} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_6} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.146}$$

$$Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_9a_4} Z^{a_9a_{10}}_{a_1a_8} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.147}$$

$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_6} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_8} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.148}$$

$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_9a_6} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.149}$$

$$Z^{a_1a_2}_{a_9a_{12}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.150}$$

$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.151}$$

$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_4} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_{12}} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.152}$$

$$Z^{a_1a_2}_{a_7a_{12}} Z^{a_3a_4}_{a_9a_6} Z^{a_5a_6}_{a_5a_8} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.153}$$

$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_7a_6} Z^{a_5a_6}_{a_5a_{12}} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.154}$$

$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{12}} Z^{a_5a_6}_{a_7a_6} Z^{a_7a_8}_{a_{11}a_4} Z^{a_9a_{10}}_{a_1a_{10}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.155}$$

$$Z^{a_1a_2}_{a_7a_{10}} Z^{a_3a_4}_{a_9a_4} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_5a_8} Z^{a_9a_{10}}_{a_1a_{12}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.156}$$

$$Z^{a_1a_2}_{a_{11}a_{10}} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_9a_6} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{12}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.157}$$

$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_9a_6} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{12}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.158}$$

$$Z^{a_1a_2}_{a_9a_8} Z^{a_3a_4}_{a_5a_{10}} Z^{a_5a_6}_{a_{11}a_6} Z^{a_7a_8}_{a_7a_4} Z^{a_9a_{10}}_{a_1a_{12}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.159}$$

$$Z^{a_1a_2}_{a_{11}a_8} Z^{a_3a_4}_{a_9a_4} Z^{a_5a_6}_{a_5a_{10}} Z^{a_7a_8}_{a_7a_6} Z^{a_9a_{10}}_{a_1a_{12}} Z^{a_{11}a_{12}}_{a_3a_2} \tag{7.160}$$

$$\tag{7.161}$$

7.3.4 Invariants with $\mathbf{n_Y \neq 0}$ not listed in the main text

$n_Y = 2, n_Z = 3$ invariants with self-loops

$$Y^{a_1}_{a_7} Y^{a_2}_{a_5} Z^{a_3a_4}_{a_3a_8} Z^{a_5a_6}_{a_4a_6} Z^{a_7a_8}_{a_1a_2} \tag{7.162}$$

$$Y^{a_1}_{a_7} Y^{a_2}_{a_5} Z^{a_3a_4}_{a_3a_6} Z^{a_5a_6}_{a_4a_8} Z^{a_7a_8}_{a_1a_2} \tag{7.163}$$

$$Y^{a_1}_{a_5} Y^{a_2}_{a_3} Z^{a_3a_4}_{a_7a_8} Z^{a_5a_6}_{a_4a_6} Z^{a_7a_8}_{a_1a_2} \tag{7.164}$$

$$Y^{a_1}_{a_5} Y^{a_2}_{a_3} Z^{a_3a_4}_{a_4a_7} Z^{a_5a_6}_{a_6a_8} Z^{a_7a_8}_{a_1a_2} \tag{7.165}$$

$$Y^{a_1}_{a_7} Y^{a_2}_{a_3} Z^{a_3a_4}_{a_5a_8} Z^{a_5a_6}_{a_2a_6} Z^{a_7a_8}_{a_1a_4} \tag{7.166}$$

$$Y^{a_1}_{a_5} Y^{a_2}_{a_3} Z^{a_3a_4}_{a_6a_7} Z^{a_5a_6}_{a_2a_4} Z^{a_7a_8}_{a_1a_8} \tag{7.167}$$

$$Y^{a_1}_{a_2} Y^{a_2}_{a_5} Z^{a_3a_4}_{a_7a_8} Z^{a_5a_6}_{a_3a_6} Z^{a_7a_8}_{a_1a_4} \tag{7.168}$$

$$Y^{a_1}_{a_2} Y^{a_2}_{a_5} Z^{a_3a_4}_{a_3a_7} Z^{a_5a_6}_{a_6a_8} Z^{a_7a_8}_{a_1a_4} \tag{7.169}$$

$n_Y = 3, n_Z = 3$ Invariants with self-loops

$$Y^{a_1}_{a_8} Y^{a_2}_{a_6} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_3a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.170}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_6} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_3a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.171}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_7} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_3a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.172}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_4} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_3a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.173}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_4} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_3a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.174}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_3a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.175}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_3a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.176}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_7} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_3a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.177}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_3} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_4a_9} Z^{a_6a_7}_{a_5a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.178}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_3} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_4a_7} Z^{a_6a_7}_{a_5a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.179}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_4a_9} Z^{a_6a_7}_{a_5a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.180}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_4a_7} Z^{a_6a_7}_{a_5a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.181}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_5a_7} Z^{a_8a_9}_{a_1a_2} \tag{7.182}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_7a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.183}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_8a_9} Z^{a_8a_9}_{a_1a_2} \tag{7.184}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_4} Y^{a_3}_{a_3} Z^{a_4a_5}_{a_6a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.185}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_4} Y^{a_3}_{a_3} Z^{a_4a_5}_{a_7a_8} Z^{a_6a_7}_{a_2a_5} Z^{a_8a_9}_{a_1a_9} \tag{7.186}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_3} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_4a_7} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_5} \tag{7.187}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_8} Z^{a_4a_5}_{a_4a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.188}$$
$$Y^{a_1}_{a_8} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_6a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.189}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.190}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_7a_8} Z^{a_6a_7}_{a_2a_5} Z^{a_8a_9}_{a_1a_9} \tag{7.191}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_7} \tag{7.192}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_9} \tag{7.193}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_4} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_2a_8} Z^{a_8a_9}_{a_1a_9} \tag{7.194}$$
$$Y^{a_1}_{a_4} Y^{a_2}_{a_3} Y^{a_3}_{a_6} Z^{a_4a_5}_{a_7a_8} Z^{a_6a_7}_{a_2a_5} Z^{a_8a_9}_{a_1a_9} \tag{7.195}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_2} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_4a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.196}$$
$$Y^{a_1}_{a_6} Y^{a_2}_{a_3} Y^{a_3}_{a_2} Z^{a_4a_5}_{a_4a_8} Z^{a_6a_7}_{a_7a_9} Z^{a_8a_9}_{a_1a_5} \tag{7.197}$$
$$Y^{a_1}_{a_3} Y^{a_2}_{a_6} Y^{a_3}_{a_2} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_4a_7} Z^{a_8a_9}_{a_1a_5} \tag{7.198}$$
$$Y^{a_1}_{a_3} Y^{a_2}_{a_6} Y^{a_3}_{a_2} Z^{a_4a_5}_{a_4a_8} Z^{a_6a_7}_{a_7a_9} Z^{a_8a_9}_{a_1a_5} \tag{7.199}$$

$n_y = 1, n_Z = 4$ Invariants with self-loops

$$Y^{a_1}_{a_8} Z^{a_2a_3}_{a_6a_7} Z^{a_4a_5}_{a_4a_9} Z^{a_6a_7}_{a_2a_5} Z^{a_8a_9}_{a_1a_3} \tag{7.200}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_8a_9} Z^{a_4a_5}_{a_4a_7} Z^{a_6a_7}_{a_2a_5} Z^{a_8a_9}_{a_1a_3} \tag{7.201}$$
$$Y^{a_1}_{a_8} Z^{a_2a_3}_{a_4a_6} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.202}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_8} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.203}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_8} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_3} \tag{7.204}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_7} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_3} \tag{7.205}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_5} Z^{a_4a_5}_{a_8a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.206}$$
$$Y^{a_1}_{a_4} Z^{a_2a_3}_{a_8a_9} Z^{a_4a_5}_{a_5a_6} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.207}$$
$$Y^{a_1}_{a_4} Z^{a_2a_3}_{a_6a_8} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_2a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.208}$$
$$Y^{a_1}_{a_4} Z^{a_2a_3}_{a_6a_8} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_3} \tag{7.209}$$
$$Y^{a_1}_{a_4} Z^{a_2a_3}_{a_6a_7} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_2a_9} Z^{a_8a_9}_{a_1a_3} \tag{7.210}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_8} Z^{a_4a_5}_{a_5a_9} Z^{a_6a_7}_{a_2a_3} Z^{a_8a_9}_{a_1a_7} \tag{7.211}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_4a_8} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_2a_3} Z^{a_8a_9}_{a_1a_9} \tag{7.212}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_2a_8} Z^{a_4a_5}_{a_4a_9} Z^{a_6a_7}_{a_5a_7} Z^{a_8a_9}_{a_1a_3} \tag{7.213}$$
$$Y^{a_1}_{a_6} Z^{a_2a_3}_{a_2a_4} Z^{a_4a_5}_{a_5a_8} Z^{a_6a_7}_{a_7a_9} Z^{a_8a_9}_{a_1a_3} \tag{7.214}$$

7.3.5 Lists of spontaneous CP-odd invariants

$n_v = 1, n_Z = 3$

$$v_{a_4} v^{*a_1} Z^{a_2a_3}_{a_6a_7} Z^{a_4a_5}_{a_2a_5} Z^{a_6a_7}_{a_1a_3} \tag{7.215}$$
$$v_{a_4} v^{*a_1} Z^{a_2a_3}_{a_2a_6} Z^{a_4a_5}_{a_5a_7} Z^{a_6a_7}_{a_1a_3} \tag{7.216}$$

$n_v = 2, n_Z = 3$

$$v_{a_5} v_{a_7} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_6 a_8} Z^{a_5 a_6}_{a_3 a_4} Z^{a_7 a_8}_{a_1 a_2} \tag{7.217}$$
$$v_{a_5} v_{a_7} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_3 a_8} Z^{a_5 a_6}_{a_4 a_6} Z^{a_7 a_8}_{a_1 a_2} \tag{7.218}$$
$$v_{a_5} v_{a_7} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_3 a_6} Z^{a_5 a_6}_{a_4 a_8} Z^{a_7 a_8}_{a_1 a_2} \tag{7.219}$$
$$v_{a_5} v_{a_7} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_3 a_4} Z^{a_5 a_6}_{a_6 a_8} Z^{a_7 a_8}_{a_1 a_2} \tag{7.220}$$
$$v_{a_3} v_{a_5} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_7 a_8} Z^{a_5 a_6}_{a_4 a_6} Z^{a_7 a_8}_{a_1 a_2} \tag{7.221}$$
$$v_{a_3} v_{a_5} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_6 a_7} Z^{a_5 a_6}_{a_4 a_8} Z^{a_7 a_8}_{a_1 a_2} \tag{7.222}$$
$$v_{a_3} v_{a_5} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_4 a_7} Z^{a_5 a_6}_{a_6 a_8} Z^{a_7 a_8}_{a_1 a_2} \tag{7.223}$$
$$v_{a_3} v_{a_7} v^{*a_1} v^{*a_2} Z^{a_3 a_4}_{a_5 a_8} Z^{a_5 a_6}_{a_1 a_6} Z^{a_7 a_8}_{a_2 a_4} \tag{7.224}$$

$n_v = 1, n_Z = 4$

$$v_{a_8} v^{*a_1} Z^{a_2 a_3}_{a_6 a_7} Z^{a_4 a_5}_{a_4 a_9} Z^{a_6 a_7}_{a_2 a_5} Z^{a_8 a_9}_{a_1 a_3} \tag{7.225}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_8 a_9} Z^{a_4 a_5}_{a_4 a_7} Z^{a_6 a_7}_{a_2 a_5} Z^{a_8 a_9}_{a_1 a_3} \tag{7.226}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_7} Z^{a_4 a_5}_{a_8 a_9} Z^{a_6 a_7}_{a_2 a_5} Z^{a_8 a_9}_{a_1 a_3} \tag{7.227}$$
$$v_{a_8} v^{*a_1} Z^{a_2 a_3}_{a_4 a_6} Z^{a_4 a_5}_{a_5 a_9} Z^{a_6 a_7}_{a_2 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.228}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_8} Z^{a_4 a_5}_{a_5 a_9} Z^{a_6 a_7}_{a_2 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.229}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_8} Z^{a_4 a_5}_{a_5 a_7} Z^{a_6 a_7}_{a_2 a_9} Z^{a_8 a_9}_{a_1 a_3} \tag{7.230}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_7} Z^{a_4 a_5}_{a_5 a_8} Z^{a_6 a_7}_{a_2 a_9} Z^{a_8 a_9}_{a_1 a_3} \tag{7.231}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_5} Z^{a_4 a_5}_{a_8 a_9} Z^{a_6 a_7}_{a_2 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.232}$$
$$v_{a_4} v^{*a_1} Z^{a_2 a_3}_{a_8 a_9} Z^{a_4 a_5}_{a_5 a_6} Z^{a_6 a_7}_{a_2 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.233}$$
$$v_{a_4} v^{*a_1} Z^{a_2 a_3}_{a_6 a_8} Z^{a_4 a_5}_{a_5 a_9} Z^{a_6 a_7}_{a_2 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.234}$$
$$v_{a_4} v^{*a_1} Z^{a_2 a_3}_{a_6 a_8} Z^{a_4 a_5}_{a_5 a_7} Z^{a_6 a_7}_{a_2 a_9} Z^{a_8 a_9}_{a_1 a_3} \tag{7.235}$$
$$v_{a_4} v^{*a_1} Z^{a_2 a_3}_{a_6 a_7} Z^{a_4 a_5}_{a_5 a_8} Z^{a_6 a_7}_{a_2 a_9} Z^{a_8 a_9}_{a_1 a_3} \tag{7.236}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_8} Z^{a_4 a_5}_{a_5 a_9} Z^{a_6 a_7}_{a_2 a_3} Z^{a_8 a_9}_{a_1 a_7} \tag{7.237}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_4 a_8} Z^{a_4 a_5}_{a_5 a_7} Z^{a_6 a_7}_{a_2 a_3} Z^{a_8 a_9}_{a_1 a_9} \tag{7.238}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_2 a_8} Z^{a_4 a_5}_{a_4 a_9} Z^{a_6 a_7}_{a_5 a_7} Z^{a_8 a_9}_{a_1 a_3} \tag{7.239}$$
$$v_{a_6} v^{*a_1} Z^{a_2 a_3}_{a_2 a_4} Z^{a_4 a_5}_{a_5 a_8} Z^{a_6 a_7}_{a_7 a_9} Z^{a_8 a_9}_{a_1 a_3} \tag{7.240}$$

7.3.6 Larger CP-odd invariants

In addition to the smaller invariants discussed previously, we also found some larger CPIs with up to 9 Z tensors. Using the notation established in Eq. (5.40), we show only half of the CPI , which is sufficient to uniquely define it.

$$I_1^{(7,2)} = Z^{b_3 b_4}_{b_1 b_2}\, Z^{b_1 a_5}_{b_3 a_1}\, Z^{b_2 a_6}_{b_4 a_2}\, Z^{c_3 c_4}_{c_1 c_2}\, Z^{c_1 a'_1}_{c_3 a_3}\, Z^{c_2 a_2}_{c_4 a_4}\, Z^{a'_3 a_4}_{a_5 a_6}\, Y^{a_1}_{a'_1}\, Y^{a_3}_{a'_3}\,, \tag{7.241}$$

$$I_1^{(8)} = Z^{b_3 b_4}_{b_1 b_2}\, Z^{b_1 b_6}_{b_3 b_5}\, Z^{b_2 a_5}_{b_4 a_1}\, Z^{b_5 a_6}_{b_6 a_2}\, Z^{c_3 c_4}_{c_1 c_2}\, Z^{c_1 a_1}_{c_3 a_3}\, Z^{c_2 a_2}_{c_4 a_4}\, Z^{a_3 a_4}_{a_5 a_6}\,, \tag{7.242}$$

$$I_1^{(9)} = Z^{b_3 b_4}_{b_1 b_2}\, Z^{b_1 b_6}_{b_3 b_5}\, Z^{b_2 b_8}_{b_4 b_7}\, Z^{b_5 a_5}_{b_6 a_1}\, Z^{b_7 a_6}_{b_8 a_2}\, Z^{c_3 c_4}_{c_1 c_2}\, Z^{c_1 a_1}_{c_3 a_3}\, Z^{c_2 a_2}_{c_4 a_4}\, Z^{a_3 a_4}_{a_5 a_6}\,. \tag{7.243}$$

The respective CPIs $\mathcal{I}_1^{(7,2)}$, $\mathcal{I}_1^{(8)}$, $\mathcal{I}_1^{(9)}$ can be obtained by subtracting from the I above the I^* obtained by swapping the upper and lower indices, as described in general in Section 5.2.2.

7.4 More group theory of $\Delta(6n^2)$

$\Delta(6n^2)$ is non-abelian finite subgroup of $SU(3)$. The $\Delta(6n^2)$ is isomorphic to $(Z_n \times Z_n) \rtimes S_3$, where S_3 is isomorphic to $Z_3 \rtimes Z_2$, and it can be conveniently defined by four generators a, b, c and d obeying the relations [49]:

$$\begin{gathered} a^3 = b^2 = (ab)^2 = 1, \\ c^n = d^n = 1, \qquad cd = dc\,, \\ aca^{-1} = c^{-1}d^{-1}, \quad ada^{-1} = c, \quad bcb^{-1} = d^{-1}, \quad bdb^{-1} = c^{-1}\,. \end{gathered} \tag{7.244}$$

The elements a and b are the generators of S_3 while c and d generate $Z_n \times Z_n$, and the last line defines the semidirect product " $\rtimes$ ". Note that the generator $d = bc^{-1}b^{-1}$ is not independent. All the group elements can be written into the form

$$g = a^\alpha b^\beta c^\gamma d^\delta\,, \tag{7.245}$$

where $\alpha = 0, 1, 2$, $\beta = 0, 1$, $\gamma, \delta = 0, 1, 2, \ldots n-1$. In the following we list the elements of $\Delta(6n^2)$ by order of the generated cyclic subgroup.

- Elements of order 2, if n even:

$$c^{n/2}, d^{n/2}, c^{n/2}d^{n/2} \tag{7.246}$$

- Elements of order 2, always:

$$bc^{\epsilon}d^{\epsilon}, abc^{\gamma}, a^2bd^{\delta} \tag{7.247}$$

with $\epsilon, \gamma, \delta = 0, \ldots, n-1$.

- Elements of order 3, if 3 divides n:

$$c^{n/3}, d^{n/3}, \ldots \tag{7.248}$$

where the dots indicate all possible products and powers of the two first elements.

- Elements of order 3, always:

$$ac^{\gamma}d^{\delta}, a^2c^{\gamma}d^{\delta} \tag{7.249}$$

with $\gamma, \delta = 0, \ldots, n-1$

- Elements of order m where m divides n, if m and n are even:

$$bc^{\delta+2kn/m}d^{\delta}, abc^{\gamma}d^{2kn/m}, a^2bc^{2kn/m}d^{\delta} \tag{7.250}$$

with $\gamma, \delta = 0, \ldots, n-1$ and $0 \leq k \leq m/2$

- Elements of order m where m divides n, always:

$$c^{kn/m}d^{ln/m} \tag{7.251}$$

with $k, l = 0, \ldots, n-1$.

The $\Delta(6n^2)$ group have been thoroughly studied in Ref. [49]. In the following, we shall review the basic aspects, which is relevant to our present work. The conjugacy classes of $\Delta(6n^2)$ group are of the following forms:

- $n \neq 3\,\mathbb{Z}$

$$1 \;:\; 1C_1 = \{1\}\,, \tag{7.252a}$$

$$n-1 \;:\; 3C_1^{(\rho)} = \left\{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\right\}, \quad \rho = 1,2,...,n-1, \tag{7.252b}$$

$$\frac{n^2-3n+2}{6} \;:\; 6C_1^{(\rho,\sigma)} = \left\{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\right\}, \tag{7.252c}$$

$$1 \;:\; 2n^2 C_2 = \left\{ac^z d^y,\; a^2 c^{-y} d^{-z} | z, y = 0,1,\ldots,n-1\right\}, \tag{7.252d}$$

$$n : 3nC_3^{(\rho)} = \left\{bc^{\rho+x} d^x,\; a^2 bc^{-\rho} d^{-x-\rho},\; abc^{-x} d^\rho | x = 0,1,\ldots,n-1\right\}, \rho = 0,1,\ldots,n-1. \tag{7.252e}$$

The convention used here is that the quantity left of the colon is the number of classes of the kind on the right of the colon. In Eq. (7.252c), the parameter $\rho, \sigma = 0,1,...,n-1$, but excluding possibilities given by

$$\rho + \sigma = \; 0 \bmod n, \quad 2\rho - \sigma = \; 0 \bmod n, \quad \rho - 2\sigma = \; 0 \bmod n\,. \tag{7.253}$$

- $n = 3\,\mathbb{Z}$

$$1 \;:\; 1C_1 = \{1\}\,, \tag{7.254a}$$

$$2 \;:\; 1C_1^{(\nu)} = \left\{c^\nu d^{2\nu}\right\}, \quad \nu = \tfrac{n}{3}, \tfrac{2n}{3}, \tag{7.254b}$$

$$n-3 \;:\; 3C_1^{(\rho)} = \left\{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\right\}, \quad \rho \neq \tfrac{n}{3}, \tfrac{2n}{3}, \tag{7.254c}$$

$$\frac{n^2-3n+6}{6} \;:\; 6C_1^{(\rho,\sigma)} = \left\{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\right\}, \tag{7.254d}$$

$$3 : \frac{2n^2}{3} C_2^{(\tau)} = \{ac^{\tau-y-3x} d^y, a^2 c^{-y} d^{y+3x-\tau} | y = 0,1,...,n-1, x = 0,1,\ldots,\tfrac{n-3}{3}\}, \tau = 0,1,2\,,$$
$$n : 3nC_3^{(\rho)} = \left\{bc^{\rho+x} d^x, a^2 bc^{-\rho} d^{-x-\rho}, abc^{-x} d^\rho | x = 0,1,\ldots,n-1\right\}, \rho = 0,1,\ldots,n-1. \tag{7.254e}$$

In Eq. (7.254d), $\rho, \sigma = 0,1,...,n-1$, again excluding possibilities given by Eq. (7.253).

The irreducible representations and their representation matrices of the $\Delta(6n^2)$ group are as follows [49]:

(i) $n \neq 3\,\mathbb{Z}$

- One-dimensional representations

$$\mathbf{1_1} \;:\; a = b = c = d = 1, \tag{7.255a}$$

$$\mathbf{1_2} \;:\; a = c = d = 1,\; b = -1, \tag{7.255b}$$

- Two-dimensional representation

$$\mathbf{2}: a = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7.256)$$

which is related to the basis chosen in Ref. [49] by a unitary transformation U with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \quad (7.257)$$

In our new basis, all the Clebsch-Gordan (CG) coefficients are real, as is shown in the Appendix 7.4.1. Hence our basis is the so-called the "CP" basis. The conventional CP transformation $\varphi \to \varphi^*$ can be consistently imposed onto the theory in our basis, and all the coupling constant would be constrained to be real.

- Three-dimensional representations

$$\mathbf{3}_{1,k}: a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (7.258a)$$

$$\mathbf{3}_{2,k}: a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ b = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (7.258b)$$

where $\eta \equiv e^{2\pi i/n}$ and $k = 1, 2, \ldots n-1$.

- Six-dimensional representations

$$\mathbf{6}_{\widetilde{(k,l)}}: a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad (7.259)$$

with

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7.260}$$

$$c_1 = d_2^{-1} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-l-k} \end{pmatrix}, \qquad c_2 = d_1^{-1} = \begin{pmatrix} \eta^{l+k} & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix} \tag{7..261}$$

Here $\widetilde{\ }$ denotes the mapping

$$\widetilde{\begin{pmatrix} k \\ l \end{pmatrix}} \longmapsto \text{either} \begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} -k-l \\ k \end{pmatrix}, \begin{pmatrix} l \\ -k-l \end{pmatrix}, \begin{pmatrix} -l \\ -k \end{pmatrix}, \begin{pmatrix} k+l \\ -l \end{pmatrix}, \text{or} \begin{pmatrix} -k \\ k+l \end{pmatrix}, \tag{7.262}$$

$k, l = 0, 1, \dots n-1$, and the following cases are forbidden.

$$l = 0, \quad k = 0, \quad k + l = 0 \bmod n\,. \tag{7.263}$$

(ii) $n = 3\,\mathbb{Z}$

- One-dimensional representations

$$\mathbf{1_1} \;:\; a = b = c = d = 1, \tag{7.264a}$$

$$\mathbf{1_2} \;:\; a = c = d = 1,\ b = -1, \tag{7.264b}$$

- Two-dimensional representation

$$\mathbf{2_1} : a = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.265a}$$

$$\mathbf{2_2} : a = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \tag{7.265b}$$

$$\mathbf{2_3} : a = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \tag{7.265c}$$

$$\mathbf{2_4} : a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \tag{7.265d}$$

They are related to the representation matrices of Ref. [49] by the unitary transformation U in Eq. (7.257).

- Three-dimensional representations

$$\mathbf{3}_{1,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \; b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \; c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \; d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix} , \tag{7.266a}$$

$$\mathbf{3}_{2,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \; b = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \; c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \; d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix} , \tag{7.266b}$$

where $k = 1, 2, \ldots n-1$.

- Six-dimensional representations

$$\mathbf{6}_{\widetilde{(k,l)}} \; : \; a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} , \quad b = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix} , \quad c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} , \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} . \tag{7.267}$$

The 3×3 unitary matrices $a_{1,2}$, $c_{1,2}$ and $d_{1,2}$ are given in Eq. (7.260). Again the following values are prohibited:

$$l = 0, \quad k = 0, \quad k = l = n/3, \quad k = l = 2n/3, \quad k + l = 0 \bmod n \, . \tag{7.268}$$

7.4.1 Clebsch-Gordan coefficients for $\Delta(6n^2)$ group with $n \neq 3\mathbb{Z}$

In the following, we shall present all the CG coefficients in the form of $x \otimes y$ in our chosen basis, x_i denotes the element of the left base vectors x, and y_i is the element of the right base vectors y. We shall see explicitly that all the CG coefficients are real.

- $\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{2}$

$$\mathbf{2} \sim \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}, \quad \mathbf{1_1} \sim x_1y_1 + x_2y_2, \quad \mathbf{1_2} \sim x_1y_2 - x_2y_1\,. \tag{7.269}$$

- $\mathbf{2} \otimes \mathbf{3}_{1,k} = \mathbf{3}_{1,k} \oplus \mathbf{3}_{2,k}$

$$\mathbf{3}_{1,k} \sim \begin{pmatrix} \left(x_1 - \sqrt{3}\,x_2\right) y_1 \\ -2x_1y_2 \\ \left(x_1 + \sqrt{3}\,x_2\right) y_3 \end{pmatrix}, \quad \mathbf{3}_{2,k} \sim \begin{pmatrix} \left(\sqrt{3}\,x_1 + x_2\right) y_1 \\ -2x_2y_2 \\ \left(-\sqrt{3}\,x_1 + x_2\right) y_3 \end{pmatrix}. \tag{7.270}$$

- $\mathbf{2} \otimes \mathbf{3}_{2,k} = \mathbf{3}_{1,k} \oplus \mathbf{3}_{2,k}$

$$\mathbf{3}_{1,k} \sim \begin{pmatrix} \left(\sqrt{3}\,x_1 + x_2\right) y_1 \\ -2x_2y_2 \\ \left(-\sqrt{3}\,x_1 + x_2\right) y_3 \end{pmatrix}, \quad \mathbf{3}_{2,k} \sim \begin{pmatrix} \left(x_1 - \sqrt{3}\,x_2\right) y_1 \\ -2x_1y_2 \\ \left(x_1 + \sqrt{3}\,x_2\right) y_3 \end{pmatrix}. \tag{7.271}$$

- $\mathbf{2} \otimes \mathbf{6}_{(k,l)} = \mathbf{6}_{(k,l)} \oplus \mathbf{6}_{(k,l)}$

$$\mathbf{6}_{(k,l)} \sim \begin{pmatrix} \left(\sqrt{3}\,x_1 + x_2\right) y_1 \\ -2x_2y_2 \\ \left(-\sqrt{3}\,x_1 + x_2\right) y_3 \\ \left(\sqrt{3}\,x_1 - x_2\right) y_4 \\ 2x_2y_5 \\ -\left(\sqrt{3}\,x_1 + x_2\right) y_6 \end{pmatrix}, \quad \mathbf{6}_{(k,l)} \sim \begin{pmatrix} 2x_2y_1 \\ \left(\sqrt{3}\,x_1 - x_2\right) y_2 \\ -\left(\sqrt{3}\,x_1 + x_2\right) y_3 \\ -2x_2y_4 \\ \left(\sqrt{3}\,x_1 + x_2\right) y_5 \\ \left(-\sqrt{3}\,x_1 + x_2\right) y_6 \end{pmatrix}. \tag{7.272}$$

- $\mathbf{3}_{1,l} \otimes \mathbf{3}_{1,l'} = \mathbf{3}_{1,l+l'} \oplus \mathbf{6}_{\widetilde{(l,-l')}}$

$$\mathbf{3}_{1,l+l'} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_2 \\ x_2y_1 \\ x_1y_3 \end{pmatrix}, \tag{7.273}$$

- $\mathbf{3}_{1,l} \otimes \mathbf{3}_{2,l'} = \mathbf{3}_{2,l+l'} \oplus \mathbf{6}_{\widetilde{(l,-l')}}$

$$\mathbf{3}_{2,l+l'} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_2 \\ -x_2y_1 \\ -x_1y_3 \end{pmatrix}, \tag{7.274}$$

- $\mathbf{3}_{1,l} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\widetilde{\begin{pmatrix} k' \\ l'-l \end{pmatrix}}} \oplus \mathbf{6}_{\widetilde{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}}} \oplus \mathbf{6}_{\widetilde{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}}$

$$\mathbf{6}_{\begin{pmatrix} l'-l \\ l-k'-l' \end{pmatrix}} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_3y_6 \\ x_2y_4 \\ x_1y_5 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_3y_4 \\ x_2y_5 \\ x_1y_6 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} -l-k'-l' \\ l+k' \end{pmatrix}} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_5 \\ x_2y_6 \\ x_1y_4 \end{pmatrix}. \tag{7.275}$$

- $\mathbf{3}_{2,l} \otimes \mathbf{3}_{2,l'} = \mathbf{3}_{1,l+l'} \oplus \mathbf{6}_{\widetilde{(l,-l')}}$

$$\mathbf{3}_{1,l+l'} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_2 \\ x_2y_1 \\ x_1y_3 \end{pmatrix}. \tag{7.276}$$

- $\mathbf{3}_{2,l} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\widetilde{\begin{pmatrix} k' \\ l'-l \end{pmatrix}}} \oplus \mathbf{6}_{\widetilde{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}}} \oplus \mathbf{6}_{\widetilde{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}}$

$$\mathbf{6}_{\begin{pmatrix} l'-l \\ l-k'-l' \end{pmatrix}} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ -x_3y_6 \\ -x_2y_4 \\ -x_1y_5 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ -x_3y_4 \\ -x_2y_5 \\ -x_1y_6 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} -l-k'-l' \\ l+k' \end{pmatrix}} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_5 \\ -x_2y_6 \\ -x_1y_4 \end{pmatrix}. \tag{7.277}$$

- $\mathbf{6}_{(k,l)} \otimes \mathbf{6}_{(k',l')} = \sum_{p,s} \mathbf{6}_{\widetilde{\left(\begin{pmatrix} k \\ l \end{pmatrix} + M_s^p \begin{pmatrix} k' \\ l' \end{pmatrix}\right)}}$

$$\mathbf{6}_{\left(\begin{smallmatrix} k+k' \\ l+l' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_4y_4 \\ x_5y_5 \\ x_6y_6 \end{pmatrix}, \quad \mathbf{6}_{\left(\begin{smallmatrix} k-k'-l' \\ l+k' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_4y_5 \\ x_5y_6 \\ x_6y_4 \end{pmatrix}, \quad \mathbf{6}_{\left(\begin{smallmatrix} k+l' \\ l-l'-k' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_4y_6 \\ x_5y_4 \\ x_6y_5 \end{pmatrix},$$
$$\mathbf{6}_{\left(\begin{smallmatrix} k-k' \\ l+k'+l' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_4 \\ x_2y_6 \\ x_3y_5 \\ x_4y_1 \\ x_5y_3 \\ x_6y_2 \end{pmatrix}, \quad \mathbf{6}_{\left(\begin{smallmatrix} k+k'+l' \\ l-l' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_5 \\ x_2y_4 \\ x_3y_6 \\ x_4y_2 \\ x_5y_1 \\ x_6y_3 \end{pmatrix}, \quad \mathbf{6}_{\left(\begin{smallmatrix} k-l' \\ l-k' \end{smallmatrix}\right)} \sim \begin{pmatrix} x_1y_6 \\ x_2y_5 \\ x_3y_4 \\ x_4y_3 \\ x_5y_2 \\ x_6y_1 \end{pmatrix}. \tag{7.278}$$

For the case of $n = 3\mathbb{Z}$, the CG-coefficients can be calculated although it is somewhat lengthy. Part of the CG coefficients are complex numbers in our chosen basis, the explicit form would not be reported here since general CP transformations can not be consistently defined in generic settings based on such groups unless the doublet representations $\mathbf{2_2}$, $\mathbf{2_3}$ and $\mathbf{2_4}$ are not introduced in a specific model.

7.4.2 $\mathbf{\Delta(6n^2)}$ potentials as particular cases of $\mathbf{\Delta(3n^2)}$ potentials

The triplet generators of $\Delta(6n^2)$ with $n \in \mathbb{N}$ are [49]

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad b = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.279}$$

where $\eta = e^{2\pi i/n}$ and $l \in \mathbb{N}$. If the field transforms as a faithful triplet, any c-invariant operator $\mathcal{O}$ will also be invariant under the phase transformation[3]

$$c_0 = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{7.280}$$

and therefore also under

$$c_0^{-l} = \begin{pmatrix} \eta^{-l} & 0 & 0 \\ 0 & \eta^{l} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad c_0^{k} = \begin{pmatrix} \eta^{k} & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7.281}$$

Imposing additionally invariance under a, we quickly find that the operator $\mathcal{O}$ is also invariant under

$$a c_0^{-l} a^2 = \begin{pmatrix} \eta^{l} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix} \qquad \text{and} \qquad a^2 c_0^{k} a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^{k} & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix} . \tag{7.282}$$

As a result, the operator $\mathcal{O}$ is symmetric under the successive application of $a c_0^{-l} a^2$ and $a^2 c_0^k a$, i.e.

$$\begin{pmatrix} \eta^{l} & 0 & 0 \\ 0 & \eta^{k} & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix} . \tag{7.283}$$

Demanding invariance under a and c of Eq. (7.279) therefore leads to the set of $\Delta(3n^2)$ invariant operators where the triplet generators are given by [156]

$$a' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \qquad c' = \begin{pmatrix} \eta^{l} & 0 & 0 \\ 0 & \eta^{k} & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix} . \tag{7.284}$$

We thus conclude that the $\Delta(6n^2)$ symmetric potential can be deduced from the $\Delta(3n^2)$ invariant potential by simply dropping all terms which are *not* symmetric under b of Eq. (7.279). Therefore, in each of these cases it is sufficient to use the already obtained expressions for the CPIs and set constraints on the coefficients to make all the terms in the potential invariant under the b generator.

[3]For faithful representations, l and n have to be coprime. As a consequence, there must be an integer p such that $c^p = c_0$.

References

[1] S. F. King, T. Neder and A. J. Stuart, Phys. Lett. B **726** (2013) 312 doi:10.1016/j.physletb.2013.08.052 [arXiv:1305.3200 [hep-ph]].

[2] S. F. King and T. Neder, Phys. Lett. B **736** (2014) 308 doi:10.1016/j.physletb.2014.07.043 [arXiv:1403.1758 [hep-ph]].

[3] T. Neder, J. Phys. Conf. Ser. **598** (2015) no.1, 012025 doi:10.1088/1742-6596/598/1/012025 [arXiv:1405.0019 [hep-ph]].

[4] G. J. Ding, S. F. King and T. Neder, JHEP **1412** (2014) 007 doi:10.1007/JHEP12(2014)007 [arXiv:1409.8005 [hep-ph]].

[5] T. Neder, J. Phys. Conf. Ser. **631** (2015) no.1, 012019 doi:10.1088/1742-6596/631/1/012019 [arXiv:1503.09041 [hep-ph]].

[6] I. de Medeiros Varzielas, S. F. King, C. Luhn and T. Neder, arXiv:1603.06942 [hep-ph].

[7] Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81** (1998) 1562 doi:10.1103/PhysRevLett.81.1562 [hep-ex/9807003].

[8] S. L. Glashow, Nucl. Phys. **22** (1961) 579. doi:10.1016/0029-5582(61)90469-2

[9] S. Weinberg, Phys. Rev. Lett. **19** (1967) 1264. doi:10.1103/PhysRevLett.19.1264

[10] E. P. Wigner, Annals Math. **40** (1939) 149 [Nucl. Phys. Proc. Suppl. **6** (1989) 9]. doi:10.2307/1968551

[11] S. R. Coleman and J. Mandula, Phys. Rev. **159** (1967) 1251. doi:10.1103/PhysRev.159.1251

[12] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B **88** (1975) 257. doi:10.1016/0550-3213(75)90279-5

[13] P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **594** (2016) A13 doi:10.1051/0004-6361/201525830 [arXiv:1502.01589 [astro-ph.CO]].

[14] J. M. Cline, hep-ph/0609145.

[15] A. D. Sakharov, Pisma Zh. Eksp. Teor. Fiz. **5** (1967) 32 [JETP Lett. **5** (1967) 24] [Sov. Phys. Usp. **34** (1991) 392] [Usp. Fiz. Nauk **161** (1991) 61]. doi:10.1070/PU1991v034n05ABEH002497

[16] G. Luders, Kong. Dan. Vid. Sel. Mat. Fys. Med. **28N5** (1954) 1.

[17] S. Weinberg,

[18] M. E. Peskin and D. V. Schroeder, Reading, USA: Addison-Wesley (1995) 842 p

[19] C. Giunti and C. W. Kim, Oxford, UK: Univ. Pr. (2007) 710 p

[20] G. C. Branco, L. Lavoura and J. P. Silva, Int. Ser. Monogr. Phys. **103** (1999) 1.

[21] O. Eberhardt, G. Herbert, H. Lacker, A. Lenz, A. Menzel, U. Nierste and M. Wiebusch, Phys. Rev. Lett. **109** (2012) 241802 doi:10.1103/PhysRevLett.109.241802 [arXiv:1209.1101 [hep-ph]].

[22] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531. doi:10.1103/PhysRevLett.10.531

[23] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652. doi:10.1143/PTP.49.652

[24] S. F. King and P. O. Ludl, JHEP **1606** (2016) 147 doi:10.1007/JHEP06(2016)147 [arXiv:1605.01683 [hep-ph]].

[25] S. F. King and C. Luhn, Rept. Prog. Phys. **76** (2013) 056201 doi:10.1088/0034-4885/76/5/056201 [arXiv:1301.1340 [hep-ph]].

[26] W. Grimus and P. O. Ludl, J. Phys. A **43** (2010) 445209 doi:10.1088/1751-8113/43/44/445209 [arXiv:1006.0098 [hep-ph]].

[27] W. Grimus and P. O. Ludl, J. Phys. A **45** (2012) 233001 doi:10.1088/1751-8113/45/23/233001 [arXiv:1110.6376 [hep-ph]].

[28] W. Grimus and P. O. Ludl, J. Phys. A **47** (2014) no.7, 075202 doi:10.1088/1751-8113/47/7/075202 [arXiv:1310.3746 [math-ph]].

[29] P. O. Ludl, J. Phys. A **43** (2010) 395204 Erratum: [J. Phys. A **44** (2011) 139501] doi:10.1088/1751-8113/44/13/139501, 10.1088/1751-8113/43/39/395204 [arXiv:1006.1479 [math-ph]].

[30] F. Feruglio, C. Hagedorn and R. Ziegler, JHEP **1307** (2013) 027 doi:10.1007/JHEP07(2013)027 [arXiv:1211.5560 [hep-ph]].

[31] F. P. An *et al.* [Daya Bay Collaboration], Phys. Rev. Lett. **108** (2012) 171803 doi:10.1103/PhysRevLett.108.171803 [arXiv:1203.1669 [hep-ex]].

[32] J. K. Ahn *et al.* [RENO Collaboration], Phys. Rev. Lett. **108** (2012) 191802 doi:10.1103/PhysRevLett.108.191802 [arXiv:1204.0626 [hep-ex]].

[33] Y. Abe *et al.* [Double Chooz Collaboration], Phys. Lett. B **723** (2013) 66 doi:10.1016/j.physletb.2013.04.050 [arXiv:1301.2948 [hep-ex]].

[34] C. S. Lam, Phys. Rev. D **78** (2008) 073015 doi:10.1103/PhysRevD.78.073015 [arXiv:0809.1185 [hep-ph]].

[35] C. S. Lam, Phys. Rev. Lett. **101** (2008) 121602 doi:10.1103/PhysRevLett.101.121602 [arXiv:0804.2622 [hep-ph]].

[36] J. Beringer *et al.* [Particle Data Group Collaboration], Phys. Rev. D **86** (2012) 010001. doi:10.1103/PhysRevD.86.010001

[37] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, Phys. Lett. B **703** (2011) 447 doi:10.1016/j.physletb.2011.08.013 [arXiv:1107.3486 [hep-ph]].

[38] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, Nucl. Phys. B **858** (2012) 437 doi:10.1016/j.nuclphysb.2012.01.017 [arXiv:1112.1340 [hep-ph]].

[39] G. J. Ding, Nucl. Phys. B **862** (2012) 1 doi:10.1016/j.nuclphysb.2012.04.002 [arXiv:1201.3279 [hep-ph]].

[40] S. F. King, C. Luhn and A. J. Stuart, Nucl. Phys. B **867** (2013) 203 doi:10.1016/j.nuclphysb.2012.09.021 [arXiv:1207.5741 [hep-ph]].

[41] C. S. Lam, Phys. Rev. D **87** (2013) no.1, 013001 doi:10.1103/PhysRevD.87.013001 [arXiv:1208.5527 [hep-ph]].

[42] C. S. Lam, Phys. Rev. D **87** (2013) no.5, 053012 doi:10.1103/PhysRevD.87.053012 [arXiv:1301.1736 [hep-ph]].

[43] M. Holthausen, K. S. Lim and M. Lindner, Phys. Lett. B **721** (2013) 61 doi:10.1016/j.physletb.2013.02.047 [arXiv:1212.2411 [hep-ph]].

[44] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008. http://www.gap-system.org.

[45] V. Dabbaghian, *REPSN - for constructing representations of finite groups, GAP package, Version 3.0.2*. The GAP Group, 2011. http://www.gap-system.org/Packages/repsn.html.

[46] H.U.Besche, B.Eick, and E.O'Brien, *SmallGroups - library of all 'small' groups, GAP package, Version included in GAP 4.4.12*. The GAP Group, 2002. http://www.gap-system.org/Packages/sgl.html.

[47] E. Aichinger, F. Binder, J. Ecker, P. Mayr, and C. Nöbauer, *SONATA - system of near-rings and their applications, GAP package, Version 2*, 2003. http://www.algebra.uni-linz.ac.at/Sonata/.

[48] S. F. King and C. Luhn, JHEP **0910** (2009) 093 doi:10.1088/1126-6708/2009/10/093 [arXiv:0908.1897 [hep-ph]].

[49] J. A. Escobar and C. Luhn, J. Math. Phys. **50** (2009) 013524 doi:10.1063/1.3046563 [arXiv:0809.0639 [hep-th]].

[50] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. B **437** (1998) 107 doi:10.1016/S0370-2693(98)00880-6 [hep-ph/9806387].

[51] G. Altarelli, F. Feruglio and L. Merlo, JHEP **0905** (2009) 020 doi:10.1088/1126-6708/2009/05/020 [arXiv:0903.1940 [hep-ph]].

[52] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **349** (1995) 137. doi:10.1016/0370-2693(95)00213-5

[53] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **458** (1999) 79 doi:10.1016/S0370-2693(99)00438-4 [hep-ph/9904297].

[54] N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. **97** (2006) 041601 doi:10.1103/PhysRevLett.97.041601 [hep-ph/0603116].

[55] X. G. He and A. Zee, Phys. Lett. B **645** (2007) 427 doi:10.1016/j.physletb.2006.11.055 [hep-ph/0607163].

[56] W. Grimus and L. Lavoura, JHEP **0809** (2008) 106 doi:10.1088/1126-6708/2008/09/106 [arXiv:0809.0226 [hep-ph]].

[57] S. F. King, Phys. Lett. B **659** (2008) 244 doi:10.1016/j.physletb.2007.10.078 [arXiv:0710.0530 [hep-ph]].

[58] P. F. Harrison and W. G. Scott, Phys. Lett. B **628** (2005) 93 doi:10.1016/j.physletb.2005.09.009 [hep-ph/0508012].

[59] D. Adams, “The Hitchhiker’s Guide to the Galaxy”, Pan Books (1979)

[60] R. M. Fonseca and W. Grimus, JHEP **1409** (2014) 033 doi:10.1007/JHEP09(2014)033 [arXiv:1405.3678 [hep-ph]].

[61] I. de Medeiros Varzielas and G. G. Ross, JHEP **1212** (2012) 041 doi:10.1007/JHEP12(2012)041 [arXiv:1203.6636 [hep-ph]].

[62] R. Krishnan, J. Phys. Conf. Ser. **447** (2013) 012043 doi:10.1088/1742-6596/447/1/012043 [arXiv:1211.3364 [hep-ph]].

[63] M. Holthausen and K. S. Lim, Phys. Rev. D **88** (2013) 033018 doi:10.1103/PhysRevD.88.033018 [arXiv:1306.4356 [hep-ph]].

[64] M. C. Chen, M. Fallbacher, K. T. Mahanthappa, M. Ratz and A. Trautner, Nucl. Phys. B **883** (2014) 267 doi:10.1016/j.nuclphysb.2014.03.023 [arXiv:1402.0507 [hep-ph]].

[65] I. Girardi, A. Meroni, S. T. Petcov and M. Spinrath, JHEP **1402** (2014) 050 doi:10.1007/JHEP02(2014)050 [arXiv:1312.1966 [hep-ph]].

[66] G. J. Ding and Y. L. Zhou, Chin. Phys. C **39** (2015) no.2, 021001 doi:10.1088/1674-1137/39/2/021001 [arXiv:1312.5222 [hep-ph]].

[67] F. Feruglio, C. Hagedorn and R. Ziegler, Eur. Phys. J. C **74** (2014) 2753 doi:10.1140/epjc/s10052-014-2753-2 [arXiv:1303.7178 [hep-ph]].

[68] C. C. Nishi, Phys. Rev. D **88** (2013) no.3, 033010 doi:10.1103/PhysRevD.88.033010 [arXiv:1306.0877 [hep-ph]].

[69] C. Luhn, Nucl. Phys. B **875** (2013) 80 doi:10.1016/j.nuclphysb.2013.07.003 [arXiv:1306.2358 [hep-ph]].

[70] W. Grimus and M. N. Rebelo, Phys. Rept. **281** (1997) 239 doi:10.1016/S0370-1573(96)00030-0 [hep-ph/9506272].

[71] G. J. Ding, S. F. King and A. J. Stuart, JHEP **1312** (2013) 006 doi:10.1007/JHEP12(2013)006 [arXiv:1307.4212 [hep-ph]].

[72] G. J. Ding, S. F. King, C. Luhn and A. J. Stuart, JHEP **1305** (2013) 084 doi:10.1007/JHEP05(2013)084 [arXiv:1303.6180 [hep-ph]].

[73] M. Holthausen, M. Lindner and M. A. Schmidt, JHEP **1304** (2013) 122 doi:10.1007/JHEP04(2013)122 [arXiv:1211.6953 [hep-ph]].

[74] R. Kuchimanchi, Phys. Rev. D **86** (2012) 036002 doi:10.1103/PhysRevD.86.036002 [arXiv:1203.2772 [hep-ph]].

[75] R. Kuchimanchi, Phys. Rev. D **82** (2010) 116008 doi:10.1103/PhysRevD.82.116008 [arXiv:1009.5961 [hep-ph]].

[76] R. Kuchimanchi, Eur. Phys. J. C **74** (2014) 2726 doi:10.1140/epjc/s10052-014-2726-5 [arXiv:1209.3031 [hep-ph]].

[77] M. S. Boucenna, S. Morisi, E. Peinado, Y. Shimizu and J. W. F. Valle, Phys. Rev. D **86** (2012) 073008 doi:10.1103/PhysRevD.86.073008 [arXiv:1204.4733 [hep-ph]].

[78] F. Capozzi, G. L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, Phys. Rev. D **89** (2014) 093018 doi:10.1103/PhysRevD.89.093018 [arXiv:1312.2878 [hep-ph]].

[79] M. Auger *et al.* [EXO-200 Collaboration], Phys. Rev. Lett. **109** (2012) 032505 doi:10.1103/PhysRevLett.109.032505 [arXiv:1205.5608 [hep-ex]].

[80] P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **571** (2014) A16 doi:10.1051/0004-6361/201321591 [arXiv:1303.5076 [astro-ph.CO]].

[81] S. F. King, A. Merle and A. J. Stuart, JHEP **1312** (2013) 005 doi:10.1007/JHEP12(2013)005 [arXiv:1307.2901 [hep-ph]].

[82] D. V. Forero, M. Tortola and J. W. F. Valle, Phys. Rev. D **86** (2012) 073012 doi:10.1103/PhysRevD.86.073012 [arXiv:1205.4018 [hep-ph]].

[83] M. C. Gonzalez-Garcia, M. Maltoni and T. Schwetz, JHEP **1411** (2014) 052 doi:10.1007/JHEP11(2014)052 [arXiv:1409.5439 [hep-ph]].

[84] September 7-14, 2014, Neutrino Oscillation Workshop, Conca Specchiulla (Otranto, Lecce, Italy), `http://www.ba.infn.it/~now/now2014/web-content/index.html`.

[85] J. Turner, Phys. Rev. D **92** (2015) no.11, 116007 doi:10.1103/PhysRevD.92.116007 [arXiv:1507.06224 [hep-ph]].

[86] C. C. Li and G. J. Ding, Nucl. Phys. B **881** (2014) 206 doi:10.1016/j.nuclphysb.2014.02.002 [arXiv:1312.4401 [hep-ph]].

[87] C. C. Li and G. J. Ding, JHEP **1508** (2015) 017 doi:10.1007/JHEP08(2015)017 [arXiv:1408.0785 [hep-ph]].

[88] C. Luhn, Nucl. Phys. B **875** (2013) 80 doi:10.1016/j.nuclphysb.2013.07.003 [arXiv:1306.2358 [hep-ph]].

[89] R. Krishnan, P. F. Harrison and W. G. Scott, JHEP **1304** (2013) 087 doi:10.1007/JHEP04(2013)087 [arXiv:1211.2000 [hep-ph]].

[90] R. N. Mohapatra and C. C. Nishi, Phys. Rev. D **86** (2012) 073007 doi:10.1103/PhysRevD.86.073007 [arXiv:1208.2875 [hep-ph]].

[91] G. C. Branco, J. M. Gerard and W. Grimus, Phys. Lett. B **136** (1984) 383. doi:10.1016/0370-2693(84)92024-0

[92] M. C. Chen and K. T. Mahanthappa, Phys. Lett. B **681** (2009) 444 doi:10.1016/j.physletb.2009.10.059 [arXiv:0904.1721 [hep-ph]].

[93] S. Antusch, S. F. King, C. Luhn and M. Spinrath, Nucl. Phys. B **850** (2011) 477 doi:10.1016/j.nuclphysb.2011.05.005 [arXiv:1103.5930 [hep-ph]].

[94] G. J. Ding and S. F. King, Phys. Rev. D **89** (2014) no.9, 093020 doi:10.1103/PhysRevD.89.093020 [arXiv:1403.5846 [hep-ph]].

[95] C. Hagedorn, A. Meroni and E. Molinaro, Nucl. Phys. B **891** (2015) 499 doi:10.1016/j.nuclphysb.2014.12.013 [arXiv:1408.7118 [hep-ph]].

[96] G. Ecker, W. Grimus and W. Konetschny, Nucl. Phys. B **191** (1981) 465. doi:10.1016/0550-3213(81)90309-6

[97] L. Lavoura and P. O. Ludl, Phys. Lett. B **731** (2014) 331 doi:10.1016/j.physletb.2014.03.001 [arXiv:1401.5036 [hep-ph]].

[98] Talk with professor Wei Wang. JUNO experiment, `http://english.ihep.cas.cn/rs/fs/juno0815/`.

[99] C. H. Albright and W. Rodejohann, Eur. Phys. J. C **62** (2009) 599 doi:10.1140/epjc/s10052-009-1074-3 [arXiv:0812.0436 [hep-ph]].

[100] J. B. Albert *et al.* [EXO-200 Collaboration], Nature **510** (2014) 229 doi:10.1038/nature13432 [arXiv:1402.6956 [nucl-ex]].

[101] A. Gando *et al.* [KamLAND-Zen Collaboration], Phys. Rev. Lett. **110** (2013) no.6, 062502 doi:10.1103/PhysRevLett.110.062502 [arXiv:1211.3863 [hep-ex]].

[102] C. Jarlskog, Phys. Rev. Lett. **55** (1985) 1039. doi:10.1103/PhysRevLett.55.1039

[103] G. Aad *et al.* [ATLAS Collaboration], Phys. Lett. B **716** (2012) 1 doi:10.1016/j.physletb.2012.08.020 [arXiv:1207.7214 [hep-ex]].

[104] S. Chatrchyan *et al.* [CMS Collaboration], Phys. Lett. B **716** (2012) 30 doi:10.1016/j.physletb.2012.08.021 [arXiv:1207.7235 [hep-ex]].

[105] T. D. Lee, Phys. Rev. D **8** (1973) 1226. doi:10.1103/PhysRevD.8.1226

[106] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **155** (1985) 36. doi:10.1016/0370-2693(85)91028-7

[107] D. V. Forero, M. Tortola and J. W. F. Valle, Phys. Rev. D **90** (2014) no.9, 093006 doi:10.1103/PhysRevD.90.093006 [arXiv:1405.7540 [hep-ph]].

[108] M. Fukugita and T. Yanagida, Phys. Lett. B **174** (1986) 45. doi:10.1016/0370-2693(86)91126-3

[109] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82** (2010) 2701 doi:10.1103/RevModPhys.82.2701 [arXiv:1002.0211 [hep-ph]].

[110] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, Prog. Theor. Phys. Suppl. **183** (2010) 1 doi:10.1143/PTPS.183.1 [arXiv:1003.3552 [hep-th]].

[111] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, New J. Phys. **16** (2014) 045018 doi:10.1088/1367-2630/16/4/045018 [arXiv:1402.4271 [hep-ph]].

[112] S. F. King, J. Phys. G **42** (2015) 123001 doi:10.1088/0954-3899/42/12/123001 [arXiv:1510.02091 [hep-ph]].

[113] J. Bernabeu, G. C. Branco and M. Gronau, Phys. Lett. B **169** (1986) 243. doi:10.1016/0370-2693(86)90659-3

[114] G. C. Branco, L. Lavoura and M. N. Rebelo, Phys. Lett. B **180** (1986) 264. doi:10.1016/0370-2693(86)90307-2

[115] G. C. Branco, I. de Medeiros Varzielas and S. F. King, Phys. Rev. D **92** (2015) no.3, 036007 doi:10.1103/PhysRevD.92.036007 [arXiv:1502.03105 [hep-ph]].

[116] G. C. Branco, I. de Medeiros Varzielas and S. F. King, Nucl. Phys. B **899** (2015) 14 doi:10.1016/j.nuclphysb.2015.07.024 [arXiv:1505.06165 [hep-ph]].

[117] V. Keus, S. F. King, S. Moretti and K. Yagyu, JHEP **1604** (2016) 048 doi:10.1007/JHEP04(2016)048 [arXiv:1510.04028 [hep-ph]].

[118] A. Mendez and A. Pomarol, Phys. Lett. B **272** (1991) 313. doi:10.1016/0370-2693(91)91836-K

[119] L. Lavoura and J. P. Silva, Phys. Rev. D **50** (1994) 4619 doi:10.1103/PhysRevD.50.4619 [hep-ph/9404276].

[120] F. J. Botella and J. P. Silva, Phys. Rev. D **51** (1995) 3870 doi:10.1103/PhysRevD.51.3870 [hep-ph/9411288].

[121] G. C. Branco, M. N. Rebelo and J. I. Silva-Marcos, Phys. Lett. B **614** (2005) 187 doi:10.1016/j.physletb.2005.03.075 [hep-ph/0502118].

[122] S. Davidson and H. E. Haber, Phys. Rev. D **72** (2005) 035004 Erratum: [Phys. Rev. D **72** (2005) 099902] doi:10.1103/PhysRevD.72.099902, 10.1103/PhysRevD.72.035004 [hep-ph/0504050].

[123] J. F. Gunion and H. E. Haber, Phys. Rev. D **72** (2005) 095002 doi:10.1103/PhysRevD.72.095002 [hep-ph/0506227].

[124] G. J. Ding and Y. L. Zhou, JHEP **1406** (2014) 023 doi:10.1007/JHEP06(2014)023 [arXiv:1404.0592 [hep-ph]].

[125] F. Bjrkeroth, F. J. de Anda, I. de Medeiros Varzielas and S. F. King, JHEP **1506** (2015) 141 doi:10.1007/JHEP06(2015)141 [arXiv:1503.03306 [hep-ph]].

[126] C. C. Li and G. J. Ding, JHEP **1505** (2015) 100 doi:10.1007/JHEP05(2015)100 [arXiv:1503.03711 [hep-ph]].

[127] A. Di Iura, C. Hagedorn and D. Meloni, JHEP **1508** (2015) 037 doi:10.1007/JHEP08(2015)037 [arXiv:1503.04140 [hep-ph]].

[128] P. Ballett, S. Pascoli and J. Turner, Phys. Rev. D **92** (2015) no.9, 093008 doi:10.1103/PhysRevD.92.093008 [arXiv:1503.07543 [hep-ph]].

[129] G. J. Ding and S. F. King, Phys. Rev. D **93** (2016) 025013 doi:10.1103/PhysRevD.93.025013 [arXiv:1510.03188 [hep-ph]].

[130] F. Bjrkeroth, F. J. de Anda, I. de Medeiros Varzielas and S. F. King, Phys. Rev. D **94** (2016) no.1, 016006 doi:10.1103/PhysRevD.94.016006 [arXiv:1512.00850 [hep-ph]].

[131] C. C. Li, C. Y. Yao and G. J. Ding, JHEP **1605** (2016) 007 doi:10.1007/JHEP05(2016)007 [arXiv:1601.06393 [hep-ph]].

[132] R. de Adelhart Toorop, F. Bazzocchi, L. Merlo and A. Paris, JHEP **1103** (2011) 035 Erratum: [JHEP **1301** (2013) 098] doi:10.1007/JHEP03(2011)035, 10.1007/JHEP01(2013)098 [arXiv:1012.1791 [hep-ph]].

[133] R. de Adelhart Toorop, F. Bazzocchi, L. Merlo and A. Paris, JHEP **1103** (2011) 040 doi:10.1007/JHEP03(2011)040 [arXiv:1012.2091 [hep-ph]].

[134] I. de Medeiros Varzielas and D. Emmanuel-Costa, Phys. Rev. D **84** (2011) 117901 doi:10.1103/PhysRevD.84.117901 [arXiv:1106.5477 [hep-ph]].

[135] I. de Medeiros Varzielas, D. Emmanuel-Costa and P. Leser, Phys. Lett. B **716** (2012) 193 doi:10.1016/j.physletb.2012.08.008 [arXiv:1204.3633 [hep-ph]].

[136] G. Bhattacharyya, I. de Medeiros Varzielas and P. Leser, Phys. Rev. Lett. **109** (2012) 241603 doi:10.1103/PhysRevLett.109.241603 [arXiv:1210.0545 [hep-ph]].

[137] I. P. Ivanov and E. Vdovin, Eur. Phys. J. C **73** (2013) no.2, 2309 doi:10.1140/epjc/s10052-013-2309-x [arXiv:1210.6553 [hep-ph]].

[138] I. P. Ivanov and E. Vdovin, Phys. Rev. D **86** (2012) 095030 doi:10.1103/PhysRevD.86.095030 [arXiv:1206.7108 [hep-ph]].

[139] I. de Medeiros Varzielas and D. Pidt, J. Phys. G **41** (2014) 025004 doi:10.1088/0954-3899/41/2/025004 [arXiv:1307.0711 [hep-ph]].

[140] I. de Medeiros Varzielas and D. Pidt, JHEP **1311** (2013) 206 doi:10.1007/JHEP11(2013)206 [arXiv:1307.6545 [hep-ph]].

[141] V. Keus, S. F. King and S. Moretti, JHEP **1401** (2014) 052 doi:10.1007/JHEP01(2014)052 [arXiv:1310.8253 [hep-ph]].

[142] V. Keus, S. F. King, S. Moretti and D. Sokolowska, JHEP **1411** (2014) 016 doi:10.1007/JHEP11(2014)016 [arXiv:1407.7859 [hep-ph]].

[143] I. P. Ivanov and C. C. Nishi, JHEP **1501** (2015) 021 doi:10.1007/JHEP01(2015)021 [arXiv:1410.6139 [hep-ph]].

[144] V. Keus, S. F. King, S. Moretti and D. Sokolowska, JHEP **1511** (2015) 003 doi:10.1007/JHEP11(2015)003 [arXiv:1507.08433 [hep-ph]].

[145] M. Fallbacher and A. Trautner, Nucl. Phys. B **894** (2015) 136 doi:10.1016/j.nuclphysb.2015.03.003 [arXiv:1502.01829 [hep-ph]].

[146] I. P. Ivanov and J. P. Silva, Phys. Rev. D **93** (2016) no.9, 095014 doi:10.1103/PhysRevD.93.095014 [arXiv:1512.09276 [hep-ph]].

[147] D. Emmanuel-Costa, O. M. Ogreid, P. Osland and M. N. Rebelo, JHEP **1602** (2016) 154 Erratum: [JHEP **1608** (2016) 169] doi:10.1007/JHEP08(2016)169, 10.1007/JHEP02(2016)154 [arXiv:1601.04654 [hep-ph]].

[148] I. P. Ivanov and L. Lavoura, Eur. Phys. J. C **73** (2013) no.4, 2416 doi:10.1140/epjc/s10052-013-2416-8 [arXiv:1302.3656 [hep-ph]].

[149] I. P. Ivanov and C. C. Nishi, Phys. Rev. D **82** (2010) 015014 doi:10.1103/PhysRevD.82.015014 [arXiv:1004.1799 [hep-th]].

[150] I. P. Ivanov, JHEP **1007** (2010) 020 doi:10.1007/JHEP07(2010)020 [arXiv:1004.1802 [hep-th]].

[151] A. E. Carcamo Hernandez, I. de Medeiros Varzielas, S. G. Kovalenko, H. Ps and I. Schmidt, Phys. Rev. D **88** (2013) no.7, 076014 doi:10.1103/PhysRevD.88.076014 [arXiv:1307.6499 [hep-ph]].

[152] V. Keus, S. F. King and S. Moretti, Phys. Rev. D **90** (2014) no.7, 075015 doi:10.1103/PhysRevD.90.075015 [arXiv:1408.0796 [hep-ph]].

[153] C. C. Nishi, JHEP **1503** (2015) 034 doi:10.1007/JHEP03(2015)034 [arXiv:1411.4909 [hep-ph]].

[154] I. de Medeiros Varzielas, O. Fischer and V. Maurer, JHEP **1508** (2015) 080 doi:10.1007/JHEP08(2015)080 [arXiv:1504.03955 [hep-ph]].

[155] E. Ma and G. Rajasekaran, Phys. Rev. D **64** (2001) 113012 doi:10.1103/PhysRevD.64.113012 [hep-ph/0106291].

[156] C. Luhn, S. Nasri and P. Ramond, J. Math. Phys. **48** (2007) 073501 doi:10.1063/1.2734865 [hep-th/0701188].

[157] E. Ma, Phys. Lett. B **632** (2006) 352 doi:10.1016/j.physletb.2005.10.019 [hep-ph/0508231].

[158] E. Ma, Mod. Phys. Lett. A **21** (2006) 1917 doi:10.1142/S0217732306021190 [hep-ph/0607056].

[159] H. Ishimori, T. Kobayashi, H. Okada, Y. Shimizu and M. Tanimoto, JHEP **0904** (2009) 011 doi:10.1088/1126-6708/2009/04/011 [arXiv:0811.4683 [hep-ph]].

[160] G. C. Branco and I. P. Ivanov, JHEP **1601** (2016) 116 doi:10.1007/JHEP01(2016)116 [arXiv:1511.02764 [hep-ph]].

[161] M. Punturo *et al.*, Class. Quant. Grav. **27** (2010) 194002. doi:10.1088/0264-9381/27/19/194002