PTEP

Prog. Theor. Exp. Phys. **2022** 043B05(13 pages)
DOI: 10.1093/ptep/ptac054

Conditions of naturalness and fine-tuning for the type-I seesaw mechanism with four-zero texture

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Received February 7, 2022; Revised March 19, 2022; Accepted March 20, 2022; Published March 23, 2022

In this paper, we search for conditions under which the mass matrix of light neutrinos m_ν is not a result of large cancellations for the type-I seesaw mechanism with four-zero texture. For the Yukawa matrix of neutrinos Y_ν and heavy Majorana mass matrix M_R, these conditions are written as $(Y_\nu)_{i2} \propto (m_\nu)_{i2} \Rightarrow (Y_\nu)_{i2} \propto (M_R)_{i2}$. We call them *alignment* conditions because they align certain rows or columns of the three neutrino mass matrices. If these conditions do not hold, the large mixing in m_ν is a result of fine-tuning due to the cancellation of several terms. Then they are required from the viewpoint of naturalness. They give an explanation of the seesaw invariance of four-zero texture, and place rough restrictions on flavor structures of neutrinos. Under these conditions, Y_ν must have a cascade hierarchy. For M_R, the 12 submatrix has a similar hierarchy to Y_ν and m_ν. However, the 23 submatrix has a waterfall hierarchy without some fine-tuning. Therefore, it is likely that Y_ν and M_R have qualitatively different flavor structures. Furthermore, since the conditions restrict CP phases of the matrix elements, they imply the existence of a universal generalized CP symmetry in the neutrino sector.

Subject Index B52, B54, B55

1. Introduction

Research on the peculiar flavor structure of the Standard Model may provide some hints to the flavor puzzle and the theoretical origin of the Higgs boson. On the other hand, treatments of the flavor strongly depend on what we consider the Higgs boson to be. Therefore, model-independent texture studies of mass matrices M_f are a dominant approach.

Among these, the four-zero texture $(M_f)_{11} = (M_f)_{13,31} = 0$ [1] is still viable [2–26]. This system has a nice property called seesaw invariance [6,8] in the type-I seesaw mechanism [27–29]. By imposing the four-zero texture on the right-handed neutrino mass M_R and the Yukawa matrix Y_ν, the mass of light neutrinos m_ν also becomes the four-zero texture. However, the type-I seesaw relation of four-zero texture has many terms and its physical meaning is unclear.

Several studies have attempted to grasp the naturalness [30] of the seesaw parameters [31–33], in a meaning in which there is no significant cancellation between terms. In this paper, we explore the conditions under which no large cancellation occurs in this type-I seesaw relation with four-zero texture, their properties, and consequences.

This paper is organized as follows. In the next section, we explore the conditions under which the seesaw relation with four-zero texture does not have large cancellations. Section 3 gives a survey of the properties of such conditions and their restrictions on the flavor structure of neu-

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trinos. In Sect. 4, we discuss a realization of the conditions and the consequences of imposing them on m_ν with the trimaximal mixing condition. The final section is devoted to the summary.

2. Type-I seesaw mechanism and four-zero texture

In this section, we discuss cancellations between terms of neutrino mass matrices in the type-I seesaw mechanism with four-zero texture. Suppose that the Dirac neutrino mass matrix M_D and the Majorana mass matrix of right-handed neutrinos M_R have the following four-zero texture:

$$M_D = \frac{v}{\sqrt{2}} Y_\nu = \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu^* & D_\nu & B_\nu \\ 0 & B_\nu^* & A_\nu \end{pmatrix}, \; M_R = \begin{pmatrix} 0 & C_R & 0 \\ C_R & D_R & B_R \\ 0 & B_R & A_R \end{pmatrix}. \tag{1}$$

The Hermiticity of M_D can be justified by the parity symmetry of the left–right symmetric model [34–36]. Since the phases of B_ν and C_ν can be removed by redefinition of fields, M_D and Y_ν are set to be real matrices without loss of generality. Thus, let $A_\nu \sim D_\nu$ be real parameters and $A_R \sim D_R$ be complex ones.

By the type-I seesaw mechanism, the mass matrix of light neutrinos m_ν also has a four-zero texture [6,8]:

$$m_\nu = M_D M_R^{-1} M_D^T \tag{2}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{F_\nu^2}{A_R} & \frac{A_\nu F_\nu}{A_R} \\ 0 & \frac{A_\nu F_\nu}{A_R} & \frac{A_\nu^2}{A_R} \end{pmatrix} + \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & 0 \\ \frac{C_\nu^2}{C_R} & \frac{2C_\nu D_\nu}{C_R} - \frac{C_\nu^2 D_R}{C_R^2} & \frac{B_\nu C_\nu}{C_R} \\ 0 & \frac{B_\nu C_\nu}{C_R} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}, \tag{3}$$

where $F_\nu = (B_\nu C_R - B_R C_\nu)/C_R$. In general, $a \sim d$ are also complex parameters. Note that the first matrix has determinant zero and its rank is equal to one. If $|A_\nu| \gtrsim |F_\nu|$ holds, the contributions of the first matrix in Eq. (3) can be neglected except for the 33 element, because the parameters a, b, and d are expected to be of similar magnitude from the bimaximal mixing of θ_{23}. By assuming that Y_ν and M_D have the following hierarchy:

$$|A_\nu| \gg |B_\nu|, |D_\nu|, |C_\nu|, \tag{4}$$

this requirement is rewritten as

$$|A_\nu| \gtrsim |F_\nu| = \left| B_\nu - C_\nu \frac{B_R}{C_R} \right| \quad \Leftrightarrow \quad \left| \frac{A_\nu}{C_\nu} \right| \gtrsim \left| \frac{B_R}{C_R} \right|. \tag{5}$$

The condition (5) compares degrees of hierarchies between M_R and M_D. Under Eq. (5), the form of m_ν becomes rather concise:

$$m_\nu \simeq \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & 0 \\ \frac{C_\nu^2}{C_R} & \frac{C_\nu D_\nu}{C_R} + \frac{C_\nu}{C_R}(D_\nu - C_\nu \frac{D_R}{C_R}) & \frac{B_\nu C_\nu}{C_R} \\ 0 & \frac{B_\nu C_\nu}{C_R} & \frac{A_\nu^2}{A_R} \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}. \tag{6}$$

In this case, we obtain

$$\frac{b}{c} \simeq \frac{B_\nu}{C_\nu}, \tag{7}$$

a relation of ratios between elements of m_ν and Y_ν.

On the other hand, when Eqs. (5) and (7) do not hold and the hierarchy of M_R is so strong $|A_\nu/C_\nu| \ll |B_R/C_R|$, the terms containing F_ν in Eq. (3) are noticeably larger than the other matrix elements $m_\nu \sim C_\nu^2/C_R \sim A_\nu^2/A_R$. In such a situation, the large mixings of neutrinos are a result of fine-tuning between terms with large magnitudes. Therefore, the condition (7) is also required from the naturalness in the meaning of no fine-tuning.

When the conditions (5) and (7) hold, the smallness of F_ν itself is also not a consequence of a large cancellation:

$$\left|\frac{A_\nu F_\nu}{A_R}\right| \ni \left|-\frac{A_\nu}{A_R}\frac{C_\nu B_R}{C_R}\right| \lesssim \left|\frac{A_\nu^2}{A_R}\right|. \tag{8}$$

Thus, the absolute values of the terms in $A_\nu F_\nu/A_R$ are also smaller than those in m_ν, respectively.

A similar relation holds for the 22 element of m_ν. From $|b| \sim |d|$ in Eq. (6), a condition for a large cancellation becomes

$$\frac{2C_\nu D_\nu}{C_R} \simeq \frac{C_\nu^2 D_R}{C_R^2}, \quad \left|\frac{C_\nu D_\nu}{C_R}\right| \gg |\frac{C_\nu B_\nu}{C_R}|. \tag{9}$$

This can be rewritten as the following:

$$\left|D_\nu - C_\nu \frac{D_R}{C_R}\right| \simeq |D_\nu| \gg |B_\nu|. \tag{10}$$

In contrast, a condition for such cancellation not to occur is

$$\left|D_\nu - C_\nu \frac{D_R}{C_R}\right| \lesssim |B_\nu|. \tag{11}$$

In this case $d \simeq C_\nu D_\nu/C_R$ holds and we obtain

$$\frac{d}{c} \simeq \frac{D_\nu}{C_\nu}. \tag{12}$$

The same condition as Eq. (8) can hold for D_ν. The 22 element of m_ν is

$$(m_\nu)_{22} = d \simeq c\frac{D_\nu}{C_\nu} = \frac{C_\nu D_\nu}{C_R}, \tag{13}$$

and each term in Eqs. (3) and (6) has an absolute value of only about $(m_\nu)_{22}$. Therefore, Eqs. (7) and (12) are sufficient conditions for the naturalness.

Note that these relations include phases of b, c, and d. Since B_ν, C_ν, and D_ν are taken as real parameters, the relative phases of b, c, and d must be almost zero (or π) in this basis. The phases of A_ν and A_R are not restricted by these conditions. If these two phases are aligned as well as the other parameters B_ν, C_ν, and D_ν, such a situation is realized by the diagonal reflection symmetries (DRS) [22–24]:

$$R m_\nu^* R = m_\nu, \quad R M_{D,R}^* R = M_{D,R}, \tag{14}$$

where

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (m_\nu, M_{D,R}) = \begin{pmatrix} F_{\nu,D,R} & iC_{\nu,D,R} & iE_{\nu,D,R} \\ iC_{\nu,D,R} & D_{\nu,D,R} & B_{\nu,D,R} \\ iE_{\nu,D,R} & B_{\nu,D,R} & A_{\nu,D,R} \end{pmatrix}. \tag{15}$$

By redefining the phases of l_L and ν_R, all the mass matrices m_ν and $M_{D,R}$ in the neutrino sector have only real parameters, and indeed the conditions (7) and (12) are real. More generally, this situation can be achieved by an imposition of the same generalized CP symmetry (GCP) [37–40] on the neutrino sector. For this purpose, it is sufficient that the GCPs of M_D and M_R are the same in the seesaw mechanism,

$$X M_D^* X^\dagger = M_D, \quad X^* M_R^* X^\dagger = M_R, \quad \Rightarrow \quad X m_\nu^* X^T = m_\nu, \tag{16}$$

with a Hermitian unitary matrix $X = X^\dagger$.

2.1 *Reasons for the conditions*

Here, we discuss why such conditions are necessary. When the mass matrices (1) are expressed as $M_D = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ and $M_R = (\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3)$ with 3D vectors $\boldsymbol{y}_i$ and $\boldsymbol{M}_i$, the inverse matrix of M_R can be written as

$$M_R^{-1} = \frac{1}{\mathrm{Det} M_R} \begin{pmatrix} A_R D_R - B_R^2 & -A_R C_R & B_R C_R \\ -A_R C_R & 0 & 0 \\ B_R C_R & 0 & -C_R^2 \end{pmatrix} = \frac{1}{\mathrm{Det} M_R} \begin{pmatrix} (\boldsymbol{M}_2 \times \boldsymbol{M}_3)^T \\ (\boldsymbol{M}_3 \times \boldsymbol{M}_1)^T \\ (\boldsymbol{M}_1 \times \boldsymbol{M}_2)^T \end{pmatrix}. \tag{17}$$

The first matrix in m_ν (3) is generated from part of M_R^{-1}:

$$m_\nu \ni \frac{1}{\mathrm{Det} M_R} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} \begin{pmatrix} -B_R^2 & 0 & B_R C_R \\ 0 & 0 & 0 \\ B_R C_R & 0 & -C_R^2 \end{pmatrix} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{F_\nu^2}{A_R} & \frac{A_\nu F_\nu}{A_R} \\ 0 & \frac{A_\nu F_\nu}{A_R} & \frac{A_\nu^2}{A_R} \end{pmatrix}. \tag{18}$$

By comparing the 32 and 33 elements of this matrix, the restriction (5) on F_ν is rewritten as a condition on the third row of $\frac{1}{M_R}$ as follows:

$$|F_\nu| \lesssim |A_\nu| \quad \Rightarrow \quad |\mathrm{Det}(\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{y}_2)| \lesssim |\mathrm{Det}(\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{y}_3)|. \tag{19}$$

The rest of m_ν comes from

$$m_\nu \ni \frac{1}{\mathrm{Det} M_R} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} \begin{pmatrix} A_R D_R & -A_R C_R & 0 \\ -A_R C_R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & 0 \\ \frac{C_\nu^2}{C_R} & \frac{2C_\nu D_\nu}{C_R} - \frac{C_\nu^2 D_R}{C_R^2} & \frac{B_\nu C_\nu}{C_R} \\ 0 & \frac{B_\nu C_\nu}{C_R} & 0 \end{pmatrix}. \tag{20}$$

In a similar way, under the assumption that Eq. (19) holds, the condition (11) for D_R can be rewritten as:

$$\left| D_\nu - C_\nu \frac{D_R}{C_R} \right| \lesssim |B_\nu| \quad \Rightarrow \quad |\mathrm{Det}(\boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{y}_2)| \quad \lesssim |\mathrm{Det}(\boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{y}_3)|. \tag{21}$$

If the vectors $\boldsymbol{M}_{1,2,3}$ are linearly independent, the conditions that the two determinants are small can be rewritten as

$$|\mathrm{Det}(\boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{y}_2)|, |\mathrm{Det}(\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{y}_2)| \simeq 0 \quad \Leftrightarrow \quad \boldsymbol{y}_2 \simeq \alpha \boldsymbol{M}_2 \simeq \beta \boldsymbol{m}_2. \tag{22}$$

Here α, β are some complex coefficients and $m_\nu \equiv (\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$. In equivalent notation, we arrive at the following result:

$$(M_D)_{i2} \simeq \alpha (M_R)_{i2} \simeq \beta (m_\nu)_{i2}. \tag{23}$$

These conditions align the certain rows (or columns) of the three mass matrices of neutrinos. Therefore, we call Eqs. (7) and (12), or Eq. (23), *alignment conditions*. Similar conditions are expected to hold for a wide range of textures other than the four-zero texture.

In the four-zero texture (1), the first row and column satisfy the exact alignment conditions $(M_D)_{i1} = \alpha (M_R)_{i1} = \beta (m_\nu)_{i1}$, which is the cause of the seesaw invariance [6,8]. Since all the other seesaw-invariant four-zero textures [6] have two texture zeros in the same row or column, we can also consider the invariance a result of the alignment conditions.

The discussion here is similar to sequential dominance [41–45]. However, it is different in that M_R is not diagonalized. The result obtained on the trimaximal condition later is also different from the constrained sequential dominance [46–49].

3. Properties of alignment conditions

In this section, some properties of the alignment conditions are explored. First of all, we will consider the parameter region where the conditions (7) and (12) hold. The Majorana mass matrix of ν_R is represented by M_D and m_ν as

$$M_R = M_D^T m_\nu^{-1} M_D \tag{24}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{F_R^2}{a} & \frac{A_\nu F_R}{a} \\ 0 & \frac{A_\nu F_R}{a} & \frac{A_\nu^2}{a} \end{pmatrix} + \begin{pmatrix} 0 & \frac{C_\nu^2}{c} & 0 \\ \frac{C_\nu^2}{c} & \frac{C_\nu D_\nu}{c} + \frac{C_\nu}{c}(D_\nu - \frac{d}{c}C_\nu) & \frac{B_\nu C_\nu}{c} \\ 0 & \frac{B_\nu C_\nu}{c} & 0 \end{pmatrix} = \begin{pmatrix} 0 & C_R & 0 \\ C_R & D_R & B_R \\ 0 & B_R & A_R \end{pmatrix}, \tag{25}$$

where $F_R = (cB_\nu - bC_\nu)/c$. Imposing the inequality (5) on the 12 and 23 elements of M_R, we obtain

$$\left|\frac{B_R}{C_R}\right| = \left|\frac{A_\nu(cB_\nu - bC_\nu) + aB_\nu C_\nu}{aC_\nu^2}\right| \lesssim \left|\frac{A_\nu}{C_\nu}\right|. \tag{26}$$

Since the term $aB_\nu C_\nu$ can be neglected from the hierarchy (4), Eq. (26) restricts a deviation ϵ of B_ν from Eq. (7) as

$$B_\nu \equiv \frac{b}{c}C_\nu + \epsilon, \quad |\epsilon| = \left|B_\nu - \frac{b}{c}C_\nu\right| \lesssim \left|\frac{aC_\nu}{c}\right|. \tag{27}$$

Thus, this ϵ is restricted to a region with a width of about C_ν. For example, by assuming $Y_\nu \simeq Y_e$, the range of this limit is about $C_e \simeq \sqrt{m_\mu m_e} \sim 7$MeV with $B_e \simeq m_\mu \sim 100$MeV. The range is sufficiently large to allow renormalizations and threshold corrections. From $|a| \sim |b|$, the magnitude of B_ν for this range of $|\epsilon|$ becomes $0 \lesssim |B_\nu| \lesssim 2bC_\nu/c$.

Similarly, we define a complex parameter δ for the 22 element:

$$D_\nu \equiv \frac{d}{c}C_\nu + \delta. \tag{28}$$

Equations (11) and (25) lead to

$$\left|D_\nu - C_\nu\frac{D_R}{C_R}\right| = \left|-\left(D_\nu - \frac{d}{c}C_\nu\right)\right| = |\delta| \lesssim |B_\nu| \simeq \left|\frac{bC_\nu}{c}\right|. \tag{29}$$

From $|a| \sim |b| \sim |d|$, the allowed ranges of ϵ and δ are comparable.

3.1 *Flavor structure and alignment conditions*

The alignment conditions place rough constraints on the hierarchies of Y_ν and M_R. If the conditions (7) and (12) hold, the hierarchy of Y_ν becomes a cascade type. Here, waterfall (geometric) and cascade textures are matrices of the following forms [50,51]:

$$M_{\text{waterfall}} \sim \begin{pmatrix} \delta^2 & \lambda\delta & \delta \\ \lambda\delta & \lambda^2 & \lambda \\ \delta & \lambda & 1 \end{pmatrix}, \quad M_{\text{cascade}} \sim \begin{pmatrix} \delta & \delta & \delta \\ \delta & \lambda & \lambda \\ \delta & \lambda & 1 \end{pmatrix}. \tag{30}$$

Although Ref. [51] assumed $1 \gg \lambda \gg \delta$, this paper includes a situation with $\lambda \gtrsim \delta$. If the four-zero texture arises from sequential breaking of flavor symmetry such as $SU(3)_F$ (or its subgroup), the cascade type is desirable for flavor structures of Y_ν and M_D. The observation of the CKM matrix and some unification also suggest the cascade type for Hermitian Y_d and Y_e [23].

Furthermore, the bimaximal mixing of θ_{23} implies $|a| \sim |b| \sim |d|$ in m_ν (6). Bimaximality places constraints on the magnitudes of matrix elements. First, the condition $|a| \sim |b|$ in Eq. (6) leads to

$$\frac{B_\nu C_\nu}{C_R} \sim \frac{A_\nu^2}{A_R} \quad \Leftrightarrow \quad \frac{A_R}{C_R} \sim \frac{A_\nu}{B_\nu}\frac{A_\nu}{C_\nu}. \tag{31}$$

Thus, the magnitude of M_R for the third generation is enhanced by A_ν/B_ν and is much larger than that of Y_ν. If there is also no fine-tuning between the terms in the 22 element, $d \sim b$ also leads to

$$\frac{D_R}{C_R} \sim \frac{D_\nu}{C_\nu} \sim \frac{B_\nu}{C_\nu} \simeq \frac{b}{c}. \tag{32}$$

This suggests a mild hierarchical structure for light generations of M_R, similar to those of Y_ν and m_ν.

These facts can also be seen from the reconstruction of M_R. Substituting Eqs. (27) and (28) into Eq. (25), M_R for a given Y_ν and an upper limit of its absolute value are

$$M_R = \begin{pmatrix} 0 & \frac{C_\nu^2}{c} & 0 \\ \frac{C_\nu^2}{c} & \frac{\epsilon^2}{a} + \frac{dC_\nu^2}{c^2} + \frac{2\delta C_\nu}{c} & \frac{\epsilon A_\nu}{a} + \frac{bC_\nu^2}{c^2} + \frac{\epsilon C_\nu}{c} \\ 0 & \frac{\epsilon A_\nu}{a} + \frac{bC_\nu^2}{c^2} + \frac{\epsilon C_\nu}{c} & \frac{A_\nu^2}{a} \end{pmatrix} \lesssim \frac{C_\nu}{c}\begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & 4D_\nu & A_\nu \\ 0 & A_\nu & \frac{cA_\nu^2}{aC_\nu} \end{pmatrix}. \tag{33}$$

In this case, M_R has a partial cascade hierarchy and satisfies Eqs. (31) and (32). On the other hand, in a situation like $|B_\nu|, |D_\nu| \gg |bC_\nu/c|$, Eqs. (31) and (32) do not hold and M_R will be a waterfall-type matrix [20]:

$$M_R \simeq \begin{pmatrix} 0 & \frac{C_\nu^2}{c} & 0 \\ \frac{C_\nu^2}{c} & \frac{B_\nu^2}{a} + \frac{2D_\nu C_\nu}{c} & \frac{A_\nu B_\nu}{a} \\ 0 & \frac{A_\nu B_\nu}{a} & \frac{A_\nu^2}{a} \end{pmatrix}. \tag{34}$$

In particular, if equalities hold for the conditions (7) and (12) with $\epsilon = \delta = 0$, the conditions also hold for M_R (33):

$$\frac{D_\nu}{C_\nu} = \frac{d}{c} \quad \Leftrightarrow \quad \frac{D_R}{C_R} = \frac{D_\nu}{C_\nu}. \tag{35}$$

$$\frac{B_\nu}{C_\nu} = \frac{b}{c} \quad \Leftrightarrow \quad \frac{B_R}{C_R} = \frac{B_\nu}{C_\nu}. \tag{36}$$

Equivalently, the structure of M_R becomes

$$M_R = \frac{C_\nu}{c}\begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & \frac{c}{a}\frac{A_\nu^2}{C_\nu} \end{pmatrix} = \begin{pmatrix} 0 & C_R & 0 \\ C_R & D_R & B_R \\ 0 & B_R & A_R \end{pmatrix}. \tag{37}$$

In this case, a partially universal texture of the lighter generations emerges in the neutrino sector m_ν, M_D, and M_R. This is rather attractive from the viewpoint of model building.

Table 1. Charge assignments of fields under gauge and flavor symmetries.

	$SU(2)_L$	$U(1)_Y$	$SU(3)_F$	$U(1)$
l_{Li}	**2**	$-1/2$	**3**	$+1$
ν_{Ri}	**1**	0	$\mathbf{3}^*$	-1
H	**2**	1/2	**1**	0
Δ	**1**	0	**1**	0
η	**1**	1	**3**	$+1$
θ	**1**	1	**6**	$+2$

3.2 *The conditions for M_R*

Let us investigate the range where these conditions (35) and (36) hold for D_R and B_R. First, by taking a ratio of the 12 and 22 elements in Eq. (33),

$$\frac{(M_R)_{22}}{(M_R)_{12}} = \frac{d}{c} + \frac{2\delta}{C_\nu} + \frac{\epsilon^2 c}{aC_\nu^2}. \tag{38}$$

From Eqs. (27)–(29), $|\delta|, |\epsilon| \lesssim |bC_\nu/c|$ holds. Then the upper limit of the absolute value of Eq. (38) in this range will be about $4d/c$, which is comparable to the allowed range $0 \lesssim |D_\nu/C_\nu| \lesssim 2d/c$ for D_ν/C_ν.

However, the condition for B_R (36) holds in a very narrow region. In the same way as in Eq. (38), the ratio of matrix elements of M_R (33) becomes

$$\frac{(M_R)_{23}}{(M_R)_{12}} = \frac{b}{c} + \frac{\epsilon}{C_\nu} + \frac{\epsilon c A_\nu}{aC_\nu^2}. \tag{39}$$

From this, the range of ϵ for which Eq. (36) is approximately valid is

$$\left|\epsilon \frac{cA_\nu}{aC_\nu^2}\right| \lesssim \frac{b}{c} \quad \Rightarrow \quad |\epsilon| \lesssim \frac{abC_\nu^2}{c^2 A_\nu}. \tag{40}$$

This means that the magnitude of ϵ is constrained to about C_ν^2/A_ν. For example, a unified relation $Y_\nu \simeq Y_e$ yields a very narrow range, $C_e^2/A_e \simeq m_e m_\mu/m_\tau \simeq 0.03\mathrm{MeV}$. Such a strict restriction is not realistic for renormalizations and/or threshold corrections. Conversely, if we consider the upper limit of ϵ as Eq. (27), the third term in Eq. (39) for $\epsilon \sim C_\nu$ is as large as A_ν/C_ν. Therefore, the condition (36) for B_R seems to be less rigid than that of Y_ν.

In conclusion, the flavor structure of M_R with four-zero texture and alignment conditions can be similar to Y_ν for the 12 submatrix. By contrast, it is difficult to make the same structures for the 23 submatrix of M_R and Y_ν without fine-tuning. As can be seen in Eq. (33), Y_ν and M_R are likely to have qualitatively different flavor structures. However, even if B_ν at low energy does not satisfy the strict limit (40), it is possible that the universality (36) $B_R/C_R = B_\nu/C_\nu$ is valid in a high-energy region such as the GUT scale.

4. Realization and application of the conditions

In this section, we discuss a field theoretical realization of such alignment conditions and apply them to realistic textures. Let us consider fields that transform under gauge symmetry and $SU(3)_F \times U(1)$ flavor symmetry, as shown in Table 1. In the context of $SU(3)_F$ [52–55], a three-dimensional representation of flavon η is used to generate a four-zero texture. However, this is an antisymmetric representation and cannot be used for M_R. Then, a six-dimensional representation of flavon θ is adopted. The $U(1)$ symmetry prohibits the $\mathbf{3}(\mathbf{6})$ representation $\eta(\theta)$ from being used an odd (even) number of times[1].

Under these symmetries, the general invariant Yukawa interactions up to dimension six are

$$\mathcal{L} = -\left(y_\nu^\eta \frac{\eta_i\eta_j}{\Lambda^2} + y_\nu^\theta \frac{\theta_{ij}}{\Lambda}\right)\bar{l}_{Li}\nu_{Rj}\tilde{H} - \left(y_R^\eta \frac{\eta_i\eta_j}{\Lambda^2} + y_R^\theta \frac{\theta_{ij}}{\Lambda}\right)\bar{\nu}_{Ri}^c \nu_{Rj}\Delta + \text{h.c.}, \tag{41}$$

where Λ is a cut-off scale. The general vacuum expectation values (vevs) for the three and six representations are

$$\langle\eta\rangle = \begin{pmatrix} C \\ B \\ A \end{pmatrix}, \quad \langle\theta\rangle = \begin{pmatrix} f' & e' & d' \\ e' & c' & b' \\ d' & b' & a' \end{pmatrix}, \tag{42}$$

with complex values $A \sim C$ and $a \sim f$.

By redefining the neutrino fields (or weak basis transformations (WBTs) [57,58]), there is no loss of generality in rewriting $\langle\eta\rangle = (0, 0, A)^T$. Furthermore, the 11 element of $\langle\theta\rangle$ can be set to zero by a WBT for 12 mixing. This leads to

$$\langle\eta\rangle = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix}, \quad \langle\theta\rangle = \begin{pmatrix} 0 & e & d \\ e & c & b \\ d & b & a \end{pmatrix}. \tag{43}$$

For hierarchical flavor structures, we assume that $y_{\nu,R}^\eta A^2/\Lambda^2 \gg y_{\nu,R}^\theta \langle\theta\rangle_{ij}/\Lambda$ in this basis.

As Δ and H acquire vevs, the neutrino mass matrices are respectively

$$\tilde{M}_D = \frac{y_\nu^\theta}{\Lambda}\langle H\rangle \begin{pmatrix} 0 & e & d \\ e & c & b \\ d & b & a + \frac{y_\nu^\eta}{y_\nu^\theta}\frac{A^2}{\Lambda} \end{pmatrix}, \quad \tilde{M}_R = \frac{y_R^\theta}{\Lambda}\langle\Delta\rangle \begin{pmatrix} 0 & e & d \\ e & c & b \\ d & b & a + \frac{y_R^\eta}{y_R^\theta}\frac{A^2}{\Lambda} \end{pmatrix}. \tag{44}$$

In this basis, the first and second generations satisfy the exact alignment conditions:

$$(\tilde{M}_D)_{i1} = \alpha(\tilde{M}_R)_{i1} = \beta(m_\nu)_{i1}, \quad (\tilde{M}_D)_{i2} = \alpha'(\tilde{M}_R)_{i2} = \beta'(m_\nu)_{i2}. \tag{45}$$

Furthermore, when the 13 and 31 elements are set to zero by WBTs of 23 mixing with $\tan\phi = d/e$, a four-zero texture with broken alignment is realized. The 22 elements of mass matrices in such a basis are

$$(\tilde{M}_D)_{22} = c\cos^2\phi + 2b\sin\phi\cos\phi + \left(a + \frac{y_\nu^\eta}{y_\nu^\theta}\frac{A^2}{\Lambda}\right)\sin^2\phi, \tag{46}$$

$$(\tilde{M}_R)_{22} = c\cos^2\phi + 2b\sin\phi\cos\phi + \left(a + \frac{y_R^\eta}{y_R^\theta}\frac{A^2}{\Lambda}\right)\sin^2\phi. \tag{47}$$

Breaking effects come from the difference in the ratio of y_ν^η/y_ν^θ and y_R^η/y_R^θ. However, if the mixing angle ϕ is small (i.e., $d \ll e$), the effects $\frac{y_{\nu,R}^\eta}{y_{\nu,R}^\theta}\frac{A^2}{\Lambda}\sin^2\phi$ can also be small. For example, if $\frac{y_{\nu,R}^\eta}{y_{\nu,R}^\theta}\frac{A^2}{\Lambda} : c \simeq m_t : m_c \simeq 300 : 1$ holds like up-type quarks, conditions to prevent a misalignment $|\frac{y_{\nu,R}^\eta}{y_{\nu,R}^\theta}\frac{A^2}{\Lambda}\sin^2\phi| \lesssim c$ lead to an allowed region of ϕ as $|\sin\phi| \lesssim \sqrt{1/300} \simeq 0.06$. By contrast, if $\frac{y_{\nu,R}^\eta}{y_{\nu,R}^\theta}\frac{A^2}{\Lambda} : c \simeq 16 : 1$ holds as in charged leptons, the allowed region of ϕ becomes $|\sin\phi| \lesssim \sqrt{1/16} \simeq 0.25$ and it has a wide parameter range.

[1]From an irreducible decomposition of $SU(3)$, $6 \times 6 = \bar{6}_s + 15_a + 15'_s$, $6 \times \bar{6} = 1 + 8 + 27$ [56], it is possible to write an invariant interaction using (the complex conjugate of) the six representation twice.

Although the effect on the 23 element b is about $\frac{y_R^\eta}{y_R^\theta}\frac{A^2}{\Lambda}\sin\phi\cos\phi$, the alignment condition does not have to be so rigid by Eq. (39). In this way, the alignment condition (35) is not difficult to achieve if there is a flavon that generates the first and second generations of M_D and M_R simultaneously.

4.1 *Application to trimaximal mixing and magic symmetry*

Here we will analyze a neutrino mass matrix with four-zero texture that satisfies the alignment conditions and the trimaximal mixing [59–65]. The four-zero texture with a trimaximal condition is given by [66]

$$m_{\nu T} = \begin{pmatrix} 0 & c & 0 \\ c & c+a & c+a \\ 0 & c+a & a \end{pmatrix}, \quad m_{\nu T}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \tag{48}$$

This matrix satisfies the following magic symmetry [62], which is equivalent to the eigenvector condition (48):

$$S_2 m_{\nu T} S_2 = m_{\nu T}, \quad S_2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad S_2^2 = 1_3. \tag{49}$$

However, even if we impose the trimaximal condition (48) on the general m_ν (3), there are so many terms and the physical meanings are unclear. Besides, if the trimaximal mixing is a consequence of fine-tuning, it is difficult to derive relations for the matrix elements of Y_ν and M_R.

Therefore, here we impose the alignment conditions (23), or equivalently Eqs. (7) and (12). Similar results have been obtained in several studies [31,32,51]. The relations $B_\nu/D_\nu = B_R/D_R = 1$ are relatively easy to realize in an A_4 model with a flavon with vev $\langle\varphi\rangle = (0, 1, 1)$. A specific realization can be seen in Ref. [24]. Although the other one $C_\nu/D_\nu = C_R/D_R$ is non-trivial, it is possible if we require some kind of unification between l_l and ν_R (or Y_ν and M_R) when constructing the flavor structure.

From Eqs. (27) and (28), or $b = \frac{c}{C_\nu}(B_\nu - \epsilon)$ and $d = \frac{c}{C_\nu}(D_\nu - \delta)$, the mass matrix m_ν (3) is represented by the deviations ϵ and δ as

$$m_\nu = \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & 0 \\ \frac{C_\nu^2}{C_R} & \frac{C_\nu}{C_R}(D_\nu - \delta) & \frac{C_\nu}{C_R}(B_\nu - \epsilon) \\ 0 & \frac{C_\nu}{C_R}(B_\nu - \epsilon) & \frac{A_\nu^2}{A_R} \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}. \tag{50}$$

By identifying Eq. (50) with $m_{\nu T}$ (48), two new relations arise:

$$c + a = \frac{C_\nu^2}{C_R} + \frac{A_\nu^2}{A_R} = \frac{C_\nu}{C_R}(B_\nu - \epsilon) = \frac{C_\nu}{C_R}(D_\nu - \delta), \tag{51}$$

or, equivalently,

$$C_\nu\left(1 + \frac{a}{c}\right) = D_\nu - \delta = B_\nu - \epsilon. \tag{52}$$

By the alignment conditions $(Y_\nu)_{2i} \simeq \alpha(m_\nu)_{2i}$, the second row (or column) $(Y_\nu)_{2i}$ also satisfies the trimaximal condition approximately. Without the conditions, it is difficult to extract such a relation from only the trimaximal condition (48).

Under the alignment conditions, the form of M_D is found to be

$$M_D = \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & \frac{(a+c)C_\nu}{c} + \delta & \frac{(a+c)C_\nu}{c} + \epsilon \\ 0 & \frac{(a+c)C_\nu}{c} + \epsilon & A_\nu \end{pmatrix}. \tag{53}$$

By further imposing DRS (14), $m_{\nu T}$ (48) reproduces the MNS matrix with an accuracy of $O(10^{-2})$ and predicts $a \simeq 2c$ [24]. In this case, these parameters are evaluated to $B_\nu \sim D_\nu \sim 3C_\nu$. A reconstructed M_R (33) from m_ν and M_D becomes

$$M_R = \begin{pmatrix} 0 & \frac{C_\nu^2}{c} & 0 \\ \frac{C_\nu^2}{c} & \frac{\epsilon^2}{a} + \frac{(a+c)C_\nu^2}{c^2} + \frac{2\delta C_\nu}{c} & \frac{\epsilon A_\nu}{a} + \frac{(a+c)C_\nu^2}{c^2} + \frac{\epsilon C_\nu}{c} \\ 0 & \frac{\epsilon A_\nu}{a} + \frac{(a+c)C_\nu^2}{c^2} + \frac{\epsilon C_\nu}{c} & \frac{A_\nu^2}{a} \end{pmatrix}. \tag{54}$$

Thus, we expect a universal mild hierarchy for the lighter generations among m_ν, Y_ν, and M_R.

Furthermore, the relations $A_{\nu,\,R} + C_{\nu,\,R} = B_{\nu,\,R}$ lead to the magic symmetry (49) for M_D and M_R, and the matrix elements of m_ν, M_D, and M_R are concisely related as in the previous study [24]. In this case, at the cost of losing the hierarchy (4), the alignment conditions (7) and (12) are not necessarily needed.

Perhaps a solution $F_\nu = A_\nu$ or, equivalently, $\epsilon C_\nu / C_R = a$ is similar to the sequential dominance [41,42] and seems to be natural because it separates the contributions of the two mass scales a and c in Eq. (48). In this case, $B_\nu = C_\nu$ holds and further imposition of $B_\nu = D_\nu$ leads to $D_R = C_R$. This is also similar to the constrained sequential dominance (CSD) [46].

5. Summary

In this paper, we search for conditions that the mass matrix of light neutrinos m_ν is not a result of large cancellations between terms for the type-I seesaw mechanism with four-zero texture. For the Yukawa matrix of neutrinos Y_ν and heavy Majorana mass matrix M_R, these conditions have the form $(Y_\nu)_{i2} \propto (m_\nu)_{i2} \Rightarrow (Y_\nu)_{i2} \propto (M_R)_{i2}$. We call them *alignment* conditions because they align certain rows or columns of the three neutrino mass matrices. If these conditions do not hold, the large mixing in m_ν is a result of fine-tuning due to the cancellation of terms with large magnitudes. Then they are required from the viewpoint of naturalness. Similar conditions are expected to hold for a wide range of textures other than the four-zero texture. In particular, if there are two zero textures in the same row and column of Y_ν and M_R, the mass of the light neutrinos m_ν has the same property. Some kinds of seesaw-invariant zero textures can also be explained from the conditions.

They place rough restrictions on the flavor structures of neutrinos. Under these conditions, Y_ν must have a cascade hierarchy. For M_R, the 12 submatrix has a similar hierarchy to Y_ν and m_ν. However, the 23 submatrix has a waterfall hierarchy without some fine-tuning for matrix elements of Y_ν. Therefore, it is likely that Y_ν and M_R have qualitatively different flavor structures. Furthermore, since the conditions incorporate *CP* phases of the matrix elements, they imply the existence of a universal generalized *CP* symmetry in the neutrino sector.

If the conditions are satisfied, information on Y_ν can be immediately extracted from a reconstructed m_ν. By combining the alignment conditions with the trimaximal mixing condition, the second row (or column) $(Y_\nu)_{2i}$ also satisfies the trimaximal condition. This yields two relations

$(Y_\nu)_{22} \simeq (Y_\nu)_{23} \simeq (Y_\nu)_{12}[(m_\nu)_{12} + (m_\nu)_{33}]/(m_\nu)_{12}$. Without the conditions, it is generally difficult to translate the trimaximal condition into such a relation for matrix elements in the type-I seesaw mechanism.

Acknowledgements

This study is financially supported by JSPS Grants-in-Aid for Scientific Research No. 18H01210 and No. 20K14459 and a MEXT KAKENHI Grant No. 18H05543.

Funding

Open Access funding: SCOAP^3.

REFERENCES

[1] H. Fritzsch and Z.-z. Xing, Phys. Lett. B **353**, 114 (1995) [arXiv:hep-ph/9502297] [Search INSPIRE].
[2] H. Lehmann, C. Newton, and T. T. Wu, Phys. Lett. B **384**, 249 (1996).
[3] K. Kang and S. K. Kang, Phys. Rev. D **56**, 1511 (1997) [arXiv:hep-ph/9704253] [Search INSPIRE].
[4] A. Mondragon and E. Rodriguez-Jauregui, Phys. Rev. D **59**, 093009 (1999) [arXiv:hep-ph/9807214] [Search INSPIRE].
[5] J. L. Chkareuli and C. D. Froggatt, Phys. Lett. B **450**, 158 (1999) [arXiv:hep-ph/9812499] [Search INSPIRE].
[6] H. Nishiura, K. Matsuda, and T. Fukuyama, Phys. Rev. D **60**, 013006 (1999) [arXiv:hep-ph/9902385] [Search INSPIRE].
[7] K. Matsuda, T. Fukuyama, and H. Nishiura, Phys. Rev. D **61**, 053001 (2000) [arXiv:hep-ph/9906433] [Search INSPIRE].
[8] H. Fritzsch and Z.-z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000) [arXiv:hep-ph/9912358] [Search INSPIRE].
[9] H. Fritzsch and Z.-Z. Xing, Phys. Lett. B **555**, 63 (2003) [arXiv:hep-ph/0212195] [Search INSPIRE].
[10] Z.-z. Xing and H. Zhang, Phys. Lett. B **569**, 30 (2003) [arXiv:hep-ph/0304234] [Search INSPIRE].
[11] Z.-Z. Xing and H. Zhang, J. Phys. G **30**, 129 (2004) [arXiv:hep-ph/0309112] [Search INSPIRE].
[12] M. Bando, S. Kaneko, M. Obara, and M. Tanimoto, Prog. Theor. Phys. **112**, 533 (2004) [arXiv:hep-ph/0405071] [Search INSPIRE].
[13] K. Matsuda and H. Nishiura, Phys. Rev. D **74**, 033014 (2006) [arXiv:hep-ph/0606142] [Search INSPIRE].
[14] G. C. Branco, M. N. Rebelo, and J. I. Silva-Marcos, Phys. Rev. D **76**, 033008 (2007) [arXiv:hep-ph/0612252] [Search INSPIRE].
[15] G. Ahuja, S. Kumar, M. Randhawa, M. Gupta, and S. Dev, Phys. Rev. D **76**, 013006 (2007) [arXiv:hep-ph/0703005] [Search INSPIRE].
[16] J. Barranco, F. Gonzalez Canales, and A. Mondragon, Phys. Rev. D **82**, 073010 (2010) [arXiv:1004.3781 [hep-ph]] [Search INSPIRE].
[17] R. Verma, G. Ahuja, N. Mahajan, M. Gupta, and M. Randhawa, J. Phys. G **37**, 075020 (2010) [arXiv:1004.5452 [hep-ph]] [Search INSPIRE].
[18] M. Gupta and G. Ahuja, Int. J. Mod. Phys. A **26**, 2973 (2011) [arXiv:1206.3844 [hep-ph]] [Search INSPIRE].
[19] Z.-Z. Xing and Z.-H. Zhao, Nucl. Phys. B **897**, 302 (2015) [arXiv:1501.06346 [hep-ph]] [Search INSPIRE].
[20] M. J. S. Yang, Phys. Rev. D **95**, 055029 (2017) [arXiv:1612.09049 [hep-ph]] [Search INSPIRE].
[21] M. J. S. Yang, Phys. Lett. B **806**, 135483 (2020) [arXiv:2002.09152 [hep-ph]] [Search INSPIRE].
[22] M. J. S. Yang, Chin. Phys. C **45**, 043103 (2021) [arXiv:2003.11701 [hep-ph]] [Search INSPIRE].
[23] M. J. S. Yang, Nucl. Phys. B **972**, 115549 (2021) [arXiv:2103.12289 [hep-ph]] [Search INSPIRE].

[24] M. J. S. Yang, Prog. Theor. Exp. Phys. **2022**, 013 (2021) [arXiv:2104.12063 [hep-ph]] [Search INSPIRE].
[25] A. Bagai, A. Vashisht, N. Awasthi, G. Ahuja, and M. Gupta, [arXiv:2110.05065 [hep-ph]] [Search INSPIRE].
[26] H. Fritzsch, Z.-Z. Xing, and D. Zhang, Nucl. Phys. B **974**, 115634 (2022) [arXiv:2111.06727 [hep-ph]]Search INSPIRE].
[27] P. Minkowski, Phys. Lett. B **67**, 421 (1977).
[28] M. Gell-Mann, P. Ramond, and R. Slansky, Conf. Proc. **C790927**, 315 (1979).
[29] T. Yanagida, Conf. Proc. **C7902131**, 95 (1979).
[30] G. 't Hooft, NATO Adv. Study Inst. Ser. B Phys. **59**, 135 (1980).
[31] R. Dermisek, Phys. Rev. D **70**, 073016 (2004) [arXiv:hep-ph/0406017] [Search INSPIRE].
[32] J. Sayre and S. Wiesenfeldt, Phys. Rev. D **77**, 053005 (2008) [arXiv:0711.1687 [hep-ph]] [Search INSPIRE].
[33] D. Meloni and G. Blankenburg, Nucl. Phys. B **867**, 749 (2013) [arXiv:1204.2706[hep-ph]] [Search INSPIRE].
[34] J. C. Pati and A. Salam, Phys. Rev. D 10, 275 (1974); **11**, 703 (1975) [erratum].
[35] G. Senjanovic and R. N. Mohapatra, Phys. Rev. D **12**, 1502 (1975).
[36] R. N. Mohapatra and J. C. Pati, Phys. Rev. D **11**, 566 (1975).
[37] G. Ecker, W. Grimus, and H. Neufeld, Nucl. Phys. B **247**, 70 (1984).
[38] M. Gronau and R. N. Mohapatra, Phys. Lett. B **168**, 248 (1986).
[39] F. Feruglio, C. Hagedorn, and R. Ziegler, J. High Energy Phys. **1307**, 027 (2013) [arXiv:1211.5560 [hep-ph]] [Search INSPIRE].
[40] M. Holthausen, M. Lindner, and M. A. Schmidt, J. High Energy Phys. **1304**, 122 (2013) [arXiv:1211.6953 [hep-ph]] [SearchINSPIRE].
[41] S. F. King, Phys. Lett. B **439**, 350 (1998) [arXiv:hep-ph/9806440] [Search INSPIRE].
[42] S. F. King, Nucl. Phys. B **562**, 57 (1999) [arXiv:hep-ph/9904210] [Search INSPIRE].
[43] S. F. King, Nucl. Phys. B **576**, 85 (2000) [arXiv:hep-ph/9912492] [Search INSPIRE].
[44] S. F. King, J. High Energy Phys. **0209**, 011 (2002) [arXiv:hep-ph/0204360] [Search INSPIRE].
[45] S. Antusch and S. F. King, New J. Phys. **6**, 110 (2004) [arXiv:hep-ph/0405272] [Search INSPIRE].
[46] S. F. King, J. High Energy Phys. **0508**, 105 (2005) [arXiv:hep-ph/0506297] [Search INSPIRE].
[47] S. Antusch, S. F. King, C. Luhn, and M. Spinrath, Nucl. Phys. B **856**, 328 (2012) [arXiv:1108.4278 [hep-ph]] [Search INSPIRE].
[48] S. Antusch, S. F. King, and M. Spinrath, Phys. Rev. D **87**, 096018 (2013) [arXiv:1301.6764 [hep-ph]] [Search INSPIRE].
[49] F. Björkeroth and S. F. King, J. Phys. G **42**, 125002 (2015) [arXiv:1412.6996 [hep-ph]] [SearchINSPIRE].
[50] I. Dorsner and S. M. Barr, Nucl. Phys. B **617**, 493 (2001) [arXiv:hep-ph/0108168] [Search INSPIRE].
[51] N. Haba, R. Takahashi, M. Tanimoto, and K. Yoshioka, Phys. Rev. D **78**, 113002 (2008) [arXiv:0804.4055 [hep-ph]] [Search INSPIRE].
[52] Z. G. Berezhiani, Phys. Lett. B **150**, 177 (1985).
[53] S. F. King and G. G. Ross, Phys. Lett. B **520**, 243 (2001) [arXiv:hep-ph/0108112] [Search INSPIRE].
[54] J. L. Chkareuli, C. D. Froggatt, and H. B. Nielsen, Nucl. Phys. B **626**, 307 (2002) [arXiv:hep-ph/0109156] [Search INSPIRE].
[55] S. F. King and G. G. Ross, Phys. Lett. B **574**, 239 (2003) [arXiv:hep-ph/0307190] [Search INSPIRE].
[56] R. Slansky, Phys. Rept. **79**, 1 (1981).
[57] G. Branco, D. Emmanuel-Costa, and R. Gonzalez Felipe, Phys. Lett. B **477**, 147 (2000) [arXiv:hep-ph/9911418] [Search INSPIRE].
[58] G. Branco, D. Emmanuel-Costa, R. Gonzalez Felipe, and H. Serodio, Phys. Lett. B **670**, 340 (2009) [arXiv:0711.1613 [hep-ph]] [Search INSPIRE].
[59] P. F. Harrison and W. G. Scott, Phys. Lett. B **535**, 163 (2002) [arXiv:hep-ph/0203209] [Search INSPIRE].
[60] P. F. Harrison and W. G. Scott, Phys. Lett. B **594**, 324 (2004) [arXiv:hep-ph/0403278] [Search INSPIRE].
[61] R. Friedberg and T. D. Lee, HEPNP **30**, 591 (2006) [arXiv:hep-ph/0606071] [Search INSPIRE].

[62] C. S. Lam, Phys. Lett. B **640**, 260 (2006) [arXiv:hep-ph/0606220] [Search INSPIRE].
[63] J. D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. D **74**, 073012 (2006) [arXiv:hep-ph/0511201] [Search INSPIRE].
[64] X.-G. He and A. Zee, Phys. Lett. B **645**, 427 (2007) [arXiv:hep-ph/0607163] [Search INSPIRE].
[65] W. Grimus and L. Lavoura, J. High Energy Phys. **0809**, 106 (2008) [arXiv:0809.0226 [hep-ph]] [Search INSPIRE].
[66] R. R. Gautam and S. Kumar, Phys. Rev. D **94**, 036004 (2016) [arXiv:1607.08328 [hep-ph]] [Search INSPIRE].