Boundary conditions and vacuum fluctuations in AdS₄

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Abstract

Initial conditions given on a spacelike, static slice of a non-globally hyperbolic space-time may not define the fates of classical and quantum fields uniquely. Such lack of global hyperbolicity is a well-known property of the anti-de Sitter solution and led many authors to question how is it possible to develop a quantum field theory on this spacetime. Wald and Ishibashi took a step towards the healing of that causal issue when considering the propagation of scalar fields on AdS. They proposed a systematic procedure to obtain a physically consistent dynamical evolution. Their prescription relies on determining the self-adjoint extensions of the spatial component of the differential wave operator. Such a requirement leads to the imposition of a specific set of boundary conditions at infinity. We employ their scheme in the particular case of the four-dimensional AdS spacetime and compute the expectation values of the field squared and the energy-momentum tensor, which will then bear the effects of those boundary conditions. We are not aware of any laws of nature constraining us to prescribe the same boundary conditions to all modes of the wave equation. Thus, we formulate a physical setup in which one of those modes satisfy a Robin boundary condition, while all others satisfy the Dirichlet condition. Due to our unusual settings, the resulting contributions to the fluctuations of the expectation values will not respect AdS invariance. As a consequence, a back-reaction procedure would yield a non-maximally symmetric spacetime. Furthermore, we verify the violation of weak energy condition as a direct consequence of our prescription for dynamics.

Keywords Anti-de Sitter · Boundary conditions · Non-globally hyperbolic spacetimes · Self-adjointness

1 Introduction

One of the most remarkable outcomes of string theory was the proposition of the AdS/CFT correspondence [1]. It is conjectured that a theory of quantum gravity

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on n-dimensional AdS displays an underlying equivalent conformal quantum field theory without gravity, taking place at the $(n-1)$-dimensional conformal boundary of AdS. Accordingly, applications to high energy and condensed matter physics appeared within the efforts to test the limits of this new conjecture, placing the anti-de Sitter spacetime under the scientific spotlight.

Although most of the developments in AdS rely on string theory techniques, on a recent work [2], the authors have focused on studying semiclassical properties of the spacetime. Using the mathematical apparatus of quantum field theory (QFT) in curved spaces, they have found the fluctuations of the expectation values of the energy-momentum tensor and the field squared in AdS_n. However, they did not discuss in depth the implications of the causal structure of the spacetime, i.e., the effects of non-globally hyperbolicity.

Since AdS has a conformal boundary, we may not be able to determine much about the history of a physical quantity without specifying its behavior at infinity. Such a circumstance poses a fundamental issue on the quantization procedure, namely the solutions of the wave equation will not be uniquely defined by initial conditions in AdS, i.e., the Cauchy problem is not well-posed. Thus, unless we give extra information at the conformal boundary, the lack of predictability makes it impracticable to build a quantized field whose dynamical evolution comprises the entire history of the spacetime.

Avis et al. [3] were the first ones to address the causal pathology of AdS when solving field equations. They developed QFT on AdS_4 by regulating information leaving or entering the spacetime *by hand*. Their approach proposes the imposition of boundary conditions at the spatial infinity in order to control whether information flows through (or is reflected by) the conformal boundary. Even though Avis et al. provide us with physically consistent solutions to the wave equation, works by Wald [4] and Ishibashi [5,6] reveal that a broader category of boundary conditions might be employed to obtain a physical dynamical evolution.

In [5], the authors present a prescription for dynamics of fields in general non-globally hyperbolic spacetimes based on the grounds of physical consistency. In order to fulfill some reasonable physical requirements (to be explained later), they argue that the spatial component of the differential wave operator must be self-adjoint. Besides, in [6], they show that the prescription for dynamics in AdS translates into specifying boundary conditions at the conformal boundary. While Kent and Winstanley [2], impose the Dirichlet boundary condition at infinity, they are neglecting an entire set of non-equivalent dynamical outcomes. According to Ishibashi and Wald [6], those outcomes would correspond to the various boundary conditions that one could have specified at infinity.

In this paper, we study physical effects that may arise due to non-Dirichlet boundary conditions at the conformal boundary. We investigate those effects by computing the vacuum fluctuations of the expectation values of the quadratic field and the energy-momentum tensor for conformally coupled scalar fields in AdS_4. Also, we will keep Ref. [2] as a basis for our results and shall return to it for further comparison.

An arbitrary Robin boundary condition (which is neither Dirichlet nor Neumann) introduces an energy scale. We expect that this should be responsible for the break of AdS invariance. In two recent papers [7,8], the authors showed that this is indeed

the case, at least for AdS$_2$ in Poncaré coordinates, where there is only one boundary condition parametrized by $\beta \in \mathbb{R}$. In particular, it was shown that the Green's function does not respect the spacetime symmetries since it does not depend exclusively on the geodesic distance.

Is it possible to obtain a similar effect in global AdS$_n$? The answer is yes, as we will see later in this paper. However, as can be seen in Ref. [9], the expansion of the Green's functions in terms of the normal modes is highly nontrivial for a general Robin boundary condition and the search of a closed form seem to be hopeless.

Nevertheless, it is still possible to break AdS invariance by choosing a nontrivial boundary condition. Since a scalar field in global AdS$_n$ has a simple decomposition in terms of angular momentum modes and each mode has its own dynamics, it is possible, as a first attempt, to change the boundary condition in a finite number of these l modes. Clearly, each l mode with a nontrivial boundary condition will have its frequencies spectrum modified. The resulting system becomes numerically tractable, but is this choice physical? Since this question does not have an immediate answer, we will try to set up at least the compatibility of our model with basic concepts from quantum field theory.

In this way, we consider ou model as a first step in showing how non trivial boundary conditions may bring highly nontrivial effects to quantum field theory. Besides, by considering backreaction, we expect to see non trivial effects even in the classical theory (future research should consider these effects more carefully).

We have organized this article as follows. In Sect. 2, we briefly review some of the fundamental aspects of the anti-de Sitter solution. Then, in Sect. 3, we display the systematic procedure that describes the dynamics of scalar fields in non-globally hyperbolic spacetimes—such as AdS—first presented by Wald and Ishibashi. With that scheme in hands, we show the implications their prescription has on scalar fields propagating on AdS, in Sect. 4. Our next step is to build the proper Green's functions in Sect. 5, and employ them in the computations of the renormalized quantities of interest, namely the fluctuations of the expectation values of the field squared and the energy-momentum tensor, both shown in Sect. 6. Finally, we discuss our results in Sect. 7.

2 Anti-de Sitter spacetime

Surfaces of constant negative curvature are well-known in geometry and comprise the set of hyperbolic spaces. In the context of General Relativity, the equivalent to those spaces is the n-dimensional anti-de Sitter space, which appears as a solution to Einstein equations when choosing a negative cosmological constant ($\Lambda < 0$) in the absence of matter and energy. Setting $\Lambda := -\frac{(n-1)(n-2)}{2H^2}$, we may write the Einstein equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{(n-1)(n-2)}{2H^2}g_{\mu\nu} = 0. \tag{1}$$

The outcome is an n-dimensional maximally symmetric pseudo-Riemmanian metric defined over a Lorentzian manifold with constant negative curvature, i.e., the AdS$_n$ spacetime. In a suitable set of parametrized coordinates $\{x^\mu\}$,[1] the line element for the induced metric $g_{\mu\nu}$ on AdS$_n$ is

$$ds^2 = H^2(\sec\rho)^2[-d\tau^2 + d\rho^2 + (\sin\rho)^2 d\Omega_{n-2}^2], \tag{2}$$

where $d\Omega_{n-2}^2$ is the line element on a unit $(n-2)$-sphere.

2.1 Topology

We may understand AdS$_n$ as an isometric embedding of a single sheeted n-dimensional hyperboloid in an $(n+1)$-dimensional flat space provided with metric diag$(-1, 1, \ldots 1, -1)$. Timelike curves in AdS are transverse sections of the hyperboloid, and they are always closed. The periodicity of the timelike coordinate, τ, suggests that given a point in spacetime, we can return to it by only traveling along a timelike geodesic of length 2π in τ. Accordingly, the topology of AdS$_n$ becomes apparent, namely $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, which is compatible with the existence of closed timelike curves. Thus, unphysical events can take place in the spacetime, such as a particle returning to the same position through a periodic motion in time.

2.2 Causal structure

Wald remarks in [10] that observers following closed timelike geodesics would have no difficulty altering past events hence breaking causality. In an attempt to solve this primary issue, we can 'unwrap' the hyperboloid along the timelike direction, and patch together unwrapped hyperboloids one after the other. In other words, we construct a spacetime spatially identical to AdS but extended in time, i.e., the temporal coordinate no longer ranges from $-\pi$ to π but from $-\infty$ to ∞. We refer to such procedure as the universal covering of AdS, and the resulting spacetime as CAdS.

Even though the unwrapping of AdS prevents the existence of closed timelike curves, another fundamental causality issue remains, namely the lack of predictability associated with fields propagating on the spacetime. Indeed, no Cauchy hypersurfaces exist in AdS (and CAdS) hence portraying it as a non-globally hyperbolic spacetime. The Cauchy problem will not be well-posed, yielding non-unique dynamics for a given set of initial conditions. We can understand this scenario as a result of information leaking through the spatial infinity of the spacetime, i.e., flowing in (out) from (through) the boundary. In order to solve such a pathological behavior, we shall discuss in the next

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[1] The radial coordinate, ρ, is defined over the interval $[0, \pi/2)$. The polar and azimuthal coordinates on the unit $(n-2)$-sphere are θ_j $(j = 1, \ldots, n-3)$ and $\varphi := \theta_{n-2}$, respectively, each satisfying $0 \leq \theta_j \leq \pi$ and $0 \leq \varphi < 2\pi$. The timelike coordinate, τ, ranges from $-\pi$ to π.

sections how to adequately address causality issues associated with field equations in non-globally hyperbolic spacetimes.

3 Scalar fields in non-globally hyperbolic static spacetimes

An extensive literature (see, for instance, [11] and references therein) provides a complete guide on QFT in curved spaces, and conduct us through a generalized quantization procedure based on that of QFT in Minkowski spacetime. Nevertheless, several researchers developed most of it in a category of spacetimes whose causal structure is thoroughly well-defined, namely globally hyperbolic spacetimes. Indeed, as we discussed previously if a spacetime does not feature global hyperbolicity, then basic field equations might not have causal solutions, which jeopardizes the quantization of fields. On what follows, we use works by Wald [4] and Ishibashi [5,6] to prescribe the appropriate dynamics of scalar fields in non-globally hyperbolic spacetimes.

Let us consider a static spacetime $(\mathcal{M}, g_{\mu\nu})$, which admits the following decomposition of its metric [12]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -V^2 dt^2 + h_{ij}dx^i dx^j. \tag{3}$$

In Eq. (3), h_{ij} is the metric induced on a spacelike hypersurface Σ orthogonal to a given timelike Killing field τ^μ of the metric, and we define $V^2 = -\tau^\mu \tau_\mu$. In this particular case, Klein–Gordon equation,

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi - \xi R \phi = 0, \tag{4}$$

reduces to

$$\partial_t^2 \phi = -A\phi, \tag{5}$$

in which $A := -V D^i(V D_i \phi) + m^2 V^2 + \xi R V^2$ is the spatial component of the wave operator, and D_i is the covariant derivative in a spatial slice of Σ.

Since Σ is a subset of the spacetime, singular or boundary points cannot be part of Σ. However, it is clear that these points may be in causal contact with some event in $\mathcal{M}$. Wald's strategy [4] relies on the fact that, since we do not know what happens at the boundary of Σ, a convenient choice for the domain of the operador A is $\mathcal{D}_0(A) = C_0^\infty(\Sigma)$, i.e., our "initial" domain is insensitive to the boundary points. By choosing the Hilbert space $\mathcal{H} = L^2(\Sigma, V^{-1}d\Sigma)$, where $d\Sigma$ is the induced metric on Σ we have: (i) $\mathcal{D}_0(A)$ is dense in $\mathcal{H}$; (ii) $(A, \mathcal{D}_0(A))$ is a symmetric operator; (iii) as a consequence there is at least one positive self-adjoint extension for $(A, \mathcal{D}_0(A))$.[2]

An extensive literature on Functional Analysis (e.g., see [13,14]) discusses the properties of such operators and present a systematic procedure for obtaining their self-adjoint extensions, accredited to Weyl and von Neumann. Let us first define the adjoint operator $(A^\dagger, \mathcal{D}(A^\dagger))$ by:

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[2] We should emphasize that the choice $\mathcal{D}_0(A) = C_0^\infty(\Sigma)$ makes perfect sense since the classical theory does not say which effects may arise from boundary points. However this is not the only possible option. In fact there are infinitely many dense domains $\mathcal{D}(A)$ where A is symmetric which would also fit to our purposes.

$$\mathcal{D}(A^\dagger) = \{\xi \in \mathcal{H} | \exists \, \chi \text{ such that } \langle \xi, A\psi \rangle = \langle \chi, \psi \rangle \, \forall \, \psi \in \mathcal{D}_0(A)\},$$
$$A^\dagger \xi = \chi, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathcal{H}$. It can be easily checked (simply using integration by parts) that $\mathcal{D}_0(A) \subsetneq \mathcal{D}(A^\dagger)$ so that $(A, \mathcal{D}_0(A))$ is clearly not self-adjoint. However, as mentioned above, $(A, \mathcal{D}_0(A))$ may be extended to a self-adjoint operator.

In order to find these extensions, let us define the *deficiency subspaces* of A, denoted $\mathcal{N}_\pm \subset \mathcal{H}$, by

$$\mathcal{N}_\pm = \{\psi_\pm \in \mathcal{D}(A^\dagger) | \, A^\dagger \psi_\pm = \pm i\lambda\psi_\pm, \lambda \in \mathbb{R}^+\}, \tag{7}$$

and the *deficiency indices* as $n_\pm = \dim(\mathcal{N}_\pm)$. For n-th order ordinary differential equations, there are three cases to be considered:

1. If $n_+ \neq n_-$, then A has no self-adjoint extension.
2. If $n_+ = n_- = 0$, then A is essentially self-adjoint, and we obtain it by taking the closure, $\bar{A}$, of A.
3. If $n_+ = n_- = n \geq 1$, then infinitely many self-adjoint extensions of A exist. They are in one-to-one correspondence to the isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$ parametrized by an $n \times n$ unitary matrix U.

For partial differential equations in which the Hilbert space can be split in the form $\mathcal{H} = \bigoplus_{n_i} \mathcal{H}_{n_i}$ (as in the case to be considered in this paper) we must consider each subspace n_i individualy.

Certainly, the third case is more complex than the others, and we must follow a method for obtaining the self-adjoint extensions (see [14] for a proper description of it). They are given by A_E, with E being a parameter labeling the extension, defined by

$$\mathcal{D}(A_E) = \{\Phi_0 + \Phi_+ + U_E\Phi_+ | \, \Phi_0 \in \mathcal{D}(A), \Phi_+ \in \mathcal{N}_+\}, \tag{8}$$

and

$$A_E\Phi = A\Phi_0 + i\Phi_+ - iU_E\Phi_+, \tag{9}$$

for all $\Phi \in \mathcal{D}(A_E)$. This procedure can always be followed to find whether an operator has self-adjoint extensions and identify them, in case they exist.

Wald [4] proved that the one parameter family

$$\phi_t = \cos(A_E^{1/2}t)\phi_0 + A_E^{-1/2}\sin(A_E^{1/2}t)\dot{\phi}_0, \tag{10}$$

which is a solution of the equation

$$\frac{d^2\phi_t}{dt^2} = -A_E\phi_t, \tag{11}$$

reproduces the solution of Eq. (4) determined by ordinary Cauchy evolution [with initial data in $\mathcal{D}(A_E)$].

It is straightforward to notice that for each extension A_E there will be an associated dynamical evolution of Eq. (10). Consequently, the dynamics of the field is not uniquely determined by initial conditions. We identify those non-equivalent solutions as a result of various boundary conditions that one can impose at a region in space, such as a singularity or a boundary [4]. Ishibashi and Wald [5], argue that Eq. (10) is the only one that prescribes a physically sensible dynamics of scalar fields in non-globally hyperbolic static spacetimes. By comparison with the globally hyperbolic case, they establish a set of conditions that determine whether a time evolution is consistent or not, namely:

1. Solutions of the wave equation must be causal;
2. The prescription for dynamics must be invariant under time translation and reflection;
3. There exists a conserved energy functional also respecting time translation and reflection invariance, in agreement with the globally-hyperbolic case;
4. Solutions satisfy a convergence condition, as proposed in [4].

We should emphasize that Wald's prescription gives us a reasonable evolution for the classical field. By reasonable we mean respecting the four requirements listed above. In particular, the conserved energy functional is consistent with the unitarity of the field in the corresponding Hilbert space. This is, of course, a necessary requirement in the construction of a "closed" quantum theory—not loosing or gaining information from spatial infinty.

Nevertheless, there are other ways to evolve and quantize the classical field. In Ref. [3], for example, the authors studied the quantization of a classical scalar field respecting the so called "transparent" boundary condition. In this scheme the AdS spacetime becomes transparent when mapped into the Einsteins static universe. In this way, the energy is not time independent (there is an exergy flux due to transparency). Besides, the resulting quantization is not "closed" in the sense that the one particle state becomes a mixture of particle states of the closed AdS (obtained through the reflective Dirichlet and Neumann boundary conditons). We therefore choose Wald's prescription since it leads to a very concise quantization. We must, however, stress out that other works address quantum field theory in spacetimes with timelike boundaries differently, e.g., see [15–17].

4 Boundary conditions at infinity of anti-de Sitter

Let us now consider Klein–Gordon equation (5) in AdS$_n$, as follows

$$\partial_\tau^2 \phi = -(\sec \rho)^2 \Big\{ (\cot \rho)^2 \Big[-(n-2)\tan \rho \, \partial_\rho^2 - \Delta_S \Big] - H^2 m_\xi^2 \Big\} \phi, \qquad (12)$$

where m_ξ is the effective mass of the field defined by $m_\xi^2 = m^2 - \xi n(n-1)H^{-2}$, and

$$\Delta_S = \sum_{j=1}^{n-3} \left[(n-2) \cot \theta_j \partial_{\theta_j} + \prod_{k=1}^{j-1} (\csc \theta_k)^2 \partial_{\theta_j}^2 \right] + \prod_{j=1}^{n-3} (\csc \theta_j)^2 \partial_\varphi^2 \qquad (13)$$

is the Laplace–Beltrami operator on the unit $(n-2)$-sphere whose eigenfunctions are Generalized Spherical Harmonic functions, $Y_l(\theta_j, \phi)$, with eigenvalues $l(l+n-3)$. We may recall that a static slice of AdS_n can be decomposed into a real interval $[0, \pi/2)$, labeled by the radial coordinate ρ, and an $(n-2)$-dimensional unit sphere $\mathbb{S}^{n-2}$, parametrized by the angular coordinates θ_j and φ. It is also worth pointing out that, as the spacetime is static, there exists a timelike Killing field ∂_τ, whose eigenfunctions $e^{-i\omega\tau}$ with positive energy, $\omega > 0$, can be used to expand the solution ϕ. Thus, ϕ will be an eigenfunction of the quadratic operator ∂_τ^2 with eigenvalue $-\omega^2$. With those considerations in hand, let us write the solution as

$$\phi(\tau, \rho, \theta_j, \varphi) = \sum_{\omega, l} e^{-i\omega\tau} \tilde{f}_{\omega,l}(\rho) Y_l(\theta_j, \phi). \qquad (14)$$

Under the transformation

$$\tilde{f}_{\omega,l}(\rho) = (\cot \rho)^{\frac{n-2}{2}} f_{\omega,l}(\rho), \qquad (15)$$

and omitting temporal and angular dependence, Eq. (12) reduces to

$$A f_{\omega,l}(\rho) = \omega^2 f_{\omega,l}(\rho), \qquad (16)$$

upon the identification [6][3]

$$A \equiv -\frac{d^2}{d\rho^2} + \frac{\nu^2 - 1/4}{(\cos \rho)^2} + \frac{\sigma^2 - 1/4}{(\sin \rho)^2}, \qquad (17)$$

which is a differential operator whose domain is $C_0^\infty(0, \pi/2)$ defined over a Hilbert space $\mathcal{H} = \mathcal{L}^2([0, \pi/2], d\rho)$, and the coefficients of the equation are defined as

$$\nu^2 - 1/4 = \frac{n(n-2)}{4} + H^2 m^2 - n(n-1)\xi, \qquad (18)$$

and

$$\sigma^2 - 1/4 = \frac{(n-2)(n-4)}{4} + l(l+n-3). \qquad (19)$$

From Eq. (19), it is straightforward to check that

$$\sigma = l + \frac{n-3}{2}. \qquad (20)$$

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[3] Ishibashi and Wald define the radial coordinate x for the spatial infinity to be located at $x = 0$ [6]. It relates to our radial coordinate ρ by $x = \pi/2 - \rho$.

The coefficient ν is taken to be the positive square root of ν^2 and will depend on the mass and coupling factor of the field. In such conditions, there are four relevant cases to be analyzed, namely

(i) $\nu^2 \geq 1$: in this case, the effective mass of the field satisfies the relation $H^2 m_\xi^2 \geq -(n+1)(n-3)/4$, which comprise the minimally coupled, massless scalar field for $n \geq 3$.

(ii) $0 < \nu^2 < 1$: this case occurs for $-(n-1)^2/4 < H^2 m_\xi^2 < -(n+1)(n-3)/4$, and includes conformally invariant scalar fields in all dimensions.

(iii) $\nu^2 = 0$: this is the case when the effective mass squared reaches a critical value, namely $H^2 m_\xi^2 \equiv -(n-1)^2/4$.

(iv) $\nu^2 < 0$: in this case, the effective mass squared is lower than the critical mass, i.e., $H^2 m_\xi^2 < -(n-1)^2/4$.

We notice that, when condition (iv) is satified, we have normalizable negative-energy states. This states correpond to instabilities of the classical AdS wave equation. This condition is the well known Breitenlohner–Freedman bound [18,19].

In [6], the authors examine the positivity of the operator A in terms of ν. They demonstrate that, in all cases in which $\nu^2 \geq 0$—i.e., in (i), (ii) and (iii)—A is a positive operator. Meanwhile, in case (iv), the operator is unbounded bellow. Consequently, A has no positive, self-adjoint extensions in case (iv). On the other hand, at least one self-adjoint extension to A exists—that is, the Friedrichs extension [13]—in all other cases: (i), (ii) and (iii).

According to Eq. (7), to construct the deficiency subspaces $\mathcal{N}_\pm$, we must take $\omega^2 = \pm i$, so $\omega \in \mathbb{C}$. In the case (i), i.e., $\nu \geq 1$, Eq. (17) has no solution in the Hilbert space so that the deficiency indices are null. In this case and the operator admits a unique self-adjoint extension for each value of l. In other words, the repulsive effective potential in A, i.e., $(\cos \rho)^{-2}$, prevents the fields from reaching spatial infinity. Hence, they vanish there, and no additional boundary conditions are required. Conversely, in cases (ii) and (iii), the square integrable solution to Eq. (16) with $\omega = \pm i$ is given by

$$f_{\omega,l}(\rho) = \mathbf{C} \cdot (\cos \rho)^{\nu+1/2} \cdot (\sin \rho)^{\sigma+1/2}$$
$$\times {}_2F_1\left(\frac{\nu+\sigma+\omega+1}{2}, \frac{\nu+\sigma-\omega+1}{2}; 1+\sigma, (\sin \rho)^2\right). \quad (21)$$

The other linear independent solution is not square-integrable, i.e. it does not belong to the Hilbert space, so we neglect it here. The deficiency indices in these cases are $n_+ = n_- = 1$, so infinitely many positive self-adjoint extensions of A exist. Now, the effective potential is not as strong as in case (i); hence we may associate the extensions to boundary conditions prescribed at infinity.

A one-parameter family of self-adjoint extensions, A_β, of A exists for $0 \leq \nu^2 < 1$ [cases (ii) and (iii)]. Equation (8) provides us with the appropriate domain of A_β. Since the domain of A consists of functions in C_0^∞, all additional information needed to prescribe a physically consistent dynamical evolution must come from the asymptotic behavior of f_+ and Uf_+, for all isometries U.

Let U_β denote the isometries between $\mathcal{N}_+$ and $\mathcal{N}_-$, given by

$$U_\beta f_+ = e^{i\beta} f_-, \tag{22}$$

for $\beta \in (-\pi, \pi]$. Let us consider the function

$$f_\beta := f_+ + U_\beta f_+ \equiv f_+ + e^{i\beta} f_-, \tag{23}$$

whose behavior near infinity ($\rho = \pi/2$) dictates the boundary conditions satisfied by all solutions ϕ_t of the form (10). For $0 < \nu < 1$, the asymptotic behavior at $\rho = \pi/2$ is

$$f_\beta \propto (\sin \rho)^{\sigma+1/2} \cdot (\cos \rho)^{-\nu+1/2} \times (a_\nu + b_\nu (\cos \rho)^{2\nu} + c_\nu (\cos \rho)^2 + \cdots), \tag{24}$$

where the coefficients of the leading terms, a_ν and b_ν, are functions of ν, σ, the spacetime dimension n and the parameter β. The leading powers in ρ of f_+ are

$$f_\beta \approx b_\nu \left(\frac{\pi}{2} - \rho\right)^{\nu+1/2} \left\{1 + \frac{a_\nu}{b_\nu} \left(\frac{\pi}{2} - \rho\right)^{-2\nu}\right\}, \tag{25}$$

from which we can see that the asymptotic boundary condition depends on the ratio a_ν/b_ν, which may take any real value. For $\nu = 0$, we have

$$f_\beta \propto (\sin \rho)^{\sigma+1/2} \cdot (\cos \rho)^{1/2} \times (a_0 \log(\cos^2 \rho) + b_0 + c_0 (\cos \rho)^2 \log(\cos^2 \rho) + \cdots), \tag{26}$$

and an analogous procedure reveals that the asymptotic boundary condition depends on a_0/b_0 also in this case. However, the function $(\sin \rho)^{-\sigma-1/2} \cdot (\cos \rho)^{-1/2} \cdot f_\beta$ and its first derivative in ρ both scale with a_0 when approaching infinity $\rho = \pi/2$. Setting $a_0 = 0$, we recover Dirichlet and Neumann boundary condition imposed simultaneously, which is precisely Friedrichs extension.

On what follows, we shall denote the ratio a_ν/b_ν by α_ν, hence all self-adjoint extensions of the operator will be parametrized by α instead of β, although $\alpha \equiv \alpha(\beta)$. From Eq. (25), we can check that[4]

$$\frac{\frac{d}{d\rho}[(\sin \rho)^{-\sigma-1/2} \cdot (\cos \rho)^{\nu-1/2} \cdot f_\alpha]}{[(\sin \rho)^{-\sigma-1/2} \cdot (\cos \rho)^{3\nu-3/2} \cdot f_\alpha]}\bigg|_{\rho=\pi/2} = -2\nu \frac{1}{\alpha_\nu}, \tag{27}$$

which we identify as generalized Robin boundary conditions for $0 < \nu < 1$. One recovers generalized Dirichlet or Neumann boundary conditions by setting α_ν equals

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[4] We exchanged all indices β for α.

to 0 and $\pm\infty$, respectively. In the particular case $\nu = 1/2$, Eq. (25) reduces to an even simpler form of the boundary conditions given by[5]

$$\left[\frac{df_\alpha}{d\rho} \Big/ f_\alpha\right]_{\rho=\pi/2} = -\frac{1}{\alpha}, \tag{28}$$

which is the usual Robin boundary condition, hence mixing Dirichlet ($\alpha = 0$) and Neumann ($\alpha = \pm\infty$) conditions.

Even though the extensions A_α are now parametrized by a real parameter α_ν, not all of them are positive. Except for $\nu^2 \geq 1$, whose unique self-adjoint extension is already positive, the remaining cases satisfy the positivity conditions shown in [6]:

For $0 < \nu^2 < 1$, we have

$$\frac{b_\nu}{a_\nu} \equiv \frac{1}{\alpha_\nu} \geq -\left|\frac{\Gamma(-\nu)}{\Gamma(\nu)}\right| \frac{\Gamma(\frac{\sigma+\nu+1}{2})^2}{\Gamma(\frac{\sigma-\nu+1}{2})^2}. \tag{29}$$

For $\nu^2 = 0$, we have

$$\frac{b_0}{a_0} \leq 2\gamma + 2\psi\left(\frac{\sigma+1}{2}\right), \tag{30}$$

where γ is the Euler gamma and ψ is the digamma function.

It is worth pointing out that Eqs. (27) and (28) must be satisfied mode by mode, i.e., for each spherical label l—and for each σ, indirectly (see Eq. 19)—, the conditions are satisfied by $f_{\beta,\omega,l}$. Accordingly, there are infinitely many parameters $\alpha_{\nu,l}$ associated to each $f_{\beta,\omega,l}$, and they all satisfy different positivity conditions, given in Eqs. (29) and (30).

5 Green's functions in AdS

In [20], Allen and Jacobson show that, in a maximally symmetric spacetime, two-point functions such as $G_F(x, x') = -i\langle\psi|T\{\phi(x)\phi(x')\}|\psi\rangle$, where $|\psi\rangle$ is a maximally symmetric state, may be written in terms of the geodetic interval $s(x, x')$,[6] i.e.,

$$G_F(x, x') := G_F(s(x, x')) \equiv G_F^{(AJ)}(s). \tag{31}$$

Their proposition simplifies the computations considerably since the wave equation becomes an ODE of the variable s. They also require that the Green's function falls off as fast as possible at spatial infinity, which in AdS translates into: $G_F \to 0$ as $s \to \infty$. In other words, they are choosing Dirichlet boundary condition for the field ϕ. Kent and Winstanley [2], exploit this simplicity to find the fluctuation of the field squared and the energy-momentum tensor in all spacetime dimensions of AdS. They

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[5] In case $\nu = 1/2$, we drop the index of $\alpha_{1/2}$ and replace it simply by α.

[6] In AdS, s is constructed so that it goes to zero as $x' \to x$ and goes to infinity as we approach the boundary.

also verify that their results are compatible with the ones of Burgess and Lütken, whose approach in [21] was to perform a summation of modes of the wave solutions.

We are not aware of any law of nature that restricts the boundary conditions of all modes to Dirichlet ones. Indeed, Ishibashi and Wald showed in [5] that there is an entire category of boundary conditions that prescribe a physically consistent dynamical evolution. Additionally, there is no guarantee that all modes must satisfy the same boundary condition.

Let us then consider a setup in which one of the modes of the wave equation, $u_{\omega_\alpha, l_\alpha}$, is chosen so that its radial component $f_{\omega_\alpha, l_\alpha}(\rho)$ satisfies a generalized Robin boundary condition with parameter α. Meanwhile, the components $f_{\omega, l}(\rho)$ of all other modes $u_{\omega, l}(x)$ ($l \neq l_\alpha$) satisfy Dirichlet boundary conditions.

The Green's function in this case is given by mode sum (from now on, we consider $\tau > \tau'$)

$$
G_F(x, x') = -i \left(\cot \rho \cot \rho' \right)^{\frac{n-2}{2}}
$$
$$
\times \left\{ \sum_{\omega_\alpha} |\mathcal{N}_{\omega_\alpha, l_\alpha}|^2 Y_{l_\alpha}(\theta_j, \varphi) Y_{l_\alpha}^*(\theta_j', \varphi') f_{\omega_\alpha, l_\alpha}(\rho) f_{\omega_\alpha, l_\alpha}(\rho') e^{-i\omega_\alpha(\tau-\tau')} \right.
$$
$$
\left. + \sum_{\substack{l \geq 0 \\ l \neq l_\alpha}} \sum_{\omega} |\mathcal{N}_{\omega, l}|^2 Y_l(\theta_j, \varphi) Y_l^*(\theta_j', \varphi') f_{\omega, l}(\rho) f_{\omega, l}(\rho') e^{-i\omega(\tau-\tau')} \right\},
$$

$$(32)$$

where $\mathcal{N}_{\omega, l}$ are normalization constants. We may complete the last term in the summation for all Dirichlet modes by adding them to and subtracting them off Eq. (32), i.e.,

$$
G_F(x, x') = -i \left(\cot \rho \cot \rho' \right)^{\frac{n-2}{2}}
$$
$$
\times \left\{ \sum_{\omega_\alpha} |\mathcal{N}_{\omega_\alpha, l_\alpha}|^2 Y_{l_\alpha}(\theta_j, \varphi) Y_{l_\alpha}^*(\theta_j', \varphi') f_{\omega_\alpha, l_\alpha}(\rho) f_{\omega_\alpha, l_\alpha}(\rho') e^{-i\omega_\alpha(\tau-\tau')} \right.
$$
$$
- \sum_{\omega} |\mathcal{N}_{\omega, l_\alpha}|^2 Y_{l_\alpha}(\theta_j, \varphi) Y_{l_\alpha}^*(\theta_j', \varphi') f_{\omega, l_\alpha}(\rho) f_{\omega, l_\alpha}(\rho') e^{-i\omega(\tau-\tau')}
$$
$$
\left. + \sum_{l, \omega} |\mathcal{N}_{\omega, l}|^2 Y_l(\theta_j, \varphi) Y_l^*(\theta_j', \varphi') f_{\omega, l}(\rho) f_{\omega, l}(\rho') e^{-i\omega(\tau-\tau')} \right\}. \quad (33)
$$

Let us denote the last term in Eq. (33) by $G_F^{(D)}$, and the first two terms by $G_F^{(\alpha)}$. The Green's function $G^{(D)}$ is obtained by the summation of Dirichlet modes purely. Thus, in the coincidence limit, it recovers the same results as $G^{(BL)}$, by Burgess and Lütken, and $G_F^{(AJ)}$, by Allen and Jacobson. On the other hand, $G_F^{(\alpha)}$ lacks contributions from all spherical components, since it is not summed over all angular modes l. Hence, $G_F^{(\alpha)}$ may not be a maximally symmetric function. It seems reasonable for us to write that

$$
G_F(x, x') \equiv G_F^{(\alpha)}(x, x') + G_F^{(D)}(s(x, x')). \quad (34)
$$

Equation (34) illustrates the break of AdS invariance of the Green's function, as it may not depend on the geodetic interval s entirely anymore. We attribute the break on the maximal symmetry of G_F to the imposition of different boundary conditions for each angular mode.

Given the decomposition (34), one question arises.[7] Is it compatible with quantum field theory by means of the canonical commutation relations

$$
\begin{aligned}
[\phi(\tau, x^a), \phi(\tau, x'^a)] &= 0, \\
[\phi(\tau, x^a), \pi(\tau, x'^a)] &= i\delta(x^a, x'^a), \\
[\pi(\tau, x^a), \pi(\tau, x'^a)] &= 0,
\end{aligned}
\tag{35}
$$

where $x^a = (\rho, \theta, \varphi)$? The answer is yes, and to prove this, consider the Green's function $G(x, x') = G^+(x, x') - G^-(x, x')$. Without loss of generality, let us consider a nontrivial boundary condition in the $l = 0$ mode only. This will also be our choice—for simplicity—later in the paper.

Within our notation, it is given by

$$
G(x, x') = G_0^\alpha(x, x') - G_0^{\text{Dirichlet}}(x, x') + G^{\text{Dirichlet}}(x, x').
\tag{36}
$$

The first term in the above equation is the zero-th mode corresponding to an arbitrary Robin boundary condition, the second one is zero-th mode corresponding to Dirichlet boundary condition and the third one is the complete Green's function corresponding Dirichlet boundary condition (the one respecting the spacetime symmetries). The first and second term have the form

$$
G_0(\tau, \rho; \tau'\rho') = \sum_\omega u_\omega(\rho)u_\omega(\rho')\left(e^{i\omega(\tau-\tau')} - e^{-i\omega(\tau-\tau')}\right).
\tag{37}
$$

Now, given the normalization of $u_\omega(\rho)$ and the fact that they form a complete set of a self-adjoint operator (corresponding to $l = 0$) we have

$$
\partial_t G_0(\tau, \rho; \tau'\rho')\big|_{t=t'} = \sum_\omega 2i\omega u_\omega(\rho)u_\omega(\rho') = i\delta(\rho - \rho').
\tag{38}
$$

In this way

$$
\begin{aligned}
\partial_t G(x, x')\big|_{t=t'} &= i\delta(\rho - \rho') - i\delta(\rho - \rho') + \partial_t G^{\text{Dirichlet}}(x, x')\big|_{t=t'} \\
&= -i\delta(x^a - x'^a),
\end{aligned}
\tag{39}
$$

since $G^{\text{Dirichlet}}$ is obviously consistent with the commutation relations. This proves that our complete solution is also consistent with the commutation relations.

---

[7] We thank an anomymous referee for raising this point.

Another important question is whether the Green's function has the Hadamard form. First of all, since we are dealing with a non-globally hyperbolic spacetime, we are interested in the local Hadamard form only. Global Hadamard form is not equivalent in this case since we can face reflecting singularities of the Green's function [16]. Besides, by choosing the positive self-adjoint extensions and constructing the Green's function with the positive frequency modes with respect to the Killing field $\partial/\partial\tau$, we are constructing a Green's function of a ground state. Using Appendix A of Ref. [22] (where it is proved that such construction always leads to a ground state with the correct Hadamard form), we can state that our Green's function has the expected Hadamard form.

6 Renormalized quantities for a conformal massless scalar field in AdS$_4$

In order to shed light on what we have discussed so far, we shall specialize to four spacetime dimensions, AdS$_4$. For simplicity on the computation of quantities of interest, let us restrict ourselves to a conformally invariant, massless scalar field, ϕ, i.e., $m = 0$ and $\xi = \frac{1}{6}$. In this case, from Eq. (18), we get $\nu = 1/2$, and from Eq. (19), we find that $\sigma = (2l + 1)/2$. Equation (16) becomes

$$\left(-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\sin^2\rho}\right) f = \omega^2 f, \tag{40}$$

and its solutions are

$$f = \sqrt{\sin\rho} \left(\mathbf{C}_1 \cdot P_{\omega-1/2}^{l+1/2}(\cos\rho) + \mathbf{C}_2 \cdot Q_{\omega-1/2}^{l+1/2}(\cos\rho)\right), \tag{41}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are constants to be determined, and P and Q are the associated Legendre functions of the First and Second kinds, respectively. Square integrability requires f to fall off at the origin $\rho = 0$, hence $\mathbf{C}_1 \to 0$.[8] A complete set of eigenfunctions is then

$$f_{\omega,l}(\rho) = \mathcal{N}_{\omega,l} \cdot \sqrt{\sin\rho} \cdot Q_{\omega-1/2}^{l+1/2}(\cos\rho), \tag{42}$$

for normalization constants $\mathcal{N}_{\omega,l}$ to be determined.

As discussed in Sect. 4, boundary conditions at infinity are necessary to prescribe the dynamical evolution of the field in AdS$_n$. In case $\nu = 1/2$, Robin boundary conditions (28) are the appropriate ones. We aim to provide an example of the setups discussed in the last section. For that, we will consider that all non-spherically symmetric modes respect Dirichlet boundary conditions. However, the $l = 0$ mode will be chosen to satisfy Robin condition with a parameter α. As discussed above, the vacuum will not be AdS invariant in this case. However, since the non-trivial boundary condition is on $l = 0$ mode, we still preserve spherical symmetry.

---

[8] Formula 14.8.1 of Ref. [23] shows the divergence of P at $\rho = 0$.

Formulas 14.5.3 and 14.5.4 in Ref. [23] allow us to describe the behavior of $f_{\omega,l}$ and its derivative at the boundary, as follows

$$f_{\omega,l}(\rho \to \pi/2) \sim -\mathcal{N}_{\omega,l} \frac{2^{l-1/2}\sqrt{\pi} \sin\left(\frac{(l+\omega)\pi}{2}\right) \Gamma\left(\frac{l+\omega+1}{2}\right)}{\Gamma\left(\frac{-l+\omega+1}{2}\right)}, \tag{43}$$

$$\frac{df_{\omega,l}}{d\rho}\bigg|_{\rho \to \pi/2} \sim -\mathcal{N}_{\omega,l} \frac{2^{l+1/2}\sqrt{\pi} \cos\left(\frac{(l+\omega)\pi}{2}\right) \Gamma\left(\frac{l+\omega+2}{2}\right)}{\Gamma\left(\frac{-l+\omega}{2}\right)} \tag{44}$$

For $l > 0$, all modes satisfy $f_{\omega,l}(\rho \to \pi/2) = 0$ (Dirichlet boundary condition), thus the quantized frequencies are given by

$$\omega = 0, \pm 2, \pm 4, \ldots, \pm l, \pm(l+2), \pm(l+4), \pm(l+6), \ldots. \tag{45}$$

These frequencies come from the relations

$$\sin\left(\frac{(l+\omega)\pi}{2}\right) = 0; \quad \Gamma\left(\frac{-l+\omega+1}{2}\right) = -\infty. \tag{46}$$

(clearly, for $l = 0$, the quantized frequencies for Dirichlet boundary condition are given by the set $2\mathbb{Z}$).

For $l = 0$, we calculate the ratio between derivative (44) and function (43) to use it in (28), i.e.,

$$\left[\frac{df_{\omega,0}}{d\rho} \bigg/ f_{\omega,0}\right]_{\rho=\pi/2} = 2\cot\left(\omega\frac{\pi}{2}\right)\frac{\Gamma\left(1+\frac{\omega}{2}\right)}{\Gamma\left(\frac{\omega}{2}\right)}$$

$$= \omega\cot\left(\omega\frac{\pi}{2}\right) = -\frac{1}{\alpha}. \tag{47}$$

Positivity condition (29) requires that

$$\frac{1}{\alpha} \geq -\left|\frac{\Gamma(-1/2)}{\Gamma(1/2)}\right|\frac{\Gamma(1)^2}{\Gamma(1/2)^2} = -\frac{2}{\pi} \Rightarrow \alpha \leq -\frac{\pi}{2} \text{ or } \alpha \geq 0. \tag{48}$$

In our analysis, we consider $\alpha \geq 0$, which includes Dirichlet, $\alpha = 0$, and Neumann, $\alpha \to \infty$, cases.

Equation (47) imposes a quantization condition for the frequencies ω in terms of the parameter α. Except for $\alpha = 0$ and $\alpha = \infty$, it cannot be solved analytically for an arbitrary value α. One can readily verify that, in the Neumann case ($\alpha \to \infty$), the frequencies are odd integers. Meanwhile, for Dirichlet, they are even integers, which is consistent with Eq. (45).

In our procedure, we employed the software Mathematica [24] to solve Eq. (47) numerically in a determined range of ω for several values of α. As shown in Fig. 1, the
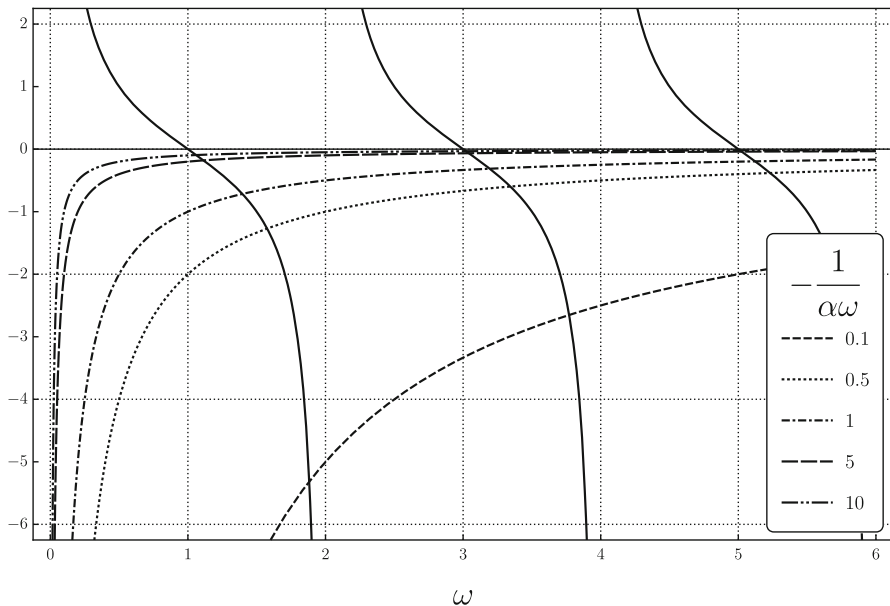
Fig. 1 Quantization condition for ω imposed by (47). The solid lines show the function $\cot(\omega\pi/2)$ and all other curves are $-1/(\alpha\omega)$, for a few values of α

solutions of (47) are given by the intersection points between the two functions. We can see that ω values for arbitrary α always lie between an odd number and its next even integer, which are precisely the frequencies for Neumann and Dirichlet conditions, respectively. Thus, given a Neumann frequency, $\omega_{N,r} = 2r - 1$, and a Dirichlet one, $\omega_{D,r} = 2r$, for $r > 0$, we may denote an α frequency between them as $\omega_{\alpha,r}$, even though it is not an integer number.

6.1 Quadratic field fluctuations $\langle \phi^2 \rangle$

Before computing the Green's function, it is useful to write solution $f_{\omega_{\alpha,r},0}$ in a more convenient form and normalize it accordingly. Using Ref. [23], we find[9]

$$f_{r,0}^{(\alpha)}(\rho) = H^{-1} \sqrt{\frac{2}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}} \sin(\omega_{\alpha,r}\rho). \tag{49}$$

Now, we recall our discussion from last section to construct the appropriate Green's function. We can decompose our Green's functions in two parts, i.e.,

$$G_F^{(\alpha)}(x, x') = -i \frac{\cot \rho \cot \rho'}{4\pi H^2}$$

---

[9] For convenience, we change the lower label in $f_{\omega,0}$ from $\omega_{\alpha,r}$ to r simply, and add an upper index α to denote our choice of boundary condition.

$$\times \sum_{r>0} \left(\frac{2}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)} \sin(\omega_{\alpha,r}\rho)\sin(\omega_{\alpha,r}\rho')e^{-i\omega_{\alpha,r}(\tau-\tau')} \right.$$
$$\left. - \frac{2}{2r\pi}\sin(2r\rho)\sin(2r\rho')e^{-i2r(\tau-\tau')} \right), \tag{50}$$

and

$$G_F^{(D)}(x,x') = -iH^{-2}\cot\rho\cot\rho'$$
$$\times \sum_{\omega\geq 0}\sum_{l\geq 0}\sum_{m=-l}^{l} |\mathcal{N}_{r,l}|^2 Y_l^m(\theta_j,\varphi)[Y_l^m(\theta_j',\varphi')]^* f_{r,l}(\rho)f_{r,l}(\rho')e^{-i\omega(\tau-\tau')}, \tag{51}$$

where $\omega \geq 0$ means that we are summing over the positive frequencies in Eq. (45).

Our 'Dirichlet' Green's function (51) is obtained from a summation of AdS invariant modes of the wave equation. Hence, it respects maximal symmetry and recovers the results of Burgess and Lütken, $G_F^{(BL)}$, and Allen and Jacobson, $G_F^{(AJ)}$, i.e., $G_F^{(D)}(x,x') \equiv G_F^{(D)}(s(x,x'))$. As Kent and Winstanley show in [2], approaching the coincidence limit $s \to 0$, the function $G_F^{(D)}$ diverges according to the Hadamard form. Thus, point-splitting renormalization can be employed to compute finite quantities. Furthermore, they obtain the Hadamard forms in AdS for any spacetime dimension through a systematic method, based on [25].

In the particular case of AdS$_4$, for a conformally invariant field, the Green's function $G_F^{(D)}$ has the Hadamard form given by

$$G_F^{(D)}(s) \sim -\frac{i}{4\pi^2 s^2}, \quad s \to 0. \tag{52}$$

After renormalization, it may be written as [2]

$$\left[G_F^{(D)} \right]_{\text{ren}}(s) = -\frac{i}{8\pi^2 H^2}\left\{ -\frac{1}{6} + \frac{13}{240}\frac{s^2}{H^2} + \mathcal{O}(s^4) \right\}. \tag{53}$$

We may find the expectation value of the quadratic field fluctuations as follows

$$\langle\phi^2\rangle^{(D)} = i\lim_{s\to 0}\left[G_F^{(D)} \right]_{\text{ren}}(s) = -\frac{1}{48\pi^2 H^2}, \tag{54}$$

which is naturally in accordance with the results in Ref. [2]. Analogously, the effect of our Green's function (50) on $\langle\phi^2\rangle$ appears when taking the coincidence limit $x' \to x$. However, calculating $G_F^{(\alpha)}$ analytically is impossible, since the summation is taken over numerical values of frequencies. Hence, we adopt a numerical approach to find our results.

We expect $G_F^{(\alpha)}$ to be finite, since the Hadamard form took care of the divergences in $G_F^{(D)}$. On the other hand, we cannot perform the infinite sum in (50) numerically, so a

residual divergent behavior might appear. Through our computations, we noted it was convenient to take the coincidence limit in the radial coordinate first, i.e., $\rho' \to \rho$, and then in the time coordinate. Thus, our final step would be to take the limit of $\tau' \to \tau$. It is more convenient though, to analytically extend the function on the complex plane and take the limit through the imaginary axis, i.e., $\tau' \to \tau + i\epsilon$, hence $\tau - \tau' \to -i\epsilon$. Finally, by multiplying $G_F^{(\alpha)}$ by i, we will have an entirely real-valued function that, in the limit $\epsilon \to 0$, yields directly the quadratic fluctuations of the field, and it is much simpler for us to handle it numerically.

Before implementing the numerical routine, we considered the only case that can be treated analytically, which is the Neumann condition, $\alpha \to \infty$. In this situation, the frequencies are $\omega_{\infty,r} = 2r - 1$, for $r > 0$, and the Green's function reduces to the following summation

$$iG_F^{(\infty)}(\epsilon, \rho, \rho) = \frac{\cot^2 \rho}{2\pi^2 H^2} \times \sum_{r>0} \left(\frac{\sin^2((2r-1)\rho)}{2r-1} e^{-(2r-1)\epsilon} - \frac{\sin^2(2r\rho)}{2r} e^{-2r\epsilon} \right),$$

(55)

which we calculated using Mathematica [24], resulting

$$iG_F^{(\infty)}(\epsilon, \rho, \rho) = \frac{\cot^2 \rho}{16\pi^2 H^2} \times \log \left[\cosh \left(\frac{\epsilon}{2} \right)^4 \sec \left(\rho - i\frac{\epsilon}{2} \right)^2 \sec \left(\rho + i\frac{\epsilon}{2} \right)^2 \right].$$

(56)

It is straightforward to find the expectation value $\langle \phi^2 \rangle^{(N)}$ by simply taking $\epsilon \to 0$, i.e.,

$$\langle \phi^2 \rangle^{(N)}(\rho) = \frac{\cot^2 \rho}{4\pi^2 H^2} \log \left[\sec(\rho) \right].$$

(57)

First of all, notice that $\langle \phi^2 \rangle^{(N)}(\rho)$ fails to be a constant so that the spacetime is not homogenous anymore—at least for the quantum field. Therefore, there is a preferred frame—the one in which Neumman boundary condition has been chosen for tha particular mode $l = 0$. What if we had chosen Neumann boundary condition in every l mode, instead? Since this particular boundary condition does not introduces an energy scale, we expect that the relevant physical quantities will respect the symmetries of the spacetime and $\langle \phi^2 \rangle(\rho)$ will be constant in this case (see Ref. [3], where it was shown that the Green's function respects the spacetime symmetries for reflective Neumann boundary condition).

The function $\langle \phi^2 \rangle^{(N)}$ is finite because both terms inside the sum in Eq. (55) diverge with same strength. Naturally, their subtraction eliminates the infinities. In particular, the last term in Eq. (55), the Dirichlet counterpart of $G_F^{(\alpha)}$, denoted $G_F^{(\alpha,D)}$, appears for all values of α and dictates the divergent behavior at $\epsilon \to 0$. We find its form by calculating the infinite summation and expanding it in powers of ϵ, i.e.,

$$iG_F^{(\alpha,D)}(\epsilon,\rho) = \frac{\cot^2\rho}{8\pi^2 H^2}\left\{-\log\epsilon + \log[\sin(2\rho)] + \mathcal{O}(\epsilon^2)\right\}. \tag{58}$$

Our numerical approach to find the expectation value $\langle\phi^2\rangle^{(\alpha)}$ proceeded as follows:

1. Given a value for α, solve Eq. (47) to find the frequencies $\omega_{\alpha,r}$ up to $r_{\max} = 5000$;
2. Given a value of ρ between 0 and $\pi/2$, compute numerically the truncated summation

$$iG_F^{(\alpha)}(\epsilon,\rho,\rho) \approx \frac{\cot^2\rho}{2\pi}\sum_{r=1}^{r_{\max}}\left(\frac{\sin^2(\omega_{\alpha,r}\rho)}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}e^{-\omega_{\alpha,r}\epsilon} - \frac{\sin^2(2r\rho)}{2r\pi}e^{-2r\epsilon}\right)$$
$$=: \mathtt{f}_\rho^{(\alpha)}[\epsilon], \tag{59}$$

for 50 values of ϵ equally spaced in the range 0.002 to 0.1.[10]
3. Fit the function $\mathtt{f}_\rho^{(\alpha)}[\epsilon]$ using a model that reproduces the divergent behavior in Eq. (58) followed by a Taylor expansion up to order ϵ^2, i.e.,

$$\mathtt{f}[\epsilon] = \mathtt{a} + \mathtt{b}\log[\epsilon] + \mathtt{c}\cdot\epsilon + \mathtt{d}\cdot\epsilon^2. \tag{60}$$

As $G_F^{(\alpha)}$ is a finite quantity, we expect the divergent behavior of $\mathtt{f}_\rho^{(\alpha)}[\epsilon]$ to be extremely attenuated. We have found coefficients $\mathtt{b}$ ranging between 10^{-9} and 10^{-12}, recovering the expected *almost-finite* behavior. The coefficients $\mathtt{c}$ and $\mathtt{d}$ were effective on reducing the residuals of the fit. Finally, $\mathtt{a}$ gives the approximated finite numerical value of $\langle\phi^2\rangle^{(\alpha)}$ at the point ρ.
4. Repeat steps 2 and 3 for as many values of ρ between 0 and $\pi/2$ as desired.
5. Repeat the entire procedure for another value of α.

We followed the scheme described above for 14 values for the parameter α. We chose 80 equally spaced points in the range $(0, \pi/2)$ to obtain a good resolution of the behavior of $\langle\phi^2\rangle^{(\alpha)}(\rho)$. Our results are plotted in Fig. 2. The curve corresponding to $\alpha = 1000$ reproduces almost perfectly the analytic Neumann result (57). Accordingly, as we approach the other extreme, $\alpha = 0$—corresponding to Dirichlet conditions— we can see the curves getting closer to zero. Consistently, if $\alpha = 0$, then $G_F^{(\alpha)}$ indeed vanishes, as one can see from Eq. (50).

From Fig. 2, we notice that the correction to $\langle\phi^2\rangle$ arising from the nontrivial boundary condition becomes more relevant close to $\rho = 0$. This may seem strange, since the boundary condition is necessary at the conformal boundary. However, since any observer (close to $\rho = 0$ or not) is infinitely far away from the boundary, this does not seem to be a contradiction.

---

[10] Our choice for $r_{\max}$ and the range of ϵ was made so the last term of the sum would be negligible with respect to the first one. Indeed, the first term is of order $e^{-2\cdot 1\cdot 0.002} \sim 10^{-1}$, while the last is $e^{-2\cdot 5000\cdot 0.002} \sim 10^{-9}$. Also, we needed ϵ small enough so the divergent behavior would appear.
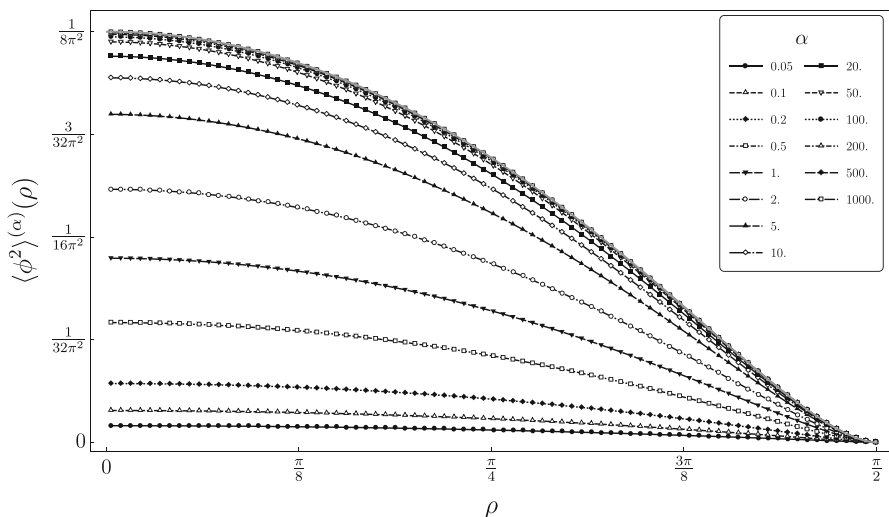
Fig. 2 Contribution to the expectation value of the quadratic field fluctuations due to Robin boundary conditions at infinity for the spherically symmetric mode. The solid gray curve shows the analytical solution of the Neumann case, $\langle\phi^2\rangle^{(N)}$. H is set to one

6.2 Energy-momentum tensor fluctuations $\langle T^\mu_\nu\rangle^{(\alpha)}$

In [2], the authors obtain the renormalized energy-momentum tensor $\langle T^\mu_\nu\rangle_{\mathrm{ren}}$ in AdS$_n$. They use the formula from Ref. [25]

$$\langle T_{\mu\nu}\rangle_{\mathrm{ren}} = -[G]_{\mu\nu} + \frac{1}{2}(1-2\xi)[G]_{;\mu\nu}$$
$$+ \frac{1}{2}\left(2\xi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\nabla^\sigma[G] + \xi R_{\mu\nu}[G]\right) + \Theta_{\mu\nu}, \qquad (61)$$

where

$$[G](x) := \lim_{x'\to x} i\,[G_F]_{\mathrm{ren}}(x,x'), \qquad (62)$$

$$[G]_{\mu\nu}(x) := \lim_{x'\to x} i\,[G_F]_{\mathrm{ren}}(x,x')_{;\mu\nu}, \qquad (63)$$

and $\Theta_{\mu\nu}$ is a purely geometric tensor constructed to be conserved. Kent and Winstanley find that the non-geometrical component of the tensor is proportional to the metric tensor, which is completely consistent with the maximal symmetry of AdS. In our particular case of a conformally invariant field in four spacetime dimensions, we have

$$\langle T_{\mu\nu}\rangle^{(D)}_{\mathrm{ren}} = -\frac{1}{960\pi^2 H^4}g_{\mu\nu}, \qquad (64)$$

and the geometric tensor $\Theta_{\mu\nu}$ is identically zero. We may obtain this renormalized expectation value from Green's function $\left[G_F^{(D)}\right]_{\mathrm{ren}}$, hence is associated with Dirichlet conditions in all modes of the wave equation.

Here, we want the contributions to the energy-momentum tensor coming from $G_F^{(\alpha)}$. Our approach will be analogous to that of the Green's functions: we decompose $\langle T^\mu_\nu \rangle_{\text{ren}}$ into two parts, one carrying the boundary condition, denoted $\langle T^\mu_\nu \rangle_{\text{ren}}^{(\alpha)}$ with 16 components T^μ_ν, and another one reproducing the Dirichlet results as in Eq. (64).

In our case, Eqs. (62) and (63) may be written as

$$[G](\rho) = \lim_{\epsilon \to 0} i\, G_F^{(\alpha)}(\epsilon, \rho, \rho), \tag{65}$$

and

$$
\begin{aligned}
[G]_{\mu\nu}(\rho) &= \lim_{\epsilon \to 0} i\, (G_F^{(\alpha)})_{;\mu\nu}(\epsilon, \rho, \rho) \\
&= \lim_{\epsilon \to 0} i\left[(G_F^{(\alpha)})_{,\mu\nu} - \Gamma_{\mu\nu}^\lambda (G_F^{(\alpha)})_{,\lambda} \right],
\end{aligned}
\tag{66}
$$

from which it follows that $[G](\rho) \equiv \langle \phi^2 \rangle^{(\alpha)}(\rho)$. According to formula (61), we have here

$$\langle T_{\mu\nu} \rangle_{\text{ren}}^{(\alpha)} = -[G]_{\mu\nu} + \frac{1}{3}[G]_{;\mu\nu} - \frac{1}{12}\nabla_\kappa \nabla^\kappa [G]g_{\mu\nu} - \frac{1}{2H^2}[G]g_{\mu\nu}. \tag{67}$$

Considering all non-vanishing Christoffel symbols, the definitions for $[G]$ and $[G]_{\mu\nu}$, and the symmetric condition $\langle T_{\mu\nu} \rangle_{\text{ren}}^{(\alpha)} = \langle T_{\nu\mu} \rangle_{\text{ren}}^{(\alpha)}$, we readily verify that the only non-vanishing components are diagonal terms and the term $T_{\tau\rho}$ ($= T_{\rho\tau}$). Let us recall the temporal inversion ($\tau \to -\tau$) symmetry of AdS, denoted I, given in four dimensions by the transformation matrix $I_\mu^{\mu'} = \text{diag}(-1, 1, 1, 1)$. As none of our quantities depend explicitly on τ, we expect this discrete symmetry to be preserved. In particular, we expect $T_{\tau x^j} = T_{-\tau x^j} = T_{\tau' x'^j}$, for $x^j = (\rho, \theta, \varphi)$. On the other hand, $\langle T^\mu_\nu \rangle_{\text{ren}}^{(\alpha)}$ transforms as a tensor, so we have

$$\langle T_{\mu'\nu'} \rangle_{\text{ren}}^{(\alpha)} = I_{\mu'}^\mu I_{\nu'}^\nu \langle T_{\mu\nu} \rangle_{\text{ren}}^{(\alpha)} \Rightarrow T_{-\tau\rho} = T_{\tau'\rho'} = I_{\tau'}^\tau I_{\rho'}^\rho T_{\tau\rho} = -T_{\tau\rho}. \tag{68}$$

That yields $T_{\tau\rho} = -T_{\tau\rho}$, which then implies $T_{\tau\rho} = T_{\rho\tau} \equiv 0$.

At this point, we have a diagonal tensor, whose remaining components may be calculated using Eq. (67). Our computational efforts were not successful when trying to compute the numerical expressions directly. However, we came up with a solution based on some properties that $\langle T^\mu_\nu \rangle_{\text{ren}}^{(\alpha)}$ must satisfy, based on the definition of $\langle T^\mu_\nu \rangle_{\text{ren}}$.

Let us first consider the effect of the trace anomaly. One can readily verify that it is respected by $\langle T^\mu_\nu \rangle_{\text{ren}}^{(D)}$ [2,25], i.e.,

$$\langle T^\mu_\mu \rangle_{\text{ren}} = \langle T^\mu_\mu \rangle_{\text{ren}}^{(D)} = -\frac{1}{240\pi^2 H^4}, \tag{69}$$

so our tensor $\langle T^\mu_\nu \rangle_{\text{ren}}^{(\alpha)}$ must be traceless,

$$\langle T^\mu_\mu \rangle_{\text{ren}}^{(\alpha)} \equiv 0 = T^\tau_\tau + T^\rho_\rho + T^\theta_\theta + T^\varphi_\varphi, \tag{70}$$

which is our first constrain on the remaining diagonal components. We may use the symmetries of AdS as well. Although our Green's function breaks AdS invariance of the radial coordinate ρ, all other symmetries should remain valid. In AdS$_4$ there exist 10 Killing fields corresponding to the following isometries: one temporal translation, three rotations, four boosts and four spatial translations. From which, we only expect the first two to be preserved after imposing Robin boundary conditions in only one of the modes.

The temporal Killing field, $t = \partial_\tau$, yields a conservation equation along with its flow, given by the Lie derivative of the tensor with respect to t, i.e.,

$$\mathcal{L}_t \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow t^\sigma \partial_\sigma \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\tau \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0, \tag{71}$$

which shows that all components of $\langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}}$ are independent of τ. Additionally, we have the generators of spherical symmetry, given by the following Killing fields

$$\chi_1 = \partial_\varphi, \tag{72}$$
$$\chi_2 = \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi, \tag{73}$$
$$\chi_3 = -\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi. \tag{74}$$

Since a combination of them is still a Killing field, we may use χ_2 and χ_3 to obtain $\chi_4 = \partial_\theta$. We can use χ_1 and χ_4 to find other two conservation equations similar to that of t, as follows

$$\mathcal{L}_{\chi_1} \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow \chi_1^\sigma \partial_\sigma \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\varphi \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0, \tag{75}$$
$$\mathcal{L}_{\chi_4} \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0 \Rightarrow \chi_4^\sigma \partial_\sigma \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = \partial_\theta \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} = 0. \tag{76}$$

These equations show us that $\langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}}$ can be a function of ρ only, i.e., $\langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} \equiv \langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}} (\rho)$.

Finally, the conservation equation,

$$\nabla^\nu \langle T^\mu_\nu \rangle_{\text{ren}} = 0, \tag{77}$$

provide us with the last set of constrains. As $\langle T^\mu_\nu \rangle^{(D)}_{\text{ren}}$ is proportional to the metric, it is automatically conserved, since $\nabla^\mu g_{\mu\nu} = 0$. Hence, for $\langle T^\mu_\nu \rangle_{\text{ren}}$ to be entirely conserved, we must impose Eq. (77) on $\langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}}$ as well, which, using the properties we have found for $\langle T^\mu_\nu \rangle^{(\alpha)}_{\text{ren}}$ so far, reduces to

$$\partial_\rho T^\rho_\rho - \tan \rho T^\tau_\tau + (4 \csc(2\rho) + \tan \rho) T^\rho_\rho - 2 \csc(2\rho)(T^\theta_\theta + T^\varphi_\varphi) = 0, \tag{78}$$
$$\cot \theta (T^\theta_\theta - T^\varphi_\varphi) = 0 \Rightarrow T^\theta_\theta = T^\varphi_\varphi. \tag{79}$$

Before discussing our numerical approach for the expectation value of the energy-momentum tensor, we treat the case $\alpha \to \infty$, i.e., Neumann boundary condition.

Again, we were able to find an analytic result only in this situation. We used Eq. (67) to find the formulas for components,[11]

$$
\mathrm{T}^\tau_{\ \tau} = \cos^2 \rho \left([G]_{\tau\tau} - \tan \rho [G]_r \right) + \frac{1}{3} \cos^2 \rho \tan \rho [G]_{,\rho}
$$
$$
- \frac{1}{12} \left(\cos^2 \rho [G]_{,\rho\rho} + 2 \cot \rho [G]_{,\rho} \right) - \frac{1}{2}[G], \tag{80}
$$

$$
\mathrm{T}^\rho_{\ \rho} = - \cos^2 \rho \left([G]_{\rho\rho} - \tan \rho [G]_r \right) + \frac{1}{3} \cos^2 \rho \left([G]_{,\rho\rho} - \tan \rho [G]_{,r} \right)
$$
$$
- \frac{1}{12} \left(\cos^2 \rho [G]_{,\rho\rho} + 2 \cot \rho [G]_{,\rho} \right) - \frac{1}{2}[G], \tag{81}
$$

$$
\mathrm{T}^\theta_{\ \theta} = \mathrm{T}^\varphi_{\ \varphi} = - \cot^2 \rho \tan \rho [G]_r + \frac{1}{3} \cot^2 \rho \tan \rho [G]_{,\rho}
$$
$$
- \frac{1}{12} \left(\cos^2 \rho [G]_{,\rho\rho} + 2 \cot \rho [G]_{,\rho} \right) - \frac{1}{2}[G]. \tag{82}
$$

Our attempts to compute $[G]_{\rho\rho}$ and $[G]_\rho$ analytically and numerically were not successful. Hence, we adopted another approach that combined the explicit formulas above and the constrains given by Eqs. (70) and (78).

Let us conveniently define a function $\mathrm{F}(\rho)$ depending exclusively on the quantities we were able to compute, namely $[G]_{\tau\tau}(\rho)$ and $[G](\rho)$, as follows

$$
\mathrm{F}(\rho) := \csc^2 \rho \mathrm{T}^\tau_{\ \tau}(\rho) - \mathrm{T}^\theta_{\ \theta}(\rho)
$$
$$
= \cot^2 \rho \left\{ [G]_{\tau\tau} - \frac{1}{12} \left(\cos^2 \rho [G]_{,\rho\rho} + 2 \cot \rho [G]_{,\rho} \right) - \frac{1}{2}[G] \right\}. \tag{83}
$$

Using Eq. (70) and recalling that $\mathrm{T}^\theta_{\ \theta} = \mathrm{T}^\varphi_{\ \varphi}$, we find that

$$
\mathrm{T}^\rho_{\ \rho}(\rho) = 2\mathrm{F}(\rho) - (1 + 2\csc^2 \rho)\mathrm{T}^\tau_{\ \tau}(\rho), \tag{84}
$$

and applying it to (78), we have

$$
\partial_\rho \mathrm{T}^\tau_{\ \tau} + 2 \frac{9 - \cos(2\rho) + 2\csc^2 \rho}{(5 - \cos(2\rho)) \cot \rho} \mathrm{T}^\tau_{\ \tau} = 2 \frac{(\sin(2\rho)\mathrm{F}' + (7 - \cos(2\rho))\mathrm{F})}{(5 - \cos(2\rho)) \cot \rho}. \tag{85}
$$

The equation above is of the form

$$
u'(\rho) + p(\rho)u(\rho) = q(\rho), \tag{86}
$$

upon the identifications $u \equiv \mathrm{T}^\tau_{\ \tau}$,

$$
p(\rho) = 2 \frac{9 - \cos(2\rho) + 2\csc^2 \rho}{(5 - \cos(2\rho)) \cot \rho} \tag{87}
$$

---

[11] We are setting $H = 1$ to clear the expressions, later on we reinsert it.

and

$$q(\rho) = 2\frac{(\sin(2\rho)F'(\rho) + (7 - \cos(2\rho))F(\rho))}{(5 - \cos(2\rho))\cot\rho}. \tag{88}$$

One can verify that

$$u(\rho) = \exp\left[-\int d\rho\ p\right]\left(\int_0^\rho d\rho'\ \exp\left[\int d\rho' p\right]\cdot q + \mathbf{C}\right) \tag{89}$$

solves the equation. In our case, we have

$$\exp\left[\int d\rho\ p\right] = \tan\rho \sec^3\rho\sqrt{5 - \cos(2\rho)}, \tag{90}$$

which vanishes at $\rho = 0$ and diverges at $\rho = \pi/2$. Naturally, the inverse function $\exp\left[-\int d\rho\ p\right]$ vanishes at the boundary, but diverges at $\rho = 0$ with strength $1/\rho$. As it is physically reasonable to ask for a finite T^τ_τ at $\rho = 0$, we set $\mathbf{C}$ to zero. Finally, we compute T^τ_τ using the following expression

$$T^\tau_\tau(\rho) = 2\frac{\cot\rho\cos^3\rho}{\sqrt{5 - \cos(2\rho)}}\int_0^\rho d\rho'\ \frac{\tan^2\rho'\left(\sin(2\rho')F'(\rho') + (7 - \cos(2\rho'))F(\rho')\right)}{\cos^3\rho'\sqrt{5 - \cos(2\rho')}}. \tag{91}$$

For the Neumann case, we used our previous analytic results and found F to be

$$F(\rho) = \frac{\cot^2\rho}{48\pi^2}\left(\csc^2\rho + 2 + 2(\csc^4\rho - 1)\log(\sec\rho)\right). \tag{92}$$

Applying it in Eq. (91), and then using (84) and (83), we find

$$\langle T^\mu_\nu\rangle^{(N)}_{\text{ren}}(\rho) = \frac{\cot^2\rho}{48\pi^2 H^4}\left\{\left(\sin^2\rho\right)\text{diag}(1, -1, 0, 0)\right.$$
$$\left. + \left(1 - 2\cot^2\rho\log(\sec\rho)\right)\text{diag}(1, 1, -1, -1)\right\}. \tag{93}$$

Now, we have a result to compare our numerical ones with.

To compute the function F numerically, we used our previous results of $\langle\phi^2\rangle^{(\alpha)}(=[G])$, but we also need $[G]_{\tau\tau}$. According to (63), we find it by taking the second derivative of $G^{(\alpha)}_F(\tau, \tau', \rho, \rho)$ with respect to τ and, then, taking the coincidence limit. In the convention we adopted, $\partial_{\tau\tau} = -\partial_{\epsilon\epsilon}$. Indeed, we find it to be

$$-\partial_{\epsilon\epsilon}G^{(\alpha,D)}_F = \frac{\cot^2\rho}{8\pi^2}\times\left\{-\frac{1}{\epsilon^2} - \frac{1}{24}(5 + \cos(4\rho))\csc^2\rho\sec^2\rho + \mathcal{O}(\epsilon^2)\right\}. \tag{94}$$

Our numerical procedure to find the expectation value of the energy-momentum tensor fluctuations was:

1. Given a value for α, use the frequencies $\omega_{\alpha,r}$ found before;
2. Given a value of ρ between 0 and $\pi/2$, compute numerically the truncated summation

$$
-i\partial_{\epsilon\epsilon}G_F^{(\alpha)} \approx -\frac{\cot^2\rho}{2\pi}\sum_{r=1}^{r_{\max}}\left(\frac{\omega_{\alpha,r}^2\,\sin^2(\omega_{\alpha,r}\rho)}{\omega_{\alpha,r}\pi - \sin(\omega_{\alpha,r}\pi)}e^{-\omega_{\alpha,r}\epsilon} - \frac{(2r)^2\sin^2(2r\rho)}{2r\pi}e^{-2r\epsilon}\right),
$$

(95)

   denoted $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$, for 50 values of ϵ equally spaced in the range 0.002 to 0.1.
3. Fit the function $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$ using a model that reproduces the divergent behavior followed by a Taylor expansion up to order ϵ^2, i.e.,

$$
\mathtt{h}[\epsilon] = \mathtt{a} + \frac{\mathtt{b}}{\epsilon^2} + \mathtt{c}\cdot\epsilon + \mathtt{d}\cdot\epsilon^2.
$$

(96)

   As expected, the divergent behavior of $\mathtt{F}_\rho^{(\alpha)}[\epsilon]$ is extremely attenuated, and the coefficient $\mathtt{b}$ is negligible compared to the others. Again, the coefficients $\mathtt{c}$ and $\mathtt{d}$ were effective on reducing the residuals of the fit. Finally, $\mathtt{a}$ gives the finite approximated numerical value of $[G]_{\tau\tau}$ at the point ρ.
4. Repeat steps 2 and 3 for as many values of ρ between 0 and $\pi/2$ as desired to obtain the complete $[G]_{\tau\tau}(\rho)$.
5. Use our previous results for $[G]$ together with $[G]_{\tau\tau}$ in Eq. (83) to find a numerical interpolation of $\mathrm{F}(\rho)$, denoted $\mathtt{F}[\rho]$.
6. Given a value of ρ between 0 and $\pi/2$, use $\mathtt{F}[\rho]$ in Eq. (91) and perform a numerical integration to obtain an approximate value of $\mathrm{T}^\tau_{\ \tau}$ at that specific ρ.
7. Repeat step 6 for several values of ρ to find a complete numerical function $\mathrm{T}^\tau_{\ \tau}$. With that in hands, compute $\mathrm{T}^\rho_{\ \rho}$ and $\mathrm{T}^\theta_{\ \theta}$ using Eqs. (84) and (83).
8. Repeat the entire procedure for a different value of α.

Similarly to our results for the expectation value of the field squared, we followed the numerical procedure for 14 values of α. We have found all components of $\langle T^\mu_\nu\rangle^{(\alpha)}_{\mathrm{ren}}$. In Fig. 3, we can see $\mathrm{T}^\tau_{\ \tau}$ for several values of α, it is clear that the form of the function follows the analytic result for the Neumann condition (plotted in gray). For all the remaining components of the tensor, the results were the same: by increasing α, they approach the expected analytic behavior of Eq. (93) (Neumann condition).

7 Discussion and further remarks

Avis, Isham, and Storey took a first-step, in Ref. [3], towards the development of a quantum field theory in anti-de Sitter spacetime. They acknowledged that the conformal infinity poses a serious causality issue to the wave equation but solve it by regulating the information flow through the boundary 'by hand.' They imposed the
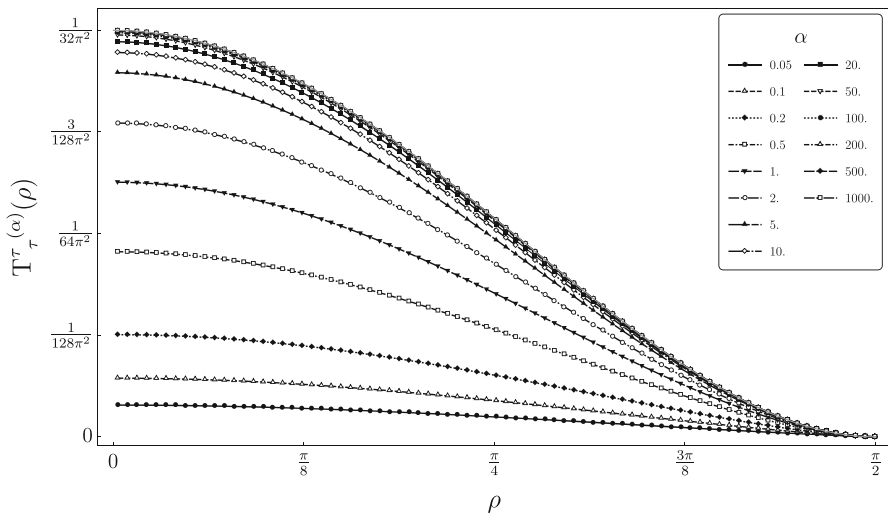
Fig. 3 Contribution to the expectation value T^τ_τ due to Robin boundary conditions at infinity for the spherically symmetric mode. H is set to one
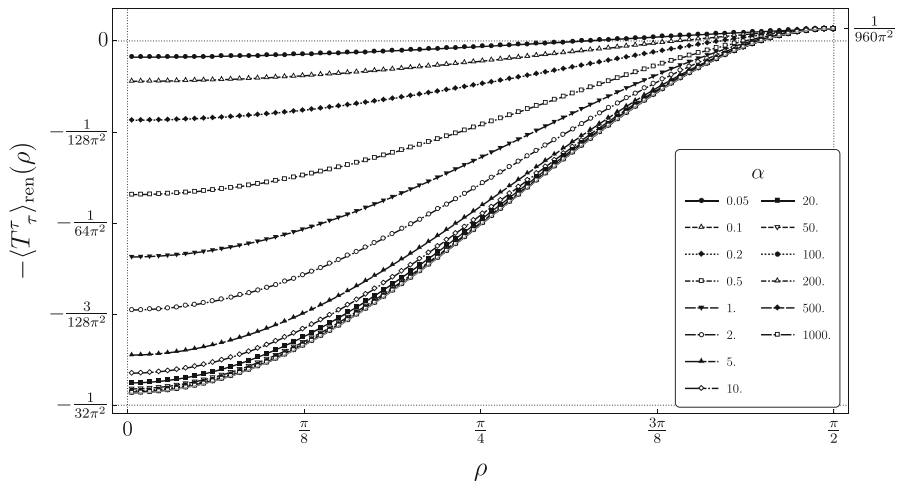
so-called 'transparent' and 'reflective' boundary conditions at infinity in analogy to a box in Minkowski spacetime. In this way, they quantized the fields in the Einstein Static Universe and restricted it to the AdS later.

Conversely, in this article, we considered the developments made by Ishibashi and Wald in [5], where they propose a physically consistent prescription for the dynamical evolution of fields. In the particular case that we have considered, they show that the imposition of mixed boundary conditions at the spatial infinity is sufficient to determine the evolution of quantum fields uniquely.

In the setup studied by Kent and Winstanley [2], all angular modes of the wave equation satisfy the same Dirichlet boundary condition at infinity. Their results are consistent with the maximal symmetry of AdS. Hence, the expectation values of field-dependent quantities fluctuate in the same way throughout spacetime, i.e., they are coordinate-independent. In the light of Wald's and Ishibashi's developments, we presented a setup here that puts up to question how necessary it is to impose the same boundary conditions to all modes of the wave equation. Indeed, we are not aware of any requirement of nature that precludes us from considering various setups in terms of boundary conditions.

Our analysis indicated a violation of AdS invariance in the Green's functions, which carried out implications on the related quantities: the quadratic fluctuations of the field and the energy-momentum tensor. Both of them are now dependent on the radial coordinate for any values of the parameter α, as shown in Figs. 2 and 3. At this stage, any attempt of obtaining a back-reacted metric using Einstein's semi-classical equations,

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu}\rangle_{\text{ren}} \equiv 8\pi G \left(\langle T_{\mu\nu}\rangle^{(D)}_{\text{ren}} + \langle T_{\mu\nu}\rangle^{(\alpha)}_{\text{ren}}\right), \tag{97}$$

Fig. 4 Energy densitiy $-\langle T^\tau_\tau \rangle_{\text{ren}}$ of a massless scalar field conformally coupled to AdS$_4$ for several Robin boundary conditions. H is set to one

would not yield a maximally symmetric metric anymore, but a spherically symmetric one. In these conditions, the coordinate system used to define the angular modes of the wave equation will be privileged. In particular, in this system, the energy density reaches its minimum at the origin $\rho = 0$, as shown in Fig. 4.

In Fig. 4, we can see a clear violation of the weak energy condition in most of the spacetime, except close to the boundary, where the Dirichlet contribution, $-\langle T^\tau_\tau \rangle^{(D)}_{\text{ren}}$, pushes the energy density back up over zero. Even though such violation is no stranger to us—as can be observed in the Casimir effect—it appeared as a consequence of the contribution from the Robin boundary condition exclusively. Indeed, the Dirichlet term, $-\langle T^\tau_\tau \rangle^{(D)}_{\text{ren}}$, of the energy density is positive throughout the entire spacetime. Thus, it is safe to assert that the violation of the weak energy condition is a direct consequence of the imposition of non-Dirichlet boundary conditions at infinity.

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