

Separability, plane wave limits and rotating black holes

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Abstract

We present a systematic construction of the Penrose coordinates and plane wave limits of spacetimes for which both the null Hamilton–Jacobi and geodesic equations separate. The method is applied to Kerr–NUT–(A)dS four-dimensional black holes. The plane wave limits of the near horizon geometry of the extreme Kerr black hole are also explored. All near horizon geometries of extreme black holes with a regular Killing horizon admit Minkowski spacetime as a plane wave limit.

Keywords: separability, Kerr black hole, plane wave limits

1. Introduction

It has been known for sometime [1–3] that the geodesic equation of many four- and higher-dimensional black holes can be separated, see also the comprehensive reviews [4, 5] and references within. This means that there is a set of coordinates on the spacetime such that sequentially the geodesic equation can be expressed as a set of ordinary differential equations each depending on a single unknown function. In principle these equations can be solved¹ to find all the geodesics of the spacetime. The separability of the geodesic equation is due to the presence of ‘hidden’ constants of motion which are in addition to those associated with the (commuting) isometries of the spacetime metric [3, 6–8]. This extends to other differential equations on the spacetime, like the Hamilton–Jacobi (HJ), Klein–Gordon and Dirac equations [2, 9].

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¹ Separability does not necessarily imply integrability provided that the latter means that the solutions can be expressed in terms of some ‘simple’ functions.



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Sometime ago Penrose has shown that on every spacetime one can define a set of coordinates adapted to a null geodesic congruence². In turn these can be used to construct a plane wave or a Penrose limit [10] of the spacetime. The plane wave limits that probe the singularities of a large class of spherically symmetric black holes, like the Schwarzschild and Reissner–Nordström, have been explored in [11–13]. It has been found that the plane wave profile in Brinkmann coordinates is $A \sim \lambda^{-2}$, where λ is the affine parameter of a null geodesic. The black hole singularity is at $\lambda = 0$. A similar plane wave profile is also exhibited by cosmological singularities.

The purpose of this paper is to first give a general description of the method that can be used to construct the Penrose coordinates (PCs) on a spacetime whenever both the null geodesic and HJ equations are separable and then utilise these coordinates to explore the associated plane wave limits. This will extend the results obtained for the plane wave limits of static black holes in [11–13] to stationary black holes. In particular, we shall apply the method to describe the PCs and plane wave limits of the Kerr-NUT-(A)dS black holes. We shall also describe the PCs and plane wave limits of the near horizon geometry of the extreme Kerr black hole. This will extend the results of [14] on the plane wave limits of Anti-de-Sitter near horizon geometries of supersymmetric black holes and branes to those of the near horizon geometries of extreme rotating black holes. It turns out that every near horizon geometry of an extreme Killing horizon has a plane wave limit which is the Minkowski spacetime.

The motivation for this work is twofold. The plane wave limits of some classic spacetimes, like those of the Kerr black hole, are interesting in their own right and to our knowledge have not been explored before. Another motivation for this work is to understand how test strings behave in black hole and cosmological spacetimes. It is well known that the equations of motion of strings have only been solved on spacetimes that exhibit a large group of symmetries, like Minkowski spacetime, group manifolds, some homogeneous spaces and some of their discrete quotients, see e.g. [15, 16] and references within. The list does not include four-dimensional black holes and cosmological spacetimes. As a result it is not known how strings behave near black hole and cosmological singularities. Though significant progress has been made towards the understanding of the thermodynamical properties of (extreme) black holes using holography and AdS/CFT techniques, see e.g. [17–19]. To get a glimpse of how strings behave in black hole and cosmological spacetimes, it has been advocated in [11, 12] to explore the propagation of strings at the plane wave limits of these spacetimes. In addition, the authors of [20] have found that there is a prescription such that the string massive modes propagating on a plane wave spacetime with profile $A \sim \lambda^{-2}$ can go through the singularity from the $\lambda > 0$ region to the $\lambda < 0$ region while particles do not. This indicates that strings exhibit a ‘softer’ behaviour near singularities than particles. Conceivably the Penrose diagrams of spacetimes probed with test strings may look different from the standard ones that exhibit the behaviour of test particles, i.e. that of geodesics. Other applications of plane wave limits are those of the AdS near horizon geometries of brane and black hole spacetimes [14]. These plane wave limits [21] have found extensive applications in string theory [22] and in AdS/CFT [23], see also [24] for a more recent work.

This paper is organised as follows. In section 2, we describe the construction of PCs for spacetimes with separable HJ and geodesic equations. In section 3, we apply the formalism that has been developed to Kerr-NUT-(A)dS black holes. In section 4, we explore the plane wave limits of near horizon geometry of the extreme Kerr black hole, and in section 5 we give our conclusions.

² We shall refer to these as Penrose coordinates.

2. Separability and PCs

2.1. HJ function and PCs

A detailed description of the construction of PCs on a spacetime associated to a null geodesic can be found in [11, 12]. Here for completeness we shall repeat some essential steps in the construction and make the computation a bit more explicit. To begin consider the null HJ equation

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = 0, \quad (1)$$

where S is the HJ function. This is a first order partial differential equation and so locally there is a unique solution provided that a boundary condition for S is specified on a spatial hypersurface $F(x^\mu) = 0$ in a spacetime. In practice though such boundary value problems are hard to solve. Instead the aim is to find the ‘complete’ solution to the null HJ equation³. Such a solution is characterized by the presence of n integration constants α , where n is the dimension of the spacetime. Two of these constants are forgetful because if S is a solution, then $pS + c$ will also be a solution for any constants p and c . Thus the complete solutions of the null HJ equation is specified by $n - 2$ independent integration constants.

Next observe that the solutions of

$$\dot{x}^\mu = g^{\mu\nu} \partial_\nu S \quad (2)$$

are null geodesics, where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and λ is the affine parameter. Indeed $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$ as a consequence of (1) and

$$\nabla_\lambda \dot{x}^\mu = g^{\mu\nu} \dot{x}^\rho \nabla_\rho \partial_\nu S = \frac{1}{2} g^{\mu\nu} \partial_\nu (g^{\rho\sigma} \partial_\rho S \partial_\sigma S) = 0. \quad (3)$$

The system of first order equations conditions (2) has a unique solution $x^\mu(\lambda, x_0^\nu, \alpha)$ for small λ subject to a boundary condition at $\lambda = 0$, $x^\mu(0, x_0^\nu, \alpha) = x_0^\mu$, where α are the integration constants of a complete solution to the HJ equation. Next consider the solutions of (2) for different boundary conditions x_0^μ and assume that they define a null geodesic congruence on the spacetime such that they intersect a hypersurface \mathcal{H} only once. In such a case after a possible redefinition of the affine parameter of each geodesic, the boundary conditions for the solutions of (2) at $\lambda = 0$ can be taken to be the points of \mathcal{H} .

Furthermore observe that given a solution of (2), one has that

$$S(x^\mu(\lambda, x_0^\nu, \alpha), \alpha) = S(x_0^\nu, \alpha), \quad (4)$$

i.e. S does not depend on the affine parameter λ . Indeed

$$\dot{S} = \dot{x}^\mu \partial_\mu S = g^{\mu\nu} \partial_\mu S \partial_\nu S = 0. \quad (5)$$

Given a null geodesic congruence as described above, the definition of a PC system (λ, u, w^i) , $i = 1, \dots, n - 2$, on a spacetime is as follows. First λ is the affine parameter of the null geodesic congruence. The coordinate u is identified with the HJ function as $u = S(x_0^\mu, \alpha)$. Assuming that the level sets of $S(x_0^\mu, \alpha)$ are transversal to the initial value hypersurface \mathcal{H} given by $H(x_0^\mu) = 0$, the coordinates w^i are identified with independent coordinates x_0^μ which remain after solving both $H(x_0^\mu) = 0$ and $u = S(x_0^\mu, \alpha)$ for x_0^μ . Formally, one has

$$\{u, w^i\} = \{x_0^\mu\} / \{H(x_0^\mu), u - S(x_0^\mu, \alpha)\}. \quad (6)$$

³ However note that if S is a solution to the null HJ equation, then $G(S)$ for any (smooth) function G is also a solution.

An explicit computation that utilises the above procedure for constructing PCs can be found in section 2.2. The choice of H is not essential as a different choice will give rise to a different parameterisation of the same null geodesic congruence. Note however that a different choice of integration constants α may lead to a different choice of PCs on a spacetime. This is because a choice of different set of integration constants α may lead to a different null geodesic congruence.

To write explicitly the change of coordinate transformation consider a solution $x^\mu(\lambda, x_0^\mu, \alpha)$ to (2), take the differential and express it in the new coordinates as

$$\begin{aligned} dx^\mu &= \dot{x}^\mu d\lambda + \frac{\partial x^\mu}{\partial x_0^\nu} \left(\frac{\partial x_0^\nu}{\partial u} du + \frac{\partial x_0^\nu}{\partial w^i} dw^i \right) \\ &= g^{\mu\nu} \partial_\nu S d\lambda + \frac{\partial x^\mu}{\partial x_0^\nu} \left(\frac{\partial x_0^\nu}{\partial u} du + \frac{\partial x_0^\nu}{\partial w^i} dw^i \right), \end{aligned} \quad (7)$$

where the integration constants α are considered fixed and they will be suppressed from now on. In these new coordinates⁴ the metric reads as

$$g = 2du \left(d\lambda + h_i(\lambda, u, w) dw^i + \frac{1}{2} \Delta(\lambda, u, w) du \right) + \gamma_{ij}(\lambda, u, w) dw^i dw^j, \quad (8)$$

where we have used

$$g_{\lambda\lambda} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g_{\mu\nu} g^{\mu\rho} \partial_\rho S g^{\nu\sigma} \partial_\sigma S = 0, \quad (9)$$

as a consequence of (1) and

$$\begin{aligned} g_{\mu\nu} g^{\mu\rho} \partial_\rho S \frac{\partial x^\nu}{\partial x_0^\sigma} \left(\frac{\partial x_0^\sigma}{\partial u} du + \frac{\partial x_0^\sigma}{\partial w^i} dw^i \right) d\lambda &= \partial_\nu S \frac{\partial x^\nu}{\partial x_0^\sigma} \left(\frac{\partial x_0^\sigma}{\partial u} du + \frac{\partial x_0^\sigma}{\partial w^i} dw^i \right) d\lambda \\ &= \partial_{x_0^\nu} S(x_0^\rho) \frac{\partial x_0^\nu}{\partial u} d\lambda du \\ &\quad + \partial_{x_0^\nu} S(x_0^\rho) \frac{\partial x_0^\nu}{\partial w^i} d\lambda dw^i \\ &= \frac{\partial u}{\partial u} d\lambda du + \frac{\partial u}{\partial w^i} dw^i d\lambda \\ &= du d\lambda. \end{aligned} \quad (10)$$

The remaining components of the metric can be easily evaluated leading to (8).

The plane wave limit of a spacetime is defined as follows. First one adapts a set of PCs on a spacetime as in (8), scales the spacetime metric as $g \rightarrow \ell^{-2}g$ and simultaneously performs the coordinate transformation $u \rightarrow \ell^2 u$, $w^i \rightarrow \ell w^i$ and $\lambda \rightarrow \lambda$, where ℓ is a constant parameter. Then one takes the limit $\ell \rightarrow 0$. The metric at the limit takes the form

$$g_{pw} = 2du d\lambda + \gamma_{ij}(\lambda) dw^i dw^j. \quad (11)$$

This is a plane-wave metric written in Rosen coordinates. The limit $\ell \rightarrow 0$ restricts the non-vanishing components of the metric at the null geodesic with boundary conditions $u = w^i = 0$. So the limit depends on the choice of this null geodesic.

⁴ Similar coordinates can be adapted to time-like ($m^2 > 0$) and space-like ($m^2 < 0$) geodesics, m constant. The metric in the new coordinates reads as $g = m^2 d\lambda^2 + 2du \left(d\lambda + h_i dw^i + \frac{1}{2} \Delta du \right) + \gamma_{ij} dw^i dw^j$, where $g^{\mu\nu} \partial_\mu S \partial_\nu S = m^2$.

The plane wave metric g_{pw} can be rewritten in Brinkmann coordinates as

$$g_{pw} = 2d\lambda \left(dv + \frac{1}{2}A_{ab}(\lambda)x^a x^b d\lambda \right) + \delta_{ab}dx^a dx^b, \quad (12)$$

where $w^i = e_a^i(\lambda)x^a$, $u = v - \frac{1}{2}\gamma_{ij}\partial_\lambda e_a^i e_b^j x^a x^b$, with $\gamma_{ij}e_a^i e_b^j = \delta_{ab}$ and

$$A_{ab} = \gamma_{ij}\partial_\lambda e_a^i \partial_\lambda e_b^j - \partial_\lambda \left(\gamma_{ij}\partial_\lambda e_a^i e_b^j \right), \quad (13)$$

is the plane wave profile mentioned in the introduction.

2.2. Construction PCs for separable systems

In gravitational systems that both the null geodesic and HJ equations are separable, there is a simplification in the description of PCs of the spacetime. The separability of the HJ equation means that there is a coordinate system $\{x^\mu\}$ on the spacetime such that the HJ function S can be written as

$$S = \sum_{\mu} S_{(\mu)}(x^\mu) \quad (14)$$

where each function $S_{(\mu)}$ depends only on the coordinate x^μ , and the HJ equation reduces to a set of first order ordinary differential equations one for each $S_{(\mu)}$. Such differential equations can be integrated to determine $S_{(\mu)}$ in terms of x^μ up to an integration constant α .

The integrability of the geodesic equations means that among the coordinates $\{x^\mu\}$ that separate the null HJ equation, there is a coordinate, say $x^1 \in \{x^\mu\}$, such that the geodesic equation $\dot{x}^1 = g^{1\nu}\partial_\nu S$ is an equation that involves only x^1 and the affine parameter λ , i.e. it does not depend on the remaining spacetime coordinates. As a result this can be integrated to express x^1 in terms of λ as $x^1 = x^1(\lambda, x_0^1)$. Then the remaining geodesic equations can be solved sequentially. This means that if the solution for the first k -coordinates is known, say $x^\mu = x^\mu(\lambda, x_0^1, \dots, x_0^k)$, $\mu = 1, \dots, k$, then after a substitution of these solutions into the geodesic equation for the $k+1$ coordinate, x^{k+1} will lead to an equation which will involve only the coordinate x^{k+1} and λ . As a result, it can also be integrated to give x^{k+1} in terms of λ , and so on.

It is clear that the separability of the geodesic equations will lead to a solution of the geodesic equations which can be arranged, after a possible permutation of the coordinates, to be of the form

$$x^\mu = x^\mu(\lambda, x_0^\rho), \quad \rho \leq \mu, \quad \mu = 1, 2, 3, \dots, n. \quad (15)$$

Note that x^μ depends only on the first μ integration constants, x_0^ν , $\nu = 1, \dots, \mu$. To proceed, it is convenient to take $H(x_0^\nu) = x_0^1 = 0$. In such a case, the change of coordinate transformation to the PCs reads

$$dx^1 = \dot{x}^1 d\lambda; \quad dx^\mu = \dot{x}^\mu d\lambda + \sum_{\nu=2}^{\mu} \frac{\partial x^\mu}{\partial x_0^\nu} dx_0^\nu, \quad \mu > 1; \quad u = S(x_0^\nu) = \sum_{\mu>1} S_{(\mu)}(x_0^\mu). \quad (16)$$

For spacetimes with isometries generated by Killing vectors ξ , there is a further simplification. Typically the separability coordinates $\{x^\mu\}$ are adapted to the commuting isometries of the spacetime. If there is such an isometry ξ which is adapted to any coordinate x^μ , $\mu > 1$, say for simplicity x^2 , i.e. $\xi = \partial_{x^2}$, then the HJ function reads as

$$S = L_2 x^2 + \sum_{\mu \neq 2} S_{(\mu)}(x^\mu), \quad (17)$$

where L_2 is a constant. L_2 is one of the integration constants α of the HJ equation. For null geodesic congruences for which $L_2 \neq 0$, the coordinate transformation $u = S(x_0^\nu)$ can be solved as

$$x_0^2 = \frac{1}{L_2} \left(u - \sum_{\mu \neq 2} S_{(\mu)}(x_0^\mu) \right). \quad (18)$$

Combining the equation above with $H(x_0^\nu) = x_0^1 = 0$, one concludes that the coordinates w^i in (6) can be identified with x_0^μ , $\mu > 2$.

There may be several commuting Killing vector fields ξ on a spacetime. In such a case the HJ function can be written as

$$S = \sum_{p=2}^{k+1} L_p x^p + \sum_{\mu \neq 2, \dots, k+1} S_{(\mu)}(x^\mu), \quad (19)$$

where x^p are the coordinates adapted to the k commuting Killing vector fields and L_p are integration constants of the HJ equation. Therefore for each $L_p \neq 0$, one can solve $u = S(x_0^\nu)$ as in (18). This allows one to explore different plane wave limits of a spacetime. The constants L_p are conserved charges of the geodesic equation which label different kinematic regimes of the system. So different choices for the constants L_p may lead to different null geodesic congruences which in turn may lead to different plane wave limits for the spacetime.

3. The Kerr-Nut-(A)dS black hole

The PCs and associated limits for spherically symmetric black holes have been extensively investigated in [11, 12]. Here we shall focus on rotating black holes as described by the Kerr-NUT-(A)dS family. Later we shall specialise on the Kerr black hole and in the next section on the near horizon geometry of the extreme Kerr black hole.

The metric of the Kerr-Nut-(A)dS black hole written in the coordinates that both the geodesic and HJ equations separate can be expressed as

$$g = -\frac{\Delta_r}{\Sigma} (d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{\Sigma} (d\tau - r^2 d\psi)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_y} dy^2, \quad (20)$$

where

$$\begin{aligned} \Sigma &= r^2 + y^2, \quad \Delta_r = (r^2 + a^2)(1 - \frac{\Lambda}{3} r^2) - 2Mr, \\ \Delta_y &= (-y^2 + a^2)(1 + \frac{\Lambda}{3} y^2) + 2Ny, \end{aligned} \quad (21)$$

where M is the mass, a is the angular momentum, and N is the NUT charge of the black hole. The metric above solves the Einstein equations with a cosmological constant Λ . For dS and AdS black holes the cosmological constant is $\Lambda > 0$ and $\Lambda < 0$, respectively.

The parameters of the Kerr black hole are the mass M and the angular momentum a as $\Lambda = N = 0$. We also take $|y| < |a|$, $r > 0$ and $-\infty < \tau < +\infty$ and restrict the parameters as $M \geq |a|$.

In these coordinates the HJ equation for null geodesics for the metric (20) can be separated [25] as

$$S = -E\tau + L_\psi\psi + S_{(r)}(r) + S_{(y)}(y), \quad (22)$$

where

$$\Delta_r(\partial_r S_{(r)})^2 = \frac{\mathcal{X}_r}{\Delta_r}, \quad \Delta_y(\partial_y S_{(y)})^2 = \frac{\mathcal{X}_y}{\Delta_y}, \quad (23)$$

with

$$\mathcal{X}_r = (Er^2 - L_\psi)^2 - \Delta_r K, \quad \mathcal{X}_y = -(Ey^2 + L_\psi)^2 + \Delta_y K, \quad (24)$$

and E , L_ψ and K are integration constants α that parameterise a complete solution to the HJ equation. For this, we have used that

$$g^{-1} = \frac{1}{\Sigma} [-\Delta_r^{-1}(r^2\partial_\tau + \partial_\psi)^2 + \Delta_y^{-1}(y^2\partial_\tau - \partial_\psi)^2 + \Delta_r\partial_r^2 + \Delta_y\partial_y^2]. \quad (25)$$

Observe that ∂_τ and ∂_ψ are commuting isometries and so their adapted coordinates appear linearly in the HJ function S .

For the Kerr black hole, the HJ equation does not have solutions if the separation constant $K = 0$ and either E or L_ψ are non-vanishing. As there are no non-trivial solutions for $K = E = L_\psi = 0$, we shall take $K \neq 0$.

Returning to the Kerr-Nut-(A)dS black hole, the null geodesic equations read as

$$\dot{\tau} = -Eg^{\tau\tau} + L_\psi g^{\tau\psi}, \quad \dot{\psi} = g^{\psi\psi}L_\psi - g^{\psi\tau}E, \quad \dot{r} = g^{rr}\partial_r S_{(r)}, \quad \dot{y} = g^{yy}\partial_y S_{(y)}. \quad (26)$$

The last two equations can be solved to express both r and y coordinates in terms of the affine parameter λ . One can choose the hypersurface \mathcal{H} either as $H = r_0 = 0$ or as $H = y_0 = 0$. The two choices are equivalent. Let us take $H = r_0 = 0$. In such a case, the null geodesic equations can be formally integrated to yield

$$\tau = \tau(\lambda, y_0) + \tau_0, \quad \psi = \psi(\lambda, y_0) + \psi_0, \quad y = y(\lambda, y_0), \quad r = r(\lambda). \quad (27)$$

As a consequence the change of coordinates transformation is

$$\begin{aligned} d\tau &= \dot{\tau} d\lambda + \partial_{y_0}\tau dy_0 + d\tau_0, & d\psi &= \dot{\psi} d\lambda + \partial_{y_0}\psi dy_0 + d\psi_0, \\ dy &= \dot{y} d\lambda + \partial_{y_0}y dy_0, & dr &= \dot{r} d\lambda, & u = S &= -E\tau_0 + L_\psi\psi_0 + S_{(y_0)}(y_0), \end{aligned} \quad (28)$$

To continue the last condition should be solved to express one of the coordinates τ_0 , ψ_0 or y_0 in terms of u . For example, if either $E \neq 0$ or $L_\psi \neq 0$, then one can easily express τ_0 or ψ_0

in terms of u and the remaining coordinates, respectively. If both $E, L_\psi \neq 0$, the two choices are equivalent. Otherwise, the kinematic regimes are different. In particular if $E \neq 0$, then

$$\tau_0 = \frac{1}{E} (-u + L_\psi \psi_0 + S_{(y_0)}(y_0)). \quad (29)$$

Clearly the w coordinates in (6) are y_0, ψ_0 . Using these, the metric can be put in PCs with components

$$\begin{aligned} \Delta &= \frac{1}{E^2 \Sigma} (\Delta_y - \Delta_r), \\ h_{\psi_0} &= \frac{1}{E \Sigma} \left(\Delta_r \left(\frac{1}{E} L_\psi + y^2 \right) - \Delta_y \left(\frac{1}{E} L_\psi - r^2 \right) \right), \\ h_{y_0} &= \frac{1}{E \Sigma} (\Delta_r Q_y - \Delta_y Q_r), \\ \gamma_{y_0 y_0} &= -\frac{\Delta_r}{\Sigma} Q_y^2 + \frac{\Delta_y}{\Sigma} Q_r^2 + \frac{\Sigma}{\Delta_y} \left(\frac{\partial y}{\partial y_0} \right)^2, \\ \gamma_{\psi_0 \psi_0} &= -\frac{\Delta_r}{\Sigma} \left(\frac{1}{E} L_\psi + y^2 \right)^2 + \frac{\Delta_y}{\Sigma} \left(\frac{1}{E} L_\psi - r^2 \right)^2, \\ \gamma_{\psi_0 y_0} &= -\frac{\Delta_r}{\Sigma} Q_y \left(\frac{1}{E} L_\psi + y^2 \right) + \frac{\Delta_y}{\Sigma} Q_r \left(\frac{1}{E} L_\psi - r^2 \right), \end{aligned} \quad (30)$$

where

$$Q_y = \left(\frac{\partial \tau}{\partial y_0} + E^{-1} \frac{\partial S_{(y_0)}(y_0)}{\partial y_0} + y^2 \frac{\partial \psi}{\partial y_0} \right), \quad Q_r = \left(\frac{\partial \tau}{\partial y_0} + E^{-1} \frac{\partial S_{(y_0)}(y_0)}{\partial y_0} - r^2 \frac{\partial \psi}{\partial y_0} \right). \quad (31)$$

One can continue in the same way for $L_\psi \neq 0$ to express ψ_0 in terms of u, τ_0 and y_0 . The computation is straightforward and the result will not be stated here.

Finally if both $E = L_\psi = 0$, then $u = S_{(y_0)}(y_0)$. y_0 can be expressed in terms of u by inverting this equation and the w coordinates in (6) are τ_0 and ψ_0 . For the Kerr black hole in this kinematic regime, one has that

$$S_{(r_0)} = \int^{r_0} dp \frac{K^{\frac{1}{2}}}{|\Delta_p|}, \quad S_{(y_0)}(y_0) = \int^{y_0} dp \frac{K^{\frac{1}{2}}}{a^2 - p^2}. \quad (32)$$

where $K > 0$ and $r_- < r < r_+$, i.e. the region is between the two horizons of the Kerr black hole. The components of the metric in PCs are

$$\begin{aligned} \Delta &= -\frac{\Delta_r}{\Sigma} Q_y^2 + \frac{\Delta_y}{\Sigma} Q_r, \quad h_{\tau_0} = -\frac{\Delta_r}{\Sigma} Q_y + \frac{\Delta_y}{\Sigma} Q_r, \quad h_{\psi_0} = -\frac{\Delta_r}{\Sigma} y^2 Q_y - \frac{\Delta_y}{\Sigma} r^2 Q_r, \\ \gamma &= -\frac{\Delta_r}{\Sigma} (d\tau_0 + y^2 d\psi_0)^2 + \frac{\Delta_y}{\Sigma} (d\tau_0 - r^2 d\psi_0)^2, \end{aligned} \quad (33)$$

where now

$$Q_y = \frac{\partial \tau}{\partial u} + y^2 \frac{\partial \psi}{\partial u}, \quad Q_r = \frac{\partial \tau}{\partial u} - r^2 \frac{\partial \psi}{\partial u}. \quad (34)$$

The description of a closed expression for the metric in (30) and its associated plane wave limits depends on the construction of explicit solutions of the geodesic equation in the various kinematic regimes. This is not straightforward as the associated differential equations, although separated, are non-linear. For this we shall explore a simpler related problem that of the PCs and plane wave limits of the near horizon geometry of the extreme Kerr black hole.

4. Near horizon geometry of extreme Kerr black hole

4.1. The metric

The metric [26] of the near horizon geometry of the extreme Kerr black hole is

$$g = (1 + z^2)(-\rho^2 d\tau^2 + \rho^{-2} d\rho^2) + \frac{1 + z^2}{1 - z^2} dz^2 + \frac{1 - z^2}{1 + z^2} (2\rho d\tau + d\varphi)^2. \quad (35)$$

Next introduce the coordinates

$$v = \tau - \rho^{-1}, \quad \chi = \varphi - 2d \log \rho, \quad (36)$$

to express the above metric as

$$g = (1 + z^2)(-\rho^2 dv^2 + 2dv d\rho) + \frac{1 + z^2}{1 - z^2} dz^2 + \frac{1 - z^2}{1 + z^2} (2\rho dv + d\chi)^2. \quad (37)$$

In these coordinates the HJ equation is separable. In particular set $S = Ev + L\chi + R(\rho) + Z(z)$, E and L constants, to find that

$$\rho^2 \left(\frac{dR}{d\rho} \right)^2 + (2E - 4L\rho) \frac{dR}{d\rho} = -Q^2, \quad \left(\frac{dZ}{dz} \right)^2 + \frac{(1 + z^2)^2}{(1 - z^2)^2} L^2 = \frac{Q^2}{1 - z^2}, \quad (38)$$

where Q is a separation constant. We have also used that

$$g^{-1} = \frac{2}{1 + z^2} \partial_v \partial_\rho + \frac{\rho^2}{1 + x^2} \partial_\rho^2 - \frac{4\rho}{1 + z^2} \partial_\rho \partial_\chi + \frac{1 - z^2}{1 + z^2} \partial_z^2 + \frac{1 + z^2}{1 - z^2} \partial_\chi^2. \quad (39)$$

Further progress depends on exploring (38) and the geodesic equations.

4.2. Null Gaussian coordinates

It is well known that all regular (extreme) Killing horizons can be expressed in null Gaussian coordinates, see [27, 28]. For the near horizon Kerr geometry, these is a special case of PCs which are associated with null geodesics with $Q = L = 0$; for a different derivation see [29]. In this kinematic regime the HJ equation can be integrated to yield

$$S = Ev + 2E\rho^{-1}. \quad (40)$$

Moreover, the associated null geodesic equations can be solved to find

$$z = z_0, \quad \rho = -\frac{E}{1+z_0^2}\lambda, \quad u = \frac{2}{E}(1+z_0^2)\lambda^{-1} + u_0, \quad \chi = -4 \log \lambda + \chi_0, \quad (41)$$

where λ is the affine parameter of the geodesics and no integration constant has been introduced for ρ . Adapting PCs to this null geodesic congruence, one finds that

$$\begin{aligned} h &= \lambda \left(\frac{4(1-x^2)}{(1+x^2)^2} d\phi - \frac{2x}{1+x^2} dx \right), \\ \Delta &= \lambda^2 \left(\frac{3-6x^2-x^4}{a^2(1+x^2)^3} \right), \\ \gamma &= a^2 \frac{1+x^2}{1-x^2} dx^2 + 4a^2 \frac{1-x^2}{1+x^2} d\phi^2, \end{aligned} \quad (42)$$

with $a^2 = 1$, where we have set $x = z_0$, $-2\phi = \chi_0 - 4 \log(1+z_0^2)$ and $u = S = Ev_0$. A non-trivial a^2 factor can be introduced after rescaling the whole metric with a^2 and changing coordinates as $\lambda \rightarrow a^2\lambda$. The metric of the near horizon geometry has been put in Gaussian null coordinates, i.e. it is of the form

$$g = 2du(d\lambda + \lambda \tilde{h}_i dx^i + \frac{1}{2}\lambda^2 \tilde{\Delta} du) + \gamma_{ij} dx^i dx^j, \quad (43)$$

where \tilde{h} , $\tilde{\Delta}$ and γ depend only on the coordinate x , and where $h = \lambda\tilde{h}$ and $\Delta = \lambda^2\tilde{\Delta}$.

It has already been mentioned in the beginning of this section that all near horizon geometries of extreme regular Killing horizons can be written in null Gaussian coordinates. This means that the metric takes the form (43) and its components \tilde{h} , $\tilde{\Delta}$ and γ do not depend on the coordinates u and λ but they may depend on all the rest of the coordinates. As γ does not depend of λ , the plane wave limit is the Minkowski spacetime. Therefore all near horizon geometries of extreme Killing horizons admit Minkowski spacetime as a plane wave limit. In particular, the near horizon geometry of the Kerr black hole admits Minkowski space as plane wave limit. Typically near horizon geometries of extreme regular Killing horizons may also admit other plane wave limits adapted to a different set of PCs. For example the $\text{AdS}_2 \times S^2$ near horizon geometry of the extreme Reissner–Nordström black hole admits two different plane wave limits [20] and only one of them is the Minkowski spacetime.

4.3. The $L = 0$ congruence

Next consider the kinematic regime for which $E, Q \neq 0$ and $L = 0$. The HJ function is $S = Ev + R(\rho) + Z(z)$, where

$$R(\rho) = \frac{Q}{\cos \theta} - \epsilon_\rho Q \tan \theta + \epsilon_\rho Q \theta, \quad \rho = \frac{E}{Q} \cos \theta, \quad \epsilon_\rho = \pm 1, \quad (44)$$

and

$$Z(z) = -\epsilon_z Q \psi, \quad z = \cos \psi, \quad \epsilon_z = \pm 1, \quad (45)$$

which follow from (38). In turn, the solution to the geodesic equation can be expressed as

$$\frac{3}{2}\psi + \frac{1}{4}\sin(2\psi) = -\epsilon_z Q \lambda + \psi_0, \quad \theta = \epsilon_\rho \epsilon_z \psi + \theta_0,$$

$$\begin{aligned}
v &= -\frac{Q}{E} \left[\tan(-\epsilon_z \psi - \epsilon_\rho \theta_0) + \cos^{-1}(-\epsilon_z \psi - \epsilon_\rho \theta_0) \right] + v_0, \\
\chi &= 2 \left[\log \tan \left(\frac{-\epsilon_z \psi - \epsilon_\rho \theta_0}{2} + \frac{\pi}{4} \right) - \log \cos(-\epsilon_z \psi - \epsilon_\rho \theta_0) \right] + \chi_0.
\end{aligned} \quad (46)$$

The first equation above can be used to determine ψ in terms of the affine parameter λ . Following the general description of the PCs for systems with separable HJ and geodesic equations, one writes

$$u = S = \epsilon_\rho Q \theta_0 + E v_0, \quad \lambda = \lambda, \quad \phi = \chi_0, \quad x = \theta_0, \quad (47)$$

where ψ_0 is fixed, i.e. ψ_0 is set to zero. The metric in these coordinates reads as

$$\begin{aligned}
g = du & \left[2d\lambda + \epsilon_\rho \frac{2}{Q} \left((1 + \cos^2 \psi) + Q^2 Y_Q \cos^2(\epsilon_z \psi + \epsilon_\rho x) \right) dx \right. \\
& + \frac{4}{Q} \frac{\sin^2 \psi}{1 + \cos^2 \psi} \cos(\epsilon_z \psi + \epsilon_\rho x) d\phi + Y_Q \cos^2(\epsilon_z + \epsilon_\rho x) du \left. \right] \\
& + \frac{\sin^2 \psi}{1 + \cos^2 \psi} (d\phi - 2\epsilon_\rho \cos(\epsilon_z \psi + \epsilon_\rho x) dx)^2 \\
& + (1 + \cos^2 \psi) \sin^2(\epsilon_z \psi + \epsilon_\rho x) dx^2,
\end{aligned} \quad (48)$$

where $\psi = \psi(\lambda)$ is a solution of the first equation in (46) and

$$Y_Q = \frac{3 - 6 \cos^2 \psi - \cos^4 \psi}{Q^2 (1 + \cos^2 \psi)}. \quad (49)$$

The plane wave limit metric in Rosen coordinates is

$$g_{pw} = 2du d\lambda + \frac{\sin^2 \psi}{1 + \cos^2 \psi} (d\phi - 2\epsilon_\rho \cos(\psi) dx)^2 + (1 + \cos^2 \psi) \sin^2(\psi) dx^2. \quad (50)$$

This can be written in (u, ψ, x, ϕ) coordinates using $(1 + \cos^2 \psi) d\psi = -\epsilon_z Q d\lambda$.

Despite the simplifications, the plane wave metric (50) is implicitly described. However notice that for small ψ , one has that $2\psi = -\epsilon_z Q \lambda$ and the plane wave limit is Minkowski spacetime. Similarly for large ψ , one has $\frac{3}{2}\psi = -\epsilon_z Q \lambda$ and the plane wave metric (50) can be explicitly expressed in terms of λ as

$$\begin{aligned}
ds^2 &= 2du d\lambda + \frac{\sin^2 \left(\frac{2}{3} Q \lambda \right)}{1 + \cos^2 \left(\frac{2}{3} Q \lambda \right)} \left(d\phi - 2\epsilon_\rho \cos \left(\frac{2}{3} Q \lambda \right) dx \right)^2 \\
&+ \left(1 + \cos^2 \left(\frac{2}{3} Q \lambda \right) \right) \sin^2 \left(\frac{2}{3} Q \lambda \right) dx^2.
\end{aligned} \quad (51)$$

This completes the description of the plane wave limits of the near horizon geometry of extreme Kerr black hole in the kinematic regime $E, Q \neq 0$ and $L = 0$.

4.4. General case

Next consider the general case with $E, Q, L \neq 0$. The HJ equation (38) for R can be solved to yield

$$R = 2L \log \rho + \frac{E}{\rho} + \tilde{R}(\rho), \quad (52)$$

where

$$\tilde{R}(\rho) = \epsilon_\rho \int^\rho dp \sqrt{\left(\frac{E}{p^2} - \frac{2L}{p}\right)^2 - \frac{Q^2}{p^2}}. \quad (53)$$

Furthermore setting $z = \cos \psi$, one finds that

$$Z(z) = \Psi(\psi) = \epsilon_z \int^\psi d\beta \frac{1}{\sin \beta} \sqrt{Q^2 \sin^2 \beta - L^2(1 + \cos^2 \beta)^2}. \quad (54)$$

The geodesic equations can be expressed as

$$\begin{aligned} \dot{v} &= \frac{1}{\cos^2 \psi} \frac{dR}{d\rho}, \\ \dot{\psi} &= \frac{1}{1 + \cos^2 \psi} \frac{d\Psi}{d\psi}, \\ \dot{\chi} &= \frac{1 + \cos^2 \psi}{\sin^2 \psi} L - \frac{2\rho}{1 + \cos^2 \psi} \frac{dR}{d\rho}, \\ \dot{\rho} &= \frac{E}{1 + \cos^2 \psi} + \frac{\rho^2}{1 + \cos^2 \psi} \frac{dR}{d\rho} - \frac{2L\rho}{1 + \cos^2 \psi}. \end{aligned} \quad (55)$$

The second equation above can be solved to determine ψ in terms of the affine parameter λ as $\psi = \psi(\lambda)$. As in the previous case, we do not introduce a boundary condition for ψ . Then the last geodesic equation can be solved as $\rho = \rho(\lambda, \rho_0)$. Using this, one can solve the geodesic equation for v and χ as $v = V(\lambda, \rho_0) + v_0$ and $\chi = X(\lambda, \rho_0) + \chi_0$, respectively, where

$$V = \int^\lambda d\mu \frac{1}{1 + \cos^2 \psi} \frac{dR}{d\rho}, \quad X = \int^\lambda d\mu \left(\frac{1 + \cos^2 \psi}{\sin^2 \psi} L - \frac{2\rho}{1 + \cos^2 \psi} \frac{dR}{d\rho} \right). \quad (56)$$

To express the metric in PCs, set $x = \rho_0$, $\phi = \chi_0$, $\lambda = \lambda$ and $u = S(\rho_0, \chi_0, v_0)$. Eliminating v_0 in favour of the u coordinate, one finds that the components of the metric in the new coordinates can be expressed as

$$\begin{aligned} \Delta &= \frac{3 - 6 \cos^2 \phi - \cos^4 \psi}{E^2(1 + \cos^2 \psi)}, \quad h_\phi = -L\lambda \Delta + \frac{2\epsilon^2 \psi}{E(1 + \cos^2 \psi)} \\ h_x &= \frac{2 \sin^2 \psi}{E(1 + \cos^2 \psi)} \partial_{\rho_0} X + \frac{1}{E\lambda} (1 + \cos^2 \psi) \partial_{\rho_0} \rho - \Delta \lambda \partial_{\rho_0} R(\rho_0) + E\Delta \lambda \partial_{\rho_0} V, \\ \gamma_{\phi\phi} &= \frac{\sin^2 \psi}{1 + \cos^2 \psi} - \frac{4\lambda L \sin^2 \psi}{E(1 + \cos^2 \psi)} + L^2 \Delta \lambda^2, \end{aligned}$$

$$\begin{aligned}
\gamma_{xx} &= E^2 \Delta (E^{-1} \partial_{\rho_0} R(\rho) - \partial_{\rho_0} V)^2 - 2\lambda \frac{\sin^2 \psi}{1 + \cos^2 \psi} \partial_{\rho_0} X (E^{-1} \partial_{\rho_0} R(\rho) - \partial_{\rho_0} V) \\
&\quad - 2 \partial_{\rho_0} \rho (1 + \cos^2 \psi) (E^{-1} \partial_{\rho_0} R(\rho) - \partial_{\rho_0} V) + \frac{\sin^2 \psi}{1 + \cos^2 \psi} (\partial_{\rho_0} X)^2, \\
\gamma_{x\phi} &= -\frac{L}{E} (1 + \cos^2 \psi) \partial_{\rho_0} \rho + E^2 \Delta \lambda^2 (E^{-1} \partial_{\rho_0} R(\rho) - \partial_{\rho_0} V) \\
&\quad + \frac{\sin^2 \psi}{1 + \cos^2 \psi} ((1 - 2\lambda L E^{-1}) \partial_{\rho_0} X + 2\lambda \partial_{\rho_0} V - 2\lambda E^{-1} \partial_{\rho_0} R(\rho_0)). \quad (57)
\end{aligned}$$

The plane wave limit of this metric can be easily taken and it will not be stated here. The above expression of the metric can become explicit provided one can carry out the integrals in (56) and so integrate the null geodesic equation. It may not be possible to express such integrals in terms of ‘simple’ functions, see for example [30] for related studies.

5. Concluding remarks

We have demonstrated that for gravitational backgrounds for which both null HJ and geodesic equations separate, there is a systematic way to express the spacetime metric in PCs and proceed to take their plane wave limits. A further simplification of this construction is possible whenever the spacetime admits one or more commuting Killing vector fields. We applied this method to write the metric of Kerr-NUT-(A)dS four-dimensional black holes and that of the near horizon geometry of the extreme Kerr black hole in PCs. Then we proceeded to explore the plane wave limits of the latter.

Although the above separability properties of a spacetime allow one to give a close expression for the spacetime metric in PCs and derive the plane wave limits of these spacetimes, it does not necessarily lead to an explicit expression for the spacetime metric in PCs and its plane wave limits. This is because the latter requires an explicit solution of null geodesic equation in the HJ equation separation coordinates. Such a solution may not be written in terms of ‘simple’ functions leading to rather involved expressions for the spacetime metric. Nevertheless some plane wave limits have been found and it has been emphasized that the near horizon geometries of all extreme regular Killing horizons admit Minkowski spacetime as a plane wave limit. The propagation of strings in these plane wave limits will be presented elsewhere.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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