Notes on A_∞-Algebras, A_∞-Categories and Non-Commutative Geometry

M. Kontsevich[1] and Y. Soibelman[2]

[1] IHES, 35 route de Chartres, F-91440, France
`maxim@ihes.fr`
[2] Department of Mathematics, KSU, Manhattan, KS 66506, USA
`soibel@math.ksu.edu`

Abstract. We develop a geometric approach to A-infinity algebras and A-infinity categories based on the notion of formal scheme in the category of graded vector spaces. The geometric approach clarifies several questions, e.g. the notion of homological unit or A-infinity structure on A-infinity functors. We discuss Hochschild complexes of A-infinity algebras from geometric point of view. The chapter contains homological versions of the notions of properness and smoothness of projective varieties as well as the non-commutative version of the Hodge-to-de Rham degeneration conjecture. We also discuss a generalization of Deligne's conjecture which includes both Hochschild chains and cochains. We conclude the chapter with the description of an action of the PROP of singular chains of the topological PROP of two-dimensional surfaces on the Hochschild chain complex of an A-infinity algebra with scalar product (this action is more or less equivalent to the structure of two-dimensional Topological Field Theory associated with an "abstract" Calabi–Yau manifold).

1 Introduction

1.1 A_∞-Algebras as Spaces

The notion of A_∞-algebra introduced by Stasheff (or the notion of A_∞-category introduced by Fukaya) has two different interpretations. First one is operadic: an A_∞-algebra is an algebra over the A_∞-operad (one of its versions is the operad of singular chains of the operad of intervals in the real line). Second one is geometric: an A_∞-algebra is the same as a non-commutative formal graded manifold X over, say, field k, having a marked k-point pt and equipped with a vector field d of degree $+1$ such that $d|_{pt} = 0$ and $[d, d] = 0$ (such vector fields are called *homological*). By definition the algebra of functions on the non-commutative formal pointed graded manifold is isomorphic to the algebra of formal series $\sum_{n\geq 0}\sum_{i_1,i_2,\dots,i_n\in I} a_{i_1\dots i_n}x_{i_1}\dots x_{i_n} := \sum_M a_M x^M$ of free graded variables $x_i, i \in I$ (the set I can be infinite). Here

Kontsevich, M., Soibelman Y.: *Notes on A_∞-Algebras, A_∞-Categories and Non-Commutative Geometry*. Lect. Notes Phys. **757**, 153–219 (2009)
DOI 10.1007/978-3-540-68030-7_6

$M = (i_1, ..., i_n), n \geq 0$ is a non-commutative multi-index, i.e. an element of the free monoid generated by I. Homological vector field makes the above graded algebra into a complex of vector spaces. The triple (X, pt, d) is called a *non-commutative formal pointed differential-graded (or simply dg-) manifold.*

It is an interesting problem to make a dictionary from the pure algebraic language of A_∞-algebras and A_∞-categories to the language of non-commutative geometry.[3] One purpose of these notes is to make few steps in this direction.

From the point of view of Grothendieck's approach to the notion of "space," our formal pointed manifolds are given by functors on graded associative Artin algebras commuting with finite projective limits. It is easy to see that such functors are represented by graded coalgebras. These coalgebras can be thought of as coalgebras of distributions on formal pointed manifolds. The above-mentioned algebras of formal power series are dual to the coalgebras of distributions.

In the case of (small) A_∞-categories considered in the subsequent paper we will slightly modify the above definitions. Instead of one marked point one will have a closed subscheme of disjoint points (objects) in a formal graded manifold and the homological vector field d must be compatible with the embedding of this subscheme as well as with the projection onto it.

1.2 Some Applications of Geometric Language

Geometric approach to A_∞-algebras and A_∞-categories clarifies several longstanding questions. In particular one can obtain an explicit description of the A_∞-structure on A_∞-functors. This will be explained in detail in the subsequent paper. Here we make few remarks. In geometric terms A_∞-functors are interpreted as maps between non-commutative formal dg-manifolds commuting with homological vector fields. We will introduce a non-commutative formal dg-manifold of maps between two such spaces. Functors are just "commutative" points of the latter. The case of A_∞-categories with one object (i.e., A_∞-algebras) is considered in this chapter. The general case reflects the difference between quivers with one vertex and quivers with many vertices (vertices correspond to objects).[4] As a result of the above considerations one can describe explicitly the A_∞-structure on functors in terms of sums over sets of trees. Among other applications of our geometric language we mention an interpretation of the Hochschild chain complex of an A_∞-algebra in terms of cyclic differential forms on the corresponding formal pointed dg-manifold (Sect. 7.2).

Geometric language simplifies some proofs as well. For example, Hochschild cohomology of an A_∞-category $\mathcal{C}$ is isomorphic to $Ext^\bullet(Id_\mathcal{C}, Id_\mathcal{C})$ taken in the

---

[3] We use "formal" non-commutative geometry in tensor categories, which is different from the non-commutative geometry in the sense of Alain Connes.

[4] Another, purely algebraic approach to the A_∞-structure on functors was suggested in [39].

A_∞-category of endofunctors $\mathcal{C} \to \mathcal{C}$. This result admits an easy proof, if one interprets Hochschild cochains as vector fields and functors as maps (the idea to treat $Ext^\bullet(Id_\mathcal{C}, Id_\mathcal{C})$ as the tangent space to deformations of the derived category $D^b(\mathcal{C})$ goes back to A.Bondal).

1.3 Content of the Paper

Present paper contains two parts out of three (the last one is devoted to A_∞-categories and will appear later). Here we discuss A_∞-algebras (=non-commutative formal pointed dg-manifolds with fixed affine coordinates). We have tried to be precise and provide details of most of the proofs.

Part I is devoted to the geometric description of A_∞-algebras. We start with basics on formal graded affine schemes, then add a homological vector field, thus arriving to the geometric definition of A_∞-algebras as formal pointed dg-manifolds. Most of the material is well-known in algebraic language. We cannot completely avoid A_∞-categories (subject of the subsequent paper). They appear in the form of categories of A_∞-modules and A_∞-bimodules, which can be defined directly.

Since in the A_∞-world many notions are defined "up to quasi-isomorphism", their geometric meaning is not obvious. As an example we mention the notion of *weak unit*. Basically, this means that the unit exists at the level of cohomology only. In Sect. 4 we discuss the relationship of weak units with the "differential-graded" version of the affine line.

We start Part II with the definition of the Hochschild complexes of A_∞-algebras. As we already mentioned, Hochschild cochain complex is interpreted in terms of graded vector fields on the non-commutative formal affine space. Dualizing, Hochschild chain complex is interpreted in terms of degree one cyclic differential forms. This interpretation is motivated by [30]. It differs from the traditional picture (see e.g. [7, 11]) where one assigns to a Hochschild chain $a_0 \otimes a_1 \otimes ... \otimes a_n$ the differential form $a_0 da_1 ... da_n$. In our approach we interepret a_i as the dual to an affine coordinate x_i and the above expression is dual to the cyclic differential 1-form $x_1 ... x_n dx_0$. We also discuss graphical description of Hochschild chains, the differential, etc.

After that we discuss homologically smooth compact A_∞-algebras. Those are analogs of smooth projective varieties in algebraic geometry. Indeed, the derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X is A_∞-equivalent to the category of perfect modules over a homologically smooth compact A_∞-algebra (this can be obtained using the results of [5]). The algebra contains as much information about the geometry of X as the category $D^b(X)$ does. A good illustration of this idea is given by the "abstract" version of Hodge theory presented in Sect. 9. It is largely conjectural topic, which eventually should be incorporated in the theory of "non-commutative motives." Encoding smooth proper varieties by homologically smooth compact A_∞-algebras we can forget about the underlying commutative geometry and try to develop a theory of "non-commutative smooth projective varieties" in an abstract form. Let us briefly explain what does it mean for the Hodge theory. Let $(C_\bullet(A,A),b)$ be the Hochschild chain complex of a (weakly unital) homologically smooth compact A_∞-algebra A. The corresponding negative cyclic complex $(C_\bullet(A,A)[[u]], b+uB)$ gives rise to a family of complexes over the formal affine line $\mathbf{A}^1_{form}[+2]$ (shift of the grading reflects the fact that the variable u has degree $+2$, cf. [7, 11]). We conjecture that the corresponding family of cohomology groups gives rise to a vector bundle over the formal line. The generic fiber of this vector bundle is isomorphic to periodic cyclic homology, while the fiber over $u=0$ is isomorphic to the Hochschild homology. If compact homologically smooth A_∞-algebra A corresponds to a smooth projective variety as explained above, then the generic fiber is just the algebraic de Rham cohomology of the variety, while the fiber over $u=0$ is the Hodge cohomology. Then our conjecture becomes the classical theorem which claims degeneration of the spectral sequence Hodge-to-de Rham.[5]

Last section of Part II is devoted to the relationship between moduli spaces of points on a cylinder and algebraic structures on the Hochschild complexes. In Sect. 11.3 we formulate a generalization of Deligne's conjecture. Recall that Deligne's conjecture says (see e.g., [35]) that the Hochschild cochain complex of an A_∞-algebra is an algebra over the operad of chains on the topological operad of little discs. In the conventional approach to non-commutative geometry Hochschild cochains correspond to polyvector fields, while Hochschild chains correspond to de Rham differential forms. One can contract a form with a polyvector field or take a Lie derivative of a form with respect to a polyvector field. This geometric point of view leads to a generalization of Deligne's conjecture which includes Hochschild chains equipped with the structure of (homotopy) module over cochains and to the "Cartan type" calculus which involves both chains and cochains (cf. [11, 48]). We unify both approaches under one roof formulating a theorem which says that the pair consisting of the Hochschild chain and Hochschild cochain complexes of the same A_∞-algebra is an algebra over the colored operad of singular chains on configurations of discs on a cylinder with marked points on each of the boundary circles.[6]

Sections 10 and 11.6 are devoted to A_∞-algebras with scalar product, which is the same as non-commutative formal symplectic manifolds. In Sect. 10 we also discuss a homological version of this notion and explain that it corresponds to the notion of Calabi–Yau structure on a manifold. In Sect. 11.6 we define an action of the PROP of singular chains of the topological PROP of smooth oriented two-dimensional surfaces with boundaries on the Hochschild chain complex of an A_∞-algebra with scalar product. If in addition A is homologically smooth and the spectral sequence Hodge-to-de Rham degenerates, then the above action extends to the action of the PROP of singular chains

---

[5] In a recent preprint [24], D. Kaledin claims the proof of our conjecture. He uses a different approach.

[6] After our paper was finished we received the paper [49] where the authors proved an equivalent result.

on the topological PROP of stable two-dimensional surfaces. This is essentially equivalent to a structure of two-dimensional Cohomological TFT (similar ideas have been developed by Kevin Costello, see [8]). More details and an application of this approach to the calculation of Gromov–Witten invariants will be given in [22].

1.4 Generalization to A_∞-Categories

Let us say few words about the subsequent paper which is devoted to A_∞-categories. The formalism of present paper admits a straightforward generalization to the case of A_∞-categories. The latter should be viewed as noncommutative formal dg-manifolds with a closed marked subscheme of objects. Although some parts of the theory of A_∞-categories admit nice interpretation in terms of non-commutative geometry, some other still wait for it. This includes e.g. triangulated A_∞-categories. We will present the theory of triangulated A_∞-categories from the point of view of A_∞-functors from "elementary" categories to a given A_∞-category (see a summary in [33, 46, 47]). Those "elementary" categories are, roughly speaking, derived categories of representations of quivers with small number of vertices. Our approach has certain advantages over the traditional one. For example the complicated "octahedron axiom" admits a natural interpretation in terms of functors from the A_∞-category associated with the quiver of the Dynkin diagram A_2 (there are six indecomposible objects in the category $D^b(A_2 - mod)$ corresponding to six vertices of the octahedron). In some sections of the paper on A_∞-categories we have not been able to provide pure geometric proofs of the results, thus relying on less flexible approach which uses differential-graded categories (see [14]). As a compromise, we will present only part of the theory of A_∞-categories, with sketches of proofs, which are half-geomeric and half-algebraic, postponing more coherent exposition for future publications.

In the present and subsequent studies we mostly consider A_∞-algebras and categories over a field of characteristic zero. This assumption simplifies many results, but also makes some other less general. We refer the reader to [39, 40] for a theory over a ground ring instead of ground field (the approach of [39, 40] is pure algebraic and different from ours). Most of the results of present paper are valid for an A_∞-algebra A over the unital commutative associative ring k, as long as the graded module A is flat over k. More precisely, the results of Part I remain true except of the results of Sect. 3.2 (the minimal model theorem). In these two cases we assume that k is a field of characteristic zero. *Constructions* of Part II work over a commutative ring k. The results of Sect. 10 are valid (and the conjectures are expected to be valid) over a field of characteristic zero. Algebraic version of Hodge theory from Sect. 9 and the results of Sect. 11 are formulated for an A_∞-algebra over the field of characteristic zero, although the Conjecture 2 is expected to be true for any $\mathbf{Z}$-flat A_∞-algebra.

Part I: A_∞-Algebras and Non-commutative dg-Manifolds

2 Coalgebras and Non-commutative Schemes

Geometric description of A_∞-algebras will be given in terms of geometry of non-commutative ind-affine schemes in the tensor category of graded vector spaces (we will use $\mathbf{Z}$-grading or $\mathbf{Z}/2$-grading). In this section we are going to describe these ind-schemes as functors from finite-dimensional algebras to sets (cf. with the description of formal schemes in [20]). More precisely, such functors are represented by counital coalgebras. Corresponding geometric objects are called *non-commutative thin schemes.*

2.1 Coalgebras as Functors

Let k be a field and $\mathcal{C}$ be a k-linear Abelian symmetric monoidal category (we will call such categories *tensor*), which admits infinite sums and products (we refer to [13] about all necessary terminology of tensor categories). Then we can do simple linear algebra in $\mathcal{C}$, in particular, speak about associative algebras or coassociative coalgebras. For the rest of the paper, unless we say otherwise, we will assume that either $\mathcal{C} = Vect_k^{\mathbf{Z}}$, which is the tensor category of $\mathbf{Z}$-graded vector spaces $V = \oplus_{n\in\mathbf{Z}} V_n$, or $\mathcal{C} = Vect_k^{\mathbf{Z}/2}$, which is the tensor category of $\mathbf{Z}/2$-graded vector spaces (then $V = V_0 \oplus V_1$), or $\mathcal{C} = Vect_k$, which is the tensor category of k-vector spaces. Associativity morphisms in $Vect_k^{\mathbf{Z}}$ or $Vect_k^{\mathbf{Z}/2}$ are identity maps and commutativity morphisms are given by the Koszul rule of signs: $c(v_i \otimes v_j) = (-1)^{ij} v_j \otimes v_i$, where v_n denotes an element of degree n.

We will denote by $\mathcal{C}^f$ the Artinian category of finite-dimensional objects in $\mathcal{C}$ (i.e. objects of finite length). The category $Alg_{\mathcal{C}^f}$ of unital finite-dimensional algebras is closed with respect to finite projective limits. In particular, finite products and finite fiber products exist in $Alg_{\mathcal{C}^f}$. One has also the categories $Coalg_{\mathcal{C}}$ (resp. $Coalg_{\mathcal{C}^f}$) of coassociative counital (resp. coassociative counital finite-dimensional) coalgebras. In the case $\mathcal{C} = Vect_k$ we will also use the notation Alg_k, Alg_k^f, $Coalg_k$ and $Coalg_k^f$ for these categories. The category $Coalg_{\mathcal{C}^f} = Alg_{\mathcal{C}^f}^{op}$ admits finite inductive limits.

We will need simple facts about coalgebras. We will present proofs in the Appendix for completness.

Theorem 2.1 *Let $F : Alg_{\mathcal{C}^f} \to Sets$ be a covariant functor commuting with finite projective limits. Then it is isomorphic to a functor of the type $A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^*, B)$ for some counital coalgebra B. Moreover, the category of such functors is equivalent to the category of counital coalgebras.*

Proposition 2.2 If $B \in Ob(Coalg_{\mathcal{C}})$, then B is a union of finite-dimensional counital coalgebras.

Objects of the category $Coalg_{\mathcal{C}^f} = Alg^{op}_{\mathcal{C}^f}$ can be interpreted as "very thin" non-commutative affine schemes (cf. with finite schemes in algebraic geometry). Proposition 1 implies that the category $Coalg_{\mathcal{C}}$ is naturally equivalent to the category of ind-objects in $Coalg_{\mathcal{C}^f}$.

For a counital coalgebra B we denote by $Spc(B)$ (the "spectrum" of the coalgebra B) the corresponding functor on the category of finite-dimensional algebras. A functor isomorphic to $Spc(B)$ for some B is called a *non-commutative thin scheme*. The category of non-commutative thin schemes is equivalent to the category of counital coalgebras. For a non-commutative scheme X we denote by B_X the corresponding coalgebra. We will call it the coalgebra of *distributions* on X. The *algebra of functions* on X is by definition $\mathcal{O}(X) = B^*_X$.

Non-commutative thin schemes form a full monoidal subcategory $NAff^{th}_{\mathcal{C}} \subset Ind(NAff_{\mathcal{C}})$ of the category of non-commutative ind-affine schemes (see Appendix). Tensor product corresponds to the tensor product of coalgebras.

Let us consider few examples.

Example 2.3 Let $V \in Ob(\mathcal{C})$. Then $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ carries a structure of counital cofree coalgebra in $\mathcal{C}$ with the coproduct $\Delta(v_0 \otimes ... \otimes v_n) = \sum_{0\leq i\leq n} (v_0 \otimes ... \otimes v_i) \otimes (v_{i+1} \otimes ... \otimes v_n)$. The corresponding non-commutative thin scheme is called non-commutative formal affine space V_{form} (or formal neighborhood of zero in V).

Definition 2.4 A non-commutative formal manifold X is a non-commutative thin scheme isomorphic to some $Spc(T(V))$ from the example above. The dimension of X is defined as $dim_k V$.

The algebra $\mathcal{O}(X)$ of functions on a non-commutative formal manifold X of dimension n is isomorphic to the topological algebra $k\langle\langle x_1, ..., x_n\rangle\rangle$ of formal power series in free graded variables $x_1, ..., x_n$.

Let X be a non-commutative formal manifold and $pt\colon k \to B_X$ a k-point in X,

Definition 2.5 The pair (X, pt) is called a non-commutative formal pointed manifold. If $\mathcal{C} = Vect^{\mathbf{Z}}_k$ it will be called non-commutative formal pointed graded manifold. If $\mathcal{C} = Vect^{\mathbf{Z}/2}_k$ it will be called non-commutative formal pointed supermanifold.

The following example is a generalization of the Example 1 (which corresponds to a quiver with one vertex).

Example 2.6 Let I be a set and $B_I = \oplus_{i\in I} \mathbf{1}_i$ be the direct sum of trivial coalgebras. We denote by $\mathcal{O}(I)$ the dual topological algebra. It can be thought of as the algebra of functions on a discrete non-commutative thin scheme I.

A quiver Q in C with the set of vertices I is given by a collection of objects $E_{ij} \in \mathcal{C}, i, j \in I$ called spaces of arrows from i to j. The coalgebra of Q is the coalgebra B_Q generated by the $\mathcal{O}(I)-\mathcal{O}(I)$-bimodule $E_Q=\oplus_{i,j\in I}E_{ij}$, i.e. $B_Q\simeq\oplus_{n\geq 0}\oplus_{i_0,i_1,\dots,i_n\in I}E_{i_0i_1}\otimes\dots\otimes E_{i_{n-1}i_n}:=\oplus_{n\geq 0}B_Q^n$, $B_Q^0:=B_I$. Elements of B_Q^0 are called *trivial paths*. Elements of B_Q^n are called paths of the length n. Coproduct is given by the formula

$$\Delta(e_{i_0i_1}\otimes\dots\otimes e_{i_{n-1}i_n})=\oplus_{0\leq m\leq n}(e_{i_0i_1}\otimes\dots\otimes e_{i_{m-1}i_m})\otimes(e_{i_mi_{m+1}}\dots\otimes\dots\otimes e_{i_{n-1}i_n}),$$

where for $m=0$ (resp. $m=n$) we set $e_{i_{-1}i_0}=1_{i_0}$ (resp. $e_{i_ni_{n+1}}=1_{i_n}$).

In particular, $\Delta(1_i)=1_i\otimes 1_i, i\in I$ and $\Delta(e_{ij})=1_i\otimes e_{ij}+e_{ij}\otimes 1_j$, where $e_{ij}\in E_{ij}$ and $1_m\in B_I$ corresponds to the image of $1\in\mathbf{1}$ under the natural embedding into $\oplus_{m\in I}\mathbf{1}$.

The coalgebra B_Q has a counit ε such that $\varepsilon(1_i)=1_i$ and $\varepsilon(x)=0$ for $x\in B_Q^n, n\geq 1$.

Example 2.7 (Generalized quivers). Here we replace $\mathbf{1}_i$ by a unital simple algebra A_i (e.g. $A_i=Mat(n_i,D_i)$, where D_i is a division algebra). Then E_{ij} are $A_i-mod-A_j$-bimodules. We leave as an exercise to the reader to write down the coproduct (one uses the tensor product of bimodules) and to check that we indeed obtain a coalgebra.

Example 2.8 Let I be a set. Then the coalgebra $B_I=\oplus_{i\in I}\mathbf{1}_i$ is a direct sum of trivial coalgebras, isomorphic to the unit object in $\mathcal{C}$. This is a special case of Example 2. Note that in general B_Q is a $\mathcal{O}(I)-\mathcal{O}(I)$-bimodule.

Example 2.9 Let A be an associative unital algebra. It gives rise to the functor $F_A: Coalg_{\mathcal{C}^f}\to Sets$ such that $F_A(B)=Hom_{Alg_{\mathcal{C}}}(A,B^*)$. This functor describes finite-dimensional representations of A. It commutes with finite direct limits, hence it is representable by a coalgebra. If $A=\mathcal{O}(X)$ is the algebra of regular functions on the affine scheme X, then in the case of algebraically closed field k the coalgebra representing F_A is isomorphic to $\oplus_{x\in X(k)}\mathcal{O}_{x,X}^*$, where $\mathcal{O}_{x,X}^*$ denotes the topological dual to the completion of the local ring $\mathcal{O}_{x,X}$. If X is smooth of dimension n, then each summand is isomorphic to the topological dual to the algebra of formal power series $k[[t_1,\dots,t_n]]$. In other words, this coalgebra corresponds to the disjoint union of formal neighborhoods of all points of X.

Remark 2.10 One can describe non-commutative thin schemes more precisely by using structure theorems about finite-dimensional algebras in $\mathcal{C}$. For example, in the case $\mathcal{C}=Vect_k$ any finite-dimensional algebra A is isomorphic to a sum $A_0\oplus r$, where A_0 is a finite sum of matrix algebras $\oplus_i Mat(n_i,D_i)$, D_i are division algebras and r is the radical. In $\mathbf{Z}$-graded case a similar decomposition holds, with A_0 being a sum of algebras of the type $End(V_i)\otimes D_i$,where V_i are some graded vector spaces and D_i are division algebras of degree zero. In $\mathbf{Z}/2$-graded case the description is slightly more complicated. In particular A_0 can contain summands isomorphic to $(End(V_i)\otimes D_i)\otimes D_\lambda$, where V_i and D_i are $\mathbf{Z}/2$-graded analogs of the above-described objects and D_λ is a $1|1$-dimensional superalgebra isomorphic to $k[\xi]/(\xi^2=\lambda)$, $deg\,\xi=1, \lambda\in k^*/(k^*)^2$.

2.2 Smooth Thin Schemes

Recall that the notion of an ideal has meaning in any abelian tensor category. A two-sided ideal J is called *nilpotent* if the multiplication map $J^{\otimes n} \to J$ has zero image for a sufficiently large n.

Definition 2.11 Counital coalgebra B in a tensor category $\mathcal{C}$ is called smooth if the corresponding functor $F_B : Alg_{\mathcal{C}^f} \to Sets, F_B(A) = Hom_{Coalg_{\mathcal{C}}}(A^*, B)$ satisfies the following lifting property: for any two-sided nilpotent ideal $J \subset A$ the map $F_B(A) \to F_B(A/J)$ induced by the natural projection $A \to A/J$ is surjective. Non-commutative thin scheme X is called smooth if the corresponding counital coalgebra $B = B_X$ is smooth.

Proposition 2.12 For any quiver Q in $\mathcal{C}$ the corresponding coalgebra B_Q is smooth.

Proof. First let us assume that the result holds for all finite quivers. We remark that if A is finite-dimensional and Q is an infinite quiver then for any morphism $f\colon A^* \to B_Q$ we have: $f(A^*)$ belongs to the coalgebra of a finite sub-quiver of Q. Since the lifting property holds for the latter, the result follows. Finally, we need to prove the Proposition for a finite quiver Q . Let us choose a basis $\{e_{ij,\alpha}\}$ of each space of arrows E_{ij}. Then for a finite-dimensional algebra A the set $F_{B_Q}(A)$ is isomorphic to the set $\{((\pi_i), x_{ij,\alpha})_{i,j\in I}\}$, where $\pi_i \in A, \pi_i^2 = \pi_i, \pi_i\pi_j = \pi_j\pi_i$, if $i \neq j, \sum_{i\in I} \pi_i = 1_A$ and $x_{ij,\alpha} \in \pi_i A \pi_j$ satisfy the condition: there exists $N \geq 1$ such that $x_{i_1j_1,\alpha_1} ... x_{i_mj_m,\alpha_m} = 0$ for all $m \geq N$. Let now $J \subset A$ be the nilpotent ideal from the definition of smooth coalgebra and $(\pi_i', x_{ij,\alpha}')$ be elements of A/J satisfying the above constraints. Our goal is to lift them to A. We can lift the them to the projectors π_i and elements $x_{ij,\alpha}$ for A in such a way that the above constraints are satisfied except of the last one, which becomes an inclusion $x_{i_1j_1,\alpha_1} ... x_{i_mj_m,\alpha_m} \in J$ for $m \geq N$. Since $J^n = 0$ in A for some n we see that $x_{i_1j_1,\alpha_1} ... x_{i_mj_m,\alpha_m} = 0$ in A for $m \geq nN$. This proves the result. ■

Remark 2.13 (a) According to Cuntz and Quillen [10] a non-commutative *algebra* R in $Vect_k$ is called *smooth* if the functor $Alg_k \to Sets, F_R(A) = Hom_{Alg_k}(R, A)$ satisfies the lifting property from the Definition 3 applied to all (not only finite-dimensional) algebras. We remark that if R is smooth in the sense of Cuntz and Quillen then the coalgebra R_{dual} representing the functor $Coalg_k^f \to Sets, B \mapsto Hom_{Alg_k^f}(R, B^*)$ is smooth. One can prove that any smooth coalgebra in $Vect_k$ is isomorphic to a coalgebra of a generalized quiver (see Example 3).

(b) Almost all examples of non-commutative smooth thin schemes considered in this paper are formal pointed manifolds, i.e. they are isomorphic to $Spc(T(V))$ for some $V \in Ob(\mathcal{C})$. It is natural to try to "globalize" our results to the case of non-commutative "smooth" schemes X which satisfy the property that the completion of X at a "commutative" point gives rise to a formal pointed manifold in our sense. An example of the space of maps is considered in the next subsection.

(c) The tensor product of non-commutative smooth thin schemes is typically non-smooth, since it corresponds to the *tensor product* of coalgebras (the latter is not a categorical product).

Let now x be a k-point of a non-commutative smooth thin scheme X. By definition x is a homomorphism of counital coalgebras $x\colon k \to B_X$ (here $k = \mathbf{1}$ is the trivial coalgebra corresponding to the unit object). The completion $\widehat{X}_x$ of X at x is a formal pointed manifold which can be described such as follows. As a functor $F_{\widehat{X}_x} : Alg_{\mathcal{C}}^f \to Sets$ it assigns to a finite-dimensional algebra A the set of such homomorphisms of counital colagebras $f\colon A^* \to B_X$ which are compositions $A^* \to A_1^* \to B_X$, where $A_1^* \subset B_X$ is a conilpotent extension of x (i.e., A_1 is a finite-dimensional unital nilpotent algebra such that the natural embedding $k \to A_1^* \to B_X$ coinsides with $x\colon k \to B_X$).

Description of the coalgebra $B_{\widehat{X}_x}$ is given in the following Proposition.

Proposition 2.14 The formal neighborhood $\widehat{X}_x$ corresponds to the counital sub-coalgebra $B_{\widehat{X}_x} \subset B_X$ which is the preimage under the natural projection $B_X \to B_X/x(k)$ of the sub-coalgebra consisting of conilpotent elements in the non-counital coalgebra $B/x(k)$. Moreover, $\widehat{X}_x$ is universal for all morphisms from nilpotent extensions of x to X.

We discuss in Appendix a more general construction of the completion along a non-commutative thin subscheme.

We leave as an exercise to the reader to prove the following result.

Proposition 2.15 Let Q be a quiver and $pt_i \in X = X_{B_Q}$ corresponds to a vertex $i \in I$. Then the formal neighborhood $\widehat{X}_{pt_i}$ is a formal pointed manifold corresponding to the tensor coalgebra $T(E_{ii}) = \oplus_{n\geq 0} E_{ii}^{\otimes n}$, where E_{ii} is the space of loops at i.

2.3 Inner Hom

Let X, Y be non-commutative thin schemes and B_X, B_Y the corresponding coalgebras.

Theorem 2.16 *The functor $Alg_{\mathcal{C}^f} \to Sets$ such that*

$$A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^* \otimes B_X, B_Y)$$

is representable. The corresponding non-commutative thin scheme is denoted by $Maps(X, Y)$.

Proof. It is easy to see that the functor under consideration commutes with finite projective limits. Hence it is of the type $A \mapsto Hom_{Coalg_{\mathcal{C}}}(A^*, B)$, where

B is a counital coalgebra (Theorem 1). The corresponding non-commutative thin scheme is the desired $Maps(X, Y)$. ■

It follows from the definition that $Maps(X, Y) = \underline{Hom}(X, Y)$, where the inner Hom is taken in the symmetric monoidal category of non-commutative thin schemes. By definition $\underline{Hom}(X, Y)$ is a non-commutative thin scheme, which satisfies the following functorial isomorphism for any $Z \in Ob(NAff_{\mathcal{C}}^{th})$:

$$Hom_{NAff_{\mathcal{C}}^{th}}(Z, \underline{Hom}(X, Y)) \simeq Hom_{NAff_{\mathcal{C}}^{th}}(Z \otimes X, Y).$$

Note that the monoidal category $NAff_{\mathcal{C}}$ of all non-commutative affine schemes does not have inner $Hom's$ even in the case $\mathcal{C} = Vect_k$. If $\mathcal{C} = Vect_k$ then one can define $\underline{Hom}(X, Y)$ for $X = Spec(A)$, where A is a *finite-dimensional* unital algebra and Y is arbitrary. The situation is similar to the case of "commutative" algebraic geometry, where one can define an affine scheme of maps from a scheme of finite length to an arbitrary affine scheme. On the other hand, one can show that the category of non-commutative ind-affine schemes admit inner Hom's (the corresponding result for commutative ind-affine schemes is known).

Remark 2.17 The non-commutative thin scheme $Maps(X, Y)$ gives rise to a quiver, such that its vertices are k-points of $Maps(X, Y)$. In other words, vertices correspond to homomorphisms $B_X \to B_Y$ of the coalgebras of distributions. Taking the completion at a k-point we obtain a formal pointed manifold. More generally, one can take a completion along a subscheme of k-points, thus arriving to a non-commutative formal manifold with a marked closed subscheme (rather than one point). This construction will be used in the subsequent paper for the desription of the A_∞-structure on A_∞-functors. We also remark that the space of arrows E_{ij} of a quiver is an example of the geometric notion of bitangent space at a pair of k-points i, j. It will be discussed in the subsequent paper.

Example 2.18 Let $Q_1 = \{i_1\}$ and $Q_2 = \{i_2\}$ be quivers with one vertex such that $E_{i_1 i_1} = V_1, E_{i_2 i_2} = V_2$, $dim\, V_i < \infty, i = 1, 2$. Then $B_{Q_i} = T(V_i), i = 1, 2$ and $Maps(X_{B_{Q_1}}, X_{B_{Q_2}})$ corresponds to the quiver Q such that the set of vertices $I_Q = Hom_{Coalgc}(B_{Q_1}, B_{Q_2}) = \prod_{n \geq 1} \underline{Hom}(V_1^{\otimes n}, V_2)$ and for any two vertices $f, g \in I_Q$ the space of arrows is isomorphic to $E_{f,g} = \prod_{n \geq 0} \underline{Hom}(V_1^{\otimes n}, V_2)$.

Definition 2.19 Homomorphism $f \colon B_1 \to B_2$ of counital coalgebras is called a minimal conilpotent extension if it is an inclusion and the induced coproduct on the non-counital coalgebra $B_2/f(B_1)$ is trivial.

Composition of minimal conilpotent extensions is simply called a conilpotent extension. Definition 2.2.1 can be reformulated in terms of finite-dimensional coalgebras. Coalgebra B is smooth if the functor $C \mapsto Hom_{Coalgc}(C, B)$ satisfies the lifting property with respect to conilpotent extensions of finite-dimensional counital coalgebras. The following proposition shows that we can drop the condition of finite-dimensionality.

Proposition 2.20 If B is a smooth coalgebra then the functor $Coalg_{\mathcal{C}} \to Sets$ such that $C \mapsto Hom_{Coalg_{\mathcal{C}}}(C, B)$ satisfies the lifting property for conilpotent extensions.

Proof. Let $f\colon B_1 \to B_2$ be a conilpotent extension, and $g\colon B_1 \to B$ and be an arbitrary homomorphism of counital algebras. It can be thought of as homomorphism of $f(B_1) \to B$. We need to show that g can be extended to B_2. Let us consider the set of pairs (C, g_C) such $f(B_1) \subset C \subset B_2$ and $g_C\colon C \to B$ defines an extension of counital coalgebras, which coincides with g on $f(B_1)$. We apply Zorn lemma to the partially ordered set of such pairs and see that there exists a maximal element (B_{max}, g_{max}) in this set. We claim that $B_{max} = B_2$. Indeed, let $x \in B_2 \setminus B_{max}$. Then there exists a finite-dimensional coalgebra $B_x \subset B_2$ which contains x. Clearly B_x is a conilpotent extension of $f(B_1) \cap B_x$. Since B is smooth we can extend $g_{max} : f(B_1) \cap B_x \to B$ to $g_x : B_x \to B$ and,finally to $g_{x,max} : B_x + B_{max} \to B$. This contradicts to maximality of (B_{max}, g_{max}). Proposition is proved. ■

Proposition 2.21 If X, Y are non-commutative thin schemes and Y is smooth then $Maps(X, Y)$ is also smooth.

Proof. Let $A \to A/J$ be a nilpotent extension of finite-dimensional unital algebras. Then $(A/J)^* \otimes B_X \to A^* \otimes B_X$ is a conilpotent extension of counital coalgebras. Since B_Y is smooth then the previous Proposition implies that the induced map $Hom_{Coalg_{\mathcal{C}}}(A^* \otimes B_X, B_Y) \to Hom_{Coalg_{\mathcal{C}}}((A/J)^* \otimes B_X, B_Y)$ is surjective. This concludes the proof. ■

Let us consider the case when (X, pt_X) and (Y, pt_Y) are non-commutative formal pointed manifolds in the category $\mathcal{C} = Vect_k^{\mathbf{Z}}$. One can describe "in coordinates" the non-commutative formal pointed manifold, which is the formal neighborhood of a k-point of $Maps(X, Y)$. Namely, let $X = Spc(B)$ and $Y = Spc(C)$, and let $f \in Hom_{NAff_{\mathcal{C}}^{th}}(X, Y)$ be a morphism preserving marked points. Then f gives rise to a k-point of $Z = Maps(X, Y)$. Since $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic to the topological algebras of formal power series in free graded variables, we can choose sets of free topological generators $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ for these algebras. Then we can write for the corresponding homomorphism of algebras $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$:

$$f^*(y_j) = \sum_I c^0_{j,M} x^M,$$

where $c^0_{j,M} \in k$ and $M = (i_1, ..., i_n), i_s \in I$ is a non-commutative multi-index (all the coefficients depend on f, hence a better notation should be $c^{f,0}_{j,M}$). Notice that for $M = 0$ one gets $c^0_{j,0} = 0$ since f is a morphism of pointed schemes. Then we can consider an "infinitesimal deformation" f_{def} of f

$$f^*_{def}(y_j) = \sum_M (c^0_{j,M} + \delta c^0_{j,M}) x^M,$$

where $\delta c^0_{j,M}$ are *new variables* commuting with all x_i. Then $\delta c^0_{j,M}$ can be thought of as coordinates in the formal neighborhood of f. More pedantically it can be spelled out such as follows. Let $A = k \oplus m$ be a finite-dimensional graded unital algebra, where m is a graded nilpotent ideal of A. Then an A-point of the formal neighborhood U_f of f is a morphism $\phi \in Hom_{NAff^{th}_{\mathcal{C}}}(Spec(A) \otimes X, Y)$, such that it reduces to f modulo the nilpotent ideal m. We have for the corresponding homomorphism of algebras:

$$\phi^*(y_j) = \sum_M c_{j,M} x^M,$$

where M is a non-commutative multi-index, $c_{j,M} \in A$, and $c_{j,M} \mapsto c^0_{j,M}$ under the natural homomorphism $A \to k = A/m$. In particular $c_{j,0} \in m$. We can treat coefficients $c_{j,M}$ as A-points of the formal neighborhood U_f of $f \in Maps(X, Y)$.

Remark 2.22 The above definitions will play an important role in the subsequent paper, where the non-commutative smooth thin scheme $Spc(B_Q)$ will be assigned to a (small) A_∞-category, the non-commutative smooth thin scheme $Maps(Spc(B_{Q_1}), Spc(B_{Q_2}))$ will be used for the description of the category of A_∞-functors between A_∞-categories and the formal neighborhood of a point in the space $Maps(Spc(B_{Q_1}), Spc(B_{Q_2}))$ will correspond to natural transformations between A_∞-functors.

3 A_∞-Algebras

3.1 Main Definitions

From now on assume that $\mathcal{C} = Vect_k^{\mathbf{Z}}$ unless we say otherwise. If X is a thin scheme then a vector field on X is, by definition, a derivation of the coalgebra B_X. Vector fields form a graded Lie algebra $Vect(X)$.

Definition 3.1 A non-commutative thin differential-graded (dg for short) scheme is a pair (X, d) where X is a non-commutative thin scheme and d is a vector field on X of degree $+1$ such that $[d, d] = 0$.

We will call the vector field d *homological vector field*.

Let X be a formal pointed manifold and x_0 be its unique k-point. Such a point corresponds to a homomorphism of counital coalgebras $k \to B_X$. We say that the vector field d vanishes at x_0 if the corresponding derivation kills the image of k.

Definition 3.2 A non-commutative formal pointed dg-manifold is a pair $((X, x_0), d)$ such that (X, x_0) is a non-commutative formal pointed graded manifold and $d = d_X$ is a homological vector field on X such that $d|_{x_0} = 0$.

Homological vector field d has an infinite Taylor decomposition at x_0. More precisely, let $T_{x_0}X$ be the tangent space at x_0. It is canonically isomorphic to the graded vector space of primitive elements of the coalgebra B_X, i.e. the set of $a \in B_X$ such that $\Delta(a) = 1 \otimes a + a \otimes 1$ where $1 \in B_X$ is the image of $1 \in k$ under the homomorphism of coalgebras $x_0 : k \to B_X$ (see Appendix for the general definition of the tangent space). Then $d := d_X$ gives rise to a (non-canonically defined) collection of linear maps $d_X^{(n)} := m_n : T_{x_0}X^{\otimes n} \to T_{x_0}X[1], n \geq 1$ called *Taylor coefficients of* d which satisfy a system of quadratic relations arising from the condition $[d, d] = 0$. Indeed, our non-commutative formal pointed manifold is isomorphic to the formal neighborhood of zero in $T_{x_0}X$, hence the corresponding non-commutative thin scheme is isomorphic to the cofree tensor coalgebra $T(T_{x_0}X)$ generated by $T_{x_0}X$. Homological vector field d is a derivation of a cofree coalgebra, hence it is uniquely determined by a sequence of linear maps m_n.

Definition 3.3 Non-unital A_∞-algebra over k is given by a non-commutative formal pointed dg-manifold (X, x_0, d) together with an isomorphism of counital coalgebras $B_X \simeq T(T_{x_0}X)$.

Choice of an isomorphism with the tensor coalgebra generated by the tangent space is a non-commutative analog of a choice of affine structure in the formal neighborhood of x_0.

From the above definitions one can recover the traditional one. We present it below for convenience of the reader.

Definition 3.4 A structure of an A_∞-algebra on $V \in Ob(Vect_k^{\mathbf{Z}})$ is given by a derivation d of degree $+1$ of the non-counital cofree coalgebra $T_+(V[1]) = \oplus_{n \geq 1} V^{\otimes n}$ such that $[d, d] = 0$ in the differential-graded Lie algebra of coalgebra derivations.

Traditionally the Taylor coefficients of $d = m_1 + m_2 + \cdots$ are called (higher) multiplications for V. The pair (V, m_1) is a complex of k-vector spaces called the *tangent complex*. If $X = Spc(T(V))$ then $V[1] = T_0X$ and $m_1 = d_X^{(1)}$ is the first Taylor coefficient of the homological vector field d_X. The tangent cohomology groups $H^i(V, m_1)$ will be denoted by $H^i(V)$. Clearly $H^\bullet(V) = \oplus_{i \in \mathbf{Z}} H^i(V)$ is an associative (non-unital) algebra with the product induced by m_2.

An important class of A_∞-algebras consists of *unital* (or strictly unital) and *weakly unital* (or homologically unital) ones. We are going to discuss the definition and the geometric meaning of unitality later.

Homomorphism of A_∞-algebras can be described geometrically as a morphism of the corresponding non-commutative formal pointed dg-manifolds. In the algebraic form one recovers the following traditional definition.

Definition 3.5 A homomorphism of non-unital A_∞-algebras (A_∞-morphism for short) $(V, d_V) \to (W, d_W)$ is a homomorphism of differential-graded coalgebras $T_+(V[1]) \to T_+(W[1])$.

A homomorphism f of non-unital A_∞-algebras is determined by its Taylor coefficients $f_n\colon V^{\otimes n} \to W[1-n], n \geq 1$ satisfying the system of equations

$\sum_{1\leq l_1<...,<l_i=n}(-1)^{\gamma_i} m_i^W(f_{l_1}(a_1,...,a_{l_1}),$
$f_{l_2-l_1}(a_{l_1+1},...,a_{l_2}),...,f_{n-l_{i-1}}(a_{n-l_{i-1}+1},...,a_n)) =$
$\sum_{s+r=n+1}\sum_{1\leq j\leq s}(-1)^{\epsilon_s} f_s(a_1,...,a_{j-1},m_r^V(a_j,...,a_{j+r-1}),a_{j+r},...,a_n).$

Here $\epsilon_s = r\sum_{1\leq p\leq j-1} deg(a_p)+j-1+r(s-j)$, $\gamma_i = \sum_{1\leq p\leq i-1}(i-p)(l_p - l_{p-1}-1) + \sum_{1\leq p\leq i-1}\nu(l_p)\sum_{l_{p-1}+1\leq q\leq l_p} deg(a_q)$, where we use the notation $\nu(l_p) = \sum_{p+1\leq m\leq i}(1-l_m+l_{m-1})$ and set $l_0 = 0$.

Remark 3.6 All the above definitions and results are valid for $\mathbf{Z}/2$-graded A_∞-algebras as well. In this case we consider formal manifolds in the category $Vect_k^{\mathbf{Z}/2}$ of $\mathbf{Z}/2$-graded vector spaces. We will use the correspodning results without further comments. In this case one denotes by ΠA the $\mathbf{Z}/2$-graded vector space $A[1]$.

3.2 Minimal Models of A_∞-Algebras

One can do simple differential geometry in the symmetric monoidal category of non-commutative formal pointed dg-manifolds. New phenomenon is the possibility to define some structures up to a quasi-isomorphism.

Definition 3.7 Let $f : (X, d_X, x_0) \to (Y, d_Y, y_0)$ be a morphism of non-commutative formal pointed dg-manifolds. We say that f is a quasi-isomorphism if the induced morphism of the tangent complexes $f_1 : (T_{x_0}X, d_X^{(1)}) \to (T_{y_0}Y, d_Y^{(1)})$ is a quasi-isomorphism. We will use the same terminology for the corresponding A_∞-algebras.

Definition 3.8 An A_∞-algebra A (or the corresponding non-commutative formal pointed dg-manifold) is called minimal if $m_1 = 0$. It is called contractible if $m_n = 0$ for all $n \geq 2$ and $H^\bullet(A, m_1) = 0$.

The notion of minimality is coordinate independent, while the notion of contractibility is not.

It is easy to prove that any A_∞-algebra A has a *minimal model* M_A, i.e. M_A is minimal and there is a quasi-isomorphism $M_A \to A$ (the proof is similar to the one from [29, 36]). The minimal model is unique up to an A_∞-isomorphism. We will use the same terminology for non-commutative formal pointed dg-manifolds. In geometric language a non-commutative formal pointed dg-manifold X is isomorphic to a *categorical* product (i.e. corresponding to the completed free product of algebras of functions) $X_m \times X_{lc}$, where X_m is minimal and X_{lc} is linear contractible. The above-mentioned quasi-isomorphism corresponds to the projection $X \to X_m$.

The following result (homological inverse function theorem) can be easily deduced from the above product decomposition.

Proposition 3.9 If $f\colon A \to B$ is a quasi-isomorphism of A_∞-algebras then there is a (non-canonical) quasi-isomorphism $g : B \to A$ such that fg and gf induce identity maps on zero cohomologies $H^0(B)$ and $H^0(A)$ respectively.

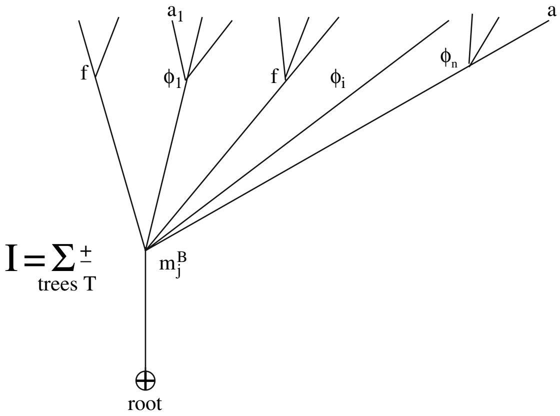
3.3 Centralizer of an A_∞-Morphism

Let A and B be two A_∞-algebras, and (X, d_X, x_0) and (Y, d_Y, y_0) be the corresponding non-commutative formal pointed dg-manifolds. Let $f : A \to B$ be a morphism of A_∞-algebras. Then the corresponding k-point $f \in Maps(Spc(A), Spc(B))$ gives rise to the formal pointed manifold $U_f = \widehat{Maps}(X, Y)_f$ (completion at the point f). Functoriality of the construction of $Maps(X, Y)$ gives rise to a homomorphism of graded Lie algebras of vector fields $Vect(X) \oplus Vect(Y) \to Vect(Maps(X, Y))$. Since $[d_X, d_Y] = 0$ on $X \otimes Y$, we have a well-defined homological vector field d_Z on $Z = Maps(X, Y)$. It corresponds to $d_X \otimes 1_Y - 1_X \otimes d_Y$ under the above homomorphism. It is easy to see that $d_Z|_f = 0$ and in fact morphisms $f\colon A \to B$ of A_∞-algebras are exactly zeros of d_Z. We are going to describe below the A_∞-algebra $Centr(f)$ (centralizer of f) which corresponds to the formal neighborhood U_f of the point $f \in Maps(X, Y)$. We can write (see Sect. 2.3 for the notation)

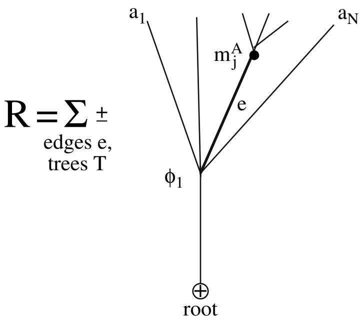
$$c_{j,M} = c^0_{j,M} + r_{j,M},$$

where $c^0_{j,M} \in k$ and $r_{j,M}$ are formal non-commutative coordinates in the neighborhood of f. Then the A_∞-algebra $Centr(f)$ has a basis $(r_{j,M})_{j,M}$ and the A_∞-structure is defined by the restriction of the homological vector d_Z to U_f.

As a $\mathbf{Z}$-graded vector space $Centr(f) = \prod_{n\geq 0} Hom_{Vect_k^{\mathbf{Z}}}(A^{\otimes n}, B)[-n]$. Let $\phi_1, ..., \phi_n \in Centr(f)$ and $a_1, ..., a_N \in A$. Then we have $m_n(\phi_1, ..., \phi_n)$ $(a_1, ..., a_N) = I + R$. Here I corresponds to the term $= 1_X \otimes d_Y$ and is given by the following expression



Similarly R corresponds to the term $d_X \otimes 1_Y$ and is described by the following figure



Comments on the figure describing I.

(1) We partition a sequence $(a_1, ..., a_N)$ into $l \geq n$ non-empty subsequences.
(2) We mark n of these subsequences counting from the left (the set can be empty).
(3) We apply multilinear map $\phi_i, 1 \leq i \leq n$ to the ith marked group of elements a_l.
(4) We apply Taylor coefficients of f to the remaining subsequences.

Notice that the term R appear only for m_1 (i.e. $n = 1$). For all subsequences we have $n \geq 1$.

From geometric point of view the term I corresponds to the vector field d_Y, while the term R corresponds to the vector field d_X.

Proposition 3.10 Let $d_{Centr(f)}$ be the derivation corresponding to the image of $d_X \oplus d_Y$ in $Maps(X, Y)$.

One has $[d_{Centr(f)}, d_{Centr(f)}] = 0$.

Proof. Clear. ■

Remark 3.11 The A_∞-algebra $Centr(f)$ and its generalization to the case of A_∞-categories discussed in the subsequent paper provide geometric description of the notion of natural transformaion in the A_∞-case (see [39, 40] for a pure algebraic approach to this notion).

4 Non-Commutative dg-line L and Weak Unit

4.1 Main Definition

Definition 4.1 An A_∞-algebra is called *unital* (or strictly unital) if there exists an element $1 \in V$ of degree zero, such that $m_2(1, v) = m_2(v, 1)$ and

$m_n(v_1, ..., 1, ..., v_n) = 0$ for all $n \neq 2$ and $v, v_1, ..., v_n \in V$. It is called *weakly unital* (or homologically unital) if the graded associative unital algebra $H^\bullet(V)$ has a unit $1 \in H^0(V)$.

The notion of strict unit depends on a choice of affine coordinates on $Spc(T(V))$, while the notion of weak unit is "coordinate free." Moreover, one can show that a weakly unital A_∞-algebra becomes strictly unital after an appropriate change of coordinates.

The category of unital or weakly unital A_∞-algebras are defined in the natural way by the requirement that morphisms preserve the unit (or weak unit) structure.

In this section we are going to discuss a non-commutative dg-version of the odd one-dimensional supervector space $\mathbf{A}^{0|1}$ and its relationship to weakly unital A_∞-algebras. The results are valid for both $\mathbf{Z}$-graded and $\mathbf{Z}/2$-graded A_∞-algebras.

Definition 4.2 Non-commutative formal dg-line $\mathbf{L}$ is a non-commutative formal pointed dg-manifold corresponding to the one-dimensional A_∞-algebra $A \simeq k$ such that $m_2 = id, m_{n\neq 2} = 0$.

The algebra of functions $\mathcal{O}(\mathbf{L})$ is isomorphic to the topological algebra of formal series $k\langle\langle \xi \rangle\rangle$, where $deg\, \xi = 1$. The differential is given by $\partial(\xi) = \xi^2$.

4.2 Adding a Weak Unit

Let (X, d_X, x_0) be a non-commutative formal pointed dg-manifold correspodning to a non-unital A_∞-algebra A. We would like to describe geometrically the procedure of adding a weak unit to A.

Let us consider the non-commutative formal pointed graded manifold $X_1 = \mathbf{L} \times X$ corresponding to the free product of the coalgebras $B_{\mathbf{L}} * B_X$. Clearly one can lift vector fields d_X and $d_{\mathbf{L}} := \partial/\partial\,\xi$ to X_1.

Lemma 4.3 *The vector field*

$$d := d_{X_1} = d_X + ad(\xi) - \xi^2 \partial/\partial\,\xi$$

satisfies the condition $[d, d] = 0$.

Proof. Straightforward check. ■

It follows from the formulas given in the proof that ξ appears in the expansion of d_X in quadratic expressions only. Let A_1 be an A_∞-algebras corresponding to X_1 and $1 \in T_{pt}X_1 = A_1[1]$ be the element of $A_1[1]$ dual to ξ (it corresponds to the tangent vector $\partial/\partial\,\xi$). Thus we see that $m_2^{A_1}(1, a) = m_2^{A_1}(a, 1) = a, m_2^{A_1}(1, 1) = 1$ for any $a \in A$ and $m_n^{A_1}(a_1, ..., 1, ..., a_n) = 0$ for all $n \geq 2, a_1, ..., a_n \in A$. This proves the following result.

Proposition 4.4 The A_∞-algebra A_1 has a strict unit.

Notice that we have a canonical morphism of non-commutative formal pointed dg-manifolds $e\colon X \to X_1$ such that $e^*|_X = id, e^*(\xi) = 0$.

Definition 4.5 Weak unit in X is given by a morphism of non-commutative formal pointed dg-manifolds $p : X_1 \to X$ such that $p \circ e = id$.

It follows from the definition that if X has a weak unit then the associative algebra $H^\bullet(A, m_1^A)$ is unital. Hence our geometric definition agrees with the pure algebraic one (explicit algebraic description of the notion of weak unit can be found, e.g., in [15], Sect. 20[7]).

5 Modules and Bimodules

5.1 Modules and Vector Bundles

Recall that a topological vector space is called linearly compact if it is a projective limit of finite-dimensional vector spaces. The duality functor $V \mapsto V^*$ establishes an anti-equivalence between the category of vector spaces (equipped with the discrete topology) and the category of linearly compact vector spaces. All that can be extended in the obvious way to the category of graded vector spaces.

Let X be a non-commutative thin scheme in $Vect_k^{\mathbf{Z}}$.

Definition 5.1 Linearly compact vector bundle $\mathcal{E}$ over X is given by a linearly compact topologically free $\mathcal{O}(X)$-module $\Gamma(\mathcal{E})$, where $\mathcal{O}(X)$ is the algebra of function on X. Module $\Gamma(\mathcal{E})$ is called the module of *sections* of the linearly compact vector bundle $\mathcal{E}$.

Suppose that (X, x_0) is formal graded manifold. *The fiber* of $\mathcal{E}$ over x_0 is given by the quotient space $\mathcal{E}_{x_0} = \Gamma(\mathcal{E})/\overline{m_{x_0}\Gamma(\mathcal{E})}$ where $m_{x_0} \subset \mathcal{O}(X)$ is the two-sided maximal ideal of functions vanishing at x_0 and the bar means the closure.

Definition 5.2 A dg-vector bundle over a formal pointed dg-manifold (X, d_X, x_0) is given by a linearly compact vector bundle $\mathcal{E}$ over (X, x_0) such that the corresponding module $\Gamma(\mathcal{E})$ carries a differential $d_\mathcal{E} \colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})[1], d_\mathcal{E}^2 = 0$ so that $(\Gamma(\mathcal{E}), d_\mathcal{E})$ becomes a dg-module over the dg-algebra $(\mathcal{O}(X), d_X)$ and $d_\mathcal{E}$ vanishes on $\mathcal{E}_{x_0}$.

Definition 5.3 Let A be a non-unital A_∞-algebra. A left A-module M is given by a dg-bundle E over the formal pointed dg-manifold $X = Spc(T(A[1]))$ together with an isomorphism of vector bundles $\Gamma(\mathcal{E}) \simeq \mathcal{O}(X)\widehat{\otimes}M^*$ called a trivialization of $\mathcal{E}$.

---

[7] V. Lyubashenko has informed us that the equivalence of two descriptions also follows from his results with Yu. Bespalov and O. Manzyuk.

Passing to dual spaces we obtain the following algebraic definition.

Definition 5.4 Let A be an A_∞-algebra and M be a $\mathbf{Z}$-graded vector space. A structure of a left A_∞-module on M over A (or simply a structure of a left A-module on M) is given by a differential d_M of degree $+1$ on $T(A[1])\otimes M$ which makes it into a dg-comodule over the dg-coalgebra $T(A[1])$.

The notion of *right* A_∞-module is similar. Right A-module is the same as left A^{op}-module. Here A^{op} is the *opposite* A_∞-algebra, which coincides with A as a $\mathbf{Z}$-graded vector space and for the higher multiplications one has: $m_n^{op}(a_1,...,a_n)=(-1)^{n(n-1)/2}m_n(a_n,...,a_1)$. The A_∞-algebra A carries the natural structures of the left and right A-modules. If we simply say "A-module" it will always mean "left A-module."

Taking the Taylor series of d_M we obtain a collection of k-linear maps (higher action morphisms) for any $n\geq 1$

$$m_n^M : A^{\otimes(n-1)}\otimes M\to M[2-n],$$

satisfying the compatibility conditions which can be written in exactly the same form as compatibility conditions for the higher products m_n^A (see e.g., [27]). All those conditions can be derived from just one property that the cofree $T_+(A[1])$-comodule $T_+(A[1],M)=\oplus_{n\geq 0}A[1]^{\otimes n}\otimes M$ carries a derivation $m^M=(m_n^M)_{n\geq 0}$ such that $[m^M,m^M]=0$. In particular (M,m_1^M) is a complex of vector spaces.

Definition 5.5 Let A be a weakly unital A_∞-algebra. An A-module M is called weakly unital if the cohomology $H^\bullet(M,m_1^M)$ is a unital $H^\bullet(A)$-module.

It is easy to see that left A_∞-modules over A form a dg-category $A-mod$ with morphisms being homomorphisms of the corresponding comodules. As a graded vector space

$$Hom_{A-mod}(M,N)=\oplus_{n\geq 0}\underline{Hom}_{Vect_k^{\mathbf{Z}}}(A[1]^{\otimes n}\otimes M,N).$$

It easy to see that $Hom_{A-mod}(M,N)$ is a complex.

If M is a right A-module and N is a left A-module then one has a naturally defined structure of a complex on $M\otimes_A N\coloneqq \oplus_{n\geq 0}M\otimes A[1]^{\otimes n}\otimes N$. The differential is given by the formula:

$$d(x\otimes a_1\otimes ...\otimes a_n\otimes y)=\sum \pm m_i^M(x\otimes a_1\otimes ...\otimes a_i)\otimes a_{i+1}\otimes ...\otimes a_n\otimes y)$$

$$+\sum \pm x\otimes a_1\otimes ...\otimes a_{i-1}\otimes m_k^A(a_i\otimes ...\otimes a_{i+k-1})\otimes a_{i+k}\otimes ...\otimes a_n\otimes y$$

$$+\sum \pm x\otimes a_1\otimes ...\otimes a_{i-1}\otimes m_j^N(a_i\otimes ...\otimes a_n\otimes y).$$

We call this complex the *derived tensor product* of M and N.

For any A_∞-algebras A and B we define an $A-B$-bimodule as a $\mathbf{Z}$-graded vector space M together with linear maps

$$c^M_{n_1,n_2}\colon A[1]^{\otimes n_1}\otimes M\otimes B[1]^{\otimes n_2}\to M[1]$$

satisfying the natural compatibility conditions (see e.g. [27]). If X and Y are formal pointed dg-manifolds corresponding to A and B respectively then an $A-B$-bimodule is the same as a dg-bundle $\mathcal{E}$ over $X\otimes Y$ equipped with a homological vector field $d_{\mathcal{E}}$ which is a lift of the vector field $d_X\otimes 1+1\otimes d_Y$.

Example 5.6 Let $A=B=M$. We define a structure of diagonal bimodule on A by setting $c^A_{n_1,n_2}=m^A_{n_1+n_2+1}$.

Proposition 5.7 (1) To have a structure of an A_∞-module on the complex M is the same as to have a homomorphism of A_∞-algebras $\phi\colon A\to \underline{End}_{\mathbf{K}}(M)$, where $\mathbf{K}$ is a category of complexes of k-vector spaces.

(2) To have a structure of an $A-B$-bimodule on a graded vector space M is the same as to have a structure of left A-module on M and to have a morphism of A_∞-algebras $\varphi_{A,B}\colon B^{op}\to Hom_{A-mod}(M,M)$.

Let A be an A_∞-algebra, M be an A-module and $\varphi_{A,A}\colon A^{op}\to Hom_{A-mod}$ (M, M) be the corresponding morphism of A_∞-algebras. Then the dg-algebra $Centr(\varphi)$ is isomorphic to the dg-algebra $Hom_{A-mod}(M,M)$.

If $M={}_AM_B$ is an $A-B$-bimodule and $N={}_BN_C$ is a $B-C$-bimodule then the complex ${}_AM_B\otimes_B {}_BN_C$ carries an $A-C$-bimodule structure. It is called the *tensor product* of M and N.

Let $f\colon X\to Y$ be a homomorphism of formal pointed dg-manifolds corresponding to a homomorphism of A_∞-algebras $A\to B$. Recall that in Sect. 4 we constructed the formal neighborhood U_f of f in $Maps(X,Y)$ and the A_∞-algebra $Centr(f)$. On the other hand, we have an $A-mod-B$ bimodule structure on B induced by f. Let us denote this bimodule by $M(f)$. We leave the proof of the following result as an exercise to the reader. It will not be used in the paper.

Proposition 5.8 If B is weakly unital then the dg-algebra $End_{A-mod-B}$ $(M(f))$ is quasi-isomorphic to $Centr(f)$.

A_∞-bimodules will be used in Part II for study of homologically smooth A_∞-algebras. In the subsequent paper devoted to A_∞-categories we will explain that bimodules give rise to A_∞-functors between the corresponding categories of modules. Tensor product of bimodules corresponds to the composition of A_∞-functors.

5.2 On the Tensor Product of A_∞-Algebras

The tensor product of two dg-algebras A_1 and A_2 is a dg-algebra. For A_∞-algebras there is no canonical simple formula for the A_∞-structure on $A_1\otimes_k A_2$

which generalizes the one in the dg-algebras case. Some complicated formulas were proposed in [44]. They are not symmetric with respect to the permutation $(A_1, A_2) \mapsto (A_2, A_1)$. We will give below the definition of the dg-algebra which is quasi-isomorphic to the one from [44] in the case when both A_1 and A_2 are weakly unital. Namely, we define the A_∞-tensor product

$$A_1 \text{``} \otimes'' A_2 = End_{A_1-mod-A_2}(A_1 \otimes A_2).$$

Note that it is a unital dg-algebra. One can show that the dg-category $A-mod-B$ is equivalent (as a dg-category) to $A_1 \text{``} \otimes'' A_2^{op} - mod$.

6 Yoneda Lemma

6.1 Explicit Formulas for the Product and Differential on *Centr*(*f*)

Let A be an A_∞-algebra and $B = End_{\mathbf{K}}(A)$ be the dg-algebra of endomorphisms of A in the category $\mathbf{K}$ of complexes of k-vector spaces. Let $f = f_A : A \to B$ be the natural A_∞-morphism coming from the left action of A on itself. Notice that B is always a unital dg-algebra, while A can be non-unital. The aim of this Section was to discuss the relationship between A and $Centr(f_A)$. This is a simplest case of the A_∞-version of Yoneda lemma (the general case easily follows from this one. See also [39, 40]).

As a graded vector space $Centr(f_A)$ is isomorphic to $\prod_{n\geq 0} \underline{Hom}(A^{\otimes(n+1)}, A)[-n]$.

Let us describe the product in $Centr(f)$ for $f = f_A$. Let ϕ, ψ be two homogeneous elements of $Centr(f)$. Then

$$(\phi \cdot \psi)(a_1, a_2, \dots, a_N) = \pm\phi(a_1, \dots, a_{p-1}, \psi(a_p, \dots, a_N)).$$

Here ψ acts on the last group of variables $a_p, \dots, a_N$ and we use the Koszul sign convention for A_∞-algebras in order to determine the sign.

Similarly one has the following formula for the differential (see Sect. 3.3):

$$(d\phi)(a_1, \dots, a_N) = \sum \pm\phi(a_1, \dots, a_s, m_i(a_{s+1}, \dots, a_{s+i}), a_{s+i+1} \dots, a_N)$$

$$+\sum \pm m_i(a_1, \dots, a_{s-1}, \phi(a_s, \dots, a_j, ..., a_N)).$$

6.2 Yoneda Homomorphism

If M is an $A-B$-bimodule then one has a homomorphism of A_∞-algebras $B^{op} \to Centr(\phi_{A,M})$ (see Propositions 5.1.7 and 5.1.8). We would like to apply this general observation in the case of the diagonal bimodule structure on A. Explicitly, we have the A_∞-morphism $A^{op} \to End_{mod-A}(A)$ or, equivalently, the collection of maps $A^{\otimes m} \to Hom(A^{\otimes n}, A)$. By conjugation it gives us a collection of maps

$$A^{\otimes m} \otimes Hom(A^{\otimes n}, A) \to Hom(A^{\otimes (m+n)}, A).$$

In this way we get a natural A_∞-morphism $Yo\colon A^{op} \to Centr(f_A)$ called the *Yoneda homomorphism.*

Proposition 6.1 The A_∞-algebra A is weakly unital if and only if the Yoneda homomorphism is a quasi-isomorphism.

Proof. Since $Centr(f_A)$ is weakly unital, then A must be weakly unital as long as Yoneda morphism is a quasi-isomorphism.

Let us prove the opposite statement. We assume that A is weakly unital. It suffices to prove that the cone $Cone(Yo)$ of the Yoneda homomorphism has trivial cohomology. Thus we need to prove that the cone of the morphism of complexes

$$(A^{op}, m_1) \to (\oplus_{n\geq 1} Hom(A^{\otimes n}, A), m_1^{Centr(f_A)}).$$

is contractible. In order to see this, one considers the extended complex $A \oplus Centr(f_A)$. It has natural filtration arising from the tensor powers of A. The corresponding spectral sequence collapses, which gives an explicit homotopy of the extended complex to the trivial one. This implies the desired quasi-isomorphism of $H^0(A^{op})$ and $H^0(Centr(f_A))$. ■

Remark 6.2 It look like the construction of $Centr(f_A)$ is the first known canonical construction of a unital dg-algebra quasi-isomorphic to a given A_∞-algebra (canonical but not functorial). This is true even in the case of strictly unital A_∞-algebras. Standard construction via bar and cobar resolutions gives a non-unital dg-algebra.

Part II: Smoothness and Compactness

7 Hochschild Cochain and Chain Complexes of an A_∞-Algebra

7.1 Hochschild Cochain Complex

We change the notation for the homological vector field to Q, since the letter d will be used for the differential.[8] Let $((X, pt), Q)$ be a non-commutative

---

[8] We recall that the super version of the notion of formal dg-manifold was introduced by A. Schwarz under the name "Q-manifold." Here letter Q refers to the supercharge notation from Quantum Field Theory.

formal pointed dg-manifold corresponding to a non-unital A_∞-algebra A and $Vect(X)$ the graded Lie algebra of vector fields on X (i.e., continuous derivations of $\mathcal{O}(X)$).

We denote by $C^\bullet(A,A) := C^\bullet(X,X) := Vect(X)[-1]$ the *Hochschild cochain complex* of A. As a $\mathbf{Z}$-graded vector space

$$C^\bullet(A,A) = \prod_{n\geq 0} \underline{Hom}_{\mathcal{C}}(A[1]^{\otimes n}, A).$$

The differential on $C^\bullet(A,A)$ is given by $[Q,\bullet]$. Algebraically, $C^\bullet(A,A)[1]$ is a DGLA of derivations of the coalgebra $T(A[1])$ (see Sect. 3).

Theorem 7.1 *Let X be a non-commutative formal pointed dg-manifold and $C^\bullet(X,X)$ be the Hochschild cochain complex. Then one has the following quasi-isomorphism of complexes*

$$C^\bullet(X,X)[1] \simeq T_{id_X}(Maps(X,X)),$$

where T_{id_X} denotes the tangent complex at the identity map.

Proof. Notice that $Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ is the non-commutative dg ind-manifold of vector fields on X. The tangent space T_{id_X} from the theorem can be identified with the set of such $f \in Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ that $f|_{\{pt\}\otimes X} = id_X$. On the other hand the DGLA $C^\bullet(X,X)[1]$ is the DGLA of vector fields on X. The theorem follows. ∎

The Hochschild complex admits a couple of other interpretations. We leave to the reader to check the equivalence of all of them. First, $C^\bullet(A,A) \simeq Centr(id_A)$. Finally, for a weakly unital A one has $C^\bullet(A,A) \simeq Hom_{A-mod-A}(A,A)$. Both are quasi-isomorphisms of complexes.

Remark 7.2 Interpretation of $C^\bullet(A,A)[1]$ as vector fields gives a DGLA structure on this space. It is a Lie algebra of the "commutative" formal group in $Vect_k^{\mathbf{Z}}$, which is an abelianization of the non-commutative formal group of inner (in the sense of tensor categories) automorphisms $\underline{Aut}(X) \subset Maps(X,X)$. Because of this non-commutative structure underlying the Hochschild cochain complex, it is natural to expect that $C^\bullet(A,A)[1]$ carries more structures than just DGLA. Indeed, Deligne's conjecture (see e.g., [35] and the last section of this paper) claims that the DGLA algebra structure on $C^\bullet(A,A)[1]$ can be extended to a structure of an algebra over the operad of singular chains of the topological operad of little discs. Graded Lie algebra structure can be recovered from cells of highest dimension in the cell decomposition of the topological operad.

7.2 Hochschild Chain Complex

In this subsection we are going to construct a complex of k-vector spaces which is dual to the Hochschild chain complex of a non-unital A_∞-algebra.

Cyclic Differential Forms of Order Zero

Let (X, pt) be a non-commutative formal pointed manifold over k and $\mathcal{O}(X)$ the algebra of functions on X. For simplicity we will assume that X is finite-dimensional, i.e., $dim_k\, T_{pt}X < \infty$. If $B = B_X$ is a counital coalgebra corresponding to X (coalgebra of distributions on X) then $\mathcal{O}(X) \simeq B^*$. Let us choose affine coordinates $x_1, x_2, ..., x_n$ at the marked point pt. Then we have an isomorphism of $\mathcal{O}(X)$ with the topological algebra $k\langle\langle x_1, ..., x_n\rangle\rangle$ of formal series in free graded variables $x_1, ..., x_n$.

We define the space of *cyclic differential degree zero forms on* X as

$$\Omega^0_{cycl}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top},$$

where $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ denotes the topological commutator (the closure of the algebraic commutator in the adic topology of the space of non-commutative formal power series).

Equivalently, we can start with the graded k-vector space $\Omega^0_{cycl,dual}(X)$ defined as the kernel of the composition $B \to B \otimes B \to \bigwedge^2 B$ (first map is the coproduct $\Delta\colon B \to B \otimes B$, while the second one is the natural projection to the skew-symmetric tensors). Then $\Omega^0_{cycl}(X) \simeq (\Omega^0_{cycl,dual}(X))^*$ (dual vector space).

Higher Order Cyclic Differential Forms

We start with the definition of the *odd tangent bundle* $T[1]X$. This is the dg-analog of the total space of the tangent supervector bundle with the changed parity of fibers. It is more convenient to describe this formal manifold in terms of algebras rather than coalgebras. Namely, the algebra of functions $\mathcal{O}(T[1]X)$ is a unital topological algebra isomorphic to the algebra of formal power series $k\langle\langle x_i, dx_i\rangle\rangle, 1 \le i \le n$, where $deg\, dx_i = deg\, x_i + 1$ (we do not impose any commutativity relations between generators). More invariant description involves the odd line. Namely, let $t_1 := Spc(B_1)$, where $(B_1)^* = k\langle\langle\xi\rangle\rangle/(\xi^2), deg\,\xi = +1$. Then we define $T[1]X$ as the formal neighborhood in $Maps(t_1, X)$ of the point p which is the composition of pt with the trivial map of t_1 into the point $Spc(k)$.

Definition 7.3 (a) The graded vector space

$$\mathcal{O}(T[1]X) = \Omega^\bullet(X) = \prod_{m\ge 0} \Omega^m(X)$$

is called the space of de Rham differential forms on X.

(b) The graded space

$$\Omega^0_{cycl}(T[1]X) = \prod_{m\ge 0} \Omega^m_{cycl}(X)$$

is called the space of cyclic differential forms on X.

In coordinate description the grading is given by the total number of dx_i. Clearly each space $\Omega^n_{cycl}(X), n \geq 0$ is dual to some vector space $\Omega^n_{cycl,dual}(X)$ equipped with the discrete topology (since this is true for $\Omega^0(T[1]X)$).

The *de Rham differential on* $\Omega^\bullet(X)$ corresponds to the vector field $\partial/\partial\xi$ (see description which uses the odd line, it is the same variable ξ). Since Ω^0_{cycl} is given by the natural (functorial) construction, the de Rham differential descends to the subspace of cyclic differential forms. We will denote the former by d_{DR} and the latter by d_{cycl}.

The space of *cyclic 1-forms* $\Omega^1_{cycl}(X)$ is a (topological) span of expressions $x_1 x_2 ... x_l \, dx_j, x_i \in \mathcal{O}(X)$. Equivalently, the space of cyclic 1-forms consists of expressions $\sum_{1 \leq i \leq n} f_i(x_1, ..., x_n) \, dx_i$ where $f_i \in k\langle\langle x_1, ..., x_n \rangle\rangle$.

There is a map $\varphi : \Omega^1_{cycl}(X) \to \mathcal{O}(X)_{red} := \mathcal{O}(X)/k$, which is defined on $\Omega^1(X)$ by the formula $adb \mapsto [a, b]$ (check that the induced map on the cyclic 1-forms is well-defined). This map does not have an analog in the commutative case.[9]

Non-commutative Cartan Calculus

Let X be a formal graded manifold over a field k. We denote by $g := g_X$ the graded Lie algebra of continuous linear maps $\mathcal{O}(T[1]X) \to \mathcal{O}(T[1]X)$ generated by de Rham differential $d = d_{dR}$ and contraction maps $i_\xi, \xi \in Vect(X)$ which are defined by the formulas $i_\xi(f) = 0, i_\xi(df) = \xi(f)$ for all $f \in \mathcal{O}(T[1]X)$. Let us define the Lie derivative $Lie_\xi = [d, i_\xi]$ (graded commutator). Then one can easily checks the usual formulas of the Cartan calculus

$$[d, d] = 0, Lie_\xi = [d, i_\xi], [d, Lie_\xi] = 0,$$

$$[Lie_\xi, i_\eta] = i_{[\xi,\eta]}, [Lie_\xi, Lie_\eta] = Lie_{[\xi,\eta]}, [i_\xi, i_\eta] = 0,$$

for any $\xi, \eta \in Vect(X)$.

By naturality, the graded Lie algebra g_X acts on the space $\Omega^\bullet_{cycl}(X)$ as well as one the dual space $(\Omega^\bullet_{cycl}(X))^*$.

Differential on the Hochschild Chain Complex

Let Q be a homological vector field on (X, pt). Then $A = T_{pt}X[-1]$ is a non-unital A_∞-algebra.

We define the *dual Hochschild chain complex* $(C_\bullet(A, A))^*$ as $\Omega^1_{cycl}(X)[2]$ with the differential Lie_Q. Our terminology is explained by the observation that $\Omega^1_{cycl}(X)[2]$ is dual to the conventional Hochschild chain complex

---

[9] V. Ginzburg pointed out that the geometric meaning of the map φ as a "contraction with double derivation" was suggested in Sect. 5.4 of [19].

$$C_\bullet(A,A) = \oplus_{n\geq 0}(A[1])^{\otimes n}\otimes A.$$

Note that we use the cohomological grading on $C_\bullet(A,A)$, i.e. chains of degree n in conventional (homological) grading have degree $-n$ in our grading. The differential has degree $+1$.

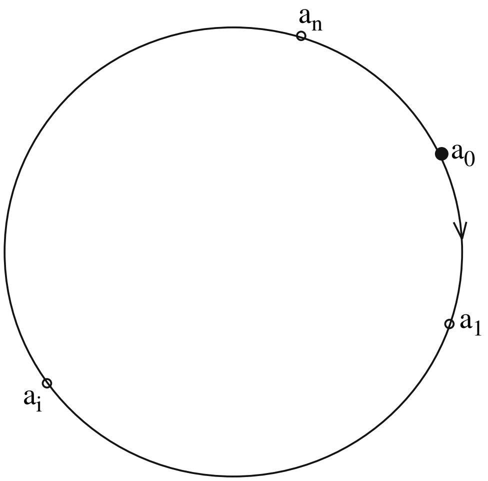
In coordinates the isomorphism identifies an element $f_i(x_1,...,x_n)\otimes x_i \in (A[1]^{\otimes n}\otimes A)^*$ with the homogeneous element $f_i(x_1,...,x_n)\,dx_i \in \Omega^1_{cycl}(X)$. Here $x_i \in (A[1])^*, 1\leq i\leq n$ are affine coordinates.

The graded Lie algebra $Vect(X)$ of vector fields of all degrees acts on any functorially defined space, in particular, on all spaces $\Omega^j(X), \Omega^j_{cycl}(X)$, etc. Then we have a differential on $\Omega^j_{cycl}(X)$ given by $b = Lie_Q$ of degree $+1$. There is an explicit formula for the differential b on $C_\bullet(A,A)$ (cf. [T]):

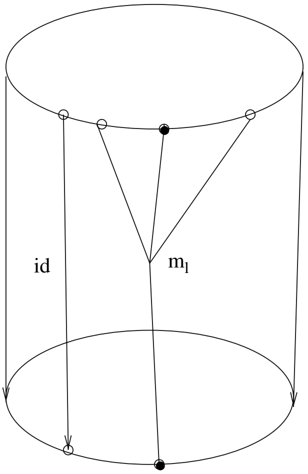
$$b(a_0\otimes ...\otimes a_n) = \sum \pm a_0\otimes ...\otimes m_l(a_i\otimes ...\otimes a_j)\otimes ...\otimes a_n$$

$$+\sum \pm m_l(a_j\otimes ...\otimes a_n\otimes a_0\otimes ...\otimes a_i)\otimes a_{i+1}\otimes ...\otimes a_{j-1}.$$

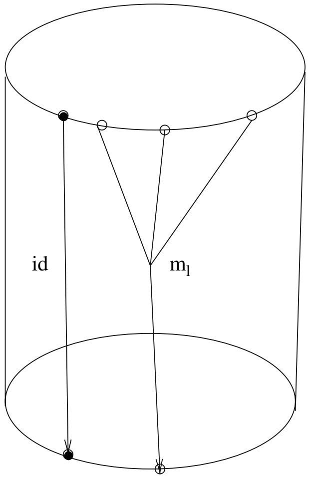
It is convenient to depict a cyclic monomial $a_0\otimes ...\otimes a_n$ in the following way. We draw a clockwise oriented circle with $n+1$ points labeled from 0 to n such that one point is marked We assign the elements $a_0, a_1, ..., a_n$ to the points with the corresponding labels, putting a_0 at the marked point.



Then we can write $b = b_1 + b_2$ where b_1 is the sum (with appropriate signs) of the expressions depicted below:



Similarly, b_2 is the sum (with appropriate signs) of the expressions depicted below:



In both cases maps m_l are applied to a consequitive cyclically ordered sequence of elements of A assigned to the points on the top circle. The identity map is applied to the remaining elements. Marked point on the top circle is the position of the element of a_0. Marked point on the bottom circle depicts the first tensor factor of the corresponding summand of b. In both the cases we start cyclic count of tensor factors clockwise from the marked point.

7.3 The Case of Strictly Unital A_∞-Algebras

Let A be a strictly unital A_∞-algebra. There is a *reduced* Hochschild chain complex

$$C_\bullet^{red}(A,A) = \oplus_{n\geq 0} A \otimes ((A/k\cdot 1)[1])^{\otimes n},$$

which is the quotient of $C_\bullet(A,A)$. Similarly there is a reduced Hochschild cochain complex

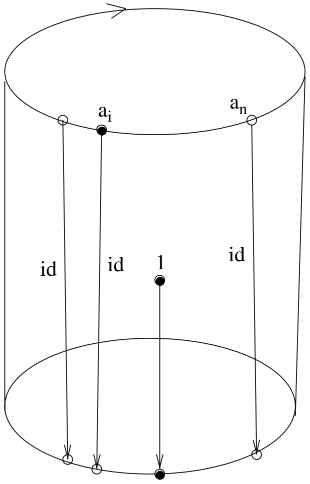
$$C_{red}^\bullet(A,A) = \prod_{n\geq 0} \underline{Hom}_{\mathcal{C}}((A/k\cdot 1)[1]^{\otimes n}, A),$$

which is a subcomplex of the Hochschild cochain complex $C^\bullet(A,A)$.

Also, $C_\bullet(A,A)$ carries also the "Connes's differential" B of degree -1 (called sometimes "de Rham differential") given by the formula (see [7], [T])

$$B(a_0\otimes ...\otimes a_n) = \sum_i \pm 1\otimes a_i \otimes ... \otimes a_n \otimes a_0 \otimes ... \otimes a_{i-1}, B^2=0, Bb+bB=0.$$

Here is a graphical description of B (it will receive an explanation in the section devoted to generalized Deligne's conjecture)



Let u be an independent variable of degree $+2$. It follows that for a strictly unital A_∞-algebra A one has a differential $b+uB$ of degree $+1$ on the graded vector space $C_\bullet(A,A)[[u]]$ which makes the latter into a complex called *negative cyclic complex* (see [7, T]). In fact $b+uB$ is a differential on a smaller complex $C_\bullet(A,A)[u]$. In the non-unital case one can use Cuntz–Quillen complex instead of a negative cyclic complex (see next subsection).

7.4 Non-unital Case: Cuntz–Quillen Complex

In this subsection we are going to present a formal dg-version of the mixed complex introduced by Cuntz and Quillen [9]. In the previous subsection we introduced the Connes differential B in the case of strictly unital A_∞-algebras. In the non-unital case the construction has to be modified. Let $X = A[1]_{form}$ be the corresponding non-commutative formal pointed dg-manifold. The algebra of functions $\mathcal{O}(X) \simeq \prod_{n\geq 0}(A[1]^{\otimes n})^*$ is a complex with the differential Lie_Q.

Proposition 7.4 If A is weakly unital then all non-zero cohomology of the complex $\mathcal{O}(X)$ are trivial and $H^0(\mathcal{O}(X)) \simeq k$.

Proof. Let us calculate the cohomology using the spectral sequence associated with the filtration $\prod_{n\geq n_0}(A[1]^{\otimes n})^*$. The term E_1 of the spectral sequence is isomorphic to the complex $\prod_{n\geq 0}((H^\bullet(A[1], m_1))^{\otimes n})^*$ with the differential induced by the multiplication m_2^A on $H^\bullet(A, m_1^A)$. By assumption $H^\bullet(A, m_1^A)$ is a unital algebra, hence all the cohomology groups vanish except of the zeroth one, which is isomorphic to k. This concludes the proof. ■.

It follows from the above Proposition that the complex $\mathcal{O}(X)/k$ is acyclic. We have the following two morphisms of complexes

$$d_{cycl} \colon (\mathcal{O}(X)/k \cdot 1, Lie_Q) \to (\Omega^1_{cycl}(X), Lie_Q)$$

and

$$\varphi \colon (\Omega^1_{cycl}(X), Lie_Q) \to (\mathcal{O}(X)/k \cdot 1, Lie_Q).$$

Here d_{cycl} and φ were introduced in the Sect. 7.2. We have: $deg(d_{cycl}) = +1$, $deg(\varphi) = -1, d_{cycl} \circ \varphi = 0, \varphi \circ d_{cycl} = 0.$.

Let us consider a *modified* Hochschild chain complex

$$C_\bullet^{mod}(A, A) := (\Omega^1_{cycl}(X)[2])^* \oplus (\mathcal{O}(X)/k \cdot 1)^*$$

with the differential
$b = \begin{pmatrix} (Lie_Q)^* & \varphi^* \\ 0 & (Lie_Q)^* \end{pmatrix}$

Let
$B = \begin{pmatrix} 0 & 0 \\ d^*_{cycl} & 0 \end{pmatrix}$ be an endomorphism of $C_\bullet^{mod}(A, A)$ of degree -1. Then $B^2 = 0$. Let u be a formal variable of degree $+2$. We define modified negative cyclic, periodic cyclic and cyclic chain complexes such as follows

$$CC_\bullet^{-,mod}(A) = (C_\bullet^{mod}(A, A)[[u]], b + uB),$$

$$CP_\bullet^{mod}(A) = (C_\bullet^{mod}(A, A)((u)), b + uB),$$

$$CC_\bullet^{mod}(A) = (CP_\bullet^{mod}(A)/CC_\bullet^{-,mod}(A))[-2].$$

For unital dg-algebras these complexes are quasi-isomorphic to the standard ones. If $char\, k = 0$ and A is weakly unital then $CC_\bullet^{-,mod}(A)$ is quasi-isomorphic to the complex $(\Omega^0_{cycl}(X), Lie_Q)^*$. Note that the $k[[u]]$-module structure on the cohomology $H^\bullet((\Omega^0_{cycl}(X), Lie_Q)^*)$ is not visible from the definition.

8 Homologically Smooth and Compact A_∞-Algebras

From now on we will assume that all A_∞-algebras are weakly unital unless we say otherwise.

8.1 Homological Smoothness

Let A be an A_∞-algebra over k and $E_1, E_2, ..., E_n$ be a sequence of A-modules. Let us consider a sequence $(E_{\le i})_{1\le i\le n}$ of A-modules together with exact triangles

$$E_i \to E_{\le i} \to E_{i+1} \to E_i[1],$$

such that $E_{\le 1} = E_1$.

We will call $E_{\le n}$ an *extension* of the sequence $E_1, ..., E_n$.

The reader also notices that the above definition can be given also for the category of $A-A$-bimodules.

Definition 8.1 (1) A perfect A-module is the one which is quasi-isomorphic to a direct summand of an extension of a sequence of modules each of which is quasi-isomorphic to $A[n], n \in \mathbf{Z}$.

(2) A perfect $A-A$-bimodule is the one which is quasi-isomorphic to a direct summand of an extension of a sequence consisting of bimodules each of which is quasi-isomorphic to $(A \otimes A)[n], n \in \mathbf{Z}$.

Perfect A-modules form a full subcategory $Perf_A$ of the dg-category $A-mod$. Perfect $A-A$-bimodules form a full subcategory $Perf_{A-mod-A}$ of the category of $A-A$-bimodules.[10]

Definition 8.2 We say that an A_∞-algebra A is homologically smooth if it is a perfect $A-A$-bimodule (equivalently, A is a perfect module over the A_∞-algebra A "$\otimes$" A^{op}).

Remark 8.3 An $A-B$-bimodule M gives rise to a dg-functor $B - mod \to A - mod$ such that $V \mapsto M \otimes_B V$. The diagonal bimodule A corresponds to the identity functor $Id_{A-mod}\colon A - mod \to A - mod$. The notion of homological smoothness can be generalized to the framework of A_∞-categories. The corresponding notion of saturated A_∞-category can be spelled out entirely in terms of the identity functor.

---

[10] Sometimes $Perf_A$ is called a thick triangulated subcategory of $A-mod$ generated by A. Then it is denoted by $\langle A\rangle$. In the case of $A-A$-bimodules we have a thik triangulated subcategory generated by $A \otimes A$, which is denoted by $\langle A \otimes A\rangle$.

Let us list few examples of homologically smooth A_∞-algebras.

Example 8.4 (a) Algebra of functions on a smooth affine scheme.

(b) $A = k[x_1, ..., x_n]_q$, which is the algebra of polynomials in variables $x_i, 1 \leq i \leq n$ subject to the relations $x_i x_j = q_{ij}\, x_j x_i$, where $q_{ij} \in k^*$ satisfy the properties $q_{ii} = 1, q_{ij} q_{ji} = 1$. More generally, all quadratic Koszul algebras, which are deformations of polynomial algebras are homologically smooth.

(c) Algebras of regular functions on quantum groups (see [37]).

(d) Free algebras $k\langle x_1, ..., x_n\rangle$.

(e) Finite-dimensional associative algebras of finite homological dimension.

(f) If X is a smooth scheme over k then the bounded derived category $D^b(Perf(X))$ of the category of perfect complexes (it is equivalent to $D^b(Coh(X))$) has a generator P (see [5]). Then the dg-algebra $A = End(P)$ (here we understand endomorphisms in the "derived sense", see [28]) is a homologically smooth algebra.

Let us introduce an $A - A$-bimodule $A^! = Hom_{A-mod-A}(A, A \otimes A)$ (cf. [18]). The structure of an $A - A$-bimodule is defined similarly to the case of associative algebras.

Proposition 8.5 If A is homologically smooth then $A^!$ is a perfect $A - A$-bimodule.

Proof. We observe that $Hom_{C-mod}(C, C)$ is a dg-algebra for any A_∞-algebra C. The Yoneda embedding $C \to Hom_{C-mod}(C, C)$ is a quasi-isomorphism of A_∞-algebras. Let us apply this observation to $C = A \otimes A^{op}$. Then using the A_∞-algebra $A``\otimes'' A^{op}$ (see Sect. 5.2) we obtain a quasi-isomorphism of $A - A$-bimodules $Hom_{A-mod-A}(A \otimes A, A \otimes A) \simeq A \otimes A$. By assumption A is quasi-isomorphic (as an A_∞-bimodule) to a direct summand in an extension of a sequence $(A \otimes A)[n_i]$ for $n_i \in \mathbf{Z}$. Hence $Hom_{A-mod-A}(A \otimes A, A \otimes A)$ is quasi-isomorphic to a direct summand in an extension of a sequence $(A \otimes A)[m_i]$ for $m_i \in \mathbf{Z}$. The result follows. ■

Definition 8.6 The bimodule $A^!$ is called the inverse dualizing bimodule.

The terminology is explained by an observation that if $A = End(P)$ where P is a generator of of $Perf(X)$ (see example 8f)) then the bimodule $A^!$ corresponds to the functor $F \mapsto F \otimes K_X^{-1}[-dim\, X]$, where K_X is the canonical class of X.[11]

Remark 8.7 In [50] the authors introduced a stronger notion of fibrant dg-algebra. Informally it corresponds to "non-commutative homologically smooth affine schemes of finite type." In the compact case (see the next section) both notions are equivalent.

---

[11] We thank Amnon Yekutieli for pointing out that the inverse dualizing module was first mentioned in the paper by M. van den Bergh "Existence theorems for dualizing complexes over non-commutative graded and filtered rings," *J. Algebra*, 195:2, 1997, 662–679.

8.2 Compact A_∞-Algebras

Definition 8.8 We say that an A_∞-algebra A is compact if the cohomology $H^\bullet(A, m_1)$ is finite-dimensional.

Example 8.9 (a) If $dim_k A < \infty$ then A is compact.

(b) Let X/k be a proper scheme of finite type. According to [5] there exists a compact dg-algebra A such that $Perf_A$ is equivalent to $D^b(Coh(X))$.

(c) If $Y \subset X$ is a proper subscheme (possibly singular) of a smooth scheme X then the bounded derived category $D^b_Y(Perf(X))$ of the category of perfect complexes on X, which are supported on Y has a generator P such that $A = End(P)$ is compact. In general it is not homologically smooth for $Y \neq X$. More generally, one can replace X by a formal smooth scheme containing Y, e.g., by the formal neighborhood of Y in the ambient smooth scheme. In particular, for $Y = \{pt\} \subset X = \mathbf{A}^1$ and the generator $\mathcal{O}_Y$ of $D^b(Perf(X))$ the corresponding graded algebra is isomorphic to $k\langle\xi\rangle/(\xi^2)$, where $deg\,\xi = 1$.

Proposition 8.10 If A is compact and homologically smooth then the Hochschild homology and cohomology of A are finite-dimensional.

Proof. (a) Let us start with Hochschild cohomology. We have an isomorphism of complexes $C^\bullet(A, A) \simeq Hom_{A-mod-A}(A, A)$. Since A is homologically smooth the latter complex is quasi-isomorphic to a direct summand of an extension of the bimodule $Hom_{A-mod-A}(A \otimes A, A \otimes A)$. The latter complex is quasi-isomorphic to $A \otimes A$ (see the proof of the Proposition 8.1.5). Since A is compact, the complex $A \otimes A$ has finite-dimensional cohomology. Therefore any perfect $A - A$-bimodule enjoys the same property. We conclude that the Hochschild cohomology groups are finite-dimensional vector spaces.

(b) Let us consider the case of Hochschild homology. With any $A - A$-bimodule E we associate a complex of vector spaces $E^\sharp = \oplus_{n \geq 0} A[1]^{\otimes n} \otimes E$ (cf. [18]). The differential on $E^\sharp$ is given by the same formulas as the Hochschild differential for $C_\bullet(A, A)$ with the only change: we place an element $e \in E$ instead of an element of A at the marked vertex (see Sect. 7). Taking $E = A$ with the structure of the diagonal $A - A$-bimodule we obtain $A^\sharp = C_\bullet(A, A)$. On the other hand, it is easy to see that the complex $(A \otimes A)^\sharp$ is quasi-isomorphic to (A, m_1), since $(A \otimes A)^\sharp$ is the quotient of the canonical free resolution (bar resolution) for A by a subcomplex A. The construction of $E^\sharp$ is functorial, hence $A^\sharp$ is quasi-isomorphic to a direct summand of an extension (in the category of complexes) of a shift of $(A \otimes A)^\sharp$, because A is smooth. Since $A^\sharp = C_\bullet(A, A)$ we see that the Hochschild homology $H_\bullet(A, A)$ is isomorphic to a direct summand of the cohomology of an extension of a sequence of k-modules $(A[n_i], m_1)$. Since the vector space $H^\bullet(A, m_1)$ is finite-dimensional the result follows. ■

Remark 8.11 For a homologically smooth compact A_∞-algebra A one has a quasi-isomorphism of complexes $C_\bullet(A, A) \simeq Hom_{A-mod-A}(A^!, A)$ Also, the complex $Hom_{A-mod-A}(M^!, N)$ is quasi-isomorphic to $(M \otimes_A N)^\sharp$ for two $A - A$-bimodules M, N, such that M is perfect. Here $M^! := Hom_{A-mod-A}(M, A \otimes A)$ Having this in mind one can offer a version of the above proof which uses the isomorphism

$$Hom_{A-mod-A}(A^!, A) \simeq Hom_{A-mod-A}(Hom_{A-mod-A}(A, A \otimes A), A).$$

Indeed, since A is homologically smooth the bimodule $Hom_{A-mod-A}(A, A \otimes A)$ is quasi-isomorphic to a direct summand P of an extension of a shift of $Hom_{A-mod-A}(A \otimes A, A \otimes A) \simeq A \otimes A$. Similarly, $Hom_{A-mod-A}(P, A)$ is quasi-isomorphic to a direct summand of an extension of a shift of $Hom_{A-mod-A}(A \otimes A, A \otimes A) \simeq A \otimes A$. Combining the above computations we see that the complex $C_\bullet(A, A)$ is quasi-isomorphic to a direct summand of an extension of a shift of the complex $A \otimes A$. The latter has finite-dimensional cohomology, since A enjoys this property.

Besides algebras of finite quivers there are two main sources of homologically smooth compact $\mathbf{Z}$-graded A_∞-algebras.

Example 8.12 (a) Combining Examples 8.1.4(f) and 8.2.2(b) we see that the derived category $D^b(Coh(X))$ is equivalent to the category $Perf_A$ for a homologically smooth compact A_∞-algebra A.

(b) According to [45] the derived category $D^b(F(X))$ of the Fukaya category of a K3 surface X is equivalent to $Perf_A$ for a homologically smooth compact A_∞-algebra A. The latter is generated by Lagrangian spheres, which are vanishing cycles at the critical points for a fibration of X over $\mathbf{CP}^1$. This result can be generalized to other Calabi–Yau manifolds.

In $\mathbf{Z}/2$-graded case examples of homologically smooth compact A_∞-algebras come from Landau–Ginzburg categories (see [42, 43]) and from Fukaya categories for Fano varieties.

Remark 8.13 Formal deformation theory of smooth compact A_∞-algebras gives a finite-dimensional formal pointed (commutative) dg-manifold. The global moduli stack can be constructed using methods of [50]). It can be thought of as a moduli stack of non-commutative smooth proper varieties.

9 Degeneration Hodge-to-de Rham

9.1 Main Conjecture

Let us assume that $char\ k = 0$ and A is a weakly unital A_∞-algebra, which can be $\mathbf{Z}$-graded or $\mathbf{Z}/2$-graded.

For any $n \geq 0$ we define the *truncated modified negative cyclic* complex $C_\bullet^{mod,(n)}(A, A) = (C_\bullet^{mod}(A, A) \otimes k[u]/(u^n), b + uB)$, where $deg\ u = +2$. Its cohomology will be denoted by $H^\bullet(C_\bullet^{mod,(n)}(A, A))$.

Definition 9.1 We say that an A_∞-algebra A satisfies the degeneration property if for any $n \geq 1$ one has: $H^\bullet(C_\bullet^{mod,(n)}(A,A))$ is a flat $k[u]/(u^n)$-module.

Conjecture 9.2 (Degeneration Hodge-to-de Rham). Let A be a weakly unital compact homologically smooth A_∞-algebra. Then it satisfies the degeneration property.

We will call the above statement the *degeneration conjecture.*

Corollary 9.3 If the A satisfies the degeneration property then the negative cyclic homology coincides with $\varprojlim_n H^\bullet(C_\bullet^{mod,(n)}(A,A))$ and it is a flat $k[[u]]$-module.

Remark 9.4 One can speak about degeneration property (modulo u^n) for A_∞-algebras which are flat over unital commutative k-algebras. For example, let R be an Artinian local k-algebra with the maximal ideal m and A be a flat R-algebra such that A/m is weakly unital, homologically smooth and compact. Then, assuming the degeneration property for A/m, one can easily see that it holds for A as well. In particular, the Hochschild homology of A gives rise to a vector bundle over $Spec(R) \times \mathbf{A}^1_{form}[-2]$.

Assuming the degeneration property for A we see that there is a $\mathbf{Z}$-graded vector bundle ξ_A over $\mathbf{A}^1_{form}[-2] = Spf(k[[u]])$ with the space of sections isomorphic to

$$\varprojlim_n H^\bullet(C_\bullet^{mod,(n)}(A,A)) = HC_\bullet^{-,mod}(A),$$

which is the negative cyclic homology of A. The fiber of ξ_A at $u = 0$ is isomorphic to the Hochschild homology $H_\bullet^{mod}(A,A) := H_\bullet(C_\bullet(A,A))$.

Note that $\mathbf{Z}$-graded $k((u))$-module $HP_\bullet^{mod}(A)$ of periodic cyclic homology can be described in terms of just one $\mathbf{Z}/2$-graded vector space $HP_{even}^{mod}(A) \oplus \Pi HP_{odd}^{mod}(A)$, where $HP_{even}^{mod}(A)$ (resp. $HP_{odd}^{mod}(A)$) consists of elements of degree zero (resp. degree $+1$) of $HP_\bullet^{mod}(A)$ and Π is the functor of changing the parity. We can interpret ξ_A in terms of ($\mathbf{Z}/2$-graded) supergeometry as a $\mathbf{G}_m$-equivariant supervector bundle over the even formal line $\mathbf{A}^1_{form}$. The structure of a $\mathbf{G}_m$-equivariant supervector bundle ξ_A is equivalent to a filtration F (called Hodge filtration) by even numbers on $HP_{even}^{mod}(A)$ and by odd numbers on $HP_{odd}^{mod}(A)$. The associated $\mathbf{Z}$-graded vector space coincides with $H_\bullet(A,A)$.

We can say few words in support of the degeneration conjecture. One is, of course, the classical Hodge-to-de Rham degeneration theorem (see Sect. 9.2 below). It is an interesting question to express the classical Hodge theory algebraically, in terms of a generator $\mathcal{E}$ of the derived category of coherent sheaves and the corresponding A_∞-algebra $A = RHom(\mathcal{E},\mathcal{E})$. The degeneration conjecture also trivially holds for algebras of finite quivers without relations.

In classical algebraic geometry there are basically two approaches to the proof of degeneration conjecture. One is analytic and uses Kähler metric, Hodge decomposition, etc. Another one is pure algebraic and uses the technique of reduction to finite characteristic (see [12]). Recently Kaledin (see [24]) suggested a proof of a version of the degeneration conjecture based on the reduction to finite characterstic.

Below we will formulate a conjecture which could lead to the definition of crystalline cohomology for A_∞-algebras. Notice that one can define homologically smooth and compact A_∞-algebras over any commutative ring, in particular, over the ring of integers $\mathbf{Z}$. We assume that A is a flat $\mathbf{Z}$-module.

Conjecture 9.5 Suppose that A is a weakly unital A_∞-algebra over $\mathbf{Z}$, such that it is homologically smooth (but not necessarily compact). Truncated negative cyclic complexes $(C_\bullet(A,A)\otimes\mathbf{Z}[[u,p]]/(u^n,p^m), b+uB)$ and $(C_\bullet(A,A)\otimes\mathbf{Z}[[u,p]]/(u^n,p^m), b-puB)$ are quasi-isomorphic for all $n,m\geq 1$ and all prime numbers p.

If, in addition, A is compact then the homology of either of the above complexes is a flat module over $\mathbf{Z}[[u,p]]/(u^n,p^m)$.

If the above conjecture is true then the degeneration conjecture, probably, can be deduced along the lines of [12]. One can also make some conjectures about Hochschild complex of an arbitrary A_∞-algebra, not assuming that it is compact or homologically smooth. More precisely, let A be a unital A_∞-algebra over the ring of p-adic numbers $\mathbf{Z}_p$. We assume that A is topologically free $\mathbf{Z}_p$-module. Let $A_0 = A\otimes_{\mathbf{Z}_p}\mathbf{Z}/p$ be the reduction modulo p. Then we have the Hochschild complex $(C_\bullet(A_0,A_0), b)$ and the $\mathbf{Z}/2$-graded complex $(C_\bullet(A_0,A_0), b+B)$.

Conjecture 9.6 For any i there is natural isomorphism of $\mathbf{Z}/2$-graded vector spaces over the field $\mathbf{Z}/p$:

$$H^\bullet(C_\bullet(A_0,A_0),b)\simeq H^\bullet(C_\bullet(A_0,A_0),b+B).$$

There are similar isomorphisms for weakly unital and non-unital A_∞-algebras, if one replaces $C_\bullet(A_0,A_0)$ by $C_\bullet^{mod}(A_0,A_0)$. Also one has similar isomorphisms for $\mathbf{Z}/2$-graded A_∞-algebras.

The last conjecture presumably gives an isomorphism used in [12], but does not imply the degeneration conjecture.

Remark 9.7 As we will explain elsewhere there are similar conjectures for saturated A_∞-categories (recall that they are generalizations of homologically smooth compact A_∞-algebras). This observation supports the idea of introducing the category $NCMot$ of non-commutative pure motives. Objects of the latter will be saturated A_∞-categories over a field and $Hom_{NCMot}(\mathcal{C}_1,\mathcal{C}_2) = K_0(Funct(\mathcal{C}_1,\mathcal{C}_2))\otimes\mathbf{Q}/equiv$ where K_0 means the K_0-group of the A_∞-category of functors and $equiv$ means numerical equivalence (i.e., the quivalence relation generated by the kernel of the Euler form $\langle E,F\rangle := \chi(RHom(E, F))$, where χ is the Euler characteristic). The above category is worth of consideration and will be discussed elsewhere (see [32]). In particular, one can formulate non-commutative analogs of Weil and Beilinson conjectures for the category $NCMot$.

9.2 Relationship with the Classical Hodge Theory

Let X be a quasi-projective scheme of finite type over a field k of characteristic zero. Then the category $Perf(X)$ of perfect sheaves on X is equivalent to $H^0(A-mod)$, where $A-mod$ is the category of A_∞-modules over a dg-algebra A. Let us recall a construction of A. Consider a complex E of vector bundles which generates the bounded derived category $D^b(Perf(X))$ (see [5]). Then A is quasi-isomorphic to $RHom(E,E)$. More explicitly, let us fix an affine covering $X = \cup_i U_i$. Then the complex $A := \oplus_{i_0,i_1,\dots,i_n} \Gamma(U_{i_0} \cap \dots \cap U_{i_n}, E^* \otimes E)[-n]$, $n = dim\, X$ computes $RHom(E,E)$ and carries a structure of dg-algebra. Different choices of A give rise to equivalent categories $H^0(A-mod)$ (derived Morita equivalence).

Properties of X are encoded in the properties of A. In particular:

(a) X is smooth iff A is homologically smooth;

(b) X is compact iff A is compact.

Moreover, if X is smooth then

$$H^\bullet(A,A) \simeq Ext^\bullet_{D^b(Coh(X\times X))}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq$$

$$\oplus_{i,j\geq 0} H^i(X, \wedge^j T_X)[-(i+j)]]$$

where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$.

Similarly

$$H_\bullet(A,A) \simeq \oplus_{i,j\geq 0} H^i(X, \wedge^j T^*_X)[j-i].$$

The RHS of the last formula is the Hodge cohomology of X. One can consider the hypercohomology $\mathbf{H}^\bullet(X, \Omega^\bullet_X[[u]]/u^n\Omega^\bullet_X[[u]])$ equipped with the differential ud_{dR}. Then the classical Hodge theory ensures degeneration of the corresponding spectral sequence, which means that the hypercohomology is a flat $k[u]/(u^n)$-module for any $n \geq 1$. Usual de Rham cohomology $H^\bullet_{dR}(X)$ is isomorphic to the generic fiber of the corresponding flat vector bundle over the formal line $\mathbf{A}^1_{form}[-2]$, while the fiber at $u=0$ is isomorphic to the Hodge cohomology $H^\bullet_{Hodge}(X) = \oplus_{i,j\geq 0} H^i(X, \wedge^j T^*_X)[j-i]$. In order to make a connection with the “abstract” theory of the previous subsection we remark that $H^\bullet_{dR}(X)$ is isomorphic to the periodic cyclic homology $HP_\bullet(A)$ while $H_\bullet(A,A)$ is isomorphic to $H^\bullet_{Hodge}(X)$.

10 A_∞-Algebras with Scalar Product

10.1 Main Definitions

Let (X, pt, Q) be a finite-dimensional formal pointed dg-manifold over a field k of characteristic zero.

Definition 10.1 A symplectic structure of degree $N \in \mathbf{Z}$ on X is given by a cyclic closed 2-form ω of degree N such that its restriction to the tangent space $T_{pt}X$ is non-degenerate.

One has the following non-commutative analog of the Darboux lemma.

Proposition 10.2 Symplectic form ω has constant coefficients in some affine coordinates at the point *pt*.

Proof. Let us choose an affine structure at the marked point and write down $\omega = \omega_0 + \omega_1 + \omega_2 + ...$, where $\omega_l = \sum_{i,j} c_{ij}(x) dx_i \otimes dx_j$ and $c_{ij}(x)$ is homogeneous of degree l (in particular, ω_0 has constant coefficients). Next we observe that the following lemma holds.

Lemma 10.3 *Let $\omega = \omega_0 + r$, where $r = \omega_l + \omega_{l+1} + ..., l \geq 1$. Then there is a change of affine coordinates $x_i \mapsto x_i + O(x^{l+1})$ which transforms ω into $\omega_0 + \omega_{l+1} + ...$*

Lemma implies the Proposition, since we can make an infinite product of the above changes of variables (it is a well-defined infinite series). The resulting automorphism of the formal neighborhood of x_0 transforms ω into ω_0.

Proof of the lemma. We have $d_{cycl}\omega_j = 0$ for all $j \geq l$ (see Sect. 7.2 for the notation). The change of variables is determined by a vector field $v = (v_1, ..., v_n)$ such that $v(x_0) = 0$. Namely, $x_i \mapsto x_i - v_i, 1 \leq i \leq n$. Moreover, we will be looking for a vector field such that $v_i = O(x^{l+1})$ for all i.

We have $Lie_v(\omega) = d(i_v\omega_0) + d(i_v r)$. Since $d\omega_l = 0$ we have $\omega_l = d\alpha_{l+1}$ for some form $\alpha_{l+1} = O(x^{l+1})$ in the obvious notation (formal Poincare lemma). Therefore in order to kill the term with ω_l we need to solve the equation $d\alpha_{l+1} = d(i_v\omega_0)$. It suffices to solve the equation $\alpha_{l+1} = i_v\omega_0$. Since ω_0 is non-degenerate, there exists a unique vector field $v = O(x^{l+1})$ solving last equation. This proves the lemma. ■

Definition 10.4 Let (X, pt, Q, ω) be a non-commutative formal pointed symplectic dg-manifold. A scalar product of degree N on the A_∞-algebra $A = T_{pt}X[-1]$ is given by a choice of affine coordinates at *pt* such that the ω becomes constant and gives rise to a non-degenerate bilinear form $A \otimes A \to k[-N]$.

Remark 10.5 Note that since $Lie_Q(\omega) = 0$ there exists a cyclic function $S \in \Omega^0_{cycl}(X)$ such that $i_Q\omega = dS$ and $\{S, S\} = 0$ (here the Poisson bracket corresponds to the symplectic form ω). It follows that the deformation theory of a non-unital A_∞-algebra A with the scalar product is controlled by the DGLA $\Omega^0_{cycl}(X)$ equipped with the differential $\{S, \bullet\}$.

We can restate the above definition in algebraic terms. Let A be a finite-dimensional A_∞-algebra, which carries a non-degenerate symmetric bilinear form $(,)$ of degree N. This means that for any two elements $a, b \in A$ such that $deg(a) + deg(b) = N$ we are given a number $(a, b) \in k$ such that:

(1) for any collection of elements $a_1, ..., a_{n+1} \in A$ the expression $(m_n(a_1, ..., a_n), a_{n+1})$ is cyclically symmetric in the graded sense (i.e., it satisfies the Koszul rule of signs with respect to the cyclic permutation of arguments);

(2) bilinear form $(\bullet, \bullet)$ is non-degenerate.

In this case we will say that A is an *A_∞-algebra with the scalar product* of degree N.

10.2 Calabi–Yau Structure

The above definition requires A to be finite-dimensional. We can relax this condition requesting that A is compact. As a result we will arrive to a homological version of the notion of scalar product. More precisely, assume that A is weakly unital compact A_∞-algebra. Let $CC_\bullet^{mod}(A) = (CC_\bullet^{mod}(A, A)[u^{-1}], b + uB)$ be the cyclic complex of A. Let us choose a cohomology class $[\varphi] \in H^\bullet(CC_\bullet^{mod}(A))^*$ of degree N. Since the complex (A, m_1) is a subcomplex of $C_\bullet^{mod}(A, A) \subset CC_\bullet^{mod}(A)$ we see that $[\varphi]$ defines a linear functional $Tr_{[\varphi]} : H^\bullet(A) \to k[-N]$.

Definition 10.6 We say that $[\varphi]$ is homologically non-degenerate if the bilinear form of degree N on $H^\bullet(A)$ given by $(a, b) \mapsto Tr_{[\varphi]}(ab)$ is non-degenerate.

Note that the above bilinear form defines a symmetric scalar product of degree N on $H^\bullet(A)$.

Theorem 10.7 *For a weakly unital compact A_∞-algebra A a homologically non-degenerate cohomology class $[\varphi]$ gives rise to a class of isomorphisms of non-degenerate scalar products on a minimal model of A.*

Proof. Since $char\, k = 0$ the complex $(CC_\bullet^{mod}(A))^*$ is quasi-isomorphic to $(\Omega^0_{cycl}(X)/k, Lie_Q)$.

Lemma 10.8 *Complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is quasi-isomorphic to the complex $(\Omega^0_{cycl}(X)/k, Lie_Q)$.*[12]

Proof. Notice that as a complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is isomorphic to the complex $\Omega^1_{cycl}(X)/d_{cycl}\, \Omega^0_{cycl}(X)$. The latter is quasi-isomorphic to $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ via $a\, db \mapsto [a, b]$ (recall that $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ denotes the topological closure of the commutator).

By definition $\Omega^0_{cycl}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top}$. We know that $\mathcal{O}(X)/k$ is acyclic, hence $\Omega^0_{cycl}(X)/k$ is quasi-isomorphic to $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$. Hence the complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ is quasi-isomorphic to $(\Omega^0_{cycl}(X)/k, Lie_Q)$. ■

---

[12] See also Proposition 5.5.1 from [19].

As a corollary we obtain an isomorphism of cohomology groups $H^\bullet(\Omega^{2,cl}_{cycl}(X)) \simeq H^\bullet(\Omega^0_{cycl}(X)/k)$. Having a non-degenerate cohomology class $[\varphi] \in H^\bullet(CC^{mod}_\bullet(A))^* \simeq H^\bullet(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ as above, we can choose its representative $\omega \in \Omega^{2,cl}_{cycl}(X)$, $Lie_Q\omega = 0$. Let us consider $\omega(x_0)$. It can be described pure algebraically such as follows. Notice that there is a natural projection $H^\bullet(\Omega^0_{cycl}(X)/k) \to (A/[A,A])^*$ which corresponds to the taking the first Taylor coefficient of the cyclic function. Then the above evaluation $\omega(x_0)$ is the image of $\varphi(x_0)$ under the natural map $(A/[A,A])^* \to (Sym^2(A))^*$ which assigns to a linear functional l the bilinear form $l(ab)$.

We claim that the total map $H^\bullet(\Omega^{2,cl}_{cycl}(X)) \to (Sym^2(A))^*$ is the same as the evaluation at x_0 of the closed cyclic 2-form. Equivalently, we claim that $\omega(x_0)(a,b) = Tr_\varphi(ab)$. Indeed, if $f \in \Omega^0_{cycl}(X)/k$ is the cyclic function corresponding to ω then we can write $f = \sum_i a_i x_i + O(x^2)$. Therefore $Lie_Q(f) = \sum_{l,i,j} a_i c_l^{ij}[x_i, x_j] + O(x^3)$, where c_l^{ij} are structure constants of $\mathcal{O}(X)$. Dualizing we obtain the claim.

Proposition 10.9 Let ω_1 and ω_2 be two symplectic structures on the finite-dimensional formal pointed minimal dg-manifold (X, pt, Q) such that $[\omega_1] = [\omega_2]$ in the cohomology of the complex $(\Omega^{2,cl}_{cycl}(X), Lie_Q)$ consisting of closed cyclic 2-forms. Then there exists a change of coordinates at x_0 preserving Q which transforms ω_1 into ω_2.

Corollary 10.10 Let (X, pt, Q) be a (possibly infinite-dimensional) formal pointed dg-manifold endowed with a (possibly degenerate) closed cyclic 2-form ω. Assume that the tangent cohomology $H^0(T_{pt}X)$ is finite-dimensional and ω induces a non-degenerate pairing on it. Then on the minimal model of (X, pt, Q) we have a canonical isomorphism class of symplectic forms modulo the action of the group $Aut(X, pt, Q)$.

Proof. Let M be a (finite-dimensional) minimal model of A. Choosing a cohomology class $[\varphi]$ as above we obtain a non-degenerate bilinear form on M, which is the restriction $\omega(x_0)$ of a representative $\omega \in \Omega^{2,cl}(X)$. By construction this scalar product depends on ω. We would like to show that in fact it depends on the cohomology class of ω, i.e., on φ only. This is the corollary of the following result.

Lemma 10.11 *Let $\omega_1 = \omega + Lie_Q(d\alpha)$. Then there exists a vector field v such that $v(x_0) = 0, [v, Q] = 0$ and $Lie_v(\omega) = Lie_Q(d\alpha)$.*

Proof. As in the proof of Darboux lemma we need to find a vector field v, satisfying the condition $di_v(\omega) = Lie_Q(d\alpha)$. Let $\beta = Lie_Q(\alpha)$. Then $d\beta = dLie_Q(\alpha) = 0$. Since ω is non-degenerate we can find v satisfying the conditions of the Proposition and such that $di_v(\omega) = Lie_Q(d\alpha)$. Using this v we can change affine coordinates transforming $\omega + Lie_Q(d\alpha)$ back to ω. This concludes the proof of the Proposition and the Theorem.■

Presumably the above construction is equivalent to the one given in [23]. We will sometimes call the cohomology class $[\varphi]$ a *Calabi–Yau structure* on A (or on the corresponding non-commutative formal pointed dg-manifold X). The following example illustrates the relation to geometry.

Example 10.12 Let X be a complex Calabi–Yau manifold of dimension n. Then it carries a nowhere vanishing holomorphic n-form vol. Let us fix a holomorphic vector bundle E and consider a dg-algebra $A = \Omega^{0,*}(X, End(E))$ of Dolbeault $(0,p)$-forms with values in $End(E)$. This dg-algebra carries a linear functional $a \mapsto \int_X Tr(a) \wedge vol$. One can check that this is a cyclic cocycle which defines a non-degenerate pairing on $H^\bullet(A)$ in the way described above.

There is another approach to Calabi–Yau structures in the case when A is homologically smooth. Namely, we say that A carries a Calabi–Yau structure of dimension N if $A^! \simeq A[N]$ (recall that $A^!$ is the $A-A$-bimodule $Hom_{A-mod-A}(A, A \otimes A)$ introduced in Sect. 8.1. Then we expect the following conjecture to be true.

Conjecture 10.13 If A is a homologically smooth compact finite-dimensional A_∞-algebra then the existence of a non-degenerate cohomology class $[\varphi]$ of degree $dim\, A$ is equivalent to the condition $A^! \simeq A[dim\, A]$.

If A is the dg-algebra of endomorphisms of a generator of $D^b(Coh(X))$ where X is Calabi–Yau then the above conjecture holds trivially.

Finally, we would like to illustrate the relationship of the non-commutative symplectic geometry discussed above with the commutative symplectic geometry of certain spaces of representations.[13] More generally we would like to associate with $X = Spc(T(A[1]))$ a collection of formal algebraic varieties, so that some "non-commutative" geometric structure on X becomes a collection of compatible "commutative" structures on formal manifolds $\mathcal{M}(X, n) := \widehat{Rep}_0(\mathcal{O}(X), Mat_n(k))$, where $Mat_n(k)$ is the associative algebra of $n \times n$ matrices over k, $\mathcal{O}(X)$ is the algebra of functions on X and $\widehat{Rep}_0(...)$ means the formal completion at the trivial representation. In other words, we would like to define a collection of compatible geometric structure on "$Mat_n(k)$-points" of the formal manifold X. In the case of symplectic structure this philosophy is illustrated by the following result.

Theorem 10.14 *Let X be a non-commutative formal symplectic manifold in $Vect_k$. Then it defines a collection of symplectic structures on all manifolds $\mathcal{M}(X, n), n \geq 1$.*

Proof. Let $\mathcal{O}(X) = A, \mathcal{O}(\mathcal{M}(X, n)) = B$. Then we can choose isomorphisms $A \simeq k\langle\langle x_1, ..., x_m\rangle\rangle$ and $B \simeq \langle\langle x_1^{\alpha,\beta}, ..., x_m^{\alpha,\beta}\rangle\rangle$, where $1 \leq \alpha, \beta \leq n$. To any $a \in A$ we can assign $\widehat{a} \in B \otimes Mat_n(k)$ such that:

---

[13] It goes back to [30] and since that time has been discussed in many papers, see e.g. [18].

$$\hat{x}_i = \sum_{\alpha,\beta} x_i^{\alpha,\beta} \otimes e_{\alpha,\beta},$$

where $e_{\alpha,\beta}$ is the $n \times n$ matrix with the only non-trivial element equal to 1 on the intersection of α-th line and β-th column. The above formulas define an algebra homomorphism. Composing it with the map $id_B \otimes Tr_{Mat_n(k)}$ we get a linear map $\mathcal{O}_{cycl}(X) \to \mathcal{O}(\mathcal{M}(X,n))$. Indeed the closure of the commutator $[A,A]$ is mapped to zero. Similarly, we have a morphism of complexes $\Omega^{\bullet}_{cycl}(X) \to \Omega^{\bullet}(\mathcal{M}(X,n))$, such that

$$dx_i \mapsto \sum_{\alpha,\beta} dx_i^{\alpha,\beta} e_{\alpha,\beta}.$$

Clearly, continuous derivations of A (i.e., vector fields on X) give rise to the vector fields on $\mathcal{M}(X,n)$.

Finally, one can see that a non-degenerate cyclic 2-form ω is mapped to the tensor product of a non-degenerate 2-form on $\mathcal{M}(X,n)$ and a nondegenerate 2-form $Tr(XY)$ on $Mat_n(k)$. Therefore a symplectic form on X gives rise to a symplectic form on $\mathcal{M}(X,n), n \geq 1$. ■

11 Hochschild Complexes as Algebras Over Operads and PROPs

Let A be a strictly unital A_∞-algebra over a field k of characteristic zero. In this section we are going to describe a colored dg-operad P such that the pair $(C^\bullet(A,A), C_\bullet(A,A))$ is an algebra over this operad. More precisely, we are going to describe $\mathbf{Z}$-graded k-vector spaces $A(n,m)$ and $B(n,m)$, $n,m \geq 0$ which are components of the colored operad such that $B(n,m) \neq 0$ for $m=1$ only and $A(n,m) \neq 0$ for $m=0$ only together with the colored operad structure and the action

(a) $A(n,0) \otimes (C^\bullet(A,A))^{\otimes n} \to C^\bullet(A,A)$,

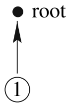
(b) $B(n,1) \otimes (C^\bullet(A,A))^{\otimes n} \otimes C_\bullet(A,A) \to C_\bullet(A,A)$.

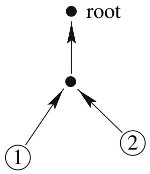
Then, assuming that A carries a non-degenerate scalar product, we are going to describe a PROP R associated with moduli spaces of Riemannian surfaces and a structure of R-algebra on $C_\bullet(A,A)$.

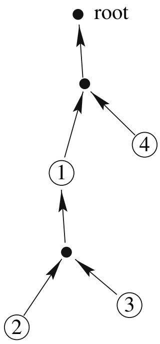
11.1 Configuration Spaces of Discs

We start with the spaces $A(n,0)$. They are chain complexes. The complex $A(n,0)$ coincides with the complex M_n of the minimal operad $M = (M_n)_{n\geq 0}$ described in [35], Sect. 5. Without going into details which can be found in loc. cit. we recall main facts about the operad M. A basis of M_n as a k-vector space is formed by n-labeled planar trees (such trees have internal vertices labeled by the set $\{1,...,n\}$ as well as other internal vertices which are non-labeled and each has the valency at least 3).

We can depict n-labeled trees such as follows





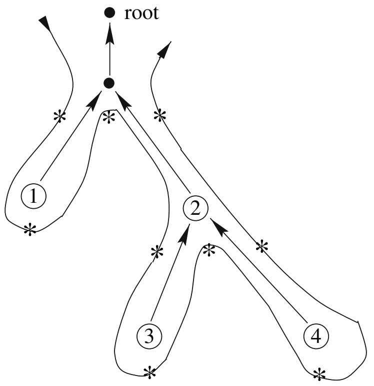


Labeled vertices are depicted as circles with numbers inscribed, non-labeled vertices are depicted as black vertices. In this way we obtain a graded operad M with the total degree of the basis element corresponding to a tree T equal to

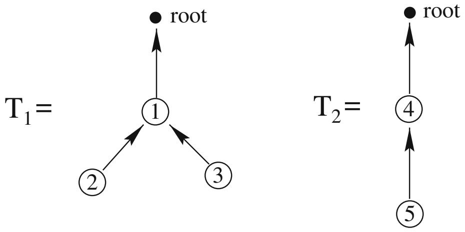
$$deg(T) = \sum_{v \in V_{lab}(T)} (1 - |v|) + \sum_{v \in V_{nonl}(T)} (3 - |v|)$$

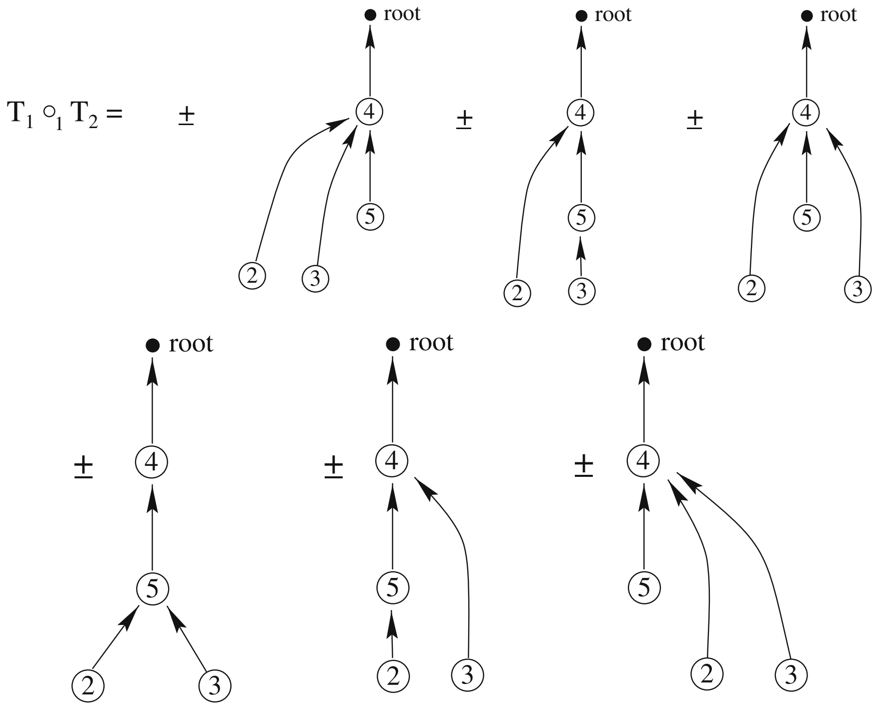
where $V_{lab}(T)$ and $V_{nonl}(T)$ denote the sets of labeled and non-labeled vertices respectively , and $|v|$ is the valency of the vertex v, i.e., the cardinality of the set of edges attached to v.

The notion of an *angle* between two edges incoming in a vertex is illustrated in the following figure (angles are marked by asteriscs).



Operadic composition and the differential are described in [35], sects. 5.2, 5.3. We borrow from there the following figure which illustrates the operadic composition of generators corresponding to labeled trees T_1 and T_2.





Informally speaking, the operadic gluing of T_2 to T_1 at an internal vertex v of T_1 is obtained by:

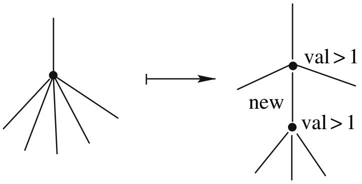
(a) Removing from T_1 the vertex v together with all incoming edges and vertices.

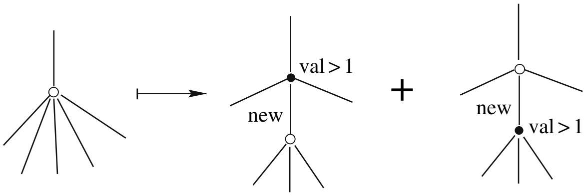
(b) Gluing T_2 to v (with the root vertex removed from T_2). Then

(c) Inserting removed vertices and edges of T_1 in all angles between incoming edges to the new vertex v_{new}.

(d) Taking the sum (with appropriate signs) over all possible inserting of edges in (c).

The differential d_M is a sum of the "local" differentials d_v, where v runs through the set of all internal vertices. Each d_v inserts a new edge into the set of edges attached to v. The following figure borrowed from [35] illustrates the difference between labeled (white) and non-labeled (black) vertices.





In this way we make M into a dg-operad. It was proved in [35], that M is quasi-isomorphic to the dg-operad $Chains(FM_2)$ of singular chains on the Fulton–Macpherson operad FM_2. The latter consists of the compactified moduli spaces of configurations of points in $\mathbf{R}^2$ (see e.g. [35], Sect. 7.2 for a description). It was also proved in [35] that $C^\bullet(A, A)$ is an algebra over the operad M (Deligne's conjecture follows from this fact). The operad FM_2 is homotopy equivalent to the famous operad $C_2 = (C_2(n))_{n\geq 0}$ of two-dimensional discs (little disc operad). Thus $C^\bullet(A, A)$ is an algebra (in the homotopy sense) over the operad $Chains(C_2)$.

11.2 Configurations of Points on the Cylinder

Let $\Sigma = S^1 \times [0, 1]$ denotes the standard cylinder.

Let us denote by $S(n)$ the set of isotopy classes of the following graphs $\Gamma \subset \Sigma$:

(a) every graph Γ is a forest (i.e., disjoint union of finitely many trees $\Gamma = \sqcup_i T_i$);

(b) the set of vertices $V(\Gamma)$ is decomposed into the union $V_{\partial\Sigma} \sqcup V_{lab} \sqcup V_{nonl} \sqcup V_1$ of four sets with the following properties:

(b1) the set $V_{\partial\Sigma}$ is the union $\{in\} \cup \{out\} \cup V_{out}$ of three sets of points which belong to the boundary $\partial\Sigma$ of the cylinder. The set $\{in\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{1\}$ while the set $\{out\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{0\}$. The set V_{out} consists of a finitely many unlableled points on the boundary circle $S^1 \times \{0\}$;

(b2) the set V_{lab} consists of n labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

(b3) the set V_{nonl} consists of a finitely many non-labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

(b4) the set V_1 is either empty or consists of only one element denoted by $\mathbf{1} \in S^1 \times (0,1)$ and called *special* vertex;

(c) the following conditions on the valencies of vertices are imposed:

(c1) the valency of the vertex *out* is ≤ 1;

(c2) the valency of each vertex from the set $V_{\partial\Sigma} \setminus V_{out}$ is equal to 1;

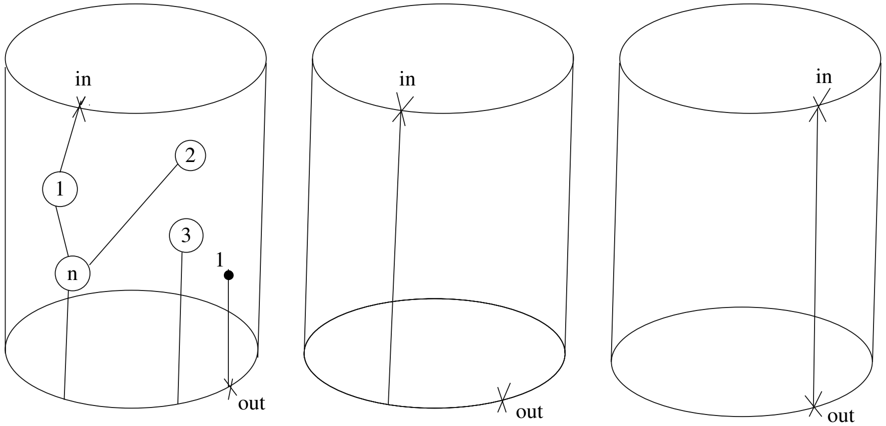
(c3) the valency of each vertex from V_{lab} is at least 1;

(c4) the valency of each vertex from V_{nonl} is at least 3;

(c5) if the set V_1 is non-empty then the valency of the special vertex is equal to 1. In this case the only outcoming edge connects $\mathbf{1}$ with the vertex *out*.

(d) Every tree T_i from the forest Γ has its root vertex in the set $V_{\partial\Sigma}$.

(e) We orient each tree T_i down to its root vertex.



Remark 11.1 Let us consider the configuration space $X_n, n \geq 0$ which consists of (modulo $\mathbf{C}^*$-dilation) equivalence classes of n points on $\mathbf{CP}^1 \backslash \{0, \infty\}$ together with two direction lines at the tangent spaces at the points 0 and ∞. One-point compactification $\widehat{X}_n$ admits a cell decomposition with cells (except of the point $\widehat{X}_n \backslash X_n$) parametrized by elements of the set $S(n)$. This can be proved with the help of Strebel differentials (cf. [35], Sect. 5.5).

Previous remark is related to the following description of the sets $S(n)$ (it will be used later in the chapter). Let us contract both circles of the boundary $\partial\Sigma$ into points. In this way we obtain a tree on the sphere. Points become vertices of the tree and lines outcoming from the points become edges. There are two vertices marked by *in* and *out* (placed at the north and south poles respectively). We orient the tree towards to the vertex *out*. An additional structure consists of:

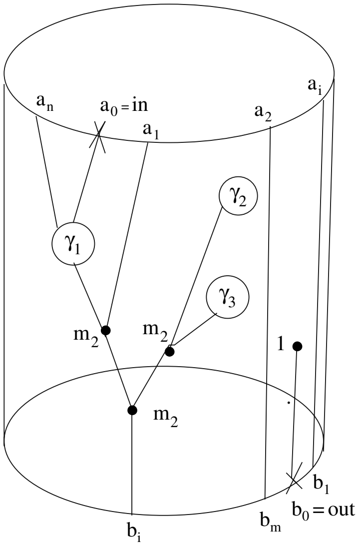
(a) Marked edge outcoming from *in* (it corresponds to the edge outcoming from *in*).

(b) Either a marked edge incoming to *out* (there was an edge incoming to *out* which connected it with a vertex not marked by $\mathbf{1}$) or an angle between two edges incoming to *out* (all edges which have one of the endpoint vertices on the bottom circle become after contracting it to a point the edges incoming to *out*, and if there was an edge connecting a point marked by **1** with *out*, we mark the angle between edges containing this line).

The reader notices that the star of the vertex *out* can be identified with a regular k-gon, where k is the number of incoming to *out* edges. For this k-gon we have either a marked point on an edge (case (a) above) or a marked angle with the vertex in *out* (case (b) above).

11.3 Generalization of Deligne's Conjecture

The definition of the operadic space $B(n,1)$ will be clear from the description of its action on the Hochschild chain complex. The space $B(n,1)$ will have a basis parametrized by elements of the set $S(n)$ described in the previous subsection. Let us describe the action of a generator of $B(n,1)$ on a pair $(\gamma_1 \otimes ... \otimes \gamma_n, \beta)$, where $\gamma_1 \otimes ... \otimes \gamma_n \in C^\bullet(A,A)^{\otimes n}$ and $\beta = a_0 \otimes a_1 \otimes ... \otimes a_l \in C_l(A,A)$. We attach elements $a_0, a_1, ..., a_l$ to points on Σ_h^{in}, in a cyclic order, such that a_0 is attached to the point *in*. We attach γ_i to the ith numbered point on the surface of Σ_h. Then we draw disjoint continuous segments (in all possible ways, considering pictures up to an isotopy) starting from each point marked by some element a_i and oriented downstairs, with the requirements (a–c) as above, with the only modification that we allow an arbitrary number of points on $S^1 \times \{1\}$. We attach higher multiplications m_j to all non-numbered vertices, so that j is equal to the incoming valency of the vertex. Reading from the top to the bottom and composing γ_i and m_j we obtain (on the bottom circle) an element $b_0 \otimes ... \otimes b_m \in C_\bullet(A,A)$ with b_0 attached to the vertex *out*. If the special vertex **1** is present then we set $b_0 = 1$. This gives the desired action.



Composition of the operations in $B(n,1)$ corresponds to the gluing of the cylinders such that the point *out* of the top cylinder is identified with the point *in* of the bottom cylinder. If after the gluing there is a line from the point marked **1** on the top cylinder which does not end at the point *out* of the bottom cylinder, we will declare such a composition to be equal to zero.

Let us now consider a topological colored operad $C_2^{col} = (C_2^{col}(n,m))_{n,m\geq 0}$ with two colors such that $C_2^{col}(n,m) \neq \emptyset$ only if $m = 0, 1$ and

(a) In the case $m = 0$ it is the little disc operad.

(b) In the case $m = 1$ $C_2^{col}(n,1)$ is the moduli space (modulo rotations) of the configurations of $n \geq 1$ discs on the cyliner $S^1 \times [0,h]$ $h \geq 0$ and two marked points on the boundary of the cylinder. We also add the degenerate circle of configurations $n = 0, h = 0$. The topological space $C_2^{col}(n,1)$ is homotopically equivalent to the configuration space X_n described in the previous subsection.

Let $Chains(C_2^{col})$ be the colored operad of singular chains on C_2^{col}. Then, similarly to [35], Sect. 7, one proves (using the explicit action of the colored operad $P = (A(n,m), B(n,m))_{n,m\geq 0}$ described above) the following result.

Theorem 11.2 *Let A be a unital A_∞-algebra. Then the pair $(C^\bullet(A,A), C_\bullet(A,A))$ is an algebra over the colored operad $Chains(C_2^{col})$ (which is quasi-isomorphic to P) such that for $h = 0, n = 0$ and coinciding points in $=$ out, the corresponding operation is the identity.*

Remark 11.3 The above Theorem generalizes Deligne's conjecture (see e.g. [35]). It is related to the abstract calculus associated with A (see [T, 48]). The reader also notices that for $h = 0, n = 0$ we have the moduli space of two points on the circle. It is homeomorphic to S^1. Thus we have an action of S^1 on $C_\bullet(A,A)$. This action gives rise to the Connes differential B.

Similarly to the case of little disc operad, one can prove the following result.

Proposition 11.4 The colored operad C_2^{col} is formal, i.e., it is quasi-isomorphic to its homology colored operad.

If A is non-unital we can consider the direct sum $A_1 = A \oplus k$ and make it into a unital A_∞-algebra. The *reduced* Hochschild chain complex of A_1 is defined as $C_\bullet^{red}(A_1, A_1) = \oplus_{n\geq 0} A_1 \otimes ((A_1/k)[1])^{\otimes n}$ with the same differential as in the unital case. One defines the reduced Hochschild cochain complex $C_{red}^\bullet(A_1, A_1)$ similarly. We define the *modified* Hochschild chain complex $C_\bullet^{mod}(A,A)$ from the following isomorphism of complexes $C_\bullet^{red}(A_1, A_1) \simeq C_\bullet^{mod}(A,A) \oplus k$. Similarly, we define the modified Hochschild cochain complex from the decomposition $C_{red}^\bullet(A_1, A_1) \simeq C_{mod}^\bullet(A,A) \oplus k$. Then, similarly to the Theorem 11.3.1 one proves the following result.

Proposition 11.5 The pair $(C_\bullet^{mod}(A,A), C_{mod}^\bullet(A,A))$ is an algebra over the colored operad which is an extension of $Chains(C_2^{col})$ by null-ary operations on Hochschild chain and cochain complexes, which correspond to the unit in A, and such that for $h = 0, n = 0$ and coinciding points $in = out$, the corresponding operation is the identity.

11.4 Remark About Gauss–Manin Connection

Let $R = k[[t_1, ..., t_n]]$ be the algebra of formal series and A be an R-flat A_∞-algebra. Then the (modified) negative cyclic complex $CC_\bullet^{-,mod}(A) = (C_\bullet(A, A)[[u]], b + uB)$ is an $R[[u]]$-module. It follows from the existense of Gauss-Manin connection (see [16]) that the cohomology $HC_\bullet^{-,mod}(A)$ is in fact a module over the ring

$$D_R(A) := k[[t_1, ..., t_n, u]][u\partial/\partial t_1, ..., u\partial/\partial t_n].$$

Inedeed, if ∇ is the Gauss–Manin connection from [16] then $u\partial/\partial t_i$ acts on the cohomology as $u\nabla_{\partial/\partial t_i}, 1 \leq i \leq n$.

The above considerations can be explained from the point of view of conjecture below. Let $g = C^\bullet(A, A)[1]$ be the DGLA associated with the Hochschild cochain complex and $M := (CC_\bullet^{-,mod}(A)$. We define a DGLA $\hat{g}$ which is the crossproduct $(g \otimes k\langle\xi\rangle) \rtimes k(\partial/\partial\xi)$, where $deg\, \xi = +1$.

Conjecture 11.6 There is a structure of an L_∞-module on M over $\hat{g}$ which extends the natural structure of a g-module and such that $\partial/\partial\xi$ acts as Connes differential B. Moreover this structure should follow from the P-algebra structure described in Sect. 11.3.

It looks plausible that the formulas for the Gauss–Manin connection from [16] can be derived from our generalization of Deilgne's conjecture. We will discuss flat connections on periodic cyclic homology later in the text.

11.5 Flat Connections and the Colored Operad

We start with **Z**-graded case. Let us interpret the **Z**-graded formal scheme $Spf(k[[u]])$ as even formal line equipped with the $\mathbf{G}_m$-action $u \mapsto \lambda^2 u$. The space $HC_\bullet^{-,mod}(A)$ can be interpreted as a space of sections of a $\mathbf{G}_m$-equivariant vector bundle ξ_A over $Spf(k[[u]])$ corresponding to the $k[[u]]$-flat module $\varprojlim_n H^\bullet(C_\bullet^{(n)}(A, A))$. The action of $\mathbf{G}_m$ identifies fibers of this vector bundle over $u \neq 0$. Thus we have a natural flat connection ∇ on the restriction of ξ_A to the complement of the point 0 which has the pole of order one at $u = 0$.

Here we are going to introduce a different construction of the connection ∇ which works also in **Z**/2-graded case. This connection will have in general a pole of degree two at $u = 0$. In particular we have the following result.

Proposition 11.7 The space of section of the vector bundle $\xi_{\mathcal{A}}$ can be endowed with a structure of a $k[[u]][[u^2\partial/\partial u]]$-module.

In fact we are going to give an explicit construction of the connection, which is based on the action of the colored dg-operad P discussed in Sect. 11.3 (more precisely, an extension P^{new} of P, see below). Before presenting an explicit formula, we will make few comments.

1. For any $\mathbf{Z}/2$-graded A_∞-algebra A one can define canonically a 1-parameter family of A_∞-algebras $A_\lambda, \lambda \in \mathbf{G}_m$, such that $A_\lambda = A$ as a $\mathbf{Z}/2$-graded vector space and $m_n^{A_\lambda} = \lambda m_n^A$.

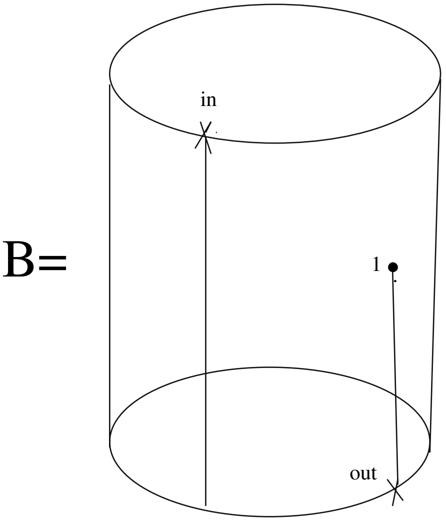
2. For simplicity we will assume that A is strictly unital. Otherwise we will work with the pair $(C_\bullet^{mod}(A,A), C^\bullet_{mod}(A,A))$ of modified Hochschild complexes.

3. We can consider an extension P^{new} of the dg-operad P allowing any non-zero valency for a non-labeled (black) vertex(in the definition of P we required that such a valency was at least three). All the formulas remain the same. But the dg-operad P^{new} is no longer formal. It contains a dg-suboperad generated by trees with all vertices being non-labeled. Action of this suboperad P^{new}_{nonl} is responsible for the flat connection discussed below.

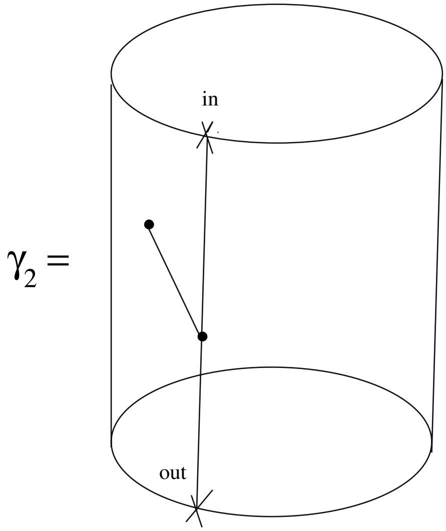
4. In addition to the connection along the variable u one has the Gauss–Manin connection which acts along the fibers of ξ_A (see Sect. 11.4). Probably one can write down an explicit formula for this connection using the action of the colored operad P^{new}. In what follows are going to describe a connection which presumably coincides with the Gauss-Manin connection.

Let us now consider a dg-algebra $k[B, \gamma_0, \gamma_2]$ which is generated by the following operations of the colored dg-operad P^{new}:

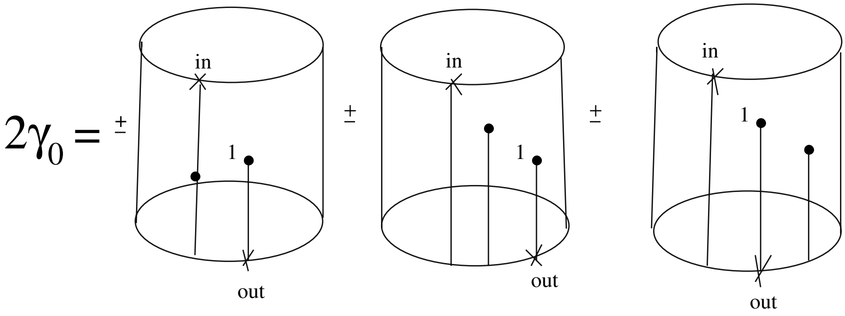
(a) Connes differential B of degree -1. It can be depicted such as follows (cf. Sect. 7.3):



(b) Generator γ_2 of degree 2, corresponding to the following figure:



(c) Generator γ_0 of degree 0, where $2\gamma_0$ is depicted below:



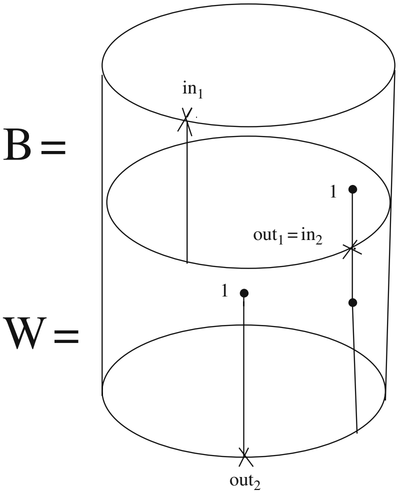
Proposition 11.8 The following identities hold in P^{new}:

$$B^2 = dB = d\gamma_2 = 0, d\gamma_0 = [B, \gamma_2],$$

$$B\gamma_0 + \gamma_0 B := [B, \gamma_0]_+ = -B.$$

Here by d we denote the Hochschild chain differential (previously it was denoted by b).

Proof. Let us prove that $[B, \gamma_0] = -B$, leaving the rest as an exercise to the reader. One has the following identities for the compositions of operations in P^{new}: $B\gamma_0 = 0$, $\gamma_0 B = B$. Let us check, for example, the last identity. Let us denote by W the first summand on the figure defining $2\gamma_0$. Then $\gamma_0 B = \frac{1}{2} WB$. The latter can be depicted in the following way:



It is easily seen equals to $2 \cdot 1/2B = B$. ■

Corollary 11.9 Hochschild chain complex $C_\bullet(A, A)$ is a dg-module over the dg-algebra $k[B, \gamma_0, \gamma_2]$.

Let us consider the truncated negative cyclic complex $(C_\bullet(A, A)[[u]]/(u^n), d_u = d + uB)$. We introduce a k-linear map ∇ of $C_\bullet(A, A)[[u]]/(u^n)$ into itself such that $\nabla_{u^2\partial/\partial u} = u^2\partial/\partial u - \gamma_2 + u\gamma_0$. Then we have:

(a) $[\nabla_{u^2\partial/\partial u}, d_u] = 0$;

(b) $[\nabla_{u^2\partial/\partial u}, u] = u^2$.

Let us denote by V the unital dg-algebra generated by $\nabla_{u^2\partial/\partial u}$ and u, subject to the relations (a), (b) and the relation $u^n = 0$. From (a) and (b) one deduces the following result.

Proposition 11.10 The complex $(C_\bullet(A, A)[[u]]/(u^n), d_u = d + uB)$ is a V-module. Moreover, assuming the degeneration conjecture, we see that the operator $\nabla_{u^2\partial/\partial u}$ defines a flat connection on the cohomology bundle

$$H^\bullet(C_\bullet(A, A)[[u]]/(u^n), d_u)$$

which has the only singularity at $u = 0$ which is a pole of second order.

Taking the inverse limit over n we see that $H^\bullet(C_\bullet(A, A)[[u]], d_u)$ gives rise to a vector bundle over $\mathbf{A}^1_{form}[-2]$ which carries a flat connection with the second order pole at $u = 0$. It is interesting to note the difference between $\mathbf{Z}$-graded and $\mathbf{Z}/2$-graded A_∞-algebras. It follows from the explicit formula for the connection ∇ that the coefficient of the second degree pole is represented by multiplication by a cocyle $(m_n)_{n\geq 1} \in C^\bullet(A, A)$. In cohomology it is trivial in $\mathbf{Z}$-graded case (because of the invariance with respect to the group action $m_n \mapsto \lambda\, m_n$), but non-trivial in $\mathbf{Z}/2$-graded case. Therefore the order of the pole of ∇ is equal to one for $\mathbf{Z}$-graded A_∞-algebras and is equal to two for

$\mathbf{Z}/2$-graded A_∞-algebras. We see that in $\mathbf{Z}$-graded case the connection along the variable u comes from the action of the group $\mathbf{G}_m$ on higher products m_n, while in $\mathbf{Z}/2$-graded case it is more complicated.

11.6 PROP of Marked Riemann Surfaces

In this section we will describe a PROP naturally acting on the Hochschild complexes of a finite-dimensional A_∞-algebra with the scalar product of degree N.

Since we have a quasi-isomorphism of complexes

$$C^\bullet(A,A) \simeq (C_\bullet(A,A))^*[-N]$$

it suffices to consider the chain complex only.

In this subsection we will assume that A is either $\mathbf{Z}$-graded (then N is an integer) or $\mathbf{Z}/2$-graded (then $N \in \mathbf{Z}/2$). We will present the results for non-unital A_∞-algebras. In this case we will consider the modified Hochschild chain complex

$$C_\bullet^{mod}(A,A) = \oplus_{n\geq 0} A \otimes (A[1])^{\otimes n} \bigoplus \oplus_{n\geq 1}(A[1])^{\otimes n},$$

equipped with the Hochschild chain differential (see Sect. 7.4).

Our construction is summarized in (i–ii) below.

(i) Let us consider the topological PROP $\mathcal{M} = (\mathcal{M}(n,m))_{n,m\geq 0}$ consisting of moduli spaces of metrics on compacts oriented surfaces with bondary consisting of $n+m$ circles and some additional marking (see precise definition below).

(ii) Let $Chains(\mathcal{M})$ be the corresponding PROP of singular chains. Then there is a structure of a $Chains(\mathcal{M})$-algebra on $C_\bullet^{mod}(A,A)$, which is encoded in a collection of morphisms of complexes

$$Chains(\mathcal{M}(n,m)) \otimes C_\bullet^{mod}(A,A)^{\otimes n} \to (C_\bullet^{mod}(A,A))^{\otimes m}.$$

In addition one has the following:

(iii) If A is homologically smooth and satisfies the degeneration property then the structure of $Chains(\mathcal{M})$-algebra extends to a structure of a $Chains(\overline{\mathcal{M}})$-algebra, where $\overline{\mathcal{M}}$ is the topological PROP of stable compactifications of $\mathcal{M}(n,m)$.

Definition 11.11 An element of $\mathcal{M}(n,m)$ is an isomorphism class of triples $(\Sigma, h, mark)$ where Σ is a compact oriented surface (not necessarily connected) with metric h and $mark$ is an orientation preserving isometry between a neighborhood of $\partial\Sigma$ and the disjoint union of $n+m$ flat semiannuli $\sqcup_{1\leq i\leq n}(S^1\times[0,\varepsilon))\sqcup\sqcup_{1\leq i\leq m}(S^1\times[-\varepsilon,0])$, where ε is a sufficiently small positive number. We will call n circle "inputs" and the rest m circles "outputs". We will assume that each connected component of Σ has at least one input and there are no discs among the connected components. Also we will add $\Sigma = S^1$ to $\mathcal{M}(1,1)$ as the identity morphism. It can be thought of as the limit of cylinders $S^1 \times [0,\varepsilon]$ as $\varepsilon \to 0$.

The composition is given by the natural gluing of surfaces.

Let us describe a construction of the action of $Chains(\mathcal{M})$ on the Hochschild chain complex. In fact, instead of $Chains(\mathcal{M})$ we will consider a quasi-isomorphic dg-PROP $R = (R(n,m)_{n,m\geq 0})$ generated by ribbon graphs with additional data. In what follows we will skip some technical details in the definition of the PROP R. They can be recovered in a more or less straightforward way.

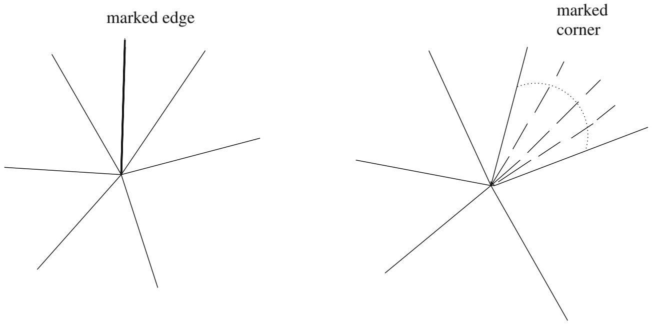
It is well-known (and can be proved with the help of Strebel differentials) that $\mathcal{M}(n,m)$ admits a stratification with strata parametrized by graphs described below. More precisely, we consider the following class of graphs.

(1) Each graph Γ is a (not necessarily connected) ribbon graph (i.e., we are given a cyclic order on the set $Star(v)$ of edges attached to a vertex v of Γ). It is well-known that replacing an edge of a ribbon graph by a thin stripe (thus getting a "fat graph") and gluing stripes in the cyclic order one gets a Riemann surface with the boundary.

(2) The set $V(\Gamma)$ of vertices of Γ is the union of three sets: $V(\Gamma) = V_{in}(\Gamma) \cup V_{middle}(\Gamma) \cup V_{out}(\Gamma)$. Here $V_{in}(\Gamma)$ consists of n numbered vertices $in_1, ..., in_n$ of the valency 1 (the outcoming edges are called tails), $V_{middle}(\Gamma)$ consists of vertices of the valency ≥ 3, and $V_{out}(\Gamma)$ consists of m numbered vertices $out_1, ..., out_m$ of valency ≥ 1.

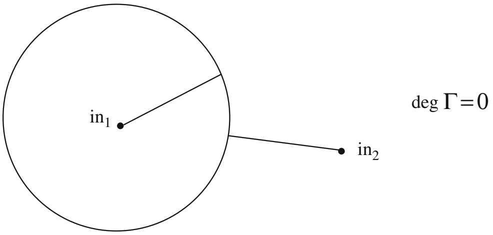
(3) We assume that the Riemann surface corresponding to Γ has n connected boundary components each of which has exactly one input vertex.

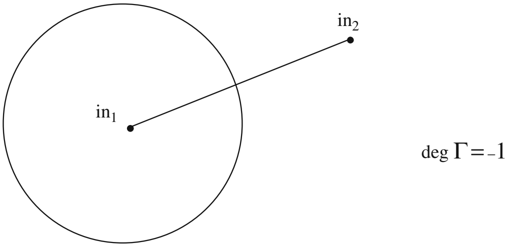
(4) For every vertex $out_j \in V_{out}(\Gamma), 1 \leq j \leq m$ we *mark* either an incoming edge or a pair of adjacent (we call such a pair of edges a *corner*).



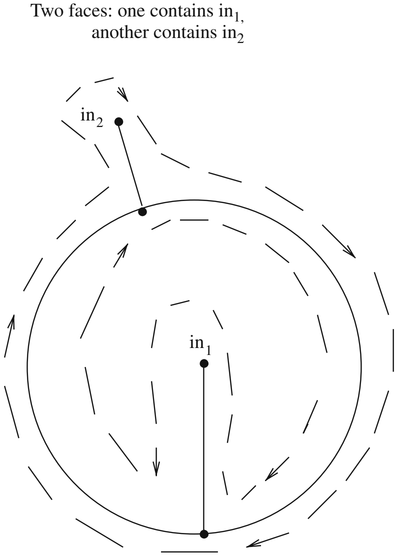
More pedantically, let $E(\Gamma)$ denotes the set of edges of Γ and $E^{or}(\Gamma)$ denotes the set of pairs (e, or) where $e \in E(\Gamma)$ and or is one of two possible orientations of e. There is an obvious map $E^{or}(\Gamma) \to V(\Gamma) \times V(\Gamma)$ which assigns to an oriented edge the pair of its endpoint vertices: source and target. The free involution σ acting on $E^{or}(\Gamma)$ (change of orientation) corresponds to the permutation map on $V(\Gamma) \times V(\Gamma)$. Cyclic order on each $Star(v)$ means that there is a bijection $\rho : E^{or}(\Gamma) \to E^{or}(\Gamma)$ such that orbits of iterations $\rho^n, n \geq 1$ are elements of $Star(v)$ for some $v \in V(\Gamma)$. In particular, the corner is given either by a pair of coinciding edges (e, e) such that $\rho(e) = e$ or by a pair edges $e, e' \in Star(v)$ such that $\rho(e) = e'$. Let us define a *face* as an orbit of $\rho \circ \sigma$. Then faces are oriented closed paths. It follows from the condition (2) that each face contains exactly one edge outcoming from some in_i.

We depict below two graphs in the case $g = 0, n = 2, m = 0$.



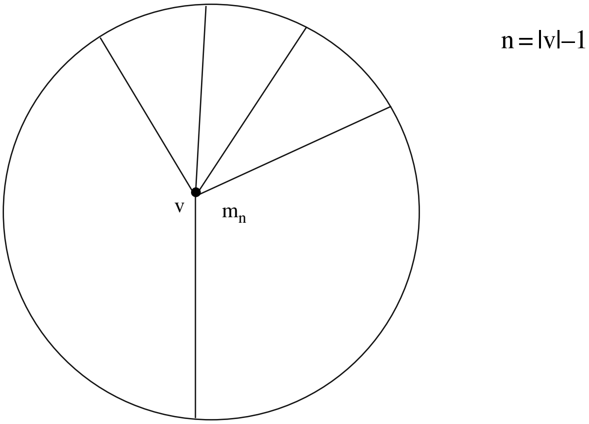


Here is a figure illustrating the notion of face



Remark 11.12 The above data (i.e., a ribbon graph with numerations of *in* and *out* vertices) have no automorphisms. Thus we can identify Γ with its isomorphism class.

The functional $(m_n(a_1, ..., a_n), a_{n+1})$ is depicted such as follows.



We define the *degree* of Γ by the formula

$$deg\,\Gamma = \sum_{v \in V_{middle}(\Gamma)} (3 - |v|) + \sum_{v \in V_{out}(\Gamma)} (1 - |v|) + \sum_{v \in V_{out}(\Gamma)} \epsilon_v - N\chi(\Gamma),$$

where $\epsilon_v = -1$, if v contains a marked corner and $\epsilon_v = 0$ otherwise. Here $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$ denotes the Euler characteristic of Γ.

Definition 11.13 We define $R(n, m)$ as a graded vector space which is a direct sum $\oplus_\Gamma \psi_\Gamma$ of 1-dimensional graded vector spaces generated by graphs Γ as above, each summand has degree $deg\,\Gamma$.

One can see that ψ_Γ is naturally identified with the tensor product of one-dimensional vector spaces (determinants) corresponding to vertices of Γ.

Now, having a graph Γ which satisfies conditions (1–3) above and Hochschild chains $\gamma_1, ..., \gamma_n \in C_\bullet^{mod}(A, A)$ we would like to define an element of $C_\bullet^{mod}(A, A)^{\otimes m}$. Roughly speaking we are going to assign the above n elements of the Hochschild complex to n faces corresponding to vertices $in_i, 1 \le i \le n$, then assign tensors corresponding to higher products m_l to internal vertices $v \in V_{middle}(\Gamma)$, then using the convolution operation on tensors given by the scalar product on A to read off the resulting tensor from $out_j, 1 \le j \le m$. More precise algorithm is described below.

(a) We decompose the modified Hochschild complex such as follows:

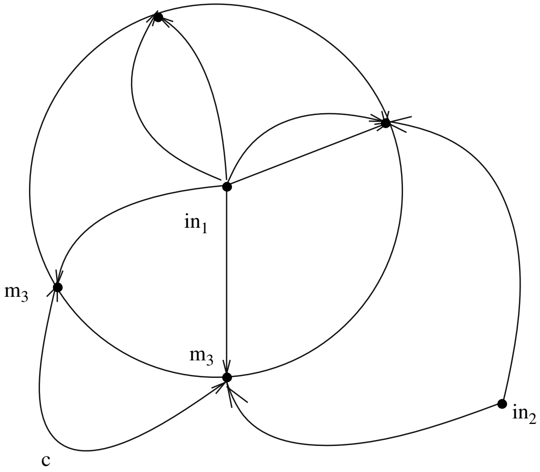
$$C_\bullet^{mod}(A, A) = \oplus_{l \ge 0, \varepsilon \in \{0,1\}} C_{l,\varepsilon}^{mod}(A, A),$$

where $C_{l,\varepsilon=0}^{mod}(A, A) = A \otimes (A[1])^{\otimes l}$ and $C_{l,\varepsilon=1}^{mod}(A, A) = k \otimes (A[1])^{\otimes l}$ according to the definition of modified Hochschild chain complex. For any choice of $l_i \ge 0, \varepsilon_i \in \{0, 1\}, 1 \le i \le n$ we are going to construct a linear map of degree zero

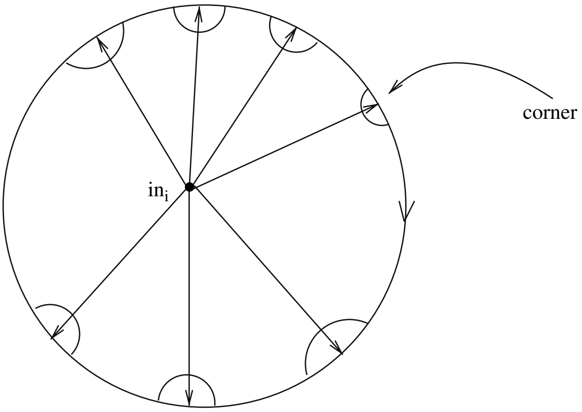
$$f_\Gamma : \psi_\Gamma \otimes C^{mod}_{l_1,\varepsilon_1}(A,A) \otimes ... \otimes C^{mod}_{l_n,\varepsilon_1}(A,A) \to (C^{mod}_\bullet(A,A))^{\otimes m}.$$

The result will be a sum $f_\Gamma = \sum_{\Gamma'} f_{\Gamma'}$ of certain maps. The description of the collection of graphs Γ' is given below.

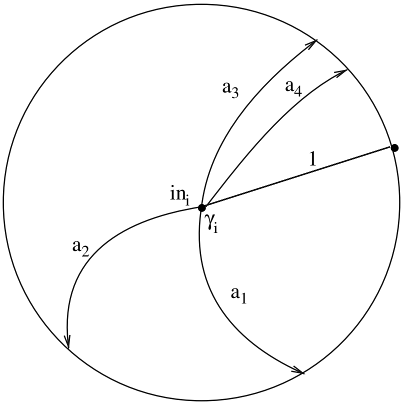
(b) Each new graph Γ' is obtained from Γ by adding new edges. More precisely one has $V(\Gamma') = V(\Gamma)$ and for each vertex $in_i \in V_{in}(\Gamma)$ we add l_i new outcoming edges. Then the valency of in_i becomes $l_i + 1$.



More pedantically, for every $i, 1 \leq i \leq n$ we have constructed a map from the set $\{1, ..., l_i\}$ to a cyclically ordered set which is an orbit of $\rho \circ \sigma$ with removed tail edge outcoming from in_i. Cyclic order on the edges of Γ' is induced by the cyclic order at every vertex and the cyclic order on the path forming the face corresponding to in_i.



(c) We assign $\gamma_i \in C_{l_i,\varepsilon_i}$ to in_i. We depict γ_i as a "wheel" representing the Hochschild cocycle. It is formed by the endpoints of the $l_i + 1$ edges outcoming from $in_i \in V(\Gamma')$ and taken in the cyclic order of the corresponding face. If $\varepsilon_i = 1$ then (up to a scalar) $\gamma_i = 1 \otimes a_1 \otimes ... \otimes a_{l_i}$ and we require that the tensor factor 1 corresponds to zero in the cyclic order.



(d) We remove from considerations graphs Γ which do not obey the following property after the step (c):

the edge corresponding to the unit $1 \in k$ *(see step c)) is of the type* (in_i, v) *where either* $v \in V_{middle}(\Gamma')$ *and* $|v| = 3$ *or* $v = out_j$ *for some* $1 \leq j \leq m$ *and the edge* (in_i, out_j) *was the marked edge for* out_j*.*

Let us call *unit edge* the one which satisfies one of the above properties. We define a new graph Γ'' which is obtained from Γ by removing unit edges.

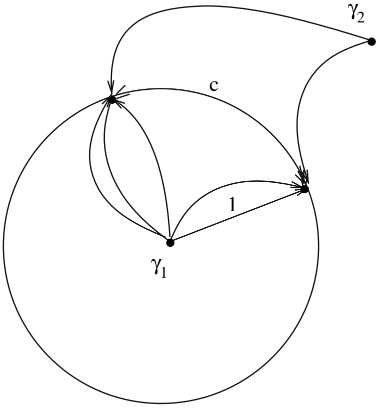
(e) Each vertex now has the valency $|v| \geq 2$. We attach to every such vertex either:

the tensor $c \in A \otimes A$ (inverse to the scalar product), if $|v| = 2$,

or

the tensor $(m_{|v|-1}(a_1, ..., a_{|v|-1}), a_{|v|})$ if $|v| \geq 3$. The latter can be identified with the element of $A^{\otimes |v|}$ (here we use the non-degenerate scalar product on A).

Let us illustrate this construction.



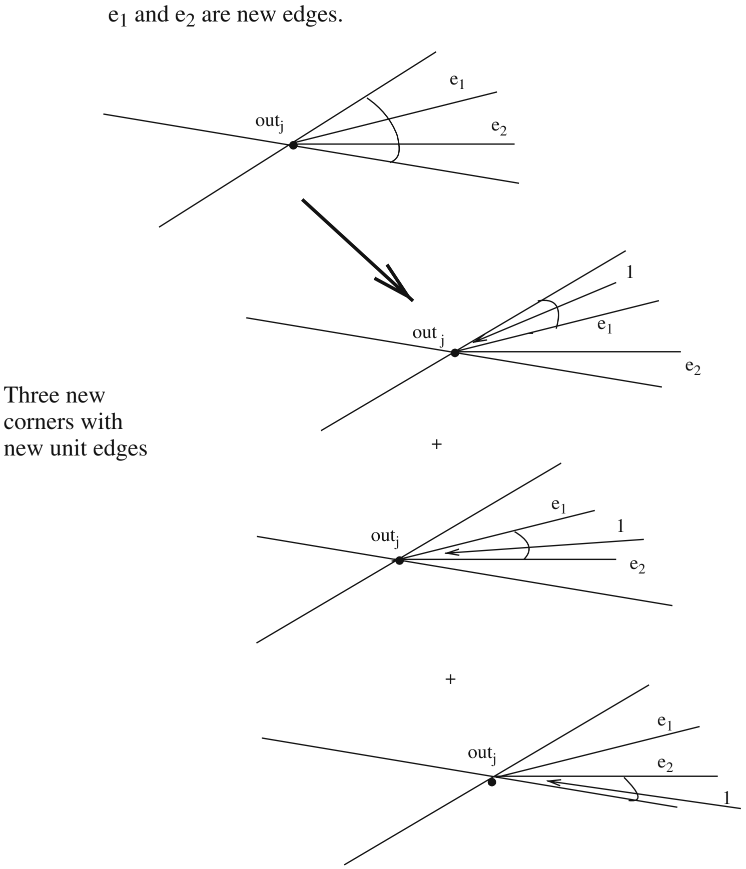
(f) Let us contract indices of tensors corresponding to $V_{in}(\Gamma'')\cup V_{middle}(\Gamma'')$ (see c, e) along the edges of Γ'' using the scalar product on A. The result will be an element a_{out} of the tensor product $\otimes_{1\le j\le m} A^{Star_{\Gamma''}(out_j)}$.

(g) Last thing we need to do is to interpret the element a_{out} as an element of $C^{mod}_{\bullet}(A,A)$. There are three cases.

Case 1. When we constructed Γ'' there was a unit edge incoming to some out_j. Then we reconstruct back the removed edge, attach $1\in k$ to it, and interpret the resulting tensor as an element of $C^{mod}_{|out_j|,\varepsilon_j=1}(A,A)$.

Case 2. There was no removed unit edge incoming to out_j and we had a marked edge (not a marked corner) at the vertex out_j. Then we have an honest element of $C^{mod}_{|out_j|,\varepsilon_j=0}(A,A)$

Case 3. Same as in Case 2, but there was a marked corner at $out_j \in V_{out}(\Gamma)$. We have added and removed new edges when constructed Γ''. Therefore the marked corner gives rise to a new set of marked corners at out_j considered as a vertex of Γ''. Inside every such a corner we insert a new edge, attach the element $1\in k$ to it and take the sum over all the corners. In this way we obtain an element of $C^{mod}_{|out_j|,\varepsilon_j=1}(A,A)$. This procedure is depicted below.



This concludes the construction of f_Γ. Notice that R is a dg-PROP with the differential given by the insertion of a new edge between two vertices from $V_{middle}(\Gamma)$.

Proof of the following Proposition will be given elsewhere.

Proposition 11.14 The above construction gives rise to a structure of a R-algebra on $C_\bullet^{mod}(A, A)$.

Remark 11.15 The above construction did not use homological smoothness of A.

Finally we would like to say few words about an extension of the R-action to the $Chains(\overline{\mathcal{M}})$-action. More details and application to Topological Field Theory will be given in [22].

If we assume the degeneration property for A, then the action of the PROP R can be extended to the action of the PROP $Chains(\overline{\mathcal{M}})$ of singular chains of the topological PROP of stable degenerations of $M_{g,n,m}^{marked}$. In order to see this, one introduces the PROP D freely generated by $R(2,0)$ and $R(1,1)$, i.e., by singular chains on the moduli space of cylinders with two inputs and zero outputs (they correspond to the scalar product on $C_\bullet(A, A)$) and by cylinders with one input and one output (they correspond to morphisms $C_\bullet(A, A) \to C_\bullet(A, A)$). In fact the (non-symmetric) bilinear form $h\colon H_\bullet(A, A) \otimes H_\bullet(A, A) \to k$ does exist for any compact A_∞-algebra A. It is described by the graph of degree zero on the figure in Sect. 11.6. This is a generalization of the bilinear form $(a, b) \in A/[A, A] \otimes A/[A, A] \mapsto Tr(axb) \in k$. It seems plausible that homological smoothness implies that h is non-degenerate. This allows us to extend the action of the dg sub-PROP $D \subset R$ to the action of the dg PROP $D' \subset R$ which contains also $R(0, 2)$ (i.e., the inverse to the above bilinear form). If we assume the degeneration property, then we can "shrink" the action of the homologically non-trivial circle of the cylinders (since the rotation around this circle corresponds to the differential B). Thus D' is quasi-isomorphic to the dg-PROP of chains on the (one-dimensional) retracts of the above cylinders (retraction contracts the circle). Let us denote the dg-PROP generated by singular chains on the retractions by D''. Thus, assuming the degeneration property, we see that the free product dg-PROP $R' = R *_D D''$ acts on $C_\bullet^{mod}(A, A)$. One can show that R' is quasi-isomorphic to the dg-PROP of chains on the topological PROP $\overline{M}_{g,n,m}^{marked}$ of stable compactifications of the surfaces from $M_{g,n,m}^{marked}$.

Remark 11.16 (a) The above construction is generalization of the construction from [31], which assigns cohomology classes of $M_{g,n}$ to a finite-dimensional A_∞-algebra with scalar product (trivalent graphs were used in [31]).

(b) Different approach to the action of the PROP R was suggested in [8]. The above Proposition gives rise to a structure of Topological Field Theory associated with a non-unital A_∞-algebra with scalar product. If the degeneration property holds for A then one can define a Cohomological Field Theory in the sense of [34].

(c) Homological smoothness of A is closely related to the existence of a non-commutative analog of the Chern class of the diagonal $\Delta \subset X \times X$ of a projective scheme X. This Chern class gives rise to the inverse to the scalar product on A. This topic will be discussed in the subsequent study devoted to A_∞-categories.

12 Appendix

12.1 Non-Commutative Schemes and Ind-Schemes

Let $\mathcal{C}$ be an Abelian k-linear tensor category. To simplify formulas we will assume that it is strict (see [41]). We will also assume that $\mathcal{C}$ admits infinite sums. To simplify the exposition we will assume below (and in the main body of the paper) that $\mathcal{C} = Vect_k^{\mathbf{Z}}$.

Definition 12.1 The category of non-commutative affine k-schemes in $\mathcal{C}$ (notation $NAff_{\mathcal{C}}$) is the one opposite to the category of associative unital k-algebras in $\mathcal{C}$.

The non-commutative scheme corresponding to the algebra A is denoted by $Spec(A)$. Conversely, if X is a non-commutative affine scheme then the corresponding algebra (algebra of regular functions on X) is denoted by $\mathcal{O}(X)$. By analogy with commutative case we call a morphism $f: X \to Y$ a *closed embedding* if the corresponding homomorphism $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ is an epimorphism.

Let us recall some terminology of ind-objects (see e.g., [1, 20, 21]). For a covariant functor $\phi: I \to \mathcal{A}$ from a small filtering category I (called filtrant in [21]) there is a notion of an inductive limit "$\varinjlim$" $\phi \in \widehat{\mathcal{A}}$ and a projective limit "$\varprojlim$" $\phi \in \widehat{\mathcal{A}}$. By definition "$\varinjlim$" $\phi(X) = \varinjlim Hom_{\mathcal{A}}(X, \phi(i))$ and "$\varprojlim$" $\phi(X) = \varinjlim Hom_{\mathcal{A}}(\phi(i), X)$. All inductive limits form a full subcategory $Ind(\mathcal{A}) \subset \widehat{\mathcal{A}}$ of ind-objects in $\mathcal{A}$. Similarly all projective limits form a full subcategory $Pro(\mathcal{A}) \subset \widehat{\mathcal{A}}$ of pro-objects in $\mathcal{A}$.

Definition 12.2 Let I be a small filtering category and $F: I \to NAff_{\mathcal{C}}$ a covariant functor. We say that "$\varinjlim$" F is a non-commutative ind-affine scheme if for a morphism $i \to j$ in I the corresponding morphism $F(i) \to F(j)$ is a closed embedding.

In other words a non-commutative ind-affine scheme X is an object of $Ind(NAff_{\mathcal{C}})$, corresponding to the projective limit $\varprojlim A_\alpha, \alpha \in I$, where each A_α is a unital associative algebra in $\mathcal{C}$ and for a morphism $\alpha \to \beta$ in I the corresponding homomorphism $A_\beta \to A_\alpha$ is a surjective homomorphism of unital algebras (i.e., one has an exact sequence $0 \to J \to A_\beta \to A_\alpha \to 0$).

Remark 12.3 Not all categorical epimorphisms of algebras are surjective homomorphisms (although the converse is true). Nevertheless one can define closed embeddings of affine schemes for an arbitrary Abelian k-linear category, observing that a surjective homomorphism of algebras $f : A \to B$ is characterized categorically by the condition that B is the cokernel of the pair of the natural projections $f_{1,2} \colon A \times_B A \to A$ defined by f.

Morphisms between non-commutative ind-affine schemes are defined as morphisms between the corresponding projective systems of unital algebras. Thus we have

$$Hom_{NAff_{\mathcal{C}}}(\varinjlim_I X_i, \varinjlim_J Y_j) = \varprojlim_I \varinjlim_J Hom_{NAff_{\mathcal{C}}}(X_i, Y_j).$$

Let us recall that an algebra $M \in Ob(\mathcal{C})$ is called nilpotent if the natural morphism $M^{\otimes n} \to M$ is zero for all sufficiently large n.

Definition 12.4 A non-commutative ind-affine scheme $\hat{X}$ is called formal if it can be represented as $\hat{X} = \varinjlim Spec(A_i)$, where $(A_i)_{i \in I}$ is a projective system of associative unital algebras in $\mathcal{C}$ such that the homomorphisms $A_i \to A_j$ are surjective and have nilpotent kernels for all morphisms $j \to i$ in I.

Let us consider few examples in the case when $\mathcal{C} = Vect_k$.

Example 12.5 In order to define the non-commutative formal affine line $\widehat{\mathbf{A}}^1_{NC}$ it suffices to define $Hom(Spec(A), \widehat{\mathbf{A}}^1_{NC})$ for any associative unital algebra A. We define $Hom_{NAff_k}(Spec(A), \widehat{\mathbf{A}}^1_{NC}) = \varinjlim Hom_{Alg_k}(k[[t]]/(t^n), A)$. Then the set of A-points of the non-commutative formal affine line consists of all nilpotent elements of A.

Example 12.6 For an arbitrary set I the non-commutative formal affine space $\widehat{\mathbf{A}}^I_{NC}$ corresponds, by definition, to the topological free algebra $k\langle\langle t_i \rangle\rangle_{i \in I}$. If A is a unital k-algebra then any homomorphism $k\langle\langle t_i \rangle\rangle_{i \in I} \to A$ maps almost all t_i to zero and the remaining generators are mapped into nilpotent elements of A. In particular, if $I = \mathbf{N} = \{1, 2, ...\}$ then $\widehat{\mathbf{A}}^{\mathbf{N}}_{NC} = \varinjlim Spec(k\langle\langle t_1, ..., t_n \rangle\rangle/(t_1, ..., t_n)^m)$, where $(t_1, ..., t_n)$ denotes the two-sided ideal generated by $t_i, 1 \leq i \leq n$ and the limit is taken over all $n, m \to \infty$.

By definition, a *closed subscheme* Y of a scheme X is defined by a two-sided ideal $J \subset \mathcal{O}(X)$. Then $\mathcal{O}(Y) = \mathcal{O}(X)/J$. If $Y \subset X$ is defined by a two-sided ideal $J \subset \mathcal{O}(X)$, then the completion of X along Y is a formal scheme corresponding to the projective limit of algebras $\varprojlim_n \mathcal{O}(X)/J^n$. This formal scheme will be denoted by $\hat{X}_Y$ or by $Spf(\mathcal{O}(X)/J)$.

Non-commutative affine schemes over a given field k form symmetric monoidal category. The tensor structure is given by the *ordinary tensor product* of unital algebras. The corresponding tensor product of non-commutative affine schemes will be denoted by $X \otimes Y$. It is not a categorical product, differently from the case of commutative affine schemes (where the tensor product of algebras corresponds to the Cartesian product $X \times Y$). For non-commutative affine schemes the analog of the Cartesian product is the *free product* of algebras.

Let A, B be free algebras. Then $Spec(A)$ and $Spec(B)$ are non-commutative manifolds. Since the tensor product $A \otimes B$ in general is not a smooth algebra, the non-commutative affine scheme $Spec(A \otimes B)$ is not a manifold.

Let X be a non-commutative ind-affine scheme in $\mathcal{C}$. A closed k-point $x \in X$ is by definition a homomorphism of $\mathcal{O}(X)$ to the tensor algebra generated by the unit object $\mathbf{1}$. Let m_x be the kernel of this homomorphism. We define the *tangent space* T_xX in the usual way as $(m_x/m_x^2)^* \in Ob(\mathcal{C})$. Here m_x^2 is the image of the multiplication map $m_x^{\otimes 2} \to m_x$.

A non-commutative ind-affine scheme with a marked closed k-point will be called *pointed*. There is a natural generalization of this notion to the case of many points. Let $Y \subset X$ be a closed subscheme of disjoint closed k-points (it corresponds to the algebra homomorphism $\mathcal{O}(X) \to \mathbf{1} \oplus \mathbf{1} \oplus ...$). Then $\hat{X}_Y$ is a formal manifold. A pair $(\hat{X}_Y, Y)$ (often abbreviated by $\hat{X}_Y$) will be called (non-commutative) *formal manifold with marked points.* If Y consists of one such point then $(\hat{X}_Y, Y)$ will be called (non-commutative) *formal pointed manifold.*

12.2 Proof of Theorem 2.1.1

In the category $Alg_{\mathcal{C}^f}$ every pair of morphisms has a kernel. Since the functor F is left exact and the category $Alg_{\mathcal{C}^f}$ is Artinian, it follows from [20], Sect. 3.1 that F is strictly pro-representable. This means that there exists a projective system of finite-dimensional algebras $(A_i)_{i \in I}$ such that, for any morphism $i \to j$ the corresponding morphsim $A_j \to A_i$ is a categorical epimorphism and for any $A \in Ob(Alg_{\mathcal{C}^f})$ one has

$$F(A) = \varinjlim_I Hom_{Alg_{\mathcal{C}^f}}(A_i, A).$$

Equivalently,

$$F(A) = \varinjlim_I Hom_{Coalg_{\mathcal{C}^f}}(A_i^*, A^*),$$

where $(A_i^*)_{i \in I}$ is an inductive system of finite-dimensional coalgebras and for any morphism $i \to j$ in I we have a categorical monomorphism $g_{ji} : A_i^* \to A_j^*$.

All what we need is to replace the projective system of algebras $(A_i)_{i \in I}$ by another projective system of algebras $(\overline{A}_i)_{i \in I}$ such that

(a) functors “$\varprojlim$”h_{A_i} and “$\varprojlim$”$h_{\overline{A}_i}$ are isomorphic (here h_X is the functor defined by the formula $h_X(Y) = Hom(X, Y)$);

(b) for any morphism $i \to j$ the corresponding homomorphism of algebras $\overline{f}_{ij} : \overline{A}_j \to \overline{A}_i$ is surjective.

Let us define $\overline{A}_i = \bigcap_{i \to j} Im(f_{ij})$, where $Im(f_{ij})$ is the image of the homomorphism $f_{ij} : A_j \to A_i$ corresponding to the morphism $i \to j$ in I. In order to prove a) it suffices to show that for any unital algebra B in $\mathcal{C}^f$ the natural map of sets

$$\varinjlim_I Hom_{\mathcal{C}^f}(A_i, B) \to \varinjlim_I Hom_{\mathcal{C}^f}(\overline{A}_i, B)$$

(the restriction map) is well-defined and bijective.

The set $\varinjlim_I Hom_{\mathcal{C}^f}(A_i, B)$ is isomorphic to $(\bigsqcup_I Hom_{\mathcal{C}^f}(A_i, B))/equiv$, where two maps $f_i : A_i \to B$ and $f_j : A_j \to B$ such that $i \to j$ are equivalent if $f_i f_{ij} = f_j$. Since $\mathcal{C}^f$ is an Artinian category, we conclude that there exists A_m such that $f_{im}(A_m) = \overline{A}_i$, $f_{jm}(A_m) = \overline{A}_j$. From this observation one easily deduces that $f_{ij}(\overline{A}_j) = \overline{A}_i$. It follows that the morphism of functors in (a) is well-defined and (b) holds. The proof that morphisms of functors biejectively correspond to homomorphisms of coalgebras is similar. This completes the proof of the theorem. ■

12.3 Proof of Proposition 2.1.2

The result follows from the fact that any $x \in B$ belongs to a finite-dimensional subcoalgebra $B_x \subset B$ and if B was counital then B_x would be also counital. Let us describe how to construct B_x. Let Δ be the coproduct in B. Then one can write

$$\Delta(x) = \sum_i a_i \otimes b_i,$$

where a_i (resp. b_i) are linearly independent elements of B.

It follows from the coassociativity of Δ that

$$\sum_i \Delta(a_i) \otimes b_i = \sum_i a_i \otimes \Delta(b_i).$$

Therefore one can find constants $c_{ij} \in k$ such that

$$\Delta(a_i) = \sum_j a_j \otimes c_{ij},$$

and

$$\Delta(b_i) = \sum_j c_{ji} \otimes b_j.$$

Applying $\Delta \otimes id$ to the last equality and using the coassociativity condition again we get

$$\Delta(c_{ji}) = \sum_n c_{jn} \otimes c_{ni}.$$

Let B_x be the vector space spanned by x and all elements a_i, b_i, c_{ij}. Then B_x is the desired subcoalgebra. ■

12.4 Formal Completion Along a Subscheme

Here we present a construction which generalizes the definition of a formal neighborhood of a k-point of a non-commutative smooth thin scheme.

Let $X = Spc(B_X)$ be such a scheme and $f\colon X \to Y = Spc(B_Y)$ be a closed embedding, i.e., the corresponding homomorphism of coalgebras $B_X \to B_Y$ is injective. We start with the category $\mathcal{N}_X$ of nilpotent extensions of X, i.e., homomorphisms $\phi\colon X \to U$, where $U = Spc(D)$ is a non-commutative thin scheme, such that the quotient $D/f(B_X)$ (which is always a non-counital coalgebra) is locally conilpotent. We recall that the local conilpotency means that for any $a \in D/f(B_X)$ there exists $n \geq 2$ such that $\Delta^{(n)}(a) = 0$, where $\Delta^{(n)}$ is the nth iterated coproduct Δ. If (X, ϕ_1, U_1) and (X, ϕ_2, U_2) are two nilpotent extensions of X then a morphism between them is a morphism of non-commutative thin schemes $t\colon U_1 \to U_2$, such that $t\phi_1 = \phi_2$ (in particular, $\mathcal{N}_X$ is a subcategory of the naturally-defined category of non-commutative relative thin schemes).

Let us consider the functor $G_f : \mathcal{N}_X^{op} \to Sets$ such that $G(X, \phi, U)$ is the set of all morphisms $\psi\colon U \to Y$ such that $\psi\phi = f$.

Proposition 12.7 Functor G_f is represented by a triple $(X, \pi, \hat{Y}_X)$ where the non-commutative thin scheme denoted by $\widehat{Y}_X$ is called the formal neighborhood of $f(X)$ in Y (or the completion of Y along $f(X)$).

Proof. Let $B_f \subset B_X$ be the counital subcoalgebra which is the preimage of the (non-counital) subcoalgebra in $B_Y/f(B_X)$ consisting of locally-conilpotent elements. Notice that $f(B_X) \subset B_f$. It is easy to see that taking $\widehat{Y}_X := Spc(B_f)$ we obtain the triple which represents the functor G_f. ∎

Notice that $\widehat{Y}_X \to Y$ is a closed embedding of non-commutative thin schemes.

Proposition 12.8 If Y is smooth then $\widehat{Y}_X$ is smooth and $\widehat{Y}_X \simeq \widehat{Y}_{\widehat{Y}_X}$.

Proof. Follows immediately from the explicit description of the coalgebra B_f given in the proof of the previous Proposition. ∎

Acknowledgments

We thank Vladimir Drinfeld for useful discussions and Victor Ginzburg and Kevin Costello for comments on the manuscript. Y.S. thanks Clay Mathematics Institute for supporting him as a Fellow during a part of the work. His work was also partially supported by an NSF grant. He is especially grateful to IHES for providing excellent research and living conditions for the duration of the project.

References

1. M. Artin, B. Mazur, Etale Homotopy, Lect. Notes Math., 100 (1969). 213
2. A. Beilinson, V. Drinfeld, Chiral algebras (in preparation)..
3. A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves (in preparation).
4. A. Bondal, M. Kapranov, Enhanced triangulated categories, Math. USSR Sbornik, 70:1, 93–107 (1991).
5. A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, math.AG/0204218. 155, 184, 185, 189
6. J. Boardman, R. Vogt, Homotopy invariant algebraic structures on topological spaces, Lect. Notes Math., 347 (1973).
7. A. Connes, Non-commutative geometry. Academic Press, 1994. 155, 156, 181
8. K. Costello, Topological conformal field theories, Calabi-Yau categories and Hochschild homology, preprint (2004). 157, 212
9. J. Cuntz, D. Quillen, Cyclic homology and non-singularity, J. Amer. Math. Soc., 8:2, 373–442 (1995). 182
10. J. Cuntz, D. Quillen, Algebra extensions and non-singularity, J. Amer. Math. Soc., 8:2, 251–289 (1995). 161
11. J. Cuntz, G. Skandalis, B. Tsygan, Cyclic homology in noncommutative geometry, Encyclopaedia of Mathematical Sciences, v. 121, Springer Verlag, p. 74–113. 155, 156
12. P. Deligne, L. Illusie, Relevements modulo p^2 et decomposition du complex de de Rham, Invent. Math. 89, 247–270 (1987). 188
13. P. Deligne, J.S. Milne, Tannakian categories, Lect. Notes Math., 900, 101–228 (1982). 158
14. V. Drinfeld, D.G. quotients of DG categories, math.KT/0210114. 157
15. K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, Lagrangian intersection Floer theory-anomaly and obstruction. Preprint, 2000. 171
16. E. Getzler, Cartan homotopy formulas and Gauss-Manin connection in cyclic homology, Israel Math. Conf. Proc., 7, 65–78. 201
17. V. Ginzburg, Non-commutative symplectic geometry, quiver varieties and operads, math.QA/0005165.
18. V. Ginzburg, Lectures on Noncommutative Geometry, math.AG/0506603. 184, 185, 193
19. V. Ginzburg, Double derivations and cyclic homology, math.KT/0505236. 178, 191
20. A. Grothendieck, Technique de descente.II. Sem. Bourbaki, 195 (1959/60). 158, 213, 215
21. M. Kashiwara, P. Schapira, Ind-sheaves, Asterisque 271 (2001). 213
22. L.Katzarkov, M. Kontsevich, T. Pantev, Calculating Gromov-Witten invariants from Fukaya category, in preparation. 157, 212
23. H. Kajiura, Noncommutative homotopy algebras associated with open strings, arXiv:math/0306332. 193
24. D. Kaledin, Non-commutative Cartier operator and Hodge-to-de Rham degeneration, math.AG/0511665. 156, 188
25. R. Kaufmann,Moduli space actions on the Hochschild co-chains of a Frobenius algebra I: Cell Operads, math.AT/0606064.
26. R. Kaufmann,Moduli space actions on the Hochschild co-chains of a Frobenius algebra II: Correlators, math.AT/0606065.

27. B. Keller, Introduction to A-infinity algebras and modules, Homology, Homotopy and Applications 3, 1–35 (2001). 172, 173
28. B. Keller, On differential graded categories, math.KT/0601185. 184
29. M. Kontsevich, Deformation quantization of Poisson manifolds, math.QA/9709040. 167
30. M. Kontsevich, Formal non-commutative symplectic geometry. In: Gelfand Mathematical Seminars, 1990–1992, p. 173–187. *Birkhävser* Boston, MA, (1993). 155, 193
31. M. Kontsevich, Feynman diagrams and low-dimensional topology. Proc. Europ. Congr. Math., 1 (1992). 212
32. M. Kontsevich, Notes on motives in finite characteristic, in preparation. 188
33. M. Kontsevich, Lecture on triangulated A_∞-categories at Max-Planck Institut für Mathematik, (2001). 157
34. M. Kontsevich, Yu.Manin, Gromov-Witten classes, quantum cohomology and enumerativ geometry, Comm. Math. Phys., 164:3, 525–562 (1994). 212
35. M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne conjecture, math.QA/0001151, published in Lett. Math. Phys., 21:1, 255–307 (2000). 156, 176, 194, 195, 196, 197, 198, 200
36. M. Kontsevich, Y. Soibelman, Deformation theory, (book in preparation). 167
37. L. Korogodski, Y. Soibelman, Algebras of functions on quantum groups.I. Math. Surveys Monogr. 56, AMS (1998). 184
38. L. Le Bruyn, Non-commutative geometry an n (book in preparation).
39. V. Lyubashenko, Category of A_∞-categories, math.CT/0210047. 154, 157, 169, 174
40. V. Lyubashenko, S. Ovsienko, A construction of A_∞-category, math.CT/0211037. 157, 169, 174
41. S. Mac Lane, Categories for the working mathematician. Springer-Verlag (1971). 213
42. D. Orlov,Triangulated categories of singularities and equivalences between Landau-Ginzburg models, math.AG/0503630. 186
43. R. Rouquier, Dimensions of triangulated categories, math.CT/0310134. 186
44. S. Sanablidze, R. Umble, A diagonal of the associahedra, math. AT/0011065. 174
45. P. Seidel, Homological mirror symmetry for the quartic surface, math.SG/0310414. 186
46. Y. Soibelman, Non-commutative geometry and deformations of A_∞-algebras and A_∞-categories, Arbeitstagung 2003, Preprint Max-PLanck Institut für Mathematik, MPIM2003-60h, (2003). 157
47. Y. Soibelman, Mirror symmetry and non-commutative geometry of A_∞-categories, J. Math. Phys., 45:10, 3742–3757 (2004). 157
48. D. Tamarkin, B. Tsygan, Non-commutative differential calculus, homotopy BV-algebras and formality conjectures, arXiv:math/0010072. 156
49. D. Tamarkin, B. Tsygan, The ring of differential operators on forms in non-commutative calculus. Proceedings of Symposia in Pure Math., 73, 105–131 (2005). 156
50. B. Toen, M. Vaquie, Moduli of objects in dg-categories, math.AG/0503269. 184, 186
51. J. L. Verdier, Categories derivees, etat 0. Lect. Notes in Math., 569, 262–312 (1977).