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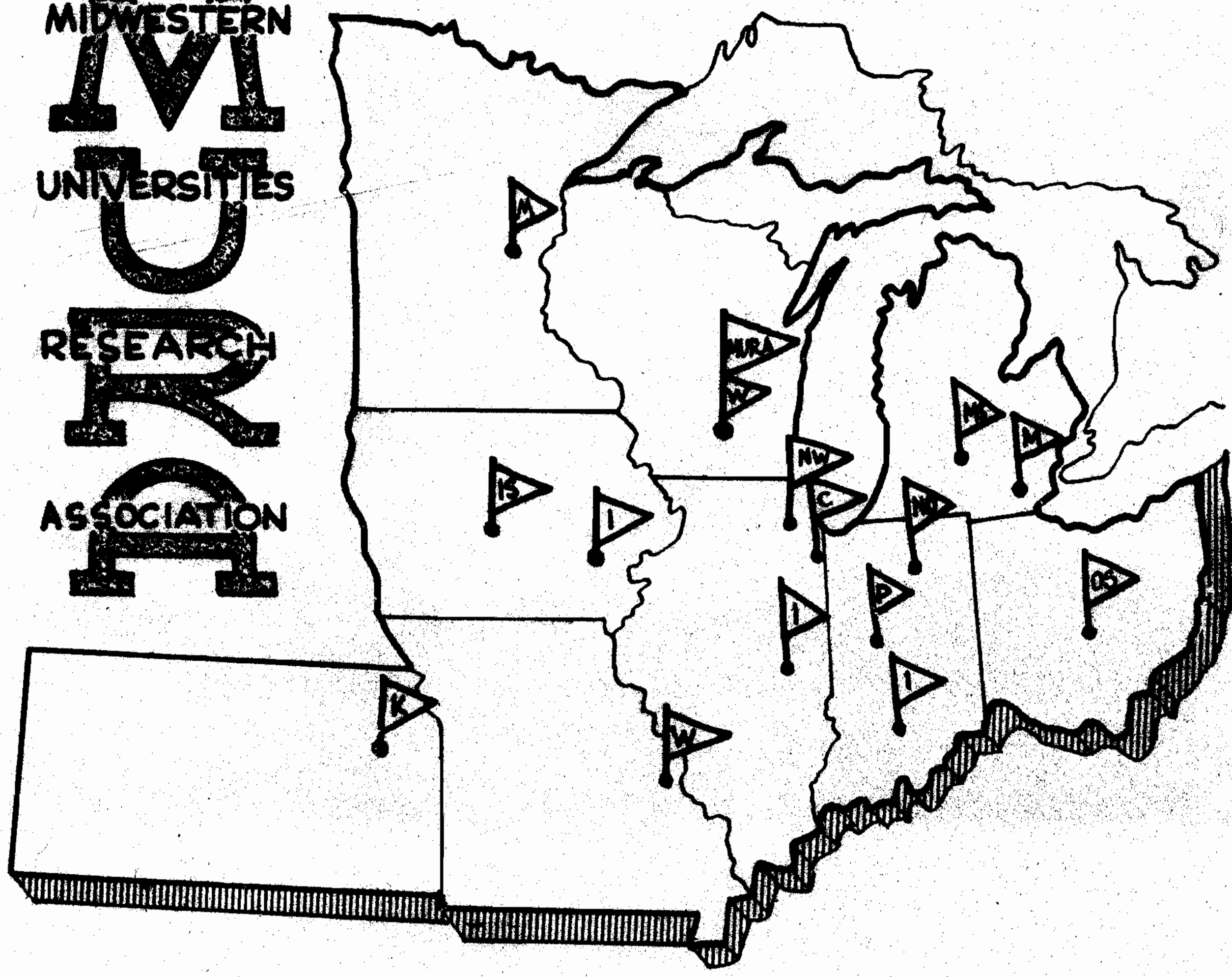


Damping of Oscillations - Requisite Energy Tolerance
At Injection - Coherent Radiation

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REPORT NUMBER 23

MURA NOTES May-June, 1954

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The following notes deal (i) with the possible supplementary damping of oscillations in a synchrotron, (ii) with the energy tolerance required at injection, and (iii) with certain aspects of coherent radiation. These provisional notes do not represent a complete analysis of these subjects, but were begun in preparation for the May 22-23 meeting of the technical group and to a small extent reflect the discussion at that meeting.

## I. DAMPING OF OSCILLATIONS

### 1. Introduction:

At recent meetings of the technical group attention has been given to the possibility of damping synchrotron oscillations, through the use of a radio-frequency E.M.F. per turn which varies across the radial aperture of the accelerator.[1] This possibility has also received attention by the Princeton group[2] and in an early Berkeley report[3] recently called to the writer's attention. Since it appears from the analysis that one may expect an undamping of betatron oscillations if the synchrotron oscillations are damped in this way, the arguments are outlined hereunder (i) as a review, (ii) as a challenge to devise (if possible) an acceptable damping mechanism, (iii) as an indication of the tolerances required in cavity construction, and (iv) with the thought that in some accelerators some additional damping of one of the oscillations may be desirable, even at the expense of a certain undamping of the other.

### 2. The Phase-Equation:

The equation governing the phase oscillations may be obtained in a manner suggested by the work of Twiss and Frank,[4] recently reviewed by Livingood,[5] by writing the equations for a general particle and for the synchronous particle as follows:[6]

We consider the E.M.F. per turn to vary in a substantially linear manner across the useful aperture of the accelerator

$$\text{E.M.F.} = V_0 \left(1 - \sigma n \frac{\Delta r}{r_s}\right) \sin\phi,$$

where $\Delta r = r - r_s$.

Introducing the vector potential of the guide field $[rA = (\text{flux})_z / 2\pi]$,

$$\frac{d}{dt}(pr + erA) = \frac{eV_0}{2\pi}\left(1 - \sigma n \frac{\Delta r}{r_s}\right)\sin\phi$$

$$\frac{d}{dt}(p_s r_s + e r_s A_s) = \frac{eV_0}{2\pi} \sin\phi_s .$$

By subtraction:

$$\frac{d}{dt}\left(p_s r_s \frac{\Delta p}{p_s}\right) = \frac{eV_o}{2\pi}(\sin\phi - \sin\phi_s) - \sigma n\left(\frac{eV_o}{2\pi}\right)\frac{\Delta r}{r_s}\sin\phi$$

$$\frac{d}{dt}\left(\frac{E_s}{-\Upsilon}\dot\phi\right) = -\omega_o^2 h\frac{eV_o}{2\pi}(\sin\phi - \sin\phi_s) + \omega_o^2 h\frac{eV_o}{2\pi}\sigma n\frac{\Delta r}{r_s}\sin\phi,$$

where $\omega_o$ represents $\frac{2\pi}{\text{period of revolution}}$ for a hypothetical particle moving on the radius $r_s$ with the speed of light.

In the traversal of several cavities (or of a single cavity several times), we write in the usual notation,

$$\frac{\Delta r}{r_s} \cong \frac{[\Delta r]_{AV}}{r_s} = a\frac{\Delta p}{p_s} = \frac{a}{\Upsilon\omega_s h}\dot\phi = \frac{a}{\beta\Upsilon h\omega_o}\dot\phi$$

and obtain:

$$\frac{d}{dt}\left(\frac{E_s}{-\Upsilon}\dot\phi\right) = -\omega_o^2 h\frac{eV_o}{2\pi}(\sin\phi - \sin\phi_s) + \frac{a\sigma n}{\beta\Upsilon}\omega_o\frac{eV_o}{2\pi}\dot\phi\sin\phi,$$

which is of the form:

$$\frac{d}{dt}(M\dot\phi) = -A(\sin\phi - \sin\phi_s) + a\dot\phi\sin\phi,$$

with $M = \frac{E_s}{-\Upsilon}$, $A = \frac{\omega_o^2 h e V_o}{2\pi}$, $a = \frac{a\sigma n}{\beta\Upsilon}\omega_o\frac{eV_o}{2\pi}$.

This result appears concordant with the non-relativistic eq. (15) of ref. 3 for a conventional synchrotron in which $P_s$ increases linearly with time, where

$$h = 1 \qquad \sigma n = -6 \qquad \omega_o = \frac{c}{r_s\left(1 + \frac{\Sigma L}{2\pi r_s}\right)}$$

$$a = \frac{1}{1-n}$$

$$\Upsilon = \frac{1}{1-n}\,\frac{1}{1 + \frac{\Sigma L}{2\pi r_s}} - 1 \qquad \phi \cong \phi_s$$

$$E_s/E_s \cong 0 \quad \text{and}$$

$$\frac{1}{E_s}\frac{\omega_o}{\beta}\frac{eV_o}{2\pi}\sin\phi_s = \frac{1}{E_s}\frac{\omega}{\beta^2}\frac{eV_o}{2\pi}\sin\phi_s = \frac{(dE/dt)_s}{\beta^2 E_s} = \frac{(dp/dt)_s}{p_s}.$$

The conclusions of ref. 3 concerning the damping of the resultant motion will thus be found to be consistent with ours for this case, as will also the results stated for the extreme relativistic situation.

## 3. Solution of the Phase Equation:

To facilitate solution of the phase equation we replace $\sin\phi$ by $\sin\phi_s$ in the damping term and, rather then proceed directly with the differential equation, note that the motion may be derived from a Lagrangian

$$L = (M/2)\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right]\dot{\phi}^2 + A\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right]\left[U(\phi) - U(\phi_s)\right],$$

where $U(\phi) = \cos\phi + \phi \sin\phi_s$.

The motion accordingly may be characterized by the Hamiltonian

$$H = \frac{1}{2M}\left[\exp\left(\sin\phi_s \int \frac{a}{M} dt'\right)\right]p^2 - A\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right]\left[U(\phi) - U(\phi_s)\right]$$

$$\cong \frac{1}{2M}\left[\exp\left(\sin\phi_s \int \frac{a}{M} dt'\right)\right]p^2 + \frac{A}{2}\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right]\cos\phi_s (\phi - \phi_s)^2,$$

with the canonically conjugate momentum $p = M\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right]\dot{\phi}$.

In adiabatic changes of the roughly periodic motion, the invariance of the action integral insures that $p_{max} \cdot (\phi - \phi_s)_{max}$ remains constant: thus $\Omega M\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right](\phi - \phi_s)^2_{max}$

$$= (AM\cos\phi_s)^{1/2}\left[\exp\left(-\sin\phi_s \int \frac{a}{M} dt'\right)\right](\phi - \phi_s)^2_{max}$$

remains constant, or

$$(\phi - \phi_s)_{max} \propto (AM\cos\phi_s)^{-1/4} \exp\left[(1/2)(\sin\phi_s)\int \frac{a}{M} dt'\right].$$

The factor $(AM\cos\phi_s)^{-1/4}$ represents the customary damping of synchrotron phase oscillations and leads to the familiar $E^{-1/4}$ damping at energies such that $\gamma$ is substantially constant. It is of interest, therefore, to estimate the exponential factor which is introduced by the variation of E.M.F. with radius.

$$\frac{a}{M}\sin\phi_s = -\frac{a\sigma n}{\beta E_s}\omega_0 \frac{eV_o}{2\pi}\sin\phi_s = -a\sigma n \frac{(dp/dt)_s}{p_s},$$

$$(1/2)(\sin\phi_s)\int \frac{a}{M} dt' = -(a\sigma n/2)\ln\frac{p_f}{p_i},$$

and $\exp\left[(1/2)(\sin\phi_s)\int \frac{a}{M} dt'\right] = (p_f/p_i)^{-a\sigma n/2}$

$$= (p_f/p_i)^{-\frac{4.81}{2}\sigma}$$

$$= (p_f/p_i)^{-2.4\sigma}$$

in agreement with the results of ref. 3 when $a = \frac{1}{1-n}$, $\sigma n = -\epsilon$, and $p \propto t$ for an alternate-gradient synchrotron operated near the center of the stability diagram.

In a typical example an increase of momentum from that corresponding to an injection energy of 50 Mev (pc = 0.31 Gev) to an energy in the neighborhood of a transition energy such that pc = 9.70 Gev then leads to the additional damping factor

$$(9.70/0.31)^{-2.4\sigma} = (31.2)^{-2.4\sigma},$$

or approximately 0.18 for $\sigma = 14.8$. It is noted that the sense of the damping is unaffected by flipping the phase at the transition energy.

It appears from the foregoing that, from the standpoint of the synchrotron oscillations, this damping mechanism would be desirable in reducing the difficulties associated with traversal of the transition energy, since the increase of amplitude resulting from an inexactly-timed change of phase would start from a lower level of amplitude. It is necessary, however, to consider the effect of this mechanism on the radial betatron oscillations.

4. Associated build-up of Betatron Oscillations:

It appears that consideration should be given to two ways in which the mechanism suggested may influence the magnitude of the radial betatron oscillations. The first of these[7] involves the $e[\bar{v} \times \bar{B}]$ forces arising from the magnetic flux-leakage within the cavity, and presumably a similar radial impulse would be expected in case a resonator with an oblique gap were employed. The second effect[3] is that resulting from the abrupt change in the equilibrium orbit at each traversal of the acceleration cavity. We proceed to consider these effects in turn.

5. Evaluation of Impulse from Leakage Flux-Density:

Writing the E.M.F. per cavity as $V_1\left[1 - \sigma n \frac{\Delta r}{r_s}\right] \cdot \sin(h\omega_s t)$,

we have
$$V_1\left[1 - \sigma n \frac{\Delta r}{r_s}\right] \cdot \sin(h\omega_s t) = -\iint \dot{B}\, dS$$
$$= -\int_0^r \int r \dot{B}(r,\theta)\, dr\, d\theta$$

$$\frac{V_1 \sigma n}{r_s} \sin(h\omega_s t) = \int r \dot{B}(r,\theta)\, d\theta$$
$$\frac{V_1}{h\omega_s} \frac{\sigma n}{r_s} \cos\phi = -\int r\, B(r,\theta)\, d\theta$$

$$= -\int B(r,\theta)\, ds,$$ integrated through the cavity.

It is realized that the R.F. electric and magnetic fields must constitute a self-consistent solution to Maxwell's equations and that difficulties could in fact arise if one attempted to achieve an E.M.F. which over an extended region were strictly independent of the path. The statements made herein appear to be satisfactory, however, for an E.M.F. of the form assumed, and considerations based on curl $\bar{H} = \dot{\bar{D}}$ suggest that neglecting the $\mu H$ so implied by a spatially constant E.M.F. affects the amplitude $\Delta r \cong \lambda \Delta r'$ by an amount negligible (5 to 30 percent in a typical case) in comparison with the term $-a\left(\frac{\Delta\rho}{\rho_s}\right)\cos\psi$ considered later.

We thus obtain for the impulse
$$\Delta p_r = \int F\, dt = e\int B \frac{ds}{dt}\, dt$$
$$= e\int B\, ds = -\frac{eV_1}{h\omega_s} \frac{\sigma n}{r_s} \cos\phi$$

$$\Delta\left(\frac{dr}{dt}\right) = -\frac{(eV_1)\sigma n}{mh\omega_s r_s}\cos\phi$$
$$= -\frac{eV_1}{p}\,\frac{\sigma n}{h}\cos\phi$$

$$\Delta\left(\frac{d(r/r_s)}{d\theta}\right) = \frac{1}{v_s}\Delta\left(\frac{dr}{dt}\right) = -\frac{eV_1}{pv_s}\,\frac{\sigma n}{h}\cos\phi \doteq -\frac{eV_1}{\beta^2 E_s}\,\frac{\sigma n}{h}\cos\phi_s$$
$$= -\left(\frac{\Delta E_1}{\beta^2 E}\right)_s \frac{\sigma n}{h}\,\mathrm{ctn}\,\phi_s = -\left(\frac{dp_1}{p}\right)_s \mathrm{ctn}\,\phi_s$$

If this impulse were the only mechanism affecting the betatron oscillations, it would be reasonable to consider the use of cavities in pairs, spaced by a half-wavelength of the radial betatron oscillations. The maximum extra relative displacement which would then be expected to arise in this way <u>between</u> the members of a <u>cavity-pair</u> would not exceed

$$2\left(\frac{\lambda}{r_s}\right)\left|\Delta\left(\frac{d(r/r_s)}{d\theta}\right)\right| = 2\,\frac{\lambda}{r_s}\left(\frac{dp_1}{p}\right)_s \frac{\sigma n}{h}\,\mathrm{ctn}\,\phi_s\,.$$

(The factor 2 is that estimated by Courant, Livingston, and Snyder[8] to allow for the non-sinusoidal character of the oscillations in an A.G.S.)

With $h_c$ cavities in all, each excited to a similar R.F. level, $\left(\frac{dp_1}{p}\right)_s$ or $\left(\frac{\Delta E_1}{\beta^2 E}\right)_s$ equals

$$\frac{2\pi r_s\left[1+\frac{\Sigma L}{2\pi r_s}\right]}{h_c\beta^2 E_s}\left(\frac{dp}{dt}\right)_s \quad \text{or substantially} \quad \frac{2\pi r_s\left[1+\frac{\Sigma L}{2\pi r_s}\right]}{h_c\beta^2 c\,(\text{acc. time})}\;\frac{E_f}{E_s};$$

in addition we may take $\lambda/r_s = 2/\sqrt{n}$ radians and obtain

$$n\left[\frac{\Delta r}{r_s}\Big|_{max}\right] \leq \frac{4(n)^{3/2}}{hh_c}\,\sigma\,\frac{2\pi r_s\left[1+\frac{\Sigma L}{2\pi r_s}\right]\mathrm{ctn}\,\phi_s}{c\cdot\text{acceleration time}}\;\frac{E_f}{\beta^2 E_s} \quad \text{from a single cavity.}$$

Typically, with $n = 400$, $h_c = h = 16$, acceleration from an injection energy of 50 Mev (kinetic) to a final energy of 25 Gev, and a rise-time of one second,

$$n\left[\frac{\Delta r}{r_s}\Big|_{max}\right] \leq \frac{4\times 8000}{16\times 16}\;\frac{2\pi\times 86.50\times 1.3\times\sqrt{3}}{3\times 10^8\times 1}\;\frac{26\times 10^9}{97\times 10^6}\,\sigma$$
$$= 0.14\,\sigma \quad \text{at injection,}$$

which is considerably less than unity (for the harmonic number assumed) with any reasonable choice of $\sigma$.

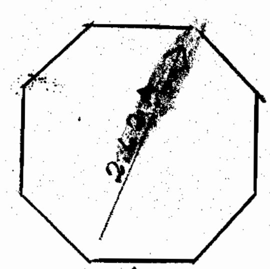
The impulse from the leakage field of such cavities also will imply an accumulative displacement in the case of particles for which the betatron wavelength is not exactly twice the separation of a cavity-pair. For estimating this effect we presume that through careful control of the magnet performance $n$ is not permitted to wander more than from the center of a small stability diamond half-way to the edge. We accordingly consider $|\delta K| = \frac{\pi/4}{N_s/2} = \frac{\pi}{8\sqrt{n}}$, or $|\delta n| = 0.2\sqrt{n}$, (where k enters as $\exp(\pm i\kappa)$ in the characteristic solutions for traversal of a sector-pair).[9] Since the increase of relative amplitude from traversal of a sector-pair spaced by $\frac{\lambda \pm \Delta\lambda}{2}$ will not exceed $\frac{|\Delta\lambda|}{r_s} \Delta \left(\frac{d(r/r_s)}{d\theta}\right)$ and typically $|\Delta\lambda| = \frac{\lambda}{\kappa}|\delta\kappa| = \frac{32 r_s}{N_s}|\delta\kappa| = \frac{16\pi r_s}{N_s 2} = \frac{\pi r_s}{n}$,

$$n\left[\frac{\Delta r}{r_s}\Big|_{max}\right] \leq \pi A\left(\frac{d(r/r_s)}{d\theta}\right) = \pi\left(\frac{\Delta E_1}{\beta^2 E}\right)_s \frac{\sigma n}{h} \operatorname{ctn}\phi_s$$

$$= \frac{\pi n}{h h_c}\,\sigma\,\frac{2\pi r_s\left[1 + \frac{\Sigma L}{2\pi r_s}\right]\operatorname{ctn}\phi_s}{c \cdot \text{acceleration time}}\;\frac{E_f}{\beta^2 E_s}$$

$$= \frac{\pi \times 400}{16 \times 16}\;\frac{2\pi \times 86.50 \times 1.3 \times \sqrt{3}}{3 \times 10^8 \times 1}\;\frac{26 \times 10^9}{97 \times 10^6}\,\sigma$$

$$= 0.0054\,\sigma.$$



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The increase of amplitude per revolution would, at the worst, be $\frac{h_c}{2}$ times the above result and after several revolutions might be about 2.62 times larger still, if we stay away from resonances by no less than the amount suggested. We accordingly write

$$n\left[\frac{\Delta r}{r_s}\Big|_{max}\right]_{overall} \leq \frac{4.1\,n}{h}\,\sigma\,\frac{2\pi r_s\left[1 + \frac{\Sigma L}{2\pi r_s}\right]\operatorname{ctn}\phi_s}{c \cdot \text{acceleration time}}\;\frac{E_f}{\beta^2 E_s}$$

$$= 0.11\,\sigma, \quad \text{for the example considered}$$
$$(h = 16,\ 50\ \text{Mev injection, etc.}).$$

With $\sigma$ somewhat less than unity, this result does not appear to be excessive in a magnet whose radial semi-aperture is comparable with $\pm r_s/n$

## 6. The Growth of Oscillations from step-wise Shifts of Equilibrium Orbits.

It has been pointed out in the Berkeley report[3] to which reference has been made earlier that in traversal of a R.F. cavity the instantaneous equilibrium orbit is suddenly displaced by an amount

$$\left(\frac{\Delta r}{r_s}\right)_{equib.} = a\frac{\Delta p_1}{p} = a\left(\frac{\Delta p_1}{p}\right)_s\left(1 - \sigma n\frac{\Delta r}{r_s}\right).$$

With this displacement there is associated an increase of the square of the relative amplitude $x_m$ of the betatron oscillations which, for sinusoids, would amount to

$$\Delta(x_m^2) \geq -2ax_m \cos\psi\,(1-\sigma n x_m \cos\psi)\left(\frac{\Delta p_1}{p}\right)_s ,$$

where $\psi$ is the phase of the oscillation at the time of traversal. If we take over this expression as roughly indicative of the behaviour in an A.G.S., we note that

$$\langle \Delta x_m^2 \rangle_{Av} = a\sigma n \left(\frac{\Delta p_1}{p}\right)_s x_m^2$$

and hence there results a growth of amplitude on this account:

$$x_m \propto p^{a\sigma n/2} \simeq p^{2.4\sigma}$$

The growth-factor so found for the betatron oscillations is thus as rapid as the attenuation factor found in section 3 for the synchrotron oscillations.

The effect just described appears definitely to detract from the utility of a system in which the E.M.F. decreases as one moves radially outward across the aperture. In some circumstances, however, the radial betatron oscillations may be of somewhat secondary importance to the synchrotron oscillations -- in such a situation consideration might be given to the use of cavities for which $\sigma$ is such that the effect in question is just sufficient to cancel the customary $1/\sqrt{p}$ adiabatic damping of the betatron oscillations:

$$\sigma < 1/4.8 \simeq 0.2$$

As has been remarked at the May 22-23 meeting of the mid-west technical group, however, a more adequate treatment of these effects would consider the betatron and synchrotron motions together in a general unified analysis.

It may be noted that Kerst has pointed out[10] that a <u>betatron</u> inherently involves an induced E.M.F. which <u>increases</u> with radius. Although "gaps" may in a sense be present, due to the shielding effect of the conducting sections of the vacuum chamber wall, phase stability is not involved and the effect on the betatron oscillations may be beneficial.

## 7. <u>Possible Statistical Growth of Induced Betatron-Oscillation Amplitude</u>:

In section 6 it was indicated that, when $\sigma = 0$, the betatron oscillation amplitude changes upon traversal of a cavity by

$$\Delta(x_m^2) \cong -2ax_m \left(\frac{\Delta p_1}{p}\right)_s \cos\psi$$

$$\text{or} \quad \Delta x_m \cong -a\left(\frac{\Delta p_1}{p}\right)_s \cos\psi .$$

Although as previously stated, the values of cos $\psi$ may be presumed to average to zero, there could conceivably be random variations of n such that the values of $\psi$ are distributed in a substantially random way statistically. In such a case we have a situation similar to the projection of a two-dimensional "ramdom walk" problem and may write for $\nu$ cavity traversals

$$\frac{d\langle(\Delta x_m)^2\rangle}{d\nu} = \frac{1}{2}\left[a\left(\frac{\Delta p_1}{p}\right)_s\right]^2 = \frac{a^2}{2}\left[\left(\frac{\Delta E_1}{\beta^2 E}\right)_s\right]^2 = \frac{a^2(\Delta E_1)^2}{2} \frac{E^2}{(E^2-E_o^2)^2}$$

$$\frac{d\langle(\Delta x_m)^2\rangle}{dE} = \frac{a^2\Delta E_1}{2} \frac{E^2}{(E^2-E_o^2)^2} .$$

Integrating,

$$\langle(\Delta x_m)^2\rangle = \frac{a^2\Delta E_1}{4E_o}\left[\mathrm{ctnh}^{-1}\frac{E_i}{E_o} - \mathrm{ctnh}^{-1}\frac{E_f}{E_o} + \frac{E_i/E_o}{(E_i/E_o)^2-1} - \frac{E_f/E_o}{(E_f/E_o)^2-1}\right.$$

$$\cong \frac{a^2}{8}\frac{\Delta E_1}{E_i-E_o} \quad (\text{for } E_i-E_o \ll E_o,\ E_f \gg E_o)$$

$$= \frac{a^2}{8h_c}\frac{2\pi r_s\left[1+\frac{\Sigma L}{2\pi r_s}\right]}{c\cdot(\text{acc. time})}\frac{E_f}{E_i-E_o}$$

$$= \left(\frac{4.81}{n}\right)^2 \frac{1}{8h_c}\frac{2\pi\times 86.50\times 1.3}{3\times10^8\times 1}\frac{26\times10^9}{50\times10^6}$$

$$= \left(\frac{4.81}{n}\right)^2 \frac{1}{h_c} 1.53\times10^{-4}$$

$$\langle(\Delta x_m)^2\rangle^{1/2} = \frac{4.81}{n h_c^{1/2}} 1.24\times10^{-2}$$

$$= \frac{0.06}{n h_c^{1/2}}$$

$$n\langle(\Delta x_m)^2\rangle^{1/2} = \frac{0.06}{h_c^{1/2}} \cong 0.015,$$

In the case of the electron synchrotron described in an earlier report,[11] we similarly write

$$\langle(\Delta x_m)^2\rangle \cong \frac{a^2\Delta E_1}{2E_i} \quad (E_f \gg E_i \gg E_o)$$

$$= \left(\frac{4.81}{n}\right)^2 \frac{1}{2h_c}\frac{0.892\times10^6}{50\times10^6}$$

$$= \left(\frac{4.81}{n}\right)^2 \frac{1}{h_c} 0.892\times10^{-2}$$

$$\langle(\Delta x_m)^2\rangle^{1/2} = \frac{4.81\times 0.094}{n h_c^{1/2}}$$

$$= \frac{0.455}{n h_c^{1/2}}$$

$$n\langle(\Delta x_m)^2\rangle^{1/2} = \frac{0.455}{h_c^{1/2}} = 0.08 \quad (\text{since } h_c = 32),$$

8. Operation with a Single Cavity:

The effects considered in the preceding sections do not appear to preclude operation of a high-energy proton synchrotron with a single cavity, since even with random phases we find from the results of page 8 that $n\langle(\Delta x_m)^2\rangle^{1/2} \cong 0.06$ when $h_c = 1$; due to the non-sinusoidal character of the oscillations, we might consider that this result could be as great as a little over twice the value found there -- say 0.14.

In a single traversal of such a cavity

$$\begin{aligned} |n \cdot \Delta x_m| &= 4.81 \left.\frac{\Delta E/v}{p}\right|_s \cos \psi \\ &\leq 4.81 \left.\frac{\Delta E}{\beta^2 E}\right|_s \\ &= 4.81 \frac{58.6 \times 10^3}{97.5 \times 10^6} \\ &= 0.0029 . \end{aligned}$$

Again, due to the non-sinusoidal character of the oscillations we may better write

$$n|\Delta x_m| \leq 0.007 .$$

## II. REQUISITE ENERGY TOLERANCE AT INJECTION

1. Motivation:

The question has been raised concerning the requisite energy tolerances at injection and whether there exists a disparity between the Linac requirements as specified at Brookhaven and those currently conceived in the mid-west group and elsewhere.

2. Acceptance into Stable Synchrotron Oscillations:

One approach to this problem has been given by K. Johnsen[12] in the CERN proton-synchrotron lectures. In this approach the requirement considered has been that the initial momentum spread shall be no greater than that acceptable into synchrotron phase oscillations. For the case of no frequency error Johnsen cites [cf. his eq. (3)] the result

$$\frac{\Delta p}{p} = \pm \frac{2}{\beta}\sqrt{\frac{e}{mc^2}\,\frac{r_m r_o \dot{B}_o [-\operatorname{ctn}\phi_o - (\phi_o - \pi/2)]}{h}}$$

$$= \pm \frac{2}{\beta}\sqrt{\frac{eV_o/2\pi}{mc^2}\,\frac{[-\cos\phi_o - (\phi_o - \pi/2)\sin\phi_o]}{h}} ,$$

which is consistent with eq. (16) of a report[6] by the present writer[13] if $|\gamma|$ in this latter equation is regarded as substantially unity.

For the similar accelerators considered in the CERN and MAC reports we list the following parameters and find

| | CERN (October 1953) | LJL(MAC)-3 | |
|---|---|---|---|
| Inj. Energy, Kinetic | 50 | 50 | Mev |
| " " Total | 0.99 | 0.99 | Gev |
| $\beta$ | 0.314 | 0.314 | |
| $\lvert\gamma\rvert$ | $\cong 1$ | 0.88 | |
| $eV_0/2\pi$ | $23.1 \times 10^3$ based on $\dot{B} = 12$ KG/sec | $18.7 \times 10^3$ | $\frac{\text{ev}}{\text{radian}}$ |
| $h$ | 38 | (38) | |
| $\Delta p/p$ | $\pm 1.80\times10^{-2}/h^{1/2} = \pm 0.29\times10^{-2}$ | $\pm 1.73 \times 10^{-2}/h^{1/2} = \pm 0.28 \times 10^{-2}$ | |
| $\frac{\Delta E}{E_{Kin}} = \frac{\beta^2}{1-\sqrt{1-\beta^2}}\left(\frac{\Delta p}{p}\right)$ | $\pm 0.56 \times 10^{-2}$ | $\pm 0.55 \times 10^{-2}$ | |

These results may be compared with what would then be a satisfactory expected performance of the linac, as reported by L. H. Johnston:[14]

$$E_{Kin} = 50 \pm 0.2 \text{ Mev}, \quad \text{or} \quad \Delta E/E_{Kin} = \pm 0.4 \times 10^{-2}.$$

The momentum spread tabulated above appears to constitute the basis of the CERN design specifications [p.6 of the CERN report [12]]. For the Brookhaven design a higher harmonic number may be under consideration -- the foregoing example with the harmonic number changed to 88 would lead to

$$\frac{\Delta p}{p} = \pm 0.19 \times 10^{-2}$$

$$\frac{\Delta E}{E_{Kin}} = \pm 0.37 \times 10^{-2}$$

Note added in proof: The Brookhaven Accelerator Development Division minutes (#57) of their March 16, 1954 meeting suggest the requirement $\pm 1/2\%$ in energy, $\pm 10^{-3}$ radian, and a width of 1/2 inch.

## 3. Avoidance of Resonances:

The change in effective $n$ due to momentum error should for safety be no greater than that which will displace the operation point from the center of a small diamond, bounded by $2\pi$- and $\pi$-resonances, half-way to the edge. The momentum spread which is tolerable on this acount has been estimated earlier[6] as $\pm 0.20/\sqrt{n}$ for operation near the center of the "necktie diagram" ($\sigma = \pi/2$), or $\pm 0.31/\sqrt{n}$ for a point situated on the diagonal but closer to the origin ($\sigma = 0.3\pi$). For a field index (n) in the neighborhood of 400, these considerations necessitate a tolerance of about $\pm 1$ per cent in momentum or $\pm 2$ percent in energy and are evidently less demanding than the requirements discussed previously cf. Fig. 9A, p. 111 of the CERN report[12] and the accompanying discussion by Adams (Sect. III-4, esp. p. 102).

4. <u>Clearance of Inflector Electrode</u>:

An additional and more severe limitation of the tolerable energy spread may arise if the beam is obliged to clear the electrode of an electrostatic inflector structure as it spirals inward during the injection interval. It should be noted that such an arrangement presents, possibly, serious difficulties in machines of the types presently under consideration, due to the very small pitch of the spiral in the presence of a linearly-rising magnetic field -- about 0.6 mm per turn. If it is intended to inject at the start of the injection interval particles whose trajectories have initially the scolloped appearance of the repetitive orbits illustrated by Courant, Livingston, and Snyder [ref. 8, Fig. 4], it may be noted that for some particles an <u>excess</u> momentum will require the superposition of a betatron oscillation (of an initially negative sign) of amplitude $(6.39/n)\,(\Delta p/p)$. In the course of a revolution, this betatron motion may come to represent a positive displacement at the inflector location. The radial error from this effect can then amount to

$$\frac{\Delta r}{r} = 2 \times \frac{6.39}{n}\,\frac{\Delta p}{p}$$

and would restrict the permissible excess momentum to

$$\frac{\Delta p}{p} = \frac{n}{2 \times 6.39}\,\frac{\Delta r}{r}.$$

With, for example, $\Delta r = 0.6$ mm $= 0.6 \times 10^{-3}$ M, $n = 400$, and $r = 86.50$ M as before, we thus find the comparatively severe limitation

$$\frac{\Delta p}{p} = \frac{400}{2 \times 6.39}\,\frac{0.6 \times 10^{-3}}{86.50} = 0.22 \times 10^{-3},$$

$$\text{or} \quad \frac{\Delta E}{E_{Kin.}} = 0.43 \times 10^{-3} = 0.043 \text{ percent}.$$

The discussion of this section presumably leads only to a rough estimate of the desired energy tolerance when an inflector is used -- to obtain a more definitive idea of the requirements it would seem appropriate to study in some detail the individual trajectories of representative particles injected with various amounts of momentum- and angular-error at various times within the injection interval.

## III. COHERENT RADIATION

1. <u>Introduction</u>:

Since there has been within the mid-west group some expression of interest in the construction of a circular electron accelerator, the attention of the Technical Group was directed to a recent report[15] by Nodvick and Saxon "On the Suppression of Coherent Radiation by Electrons in a Synchrotron". Since some general discussion of coherent radiation resulted at the May 22-23 meeting, the following comments are appended for whatever interest and reference value they may have. The mathematical notes are somewhat crude but may have the merit of affording a simple feel for the phenomenon.

2. Rough Formulation of Form-Factor for Coherent Radiation:

If the power radiated non-coherently is $P_0(\omega)\, d\omega$ per electron, the coherent radiation power from a small bunch of N electrons characterized by a symmetrical distribution density $\rho$ is

$$P = N^2 \int_0^\infty [F(\omega)]^2 P_0(\omega)\, d\omega,$$

where $$F(\omega) = \frac{\int \rho(x) \cos \frac{\omega x}{c}\, dx}{\int \rho(x)\, dx}.$$

3. List of Form-Factors:

We consider the following form-factors:

(i) For a uniform bunch of length L,

$$F = \frac{\sin \frac{L\omega}{2c}}{\frac{L\omega}{2c}}.$$

(ii) For a Guassian bunch, of width L between 1/e points:

$$F = \exp\left[-\left(\frac{L\omega}{4c}\right)^2\right].$$

(iii) For a group of particles moving with S.H.M. and with amplitudes uniformly distributed from 0 to L/2: In this case

$$\rho(x) \propto \int_x^{L/2} ds/(s^2 - x^2)^{1/2} = \cosh^{-1} \frac{L}{2x}$$ and

$$F = \frac{4}{\pi L} \int_0^{L/2} \cosh^{-1} \frac{L}{2x} \cos \frac{\omega x}{c}\, dx$$

$$= \frac{2}{\pi} \int_0^1 \cosh^{-1} \frac{1}{y} \cos \frac{L\omega y}{2c}\, dy.$$

4. Introduction of the Incoherent Spectral Distribution, $P_0(\omega)$:

If, for the low-frequency radiation important in the coherent effects, we write

$$P_0(\omega) = K\omega^{1/3},$$

the coherent radiation in the cases considered becomes

(i) For a uniform bunch,

$$P_{(1)} = \frac{2\pi}{\sqrt{3}\,\Gamma(5/3)} K N^2 \left(\frac{c}{L}\right)^{4/3}$$

$$= 4.02\, K N^2 \left(\frac{c}{L}\right)^{4/3}.$$

(ii) For a Gaussian bunch,

$$\mathcal{P}_{(2)} = 3\Gamma\left(\tfrac{5}{3}\right) K N^2 \left(\tfrac{c}{L}\right)^{4/3}$$
$$= 2.71\, K N^2 \left(\tfrac{c}{L}\right)^{4/3}$$

(iii) For S.H.M. oscillations with uniformly-distributed amplitudes the integration is more complex, but it appears safe to take

$$\mathcal{P}_{(3)} \cong 6\, K N^2 \left(\tfrac{c}{L}\right)^{4/3}.$$

The factor of 6 represents a (pessimistic) estimate of the integral

$$\left(\tfrac{2}{\pi}\right)^2 (2)^{4/3} \int_0^\infty z^{1/3}\, dz \left[\int_0^1 \cosh^{-1}\left(\tfrac{1}{y}\right) \cdot \cos zy \; dy\right]^2.$$

5. <u>Resultant Formulas for the Coherent Radiation:</u>

From eq. (II.20) of a paper by Schwinger[16] we find ($E >> E_0$)

$$\mathcal{P}_0(\omega)\, d\omega = \frac{(3)^{7/6}}{2\pi} \Gamma\left(\tfrac{5}{3}\right) \frac{e^2}{R} \left(\tfrac{R}{c}\right)^{1/3} \omega^{1/3}\, d\omega, \quad \text{or}$$

$$K = \frac{(3)^{7/6}}{2\pi} \Gamma\left(\tfrac{5}{3}\right) \frac{e^2}{R} \left(\tfrac{R}{c}\right)^{1/3}, \quad \text{e.s.u. being used.}$$

We then find

(i) for a uniform bunch

$$\mathcal{P}_{(1)} = 2\pi\, (3)^{2/3} \frac{N^2 e^2}{R} \left(\tfrac{R}{L}\right)^{4/3} \cdot \frac{c}{2\pi R}.$$

The E.M.F. loss per turn is, accordingly,

$$V_{(1)} = 2\pi\, (3)^{2/3} \frac{Ne}{R} \left(\tfrac{R}{L}\right)^{4/3} \quad \text{statvolts/turn}$$

$$= 600\pi\, (3)^{2/3} \frac{Ne}{R} \left(\tfrac{R}{L}\right)^{4/3} \quad \text{volts/turn}$$

$$= 3.92 \times 10^3 \frac{Ne}{R} \left(\tfrac{R}{L}\right)^{4/3} \quad \text{volts/turn,}$$

with $e$ still in e.s.u. and $R$ in cm.

This result may also be expressed as a "radiation resistance":

$$R_{rad.} = \frac{V}{(Ne)f} = \frac{(2\pi)^2 (3)^{2/3}}{c} \left(\tfrac{R}{L}\right)^{4/3} \text{statohms}$$

$$= 120\pi^2 \left(\tfrac{\sqrt{3}R}{L}\right)^{4/3} \text{ohms,}$$ in agreement with a result stated by Schwinger.[17]

(ii) For a Gaussian bunch

$$P(2) = 4\,(3)^{1/6}\left[\Gamma\left(\tfrac{2}{3}\right)\right]^2 \frac{N^2 e^2}{R}\left(\frac{R}{L}\right)^{4/3}\frac{c}{2\pi R}$$

in agreement with eq. (20) of a paper by Schiff.[18]

$$V(2) = 4\,(3)^{1/6}\left[\Gamma\left(\tfrac{2}{3}\right)\right]^2 \frac{Ne}{R}\left(\frac{R}{L}\right)^{4/3}$$ statvolts/turn

$$= 2.64 \times 10^3 \;\frac{Ne}{R}\left(\frac{R}{L}\right)^{4/3}$$ volts/turn, again with e in e.s.u. and R in cm.

(iii) For the S.H.M. case with distributed amplitudes we estimate

$$V(3) \cong 6 \times 10^3 \;\frac{Ne}{R}\left(\frac{R}{L}\right)^{4/3}$$ volts/turn, with e and R in the same units as before.

6. Numerical Examples:

By way of an example, first consider a single bunch of electrons for which

$N = 10^{11}$

$R = 5140$ cm, and

$L/R = 0.8$;

then $V_{(3)} \cong 6 \times 10^3 \times \dfrac{10^4 \times 4.8 \times 10^{-10}}{5.14 \times 10^3} \times \dfrac{1}{0.744} = 75 \;\dfrac{\text{volts}}{\text{turn}}$.

If, on the other hand,

$N = 3 \times 10^{11}$ per bunch,

$R = 700$ cm, and

$L/R = 0.037$, as might be expected with operation in a high harmonic,

then $V(3) \cong 6 \times 10^3 \times \dfrac{3 \times 10^{11} \times 4.8 \times 10^{-10}}{700} \times \dfrac{1}{0.0123} = 10^5$ v/turn.

For comparison, the incoherent loss, for electron-energies of 10 Gev and 2 Gev correspond in these respective cases to:

$$V_{incoh.} = 400\pi \,\frac{4.8 \times 10^{-10}}{5140}\left(\frac{10000}{0.51}\right)^4 = 17 \times 10^6 \text{ v/turn;}$$

$$V_{incoh.} = 400\pi \,\frac{4.8 \times 10^{-10}}{700}\left(\frac{2000}{0.51}\right)^4 = 2 \times 10^5 \text{ v/turn.}$$

7. Effect of Shielding:

The coherent radiation, which is of relatively long wavelength, may be reduced considerably by suitable shielding. By use of a suitably modified Green's function, Schwinger[19] has considered the case of a uniform bunch between infinite parallel conducting shields, of separation a, and obtained a shielding factor $(1/2)(1/3)^{1/6}(a/R)(R/L)^{2/3}$, for $L > a$. Saxon has reviewed the derivation of this factor, which he considers may assume the value 0.071 in a typical case ($R/a = 50$, $L/R = 0.04$), to include an estimate of the shielding effect for parallel conducting sheets of finite width.

## IV. REFERENCES

1. Eg. MAC meeting, State University of Iowa, Iowa City, Iowa (April 16, 1954); see also (a) informal note "Possible Damping of Phase Oscillations" (L.J.L., April 1, 1954), based on informal discussion with Drs. Haxby and Palfrey at Bloomington, Indiana (12 February 1954) and (b) letter to D.W. Kerst (April 9, 1954). Damping of phase oscillations was considered attractive in the interest of facilitating the phase change at the transition energy.

2. cf. W. Aron and R.F. Mozley, Minutes of Princeton meeting (July 15, 1953, p.2): "It is known that in the conventional synchrotron the use of a sloped acceleration gap gives increased phase stability at the expense of increased radial oscillation. The first question considered here was whether the same held in both phase stability regions for the strong-focusing synchrotron, and for both "positive" gaps (where the voltage gain increases with particle radius) and "negative" gaps (where the voltage decreases with particle radius). It remains true in all circumstances that the synchrotron amplitude is reduced only if the betatron amplitude is increased, and vice versa. However, since "positive" and "negative" gaps have always oppositely directed effects the use of alternating gaps as previously suggested for the transition region was not ruled out in principle.

The possibility of using such sloped gaps to damp out the betatron oscillation aroused some interest. If the energy from the betatron oscillation were coupled into a slow increase in mean radius then the problem of adjusting the radius might prove easier of solution than any other damping approach (though this was just speculation)."

3. A.A. Garren, R. L. Gluckstern, L. R. Henrich, and Lloyd Smith, "Theoretical Considerations in the Design of a Proton Synchrotron", Sect. IIID--/UCRL-547 (December 9, 1949)/. In this section the authors consider both a cavity device in which the energy gain can be a function of radius because of leakage effects and a dee-type electrode in which a radial variation may be introduced, because of the time-of-flight, by shaping the dee faces. These mechanisms are shown to give similar damping effects.

4. R. Q. Twiss and N. H. Frank, Rev. Sci. Inst. 20, 1 (1949).

5. J. J. Livingood, hectographed Argonne National Laboratory notes (November, 1953).

6. LJL(MAC)-3 (February, 1954) -- esp. ref. 7.

7. This phenomenon was suggested to the writer by his interpretation of a letter from Dr. D. W. Kerst (April 5, 1954).

8. Courant, Livingston, and Snyder, Phys. Rev. 88, 1190-1196 (December 1, 1952).

9. LJL(MAC)-2 (January, 1954) -- esp. p. 4.

10. Private telephone conversation (May 20, 1954).

11. Lawrence W. Jones and L. Jackson Laslett, LWJ-LJL/MAC-5 (September, 1953).

12. K. Johnsen, CERN proton-synchrotron lectures [M. Hildred Blewett, Ed.] Sect. III-3 (October, 1953).

13. The factor 1.17 in eq. (16) of the MAC report[6] represents the numerical value of $2\left[-\cos\phi_o - (\phi_o - \pi/2)\sin\phi_o\right]^{1/2}$, for $\phi_o = 5\pi/6$.

$\gamma \equiv -(d\omega/\omega_s) / (dp/p_s)$ and is only slightly less than unity under the conditions prevailing at injection and for which Johnsen's equation was written.

14. L. H. Johnston, Bull. of the American Physical Society, 1954 meeting in Washington D.C., paper C1.

15. John S. Nodvick and David S. Saxon, "on the Suppression of Coherent Radiation by Electrons in a Synchrotron", Technical Report No. 21, Department of Physics, University of California at Los Angeles Los Angeles, Calif., (May, 1954).

16. J. Schwinger, Phys. Rev. 75, 1912-1925 (June 15, 1949).

17. J. Schwinger, Harvard Lectures, Autumn 1945.

18. L. I. Schiff, Rev. Sci. Inst. 17, 6-14 (January, 1946) -- esp. appendix, pp. 12-14.

19. J. Schwinger, "On Radiation by Electrons in a Betatron" (1945), unpublished manuscript kindly communicated by Professor Schwinger and cited as ref. 2 of our ref. 15.