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Tighter Monogamy Relations for Concurrence and Negativity in Multiqubit Systems

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Abstract: The entanglement in multipartite quantum system is hard to characterize and quantify, although it has been intensively studied in bipartite systems. The monogamy of entanglement, as a special property of multipartite systems, shows the distribution of entanglement in the system. In this paper, we investigate the monogamy relations for multi-qubit systems. By using two entangled measures, namely the concurrence C and the negativity N_c, we establish tighter monogamy inequalities for their α-th power than those in all the existing ones. We also illustrate the tightness of our results for some classes of quantum states.

Keywords: monogamy relation; multiqubit systems; the concurrence; the negativity

MSC: 81P40, 81P42

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1. Introduction

Quantum entanglement plays an essential role in quantum information processing. The research on quantum entanglement is multi-angled and has produced many impressive results [1–8]. The entanglement in bipartite systems has been intensively studied with rich understanding, while the characterization and quantification of entanglement in multipartite quantum systems is less known. In multipartite quantum system, entangled with one subsystem limits its entanglement with the other subsystems. The monogamy of entanglement, as a special property of multipartite systems, shows the distribution of entanglement and is applied to quantum key distribution [9].

The usual monogamous relationship shows that for a tripartite system made up of A, B, and C, the entanglement between A and joint system BC is bigger than the sum of the single pair entanglement A and the other party B or C [10]. Coffman, Kundu, and Wootters (CKW) [11] first characterized the monogamy of entanglement for the three-qubit state mathematically:

$$E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC}), \tag{1}$$

where E is an entanglement measure, $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$, and $\rho_{AC} = \mathrm{Tr}_B(\rho_{ABC})$. However, not all entanglement measures satisfy this monogamous relationship, such as concurrence, negativity and entanglement of formation. Although the concurrence C does not satisfy such monogamy inequality, the squared concurrence C^2 [12,13] and the entanglement of formation E^2 [14] satisfy the monogamy relations for multiqubit states. Additionally, many monogamy relations for mulitiqubit and high-dimensional systems were established [15–24]. Recently, in [25,26], the authors gave an alternative definition of the monogamy relation with no inequality employed. The monogamous inequality is further extended to various entanglement measures, such as continuous variable entanglement [27–29], squashed entanglement [10,30,31], entanglement negativity [32–36], Tsallis-q entanglement [15,37], and Rényi entanglement [38–40].

In this paper, using the concurrence C and the negativity N_c, we derive some tighter monogamy inequalities than all the existing ones.

2. Tighter Monogamy Relations for the Concurrence

Let X and H_X denote the quantum system and its corresponding finite dimensional Hilbert space, respectively. If $|\psi\rangle_{AB}$ is a pure state of a bipartite quantum system $H_A \otimes H_B$, then the concurrence is defined as follows: [41]

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - Tr(\rho_A^2))}, \tag{2}$$

where $\rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|)$.

If ρ_{AB} is a bipartite mixed state, then the concurrence is defined as follows:

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{3}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0, \sum_i p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$.

Let $|\psi\rangle_{AB_1B_2\cdots B_{N-1}}$ be an N-qubit pure state in $H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$; its concurrence $C(|\psi\rangle_{A|B_1B_2\cdots B_{N-1}})$, viewed as a bipartite state under the partitions A and $B_1, B_2, \cdots, B_{N-1}$, satisfies the CKW inequality [12,13]:

$$C^2_{A|B_1B_2\cdots B_{N-1}} \geq C^2_{AB_1} + C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}, \tag{4}$$

where $C_{AB_i} = C(\rho_{AB_i})$ denotes the concurrence of $\rho_{AB_i} = Tr_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(|\psi\rangle_{AB_1\cdots B_{N-1}}$ $\langle\psi|), C_{A|B_1B_2\cdots B_{N-1}} = C(\rho_{A|B_1\cdots B_{N-1}})$. It is further proven that for $\alpha \geq 2$, one has [23]:

$$C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq C^\alpha_{AB_1} + C^\alpha_{AB_2} + \cdots + C^\alpha_{AB_{N-1}}. \tag{5}$$

The relation (5) is further improved so that for $\alpha \geq 2$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $1 \leq i \leq m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $m+1 \leq j \leq N-2$, $1 \leq m \leq N-3$, $N \geq 4$, then [22]:

$$\begin{aligned} C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq\ & C^\alpha_{AB_1} + \frac{\alpha}{2} C^\alpha_{AB_2} + \cdots + \left(\frac{\alpha}{2}\right)^{m-1} C^\alpha_{AB_m} \\ & + \left(\frac{\alpha}{2}\right)^{m+1} \left(C^\alpha_{AB_{m+1}} + \cdots + C^\alpha_{AB_{N-2}}\right) + \left(\frac{\alpha}{2}\right)^m C^\alpha_{AB_{N-1}}, \end{aligned} \tag{6}$$

and if for all $i = 1, 2, \cdots, N-2$, $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$, then [22]:

$$C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq C^\alpha_{AB_1} + \left(\frac{\alpha}{2}\right) C^\alpha_{AB_2} + \cdots + \left(\frac{\alpha}{2}\right)^{N-2} C^\alpha_{AB_{N-1}}. \tag{7}$$

The relations (6) and (7) are further improved in [24]: for $\alpha \geq 2$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, $0 \leq m \leq N-3$, $N \geq 4$, then:

$$\begin{aligned} C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq\ & C^\alpha_{AB_1} + \left(2^{\frac{\alpha}{2}} - 1\right) C^\alpha_{AB_2} + \cdots + \left(2^{\frac{\alpha}{2}} - 1\right)^{m-1} C^\alpha_{AB_m} \\ & + \left(2^{\frac{\alpha}{2}} - 1\right)^{m+1} \left(C^\alpha_{AB_{m+1}} + \cdots + C^\alpha_{AB_{N-2}}\right) + \left(2^{\frac{\alpha}{2}} - 1\right)^m C^\alpha_{AB_{N-1}}, \end{aligned} \tag{8}$$

and if $C_{AB_i} \geq C_{A|B_{i+1}\cdots N-1}$ for all $i = 1, 2, \cdots, N-2$, then:

$$C^\alpha_{A|B_1B_2\cdots B_{N-1}} \geq C^\alpha_{AB_1} + \left(2^{\frac{\alpha}{2}} - 1\right) C^\alpha_{AB_2} + \cdots + \left(2^{\frac{\alpha}{2}} - 1\right)^{N-3} C^\alpha_{AB_{N-2}} + \left(2^{\frac{\alpha}{2}} - 1\right)^{N-2} C^\alpha_{AB_{N-1}}. \tag{9}$$

From the proof of the above research results, we find that the above different monogamy relations actually depend on different inequalities, and the compactness of monogamy relations is exactly the compactness of these inequality relations. In fact, the monogamy relation (5) depends on the following inequality [23]:

$$(1+t)^x \geq 1 + t^x,\ t \leq 1,\ x \geq 1; \tag{10}$$

the monogamy relation (6) and (7) depends on the inequality [22]:

$$(1+t)^x \geq 1 + xt^x,\ 0 \leq t \leq 1,\ x \geq 1; \tag{11}$$

and the monogamy relation (8) and (9) depends on the inequality [24]:

$$(1+t)^x \geq 1 + (2^x - 1)t^x,\ 0 \leq t \leq 1,\ x \geq 1. \tag{12}$$

Obviously, these three inequalities show that the upper bound of the function $(1+t)^x$ for $0 \leq t \leq 1,\ x \geq 1$ is getting tighter and tighter. Then, an important idea arises: to compact the above monogamy relations, one must compact the above inequalities they rely on first. Therefore, in this paper, we first establish more compact inequalities, and then compact the existing monogamy relations.

Lemma 1. *Let $t \in [0,1]$ and $x \in [2, \infty)$, then, we have:*

$$(1+t)^x \geq 1 + (2^x - t^x)t^x. \tag{13}$$

Proof. Let $f(x,y) = (1+y)^x - y^x + y^{-x}$ with $x \geq 1$ and $y \geq 1$. Then, $\frac{\partial f}{\partial y} = x(1+y)^{x-1} - xy^{x-1} - xy^{-x-1}$ and $\frac{\partial^2 f}{\partial y^2} = x(x-1)(1+y)^{x-2} - x(x-1)y^{x-2} + x(x+1)y^{-x-2}$. We have $\frac{\partial^2 f}{\partial y^2} \geq 0$ for $x \geq 2$. Therefore, $\frac{\partial f}{\partial y}(x,y)$ is an increasing function of y, i.e., $\frac{\partial f}{\partial y}(x,y) \geq \frac{\partial f}{\partial y}(x,1) = x(2^{x-1} - 2) \geq 0$ for $x \geq 2$. Then, $f(x,y)$ is increasing with respect to y, i.e., $f(x,y) \geq f(x,1) = 2^x$. Setting $y = \frac{1}{t}$ and $0 < t \leq 1$, we obtain $(1+t)^x \geq 1 + (2^x - t^x)t^x$. The inequality is trivial for $t = 0$. □

Lemma 2. *Let ρ_{ABC} be a mixed state in $2 \otimes 2 \otimes 2^{n-2}$ quantum system $H_A \otimes H_B \otimes H_C$; if $C_{AB} \geq C_{AC}$, one has:*

$$C^\alpha_{A|BC} \geq C^\alpha_{AB} + \left[2^{\frac{\alpha}{2}} - \left(\frac{C_{AC}}{C_{AB}}\right)^\alpha\right] C^\alpha_{AC}, \tag{14}$$

for all $\alpha \geq 4$.

Proof. For arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC}, it has been shown that $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$ [12,42]. Then, for all $\alpha \geq 4$, if $C_{AB} \geq C_{AC}$, we have:

$$
\begin{aligned}
C^{\alpha}_{A|BC} &\geq \left(C^2_{AB} + C^2_{AC}\right)^{\frac{\alpha}{2}} \\
&= C^{\alpha}_{AB}\left(1 + \frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}} \\
&\geq C^{\alpha}_{AB}\left\{1 + \left[2^{\frac{\alpha}{2}} - \left(\frac{C_{AC}}{C_{AB}}\right)^{\alpha}\right]\left(\frac{C_{AC}}{C_{AB}}\right)^{\alpha}\right\} \\
&= C^{\alpha}_{AB} + \left[2^{\frac{\alpha}{2}} - \left(\frac{C_{AC}}{C_{AB}}\right)^{\alpha}\right]C^{\alpha}_{AC},
\end{aligned}
\tag{15}
$$

the second inequality is attributed to Lemma 1. Since subsystems A and B are equivalent in this case, we assume that $C_{AB} \geq C_{AC}$ without loss of generality. Moreover, if $C_{AB} = 0$, we have $C_{AB} = C_{AC} = 0$. □

Remark 1. *Lemma 2 shows that if $\alpha \geq 4$ and $C_{AB} = C_{AC}$, then Inequality (14) becomes $C^{\alpha}_{A|BC} \geq C^{\alpha}_{AC} + \left(2^{\frac{\alpha}{2}} - 1\right)C^{\alpha}_{AB}$, which is the result of Ref. [33] when $\alpha \geq 4$.*

For multipartite qubit systems, we have the following Theorem.

Theorem 1. *Let $\rho_{AB_0\cdots B_{N-1}}$ be an N+1 qubit mixed state; if $C_{AB_i} \geq C_{A|B_{i+1}B_{i+2}\cdots B_{N-1}}$ for $0 \leq i \leq m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $m+1 \leq j \leq N-2$, $0 \leq m \leq N-3$, $N \geq 3$, we have:*

$$
C^{\alpha}_{A|B_0\cdots B_{N-1}} \geq \sum_{i=0}^{m}\left(\prod_{j=0}^{i} M_j\right)C^{\alpha}_{AB_i} + \left(\prod_{i=1}^{m+1} M_i\right)\left(\sum_{j=m+1}^{N-2} Q_j C^{\alpha}_{AB_j} + C^{\alpha}_{AB_{N-1}}\right)
\tag{16}
$$

for $\alpha \geq 4$, where $M_0 = 1$, $M_{i+1} = 2^{\frac{\alpha}{2}} - \frac{C^{\alpha}_{A|B_{i+1}\cdots B_{N-1}}}{C^{\alpha}_{AB_i}}$ for $i = 0, 1, 2, \cdots, m$, and $Q_j = 2^{\frac{\alpha}{2}} - \frac{C^{\alpha}_{AB_j}}{C^{\alpha}_{A|B_{j+1}\cdots B_{N-1}}}$ for $j = m+1, \cdots, N-2$.

Proof. From Inequality (15), we have:

$$
\begin{aligned}
C^{\alpha}_{A|B_0\cdots B_{N-1}} &\geq C^{\alpha}_{AB_0} + M_1 C^{\alpha}_{A|B_1\cdots B_{N-1}} \\
&\geq C^{\alpha}_{AB_0} + M_1 C^{\alpha}_{AB_1} + M_1 M_2 C^{\alpha}_{A|B_2\cdots B_{N-1}} \\
&\geq \cdots \\
&\geq C^{\alpha}_{AB_0} + M_1 C^{\alpha}_{AB_1} + \cdots + M_1 M_2 \cdots M_m C^{\alpha}_{AB_m} \\
&\quad + M_1 M_2 \cdots M_m M_{m+1} C^{\alpha}_{A|B_{m+1}\cdots B_{N-1}},
\end{aligned}
\tag{17}
$$

Similarly, as $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $m+1 \leq j \leq N-2$, we obtain:

$$
\begin{aligned}
C^{\alpha}_{A|B_{m+1}\cdots B_{N-1}} &\geq C^{\alpha}_{A|B_{m+2}\cdots B_{N-1}} + Q_{m+1} C^{\alpha}_{AB_{m+1}} \\
&\geq \cdots \\
&\geq Q_{m+1} C^{\alpha}_{AB_{m+1}} + Q_{m+2} C^{\alpha}_{AB_{m+2}} + \cdots + Q_{N-2} C^{\alpha}_{AB_{N-2}} + C^{\alpha}_{AB_{N-1}}.
\end{aligned}
\tag{18}
$$

Combining (17) and (18), we have Theorem 1. □

As a particular case of Theorem 1, we have the following conclusion.

Theorem 2. *Let $\rho_{AB_0\cdots B_{N-1}}$ be an N+1 qubit mixed state; if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for all $i = 0, 1, 2, \cdots, N-2$, then:*

$$C^{\alpha}_{A|B_0\cdots B_{N-1}} \geq \sum_{i=0}^{N-2}\left(\prod_{j=0}^{i} M_j\right) C^{\alpha}_{AB_i} + \left(\prod_{i=0}^{N-2} M_i\right) C^{\alpha}_{AB_{N-1}}. \tag{19}$$

for $\alpha \geq 4$, where $M_0 = 1$ and $M_{i+1} = 2^{\frac{\alpha}{2}} - \frac{C^{\alpha}_{A|B_{i+1}\cdots B_{N-1}}}{C^{\alpha}_{AB_i}}$ for $0 \leq i \leq N-2$.

Actually there are many states that satisfy Theorem 1 and Theorem 2.

Example 1. *Let us consider the N-qubit GHZ state [43]:*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle), \tag{20}$$

it is easy to calculate that $C_{A|B_1\cdots B_{N-1}} = 1$ and $C_{AB_i} = 0$ for all $i = 2, ..., N-1$, and then our results in Theorem 1 and Theorem 2 obviously hold.

Example 2. *Let $|\psi\rangle$ be a three-qubit state; its generalized Schmidt decomposition form is [44]:*

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{21}$$

where $\lambda_i \geq 0$, $i = 0, 1, 2, 3, 4$, $\sum_{i=0}^{4}\lambda_i^2 = 1$. It is easy to compute that $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$, $C_{AC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_1 = \lambda_2 = \frac{1}{2}$ and $\lambda_3 = \lambda_4 = \frac{1}{2\sqrt{2}}$. One has $C_{AB} = \frac{1}{2}, C_{AC} = \frac{1}{2\sqrt{2}}$ and $C_{A|BC} = \frac{1}{\sqrt{2}}$. Then, $C^{\alpha}_{A|BC} = (\frac{1}{\sqrt{2}})^{\alpha}$, $C^{\alpha}_{AB} + [2^{\frac{\alpha}{2}} - (\frac{C^2_{AC}}{C^2_{AB}})^{\frac{\alpha}{2}}]C^{\alpha}_{AC} = (\frac{1}{2})^{\alpha-1} - (\frac{1}{4})^{\alpha}$. One can see that our result is better than $C^{\alpha}_{AB} + C^{\alpha}_{AC} = (\frac{1}{2})^{\alpha} + (\frac{1}{8})^{\frac{\alpha}{2}}$ in [23], $C^{\alpha}_{AB} + \frac{\alpha}{2}C^{\alpha}_{AC} = (\frac{1}{2})^{\alpha} + \frac{\alpha}{2}(\frac{1}{8})^{\frac{\alpha}{2}}$ in [22], and $C^{\alpha}_{AB} + \left(2^{\frac{\alpha}{2}} - 1\right)C^{\alpha}_{AC} = (\frac{1}{2})^{\alpha-1} - (\frac{1}{8})^{\frac{\alpha}{2}}$ in [24] for $\alpha \geq 4$; see Figure 1.

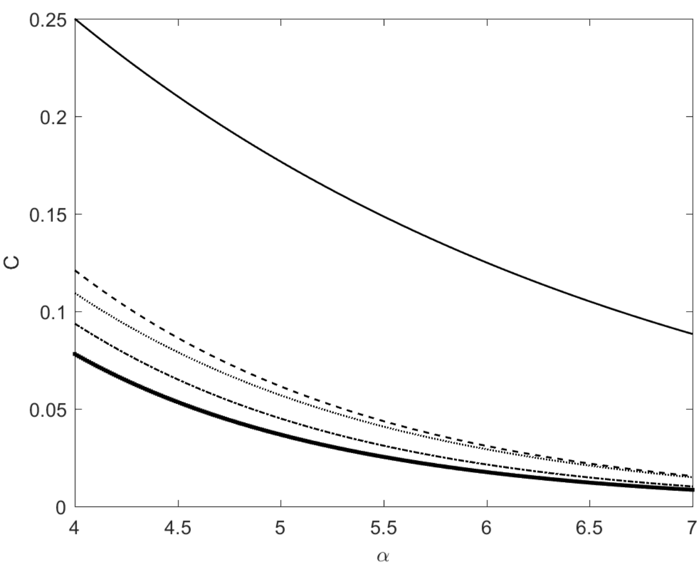

Figure 1. Axis C represents the concurrence of $|\psi\rangle_{ABC}$ and its lower bounds as functions of α. The solid line shows the concurrence of (21); the dashed line, the dotted line, the dash-dot line, and the bold solid line show lower bounds of ours and [22–24], respectively.

Remark 2. *Although the decomposition of Equation (21) is not unique, we can always select appropriate coefficients in the expression of the state to meet our results in Theorems 1 and 2.*

3. Tighter Monogamy Relations for the Negativity

In this section, we establish tighter monogamy inequalities for the negativity, which is a computable bipartite entanglement quantifier.

For a bipartite state $\rho_{AB} \in H_A \otimes H_B$, its negativity is defined as $N(\rho_{AB}) = \frac{1}{2}(\|\rho_{AB}^{T_A}\| - 1)$ [45], where $\rho_{AB}^{T_A}$ is the partial transpose with respect to the subsystem A, and $\|X\| = \mathrm{Tr}\sqrt{XX^\dagger}$ denotes the trace norm of X. To facilitate calculation, we usually remove the constant factor of 1/2 and define it as $N(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1$.

According to the above definition, the negativity of a bipartite pure state $|\psi\rangle_{AB}$ is given by:

$$N(|\psi\rangle_{AB}) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} = (Tr\sqrt{\rho_A})^2 - 1,$$

where λ_i are the eigenvalues of $\rho_A = Tr_B|\psi\rangle_{AB}\langle\psi|$ [28].

For a bipartite mixed state ρ_{AB}, there is another negativity, a convex-roof extended negativity (CREN) [21], defined as:

$$N_c(\rho_{AB}) = \min\sum_i p_i N(|\psi_i\rangle_{AB}), \tag{22}$$

where the minimum value takes all possible pure state decompositions $\{p_i, |\psi_i\rangle_{AB}\}$ of ρ_{AB}. The definition of CREN is obviously different from $N(\rho_{AB})$, and it can perfectly discriminate positive transposed bound entangled states and separable states in any bipartite quantum system [46,47].

Now we need to use a relationship between the negativity and the concurrence: for any bipartite state with Schmidt rank 2, the negativity is equivalent to the concurrence [21]. Let us first consider any bipartite pure state with Schmidt rank 2, $|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$; we can easily find that $N(|\psi\rangle_{AB}) = \||\psi\rangle\langle\psi|^{T_B}\| - 1 = 2\sqrt{\lambda_0\lambda_1} = \sqrt{2(1 - Tr\rho_A^2)} = C(|\psi\rangle_{AB})$. Consequently, for any two-qubit mixed state $\rho_{AB} = \sum p_i|\psi_i\rangle_{AB}\langle\psi_i|$, one has:

$$N_c(\rho_{AB}) = \min\sum_i p_i N(|\psi_i\rangle_{AB}) = \min\sum_i p_i C(|\psi_i\rangle_{AB}) = C(\rho_{AB}). \tag{23}$$

Consider an N-qubit state $\rho_{AB_1\cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, denoting $N_{c\ A|B_1B_2\cdots B_{N-1}} = N_c(\rho_{A|B_1\cdots B_{N-1}})$ and $N_{c\ AB_i} = N_c(\rho_{AB_i})$ for convenience. If $\alpha \geq 2$, $N_{c\ AB_i} \geq N_{c\ A|B_{i+1}\cdots B_{N-1}}$, $1 \leq i \leq m$, and $N_{c\ AB_j} \leq N_{c\ A|B_{j+1}\cdots B_{N-1}}$ for $m+1 \leq j \leq N-2$, $1 \leq m \leq N-3$, $N \geq 4$, then [24]:

$$\begin{aligned} N^\alpha_{c\ A|B_1B_2\cdots B_{N-1}} \geq\ & N^\alpha_{c\ AB_1} + \left(2^{\frac{\alpha}{2}} - 1\right)N^\alpha_{c\ AB_2} + \cdots + \left(2^{\frac{\alpha}{2}} - 1\right)^{m-1}N^\alpha_{c\ AB_m} \\ & + \left(2^{\frac{\alpha}{2}} - 1\right)^{m+1}\left(N^\alpha_{c\ AB_{m+1}} + \cdots + N^\alpha_{c\ AB_{N-2}}\right) + \left(2^{\frac{\alpha}{2}} - 1\right)^{m}N^\alpha_{c\ AB_{N-1}}, \end{aligned} \tag{24}$$

and if $N_{c\ AB_i} \geq N_{c\ A|B_{i+1}\cdots N-1}$, $i = 1, 2, \cdots, N-2$ [33]:

$$\begin{aligned} N^\alpha_{c\ A|B_1B_2\cdots B_{N-1}} \geq\ & N^\alpha_{c\ AB_1} + \left(2^{\frac{\alpha}{2}} - 1\right)N^\alpha_{c\ AB_2} + \cdots \\ & + \left(2^{\frac{\alpha}{2}} - 1\right)^{N-3}N^\alpha_{c\ AB_{N-2}} + \left(2^{\frac{\alpha}{2}} - 1\right)^{N-2}N^\alpha_{c\ AB_{N-1}}. \end{aligned} \tag{25}$$

Obviously, the above monogamy relation (24) and (25) depends on inequality (12); then, with a similar consideration to concurrence, we obtain the following result.

Theorem 3. *Let $\rho_{AB_0B_1\cdots B_{N-1}}$ be an N+1 qubit state, if $N_{c\ AB_i} \geq N_{c\ A|B_{i+1}\cdots B_{N-1}}$, $0 \leq m$, and $N_{c\ AB_j} \leq N_{c\ A|B_{j+1}\cdots B_{N-1}}$, $m+1 \leq j \leq N-2$, $0 \leq m \leq N-3$, $N \geq 3$; then we have:*

$$N^{\alpha}_{c\ A|B_0\cdots B_{N-1}} \geq \sum_{i=0}^{m}\left(\prod_{j=0}^{i}K_j\right)N^{\alpha}_{c\ AB_i} + \left(\prod_{i=1}^{m+1}K_i\right)\left(\sum_{j=m+1}^{N-2}L_jN^{\alpha}_{c\ AB_j} + N^{\alpha}_{c\ AB_{N-1}}\right), \tag{26}$$

for all $\alpha \geq 4$, *where* $K_0 = 1$, $K_{i+1} = 2^{\frac{\alpha}{2}} - \frac{N^{\alpha}_{c\ A|B_{i+1}\cdots B_{N-1}}}{N^{\alpha}_{c\ AB_i}}$ *for* $i = 0,1,2,\cdots,m$, $L_j = 2^{\frac{\alpha}{2}} - \frac{N^{\alpha}_{c\ AB_j}}{N^{\alpha}_{c\ A|B_{j+1}\cdots B_{N-1}}}$ *for* $j = m+1,\cdots,N-2$.

If $N_{c\ AB_i} \geq N_{c\ A|B_{i+1}\cdots B_{N-1}}$ for all $i = 0,1,2,\cdots,N-2$ in Theorem 3, then we have the following conclusion.

Theorem 4. *Let* $\rho_{AB_0B_1\cdots B_{N-1}}$ *be an N+1 qubit state, if* $N_{c\ AB_i} \geq N_{c\ A|B_{i+1}\cdots B_{N-1}}$ *for all* $i = 0,1,2,\cdots,N-2$*; then:*

$$N^{\alpha}_{c\ A|B_0\cdots B_{N-1}} \geq \sum_{i=0}^{N-2}\left(\prod_{j=0}^{i}K_j\right)N^{\alpha}_{c\ AB_i} + \left(\prod_{i=0}^{N-2}K_i\right)N^{\alpha}_{c\ AB_{N-1}} \tag{27}$$

for all $\alpha \geq 4$, *where* $K_0 = 1$ *and* $K_{i+1} = 2^{\frac{\beta}{2}} - \frac{N^{\alpha}_{c\ A|B_{i+1}\cdots B_{N-1}}}{N^{\alpha}_{c\ AB_i}}$ *for* $i = 0,1,2,\cdots,N-2$.

Next we show some states that satisfy Theorems 3 and 4.

Example 3. *For the N-qubit GHZ state (20) in Example 1:*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle), \tag{28}$$

it is easy to calculate that $N_{c\ A|B_1\cdots B_{N-1}} = 1$ *and* $N_{c\ AB_i} = 0$ *for all* $i = 2,...,N-1$*, and then our results in Theorem 3 and Theorem 4 obviously hold.*

Example 4. *Let* $|\psi\rangle$ *be the three-qubit state in (21); we can easily calculate that* $N_{c\ A|BC} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $N_{c\ AB} = 2\lambda_0\lambda_2$ *and* $N_{c\ AC} = 2\lambda_0\lambda_3$*. Set* $\lambda_0 = \lambda_1 = \lambda_2 = \frac{\sqrt{2}}{3}$ *and* $\lambda_3 = \lambda_4 = \frac{1}{\sqrt{6}}$*. One has* $N_{c\ A|BC} = \frac{2\sqrt{10}}{9}$ *and* $N_{c\ AB} = \frac{4}{9}, N_{c\ AC} = \frac{2\sqrt{3}}{9}$*. Hence,* $N^{\alpha}_{c\ A|BC} = (\frac{2\sqrt{10}}{9})^{\alpha}$ *and* $N^{\alpha}_{c\ AB} + \left[2^{\frac{\alpha}{2}} - (\frac{N^2_{c\ AC}}{N^2_{c\ AB}})^{\frac{\alpha}{2}}\right]N^{\alpha}_{c\ AC} = (\frac{4}{9})^{\alpha} + (\frac{2\sqrt{6}}{9})^{\alpha} - (\frac{1}{3})^{\alpha}$*. Figure 2 shows that our inequality (23) is tighter than* $N^{\alpha}_{c\ AB} + \left(2^{\frac{\alpha}{2}} - 1\right)N^{\alpha}_{c\ AC} = (\frac{4}{9})^{\alpha} + (\frac{2\sqrt{6}}{9})^{\alpha} - (\frac{1}{27})^{\frac{\alpha}{2}}$ *in [24] for* $\alpha \geq 4$*.*

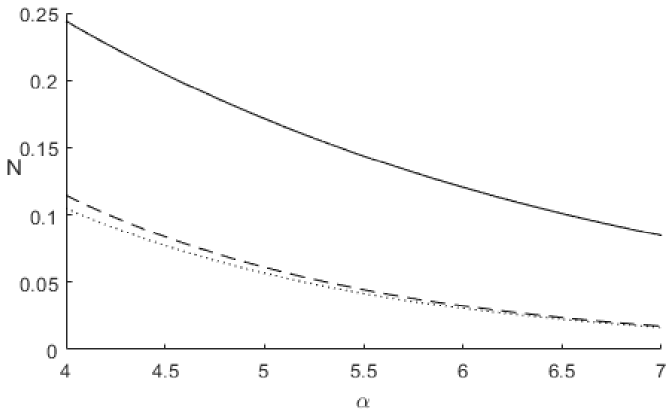

Figure 2. Axis N represents the negativity of $|\psi\rangle_{ABC}$ and its lower bounds as functions of α. The solid line shows the negativity of (21); the dashed line and the dotted line show the lower bounds of ours and [24], respectively.

4. Conclusions

In multipartite quantum systems, the monogamous entanglement relationship characterizing the quantum entanglement distribution is one of the hot issues of quantum information theory research in recent years. For example, entangled monogamy can limit the possible association between authorized users and eavesdroppers, thus tightening the security limit of quantum cryptography.

In this paper, we first proved the mathematical inequality $(1+t)^x \geq 1 + (2^x - t^x)t^x$ for $t \in [0,1]$ and $x \in [2,\infty)$, and then using it we presented monogamy relations related to the α power of the concurrence C and the negativity N_c. We also presented that they are tighter than the existing ones. The tighter monogamy relationship in this paper gives a more detailed entanglement distribution, which can enhance the research on the security of quantum cryptography in quantum key distribution. Our approach also promotes the study of monogamy related to other quantum correlations. However, our monogamy relation requires that the power must be greater than 4, and whether the monogamy relation holds for $2 < \alpha \leq 4$ needs further study.

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