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DIFFERENTIAL DISPERSIVE APPROACH TO LARGE MOMENTUM TRANSFER PROCESSES: I *

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ABSTRACT

Interpreting the renormalization group in terms of its natural differential dispersive character, we analyze systematically general large momentum transfer processes in the context of renormalizable interacting quantum fields. The possibility of a consistent treatment of such processes is thereby attained. This possibility arises mainly from the occurrence of a power series in the inverse of the scale at large momentum transfer in the respective solutions of field theory—that is to say, in the solutions as determined by the previously introduced dimensional analysis violating sources which underlie the dispersive character of the renormalization group. Hence, our results generally deviate from those of convention. They appear to be in reasonable accord with experiment.

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I. INTRODUCTION

In what follows, we shall be concerned with the implications of the violation of naive dimensional scaling in renormalized perturbation theory (four-dimensions, Minkowsky). This phenomenon was pointed out in previous Letters[1] and briefly discussed in connection with Bjorken scaling. The discussion here will in essence expand and continue the ideas introduced in Refs. 1.

Specifically, we shall primarily focus on the large momentum transfer processes in renormalizable interacting field theory. Such processes have recently received considerable attention from the theoretic standpoint, as the data have revealed several surprises.[2] Indeed, numerous attempts[3] have been made to provide a systematic treatment. The discussion below represents the basis for yet another such effort. We should remark that the other approaches are either lacking in predictive power, consistency, or both. The effort here will be seen to afford the possibility of remedying this latter situation. In particular, our approach will be seen to offer an explanation of the apparent difference in scale between

$$e + p \rightarrow e + x$$

and

$$e^{+} + e^{-} \rightarrow x' \quad ,$$

entirely within the context of renormalizable quantum field theory. Additionally, it will be seen that the fixed angle scattering data may be systematically incorporated into our formalism, naturally.

As we remarked above, the basis of our discussion will be the (perturbative) violation of dimensional analysis (in connection with the renormalization group), the effect pointed out in the previous Letters.[1] As this phenomenon is not very familiar, we shall discuss it in some detail before turning to its applications.

Its occurrence will be seen to dictate the restructuring of conventional intuition insofar as large momentum transfer processes in renormalizable interacting quantum field theory[4] are concerned. The main physical notions underlying the phenomenon (to the extent that we shall employ it) are those of long range forces and/or particle collaboration in theories with bound states or all fields massive. Specifically, it will be seen that, contrary to the notions underlying previous efforts, the present discussion will allow particles in interaction undergoing large momentum transfer to do this in large space-time volumes, as dictated by long range, binding, and other renormalizable forces. This will be effected by treating systematically the violation of dimensional analysis, the natural manifestation of the forces under discussion. To repeat, the resulting formalism will be seen to afford a systematic, consistent description of large momentum transfer processes in renormalizable quantum field theory.

The approach to small distance behavior in quantum field theory presented here will thus be seen to depart significantly from the standard theory.[4] This can be most easily seen as follows: The conventional approaches have all been formulated with the idea that, for "small" effective couplings, the main effect of interactions at asymptotic distances is to perturb naive scaling in these regions, this being made manifest by the introduction of the notion of an anomalous dimension with the interpretation as a representation of the referred-to perturbation. This physical notion was based entirely on experience with the first loops of renormalized perturbation. On the other hand, the violation of dimensional analyses will be seen to be highly nonperturbative in its consequences, although it often occurs already in perturbation theory. Thus, our predictions for asymptotic behavior will not in general be mere perturbative effects when the effective coupling is small. Rather, much like the occurrence of the positronium resonance in QED in relation to the size of e, our results will be insensitive to the behavior of the effective coupling.

Some of the more phenomenological discussions of large momentum transfer processes have already made it reasonably clear that it is the nonperturbative aspect of interactions which is determining the data being observed at high energies.[5] Indeed, it's fairly well accepted[6] that bound quark-partons afford a good fit to these data whereas several contradictions arise when it's necessary for the quark parton to be "free" at some stage in the interaction.[7] Here, we shall view this as a natural dictate for the (perturbative) violation of dimensional analysis.

We should remark that the theory to be presented actually affords a complete description of all asymptotic processes, both at small and large distances. Since the large distance behavior of the known particles is mainly electromagnetic, this regime of experiment is already well-understood, having been completely described by Feynman[8] and Schwinger.[9] Hence we shall only enter into this regime when it's necessary to discuss certain technical issues relating to boundary conditions. To repeat, we consider it already sufficiently well-understood.

At this point, the reader may wonder, "Why are they not proposing their analysis as a complete transcension of renormalized perturbation theory?" The reason we shall not is extremely transparent—we have to input the precise form of the violation phenomenon and coefficient functions. In the text below, we shall sometimes take this information just as given by the first few loops of perturbation.

However, particularly concerning bound states, we shall also have to use, on some occasions, Bethe-Salpeter[10] type equations, the precise form of which can only be known by solving the theory! Consequently, we shall not be able to transcend perturbation theory definitively. But, to be sure, we shall certainly transcend it insofar as asymptotic behavior is concerned.

The main characteristic of our results is a power series in $1/\lambda$ on the light-cone, where λ is the scale. Of course, we shall also have a power series in λ at large distances. However, as we have remarked above, this latter series is well understood. Let us mention that the data are completely consistent with such series. The coefficients of the terms in the series will be completely determined by the hard thresholds in the theory, i.e., those thresholds which violate dimensional analysis. An immediate consequence of this last statement is a rather natural argument for the correspondence principle in the Regge scaling region. As is well-known, this principle appears to be in accord with experiment.[11]

Our analysis will necessarily be somewhat technical at points, since the conventional discussions of the renormalization group equation have not considered the violation phenomenon referred to above. However, in the interest of clarity, we shall, where possible, relegate purely technical remarks to the Appendices.

The present discussion should be viewed as the first of a series of reports on our approach to the renormalization group. In this first paper we shall mainly be concerned with the dependence of the respective Green's functions on scale in various asymptotic limits. Detailed numerical results, to the extent that they are possible, will be the goal of the later works.[12]

The paper is organized as follows. In Section II, we present a reasonably detailed discussion of the renormalization group equation in connection with its differential dispersive character, always comparing our view with convention where appropriate. In Section III, we discuss, as a detailed example of our approach, the proper vertex and inverse photon propagation functions of quantum electrodynamics in connection with asymptopia, since these functions are the primary reason for belief in renormalization. In Section IV, we show how to apply our formalism to various other large momentum transfer processes. As we mentioned above, involved technical details are contained in the Appendices. And, Section V contains some concluding remarks.

II. DIFFERENTIAL DISPERSION RELATIONS

In this section, we shall discuss the theory of differential dispersion relations, the essence of which has already been introduced in the previous Letters.[1] We shall define explicitly what we mean by such relations presently. The definition will be given largely in an operative fashion. Thus, we shall initially examine various kinds of renormalizable interactions to see in which ones such relations obtain. After having accomplished this, we shall explain the general theory of application of the relations to asymptotic processes.

As the basis of our discussion will be the partial differential equations[13] (PDE) of renormalizable field theory, let us first establish a convention insofar as they are concerned, for the literature on them has recently become quite extensive. Specifically, we shall always write the renormalization group equation after the fashion of Weinberg,[14] mainly for convenience. Thus, in general we shall have (our notation is symbolic for theories with several g's and m_R's)

$$\left(\mu \frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_\theta \; m_R \frac{\partial}{\partial m_R} - \gamma_\Gamma\right) \Gamma(\{\lambda p_j\} ; \mu, m_R, g_R) = 0 \qquad (2.1)$$

where $\{\Gamma\}$ are, for example, the set of 1 PI Green's functions of the theory, and $\mu, \beta,$ $\gamma_\Gamma, \gamma_\theta$ and m_R have their usual meanings.[13] Further, we shall always consider (2.1) as applicable to all representations of the solutions of the respective theories, even though we are aware that in general it has only been explicitly verified for perturbative solutions.[13] The physical equivalence of (2.1) to other forms of PDE for renormalizable theories has been demonstrated by several authors.[15]

The dispersive aspect of (2.1) becomes manifest when one attempts to use this equation, which is a differential equation about intrinsic parameters of the respective theory, to study the behavior of $\{\Gamma\}$ as functions of the extrinsic scale λ of their momenta. For, in order to effect such a use of (2.1), its necessary to change the operator in (2.1) to one involving λ. This is most easily done by using dimensional analysis. We have

$$\left(\mu \frac{\partial}{\partial\mu} + m_R \frac{\partial}{\partial m_R} + \lambda \frac{\partial}{\partial\lambda}\right)\Gamma = D_\Gamma \Gamma + R\Gamma \tag{2.2}$$

where D_Γ is the engineering dimension of Γ and $R\Gamma$ represents the possible (perturbative) violation of naive dimensional analysis in Γ, as was pointed out in the previous Letters.[1] In those Letters we argued that in general $R\Gamma$ has the form

$$R\Gamma = -\Sigma \rho_\alpha\, \delta\left(\lambda^2 \left(\Sigma p_{j_\alpha}\right)^2 - m_\alpha^2\right) \tag{2.3}$$

where the $\{m_\alpha^2\}$ are the appropriate set of thresholds in Γ and $\{\rho_\alpha\}$ are the corresponding amplitudes. Rewriting (2.1) in terms of λ we have

$$\left(-\lambda \frac{\partial}{\partial\lambda} + \beta \frac{\partial}{\partial g} - (1+\gamma_\theta)\, m_R \frac{\partial}{\partial m_R} - \gamma_\Gamma + D_\Gamma\right)\Gamma = \sum_\alpha \rho_\alpha\, \delta\left(\lambda^2 \left(\Sigma p_{j_\alpha}\right)^2 - m_\alpha^2\right) \tag{2.4}$$

This last equation is our definition of a differential dispersion relation, since it relates a certain set of derivatives of Γ to a set of sources determined by the thresholds therein. The presence of these sources has already been emphasized in connection with Bjorken scaling in the previous Letters. As we have remarked above, below we shall show that such sources actually may permit a natural description of the present data in all large momentum transfer processes.

In the Letters, we only demonstrated (2.4) explicitly in perturbation theory in the case of quantum electrodynamics—Abelian gauge theory, considering it more or less self-evident thereafter. Thus, mainly in the interest of completing our definition (2.4), let us examine the various other renormalizable interactions in this connection.

We consider first the simplest of renormalizable interactions, that of a scalar field with quartic self-coupling. The Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right) - \frac{g}{4!} \phi^4 + \text{counter terms} \tag{2.5}$$

This is a Lagrangian with no massless exchanges, i.e., no long range force. Now, in perturbation theory, the only singularities in the $\{\Gamma\}$ corresponding to (2.5) may be characterized by

$$\theta(x) \; f(x) \quad , \tag{2.6}$$

the $\theta(x)$ coming from phase space and f(x) representing the appropriate form factor. This is a general characteristic of renormalizable field theory, as is well-known. Hence, in order for a violation of naive dimensional analysis to occur, it's necessary for f(x) to be singular at x=0, since it's clear that x is a homogeneous quadratic function of m_R, μ, and λ. It is also well-known that in perturbation theory the on-shell 1 PI Green's functions of (2.5) are not expected to have singular f's at thresholds, due to the massiveness of ϕ. However, as we shall see shortly, this last remark does not hold true for the off-shell functions in general. Thus, even the Lagrangian (2.5) violates naive dimensional scaling in perturbation theory.

And of course, nonperturbatively, one can also imagine the occurrence of a bound state pole in the theory (2.5), yielding singularities

$$\sim \delta(x) \; f(x) \quad , \tag{2.7}$$

as there would be no phase space. Then, a violation of naive scaling occurs if f(x) is as (or more) singular than log x at x=0. That this may happen can be seen by considering the production and reabsorption of the presumed pole, as shown in Fig. 1. for the 1 PI 4-point function. However, to demonstrate this explicitly in the theory (2.5) would require methods unknown to us.

From the arguments just given it's clear that if m=0 in (2.5), then violations of dimensional analysis may manifestly occur in the perturbative solutions of the theory. For example, the six-point function receives its third order contributions from the diagrams illustrated in Fig. 2. Explicit evaluation of this contribution shows that it violates dimensional analysis precisely in the form (2.3), as is discussed in Appendix I. This same result also holds for (2.5), for off-shell momenta.

Alternatively, if we add to (2.5) the field terms

$$\frac{1}{2}\,\partial_\mu A\,\partial^\mu A - g_1(A\phi^2 + \phi A^2) - g_2 A^2\phi^2 - g_3/4!\ A^4 + \text{counter terms} \tag{2.8}$$

then, as a result of the zero mass of A, it is again seen that the $\{\Gamma\}$ generally violate naive dimensional analysis in the form (2.3). (See Appendix I.) Thus, dimensional analysis is violated in all scalar renormalizable interactions, and manifestly so in the massless cases.

We consider next the interactions of scalars and vectors A_μ, adding to (2.5)

$$-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - ie\,\phi^\dagger\left(\frac{\overrightarrow{\partial}}{\partial x_\mu} - \frac{\overleftarrow{\partial}}{\partial x_\mu}\right)\phi A^\mu + e^2\phi^\dagger\phi A^2 + \text{counter terms} \tag{2.9}$$

where we have given ϕ a charge. Straightforward calculation shows that three-loop diagrams as shown in Fig. 3 violate naive dimensional analysis in the form (2.3). Further, it's clear that, in general, the other 1 PI Green's functions behave similarly (see Appendix I).

We next introduce particles of spin 1/2, adding the terms

$$\bar\psi(i\partial\!\!\!/ - m_\psi)\psi - \bar g_1\bar\psi\,\psi\phi - e\,\bar\psi\,\gamma_\mu\psi A^\mu - \bar g_2\bar\psi\,\psi A + \text{counter terms} \tag{2.10}$$

Then, from the discussion of the Letters[1] and that above it's clear that these additional interactions also give violations of naive dimensional analysis in perturbation theory. (See Appendix I.)

The introduction of internal non-Abelian local symmetry does not alter any of the above conclusions. Naive dimensional analysis certainly is violated in perturbation theory whenever there are massless fields interacting renormalizably. If all the fields are massive, violations also occur off-shell in perturbation theory as a result of the possibility of all internal lines being on-shell (free particle collaboration), as can be seen from Fig. 2 for the theory (2.5), for example.

As we remarked above, the violation of dimensional analysis may also happen nonperturbatively as a result of the occurrence of bound state poles, provided the amplitudes characterizing this occurrence are at least as singular as log x at threshold. To repeat, such amplitudes are an immediate characteristic of the existence of the bound state in any theory. We see no reason why such poles can't be a general characteristic of field theory. We shall not be concerned with the question of their existence any further here.

We should mention that from Fig. 1 it is immediate that super renormalizable theories of ϕ^3 type also violate dimensional analysis, even when all fields are massive. We shall not be concerned with such theories here.[12]

Hence, it appears that the (perturbative) violation of dimensional analysis is a natural characteristic of renormalizable field theories, especially those with long range (and/or) binding forces. This has the consequence that our notion of differential dispersion relations is operative in all such theories.

Intuitively, the meaning of the violation phenomenon is also rather natural although somewhat iconoclastic! For, as we remarked above, contrary to other intuition based on the first loops of perturbation, the violation phenomenon says that large momentum transfer does not have to occur in finite space-time volumes when there are either long range forces, forces strong enough to make bound states, or forces mediated by on-mass shell particles; rather, these transfers can occur over large space-time volumes. Hence, naive dimensional analysis need not apply, not even within perturbative corrections!

Having demonstrated the appropriateness of (2.4), let us now turn to its implications. The general solution of (2.4) has the form

$$\Gamma = \Gamma_p + \Gamma_{\text{homogeneous}} \tag{2.11}$$

with

$$\Gamma_p = \frac{1}{2}\left[-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}+D_\Gamma/2\right]^{-1}\left(\sum_\alpha \rho_\alpha \delta\left(\lambda^2\left(\Sigma p_{j_\alpha}\right)^2 - m_\alpha^2\right)+\gamma\Gamma_p\right)$$

$$= \frac{1}{4\pi}\int dk\, d\ell_1\, d\ell_2 \sum_{\alpha, j_1, j_2} \int dr_1 \ldots dr_{j_2}\; ds_1 \ldots ds_{j_2}$$

$$\bar{\rho}_\alpha\, \sigma_\alpha(r_1, s_1)\ldots\sigma_2\left(r_{j_2}, s_{j_2}\right)\frac{(ik)^{j_1+j_2}}{j_1!\, j_2!}\; \exp\left[j_1 t + i\left(\Sigma r_j + \ell_1\right)v\right.$$

$$\left. + i\left(\Sigma s_i + \ell_2\right)w\right] \Big/ \left[-j_1 + i\left(\ell_1+\ell_2+\Sigma\,(r_i+s_i)\right)+D_\Gamma/2\right]$$

$$+ \frac{1}{4\pi}\int dk\, d\ell_1 d\ell_2 \sum_j \frac{\overline{\gamma\Gamma_p}\,(ik)^j\; \exp[jt + i\ell_1 v + i\ell_2 w]}{j!\,\left[-j + i(\ell_1+\ell_2) + D_\Gamma/2\right]} \qquad (2.12)$$

and

$$\Gamma_{\text{homogeneous}} = F(t+v, t+w)\; \exp\frac{-1}{2}\int^v dx(D_\Gamma - \gamma) \quad , \qquad (2.13)$$

where we have introduced

$$v = 2\int_{g_0}^{g_R} \frac{dx}{\beta(x)}$$

$$w = -2\int_{m_0}^{m_R} \frac{dx}{x\left(1+\gamma_\theta(g(x))\right)} \qquad (2.14)$$

$$t = \log \lambda^2 \quad ,$$

as well as

$$\rho_\alpha = \int d\ell_1 d\ell_2\; e^{i(\ell_1 v + \ell_2 w)}\; \bar{\rho}_\alpha$$

$$m_\alpha^2 = -\int dr ds\; e^{i(rv+sw)}\; \sigma_\alpha \qquad (2.15)$$

$$\gamma\Gamma_p = \int dk\; d\ell_1 d\ell_2\; \overline{\gamma\Gamma_p}\; \exp i\left(k\lambda^2 + \ell_1 v + \ell_2 w\right)$$

and F is arbitrary.

Now, in order to obtain the solution of (2.4), it's necessary to impose some kind of subsidiary condition on $\{\Gamma\}$, just as it's necessary to impose Coulomb's law in order to pin down the solution of

$$\nabla^2\Phi = -4\pi e\ \delta(\vec{r}) \tag{2.16}$$

In the conventional approach, where $R\Gamma$ is ignored, the subsidiary condition is correspondence with perturbation theory in the regime where perturbation should be applicable. Of course, it is well-known that such a condition renders solutions which are applicable only in restricted asymptotic ranges of their arguments. The raison d'etre underlying the condition is the obvious, namely, the success of renormalized perturbation theory in low momentum quantum electrodynamics.[16] Here, too, we shall use correspondence with renormalized perturbation theory. We shall argue, in analogy with Coulomb's law, that in the cases of physical interest the homogeneous solution (2.13) is not allowed, except in certain cases, so that the entire solution is just (2.12). This is a drastic departure from the standard lore, where the homogeneous integrals regular at zero effective coupling are taken as the entire solution! Let us also remark that, so long as the initial coupling g_R is small, the approximation (to be described presently) which we shall employ in determining Γ_p should be valid for all momenta, again contrary to the conventional approach, where, to repeat, at most only one asymptotic regime is expected to be appropriate.

Specifically, the scheme we shall use will be based on iteration in $\gamma(g_R)$, taking g_R to be small. In such a scheme, the expansion in λ will be in powers of λ for $\lambda \to 0$ and in powers of $1/\lambda$ for $\lambda^2 \to \infty$. Thus, to repeat we shall be able to discuss large and small distance behavior.

In a general model, it is indeed possible that the homogeneous integral (2. 13) may be necessary, depending on what kind of boundary conditions are applicable, just as in classical electrodynamics, where often times one has to make use of the homogeneous solutions of (2. 16) in order to satisfy boundary conditions. The condition of correspondence is not sacred; rather, it's only convenient, in certain theories. Any boundary condition consistent with the axioms of field theory may be used, as is well-known. However, a set of boundary conditions should not be circular in the sense of assuming the existence of a limit, which is in itself a major point of the analysis. Indeed, away from the sources in (2.4), the complete characteristics are, formally,

$$-\frac{d\lambda}{\lambda} = \frac{dg}{\beta} = \frac{-dm_R}{m_R(1+\gamma_\theta)} = \frac{d\Gamma}{\gamma\Gamma} \quad , \tag{2.17}$$

so that each asymptotic limit is generally nonregular in the effective coupling, unless the theory is trivial. Hence, it would be circular to presume the existence of such asymptotic limits in effective coupling in theories of interest!

As we remarked above, bound state poles can generate violations of naive scaling if the respective residues are at least as singular as log (x) at threshold. Consequently, we shall naturally be able to handle the asymptotic interactions of the particles corresponding to such states systematically. This may be done most conveniently by employing the Bethe-Salpeter formalism. Hence, let us now consider this formalism in connection with (2.4).

Specifically, for the exclusive scattering of bound states

$$a + b \rightarrow c + d$$

the complete amplitude can be represented as

$$\mathcal{M}_{cb;ab} = \int \phi^c_{BS}\, \phi^d_{BS}\, \mathcal{M}^{irr}_{cd;ab}\, \phi^a_{BS}\, \phi^b_{BS} \prod_i d^4k \quad , \tag{2.18}$$

where $\mathcal{M}^{irr}_{cd;ab}$ is the appropriate irreducible amplitude and $\{\phi^i_{BS}\}$ are the respective Bethe-Salpeter (BS) wavefunctions. We are presuming an underlying renormalizable field theory. Thus, the irreducible vertex satisfies the differential dispersion relation

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta\frac{\partial}{\partial g_R}-(1+\gamma_\theta)\,m_R\frac{\partial}{\partial m_R}-\gamma_{\mathcal{M}}+D_{\mathcal{M}}\right)\mathcal{M}^{irr}=\sum_\alpha \rho^{\mathcal{M}}_\alpha\,\delta\left(\lambda^2\left(\Sigma p_{j_\alpha}\right)^2-m^2_\alpha\right)\ , \tag{2.19}$$

where $\{m^2_\alpha\}$ are the hard thresholds and $\{\rho^{\mathcal{M}}_\alpha\}$ are the respective amplitudes. Similarly, the Bethe-Salpeter wave functions must satisfy the analogous constraints. These constraints follow from the standard definitions in the BS formalism and are given in Appendix II. In keeping with our approach to asymptotic processes we shall take only the corresponding particular solution to (2.19) as relevant. Then depending on the details of the fundamental fields comprising a, b, c, d, (2.19) will be seen to permit a systematic discussion of fixed-angle scattering as $s\to\infty$, for example.

The next two sections will be devoted to detailed discussions of the application of the ideas in this section to QED (Section III) and high transverse momentum hadron scattering (Section IV), the latter processes being considered within the quark model for definiteness. The consistency of our approach will then be manifest.

III. ASYMPTOTIC PROPERTIES OF QUANTUM ELECTRODYNAMICS

In this section, we shall discuss the application of differential dispersion relations to quantum electrodynamics, as this theory is the main reason for belief in renormalization. We shall consider in detail the proper vertex function, since it has played such a central role in the study of the theory. Our ultimate goal here is an explanation of the recently reported large

$$e^+ + e^- \to \text{hadrons}$$

cross section. (We shall ignore in the present discussion the recently reported[17] resonances in this and related processes. The meaning of these phenomena in the context of the present theory will be taken up elsewhere.[12])

The proper vertex, Γ_μ, is also of particular interest because it permits an easy evaluation of the RHS of (2.4) in perturbation theory, as was pointed out in the previous Letters.[1] Indeed, from the result of Barbieri et al.[18] we have, by direct calculation (see Appendix VI),

$$\left(\mu\frac{\partial}{\partial\mu} + m_R\frac{\partial}{\partial m_R} + \lambda\frac{\partial}{\partial\lambda}\right)\Gamma_\mu(\lambda^2 q^2)$$

$$= \frac{g_R^5}{32\pi}\left[\gamma_\mu\left\{\frac{\pi}{2} + i\log\left(\frac{\lambda^2 q^2 - 4m_R^2}{0_+}\right)\right\} - \sigma_{\mu\nu}\frac{\lambda q^\nu}{2m_R}\right] m_R^2\,\delta\left(\lambda^2 q^2 - 4m_R^2\right)$$

$$+ \text{higher orders} \tag{3.1}$$

where 0_+ is indeterminant due to the infrared divergence. Hence, Γ_μ satisfies the differential dispersion relation

$$\left(-\lambda\frac{\partial}{\partial\lambda} + \beta\frac{\partial}{\partial g_R} - (1+\gamma_\theta)\, m_R\frac{\partial}{\partial m_R} - \gamma_\Gamma\right)\Gamma_\mu$$

$$= -\frac{g_R^5}{32\pi}\left[\gamma_\mu\left\{\frac{\pi}{2} + i\log\left(\frac{\lambda^2 q^2 - 4m^2}{0_+}\right)\right\} - \sigma_{\mu\nu}\frac{\lambda q^\nu}{2m_R}\right] m_R^2\,\delta\left(\lambda^2 q^2 - 4m_R^2\right)$$

$$+ \text{higher orders} \quad . \tag{3.2}$$

We shall make the approximation of neglecting the remaining terms in the RHS of (3.2). Further, we shall, for simplicity, take the coefficient functions as given by their lowest order forms

$$\begin{aligned} \beta(g_R) &= b_0 g_R^3 + \dots \\ \gamma_\Gamma(g_R) &= c_1 g_R^2 + \dots \\ \gamma_\theta(g_R) &= c_2 g_R^2 + \dots \end{aligned} \tag{3.3}$$

where b_0, c_1 and c_2 are given by perturbation. (We are in the Landau gauge.) Then, the particular integral for (3.2) corresponding to (2.12) is straightforward to evaluate. It is examined in the various asymptotic limits in Appendix III. A complete evaluation may be effected, for example, in terms of the transcendental functions $J^{\pm}_{s,c}$ defined by

$$\begin{aligned} J^{\pm}_s(y) &= \int_0^\infty dx \, \frac{(\sqrt{1+x^2} \pm 1)^{1/2}}{(1+x^2)^{r/2}} \sin xy \\ J^{\pm}_c(y) &= \int_0^\infty dx \, \frac{(\sqrt{1+x^2} \pm 1)^{1/2}}{(1+x^2)^{r/2}} \cos xy \quad , \end{aligned} \tag{3.4}$$

which are, of course, intimately connected with the functions of Whittaker. However, here, we shall not do this, for we are only concerned with the dependence on the scale λ in the various asymptotic limits. A complete evaluation will be the subject of later works.[12]

For $\lambda^2 \to 0$, we have (see Appendix III)

$$\Gamma^{(p)}_\mu \to \gamma_\mu \left(I_1^{(0)} + 0(\lambda^2) \right) + \frac{i\sigma_{\mu\nu} \lambda q^2}{2m_R} \left(I_2^{(0)} + 0(\lambda^2) \right) \tag{3.5}$$

and for $\lambda^2 \to \infty$ we have

$$\Gamma^{(p)}_\mu \to \gamma_\mu \left(\frac{I_1^{(0)}}{\lambda} + 0\left(\frac{1}{\lambda^3}\right) \right) + \frac{i\sigma_{\mu\nu} \lambda q^\nu}{2m_R} \left(\frac{I_2^{(0)}}{\lambda^3} + 0\left(\frac{1}{\lambda^5}\right) \right) \tag{3.6}$$

where $I_1^{(0)}$, $I_2^{(0)}$, $\overline{I}_1^{(0)}$, and $\overline{I}_2^{(0)}$ are defined in Appendix III. It is shown in Appendices III and VI that homogeneous solutions are here necessitated by the requirements of muon-electron universality as $\lambda^2 \to \infty$, as this appears to exist experimentally. The effect of this kind of solution need only be to cause the scale in (3.6) to be universal in (m_μ, m_e). Thus, (3.5) and (3.6) are complete representations of the dependence of Γ_μ on λ in the respective limits. We note that (3.5) is consistent with the conventional[8,9] perturbative representation of Γ_μ as $\lambda^2 \to 0$, since 0_+ in (3.2) is indeterminant so that Γ_μ is described by g_R, m_R and an indeterminant, just as the conventional solution is described by e_R, m, and the usual infrared cutoff. (See Appendix III for more discussion of this point.)

Of particular interest is the implication of (3.6) for the ratio

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{3.7}$$

To compute R, we need also our representation of the photon inverse propagator $D^{-1}_{\mu\nu}$. In Appendix III, we show that, in the Landau gauge,

$$D^{-1}_{\mu\nu}(\lambda q) \underset{\lambda \to \infty}{\longrightarrow} (\text{const} + \ldots)\,(g_{\mu\nu} - q_\mu q_\nu/q^2) \tag{3.8}$$

where ... is of order $m_R^2/\lambda^2 q^2$. This occurs in such away that $\Gamma D \Gamma$ retains its usual value (see Appendix V). Thus, as described in Refs. 1, we have that for $s \gtrsim s_0$,

$$R \to \frac{s\rho}{s_0} \tag{3.9}$$

where

$$\rho = \sum_i Q_i^2$$

is the quark charge content of J_{μ}^{EM} and s_0 is the energy at which (3.6) and (3.8) become operative. Assuming a gauge-theoretic view[19] of leptons, we estimate the size of s_0 to be given by the well-known $3\pi m_{\mu}^2/\alpha \simeq 13\ (\text{GeV})^2$.

As we pointed out in Ref. 1, if one takes Bjorken scaling to set-in at $Q^2 \gtrsim 1_+\ (\text{GeV})^2$, then the prediction for $\sigma(e^+e^- \rightarrow \text{hadrons})$ is that R is constant and equal to ρ for $1_+\ (\text{GeV})^2 \lesssim s \lesssim s_0 \simeq 10\ (\text{GeV})^2$ and that R rises linearly with s/s_0 with slope ρ for $10_+\ (\text{GeV})^2 \lesssim s$. These predictions are in agreement with the recently reported data[2] if ρ is taken to be as given by the fractionally charged 3-triplet model as shown by Fig. 4. In making this last remark, we are ignoring the existence of "new" physics relative to the data of Refs. 1. That is to say, as we already admitted above, we are ignoring the recently reported[17] resonances in $e^+e^- \rightarrow$ hadrons and $p+Be \rightarrow e^+e^-+x$. Therefore, while we would normally expect the rise in R to persist for some time beyond $s = 25\ (\text{GeV})^2$, the "new" physics responsible for these resonances may generate an entirely different behavior. This problem is, of course, under investigation.

In order to have a consistent picture of deep inelastic processes, one must verify that the conventional scaling prediction of Ref. 1 is not changed by (3.6) and (3.8). However, to see that this is so, it's only necessary to recall that in the "scaling" region the scattering of a fundamental fermion from a composite one, here, the proton, can be viewed as the incoherent sum of elemental scatterings between the respective constituents and the initial fundamental fermion, as shown in Fig. 5. Thus the scaling result is left unchanged by (3.6) and (3.8).

The crucial point is that the physically relevant combination $\Gamma D \Gamma$ remain unchanged from its usual value $\sim e^2/\lambda^2 q^2$ for <u>all</u> values of λ. As we indicated above, in Appendix V we argue that this is indeed the case.

Thus, the leptons, on account of their electromagnetic interactions, develop a "radius" $r \sim 1/\sqrt{s_0} \sim \left(\alpha/3\pi\, m_\mu^2\right)^{1/2} \sim \frac{1}{3.6}\ (\mathrm{GeV})^{-1}$. This radius then characterizes the very large momentum behavior of these particles until unitarity—higher order corrections and "new physics" dictate otherwise.

Returning to the general behavior of quantum electrodynamics we note that dimensional analysis will be generally violated by the Green's functions of the theory, since higher Green's functions may be constructed from $D_{\mu\nu}$ and Γ_μ. This point as well as its further implications will be discussed in more detail in later work.[12]

IV. LARGE-ANGLE SCATTERING

Aside from Bjorken scaling, perhaps the most intriguing experimental results are those of hadronic processes at large transverse momentum.[2] These data also appear to exhibit "dimensional" scaling laws for the invariant amplitudes: For example, in exclusive processes we have, generally,

$$\mathscr{M}_{\text{Had}} \rightarrow (\sqrt{s})^{4-n} \tag{4.1}$$

in the asymptotic fixed angle limit, where n is the number of elementary fields in the notation of Refs.20. In general, all large angle hadronic data have exhibited a power law falloff in scale, significantly different from what would have been expected. Several models[5,21] for these processes have been proposed, as we pointed out earlier. All such efforts have been characterized by an aspect of theoretical arbitrariness in the sense of empiricallyexcluding without fundamental reason certain graphs or types of interaction which one might otherwise expect to participate. In this section, we shall show that, upon assuming, for example, the quark model with massive renormalizable interactions, differential dispersion relations may permit a systematic treatment of the data under discussion.

For definiteness, we consider exclusive processes. Other processes will be taken up in later work. We are thus interested in

hadron a + hadron b $\rightarrow$ hadron c + hadron d

for example. The corresponding amplitude will be written as in (2.18)

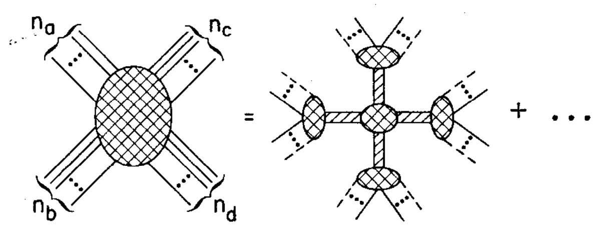
$$\mathscr{M}^{\text{Had}}_{ab;cd} = \int \phi^{c}_{BS}\,\phi^{d}_{BS}\,\mathscr{M}^{\text{irr}}_{cd;ab}\,\phi^{a}_{BS}\,\phi^{b}_{BS}\,\Pi d^4k_j \tag{4.2}$$

To establish (4.1), it is sufficient to show that (1) the $\{\phi^{i}_{BS}\}$ are finite at the origin and (2) $\mathscr{M}^{\text{irr}}$ scales "dimensionally".[20] We consider first the behavior of $\mathscr{M}^{\text{irr}}$.

Now by (2.19)

$$\left(-\lambda \frac{\partial}{\partial\lambda} + \beta\frac{\partial}{\partial g} - (1+\gamma_\theta)\, m_R \frac{\partial}{\partial m_R} - \gamma_{\mathcal{M}}\right)\mathcal{M}^{irr} = \sum \rho_\alpha\, \delta\!\left(\lambda^2\left(\Sigma_{j_\alpha}\right)^2 - m_\alpha^2\right) \qquad (4.3)$$

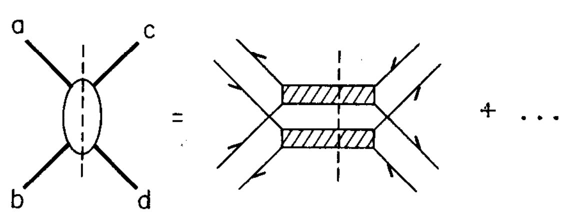
with ρ_α, m_α, β, γ_θ, γ, λ, and m_R defined as in (2.19). By assumption, a, b, c, and d are bound states of n_j fields, j=a, b, c, d, respectively. Thus, the thresholds in $\mathcal{M}^{irr}$ must necessarily reflect this fact. For we have



(4.4)

where ... stands for similar generalized interchanges.[20,21] As can be seen, the thresholds in $\mathcal{M}^{irr}$ occur in general with coefficients which are precisely the wavefunctions corresponding to the various physical states of the theory. The orthogonality and completeness properties of these states then assure that only those thresholds with the BS wavefunction residues corresponding to physical particles will contribute to $\mathcal{M}^{Had}$.

To proceed further we must pin down more precisely the form of the thresholds in (4.4). The general case, which is rather cumbersome, will not be discussed here.[12] Rather, for simplicity and illustration, we consider here the case where a=b=c=d=π, i.e., a bound state of $q\bar{q}$. Then,



(4.5)

In order to violate naive dimensional analysis, with bound poles, at least two such poles must occur, on account of the fact that every interaction of fields is here presumed to be appropriately massive (see Appendix II). It therefore follows that only discontinuities such as that exhibited in (4.5), will contribute to the RHS of (4.3) in such a way as to make a contribution to $\mathcal{M}^{\mathrm{Had}}$. Thus, from (4.5) we have

$$\left(\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\lambda\frac{\partial}{\partial\lambda}\right)\mathcal{M}^{\mathrm{irr}}(\lambda r_j) = -8\,\mathcal{M}^{\mathrm{irr}}$$
$$+\bar{\rho}\left(\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\lambda\frac{\partial}{\partial\lambda}\right)\left[m_\pi^4\,\delta\left(\lambda^2\left(\Sigma r_{i_1}\right)^2-m^2\right)\delta\left(\lambda^2\left(\Sigma r_{i_2}\right)^2-m^2\right)\right]$$
$$+\ (\text{more singular terms})\quad , \tag{4.6}$$

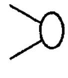
where $\bar{\rho}$ is determined by BS wavefunction residues. We therefore see from (4.6) and (4.3) that (see the discussion following (AII.15), Appendix II),

$$\mathcal{M}^{\mathrm{irr}} \to \frac{1}{\lambda^4} = \frac{1}{(\sqrt{s})^{n-4}} \tag{4.7}$$

as $\lambda \to \infty$, where n = number of elementary fields, in agreement with Brodsky and Farrar, Matveev et al., Blankenbecler et al., and Gunion.[20, 21] The result of Landshoff[22] would thus appear to be an artifact of keeping only the lowest order terms in $\mathcal{M}^{\mathrm{irr}}$; this artifact does not hold true for our complete solution for this amplitude.

We shall therefore have established dimensional counting for $\pi\pi \to \pi\pi$ if we can show that the wavefunctions are finite at the origin. So let us now turn to this issue.

As pointed out in Appendix II, the BS wavefunction satisfies

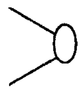
$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta(g)\frac{\partial}{\partial g}-(1+\gamma_\theta)\,m_R\frac{\partial}{\partial m_R}-\gamma_\phi\right)$$  $$= \sum_\alpha \rho_\alpha\,\delta\left(\lambda^2\left(\Sigma p_{j_\alpha}\right)^2-m_\alpha^2\right) \tag{4.8}$$

where  denotes $\phi_{BS}(\lambda r)$ with the legs amputated. Now from our arguments above, it is clear that the RHS of this last equation has the effective structure

(see Appendix II)

$$m_R^2 \left[\delta\left(\lambda^2 k^2 - (m_B + m_R)^2\right) - \delta\left(\lambda^2 k^2 - (m_B - m_R)^2\right)\right] \tag{4.9}$$

(m_B = bound state mass) so that



$$\xrightarrow[\lambda^2 \to \infty]{} 1/\lambda^4 \quad , \tag{4.10}$$

homogeneous solutions again being disallowed by our choice of <u>the</u> solution of PDE, and the relations (2.17), as discussed above. Then

$$\phi_{BS} \xrightarrow[\lambda^2 \to \infty]{} 1/\lambda^6 \quad , \tag{4.11}$$

so that $\phi(x)$ is finite at the origin as desired.[23]

We have therefore established dimensional counting for meson meson exclusive scattering. The general case will be discussed in later work.[12] We would like to emphasize that for processes just considered, the basic counting law derives from the fact that the smallest number of fields participating in a bound state is two, and the delta function resulting from the respective pole is quadratic in the scale, so that there is a power of $\lambda^{-1/2} = \lambda^{4/n - 1}$ for each elementary field. For other processes, the situation may be more involved. To repeat, we shall consider the general case in later work.

V. DISCUSSION

We have therefore presented an alternative approach to asymptotic distance behavior in renormalizable field theory. As is apparent from the general discussion, this approach is highly nonperturbative in its predictions though its foundation lies almost entirely in perturbation theory. These predictions are in reasonable accord with experiment.

Specifically, the recent high q^2-experiments tend to indicate that the results in Sections III and IV are indeed manifested in nature. In particular, the apparent difference in scale between the space-like and time-like data for the hadronic electromagnetic current has here been given a natural explanation. We should remark in our approach that the weak interactions, owing partly to the massiveness of the intermediate vector boson, behave conventionally insofar as scaling is concerned. However, violations[24] of Bjorken scaling in these experiments could be incorporated into our formalism, but not within the conventional view of leptonic weak interactions.

Our approach, as constructed, agrees with known low-momentum phenomenology. As we pointed out above, this regime is already sufficiently well understood. Thus, as we promised, we only ventured into it to "settle" certain matters relating to boundary conditions.

Our argument against the general inclusion of additional homogeneous solutions in our analysis is not strict, being essentially the existence of limits in small effective coupling as stated in the text above and Appendix III. We would like to repeat that there is in principle no sacred dictum requiring this kind of behavior, except, in general, the experiments apparently!

In the text above we have only considered explicitly the asymptotic behavior of the processes

$$e^+ + e^- \rightarrow \text{hadrons}$$

$$e + p \rightarrow e + x$$

$$\text{meson a} + \text{meson b} \rightarrow \text{meson c} + \text{meson d}$$

$$\text{at fixed angle, } s \rightarrow \infty$$

taking Bjorken scaling to be explained as briefly described in the previous Letter.[1] We have focused these processess because they are of central interest. We would like to emphasize again, however, that the ideas in the text above clearly have general applicability and represent a systematic approach to asymptotic behavior in renormalizable field theory. More detailed analyses of all of the various processes will appear elsewhere.[12]

We have thus been able to discuss the general high energy behavior of interactions in terms of the respective threshold structures. This may seem rather unintuitive at first sight. However, looked at more carefully, it becomes clear that this should be possible. For, on the intuitive scale, it's well-known that duality[25] is manifest throughout the interactions of particles, giving immediately that

$$\text{sum over resonances} \approx \text{sum over high energy.}$$

This is clearly manifest throughout the above discussion. As a corollary it follows that continuity in dynamics is also manifest in our approach, as we promised in Section I. Then, on the technical side, it's well-known[26] that already in perturbation theory the Feynman integrals may be written entirely in terms of singularities corresponding to the vanishing of one or more propagators and, thereby, to the threshold structure of the respective theory. Thus, the discussion here is nothing more than a natural systematic synthesis of these well-known physical phenomena.

Let us end by re-emphasizing that the approach presented here represents the basis for a complete discussion of the asymptotic behavior of interacting field theory under the constraint of renormalizability. More detailed implications will be taken up in subsequent works.

Acknowledgment

I would like to thank J. D. Bjorken, D. J. Gross, K. Symanzik, and S. B. Treiman for numerous critical discussions, especially K. S.

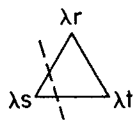
APPENDIX I

PERTURBATIVE VIOLATIONS OF DIMENSIONAL ANALYSIS

In this appendix we shall establish the violation of naive dimensional scaling in various renormalized perturbation theories.

A. $\mathscr{L}_I = g\phi^4$ (massless)

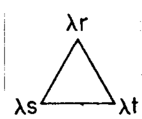
In this theory, the diagrams illustrated in Fig. 6 give, for example, the singularity



$$= \frac{\theta(\lambda^2 s^2)}{\lambda^2 s^2} f(r, s, t) \quad , \tag{AI. 1}$$

the function f being given by standard methods. From the work of Symanzik[4] it can be shown that this latter result persists to all orders in perturbation theory. Here, however, we shall think of it as being strictly perturbative. With this understanding we have the violation

$$\left(\mu \frac{\partial}{\partial\mu} + \lambda \frac{\partial}{\partial\lambda}\right)\left(\mu^2 \right.$$

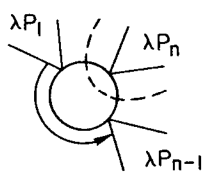


$$\left.\right) = 2\mu^2\, \delta(\lambda^2 s^2)\, f + \ldots \tag{AI. 2}$$

where ... represents the violation from the remaining thresholds. Hence, the six-point irreducible vertex violates naive dimensional analysis, for example.

For $n > 3$, the 2n-point 1PI vertex is even stronger in its violation, since it contains

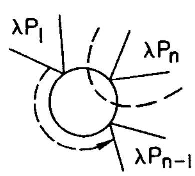
$$\Gamma_{2n} =$$



$$= \frac{\theta(\lambda^2 p_n^2)}{(\lambda p_n)^{2n-4}} f(p_1, \ldots, p_{n-1}) \tag{AI. 3}$$

giving

$$\left(\mu \frac{\partial}{\partial\mu} + \lambda \frac{\partial}{\partial\lambda}\right)\left(\mu^{2n-4} \right.$$



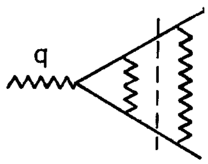
$$\left.\right) = \frac{2\mu^{2n-4}\, \delta(\lambda^2 p_n^2)}{(\lambda p_n)^{2n-6}} f_{2n} + \ldots \tag{AI. 4}$$

Hence naive dimensional analysis manifestly is violated.

The finiteness of the on-shell amplitudes in massive ϕ^4 would appear to forbid the violation of dimensional analysis in that theory's fully on-shell perturbative solutions. But, the phenomenon reappears off-shell, as can be seen from Fig. 6. It may also reappear through bound poles as pointed out in the text.

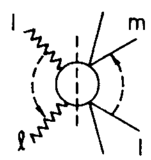
B. $\underline{\mathscr{L}_I = g_a \phi_1 \phi_2^2 + g_b \phi_2^4 ,\ m_{\phi_1} = 0 ,\ m_{\phi_2} \neq 0}$

Here, we have a dimension 3 vertex so that we expect that naive dimensional analysis will be badly violated. We have



$$\propto \frac{\theta\left(\lambda^2 q^2 - 4m_{\phi_2}^2\right)}{\lambda^2 q^2 - 4m_{\phi_2}^2}\, m_{\phi_2}^{-2} f \qquad \text{(AI. 5)}$$

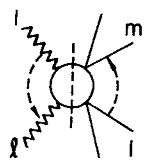
where f is dimensionless, giving a violation in analogy with (AI. 2). In general for the ℓ-ϕ_1, 2m-ϕ_2, 1PI vertex, we have



$$= m_{\phi_2}^{6-2\ell-2m} \frac{\theta\left(\lambda^2 (\Sigma q_j)^2 - 4m_{\phi_2}^2\right)}{\left(\lambda^2 (\Sigma q_j)^2 - 4m_{\phi_2}^2\right)} f_{\ell,m} \quad , \qquad \text{(AI. 6)}$$

thereby giving

$$\left(m\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial\lambda}\right)\left[m_{\phi}^{2\ell+2m-4} \right.$$



$$\left.\right] \propto m_{\phi_2}^2\, \delta\left(\lambda^2 (\Sigma q_j)^2 - 4m_{\phi_2}^2\right) f_{\ell,m} \qquad \text{(AI. 7)}$$

Now, if both ϕ's are massive, then, on account of the super-renormalizable interaction and the off-shell behavior of ϕ^4, the amplitudes will still exhibit violations of dimensional analysis in perturbation theory, but away from the usual physical regions.

C. $\underline{\mathscr{L}_I = g_s \bar{\psi}\psi\phi_1 + ig_p \bar{\psi}\gamma_5\psi\phi_2, \quad m_\psi \neq 0, \quad m_\phi = 0}$

With the usual free Lagrangian for ψ, $\bar{\psi}(i\not\partial - m_\psi)\psi$, the situation is entirely analogous to case B, except for the γ-matrix algebra resulting from

$$\frac{1}{\not p - m_\psi + i\epsilon} = \frac{\not p + m_\psi}{p^2 - m_\psi^2 + i\epsilon} \tag{AI.8}$$

and the vertices. Thus, again dimensional analysis is manifestly violated in the theory.

D. $\underline{\mathscr{L}_I = e\bar{\psi}\gamma_\mu \psi A^\mu, \quad m_A = 0, \quad m_\psi \neq 0}$

This case, Q. E. D., was already pointed out to violate dimensional analysis in Ref. 1. In Appendix III, we discuss the proper vertex Γ_μ in detail. However, again in analogy with B, it's clear that, generally, the proper vertices manifest the violation.

E. $\underline{\mathscr{L}_I = ie\,\phi^* \overleftrightarrow{\partial}_\mu \phi A^\mu + e^2\phi^2 A^2, \quad m_\phi \neq 0, \quad m_A = 0; \; m_\phi = 0, \quad m_A \neq 0}$

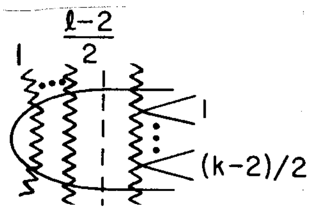
Aside from the occurrence of the momentum $p'_\mu + p_\mu$ at the derivative vertices, this theory's functions clearly behave similarly to case B (owing to the first interaction), giving a corresponding violation of dimensional analysis. The second interaction behaves analogously to the A^2B^2 terms in the next example. Thus, the violation phenomenon will again be manifest.

F. $\underline{\mathscr{L}_I = -ie[\partial_\mu B^*_\nu (A^\nu B^\mu - A^\mu B^\nu) - \partial_\mu B_\nu (A^\nu B^{*\mu} - A^\mu B^{*\nu})] + e^2(A^2 B^*_\mu B^\mu - A\cdot B\, A\cdot B^*),}$

where

$$m_A = 0, \quad m_B \neq 0 \quad .$$

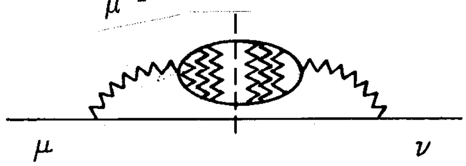
For k, $\ell \geq 4$, the ℓ-A_μ, k-B_μ 1PI vertex receives a contribution from



$$= \frac{\theta\left(\lambda^2(\Sigma q_j)^2 - 4m_B^2\right)}{\left(\lambda^2(\Sigma q_j)^2 - 4m_B^2\right)^p}\, m_B^s f \quad , \qquad \text{(AI.9)}$$

where f is dimensionless, $p = \ell/4 + (k-3)/2$, and $s = 1 - \ell/2$.

The $2B_\mu$-point 1PI vertex receives contributions from



$$= \frac{\theta\left(\lambda^2 q^2 - 9m_B^2\right)}{\lambda^2 q^2 - 9m_B^2}\, m_B^4 g \qquad \text{(AI. 10)}$$

where g is dimensionless. Both (AI. 10) and (AI. 11) are seen to violate dimensional analysis in analogy with (AI. 2).

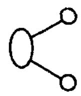
Also, a $\phi_1^2\phi_2^2$ term in case B above would behave precisely as the second term here.

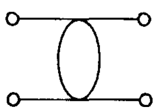
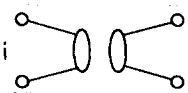
The introduction of non-Abelian local symmetry into the interactions above does not change any of the conclusions. Hence, the perturbative violation of dimensional analysis appears to be a natural aspect of long range forces. It is also generally manifest in the off-shell Green's functions of fully massive theories, due to internal particles propagating freely on-shell when the external lines are appropriately off-shell. Dimensional analysis violations therefore abound in renormalizable field theory.

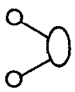
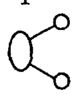
APPENDIX II

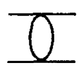
BETHE-SALPETER WAVEFUNCTION

In this appendix, we shall give the well-known derivation of the renormalization group equation for the Bethe-Salpeter (BS) wavefunction and discuss the respective dimensional analysis violating singularity structure which leads to Eq. (4.9) of the text. We restrict our attention to bound states of two elementary fields.

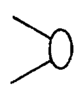
The BS wavefunction, , of the bound state of mass M_B is defined through the residue of the corresponding pole in the renormalized BS equation for the two body propagator[10,27]

 $=$  $\Big/ \, (2\pi)^4\left(k^2-m_B^2\right)$ + terms regular near the pole , (AII. 1)

where the legs are not amputated, k is the total four momentum of the system, and  is the conjugate of . As the theory is renormalizable, we have, upon amputating the legs,

$$\mu \frac{\partial}{\partial\mu} Z_B^{-1}$$  $= 0$ (AII. 2)

where μ is the renormalization point and Z_B is the appropriate renormalization constant. Thus, on substituting the RHS of (AII. 1) into (AII. 2) and extracting the various independent functions of external momenta, we have

$$\left[\mu \frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_\theta \quad m_R \frac{\partial}{\partial m_R} - \gamma_B\right]$$  $= 0$ (AII. 3)

where

$$\gamma_B = \frac{1}{2}\mu \frac{\partial}{\partial\mu} \log Z_B \tag{AII. 4}$$

and β, γ_θ, and m_R have their usual meanings. As pointed out in the text, the notations $\beta\frac{\partial}{\partial g}$ and $\gamma_\theta \; m_R \frac{\partial}{\partial m_R}$, etc. are to be understood as abbreviations for the respective sums of all such quantities in a theory with several g's and m_R's.

In the model discussed in Section IV, we consider the pion as a bound state of quark-antiquark due to renormalizable massive interactions. Therefore, the singularity structure of the BS wavefunction  is given by Fig. 7. The 1-2 (-3) singularities depicted in Fig. 7 give for the respective dimensional analysis violation amplitude ρ

$$\rho \propto \left(\mu\frac{\partial}{\partial\mu} + \lambda\frac{\partial}{\partial\lambda} + m_R\frac{\partial}{\partial m_R}\right)\left\{\frac{\theta\left(\lambda^2k_1^2-(m_B+m_f)^2\right) m_R^4}{\left[\lambda^2k_1^2-(m_B-m_f)^2\right]\left[\lambda^2k_1^2-(m_B+m_f)^2\right]} + \frac{\theta\left(\lambda^2k_1^2-(m_B-m_f)^2\right) m_R^4}{\left[\lambda^2k_1^2-(m_B-m_f)^2\right]\left[\lambda^2k_1^2-(m_B+m_f)^2\right]}\right\} f + \ldots \qquad \text{(AII. 5)}$$

where f is dimensionless, ... represents terms as or less "significant" than that exhibited, and we take $\lambda^2k_1^2=\lambda^2k_2^2$ for definiteness. Hence,

$$\rho \propto \frac{m_R^4 f}{m_f m_B}\left\{\delta\left(\lambda^2k_1^2-(m_B+m_f)^2\right) - \delta\left(\lambda^2k_1^2-(m_B-m_f)^2\right)\right\} \qquad \text{(AII. 6)}$$

in agreement with (4. 9).

APPENDIX III

PROPER VERTEX FUNCTION OF Q. E. D.

In this appendix, we shall give the details of the differential dispersive evaluation of the proper vertex function, Γ_μ, of quantum electrodynamics. Specifically, as we pointed out in Section II, this function satisfies

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta\frac{\partial}{\partial g_R}-m_R(1+\gamma_\theta)\frac{\partial}{\partial m_R}-\gamma_\Gamma\right)\Gamma_\mu(\lambda q)=-\left[\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\lambda\frac{\partial}{\partial\lambda}\right]\Gamma_\mu \tag{AIII. 1}$$

Here, we shall give a method for evaluating the particular integral (2. 12) for this equation in the regions $\lambda^2 \to 0$ and $\lambda^2 \to \infty$, under the assumption of small coupling g_R.

We begin by introducing the result of Barbieri et al.[18] for Im Γ_μ into the RHS of (AIII. 1): We find

$$\left(\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\lambda\frac{\partial}{\partial\lambda}\right)\mathrm{Im}\,\Gamma_\mu=\frac{m_R^2g_R^5}{32\pi}\left[\log\xi\;\gamma_\mu+i\sigma_{\mu\nu}\frac{q^\nu\lambda}{2m_R}\right]\delta\left(\lambda^2q^2-4m_R^2\right)+0\left(g_R^7\right) \tag{AIII. 2}$$

where we have introduced the infrared cutoff through

$$\lim_{r\to 0_+}\log\frac{\lambda^2q^2-4m_R^2}{rm_R^2}\,\delta\left(\lambda^2q^2-4m_R^2\right)\equiv\log\xi\,\delta\left(\lambda^2q^2-4m_R^2\right) \tag{AIII. 3}$$

It's well-known how to handle this infrared infinity in applications so we shall not concern ourselves with it here. The meaning of ξ will be discussed in later work.[12]

We next observe that the singularities in Re Γ_μ and Im Γ_μ are not unrelated, since Re Γ_μ may determined from Im Γ_μ through ordinary dispersion relations.

(See Ref. 18, for example.) This gives

$$\left(\lambda\frac{\partial}{\partial\lambda}+m_R\frac{\partial}{\partial m_R}+\mu\frac{\partial}{\partial\mu}\right)\Gamma_\mu=\frac{g_R^5 m_R^2}{32\pi}\left[\left(\frac{\pi}{2}+i\log\xi\right)\gamma_\mu-\sigma_{\mu\nu}\frac{\lambda q^\nu}{2m_R}\right]\delta\left(\lambda^2q^2-4m_R^2\right)$$

$$+\text{ higher orders} \tag{AIII. 4}$$

Following the discussion in the text above, we find it convenient to write (AIII. 4) in terms of the t, v, and w from (2. 14). We then have finally

$$\left(-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}-\gamma_\Gamma/2\right)\Gamma_\mu=-\frac{g_R^5 m_R^2}{64\pi}\left[\left(\frac{\pi}{2}+i\log\xi\right)\gamma_\mu-\sigma_{\mu\nu}\frac{\lambda q^\nu}{2m_R}\right]\delta\left(\lambda^2q^2-4m_R^2\right)$$

$$=-\frac{g_R^5 m_R^2}{64\pi}\left[\left(\frac{\pi}{2}+i\log\xi\right)\gamma_\mu-\sigma_{\mu\nu}\frac{\lambda q^\nu}{2m_R}\right]\left[\frac{\delta\left(\lambda\sqrt{q^2}-2m_R\right)+\delta\left(\lambda\sqrt{q^2}+2m_R\right)}{4m_R}\right] \tag{AIII. 5}$$

with the particular integral given by (2. 12).

This particular integral for (AIII. 5) becomes more transparent when we note that from (2. 14) and (3. 3) there follow

$$m_R=m_0\left[\left(1+c_2g_0^2\right)e^{-b_0w/c_2}-c_2g_0^2\right]^{c_2/2b_0} \tag{AIII. 6}$$

and

$$g_R^5=\frac{g_0^5}{\left(1-b_0g_0^2v\right)^{5/2}}=\frac{1}{b_0^{5/2}\,\Gamma(5/2)}\int_0^\infty du\,u^{3/2}\,e^{-\left(1/b_0g_0^2-v\right)u} \tag{AIII. 7}$$

where we take g_0 to be small relative to g_R, and $\Gamma(p)$ is the gamma function.

Further, we shall also for convenience replace $\log\xi$ by its coupling constant independent part (presuming this to be nonzero),[12] without loss of generality, since the numerator of the Coulomb potential, e/r, is a parameter, conventionally defined as the physical charge e.

From (AIII.6) and (AIII.7), it's clear that we may iterate (AIII.5) in γ_Γ. To wit, ignoring γ_Γ for the moment we have our first approximation ${}^P\Gamma_\mu^{(0)}$, to the particular solution (2.12) for Γ_μ

$$ {}^P\Gamma_\mu^{(0)} = f_1^{(0)} \gamma_\mu + i\sigma_{\mu\nu} \frac{\lambda q^\nu}{2m_R} f_2^{(0)} \qquad \text{(AIII.8)}$$

where

$$-64\pi f_1^{(0)} = \left(\frac{\pi}{2} + i \log\xi\right) \left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)^{-1} \int_0^\infty ds_1\, s_1^{3/2} \frac{e^{-\left(1/b_0 g_0^2 - v\right) s_1}}{b_0^{5/2}\,\Gamma(5/2)}$$

$$\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{s_2 w}\, \overline{m}_R(s_2)\, ds_2$$

$$\frac{\int dk}{8\pi} \left\{\exp ik\left(\lambda\sqrt{q^2} - 2m_R(w)\right) + \exp ik\left(\lambda\sqrt{q^2} + 2m_R(w)\right)\right\} \qquad \text{(AIII.9)}$$

and

$$-64\pi f_2^{(0)} = \frac{2m_R}{\lambda\sqrt{q^2}} \left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)^{-1} \left\{ \frac{i}{b_0^{5/2}} \int_0^\infty ds_1\, s_1^{3/2} \frac{e^{-\left(1/b_0 g_0^2 - v\right) s_1}}{\Gamma(5/2)} \right.$$

$$\frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{s_2 w}\, \overline{m}_R(s_2)\, ds_2$$

$$\left. \frac{\int dk}{8\pi} \left\{\exp ik\left(\lambda\sqrt{q^2} - 2m_R(w)\right) - \exp ik\left(\lambda\sqrt{q^2} + 2m_R(w)\right)\right\} \right\}, \qquad \text{(AIII.10)}$$

where we have introduced the familiar Laplace transform

$$\bar{m}_R(s) = \int_0^\infty e^{-sw'}\, m_R(w')\, dw' \qquad \text{(AIII. 11)}$$

and $\delta > 0$.

The first corrections, $\{f_i^{(1)}\}$, to the $\{f_i\}$ are clearly

$$f_1^{(1)} = \frac{c_1}{2b_0}\left[-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}\right]^{-1} f_1^{(0)} \int_0^\infty ds\, e^{-\left(1/b_0 g_0^2 - v\right)s} \qquad \text{(AIII. 12a)}$$

$$f_2^{(1)} = \frac{c_1}{2b_0}\,\frac{m_R}{\lambda}\left[-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}\right]^{-1} f_2^{(0)} \int_0^\infty ds\, e^{-\left(1/b_0 g_0^2 - v\right)s}\;, \qquad \text{(AIII. 12b)}$$

subsequent corrections being given by induction.

As one may verify, the corrections to (AIII. 9) and (AIII. 10) given (and implied) by (AIII. 12) are indeed small if g_R is small, so that our iteration makes sense. Thus, we shall work only with (AIII. 9) and (AIII. 10) in what follows, leaving the detailed discussion of the corrections to subsequent work.

Before proceeding further, we should mention that one may construct the particular integrals of (2.4) after Symanzik.[4] However, though this would be just as direct, we do not consider it as transparent, physically. For this reason, we have chosen to use (2. 12). This has caused no loss or gain of content.

Returning to (AIII. 9) and (AIII. 10), we now have to determine what (if any) homogeneous integrals are necessary to fully specify Γ_μ. To make this decision, we shall examine the region $\lambda^2 \to 0$, as the data in this region are well-understood. Toward this end, we first note that, as pointed out in Section II, the complete characteristics of the homogeneous version of (AIII. 5) are given by

$$-\frac{d\lambda}{\lambda} = \frac{dg}{\beta} = \frac{-dm_R}{(1+\gamma_\theta)\, m_R} = \frac{d(\Gamma_\mu)_{\alpha\beta}}{\gamma(\Gamma_\mu)_{\alpha\beta}}$$

It's well-known[13] that any homogeneous integral may be written as

$$\Gamma_{\mu,\,\text{homogeneous}} = F_\mu\left(\bar{g}(t), \bar{m}_R(t)\right) \exp -\int_0^t dt' \, \gamma(\bar{g}(t')) \quad , \qquad \text{(AIII. 13)}$$

where $\bar{g}$, $\bar{m}_R$ are the usual effective coupling and mass, respectively:

$$\frac{2d\bar{g}}{dt} = \beta(\bar{g}) \quad , \qquad \bar{g}(0, g_R) = g_R \qquad \text{(AIII. 14)}$$

$$\frac{2d\bar{m}_R}{dt} = -\left(1 + \gamma_\theta(\bar{g})\right)\bar{m}_R(t) \quad , \qquad \bar{m}_R(0, m_R) = m_R \qquad \text{(AIII. 15)}$$

Therefore, from (3.3) and (AIII. 12) it's clear that $\Gamma_{\mu,\,\text{homogeneous}}$ is singular where $\bar{g} \rightarrow 0$. Since the constant b_0 is positive for quantum electrodynamics, this function $\bar{g} \rightarrow 0$ precisely as $\lambda^2 \rightarrow 0$, as is well-known. Consequently, because we require Γ_μ to exist as a perturbation series as $\lambda^2 \rightarrow 0$, one might think that we should exclude these contributions from Γ_μ entirely! However, as one can see from (AIII. 9) and (AIII. 10), our representations of $f_{1,2}$ are highly nonperturbative as functions of g_R. Indeed, below we shall identify a certain nonperturbative function of g_R with the physical charge. Thus within our framework, the homogeneous integrals are in general allowed.

In fact, one can say more here. For the results $f^{(0)}_{1,2}$ in (AIII. 9) and (AIII. 10) do not reflect manifestly the experimentally observed muon-electron universality of quantum electrodynamics for $\lambda^2 \rightarrow 0$. It is therefore necessary to consider homogeneous integrals in addition to $f^{(0)}_{1,2}$ in order to maintain this symmetry within the approximation scheme which we are using. For, indeed, the electron's violations of dimensional analysis are trivial to the order to which we are working in g_R, since we take[19]

$$m_e \sim g_R^2 m_\mu \quad .$$

Hence, here the electron's Γ_μ is entirely homogeneous. As we shall see in Appendix VI, it would appear to be always possible to choose this vertex to satisfy the usual requirements of μ-e universality. So, let us discuss the various limits of (AIII. 9) and (AIII. 10), assuming the requirements of μ-e universality have been satisfied and, thus, that no further homogeneous integrals are necessary.

Making use of the trivial identity, for Re $a > 0$,

$$\frac{1}{a} = \int_0^\infty d\nu\ e^{-a\nu} \tag{AIII. 16}$$

we have formally

$$-64\pi f_1^{(0)} = \int_0^\infty g_R^5(v-\nu)\ m_R(w-\nu)\ \frac{\int dk}{8\pi} \left\{ \exp ik\left(e^{\frac{1}{2}(t+\nu)}\sqrt{q^2} - 2m_R(w-\nu)\right) + \exp ik\left(e^{\frac{1}{2}(t+\nu)}\sqrt{q^2} + 2m_R(w-\nu)\right)\right\} d\nu \tag{AIII. 17}$$

and

$$-64\pi f_2^{(0)} = \frac{2m_R}{\lambda\sqrt{q^2}}\ \frac{\int_0^\infty i d\nu}{b_0^{5/2}}\ g_R^5(v-\nu)\ m_R(w-\nu)\frac{\int dk}{8\pi}\left\{\exp ik\left(e^{\frac{1}{2}(t+\nu)}\sqrt{q^2} - 2m_R(w-\nu)\right) - \exp ik\left(e^{\frac{1}{2}(t+\nu)}\sqrt{q^2} + 2m_R(w-\nu)\right)\right\} \tag{AIII. 18}$$

where $g_R^5(v)$ and $m_R^2(w)$ are given by (AIII. 6) and (AIII. 7).

The limits $\lambda^2 \to 0$ of (AIII. 17) and (AIII. 18) can be seen to be independent of m_R and therefore in the combination of $\Gamma D \Gamma$ (see Appendix IV for a discussion of $D_{\mu\nu}$), which is the only one of physical significance, may be identified, in the usual way, with the physical electron charge e_R, and the electron's g-2 in the standard fashion. Of course, in the perturbation theory approach,[8, 9] the relationship between e and g-2 has been the subject of considerable theoretical effort,[28] having been computed to order α^3. In our approach, one can of course carry out the same calculation of such a coupling constant relationship, since we shall argue below that $\Gamma D \Gamma$ is unchanged from its usual value. However, the precise relationship between g-2 and e_R cannot be taken too seriously as given here, since we have driven the differential dispersion relations with only the first (violation) term of what is actually a series in coupling of such terms with singular coefficients, succeeding coefficients being more and more singular. But, the behavior of central interest is that for $\lambda^2 \to \infty$, since $\lambda^2 \to 0$ is well-understood. Succeeding terms in the driving series make contributions to Γ_μ down by powers of $1/\lambda$ relative to the first term, as $\lambda^2 \to \infty$. Thus, we may expect the $\lambda^2 \to \infty$ behavior of Γ_μ to be accurately given by our approximation.

In the limit $\lambda^2 \to 0$, we have

$$
\begin{aligned}
f_1^{(0)} &\to I_1^{(0)} + I_1^{(1)} \frac{q^2 \lambda^2}{m_R^2} + \ldots \\
f_2^{(0)} &\to I_2^{(0)} + I_2^{(1)} \frac{q^2 \lambda^2}{m_R^2} + \ldots
\end{aligned}
\tag{AIII. 19}
$$

and, for $\lambda^2 \to \infty$ we have

$$
\begin{aligned}
f_1^{(0)} &\to \frac{1}{\lambda} \bar{I}_1^{(0)} + 0\left(\frac{1}{\lambda^3}\right) \\
f_2^{(0)} &\to \frac{1}{\lambda^3} \bar{I}_2^{(0)} + 0\left(\frac{1}{\lambda^5}\right)
\end{aligned}
\tag{AIII. 20}
$$

where $I_j^{(k)}$ and $\bar{I}_j^{(k)}$ are clearly determined by (AIII. 18) and (AIII. 19) and boundary conditions. The precise meanings of $I_j^{(k)}$ and $\bar{I}_j^{(k)}$ will be the subject of later work. Here we are only interested in λ-dependence.

The implications of (AIII. 19) and (AIII. 20) are discussed in Section IV.

APPENDIX IV

PHOTON PROPAGATOR

Here, we consider the asymptotic behavior of $D^{-1}_{\mu\nu}$, the inverse photon propagator. We have,

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta\frac{\partial}{\partial g}-(1+\gamma_\theta)\,m_R\frac{\partial}{\partial m_R}-\gamma_D+2\right)D^{-1}_{\mu\nu}=-\left(\mu\frac{\partial}{\partial\mu}+m_R\frac{\partial}{\partial m_R}+\lambda\frac{\partial}{\partial\lambda}-2\right)D^{-1}_{\mu\nu}$$

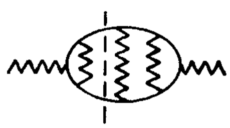
(AIV. 1)

where

$$\gamma_D = c_D g_R^2 + \dots \tag{AIV. 2}$$

We shall be interested only in the $\lambda^2 \to \infty$ behavior of $D^{-1}_{\mu\nu}$, as the $\lambda^2 \to 0$ behavior of Γ and D has been accurately described by Refs. 8 and 9.

The lowest violation of dimensional analysis in this Green function comes from

 + . . .

We have

$$\left(m_R\frac{\partial}{\partial m_R}+\mu\frac{\partial}{\partial\mu}+\lambda\frac{\partial}{\partial\lambda}-2\right)D^{-1}_{\mu\nu}(\lambda q)=\frac{ig_R^8\lambda^2}{(2\pi)^2}\left(q_\mu q_\nu-q^2g_{\mu\nu}\right)m_R^2\,\delta\!\left(\lambda^2q^2-4m_R^2\right)f$$

(AIV. 3)

where f is a dimensionless, singular, calculable function.[12]

Using (AIV. 1) we obtain, in analogy with the discussion of Γ_μ,

$$D^{-1}_{\mu\nu}(\lambda q)\;\xrightarrow[\lambda^2\to\infty]{}\;(\text{const})\left(q_\mu q_\nu-q^2g_{\mu\nu}\right) \tag{AIV. 4}$$

where const may be determined from (AIV. 3).

The evaluation of (const) will not be attempted here, as it is not necessary for the present analysis. This will be taken up elsewhere.[12] We are here only interested in the large λ dependence of $D^{-1}_{\mu\nu}(\lambda q)$. (A slightly more complete representation of $D_{\mu\nu}$ can be found in the next appendix.)

From (AIV.4) and (AIII.20) it's clear that as $\lambda \to \infty$,

$$\Gamma D \Gamma \sim 1/\lambda^2 q^2 \quad .$$

We have not shown explicitly that $\Gamma D \Gamma$ is in fact always equal to its conventional value. This is the subject of the next appendix.

APPENDIX V

BEHAVIOR OF $\Gamma D \Gamma$ IN Q. E. D.

In this appendix, we shall show that the results of Appendices III and IV imply that when $\Gamma(\lambda q)$ starts behaving as $1/\lambda$, $D^{-1}_{\mu\nu}(\lambda q)$ starts behaving as const in such a way that $\Gamma D \Gamma$ retains its usual value. To establish this result, we proceed as follows:

For clarity, we first restate the renormalization group property:

$$\left(-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}-\gamma_\Gamma/2\right)\Gamma_\mu$$

$$= g_R^5(v)\,\frac{m_R(w)}{4\sqrt{q^2}}\left\{\delta\left(\lambda+2m_R/\sqrt{q^2}\right)+\delta\left(\lambda-2m_R/\sqrt{q^2}\right)\right\}\left[\gamma_\mu a_1(\xi)+i\,a_2(\xi)\ \sigma_{\mu\nu}\lambda q^\nu/2m_R\right] \tag{AV.1a}$$

$$\left(-\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}+1-\gamma_D/2\right)D^{-1}_{\mu\nu} = \lambda^2 g_R^8(v)\ m_R^2(w)\,\delta\left(\lambda^2-4m_R^2/q^2\right)$$

$$\times\ d(\xi)\,g_{\mu\nu}+\text{gauge terms} \tag{AV. 1b}$$

where a_i and d may be determined from (AIII. 5) and (AIV. 3).

We wish to show that the solutions $\Gamma^{(0)}_\mu$ and $D^{(0)}_{\mu\nu}$ discussed in Appendices III and IV satisfy

$$\bar{U}\Gamma^{(0)}_\mu\ UD^{(0)\mu\nu}\,\bar{U}\Gamma^{(0)\mu}\ U = \frac{e_R^2\ \bar{U}\gamma_\mu U\,\bar{U}\gamma^\mu U}{\lambda^2 q^2}\quad, \tag{AV. 2}$$

where the U are the usual spinors.

In order to do this we proceed straightforwardly. We note that, from (AIII. 9), (AIII. 10) and (AV. 1b) we may write

$$f_1^{(0)} = \frac{a_1}{2}\int_0^\infty \frac{ds\; s^{3/2}}{b_0^{5/2}\,\Gamma(5/2)}\, e^{-\left(1/b_0 g_0^2 - v\right)s} \int \frac{dk}{2\pi}\, e^{ik\lambda\sqrt{q^2}} \sum_{n=0}^{\infty} \frac{-(2ik)^{2n}}{(2n)!}$$

$$\int_0^\infty d\nu\; e^{-\nu\left(s+\frac{2n+1}{2}\right)}\, m_R^{2n+1}(w-\nu)$$

$$= \frac{a_1 g_R^5}{2\Gamma(5/2)}\int_0^\infty ds\; e^{-s}\, s^{3/2} \int \frac{dk}{2\pi}\, e^{ik\lambda\sqrt{q^2}} \sum_{n=0}^{\infty} \frac{(2ik)^{2n}}{(2n)!} \int_0^\infty d\nu\; e^{-\nu\left(b_0 g_R^2 s+\frac{2n+1}{2}\right)} m_R^{2n+1}(w-\nu) \tag{AV. 3}$$

$$f_2^{(0)} = \frac{-2m_R a_2 g_R^5}{\lambda\sqrt{q^2}\;\Gamma(5/2)} \int_0^\infty ds\; e^{-s}\, s^{3/2} \int \frac{dk}{4\pi}\, e^{ik\lambda\sqrt{q^2}} \sum_{n=0}^{\infty} \frac{(2ik)^{2n+1}}{(2n+1)!}$$

$$\int_0^\infty d\nu\; e^{-\nu\left(b_0 g_R^2 s + n+1\right)}\, m_R^{2n+2}(w-\nu) \tag{AV. 4}$$

$$D_{\mu\nu}^{(0)-1} = \lambda^2 q^2 g_{\mu\nu}\, \frac{d\, g_R^8}{\Gamma(4)} \int_0^\infty ds\; s^3\, e^{-s} \int \frac{dk}{4\pi}\, e^{ik\lambda^2 q^2} \sum_{n=0}^{\infty} \frac{(-4ik)^n}{n!}$$

$$\int_0^\infty d\nu\; e^{-\nu\left(b_0 g_R^2 s+n+1\right)}\, m_R^{2n+2}(w-\nu) + \text{gauge terms} \tag{AV.5}$$

where (AV. 5) is obtained in complete analogy with (AV. 3) and (AV. 4). Notice that $\frac{\lambda}{m_R} f_2^{(0)}$ is odd in λ. Now, since $\frac{\lambda}{m_R} f_2^{(0)}$, f_1, and $D_{\mu\nu}$ are functions only of $\left\{\frac{1}{2}(t+v), \frac{1}{2}(t+w)\right\}$ almost everywhere, it follows that the change

$$\begin{aligned} \tfrac{1}{2}t &\to \tfrac{1}{2}t + r \\ v,w &\to v,w \end{aligned} \tag{AV. 6a}$$

is equivalent to

$$\begin{aligned} \frac{1}{2}t &\rightarrow \frac{1}{2}t \\ v &\rightarrow v + 2r \\ w &\rightarrow w + 2r \quad , \end{aligned} \tag{AV.6b}$$

almost everywhere. For $\lambda \rightarrow 0$, we may therefore conclude that $\lambda f_2^{(0)}/g_R^5 m_R$ can not be independent of v, since it is clearly not independent of t. Hence, the $\lambda \rightarrow 0$ limit of $f_2^{(0)}$ must be $O\left(g_R^7\right)$. But, again from (AV.6), it follows that this last statement must be true for all λ. Thus, $f_2^{(0)}$ is $O\left(g_R^2\right) \times f_1^{(0)}$. We shall therefore have established (AV.2) if we can show that $f_1^{(0)}$ is $O\left(g_R^5\right)$ and $D_{\mu\nu}^{(0)-1}$ is $O\left(g_R^8\right)$.

To convince oneself that these last estimates are indeed correct, one may simply notice that the $\lambda \rightarrow 0$ limits of $f_1^{(0)}/g_R^5$ and $D_{\mu\nu}^{(0)-1}/g_R^8\lambda^2$ (ignoring gauge terms) are clearly independent of λ and, thus, by (AV.6), necessarily independent of v and w. Hence, these limits are clearly independent of g_R. But, then, from (AV.6), it follows that $f_1^{(0)}$ is $O\left(g_R^5\right)$ and $D_{\mu\nu}^{(0)-1}$ is $O\left(g_R^8\right)$ for all λ, since, as we just observed, these estimates are true for $\lambda \rightarrow 0$. Hence, (AV.2) clearly holds with an appropriate definition of $e_R^2 \propto g_R^2$.

It's also self-evident from the discussion here that we may maintain μ-e universality, since we have not had to specify the value of m_R. We discuss this last point in more detail in the following appendix.

APPENDIX VI

MUON-ELECTRON UNIVERSALITY[29]

As we pointed out in Appendix IV, the discussion there for Γ_μ does not take into account the empirical fact that there are in nature two distinguishable fundamental fermions with charge e, namely, the electron and the muon. In particular, these particles are known to couple symmetrically to the photon up to $q^2 \gtrsim 24$ $(\mathrm{GeV})^2$.[30] Hence, any representation of electrodynamics must reflect this fact.

At present, it is as yet not known whether or not the existence of the electron necessitates the existence of the muon, for example. A priori, one can imagine a world with only one of the two. This possibility is actually manifest in Section III, since, to the order to which we are working, the dimensional analysis violation in Γ_μ can be shown to be precisely that of Fig. 8 independent of the muon, i.e., totally unconnected with Fig. 9 as this diagram does not violate dimensional analysis.

However, the possibility of the existence of the muon is reflected by the necessity to determine which homogeneous integral is necessary to completely specify Γ_μ. Indeed, this is clearly as it should be, since, as we just remarked, contributions to Γ_μ corresponding to Fig. 9 are most certainly homogeneous. We shall now show that it is actually possible to achieve the observed universality as $\lambda^2 \to \infty$.

Our starting point for the discussion of Γ_μ is (AIII. 5)

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta\frac{\partial}{\partial g}-(1+\gamma_\theta)\,m_R\,\frac{\partial}{\partial m_R}-\gamma\right)\Gamma_\mu=\frac{-g_R^5 m_R^2}{32\pi}\left[\left(\frac{\pi}{2}+i\log\xi\right)\gamma_\mu-\sigma_{\mu\nu}\,\frac{\lambda q^\nu}{2m_R}\right]\delta\left(\lambda^2q^2-4m_R^2\right)$$

$$\equiv R\Gamma_\mu \qquad \text{(AVI. 1)}$$

Thus, if $m_e \sim \alpha m_\mu$, as we are assuming, then to order g_R^5, $R\Gamma_\mu$(electron) = 0, and Γ_μ(electron) is an entirely homogeneous solution to the order to which we are working. We then determine it to the extent required by universality.

Specifically, our solutions to (AIII. 5) were the $f_{1,2}^{(0)}$ of Eqs. (AIII. 9) and (AIII. 10), ignoring γ. To these, in general, we may add a homogeneous integral, if necessary. In particular, to satisfy μ-e universality, we must choose homogeneous integrals as necessary so that $\Gamma_\mu(e) = \Gamma_\mu(\mu)$ for λ=0. To do this, we shall employ the approach of Symanzik.[4]

Specifically, it follows from (AIII. 5), that $f_{1,2}^{(0)}$ may also be described by

$$^{P}\Gamma_\mu^{(0)}(m_\mu) = \int_t^{h(t+v,\, t+w)} dt'\, R\Gamma_\mu(t',\, t+v-t',\, t+w-t') \qquad \text{(AVI. 2)}$$

for some choice of h. In particular, let h_0 be this choice.

Now since we are viewing the electron as deriving its mass from the muon,[18] $\Gamma_\mu(e)$ is homogeneous to the order to which we are working, as we already remarked. Thus, we are clearly free to choose

$$\Gamma_\mu(e) = \phi_\mu(t+v,\, t+w)\, \exp-\frac{1}{2}\int^{v} dx\, \gamma(x) \equiv {}^{P}\Gamma_\mu^{(0)}(m_\mu)(h_0) - {}^{P}\Gamma_\mu^{(0)}(m_\mu)(h{=}h_1) \qquad \text{(AVI. 3)}$$

such that μ-e universality is satisfied. A more direct argument will be the subject of later work.

APPENDIX VII

MATHEMATICAL CONSIDERATIONS

In this appendix we shall, for the sake of completeness, discuss certain mathematical aspects of the violation of dimensional analysis. Such a violation can occur only at singularities of Green's functions. Here, we shall take a closer look at such singularities.

The most common characteristic of dimensional analysis violating singularities is a discontinuity which behaves as

$$\frac{m_R^2\,\theta(x)}{x} \tag{AVII. 1}$$

where x is a quadratic function of momenta and masses; for example, in discussing Γ_μ in Appendix III we had

$$\frac{m_R^2\,\theta\left(\lambda^2 q^2 - 4m_R^2\right)}{\lambda^2 q^2 - 4m_R^2} \quad . \tag{AVII. 2}$$

The fact that x is quadratic gives

$$\left(\lambda\frac{\partial}{\partial\lambda} + m_R\frac{\partial}{\partial m_R} + \mu\frac{\partial}{\partial\mu}\right) x = 2x \quad . \tag{AVII. 3}$$

Hence

$$\left(\lambda\frac{\partial}{\partial\lambda} + m_R\frac{\partial}{\partial m_R} + \mu\frac{\partial}{\partial\mu}\right)\frac{m_R^2\,\theta(x)}{x} = -\frac{2m_R^2\,\theta(x)}{x} + \frac{2m_R^2\,\theta(x)}{x} + \frac{2m_R^2\,\delta(x)x}{x}$$

$$= 2m_R^2\,\delta(x) \tag{AVII. 4}$$

It is this kind of arithmetic which we have used throughout the manuscript. It is clearly delicate.

The delicacy involved is of course the differentiation of singularities. One might hope, for example, that since for $m^2 > 0$, $\theta(\lambda^2 t - 4m^2) = \theta\left(\frac{\lambda^2 t}{m^2} - 4\right)$, then

$$\left(m\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial\lambda}\right)\frac{m^2\theta\left(\lambda^2 t - 4m^2\right)}{\lambda^2 t - 4m^2} = \left(m\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial\lambda}\right)\frac{\theta\left(\frac{\lambda^2 t}{m^2} - 4\right)}{\frac{\lambda^2 t}{m^2} - 4}$$

$$\stackrel{?}{=} 0 \tag{AVII. 5}$$

However, the correct statement, ignoring the difference between $\theta(\lambda^2 t - 4m^2)$ and and $\theta\left(\frac{\lambda^2 t}{m^2} - 4\right)$ for $m^2 < 0$, is that the RHS of (AVII. 5) is

$$\frac{1}{\frac{\lambda^2 t}{m^2} - 4}\,\delta\left(\frac{\lambda^2 t}{m^2} - 4\right)\left(m\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial\lambda}\right)\left(\frac{\lambda^2 t}{m^2} - 4\right) = \delta\left(\frac{\lambda^2 t}{m^2} - 4\right)\left(\frac{1}{0}\right)^0 , \tag{AVII. 6}$$

which is indeterminate.

In attempting to handle this indeterminateness, we may introduce $\epsilon = \frac{\lambda^2 t}{m^2} - 4$ and write

$$\frac{1}{\frac{\lambda^2 t}{4m^2} - 4}\,\delta\left(\frac{\lambda^2 t}{m^2} - 4\right)\left(m\frac{\partial}{\partial m} + \lambda\frac{\partial}{\partial\lambda}\right)\left(\frac{\lambda^2 t}{m^2} - 4\right) = \lim_{\substack{\epsilon \to 0 \\ \frac{\epsilon_1}{\lambda} = \frac{\epsilon_2}{m} = \frac{\epsilon}{2^n}}} \frac{1}{\epsilon}\left[\frac{\lambda}{\epsilon_1}\left\{\frac{(\lambda+\epsilon_1)^2 t}{m^2} - \frac{\lambda^2 t}{m^2}\right\}\right.$$

$$\left. + \frac{m}{\epsilon_2}\left\{\frac{\lambda^2 t}{(m+\epsilon_2)^2} - \frac{\lambda^2 t}{m^2}\right\}\right]\delta\left(\frac{\lambda^2 t}{m^2} - 4\right)$$

$$= 2^{2-n}\,\delta\left(\frac{\lambda^2 t}{m^2} - 4\right) , \text{ n arbitrary} \tag{AVII. 7}$$

The result (AVII. 4) avoids this indeterminateness by L'hopital's rule.

The arithmetic (AVII.4) may be repeated by introducing the standard representation

$$\theta(x) = \frac{\lim\limits_{\epsilon \downarrow 0_+}}{2\pi i} \int_{-\infty}^{\infty} \frac{d\alpha\, e^{i\alpha x}}{\alpha - i\epsilon} \tag{AVII.8}$$

and proceeding with the strict analysis definition of derivative. Again, we find an indeterminate result, which we take to be resolved by (AVII.4). We leave the explicit demonstration of this latter indeterminateness to the reader.

Thus, it would appear that our method of handling such functions as $\theta(x)/x$ is indeed the correct one. The physical consequences which we have drawn from it would thus appear to be justified.

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FIGURE CAPTIONS

1. Virtual emission and decay of a hypothetical bound pole generating a singularity in the respective Bethe-Salpeter wavefunction.

2. Six-point proper vertex in ϕ^4 theory. For m=0, violation of dimensional analysis occurs on-shell and may be identified with longe range interaction; for m$\neq$0, violations are generated off-shell and are due to particle collaboration; i.e., to internal particles propagating on-shell freely.

3. Two-loop dimensional analysis violating contribution to the scalar-vector-scalar vertex in scalar electrodynamics.

4. Data for $e^+e^- \rightarrow$ hadrons, reproduced by permission of B. Richter, Invited Talk, 1974 London Conference. The data are roughly consistent with the fractionally charged 3-triplet model. $q_0^2 \simeq 10\ (\mathrm{GeV})^2$.

5. (a) Lepton-$\overline{\text{lepton}}$ annihilation in the one-photon exchange approximation; (b) deep inelastic lepton-proton scattering in the parton model. In (a), the amplitude is $\Gamma D < 0 | eJ^{EM} | x >$ and in (b), it is $\Gamma D \Gamma$ incoherently weighted for each constituent. Thus, we have the result (3.8) for R and the conventional result for the scattering process (ignoring the new physics associated with $\psi(3105)$).

6. Dimensional analysis violating proper vertex in ϕ^4 theory, as already exemplified by Fig. 2 above.

7. Graphical singularity structure of the pion's BS wavefunction.

8. On-shell dimensional analysis violating proper vertex Γ_μ of the electron.

9. Typical contribution to electron's Γ_μ involving the muon. Such terms are homogeneous to the order that we are working.

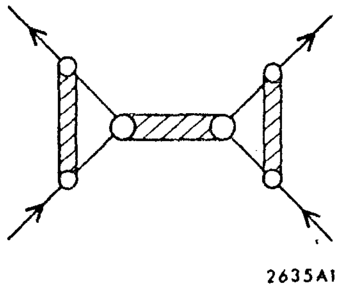


Fig. 1

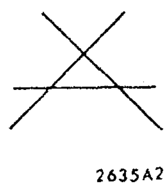


Fig. 2

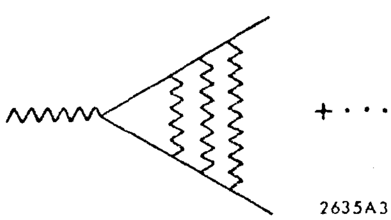
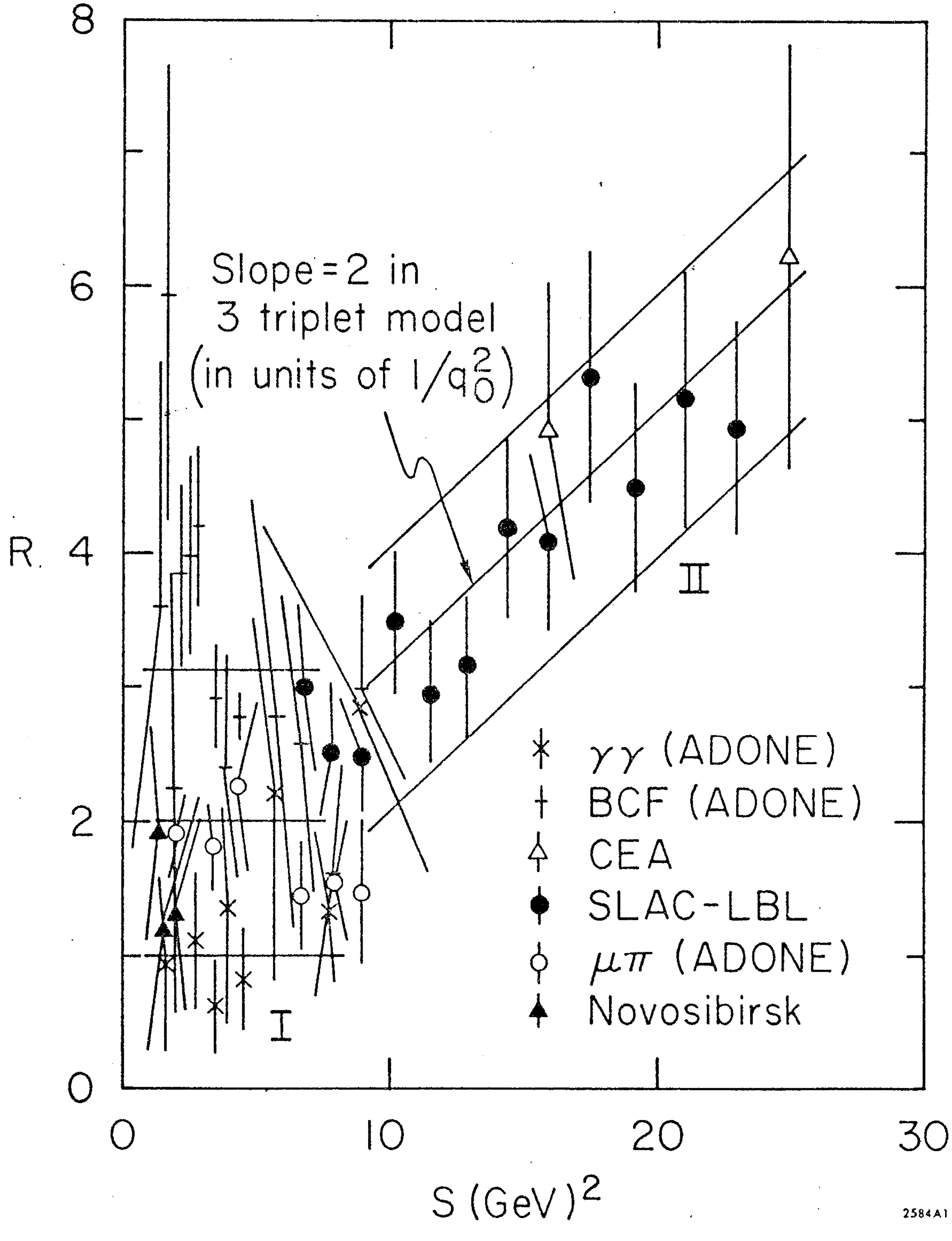


Fig. 3



8
6
R. 4
2
0
0
10
20
30
S (GeV)^2
Slope = 2 in
3 triplet model
(in units of $1/q_0^2$)
I
II
$\gamma\gamma$ (ADONE)
BCF (ADONE)
CEA
SLAC-LBL
$\mu\pi$ (ADONE)
Novosibirsk
2584A1

Fig. 4

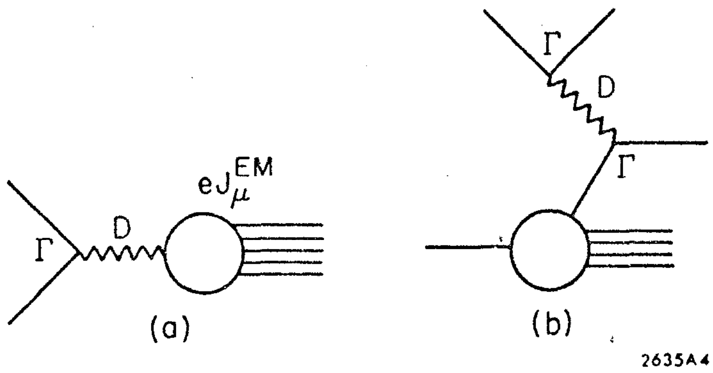


Fig. 5

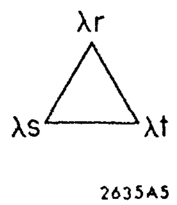


Fig. 6

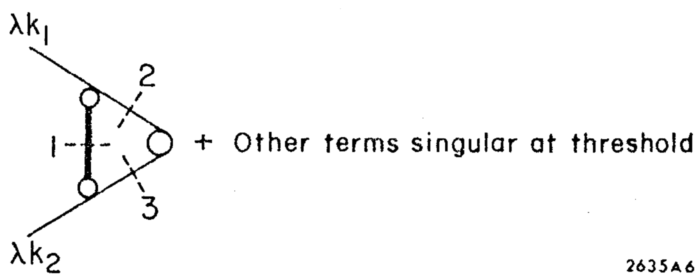


Fig. 7

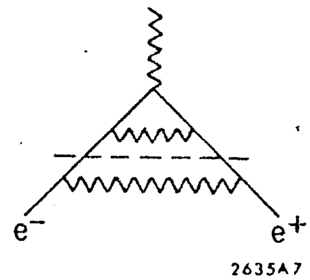


Fig. 8

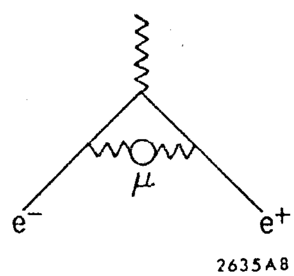


Fig. 9