

SYK MODEL AND ITS GENERALIZATIONS

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A DISSERTATION

PRESENTED TO THE FACULTY

OF PRINCETON UNIVERSITY

IN CANDIDACY FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE

BY THE DEPARTMENT OF

PHYSICS

ADVISER: EDWARD WITTEN

SEPTEMBER 2019

Abstract

We briefly review the Sachdev–Ye–Kitaev (SYK) model and study several generalizations of it. First, we consider an SYK model with global $U(1)$ charge and find its four-point function in Euclidean and real time. Then, we proceed to add $\mathcal{N} = 2$ supersymmetry. This $\mathcal{N} = 2$ SYK model is supposed to be dual to the near-horizon geometry of a stable black hole in four-dimensional supergravity. We study this model in one and two dimensions, find the four-point functions and corresponding chaos exponents. Next, we briefly review tensor models showing SYK-like behavior. We study operators unique to the tensor model (compared to the SYK) and count them using the partition function of large N gauge theory. Finally, we switch to the (approximate) gravity dual of the SYK, the Jackiw–Teitelboim theory of dilaton gravity. We study correlators of heavy operators on the boundary of the latter theory using the holographic prescription. We find some novel properties of these correlators, such as having a finite limit at large Euclidean distances. We are also able to study out-of-time ordered four-point function and find that it approaches an exponentially small limit at late times.

Acknowledgements

This work would not be possible without help of many people. First of all, I thank Edward Witten for suggesting problems and patient guidance. I also thank Igor Klebanov for in-depth discussions, support and encouragement, and of course for collaboration.

I thank Bruno Le Floch, Alexey Milekhin, Luca Iliesiu, Ho Tat Lam, Mykola Dedushenko, Joel Clingempeel, Mauricio Romo, Yale Fan, Joaquin Turiaci and other members of the literature seminar in high energy theory. I have learned from you a lot, both as a participant and as an organizer.

I thank Fedor Popov, Zhenbin Yang, Junyi Zhang, Huan He and other inhabitants of the Jadwin Hall for discussions, scientific and otherwise. Special thanks goes to Kasey Wagoner for teaching me a lot about teaching and other things.

I thank Dima Prokoptsev, Olga Kalantarov and Ilya Khayutin, Nikola Markov, Andrey Starikovskiy, Vasyl Alba, Lisa Mankovskaya, Julia Khait, Lena Khilkova, Alex Abrikosov and other people who have supported me in the dark times of my life.

I thank Justin Ripley, Olivier Belanger, Alexander Kiselev, Victor Mikhaylov, Alexander Kustov, Dina Begun, Olga Lositsky, Yury Ustinovsky, Riley Simmons–Elder, Timur Mukhamatullin and Marianna Vydrievich, Misha Ivanov, Alexander Ignatov, Vlad Vershinin, Peter and Anna Kravchuk, Dima Berezin and Lida Levkovich, Barbara Schneider, Preethi Pallegar, Ryan Amos, Anton Ivanov, Alexey Trekin, Vadim Royko, Dinara Usmanova, Nick Sukhov, Amina Kurbidaeva and Yuri Anufriev, Sergey Ryabichko and Helena Demkina, Stephane Cooperstein, Angela Radulescu, Brian Kraus, Joao Ferreira, Shutan Dong, DongWon Oh, Anastasia Morozova, Beni Snow, Chiara Toldo, Shawn Westerdale, Alexey Lavrov and all others who shared the road with me on many different occasions.

And of course, I would have never gotten this far without constant support of my husband Sergey Khilkov. Thank you for believing in me even when I do not. And thank you for simply being here. With immense gratefulness, I dedicate this work to you.

I also dedicate this work to the memory of my friend and colleague Sergey Guts, who showed me how to become a physicist while remaining human.

To Sergey Khilkov
and to the memory of
Sergey Guts

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Chapter 1

Introduction

In this Chapter we review the SYK model together with related theories of two-dimensional gravity. Many other reviews on the subject are available in the literature, see for instance [1, 2, 3].

Initially, the Sachdev–Ye model [4] has been formulated to study a quantum disordered, or “spin fluid”, state of a Heisenberg magnet with random coupling. This model has been connected to AdS_2 gravity [5], in a way analogous to the conventional AdS/CFT correspondence [6, 7, 8]. Later a model containing Majorana fermions with random Gaussian interaction was presented in [9]. It was solved in a way similar to [4], and was called Sachdev–Ye–Kitaev model, or SYK for short.

Since the SYK model is (approximately) solvable, it can be used as a laboratory to study general properties of condensed matter physics. It is actively used as a model for systems without quasi-particle excitations, in particular strange metals [10, 11, 12]. As a chaotic system, SYK presents an interesting dynamic of entanglement [13, 14]. It has been used to check the eigenstate thermalization hypothesis [15, 16]. There are also proposals for experimental realisation of SYK [17, 18, 19].

For us, the SYK model is interesting mostly as another example for holography. The canonical AdS_5/CFT_4 describes a correspondence between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Yang–Mills theory in four dimensions. In the large N limit, the super Yang–Mills theory is dominated by planar Feynman diagrams. It is a broad class of graphs, and therefore it is hard to sum diagrams directly. Although some exact answers in this theory were found with the use of integrability techniques [20, 21, 22], the theory is far from being completely solved.

In one dimension lower, there is a correspondence between critical three-dimensional $O(N)$ vector model and Vasiliev higher spin theory in AdS_4 [23, 24] (see also [25] for a review). The vector model is dominated by bubble diagrams and is solvable in the large N limit. This correspondence can be

further lowered to AdS_3/CFT_2 [26], with higher spin theory in AdS_3 on the gravitational side and large N minimal model on the CFT side.

The SYK model extends this family of theories. Its large N limit is dominated by melonic graphs, which allows to find correlation functions precisely. We discuss this solution in Section 1.2 and in more detail in Chapter 2. It has a gravitational sector governed by the Schwarzian mode, which allows for a possible duality between SYK and a two-dimensional theory of gravity. The precise gravity dual is yet to be formulated. There are many proposals for such a gravity dual, which include reconstructing it order by order using SYK correlation functions [27], going to the kinematic space for bi-local fields [28, 29] or compactifying a three-dimensional gravity theory with a scalar field [30, 31].

Although the exact answer for the gravity dual is not known, the SYK model is already used to study phenomena in gravity, such as traversable wormholes [32]. SYK is conjectured to be dual to AdS_2 gravity, which describes the near-horizon geometry of extremal black holes. Many features of black holes can be recovered in the SYK, in particular maximal exponential growth of out-of-time ordered correlators and zero-temperature entropy [29]. The model even allows to take a peek into black hole interior [33]. In the next Section, we review near-horizon geometry of black holes and related two-dimensional theories of gravity.

1.1 Near the horizon of a black hole

AdS_2 space arises as a near-horizon limit of extremal black holes, including BTZ black holes in 2+1 dimensions [34] and $\mathcal{N} = 2$ in four dimensions [35]. The simplest example involves the Reissner–Nordström solution in four-dimensional Einstein–Maxwell gravity. It is a magnetically charged static black hole with metric and electromagnetic field given by:

$$\begin{aligned} ds^2 &= -\frac{(r-r^+)(r-r^-)}{r^2} dt^2 + \frac{r^2}{(r-r^+)(r-r^-)} dr^2 + r^2 d\Omega_2^2, \\ F &= Q\epsilon_2, \\ r^\pm &= Q\ell_P + E\ell_P^2 \pm \sqrt{2QE\ell_P^3 + E^2\ell_P^4}. \end{aligned} \tag{1.1.1}$$

Here Q is the magnetic charge, ℓ_P is the Planck length, and E is the excitation energy:

$$E = M - \frac{Q}{\ell_P}. \tag{1.1.2}$$

An extremal black hole is the ground state of the system with $E = 0$. A non-extremal black hole is not stable and emits Hawking radiation until it becomes extremal.

Extremal black holes have two interesting features. First, entropy of a black hole is proportional to the horizon area. In particular, an extremal black hole has finite entropy:

$$S_{BH} = \frac{\pi (r^+)^2}{\ell_P^2} \sim \pi Q^2 \quad (\text{extremal}). \quad (1.1.3)$$

Since extremal black hole is a ground state, it has a large entropy at zero temperature. In fact, in supersymmetric realizations of black holes it is possible to find entropy from counting microscopic states [36] and connect it to that of a one-dimensional conformal field theory [37].

Second, semiclassical description of black hole thermodynamics breaks down near extremality [38, 39]. The temperature of a Reissner–Nordström black hole is:

$$T_H = \frac{r^+ - r^-}{4\pi (r^+)^2}. \quad (1.1.4)$$

Near extremality, it becomes:

$$T_H \sim \frac{1}{2\pi} \sqrt{\frac{2}{\ell_P Q^3}} E. \quad (1.1.5)$$

A thermodynamic description requires that emission of a Hawking photon does not change the energy much. The photon has energy $\sim T_H$, therefore we require:

$$E \gg T_H. \quad (1.1.6)$$

This means that the energy of excitations is bounded from below:

$$E > E_{\text{gap}} \sim \frac{1}{\ell_P Q^3}. \quad (1.1.7)$$

We interpret it as a gap between the ground state and the first excitation. In the classical limit $\ell_P \rightarrow 0$ this gap becomes infinite. When we approach extremality, energy goes to zero faster than temperature, and for the extremal black hole we are left only with ground states. These two features, large ground state entropy and a large gap above the ground state, have to be present in the SYK if we hope to interpret it as a holographic dual of near-horizon geometry. It also hints that to study non-trivial spectrum above ground states, we have to move away from extremality.

Let us now make connection with two-dimensional gravity. For an extremal black hole, $r^+ = r^-$.

Defining a new coordinate:

$$z = \frac{(r^+)^2}{r - r^+} = \frac{Q^2 \ell_P^2}{r - r^+}, \quad (1.1.8)$$

and looking near the horizon at $r \rightarrow r^+$ we can rewrite the metric in (1.1.1) as:

$$ds^2 \approx Q^2 \ell_P^2 \left(\frac{-dt^2 + dz^2}{z^2} + d\Omega_2^2 \right). \quad (1.1.9)$$

Thus we can say that the extremal black hole interpolates between flat space and $AdS_2 \times S^2$ on the horizon. We can expect to apply the logic of holography to this AdS_2 factor, expecting it to be dual to a one-dimensional conformal field theory.

The action in the gravitational theory on AdS_2 comes from dimensional reduction of the Einstein–Maxwell action in four dimensions. We start with:

$$S_{EM} = \frac{1}{\ell_P^2} \int d^4x \sqrt{-g} \left(R - \frac{\ell_P^2}{4} F^2 \right). \quad (1.1.10)$$

We are interested in a spherically symmetric solution, and therefore use an ansatz for the metric:

$$ds^2 = h_{ij} dx^i dx^j + \Phi^2 d\Omega_2^2, \quad i, j = 1, 2 \quad x^1 = r, \quad x^2 = t. \quad (1.1.11)$$

Integrating over spherical coordinates and using constant $F \sim Q$, we find the two-dimensional action:

$$S_{2d} \sim \frac{4\pi}{\ell_P^2} \int d^2x \sqrt{-h} \left(\Phi^2 R_h + 2 (\partial\Phi)^2 + 2 - \frac{1}{2} \Phi^{-2} Q^2 \ell_P^2 \right). \quad (1.1.12)$$

This action belongs to a family of dilaton gravity models studied extensively in [40]. In a pure AdS_2 space, these models suffer from the backreaction problem [39, 41]. If there is a matter action present, the equation of motion for dilaton field is:

$$-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = T_{++}^{\text{matter}}. \quad (1.1.13)$$

Here we have used the conformal gauge for the two-dimensional metric:

$$ds_{2d}^2 = -e^{2\omega} du^+ du^-, \quad u^\pm = \arctan(t \pm z). \quad (1.1.14)$$

Integrating the equation of motion along the null direction, $u^- = 0$, we find:

$$\int_0^\pi du^+ e^{-2\omega} T_{++}^{\text{matter}} = e^{-2\omega} \partial_+ \Phi^2 \Big|_{u^+=0} - e^{-2\omega} \partial_+ \Phi^2 \Big|_{u^+=\pi}. \quad (1.1.15)$$

The points with $u^+ = 0, \pi$ lie on the boundary of AdS_2 .

Classically, for any excitation the stress energy density is positive, $T_{++}^{\text{matter}} > 0$. Therefore the difference in the boundary terms in (1.1.15) also has to be positive. From the AdS metric in (1.1.9), we see that the metric components vanish quadratically near the boundary:

$$e^{-2\omega} \sim \sin^2 u^+, \quad u^+ \sim 0, \pi. \quad (1.1.16)$$

Therefore the dilaton field has to diverge at least at one boundary:

$$\Phi^2 \sim \frac{1}{u^+}, \quad u^+ \sim 0, \quad \text{or} \quad \Phi^2 \sim \frac{1}{u^+ - \pi}, \quad u^+ \sim \pi. \quad (1.1.17)$$

Hence the presence of any matter field destroys the asymptotic region.

This problem is similar to the problem of the black hole gap (1.1.7) we discussed above. To study non-trivial spectrum of an extremal black hole, we have to take a step away from criticality. Equivalently, we can take ℓ_P to be finite, or introduce a UV cutoff in the theory. On the AdS boundary, it means that we have to break the one-dimensional conformal symmetry. The scale of this symmetry breaking has to be similar to the mass gap, as it was shown in [42].

In the AdS/CFT language, introducing an ultraviolet cutoff means moving the boundary inside the AdS_2 space and considering a cut out piece, or a nearly- AdS_2 space of [43]. For the action to be consistent, we add a Gibbons–Hawking term to (1.1.12). Redefining the dilaton field:

$$\Phi^2 = \phi_0 + \phi, \quad (1.1.18)$$

and ignoring the potential term, we write the action as:

$$S = -\frac{\phi_0}{16\pi G} \left(\int \sqrt{-g} R + 2 \int_\partial K \right) - \frac{1}{16\pi G} \left(\int \sqrt{-g} \phi (R + 2) + 2 \int_\partial \phi_b K \right). \quad (1.1.19)$$

The first term here is topological and describes the ground state, with ϕ_0 being the area of the extremal black hole. The second term is the action of the two-dimensional dilaton gravity of Jackiw and Teitelboim [44, 45]. In Chapter 5 we will work with this action more closely, for now confining

ourselves with a derivation of the Schwarzian action.

The parameter ϕ_b is the boundary value of the dilaton. The condition $\phi = \phi_b$ together with the equations of motion dictates the (classical) shape of the boundary. Equation of motion for the dilaton imply constant negative curvature, $R = -2$. Therefore the bulk action in (1.1.19) vanishes, and we are left only with the boundary part:

$$S_{JT} = -\frac{\phi_b}{8\pi G} \int_{\partial} K. \quad (1.1.20)$$

We can parameterize the boundary of the nearly- AdS space as $(t(u), z(u))$, where u is the internal coordinate on the boundary. The boundary metric is taken to be constant:

$$g_{uu} = \frac{t'(u)^2 + z'(u)^2}{z^2} = \frac{1}{\epsilon^2}. \quad (1.1.21)$$

Here ϵ is a measure of how close we are to the true boundary of the pure AdS_2 space. If ϵ is small enough, we can find $z(u)$ as:

$$z(u) = \epsilon t'(u) + O(\epsilon^3). \quad (1.1.22)$$

As we have seen before, the dilaton diverges linearly near the true boundary, therefore we define the renormalized value of the boundary dilaton:

$$\phi_r(u) = \epsilon \phi_b, \quad \phi_r(u) = \text{const}. \quad (1.1.23)$$

In this notation the extrinsic curvature is:

$$K = \frac{t'(t'^2 + z'^2 + z z'') - z z' t''}{(t'^2 + z'^2)^{\frac{3}{2}}} = 1 + \epsilon^2 \text{Sch}(t, u), \quad (1.1.24)$$

where $\text{Sch}(t, u)$ denotes the Schwarzian derivative:

$$\text{Sch}(t, u) = \frac{t'''}{t''} - \frac{3}{2} \left(\frac{t''}{t'} \right)^2. \quad (1.1.25)$$

Ignoring the divergent term, we can rewrite the JT action (1.1.20) as the Schwarzian action:

$$S_{JT} \sim -\frac{1}{8\pi G} \int du \phi_r(u) \text{Sch}(t, u). \quad (1.1.26)$$

The Schwarzian action shows that the conformal symmetry of one-dimensional boundary is broken,

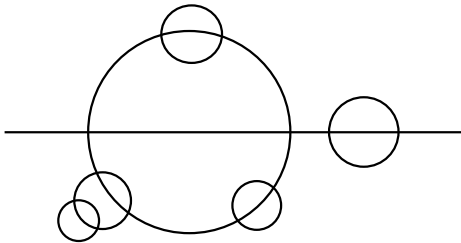


Figure 1.1: An example of a melonic diagram contributing to a two-point function.

as expected of an arbitrarily cut out piece of an AdS_2 space. In the next Section, we will encounter the same piece in the infrared limit of the SYK model, showing that the conformal symmetry is broken there as well. Therefore the low energy limit of the SYK model is closely related to dilaton gravity in the AdS_2 space. The precise connection however remains unknown.

We should also mention that there exists an extensive body of work dedicated to finding the full solution of both the Schwarzian action (see for example [46, 47, 48, 49]) and Jackiw–Teitelboim gravity (see [50, 51, 52, 53]). Many of these use the intuition coming from the SYK model, which we discuss in the next Section.

1.2 Pure SYK model

The SYK model is a close cousin of vector models [54, 55] and matrix models [56, 57]. We study it in the large N limit, which is dominated by a certain type of Feynman graphs called melonic diagrams. An example of a melonic diagram is on fig. 1.1.

The original SYK model [9, 4] is a theory of N Majorana fermions in one dimension with random four-fermion interaction. Its Hamiltonian is:

$$H = \frac{1}{12} \sum_{1 \leq i \leq j \leq k \leq l \leq N} J_{ijkl} \psi_i \psi_j \psi_k \psi_l. \quad (1.2.1)$$

This Hamiltonian can be generalized to q particle interaction [29]. The coupling is a random Gaussian variable with zero mean and variance being:

$$\langle J_{ijkl} J_{ijkl} \rangle = \frac{6J^2}{N^3} \quad (\text{no sum}). \quad (1.2.2)$$

The coupling is chosen so to make the melonic graphs dominate the large N limit. There are different ways to achieve that. One alternative to a theory with a random coupling is presented by tensor models [58, 59, 60]. We are discussing them in more detail in Chapter 4, for now focusing on the

SYK model in its original formulation.

To sum up the leading order diagrams, we introduce bilocal fields G, Σ [9, 49]. The G field is built out of fermions:

$$G(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^N \psi_i(\tau_1) \psi_i(\tau_2). \quad (1.2.3)$$

The Σ is a Lagrangian multiplier enforcing the constraint (1.2.3). Integrating out the fermions, we arrive at the effective action written purely in terms of bilocal fields:

$$\frac{I_{\text{eff}}}{N} = -\frac{1}{2} \log \det (\partial_\tau - \Sigma) + \frac{1}{2} \int d\tau_1 d\tau_2 \left(\Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2) - \frac{J^2}{4} G(\tau_1, \tau_2)^4 \right). \quad (1.2.4)$$

Instead of N fermions, we now work with two fields. Also, the translation symmetry tells us that the bilocal fields depend only on one variable:

$$G(\tau_1, \tau_2) = G(\tau_1 - \tau_2), \quad \Sigma(\tau_1, \tau_2) = \Sigma(\tau_1 - \tau_2). \quad (1.2.5)$$

At large N , the path integral is dominated by the saddle point of the effective action (1.2.4). The equations of motion for the bilocal fields are:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \quad (1.2.6)$$

$$\Sigma(\tau) = J^2 G^3(\tau). \quad (1.2.7)$$

Notice that the first equation is written in the energy space, while the second is in Euclidean time. This is a system of an algebraic and an integral equation, which is generally hard to solve precisely. It is more accessible in the infrared limit. In the right-hand side of (1.2.6), the first term represents the propagator of a free fermion:

$$G_{\text{free}}(\tau) = \frac{1}{2} \text{sgn}(\tau). \quad (1.2.8)$$

This is the ultraviolet limit of the two-point function. At strong coupling however this term is negligible. Disregarding it, we can rewrite the equations of motion as a single integral equation:

$$\int d\tau_2 G(\tau_1, \tau_2) J^2 G^3(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3). \quad (1.2.9)$$

This equation is a recurrent formula for melonic graphs contributing to the two-point function.

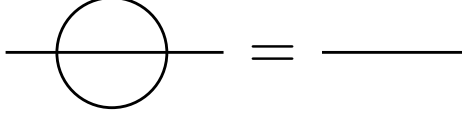


Figure 1.2: Conformal Schwinger–Dyson equations as a diagram. The bold lines represent the exact two-point function G .

Schematically, it can be written as:

$$G * \Sigma * G = G, \quad \Sigma(t) = J^2 G^3(t), \quad (1.2.10)$$

or drawn as on fig. 1.2. It means that a melonic graph built out of exact propagators remains an exact propagator.

Unlike the full equations of motion, (1.2.9) is invariant under reparameterizations:

$$\tau \rightarrow f(\tau) : \quad G(\tau_1, \tau_2) \rightarrow (f'(\tau_1) f'(\tau_2))^\Delta G(f(\tau_1), f(\tau_2)). \quad (1.2.11)$$

In particular, it means that the theory in this limit is conformally invariant. Here Δ is the conformal dimension of a fermion in the infrared. At zero temperature, the equation (1.2.9) can be solved by a conformal two-point function:

$$G(\tau) = b \frac{\text{sgn}(\tau)}{|\tau|^{2\Delta}}, \quad \Delta = \frac{1}{4}, \quad b^4 = \frac{1}{4\pi J^2}, \quad (1.2.12)$$

as well as by any function obtained as a reparameterization of (1.2.12). In particular, at finite temperature the two-point function is:

$$G_\beta = b \text{sgn}(\tau) \left(\frac{\pi}{\beta \sin \frac{\pi\tau}{\beta}} \right)^{2\Delta}. \quad (1.2.13)$$

Having an infinite-dimensional space of solutions to the equations of motion is potentially a problem because it leads to divergences in the path integral. In the conformal limit of the SYK, this divergence is nicely packed in the contribution of the $h = 2$ mode [61], [29]. We will see this happen in Chapters 2 and 3 as well. But since the original action (1.2.4) is not reparameterization-invariant, we deduce that this divergence is an effect of our approximation rather than a physical phenomenon. To regularize this divergence, we can take a step back from the conformal limit. In the full action (1.2.4) there is a term which explicitly breaks this symmetry. The effective action for

reparameterizations is the difference between full and conformal actions:

$$\frac{I_{\text{rep}}}{N} \equiv \frac{I_{\text{eff}} - I_{\text{conf}}}{N} = \int d\tau_1 d\tau_2 \delta(\tau_1 - \tau_2) (\partial_{\tau_1} - \partial_{\tau_2}) G(\tau_1, \tau_2) = G'(0). \quad (1.2.14)$$

We apply a generic reparameterization to the two-point function (1.2.12) and Taylor expand it around $\tau_{12} \rightarrow 0$, to find:

$$G_f = \frac{b \operatorname{sgn} \tau_{12}}{|t_{12}|^{2\Delta}} \left(1 + \frac{\Delta}{6} \tau_{12}^2 \operatorname{Sch}(f(\tau), \tau) + O(\tau_{12}^3) \right), \quad \tau \equiv \frac{\tau_1 + \tau_2}{2}. \quad (1.2.15)$$

Then the reparameterization action is proportional to the Schwarzian derivative:

$$\frac{I_{\text{rep}}}{N} = -\frac{\alpha_S}{J} \int d\tau \operatorname{Sch}(f, \tau), \quad \operatorname{Sch}(f, \tau) \equiv \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2. \quad (1.2.16)$$

The value of α_S requires a more refined analysis and cannot be determined from the conformal two-point function (1.2.12), except for several exactly solvable cases. In [29] it was found numerically. We can think of the reparameterization invariance as an emergent symmetry of the infrared, with the conformal propagator breaking this symmetry down to $SL(2, \mathbb{R})$. The reparameterization mode $f(\tau)$ plays the role of a Nambu-Goldstone field. We say that the reparameterization symmetry is both spontaneously and explicitly broken. The theory in which the full Virasoro symmetry holds but only approximately was called near-conformal field theory, or *NCFT*, by the authors of [29].

The $SL(2, \mathbb{R})$ symmetry can be used to find the correlation functions of the theory. To do it, we have to sum melonic diagrams. These diagrams are a small subset of four-valent graphs, and allow relatively simple recursion relations similar to (1.2.10). However, these recursion relations typically produce complicated integral equations. A way to solve them is to use the fact that correlators form a representation of $SL(2, \mathbb{R})$ and can be expanded in a basis of eigenvectors of certain operators. In Chapters 2 and 3 we find the four-point function in the eigenbasis of the two-particle (super)conformal Casimir operator. This reasoning can be extended to find all-point correlation functions in the SYK [62].

However, this technique applies only to the conformal sector of the model. There is a subspace of solutions corresponding to the Schwarzian mode which produces divergences in the correlation functions. We can view the infrared theory as the strong coupling limit of the *NCFT* and expand the correlators in powers of inverse coupling $(\beta J)^{-1}$. These corrections regularize the conformal solution. The first such correction comes from the Schwarzian action (1.2.16).

The action (1.2.14) also allows us to find the ground state entropy [29]. The free energy in the

large N limit is given by the saddle point action:

$$\beta F \sim N I_{\text{eff}}. \quad (1.2.17)$$

The partition function depends on temperature only as a part of dimensionless combination (βJ) , and therefore we can write:

$$S = (1 - \beta \partial_\beta) (\beta F) = (1 - J \partial_J) (-\beta F). \quad (1.2.18)$$

The action (1.2.4) is easy to differentiate by J , and using the equations of motion we find:

$$J \partial_J I_{\text{eff}} = -\frac{1}{4} J^2 \beta \int d\tau G^4(\tau) = \frac{1}{4} \beta G'(0). \quad (1.2.19)$$

As we mentioned before, in general $G'(0)$ is hard to find. It is an ultraviolet term and requires finding the two-point function away from the far infrared limit. However, it can be found in large q theory, where interaction (1.2.1) involves q particles instead of four. Then in the leading order in $1/q$ the entropy is:

$$\frac{S}{N} = \frac{1}{2} \ln 2 + \frac{1}{q^2} \left(-\frac{\pi^2}{4} + \frac{\pi^2}{\beta J} + \dots \right) + O(1/q^3). \quad (1.2.20)$$

Since the term (1.2.19) is also responsible for the Schwarzian action, the entropy (1.2.20) should be seen from there. Indeed, the thermal solution corresponds to:

$$f(\tau) = \tan \frac{\pi \tau}{\beta}, \quad 0 \leq \tau \leq \beta. \quad (1.2.21)$$

The Schwarzian action on such a solution is:

$$\int_0^\beta \text{Sch}(f, \tau) = \int_0^\beta \frac{2\pi^2}{\beta^2} = \frac{2\pi^2}{\beta}, \quad (1.2.22)$$

and correspondingly the entropy is:

$$S = 2\pi^2 \frac{\alpha_S}{\beta J}. \quad (1.2.23)$$

The Schwarzian action appears to be insensitive to the ground state entropy, but it correctly reproduces the term linear in temperature. We can identify this term with the entropy of a near-extremal black hole. The entropy calculations also allow to fix α_S numerically.

A similar reasoning gives exact ground state entropy for a general q -particle SYK model. Since

the low-temperature limit $\beta \rightarrow \infty$ is the same as the strong coupling limit $J \rightarrow \infty$, we can use the conformal solution (1.2.12) to find the effective action. The answer is:

$$\frac{S_0}{N} = \frac{1}{2} \ln 2 - \int_0^{1/q} dx \pi \left(\frac{1}{2} - x \right) \tan \pi x, \quad N \rightarrow \infty. \quad (1.2.24)$$

When $q = 2$, which means the interaction is quadratic, the ground state entropy is zero. For any other q there is large ground state entropy, $S_0 \sim N$, growing with q . When $q \rightarrow \infty$, the ground state entropy is the same as for N free fermions. In numerical simulations at finite N it is seen as exponentially many low-lying states with small gaps between them. As N goes to infinity, all these states merge into the ground state. Large degeneracy of the ground state is expected for a near-extremal black hole [37].

We should also mention that the SYK model is in many ways similar to random matrix models at large N , and its spectrum is accessible numerically. In particular, it was shown that the states have exponentially small gaps $\sim \exp(-\alpha N)$, which at large N merge together to form a highly degenerate ground state [63, 64, 65].

Let us summarize the properties of the SYK model which make it special:

- A one-dimensional SYK model is conjectured to be dual to dilaton gravity in AdS_2 , making it a low dimensional and relatively simple example of AdS/CFT correspondence.
- Its low-energy limit includes a Schwarzian term, which also arises in the near-horizon theory of extremal black holes. It has large ground state entropy and heat capacity growing linear in temperature. These properties are also characteristic of extremal black holes.
- It is strongly coupled, yet solvable. Typically, it is hard to find a solution for a system with strong interaction, and a weakly interacting system is not a good candidate for a theory dual to gravity. Having a reasonably simple solution means being able to study holography more closely.
- It has a chaotic mode, which is considered an important and typical feature of black hole dynamics [66, 67, 68]. The Lyapunov exponent of this mode is maximally chaotic and coincides with that of a particle near a black hole [69].
- Since it has originated as a condensed matter system, it can potentially be experimentally accessible.

- Finally, it has interesting generalizations to supersymmetric and higher dimensional systems [70, 71].

1.3 Overview

This thesis is based on [72], [73], [74], [75]. I thank Igor Klebanov, Alexey Milekhin and Grisha Tarnopolsky for collaboration on [73].

In Chapter 2, we follow the machinery developed and sharpened in [9, 29] to study the SYK model with an extra $U(1)$ symmetry. We compute the conformal four-point function as a sum of ladder diagrams. We also find an $h = 1$ mode corresponding to a $U(1)$ charge. In the large q limit, we find that this pole is removed by $(\beta J)^{-1}$ corrections.

In Chapter 3, we extend this analysis to the SYK model with $\mathcal{N} = 2$ supersymmetry. This is a model expected to be dual to an $\mathcal{N} = 2$ black hole in four dimensions. We calculate the four-point function and find that it is a linear combination of four-point functions from the previous Chapter. Supersymmetry also allows us to lift the model to two dimensions and find the four-point function there as well. The model includes a chaotic mode which has a maximal exponent in one dimension, and non-maximal exponent in two dimensions, as many other models of this family [71].

In Chapter 4 we review the tensor model similar to SYK. Apart from a tower of operators encountered in the SYK, it also includes many that are unique for tensor models. We attempt to count these operators using the partition functions of one-dimensional gauge theories and find that their number grows factorially with dimension.

Finally in Chapter 5 we go to the gravitational side and consider correlators of operators on the boundary of near- AdS_2 space. We work in a semiclassical approximation by introducing massive particles into the bulk and studying what happens to the boundary. We find in particular that the two-point function does not decay to zero when the operators are infinitely far apart. We also analytically continue correlators to real time and find that both the two-point function and the four-point function approach an exponentially small but non-zero limit in the infrared.

Chapter 2

SYK model with complex fermions

2.1 Introduction

In this Chapter we add a global $U(1)$ symmetry to the conventional SYK model. This generalization has been studied from a thermodynamical perspective to compute transport coefficients of a strange metal in [11]. An additional symmetry has also been used to construct a lattice of SYK models, effectively raising it in higher dimensions [76, 10].

Pure SYK model contains a tower of operators of the form $\psi_i \partial^{2k+1} \psi_i$. The lowest operator in the spectrum, $\psi \partial \psi$, is responsible for the Schwarzian action we discussed in Chapter 1. It has dimension of a graviton, $h = 2$. This mode creates a divergence in the four-point function. To resolve this divergence, we have to step back from the purely conformal picture and consider a term breaking the conformal symmetry. It creates corrections to the four-point function expanded in powers of inverse coupling $(\beta J)^{-1}$. These corrections are hard to access analytically except in the fully solvable case of large q . The same $h = 2$ mode also contributes to the Lyapunov exponent of the out of time ordered correlators.

A model with extra $U(1)$ global symmetry, or equivalently a model with complex fermions, has in addition a tower of operators with even number of derivatives, $\bar{\psi}_i \partial^{2k} \psi_i$. The dimensions of these operators has been found in [77]. The first operator in this tower is the $U(1)$ charge, $Q = \bar{\psi} \psi$. As a charge, it has zero dimension; however under the symmetry of the spectrum $h \leftrightarrow 1 - h$ it is identified with the $h = 1$ mode.

One might expect that the $h = 1$ mode in the complex SYK also causes divergence in the four-point function. In this Chapter, we repeat the analysis of [29] for complex fermions. We expand the

conformal four-point function in the Casimir basis and indeed find a pole at $h = 1$. Going to large q , we show that this pole is lifted by $(\beta J)^{-1}$ corrections, very similarly to the $h = 2$ divergence of the pure SYK model.

In calculating the four-point function, two discrete symmetries are important to us. The first one is time-reversal $\mathcal{T}$. In general, the eigenfunctions of the Casimir may or may not be invariant under $\mathcal{T}$. For $q = 4n$, the SYK Hamiltonian is real and therefore preserves $\mathcal{T}$; however for $q = 4n + 2$ it is imaginary and $\mathcal{T}$ -violating. But since ladder diagrams contain an even number of couplings, the four-point functions of the SYK model are nevertheless $\mathcal{T}$ -invariant. So we focus on the $\mathcal{T}$ -invariant four-point functions, although the $\mathcal{T}$ -violating states may still be relevant in next orders in $1/N$ expansion. We list corresponding eigenfunctions in Appendix 2.11.

Likewise, eigenfunctions can be even or odd under exchange of two fermions. Normalized conformal eigenfunctions depend only on the cross-ratio of the fermions' coordinates χ , and an exchange of two fermions takes $\chi \rightarrow \frac{\chi}{\chi-1}$. The four-point function of the pure SYK model is odd under exchange of two fermions and therefore even under $\chi \rightarrow \frac{\chi}{\chi-1}$. To include complex fermions we should also consider eigenfunctions which are odd under $\chi \rightarrow \frac{\chi}{\chi-1}$. The full four-point function is a linear combination of the parts odd and even under the exchange symmetry.

We can also analytically continue to real time to find an out of time ordered four-point function. We find that although the mode corresponding to the $U(1)$ charge is an eigenfunction of the retarded kernel and therefore can contribute to the chaos, its Lyapunov exponent is zero.

The structure of this Chapter is as follows. Section 2.2 sets up the Hamiltonian and outlines the calculation. Section 2.3 discusses discrete symmetries. Sections 2.4 and 2.5 apply the shadow formalism to find the four-point function. Section 2.6 discusses specifically the $h = 1$ mode, and finally Section 2.7 is devoted to continuation to real time.

The paper [78] has a considerable overlap with the analysis presented here.

2.2 Complex SYK model

We consider a one-dimensional system of N complex fermions with q -particle random interaction ($\bar{\psi}$ denotes complex conjugate):

$$H = i^{\frac{q}{2}} \sum j_{i_1 \dots i_{\frac{q}{2}} i_{\frac{q}{2}+1} \dots i_q} \bar{\psi}_{i_1} \dots \bar{\psi}_{i_{\frac{q}{2}}} \psi_{i_{\frac{q}{2}+1}} \dots \psi_{i_q}, \quad 1 \leq i_k \leq N. \quad (2.2.1)$$

The random coupling j is in general complex. It scales with N as:

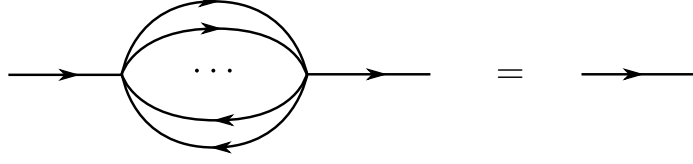


Figure 2.1: Schwinger-Dyson equation for the two-point function. Black line stands for the full conformal propagator.

$$\langle |j_{i_1} \dots \bar{j}_{\frac{q}{2}} i_{\frac{q}{2}+1} \dots i_q|^2 \rangle = \frac{J^2 (q-1)^2}{N^{q-1}}, \quad (2.2.2)$$

and N taken to be large. This theory is a straightforward generalization of the Kitaev's model [9], and by the same token it has a (near) conformal limit at large coupling $\beta J \gg 1$.

We expect the complex model to have the same pseudo-Goldstone $h = 2$ mode as the real SYK, corresponding to the operator:

$$\mathcal{O}_2 = \bar{\psi}_i \partial_t \psi_i. \quad (2.2.3)$$

In addition to that, we expect it to have the $U(1)$ charge operator. Since the $U(1)$ symmetry $\psi \rightarrow e^{i\alpha} \psi$ is conserved, it should have conformal dimension zero:

$$\mathcal{O}_0 = \bar{\psi}_i \psi_i. \quad (2.2.4)$$

In what follows, we mostly work in Euclidean time and at zero temperature, until in Section 2.7 we discuss analytic continuation to real time and out of time order correlators.

In the large N limit, the SYK model is dominated by melonic graphs. This allows us to find correlators using functional methods. The two-point function obeys the Schwinger–Dyson equation, reflecting the fact that the leading correction to the propagator comes from inserting a “melon” (see fig. 2.2.5):

$$\int d\tau' J^2 G(\tau_1, \tau') G^{q-1}(\tau', \tau_2) = -\delta(\tau_1 - \tau_2). \quad (2.2.5)$$

Solving this, one can find the propagator [61]:

$$\langle \bar{\psi}_i(\tau_1) \psi_j(\tau_2) \rangle = \delta_{ij} G(\tau_1, \tau_2) = \delta_{ij} \frac{b \operatorname{sgn}(\tau_1 - \tau_2)}{|\tau_1 - \tau_2|^{2\Delta}}, \quad (2.2.6)$$

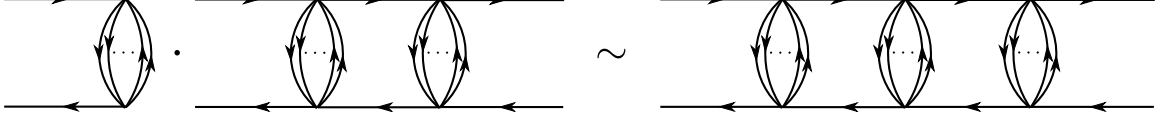


Figure 2.2: Kernel adds a rung to the ladder diagram.

where

$$\Delta = \frac{1}{q}, \quad J^2 b^q \pi = \left(\frac{1}{2} - \Delta \right) \tan \pi \Delta. \quad (2.2.7)$$

The next step is to find the four-point function of the model,

$$\langle \bar{\psi}_i(\tau_1) \psi_i(\tau_2) \bar{\psi}_j(\tau_3) \psi_j(\tau_4) \rangle. \quad (2.2.8)$$

For $i \neq j$ and in the leading order in N , this is just a product of propagators. The leading $1/N$ correction comes from the ladder diagrams, as in the fig. 2.2:

$$\langle \bar{\psi}_i(\tau_1) \psi_i(\tau_2) \bar{\psi}_j(\tau_3) \psi_j(\tau_4) \rangle = G(\tau_1, \tau_2) G(\tau_3, \tau_4) + \frac{1}{N} \mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4) + O\left(\frac{1}{N}\right). \quad (2.2.9)$$

The correction $\mathcal{F}$ is an infinite sum of ladder diagrams with different numbers of “rungs”:

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_2 + \dots \quad (2.2.10)$$

Adding a rung to the ladder can be represented by acting with a differential operator, or “kernel”, on the ladder diagram with a given number of rungs:

$$K \circ \mathcal{F}_i = \mathcal{F}_{i+1}, \quad (2.2.11)$$

We define K precisely later in Section 2.4. Then the full four point function becomes a sum of a geometric progression:

$$\mathcal{F} = \frac{1}{1 - K} \mathcal{F}_0, \quad (2.2.12)$$

where $\mathcal{F}_0$ is the zero-rung ladder, or a product of propagators. To give this expression a concrete meaning, we diagonalize the kernel:

$$K \circ \Psi_i = k_i \Psi_i. \quad (2.2.13)$$

Then in the basis of eigenfunctions of the kernel, the four-point function can be written as follows:

$$\mathcal{F} = \sum_i \frac{1}{1 - k_i} \frac{\langle \Psi_i, \mathcal{F}_0 \rangle}{\langle \Psi_i, \Psi_i \rangle} \Psi_i. \quad (2.2.14)$$

In the next sections, we proceed to find the eigenvalues k_i and eigenfunctions Ψ_i of the SYK kernel. We find that these come in two sets distinguished by their symmetry under exchange of two fermions in (2.2.8). The four-point function which is odd under $\tau_1 \leftrightarrow \tau_2$ and $\tau_3 \leftrightarrow \tau_4$ turns out to be the same as the four-point function of the SYK model with real fermions. The full four-point function also contains a piece which is even under both of these symmetries.

2.3 Discrete symmetries of the four-point function

Our goal is to find the four-point function in the conformal limit of the SYK model:

$$W(\tau_1, \tau_2, \tau_3, \tau_4) \equiv \langle \bar{\psi}_i(\tau_1) \psi_i(\tau_2) \bar{\psi}_j(\tau_3) \psi_j(\tau_4) \rangle. \quad (2.3.1)$$

Following [29] we expand this four-point function in a basis of the conformal Casimir. But before doing that, let's look at discrete symmetries of W .

For the original SYK model with Majorana fermions, the four-point function is odd under exchange of the first two fermions:

$$W^{\text{real}}(\tau_1, \tau_2, \tau_3, \tau_4) = -W^{\text{real}}(\tau_2, \tau_1, \tau_3, \tau_4). \quad (2.3.2)$$

However for the model with complex fermions this is not so. If we write a complex fermion as a sum of two real ones:

$$\psi = \xi + i\eta, \quad (2.3.3)$$

then an arbitrary correlator containing two complex conjugated fermions looks like $\langle (\dots) \rangle$ standing for the terms independent of τ_1, τ_2 :

$$\begin{aligned} \langle \bar{\psi}(\tau_1) \psi(\tau_2) (\dots) \rangle = \\ \langle (\xi(\tau_1) \xi(\tau_2) + \eta(\tau_1) \eta(\tau_2)) (\dots) \rangle + i \langle (\xi(\tau_1) \eta(\tau_2) - \eta(\tau_1) \xi(\tau_2)) (\dots) \rangle. \end{aligned} \quad (2.3.4)$$

The first term in the right-hand side is odd under $\tau_1 \leftrightarrow \tau_2$ and the second one is even:

$$(1 \leftrightarrow 2)_{\text{even}} \equiv \xi(\tau_1) \xi(\tau_2) + \eta(\tau_1) \eta(\tau_2), \quad (1 \leftrightarrow 2)_{\text{odd}} \equiv \xi(\tau_1) \eta(\tau_2) - \eta(\tau_1) \xi(\tau_2). \quad (2.3.5)$$

So in particular the four-point function is a sum of two functions, one being odd and the other even under $\tau_1 \leftrightarrow \tau_2$. The same reasoning of course applies to another pair of fermions.

So naively, regarding discrete symmetries, the four-point function should look like:

$$W(\tau_1, \tau_2, \tau_3, \tau_4) = \langle (1 \leftrightarrow 2)_{\text{even}} (3 \leftrightarrow 4)_{\text{even}} \rangle - \langle (1 \leftrightarrow 2)_{\text{odd}} (3 \leftrightarrow 4)_{\text{odd}} \rangle + \\ i \langle (1 \leftrightarrow 2)_{\text{odd}} (3 \leftrightarrow 4)_{\text{even}} \rangle + i \langle (1 \leftrightarrow 2)_{\text{even}} (3 \leftrightarrow 4)_{\text{odd}} \rangle. \quad (2.3.6)$$

However, the two terms in the second line contain i , and i is odd under the time-reversal symmetry $\mathcal{T}$. The four-point function of a model with the Hamiltonian (2.2.1) is $\mathcal{T}$ -even in the large N limit. For $q = 4k$, this is so because the Hamiltonian is manifestly $\mathcal{T}$ -even. For $q = 4k + 2$, the Hamiltonian contains i , but every ladder diagrams contains an even number of couplings, so the four-point function is again $\mathcal{T}$ -even.

In the Appendix 2.11 we consider $\mathcal{T}$ -odd four-point functions. If we assume for now that the time-reversal symmetry is preserved, we can limit ourselves to considering only the first two terms in (2.3.6). We give these terms superscripts S and A , for the symmetric and antisymmetric parts:

$$W \equiv W^S + W^A = \langle (1 \leftrightarrow 2)_{\text{even}} (3 \leftrightarrow 4)_{\text{even}} \rangle - \langle (1 \leftrightarrow 2)_{\text{odd}} (3 \leftrightarrow 4)_{\text{odd}} \rangle. \quad (2.3.7)$$

For the SYK model with Majorana fermions, the four-point function has only the antisymmetric part:

$$W^{\text{real}} = W^A. \quad (2.3.8)$$

The four-point function W is conformally covariant. To make it conformally invariant instead, we divide it by propagators:

$$\mathcal{W}(\chi) \equiv \frac{W(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)} = W^S(\chi) + W^A(\chi). \quad (2.3.9)$$

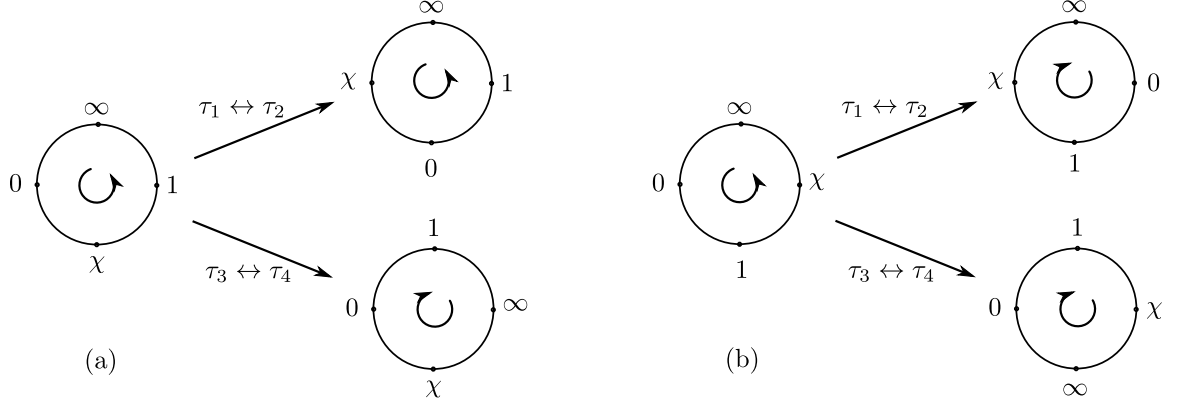


Figure 2.3: The action of exchange of coordinates on χ for (a) $0 < \chi < 1$ and (b) $\chi > 1$. We can see that exchange of coordinates can reverse orientation together with taking $\chi \rightarrow \frac{\chi}{\chi-1}$.

Here $\mathcal{W}(\chi)$ is a function of the cross-ratio:

$$\chi \equiv \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}. \quad (2.3.10)$$

We can now fix the four coordinates in the standard way:

$$\tau_1 = 0, \quad \tau_2 = \chi, \quad \tau_3 = 1, \quad \tau_4 = \infty. \quad (2.3.11)$$

In a conformal field theory invariant under time-reversal, the four-point function depends solely on the cross-ratio. However if we don't assume $\mathcal{T}$ -invariance, the four-point function may also depend on the ordering of the points $(\tau_1, \tau_2, \tau_3, \tau_4)$. The exchange of two points may reverse their cyclic ordering and therefore act as $\mathcal{T}$ on the four-point function. Let's look at this more closely.

What do discrete symmetries imply for $\mathcal{W}(\chi)$? Naively, both the exchanges $\tau_1 \leftrightarrow \tau_2$ and $\tau_3 \leftrightarrow \tau_4$ act as:

$$\chi \rightarrow \frac{\chi}{\chi-1}. \quad (2.3.12)$$

However, these transformations can reverse the orientation of time, depending on the value of χ . From the fig. 2.3 we see that the exchanges of coordinates act as:

$$\begin{array}{ccc} \chi < 1 & \chi > 1 & \\ \tau_1 \leftrightarrow \tau_2 & \chi \rightarrow \frac{\chi}{\chi-1} & \left(\chi \rightarrow \frac{\chi}{\chi-1}\right) \circ \mathcal{T} \\ \tau_3 \leftrightarrow \tau_4 & \left(\chi \rightarrow \frac{\chi}{\chi-1}\right) \circ \mathcal{T} & \left(\chi \rightarrow \frac{\chi}{\chi-1}\right) \circ \mathcal{T} \end{array} \quad (2.3.13)$$

This means that for a $\mathcal{T}$ -invariant theory, the exchange of coordinates acts exactly like $\chi \rightarrow \frac{\chi}{\chi-1}$,

$$K \circ \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} = \left(\begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \text{rung} \times \left(\frac{q}{2} - 1\right) + \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \text{rung} \times \frac{q}{2} \right) \circ \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array}$$

Figure 2.4: The kernel for complex fermions. The first term has $\frac{q}{2} - 1$ arrows pointing up and $\frac{q}{2} - 1$ arrows pointing down in the rung; the second term has $\frac{q}{2}$ arrows pointing up and only $\frac{q}{2} - 2$ arrows pointing down.

and therefore:

$$\mathcal{W}^A\left(\frac{\chi}{\chi-1}\right) = \mathcal{W}^A(\chi), \quad \mathcal{W}^S\left(\frac{\chi}{\chi-1}\right) = -\mathcal{W}^S(\chi). \quad (2.3.14)$$

Note that under the transformation (2.3.12), the antisymmetric part is even and the symmetric part is odd, because the propagators (2.2.6) are odd under the exchange of fermions.

The $\mathcal{T}$ -odd four-point functions should transform oppositely under exchanges $\tau_1 \leftrightarrow \tau_2$ and $\tau_3 \leftrightarrow \tau_4$. For $\chi < 1$, these transformations acts oppositely on a $\mathcal{T}$ -odd function. There are two options: the four-point function can be either odd or even under $\chi \rightarrow \frac{\chi}{\chi-1}$, depending on whether it is even or odd under $\tau_1 \leftrightarrow \tau_2$. For $\chi > 1$ however, these exchanges act in the same way, so the $\mathcal{T}$ -odd four-point function has to be zero in that region. In Appendix 2.11 we will see this from a direct calculation.

In the next section, we define the SYK kernel for the model with complex fermions and find its eigenfunctions using the fact that it commutes with the Casimir of the conformal group. The kernel is therefore diagonalized by three-point functions, which can also be odd or even under these discrete symmetries. Then using the shadow formalism, we construct the four-point functions out of the three-point functions with the desired symmetries.

2.4 Eigenvalues of the kernel

The SYK kernel is an integral operator acting on the four-point functions, corresponding to adding one rung to a ladder diagram (fig. 2.2). For the complex model, one can represent the kernel schematically as in fig. 2.4. There are two types of rungs which can be added, one coming with a factor of $(\frac{q}{2} - 1)$ and the other with a factor of $\frac{q}{2}$, reflecting the choice of an index going down the “rail” of the ladder.

The kernel commutes with the action of the conformal algebra. Specifically, if we define the sl_2 generators as:

$$L_0^{(\tau)} = -\tau\partial_\tau - \Delta, \quad (2.4.1)$$

$$L_{-1}^{(\tau)} = -\partial_\tau, \quad (2.4.2)$$

$$L_1^{(\tau)} = -\tau^2\partial_\tau - 2\Delta\tau, \quad (2.4.3)$$

the kernel obeys the following:

$$\left(L_i^{(1)} + L_i^{(2)}\right) K(\tau_1, \tau_2; \tau_3, \tau_4) = K(\tau_1, \tau_2; \tau_3, \tau_4) \left(L_i^{(3)} + L_i^{(4)}\right), \quad i = 1, 2, 3. \quad (2.4.4)$$

In particular, this means that the kernel commutes with the two-particle Casimir:

$$C^{(12)} \equiv \left(L_0^{(1)} + L_0^{(2)}\right)^2 - \frac{1}{2} \left\{ L_{-1}^{(1)} + L_{-1}^{(2)}, L_1^{(1)} + L_1^{(2)} \right\}, \quad (2.4.5)$$

and therefore, the kernel and the Casimir have a common basis of eigenfunctions.

The Casimir is diagonalized by conformal three-point functions. For our purposes, we are most interested in the three-point functions of two complex conjugated fermions and a bosonic operator of dimension h :

$$\langle \bar{\psi}(\tau_1) \psi(\tau_2) V_h(\tau_0) \rangle. \quad (2.4.6)$$

Depending on the operator V_h , this three-point function may be symmetric or antisymmetric in (τ_1, τ_2) . For example, if V_h is the identity, we have:

$$\langle \bar{\psi}(\tau_1) \psi(\tau_2) 1 \rangle = \frac{\text{sgn}(\tau_1 - \tau_2)}{|\tau_1 - \tau_2|^{2\Delta}}, \quad (2.4.7)$$

which is antisymmetric under exchange of fermions, while for $V_h = \bar{\psi}\psi$:

$$\langle \bar{\psi}(\tau_1) \psi(\tau_2) \bar{\psi}\psi(\tau_0) \rangle = \frac{\text{sgn}(\tau_1 - \tau_0) \text{sgn}(\tau_2 - \tau_0)}{|\tau_1 - \tau_2|^{2\Delta}}, \quad (2.4.8)$$

which is symmetric under the same exchange. (Here we have used the fact that $\bar{\psi}\psi$ is a conserved charge and hence it has dimension zero.) A generic three-point function is a sum of an antisymmetric and a symmetric part, which we call f_h^A and f_h^S . With a suitable normalization of V_h , the three-point

function looks like:

$$\langle \bar{\psi}(\tau_1) \psi(\tau_2) V_h(\tau_0) \rangle = f_h^A + i f_h^S = \frac{\text{sgn}(\tau_1 - \tau_2) + i \text{sgn}(\tau_1 - \tau_0) \text{sgn}(\tau_2 - \tau_0)}{|\tau_1 - \tau_2|^{2\Delta-h} |\tau_1 - \tau_0|^h |\tau_1 - \tau_0|^h}. \quad (2.4.9)$$

This three-point function is an eigenfunction of the Casimir with eigenvalue $h(h-1)$:

$$C^{(12)} \langle \bar{\psi}(\tau_1) \psi(\tau_2) V_h(\tau_0) \rangle = h(h-1) \langle \bar{\psi}(\tau_1) \psi(\tau_2) V_h(\tau_0) \rangle, \quad (2.4.10)$$

and therefore it is an eigenfunction of the kernel. For simplicity let's define separately the kernels acting on the symmetric and antisymmetric three-point functions. They differ by a factor of $(q-1)$:

$$K^A(\tau_1, \tau_2; \tau'_1, \tau'_2) = (q-1) K^S = -J^2 (q-1) G(\tau_1, \tau'_1) G(\tau_2, \tau'_2) G^{q-2}(\tau'_1, \tau'_2) d\tau'_1 d\tau'_2, \quad (2.4.11)$$

We want to find eigenvalues of K^A :

$$K^A \circ f_h^A = \int K^A f_h^A = k^A(h) f_h^A. \quad (2.4.12)$$

Since we already know that the three-point functions f_h^A diagonalize the kernel, we can consider a convenient limit of this expression. Taking the position of the boson to infinity,

$$f_h^A(\tau_1, \tau_2, \tau_0) \xrightarrow{\tau_0 \rightarrow \infty} \tau_0^{-2h} \frac{\text{sgn} \tau_{12}}{|\tau_{12}|^{2\Delta-h}}, \quad (2.4.13)$$

and fixing the coordinates in the kernel to be 0 and 1, we can compute the eigenvalue as follows:

$$k^A(h) = \tau_0^{2h} \int d\tau'_1 d\tau'_2 K(1, 0; \tau'_1, \tau'_2) f_h^A(\tau'_1, \tau'_2, \tau_0) \Big|_{\tau_0 \rightarrow \infty}. \quad (2.4.14)$$

Using the explicit form of the kernel and changing variables (see Appendix 2.9), we recast this integral in a symmetric form:

$$k^A(h) = \frac{1}{\alpha_0} \int d\tau \frac{\text{sgn} \tau}{|\tau|^{2\Delta} |1-\tau|^{1-h}} \int d\tau' \frac{\text{sgn} \tau'}{|\tau'|^{2\Delta} |1-\tau'|^h}, \quad (2.4.15)$$

where

$$\frac{1}{\alpha_0} = (q-1) J^2 b^q = (1-\Delta)(1-2\Delta) \frac{\tan \pi \Delta}{2\pi \Delta}. \quad (2.4.16)$$

Taking this integral, we arrive at a relatively simple expression in terms of Gamma-functions:

$$k^A(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(-2\Delta)}{\Gamma(2\Delta-2)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) (\sin \pi h - \sin 2\pi \Delta). \quad (2.4.17)$$

This eigenvalue has been found in a slightly different form in [29].

In [29], it was argued that the theory is exactly solvable for large q and for $q = 2$. In these limits, the eigenvalue of the kernel was found to be:

$$q = \infty : \quad k^A(h, 0) = \frac{2}{h(h-1)}, \quad (2.4.18)$$

$$q = 2 : \quad k^A\left(h, \frac{1}{2}\right) = -1. \quad (2.4.19)$$

For a minimal “generic” case of $q = 4$,

$$q = 4 : \quad k^A\left(h, \frac{1}{4}\right) = -\frac{3}{2} \frac{\tan \frac{\pi}{2} \left(h - \frac{1}{2}\right)}{h - \frac{1}{2}}. \quad (2.4.20)$$

To find the dimensions of the operators in the model, we have to solve the condition:

$$k^A(h, \Delta) = 1. \quad (2.4.21)$$

The eigenvalue (2.4.17) is symmetric under $h \leftrightarrow 1-h$. The dimensions of the physical operators in the spectrum are positive, so we keep only one copy with positive h . This is justified by a procedure of changing the integration contour in [29]. For $q > 2$, the eigenvalue equation has infinitely many seemingly irrational solutions, given by an asymptotic formula:

$$h = 2k + 1 + 2\Delta + O\left(\frac{1}{k}\right), \quad k > 0. \quad (2.4.22)$$

which allows us to identify them with the operators:

$$\mathcal{O}_k^A = \bar{\psi} \partial_\tau^{2k+1} \psi, \quad k \geq 0. \quad (2.4.23)$$

There is also an integer solution at $h = 2$, by (2.4.23) corresponding to the operator:

$$\mathcal{O}_{h=2}^A = \bar{\psi} \partial_\tau \psi. \quad (2.4.24)$$

This operator can be understood as a (pseudo)Goldstone boson for the broken reparameterization symmetry.

Following the same steps for the symmetric three-point function, we arrive at a very similar integral expression for the eigenvalue of the kernel:

$$k^S(h) = \frac{1}{\alpha_0} \frac{1}{q-1} \int d\tau \frac{\text{sgn } \tau \text{sgn}(1-\tau)}{|\tau|^{2\Delta} |1-\tau|^{1-h}} \int d\tau' \frac{\text{sgn } \tau' \text{sgn}(1-\tau')}{|\tau'|^{2\Delta} |1-\tau'|^h}, \quad (2.4.25)$$

which gives the following answer:

$$k^S(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(1-2\Delta)}{\Gamma(2\Delta-1)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) (\sin \pi h + \sin 2\pi \Delta). \quad (2.4.26)$$

At the same limiting solvable cases, this eigenvalue is:

$$q \rightarrow \infty : \quad k^S(h, 0) = \frac{2}{qh(h-1)} + O(q^{-2}), \quad (2.4.27)$$

$$q \rightarrow 2 : \quad k^S\left(h, \frac{1}{2}\right) = -1. \quad (2.4.28)$$

We notice that for large q , the symmetric and antisymmetric kernels differ by a factor of q . This fact will allow us to find the correction to the eigenvalue of the $h=1$ mode in Section 2.6.

The case of $q=4$ has been addressed in [60], [77].

$$k^S\left(h, \frac{1}{4}\right) = \frac{1}{2} \frac{\cot \frac{\pi}{2} \left(h - \frac{1}{2}\right)}{h - \frac{1}{2}}. \quad (2.4.29)$$

Solving the condition:

$$k^S(h, \Delta) = 1, \quad (2.4.30)$$

again gives an infinite set of operators with (positive) irrational dimensions:

$$h = 2k + 2\Delta + O\left(\frac{1}{k}\right), \quad (2.4.31)$$

which we can identify as:

$$\mathcal{O}_k^S = \bar{\psi} \partial_\tau^{2k} \psi, \quad k \geq 0. \quad (2.4.32)$$

One operator in this series has an integer dimension $h = 0$, for all $q > 2$. It is the $U(1)$ charge:

$$O^S = \bar{\psi}\psi, \quad (2.4.33)$$

As expected, the conserved charge is not renormalized and has vanishing dimension in the near-conformal limit as well. A similar procedure of changing the integration contour allows us to divide the spectrum by the $h \leftrightarrow 1 - h$ symmetry and keep only positive dimensions of the operators. In that case, the $U(1)$ charge gets identified with the $h = 1$ mode.

In the Section 2.7, we will find that the three-point functions with these two special operators with integer dimensions, $\bar{\psi}\partial_\tau\psi$ and $\bar{\psi}\psi$, are eigenfunctions of the retarded kernel and therefore can contribute to the chaotic behavior. But while the operator with $h = 2$ has a maximal Lyapunov exponent in the sense of the bound of [69], the $U(1)$ charge has a zero Lyapunov exponent.

In the next section, we find the basis for the conformal four-point functions in the shadow representation.

2.5 Four-point function in the shadow formalism

As we have seen above, the leading in $1/N$ correction to the four-point function comes from ladder diagrams. We define a conformally invariant version of this correction, dividing by propagators:

$$\mathcal{F}(\chi) = \frac{\mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}. \quad (2.5.1)$$

This function can be expanded in the basis of eigenfunctions of the two-particle Casimir. The Casimir $C^{(12)}$ can be rewritten in terms of the cross-ratio:

$$\mathcal{C}(\chi) \equiv \chi^2(1 - \chi)\partial_\chi^2 - \chi^2\partial_\chi, \quad C^{(12)}\mathcal{F}(\tau_1, \tau_2, \tau_3, \tau_4) = \mathcal{C}(\chi)\mathcal{F}(\chi). \quad (2.5.2)$$

Eigenfunctions of $\mathcal{C}$ are the sl_2 conformal blocks, which we call $F_h(\chi)$ (having chosen a convenient normalization):

$$F_h(\chi) \equiv \frac{\Gamma^2(h)}{\Gamma(2h)} \chi^h {}_2F_1(h, h; 2h; \chi), \quad \chi < 1. \quad (2.5.3)$$

The eigenvalues of the Casimir are of course the same $h(h - 1)$ we have seen when discussing three-point functions:

$$\mathcal{C}(\chi)F_h = h(h - 1)F_h. \quad (2.5.4)$$

Note that the eigenvalue is symmetric under $h \leftrightarrow 1 - h$. Given this symmetry and the fact that the Casimir is a second-order differential operator, its generic eigenfunction is a linear combination of F_h, F_{1-h} :

$$\Psi_h = a(h)F_h(\chi) + a(1-h)F_{1-h}(\chi). \quad (2.5.5)$$

For a given h , we can adjust $a(h)$ to make this combination odd or even under $\chi \rightarrow \frac{\chi}{\chi-1}$. These two options span respectively symmetric and antisymmetric four-point functions. This means that having found the contributions to the four-point functions of both types for each h , we cover all the eigenspace of the conformal Casimir.

To make the Casimir a Hermitean operator, we have to make sure its eigenvalues are real. This leaves us with two choices:

$$h = \frac{1}{2} + is, \quad \text{or} \quad h \in \mathbb{R}. \quad (2.5.6)$$

In fact, not all real values of h are allowed. This is because for real h the eigenfunction F_h has a monodromy around zero. Since we are also interested in symmetry under $\chi \rightarrow \frac{\chi}{\chi-1}$ to fix the discrete symmetries of the four-point function, we write it as a “half-monodromy”:

$$F_h\left(\frac{\chi}{\chi-1}\right) = e^{\pi i h} F_h(\chi), \quad 0 < \chi < 1. \quad (2.5.7)$$

From here we see two things. First, the continuous series $h \in \mathbb{R}$ is reduced to a discrete series of “bound states”:

$$h \in \mathbb{Z}. \quad (2.5.8)$$

Second, in this integer series the antisymmetric four-point function (in the sense of 2.3.14) should have even h :

$$\Psi_h^A = a^A(h)F_h(\chi), \quad h \in 2\mathbb{Z}, \quad (2.5.9)$$

and the symmetric four-point function should have odd h :

$$\Psi_h^S = a^S(h)F_h(\chi), \quad h \in 2\mathbb{Z} + 1, \quad (2.5.10)$$

We will confirm this later while considering normalization conditions on Ψ_h^A, Ψ_h^S .

The antisymmetric eigenfunction Ψ_h^A has been found in [29]. Our next step is to find the explicit expression for the symmetric one Ψ_h^S .

2.5.1 Shadow formalism

We have seen that the conformal Casimir is diagonalized by three-point functions of the form $\langle \bar{\psi} \psi V_h \rangle$, and it is also diagonalized by the four-point function $\Psi_h(\chi)$ (up to a product of propagators). So it seems natural to suggest that the conformal four-point function $\Psi_h(\chi)$ is constructed out of three-point functions. This idea is embodied in the shadow formalism [79],[80]. It works as follows.

Consider the four-point function to be consisting of two parts which belong to two different decoupled CFT:

$$\langle \bar{\psi}(t_1) \psi(t_2) \bar{\psi}(t_3) \psi(t_4) \rangle \rightarrow \langle \bar{\psi}(t_1) \psi(t_2) \rangle_1 \langle \bar{\psi}(t_3) \psi(t_4) \rangle_2. \quad (2.5.11)$$

Now let's add a small interaction. Let's introduce an operator V living in the first CFT and an operator V' living in the second CFT, coupled slightly via:

$$S \supset \varepsilon \int dt_0 V_h(t_0) V'_{1-h}(t_0). \quad (2.5.12)$$

To make this interaction conformal, we fix the sum of dimensions of these two operators to be one. Then to the first order in ε the four-point function is:

$$\begin{aligned} \langle \bar{\psi}(t_1) \psi(t_2) \bar{\psi}(t_3) \psi(t_4) \rangle &= \langle \bar{\psi}(t_1) \psi(t_2) \rangle \langle \bar{\psi}(t_3) \psi(t_4) \rangle + \\ &\varepsilon \sum_h \int dt_0 \langle \bar{\psi}(t_1) \psi(t_2) V_h(t_0) \rangle \langle \bar{\psi}(t_3) \psi(t_4) V'_{1-h}(t_0) \rangle. \end{aligned} \quad (2.5.13)$$

Comparing this to (2.2.9) and (2.5.1), we find an integral expression for the eigenfunctions of the Casimir:

$$\Psi_h(\chi) = \frac{1}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)} \int d\tau_0 \langle \bar{\psi}(\tau_1) \psi(\tau_2) V_h(\tau_0) \rangle \langle \bar{\psi}(\tau_3) \psi(\tau_4) V'_{1-h}(\tau_0) \rangle. \quad (2.5.14)$$

The four-point function inherits the discrete symmetries of the three-point functions. This allows us to readily write the expressions for $\mathcal{T}$ -even symmetric and antisymmetric four-point functions:

$$\Psi_h^A(\chi) = \int d\tau_0 \frac{f_h^A(\tau_1, \tau_2, \tau_0) f_{1-h}^A(\tau_3, \tau_4, \tau_0)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}, \quad (2.5.15)$$

$$\Psi_h^S(\chi) = \int d\tau_0 \frac{f_h^S(\tau_1, \tau_2, \tau_0) f_{1-h}^S(\tau_3, \tau_4, \tau_0)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}, \quad (2.5.16)$$

with the three-point functions f_h^S, f_h^A defined in (2.4.9). These expressions are manifestly symmetric under $h \leftrightarrow 1-h$, which is to be expected since the eigenvalues of the Casimir are symmetric under this too.

The antisymmetric and symmetric eigenfunctions are respectively even and odd under $\chi \rightarrow \frac{\chi}{\chi-1}$:

$$\Psi_h^A\left(\frac{\chi}{\chi-1}\right) = \Psi_h^A(\chi), \quad \Psi_h^S\left(\frac{\chi}{\chi-1}\right) = -\Psi_h^S(\chi). \quad (2.5.17)$$

Taking the two integrals (2.5.15, 2.5.16), we find the explicit form of the eigenfunctions. To do this conveniently, let's introduce one more function:

$$G_h\left(\frac{1}{\chi}\right) = \frac{2\pi}{\sin \pi h} {}_2F_1\left(h, 1-h; 1; \frac{1}{\chi}\right), \quad \chi < 0 \quad \text{or} \quad \chi > 1. \quad (2.5.18)$$

This function is proportional to the difference ($F_h - F_{1-h}$) in the regions where both are defined:

$$G_h\left(\frac{1}{\chi}\right) = \frac{1}{2\pi} \tan \pi h \left(F_h\left(\frac{\chi}{\chi-1}\right) - F_{1-h}\left(\frac{\chi}{\chi-1}\right) \right), \quad \chi < 0 \quad (2.5.19)$$

$$G_h\left(1 - \frac{1}{\chi}\right) = \frac{1}{2\pi} \tan \pi h (F_h(\chi) - F_{1-h}(\chi)), \quad 0 < \chi < 1. \quad (2.5.20)$$

$$(2.5.21)$$

Unlike the F_h defined in (2.5.3), this function is symmetric under $h \leftrightarrow 1-h$.

The even four-point function is the same as the four-point function of the real SYK model [29]:

$$\Psi_h^A(\chi) = \begin{cases} F_h(\chi) + F_{1-h}(\chi) + G_h\left(\frac{1}{\chi}\right), & \chi < 0, \\ F_h(\chi) + F_{1-h}(\chi) + G_h\left(\frac{\chi-1}{\chi}\right), & 0 < \chi < 1, \\ G_h\left(\frac{\chi-1}{\chi}\right) + G_h\left(\frac{1}{\chi}\right), & \chi > 1. \end{cases} \quad (2.5.22)$$

The odd function is as follows:

$$\Psi_h^S(\chi) = \begin{cases} F_h(\chi) + F_{1-h}(\chi) - G_h\left(\frac{1}{\chi}\right), & \chi < 0, \\ -F_h(\chi) - F_{1-h}(\chi) + G_h\left(\frac{\chi-1}{\chi}\right), & 0 < \chi < 1, \\ G_h\left(\frac{\chi-1}{\chi}\right) - G_h\left(\frac{1}{\chi}\right), & \chi > 1, \end{cases} \quad (2.5.23)$$

This form makes it clear that both these integrals are symmetric under $h \leftrightarrow 1-h$ as the shadow

integral (2.5.13) suggests, and also that Ψ^S is symmetric under $\chi \rightarrow \frac{\chi}{\chi-1}$ and Ψ^A is anti-symmetric under that.

Making use of various hypergeometric identities, we can recast these expressions as:

$$\Psi_h^A(\chi) = \begin{cases} \frac{2}{\cos \pi h} \left(\cos^2 \frac{\pi h}{2} F_h(\chi) - \sin^2 \frac{\pi h}{2} F_{1-h}(\chi) \right), & \chi < 1, \\ \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{h}{2}\right) \Gamma\left(\frac{1-h}{2}\right) {}_2F_1\left(\frac{h}{2}, \frac{1-h}{2}; \frac{1}{2}; \frac{(2-\chi)^2}{\chi^2}\right), & \chi > 1. \end{cases} \quad (2.5.24)$$

and:

$$\Psi_h^S(\chi) = \begin{cases} \frac{2}{\cos \pi h} \left(\sin^2 \frac{\pi h}{2} F_h(\chi) - \cos^2 \frac{\pi h}{2} F_{1-h}(\chi) \right), & \chi < 1, \\ \frac{4}{\sqrt{\pi}} \left(\frac{2-\chi}{\chi} \right) \Gamma\left(1 - \frac{h}{2}\right) \Gamma\left(\frac{1+h}{2}\right) {}_2F_1\left(1 - \frac{h}{2}, \frac{1+h}{2}; \frac{3}{2}; \frac{(2-\chi)^2}{\chi^2}\right), & \chi > 1. \end{cases} \quad (2.5.25)$$

2.5.2 Normalization and bound states

To express the four-point function in terms of eigenfunctions of the Casimir (2.5.24, 2.5.25), we have to find inner products between eigenfunctions. We define the inner product as:

$$\langle g, f \rangle = \int_{-\infty}^{\infty} \frac{d\chi}{\chi^2} \bar{g}(\chi) f(\chi). \quad (2.5.26)$$

The Casimir operator is Hermitean with respect to this product (up to boundary terms). The eigenvalues of the Casimir $h(h-1)$ ought to be real, hence we expect h either to be integer or to belong to the continuous series $h = \frac{1}{2} + is$. It is easy to see that for non-integer real h , the norm defined in (2.5.26) diverges.

For the continuous series, the inner product is:

$$\langle \Psi_h^A, \Psi_{h'}^A \rangle = \langle \Psi_h^S, \Psi_{h'}^S \rangle = \frac{\pi \coth \pi s}{s} 4\pi \delta(s - s'). \quad (2.5.27)$$

In this case, the main contribution to the integral (2.5.26) comes from the region $\chi \sim 0$. In this region, the four-point function looks like:

$$\Psi_h^A(\chi) \sim \left(1 + \frac{1}{\cos \pi h}\right) \frac{\Gamma^2(h)}{\Gamma(2h)} \chi^h + (h \leftrightarrow 1-h). \quad (2.5.28)$$

Using the integral form of delta-function:

$$\int_{-\infty}^{\infty} \frac{d\chi}{\chi} \left(\chi^{i(s-s')} + \chi^{-i(s-s')} \right) = 4\pi\delta(s-s'), \quad (2.5.29)$$

we find that the inner product is:

$$\langle \Psi_h^A, \Psi_{h'}^A \rangle = 4\pi\delta(s-s') \left(1 + \frac{1}{\cos \pi h} \right) \frac{\Gamma^2(h)}{\Gamma(2h)} \cdot (h \leftrightarrow 1-h) = \frac{\pi \coth \pi s}{s} 4\pi\delta(s-s'), \quad (2.5.30)$$

with the same expression for inner product of symmetric states.

The states with different symmetry are of course orthogonal:

$$\langle \Psi_h^A, \Psi_{h'}^S \rangle = 0. \quad (2.5.31)$$

For integer values of h , there are two options, either h is even positive, or odd negative:

$$h \in 2\mathbb{Z}, \quad h \geq 2 \quad \text{or} \quad h \in 1 + 2\mathbb{Z}, \quad h \leq -1. \quad (2.5.32)$$

For this series, the antisymmetric eigenfunction is normalizable,

$$\langle \Psi_{h'}^A, \Psi_h^A \rangle = \frac{2\pi^2 \delta_{hh'}}{\left| h - \frac{1}{2} \right|}, \quad (2.5.33)$$

while the symmetric function diverges, as can easily be seen from its form at $\chi > 1$. If we divide the spectrum by the $h \leftrightarrow 1-h$ symmetry, there are only even positive states left, which agrees with the expectation from the half-monodromy (2.5.9):

$$h^A \in 2\mathbb{Z}_+. \quad (2.5.34)$$

The second series is the complement of the first one:

$$h \in 1 + 2\mathbb{Z}, \quad h \geq 1 \quad \text{or} \quad h \in 2\mathbb{Z}, \quad h \leq 0. \quad (2.5.35)$$

In this case, the symmetric function is normalizable, and the antisymmetric is not:

$$\langle \Psi_{h'}^S, \Psi_h^S \rangle = \frac{2\pi^2 \delta_{hh'}}{\left| h - \frac{1}{2} \right|}. \quad (2.5.36)$$

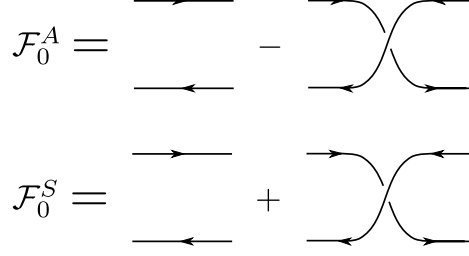


Figure 2.5: Odd and even zero-rung four-point functions.

Here we also can get rid of the non-positive dimensions in the spectrum, leaving only:

$$h^S \in 2\mathbb{Z}_+ + 1, \quad (2.5.37)$$

in agreement with the half-monodromy argument (2.5.10).

Now we can express the four-point function as a sum using the formula 2.2.14. In order to do that, we need an expression for the zero-rung four-point function. The zero-rung four-point function is a product of propagators, but it can be even or odd under exchange of fermions (see fig. 2.5):

$$\mathcal{F}_0^A(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_1, \tau_3) G(\tau_2, \tau_4) - G(\tau_1, \tau_4) G(\tau_2, \tau_3), \quad (2.5.38)$$

$$\mathcal{F}_0^S(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_1, \tau_3) G(\tau_2, \tau_4) + G(\tau_1, \tau_4) G(\tau_2, \tau_3). \quad (2.5.39)$$

Making them conformally invariant, we find an expression in terms of cross-ratio χ :

$$\Psi_0^A(\chi) = \frac{\mathcal{F}_0^A(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)} = -\operatorname{sgn} \chi \cdot |\chi|^{2\Delta} + \operatorname{sgn} \chi \operatorname{sgn}(1 - \chi) \left| \frac{\chi}{\chi - 1} \right|^{2\Delta}, \quad (2.5.40)$$

$$\Psi_0^S(\chi) = \frac{\mathcal{F}_0^S(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)} = -\operatorname{sgn} \chi \cdot |\chi|^{2\Delta} - \operatorname{sgn} \chi \operatorname{sgn}(1 - \chi) \left| \frac{\chi}{\chi - 1} \right|^{2\Delta}. \quad (2.5.41)$$

The Ψ_0^A, Ψ_0^S zero-rung four-point functions are respectively even and odd under $\chi \rightarrow \frac{\chi}{\chi - 1}$, which allows us to rewrite the inner product as follows:

$$\langle \Psi_0^S, \Psi_h^S \rangle = \int_{-\infty}^{\infty} \frac{d\chi}{\chi^2} \Psi_0^S(\chi) \Psi_h^S(\chi) = 2 \int_{-\infty}^{\infty} d\chi |\chi|^{2\Delta} \Psi_h^S(\chi) = 2\alpha_0 k^S(h), \quad (2.5.42)$$

and similarly for the anti-symmetric case.

Bringing everything together and using the formula 2.2.14, we have for the four-point function:

$$\frac{\mathcal{F}^A(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2)G(\tau_3, \tau_4)} = \alpha_0 \int_0^\infty \frac{ds}{\pi} \frac{s}{\pi \coth \pi s} \frac{k^A(\frac{1}{2} + is)}{1 - k^A(\frac{1}{2} + is)} \Psi_{\frac{1}{2} + is}^A(\chi) + \alpha_0 \sum_{n=2j>0} \frac{2j - \frac{1}{2}}{\pi^2} \frac{k^A(2j)}{1 - k^A(2j)} \Psi_{2j}^A(\chi), \quad (2.5.43)$$

which is of course the same as the sum of all ladders in [29], and:

$$\frac{\mathcal{F}^S(\tau_1, \tau_2, \tau_3, \tau_4)}{G(\tau_1, \tau_2)G(\tau_3, \tau_4)} = \alpha_0 \int_0^\infty \frac{ds}{\pi} \frac{s}{\pi \coth \pi s} \frac{k^S(\frac{1}{2} + is)}{1 - k^S(\frac{1}{2} + is)} \Psi_{\frac{1}{2} + is}^S(\chi) + \alpha_0 \sum_{n=2j+1>0} \frac{2j + \frac{1}{2}}{\pi^2} \frac{k^S(2j+1)}{1 - k^S(2j+1)} \Psi_{2j+1}^S(\chi). \quad (2.5.44)$$

The full four-point function is a combination of the symmetric and antisymmetric parts, and from 2.3.6 we find:

$$\mathcal{F} = \mathcal{F}^A - \mathcal{F}^S. \quad (2.5.45)$$

The antisymmetric four-point function (2.5.43) has a formally divergent contribution from the $h = 2$ mode. Since we are studying the conformal, or large coupling, limit $\beta J \gg 1$, we can expect that this divergence is present only in this limit and not in the exact solution. In [29] it was argued that the kernel receives corrections in $(\beta J)^{-1}$, which regularize this divergence.

Likewise, the symmetric four-point function (2.5.44) has a pole in h coming from the $h = 1$ mode. We can hope to regularize it the same way. In the next section, we consider the $h = 1$ contribution in the exactly solvable limit of $q \rightarrow \infty$. We see that at least in this case, the kernel shifted away from one, and the divergence in the four-point function is removed. Although we do not address the case of general q , we expect this regularization to be generic.

2.6 $h = 1$ mode

The $U(1)$ charge operator has dimension $h = 0$ in the UV. Since our model respects the $U(1)$ symmetry, the dimension stays the same in the conformal limit as well. The symmetry of the spectrum under $h \leftrightarrow 1 - h$ allows us to identify this $U(1)$ charge with the $h = 1$ mode in the symmetric four-point function (2.5.44).

The charge operator is present in the spectrum of the model in the conformal limit, therefore

its dimension solves the equation $k^S(\Delta, h) = 1$. This means that it produces a pole in the four-point function. We have already seen this happen for the $h = 2$ mode in the antisymmetric sector, although there is an important difference. The pole from the $h = 2$ mode is present only in the conformal limit, when the coupling is large $\beta J \gg 1$. This limit possesses full reparameterization symmetry, which gets broken down to $SL(2, \mathbb{R})$ symmetry at finite coupling.

The $h = 2$ mode corresponds to the generator of this reparameterization symmetry. Since the symmetry gets broken at finite coupling, the dimension of the corresponding operator receives corrections of the order of inverse coupling:

$$h - 2 \sim \frac{1}{\beta J}. \quad (2.6.1)$$

Therefore the eigenvalue of the kernel also gets corrected:

$$k^A(\Delta, h) \sim \frac{1}{\beta J}, \quad (2.6.2)$$

and the pole in the four-point function gets resolved.

At first glance, this resolution cannot happen for the $h = 1$ mode. Indeed, since $U(1)$ is an exact symmetry of the theory, the dimension of the charge should stay the same regardless of coupling. Therefore the eigenvalue of the kernel, which is a function of dimension, should not change either. However, this is not the case. The symmetry of the spectrum $h \leftrightarrow 1 - h$ is a feature of the conformal limit, and it can be absent in the exact solution of the model. We consider an exactly solvable limit of $q \rightarrow \infty$ to see that the $h = 1$ mode indeed receives corrections in the inverse coupling.

2.6.1 Correction to $h = 1$ at large q

At large q , the model simplifies and turns out to be solvable at any value of the coupling [29]. We consider the theory at finite temperature β , with the $\mathcal{J}$ coupling finite,

$$\mathcal{J} \equiv \sqrt{q} J 2^{1-q/2}. \quad (2.6.3)$$

and the dimensionless coupling kept large,

$$\beta \mathcal{J} \gg 1. \quad (2.6.4)$$

The propagator gets a $1/q$ correction which we denote $g(\tau)$:

$$G(\tau) = \frac{1}{2} \operatorname{sgn} \tau \left(1 + \frac{1}{q} g(\tau) + O(q^{-2}) \right), \quad \Sigma(\tau) = \frac{1}{2} \operatorname{sgn} \tau e^{g(\tau)} \left(\mathcal{J}^2 \cdot \frac{1}{q} + O(q^{-2}) \right), \quad (2.6.5)$$

where $g(\tau)$ was found in [29] to be:

$$e^{\frac{g(\tau)}{2}} = \frac{\cos \frac{\pi v}{2}}{\cos \left(\frac{\pi v}{2} - \frac{\pi v}{\beta} |\tau| \right)}, \quad \beta \mathcal{J} = \frac{\pi v}{\cos \frac{\pi v}{2}}. \quad (2.6.6)$$

At large $\beta \mathcal{J}$, $v \sim 1$. So to get a correction in inverse coupling, we consider v near 1 and find the answer as a series in $(1 - v)$.

The four-point function solves the symmetrized kernel equation (for simplicity, we omit half of the coordinates Ψ^S depends upon):

$$\int \tilde{K}^S(\theta_1, \theta_2 | \theta_3, \theta_4) \Psi^S(\theta_3, \theta_4) = k^S \Psi^S(\theta_1, \theta_2). \quad (2.6.7)$$

The symmetrized kernel is defined as follows:

$$\tilde{K}^S = |G(\theta_{12})|^{\frac{q-2}{2}} K^S(\theta_1, \theta_2 | \theta_3, \theta_4) |G(\theta_{34})|^{\frac{2-q}{2}}, \quad (2.6.8)$$

and K^S is the finite-temperature version of the kernel (2.4.11). Using the propagators (2.6.5), we write the eigenvalue equation as:

$$-\frac{1}{q} \frac{\mathcal{J}^2}{4} \int d\theta_3 d\theta_4 \operatorname{sgn}(\theta_{13}) \operatorname{sgn}(\theta_{24}) e^{\frac{1}{2}(g(\theta_{12}) + g(\theta_{34}))} \Psi^S(\theta_3, \theta_4) = k^S \Psi^S(\theta_1, \theta_2). \quad (2.6.9)$$

As in [29], we can apply an operator $\partial_{\theta_1} \partial_{\theta_2} e^{-\frac{1}{2}g(\theta_{12})}$ to both sides of this integral equation and get a differential equation instead. Using the ansatz:

$$\Psi_n^S(\theta_1, \theta_2) = \frac{e^{-in \frac{\theta_1 + \theta_2}{2}}}{\sin \frac{\tilde{x}}{2}} \psi_n^S(\tilde{x}), \quad \tilde{x} \equiv vx + (1 - v)\pi \quad (2.6.10)$$

and the eigenvalue of the kernel at large q ,

$$k^S \sim \frac{2}{h(h-1)} \frac{1}{q} + O(q^{-1}), \quad (2.6.11)$$

we get that the ψ_n^S function satisfies a hypergeometric equation:

$$\left(\tilde{n}^2 + 4\partial_{\tilde{x}}^2 - \frac{h(h-1)}{\sin^2 \frac{\tilde{x}}{2}} \right) \psi_n^S(\tilde{x}) = 0, \quad \tilde{n} \equiv \frac{n}{v}. \quad (2.6.12)$$

The solution has to satisfy the correct discrete symmetries. It has to be anti-periodic in θ ,

$$\Psi^S(\theta_1 + 2\pi, \theta_2) = \Psi^S(\theta_1, \theta_2 + 2\pi) = -\Psi^S(\theta_1, \theta_2), \quad (2.6.13)$$

and symmetric in coordinates:

$$\Psi^S(\theta_1, \theta_2) = \Psi^S(\theta_2, \theta_1). \quad (2.6.14)$$

In the first order in $(1-v)$, this translates into discrete symmetries of $\psi_n^S(\tilde{x})$ as:

$$\psi_n^S(x + 2\pi) = (-1)^n \psi_n^S(x), \quad \psi_n^S(-x) = -\psi_n^S(x), \quad (2.6.15)$$

or in particular:

$$\psi_n^S(2\pi - x) = (-1)^{n+1} \psi_n^S(x). \quad (2.6.16)$$

The solution to (2.6.12), satisfying this condition, in the vicinity of zero reads as:

$$\psi_n^S(\tilde{x}) \sim \begin{cases} \sin^h \frac{\tilde{x}}{2} {}_2F_1\left(\frac{h-\tilde{n}}{2}, \frac{h+\tilde{n}}{2}; \frac{1}{2}; \cos^2 \frac{\tilde{x}}{2}\right), & n \text{ odd}, \\ \cos \frac{\tilde{x}}{2} \sin^h \frac{\tilde{x}}{2} {}_2F_1\left(\frac{h-\tilde{n}+1}{2}, \frac{h+\tilde{n}+1}{2}; \frac{3}{2}; \cos^2 \frac{\tilde{x}}{2}\right), & n \text{ even}. \end{cases} \quad (2.6.17)$$

To make these functions convergent near zero, we have to make sure that the hypergeometric function truncates into polynomial, that is one of the first two arguments is a non-positive integer. This can be satisfied if we correct the value of h . In the vicinity of $h = 1$, we take:

$$h_n = 1 + |\tilde{n}| - |n| = 1 + |n| \frac{1-v}{v} + O(1-v) = 1 + \frac{2|n|}{\beta\mathcal{J}} + O((\beta\mathcal{J})^{-1}). \quad (2.6.18)$$

Notice that the dimension $h = 0$, even with a small correction, does not reduce the eigenfunction to a polynomial. This allows us to conclude that the dimension of the charge in the conformal limit is $h = 1$ and not $h = 0$ as might be expected from the conservation law.

To make the kernel (2.6.11) meaningful, we also include a $1/q$ correction to h :

$$h_n = 1 + \frac{2|n|}{\beta\mathcal{J}} + \frac{2}{q} + O(q^{-1}) + O((\beta\mathcal{J})^{-1}). \quad (2.6.19)$$

This shift can be found by considering the next order in $1/q$ in the eigenvalue equation (2.6.7).

Then the kernel also gets corrected:

$$k_n^S = 1 - \frac{2|n|}{\beta\mathcal{J}} + O(q^{-1}) + O((\beta\mathcal{J})^{-1}). \quad (2.6.20)$$

Now we can expand the $h = 1$ part of the symmetric four-point function in Ψ_n^S eigenfunctions, with the eigenvalue of the kernel given by (2.6.20). The $h = 1$ pole in the four-point function gets cured by $(\beta\mathcal{J})^{-1}$ corrections to the kernel.

The correction away from the $q \rightarrow \infty$ limit is harder to compute, but we expect it to be corrected by powers of the inverse coupling as well. Also, the question of effective action for $U(1)$ symmetry has been discussed in [70] in the context of $\mathcal{N} = 2$ supersymmetry, where it was found to be non-singular.

2.7 Chaos region

In this chapter, we proceed to find the chaotic exponent for the $h = 0$ mode, associated with the $U(1)$ charge. A classic result [81] connects the Lyapunov exponent with a double commutator:

$$\langle [W(t), V(0)]^2 \rangle \sim e^{2\lambda_L t}, \quad (2.7.1)$$

the intuition being that for conjugated variables p, q the commutator in the semiclassical limit becomes the Poisson bracket,

$$[q(t), p(0)] \rightarrow i\hbar \{q(t), p(0)\} = i\hbar \frac{\partial q(t)}{\partial q(0)}, \quad (2.7.2)$$

which describes the dependence of the trajectory on the initial conditions. Chaos means that trajectories diverge exponentially with time, hence the expected behavior of the double commutator. However to make this correlator sensible for local operators in a CFT, we need to regularize the correlator (2.7.1); to do that, we move one of the commutators halfway down the thermal circle:

$$\langle [W(t), V(0)] [W(t + i\beta/2), V(0 + i\beta/2)] \rangle. \quad (2.7.3)$$

All our previous calculations have been done in Euclidean time at zero temperature. We can formally pass to real time substituting $\tau \rightarrow it$, however in this way we lose information about the ordering of operators. It is more convenient to describe correlators by making time complex with a

small real part,

$$\tau = it + \epsilon. \quad (2.7.4)$$

Going back to our quantum mechanical intuition, a correlator in complex time is:

$$\langle q(\tau)q(0) \rangle = \langle q(it + \epsilon)q(0) \rangle = \left\langle q(0)e^{-H(it+\epsilon)}q(0) \right\rangle, \quad (2.7.5)$$

converging only for $\epsilon > 0$. Thus we can write a commutator as:

$$[W(t), V(0)] \rightarrow W(t) (V(i\epsilon) - V(-i\epsilon)), \quad (2.7.6)$$

implying that the right-hand side is an analytic continuation from Euclidean time. Bringing everything together, we see that to find the chaos exponent for the complex SYK model, we need the correlator:

$$\begin{aligned} \langle \bar{\psi}(\tau_1) \psi(\tau_2) \bar{\psi}(t_3) \psi(t_4) \rangle = \\ \langle (\bar{\psi}(\epsilon) - \bar{\psi}(-\epsilon)) \psi(it) (\bar{\psi}(\beta/2 + \epsilon) - \bar{\psi}(\beta/2 - \epsilon)) \psi(\beta/2 + it) \rangle. \end{aligned} \quad (2.7.7)$$

To compute it, we can apply a similar procedure as before, diagonalizing the retarded kernel instead of the conformal kernel.

2.7.1 Retarded kernel

Retarded kernel is the conformal kernel continued to complex time (see fig.2.6). Following the prescription of (2.7.7), we consider a ladder with one rail at time it and the other at time $\beta/2 + it$. This kernel consist of propagators of two types: ones that go along one rail, and ones which connect one rail to the other.

At finite (unit) temperature, the propagator in complex time is:

$$G_t(\tau_1 - \tau_2) = \frac{b \operatorname{sgn}(\tau_1 - \tau_2)}{|2 \sin \frac{1}{2}(\tau_1 - \tau_2)|^{2\Delta}} \quad \rightarrow \quad G_t(\tau_1 - \tau_2) = \frac{b (\operatorname{sgn} \operatorname{Re}(\tau_1 - \tau_2))^{2\Delta+1}}{(2 \sin \frac{1}{2}(\tau_1 - \tau_2))^{2\Delta}}. \quad (2.7.8)$$

The propagators which go along one rail of the ladder in fig. 2.6 are conventional retarded

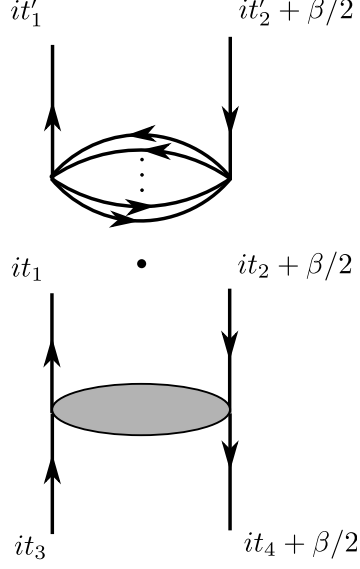


Figure 2.6: Retarded kernel acts on a four-point function in complex time.

propagators:

$$G_R(t_1, t_2) = \theta(t_1 - t_2) (G_t(it_1 + \epsilon, it_2) - G_t(it_1 - \epsilon, it_2)) = \theta(t_1 - t_2) \frac{2b \cos \pi \Delta}{(2 \sinh \frac{1}{2} (t_1 - t_2))^{2\Delta}}. \quad (2.7.9)$$

The propagator connecting left and right rails is as follows:

$$G_{lr}(t_1, t_2) = G(it_1 + \pi, it_2) = \frac{b}{(2 \cosh \frac{1}{2} (t_1 - t_2))^{2\Delta}}. \quad (2.7.10)$$

Just like the conformal kernel, the retarded kernel is diagonalized by even and odd four-point functions, with the eigenvalues differing by a factor of $(q - 1)$:

$$K_R^A = (q - 1) K_R^S = J^2 (q - 1) G_R(t_1, t_2) G_R(t_1, t'_1) G_R(t_2, t'_2) G_{lr}(t'_1, t'_2)^{q-2} dt'_1 dt'_2. \quad (2.7.11)$$

Changing variables to

$$z = e^{i\tau}, \quad (2.7.12)$$

we can rewrite the kernel as follows:

$$K_R^A = \frac{1}{\alpha_0} \theta(z'_1 - z_1) \theta(z_2 - z'_2) \frac{4 \cos^2 \pi \Delta}{|z_1 - z'_1|^{2\Delta} |z_2 - z'_2|^{2\Delta} |z'_1 - z'_2|^{2-4\Delta}} \left| \frac{z_1 z_2}{z'_1 z'_2} \right|^\Delta dz_1 dz_2. \quad (2.7.13)$$

We notice that the retarded kernel in this form is very similar to the conformal kernel, so we can easily guess its eigenfunctions:

$$f_R^S = f_R^A = \frac{|z'_1 z'_2|^\Delta}{|z'_1 - z'_2|^{2\Delta-h}}. \quad (2.7.14)$$

The eigenvalue of the antisymmetric retarded kernel is [29]:

$$k_R^A(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(-2\Delta)}{\Gamma(2\Delta-2)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) \sin \pi(2\Delta+h), \quad (2.7.15)$$

and the eigenvalue of the symmetric one is the same up to $(q-1)$,

$$k_R^S(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(1-2\Delta)}{\Gamma(2\Delta-1)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) \sin \pi(2\Delta+h). \quad (2.7.16)$$

We can rewrite them in terms of conformal kernels:

$$\frac{k_R^A(1-h)}{k^A(h)} = \frac{\cos \pi \left(\frac{h}{2} - \Delta \right)}{\cos \pi \left(\frac{h}{2} + \Delta \right)}, \quad \frac{k_R^S(1-h)}{k^S(h)} = \frac{\sin \pi \left(\frac{h}{2} - \Delta \right)}{\sin \pi \left(\frac{h}{2} + \Delta \right)}. \quad (2.7.17)$$

We see that for the series of allowed bound states with integer h , the eigenvalues of retarded and conformal kernels are equal:

$$h \in 2\mathbb{Z}_+ \quad \Rightarrow \quad k_R^A = k^A, \quad h \in 2\mathbb{Z}_+ + 1 \quad \Rightarrow \quad k_R^S = k^S, \quad (2.7.18)$$

so both $h = 2$ and $h = 0$ (or $h = 1$) modes develop chaotic behavior.

In general to find the operators contributing to chaos, we solve the equation:

$$k_R(h, \Delta) = 1. \quad (2.7.19)$$

Eigenvalues of the retarded kernel lack the symmetry $h \leftrightarrow 1-h$. The solutions of (2.7.19) for the symmetric kernel consist of the $h = -1$ mode and an infinite series of irrational dimensions:

$$k_R^S = 1 \quad \Rightarrow \quad h = -1, \quad \text{and} \quad h = k + 2\Delta + O(1/k), \quad k \in \mathbb{Z}_+. \quad (2.7.20)$$

The “minimal” case of $q = 4$ is an exception: for it, the infinite series is absent and the only mode left is $h = -1$.

Since eigenvalues for odd and even kernels are the same up to $(q-1)$, the solutions to (2.7.19)

for the anti-symmetric kernel consist of a similar infinite series plus the $h = 0$ mode:

$$k_R^A = 1 \quad \Rightarrow \quad h = 0, \quad \text{and} \quad h = k + 2\Delta + O(1/k), \quad k \in \mathbb{Z}_+, \quad (2.7.21)$$

again with the exception of $q = 4$ case.

To see chaotic behavior, we consider again the three-point function (2.7.14). Substituting back $z_1 = e^{-t_1}, z_2 = -e^{-t_2}$, we get:

$$f_R(t_1, t_2) = \frac{\exp\left(-\frac{h}{2}(t_1 + t_2)\right)}{\left(2 \cosh \frac{t_1 - t_2}{2}\right)^{2\Delta - h}}. \quad (2.7.22)$$

We see that the three-point function develops exponential growth at $h = -1$; this growth saturates the bound of chaos of [69]. However, for $h = 0$ the Lyapunov exponent in 2.7.22 is zero, hence the $U(1)$ charge does not contribute to chaotic behavior.

2.8 Discussion

In this Chapter we have found the conformal four-point function for an SYK-like model with complex fermions. This four-point function is expanded in the eigenbasis of the Casimir of the conformal group which are respectively even and odd under $\chi \rightarrow \frac{\chi}{\chi-1}$ symmetry. To find this eigenbasis, we have used the shadow formalism.

The shadow formalism naturally allows us to construct eigenfunctions of the Casimir which are odd under time-reversal $\mathcal{T}$. Although the usual four-point functions of the SYK-like models are $\mathcal{T}$ -even, at least in the large N limit, it is still an interesting possibility which may be realized for an SYK-like model at a conformal point in the next orders in $1/N$ expansion.

We have also found the eigenvalues of the SYK conformal and retarded kernels for the eigenfunctions of the Casimir. The eigenvalue equation $k^S(h) = 1$ for the symmetric kernel has a solution $h = 1$ corresponding to the $U(1)$ charge operator. The retarded kernel for this operator is equal to one as well, therefore this operator potentially contributes to chaotic behavior of the model. However, we find that the corresponding Lyapunov exponent is zero. It would be interesting to study corrections to the chaos exponent.

The $h = 1$ mode creates a divergence in the symmetric four-point function. A similar divergence is caused by an $h = 2$ mode in the usual SYK model with real Majorana fermions. This divergence is cured by non-conformal corrections when the model is studied at large but finite coupling $\beta J \gg 1$. We consider the model with $U(1)$ symmetry at large q and arbitrary coupling, and find the

corrections to the dimension of the charge operator and the eigenvalue of the kernel in this case. These corrections regularize the four-point function and make it non-singular. We expect the same happen at generic value of q .

The $U(1)$ mode is present in the $\mathcal{N} = 2$ SYK model as well, considered in [70]. In this case the $U(1)_R$ symmetry is a part of the $\mathcal{N} = 2$ superconformal symmetry, and the charge contributes to the effective action given by an $\mathcal{N} = 2$ super-Schwarzian derivative. Therefore one expects the divergence in this mode to be cured by moving away from the (super-)conformal limit, just the way the $h = 2$ divergence is removed in the non-supersymmetric real SYK. We return to this question in Chapter 3.

Another interesting question concerns SYK model with gauge symmetries (see for instance [82]). A local symmetry can be sensitive to the reparameterization invariance and its breaking when the theory is moved away from the conformal limit. We hope to discuss this and related questions elsewhere.

2.9 Appendix: Eigenvalues of the kernel

We compute the eigenvalue of the symmetric conformal kernel as an integral (2.4.14):

$$k^A(h) = \tau_0^{2h} \int d\tau'_1 d\tau'_2 K(1, 0; \tau'_1, \tau'_2) f_h^A(\tau'_1, \tau'_2, \tau_0) \Big|_{\tau_0 \rightarrow \infty} = \frac{1}{\alpha_0} \int d\tau'_1 d\tau'_2 \frac{\text{sgn}(\tau'_1 - 1) \text{sgn}(\tau'_2)}{|\tau'_1 - 1|^{2\Delta} |\tau'_2|^{2\Delta} |\tau'_1 - \tau'_2|^{2-4\Delta}} \frac{\text{sgn}(\tau'_2 - \tau'_1)}{|\tau'_1 - \tau'_2|^{2\Delta-h}}. \quad (2.9.1)$$

Changing variables

$$\tau \equiv \tau'_1 - 1, \quad \tau' \equiv \frac{\tau'_1 \tau'_2}{\tau'_2 - 1} \quad (2.9.2)$$

we find:

$$k^A(h) = \frac{1}{\alpha_0} \int d\tau \frac{\text{sgn} \tau}{|\tau|^{2\Delta} |1 - \tau|^{1-h}} \int d\tau' \frac{\text{sgn} \tau'}{|\tau'|^{2\Delta} |1 - \tau'|^h}, \quad (2.9.3)$$

Note that this expression is symmetric under $h \leftrightarrow 1 - h$. Using the integral definition of the beta function,

$$B(x, y) \equiv \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (2.9.4)$$

we can write it as:

$$k^A = \frac{1}{\alpha_0} (-B(1-2\Delta, 2\Delta-h) + B(h, 1-2\Delta) + B(h, 2\Delta-h)) \cdot (h \leftrightarrow 1-h), \quad (2.9.5)$$

which simplifies to (2.4.17).

For the eigenvalue of the anti-symmetric kernel we find:

$$k^S(h) = \frac{1}{q-1} \frac{1}{\alpha_0} \int d\tau'_1 d\tau'_2 \frac{\text{sgn}(\tau'_1-1) \text{sgn}(\tau'_2)}{|\tau'_1-1|^{2\Delta} |\tau'_2|^{2\Delta} |\tau'_1-\tau'_2|^{2-4\Delta}} \frac{1}{|\tau'_1-\tau'_2|^{2\Delta-h}}. \quad (2.9.6)$$

Making the same change of variables (2.9.2), we find:

$$k^S(h) = \frac{1}{\alpha_0} \frac{1}{q-1} \int d\tau \frac{\text{sgn} \tau \text{sgn}(1-\tau)}{|t|^{2\Delta} |1-\tau|^{1-h}} \int d\tau' \frac{\text{sgn} \tau' \text{sgn}(1-\tau')}{|\tau'|^{2\Delta} |1-\tau'|^h}, \quad (2.9.7)$$

which is again written in terms of beta functions:

$$k^S = \frac{1}{q-1} \frac{1}{\alpha_0} (-B(1-2\Delta, 2\Delta-h) + B(h, 1-2\Delta) - B(h, 2\Delta-h)) \cdot (h \leftrightarrow 1-h), \quad (2.9.8)$$

which simplifies to (2.4.26).

The same integrals appear in the inner products with zero-rung four-point functions. Writing the inner product with a symmetric function (2.5.42),

$$\langle \Psi_0^S, \Psi_h^S \rangle = 2 \int dy \frac{d\chi}{\chi^2} \frac{|\chi|^h |\text{sgn} \chi|}{|\chi-y|^h |1-y|^{1-h} |y|^h} |\chi|^{2\Delta}, \quad (2.9.9)$$

and changing variables,

$$y = \frac{1}{\tau}, \quad \chi = \frac{1}{\tau\tau'}, \quad (2.9.10)$$

we arrive at the same integral (2.9.3), giving:

$$\langle \Psi_0^S, \Psi_h^S \rangle = 2\alpha_0 k^S(h). \quad (2.9.11)$$

The same procedure gives the inner product with an anti-symmetric function:

$$\langle \Psi_0^A, \Psi_h^A \rangle = 2\alpha_0 k^A(h). \quad (2.9.12)$$

Similar integrals appear in the computations of retarded kernels. The eigenvalue of the antisym-

metric kernel is:

$$k_R^A(h) = \frac{4 \cos^2 \pi \Delta}{\alpha_0} \int \theta(-z'_1) \theta(z'_2 - 1) dz'_1 dz'_2 \frac{1}{|z'_1|^{2\Delta} |1 - z'_2|^{2\Delta} |z'_1 - z'_2|^{2-2\Delta-h}}. \quad (2.9.13)$$

Making a change of variables similar to (2.9.2):

$$z'_1 = \frac{(1-\tau)\tau'}{\tau' - 1}, \quad z'_2 = 1 - \tau, \quad (2.9.14)$$

we find that:

$$k_R^A(h) = \frac{\sin 2\pi \Delta}{\pi} \frac{(1-2\Delta)(1-\Delta)}{\Delta} \int_{-\infty}^0 dt \frac{1}{|\tau|^{2\Delta} |1-\tau|^{1-h}} \int_0^1 d\tau' \frac{1}{|\tau'|^{2\Delta} |1-\tau'|^h}. \quad (2.9.15)$$

Employing again the integral form of the beta function, we derive (2.7.15). The symmetric kernel is also proportional to this expression.

2.10 Appendix: Four-point function

The four-point function is given in (2.5.15,2.5.16) as an integral of a product of three-point functions,

$$\Psi_h = \int d\tau_0 \frac{f_h(\tau_1, \tau_2, \tau_0) f_{1-h}(\tau_3, \tau_4, \tau_0)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}. \quad (2.10.1)$$

Rewriting this expression in terms of the cross-ratio χ and using the ansatz (2.4.9), we have for instance for the symmetric eigenfunction:

$$\Psi_h^S(\chi) = \int dy \frac{|\chi|^h |1-y|^{h-1}}{|y|^h |\chi-y|^h}. \quad (2.10.2)$$

To take this integral, we employ the definition of the hypergeometric function:

$${}_2F_1(a, b; c; \chi) = B(b, c-b) \int_0^1 dx x^{b-1} (1-x)^{c-b-1} (1-\chi x)^{-a}, \quad 0 < \chi < 1, \quad (2.10.3)$$

with $B(b, c-b)$ being the Euler beta function. Changing variables and renaming parameters, we can derive a set of analogous identities for $0 < \chi < 1$ and for $\chi > 1$. The eigenfunctions for $\chi < 0$ can be restored from $\chi \rightarrow \frac{\chi}{\chi-1}$ symmetry.

Using these identities, it is easy to find the integral (2.10.2) on the four intervals for $0 < \chi < 1$:

$$y < 0 : \quad \frac{1}{2 \cos \pi h} (F_h(\chi) - F_{1-h}(\chi)) = \pi \sin \pi h G_h \left(\frac{\chi-1}{\chi} \right), \quad (2.10.4)$$

$$0 < y < \chi : \quad F_{1-h}(\chi), \quad (2.10.5)$$

$$\chi < y < 1 : \quad \frac{1}{2 \cos \pi h} (F_h(\chi) - F_{1-h}(\chi)) = \pi \sin \pi h G_h \left(\frac{\chi-1}{\chi} \right), \quad (2.10.6)$$

$$y > 1 : \quad F_h(\chi), \quad (2.10.7)$$

and for $\chi > 1$:

$$y < 0 : \quad \frac{1}{2} G_h \left(\frac{\chi-1}{\chi} \right), \quad (2.10.8)$$

$$0 < y < 1 : \quad \frac{1}{2} G_h \left(\frac{1}{\chi} \right), \quad (2.10.9)$$

$$1 < y < \chi : \quad \frac{1}{2} G_h \left(\frac{\chi-1}{\chi} \right), \quad (2.10.10)$$

$$y > \chi : \quad \frac{1}{2} G_h \left(\frac{1}{\chi} \right), \quad (2.10.11)$$

with F_h, G_h defined in (2.5.3, 2.5.18). Summing these terms up and including the sign functions, we get the even and odd eigenfunctions (2.5.22, 2.5.23).

2.11 Appendix: $\mathcal{T}$ -odd four-point functions

In addition to the symmetric and anti-symmetric eigenfunction, we can also find the functions with mixed symmetries (odd under exchange of one pair of coordinates and even under exchange of the other pair). From Section 2.3, we know that these eigenfunctions break the time-reversal symmetry. Using the shadow formalism, we can write them as follows:

$$\Psi_h^{AS} = \int d\tau_0 \frac{f_h^A(\tau_1, \tau_2, \tau_0) f_{1-h}^S(\tau_3, \tau_4, \tau_0)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}, \quad (2.11.1)$$

$$\Psi_h^{SA} = \int d\tau_0 \frac{f_h^S(\tau_1, \tau_2, \tau_0) f_{1-h}^A(\tau_3, \tau_4, \tau_0)}{G(\tau_1, \tau_2) G(\tau_3, \tau_4)}. \quad (2.11.2)$$

The first function is even under $\chi \rightarrow \frac{\chi}{\chi-1}$ and the second one is odd. Using the ansatz (2.4.9) for the three-point functions, we find:

$$\Psi_h^{AS}(\chi) = \begin{cases} 2 \sin^2 \frac{\pi h}{2} G_h \left(\frac{1}{\chi} \right), & \chi < 0, \\ 2 \sin^2 \frac{\pi h}{2} G_h \left(\frac{\chi-1}{\chi} \right), & 0 < \chi < 1, \\ 0, & \chi > 1. \end{cases} \quad (2.11.3)$$

$$\Psi_h^{SA}(\chi) = \begin{cases} -2 \cos^2 \frac{\pi h}{2} G_h \left(\frac{1}{\chi} \right), & \chi < 0, \\ 2 \cos^2 \frac{\pi h}{2} G_h \left(\frac{\chi-1}{\chi} \right), & 0 < \chi < 1, \\ 0, & \chi > 1. \end{cases} \quad (2.11.4)$$

As we have seen before from symmetry considerations, the $\mathcal{T}$ -odd four-point functions vanish for $\chi > 1$. Although this function is not continuous at $\chi = 1$, it can be checked that the eigenvalue equation for the Casimir does not have a singularity in the right-hand side:

$$\mathcal{C}\Psi_h^{AS} - h(h-1)\Psi_h^{AS} = 0. \quad (2.11.5)$$

One might wonder how it is possible for a Casimir to have four independent eigenfunctions for each eigenvalue. Indeed, the Casimir is a differential operator of the second order, so for each eigenvalue it should have two distinct eigenfunctions. The answer is that this statement only holds locally: inside each of the regions $\chi < 0$, $0 < \chi < 1$, $\chi > 1$, there are two independent eigenfunctions. When $\chi < 1$, the $\mathcal{T}$ -breaking eigenfunctions are linear combinations of the $\mathcal{T}$ -preserving ones:

$$\Psi_h^{AS}(\chi) = \begin{cases} \sin^2 \frac{\pi h}{2} (\Psi^A(\chi) - \Psi^S(\chi)), & \chi < 0, \\ \sin^2 \frac{\pi h}{2} (\Psi^A(\chi) + \Psi^S(\chi)), & 0 < \chi < 1, \end{cases} \quad (2.11.6)$$

$$\Psi_h^{SA}(\chi) = \begin{cases} -\cos^2 \frac{\pi h}{2} (\Psi^A(\chi) - \Psi^S(\chi)), & \chi < 0, \\ \cos^2 \frac{\pi h}{2} (\Psi^A(\chi) + \Psi^S(\chi)), & 0 < \chi < 1. \end{cases} \quad (2.11.7)$$

When $\chi > 1$, the basis of Casimir eigenfunctions consists of two $\mathcal{T}$ -even states. On the boundaries of these intervals the eigenfunctions can be smoothly glued together (in the sense that the Casimir equation remains non-singular), forming globally four linearly independent functions.

As we have mentioned before, the melonic limit of the SYK model does not admit $\mathcal{T}$ -odd four-point functions, but they may be relevant for next orders in $1/N$ expansion in the theory with a conformal limit.

Chapter 3

$\mathcal{N} = 2$ SYK model

3.1 Introduction

In this Chapter, we go one step further and add $\mathcal{N} = 2$ supersymmetry to the SYK model with complex fermions. As we mentioned before, the SYK model is expected to be dual to AdS_2 gravity describing the near-horizon geometry of black holes. With $\mathcal{N} = 2$ symmetry, the model allows us to study extremal black holes in four-dimensional $\mathcal{N} = 2$ supergravity [35].

As we have seen in the previous chapter, the study of the SYK models with extra symmetries largely follows the scheme developed in [29]. The two-point function of the model is found from Schwinger–Dyson equations, following immediately from the Lagrangian. The four-point function can be found directly from summing ladder diagrams, but this is rather tricky; instead, the four-point function is expanded in the basis of eigenfunctions of the Casimir of the corresponding superconformal group. The four-point function contains information about operator content of the theory; also, by means of the out-of-time ordered four-point functions we can find the chaos exponent, which is one of the main attractive features of this model. This is the scheme we are following now as well.

Supersymmetric generalizations [70] of the model are interesting for several reasons. First, as we mentioned before, they describe a model dual to an extremal $\mathcal{N} = 2$ black hole in four dimensions. Supersymmetry allows to find entropy of such black holes exactly by counting microstates [36], potentially making it possible to identify the spectra of the two dual systems directly.

Furthermore, supersymmetry makes two-dimensional versions of the SYK model consistent. In two dimensions, fermions have scaling dimension $1/2$, so a relevant interaction cannot be constructed from fermions only. In contrast, two-dimensional scalars have scaling dimension zero, but a bosonic

random potential can have negative directions. To cure that, one can consider a supersymmetric two-dimensional model of scalar superfields with a random superpotential. In an $\mathcal{N} = 2$ supersymmetric SYK model, we consider chiral superfields with a random holomorphic superpotential.

A two-dimensional $\mathcal{N} = 2$ model with a (quasi)homogeneous holomorphic superpotential is generally assumed to flow to a conformal fixed point [83]. SYK models with less supersymmetry are conformal in the infrared limit at large N , but one might expect that $1/N$ corrections induce a “slow” RG flow and drive the system away from the conformal point. Such corrections are hard to study and little is known about them to date. In contrast, we expect the $\mathcal{N} = 2$ model to flow to a true conformal point, which we can conveniently study in the large N limit with the methods designed for the usual non-supersymmetric SYK.

Although we don’t discuss this question in detail here, we notice that constructing a gravity dual of SYK is a challenging task. The similarities between SYK and AdS_2 gravity has already been noticed in the early papers on the subject [40, 29, 82, 43, 84, 30], however the full understanding of a gravity dual is still missing. We hope that adding extra supersymmetry might eventually shed some light on this question as well.

The $\mathcal{N} = 2$ SYK model has already been studied in [70] and [78]. Here, we develop the approach of [70] and work in superspace with chiral and anti-chiral fields. The $\mathcal{N} = 2$ supersymmetry allows complex superfields, and therefore we have to consider four-point functions with different parity under exchange of incoming particles. In this respect, it is very similar to the SYK model with complex fermions we have studied in the previous Chapter. Also, the $SU(1,1|1)$ superconformal group is large enough to restrict the odd coordinates in the chiral–anti-chiral four-point function to zero. We see that the eigenfunctions of the Casimir turn out to be purely bosonic, and in fact linear combinations of the $\mathcal{N} = 0$ eigenfunctions.

This Chapter is based on [74] and is a logical continuation of [72]. It relies heavily on the machinery developed in [71]. We also compare some of our results against [70] and [78] and find them in agreement.

The structure is the following. In Section 3.2 we introduce $\mathcal{N} = 2$ superspace and superfields. In Section 3.3 we write the Lagrangian of the model and discuss the conformal two-point function found from the Schwinger–Dyson equation. In Section 3.4 we discuss the two-particle superconformal Casimir and write its eigenfunctions in the shadow representation. Then we find the norm of the eigenfunctions and the eigenvalues of the SYK kernel acting on them. It allows us to write the full four-point function as a series. In Section 3.5 we find the retarded kernel and compute the Lyapunov exponent corresponding to the superconformal charge multiplet which turns out to be

maximal. Finally, in Section 3.6 we generalize some of our results to two dimensions.

3.2 $\mathcal{N} = 2$ superspace and superfields

We study the $\mathcal{N} = 2$ model at large N in the strong coupling limit. The model flows to a theory which possesses the full $SU(1,1|1)$ superconformal symmetry. To study the correlators, it is convenient to work in the one-dimensional $\mathcal{N} = 2$ superspace (with Euclidean signature), parameterized by:

$$(\tau, \theta, \bar{\theta}). \quad (3.2.1)$$

In what follows, we will often substitute this set of coordinates with a single number representing the index of the supercoordinate, for example:

$$\Phi(1) \equiv \Phi(\tau_1, \theta_1, \bar{\theta}_1). \quad (3.2.2)$$

The $SU(1,1|1)$ group has four bosonic and four fermionic coordinates. It is generated by super-translations:

$$\tau \rightarrow \tau + \epsilon + \theta \bar{\eta} + \bar{\theta} \eta, \quad \theta \rightarrow \theta + \eta, \quad \bar{\theta} \rightarrow \bar{\theta} + \bar{\eta}, \quad (3.2.3)$$

inversions:

$$\tau \rightarrow -\frac{1}{\tau}, \quad \theta \rightarrow \frac{\theta}{\tau}, \quad \bar{\theta} \rightarrow \frac{\bar{\theta}}{\tau}, \quad (3.2.4)$$

and the R -symmetry transformation:

$$\theta \rightarrow e^{i\alpha} \theta, \quad \bar{\theta} \rightarrow e^{-i\alpha} \bar{\theta}. \quad (3.2.5)$$

In Appendix 3.8, we write down the generators of the $su(1,1|1)$ superconformal group as differential operators in the superspace.

The correlators in a CFT have to be conformally covariant. In particular, they have to be invariant under translations, which in non-supersymmetric theory makes them depend only on differences of coordinates:

$$\tau_{12} = \tau_1 - \tau_2. \quad (3.2.6)$$

In the supersymmetric case, this condition gets more restrictive and correlation functions are invariant under super-translations, together with R -symmetry. We can write two combinations of

super-coordinates with conformal weight -1 which satisfy these restrictions:

$$\Delta_{12} \equiv \tau_1 - \tau_2 - \theta_1 \bar{\theta}_2 - \bar{\theta}_1 \theta_2, \quad \lambda_{12} \equiv (\theta_1 - \theta_2) (\bar{\theta}_1 - \bar{\theta}_2). \quad (3.2.7)$$

These two combinations have different symmetry under $1 \leftrightarrow 2$ permutation:

$$\Delta_{12} = -\Delta_{21}, \quad \lambda_{12} = \lambda_{21}. \quad (3.2.8)$$

The correlators should be functions of Δ, λ . In fact, we can restrict them even further using chirality constraint. The complex fermions and bosons in the model can be arranged into chiral superfields $\Psi, \bar{\Psi}$ satisfying:

$$\bar{D}\Psi = 0, \quad D\bar{\Psi} = 0, \quad (3.2.9)$$

where $D, \bar{D}$ are super-derivatives:

$$D \equiv \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \tau}, \quad \bar{D} \equiv \frac{\partial}{\partial \bar{\theta}} + \theta \frac{\partial}{\partial \tau}. \quad (3.2.10)$$

Correlators of chiral (anti-chiral) fields are also chiral (or anti-chiral):

$$\bar{D}_1 \langle \Psi(1) \dots \rangle = 0. \quad (3.2.11)$$

Therefore they should depend on a chiral (anti-chiral) combination of the super-translation invariants Δ, λ . Let us find a linear combination annihilated by D :

$$\langle 12 \rangle \equiv \Delta_{12} - \lambda_{12} = \tau_1 - \tau_2 - 2\bar{\theta}_1 \theta_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2. \quad (3.2.12)$$

This choice is unique, and the nice thing about this invariant combination is that it is both chiral in the first coordinate and anti-chiral in the second one:

$$D_1 \langle 12 \rangle = \bar{D}_2 \langle 12 \rangle = 0. \quad (3.2.13)$$

It makes writing the correlators particularly easy. For example, the two-point function can depend only on the $\langle 12 \rangle$ combination:

$$\mathcal{G}(1|2) \equiv \mathcal{G}(\tau_1, \theta_1, \bar{\theta}_1 | \tau_2, \theta_2, \bar{\theta}_2) \equiv \langle \bar{\Psi}(\tau_1, \theta_1, \bar{\theta}_1) \Psi(\tau_2, \theta_2, \bar{\theta}_2) \rangle = \mathcal{G}(\langle 12 \rangle). \quad (3.2.14)$$

Likewise, the three-point function combining a chiral and an antichiral fields with some superfield V is a function of three invariants:

$$\langle \bar{\Psi}(1) \Psi(2) V(0) \rangle = f(\langle 12 \rangle, \langle 10 \rangle, \langle 02 \rangle). \quad (3.2.15)$$

To make this three-point function non-trivial, the R -charge of the $\mathcal{V}$ operator has to vanish. It means in particular that $\mathcal{V}$ cannot be a chiral or an anti-chiral superfield.

In what follows we write all the correlation functions in terms of the $\langle ij \rangle$ invariants. This makes the correlators manifestly supersymmetric. Using the superconformal group sometimes helps us fix most of the odd variables, so that the results can be written as functions of purely bosonic variables; however, the odd variables are generally easy to reinstall back. This can be used to find the correlation functions of the component fields, although we are not following this approach here.

3.3 Two-point function

We are studying correlators of chiral superfields $\Psi, \bar{\Psi}$, written in the $\mathcal{N} = 2$ superspace. The Lagrangian of the model consists of a kinetic F -term and a holomorphic superpotential:

$$\mathcal{L} = \int d\bar{\theta} d\tau \bar{\Psi}_i D \Psi_i + i^{\frac{\hat{q}-1}{2}} \int d\theta d\tau C_{i_1 i_2 \dots i_{\hat{q}}} \Psi_{i_1} \dots \Psi_{i_{\hat{q}}} + i^{\frac{\hat{q}-1}{2}} \int d\bar{\theta} d\tau \bar{C}_{i_1 i_2 \dots i_{\hat{q}}} \bar{\Psi}_{i_1} \dots \bar{\Psi}_{i_{\hat{q}}}, \quad (3.3.1)$$

with complex random Gaussian coupling:

$$\langle C_{i_1 \dots i_{\hat{q}}} \bar{C}_{i_1 \dots i_{\hat{q}}} \rangle = (\hat{q} - 1)! \frac{J}{N^{\hat{q}-1}}, \quad (3.3.2)$$

$\hat{q}$ being an arbitrary odd integer.

Ψ is a chiral superfield annihilated by $\bar{D}$, so in components it reads as:

$$\Psi = \psi(\tau + \theta \bar{\theta}) + \theta b. \quad (3.3.3)$$

ψ, b are complex fermion and scalar. From the Lagrangian (3.3.1) we see that the scalar field is non-dynamical. We can integrate it out and find that the effective Lagrangian has the schematic form:

$$\mathcal{L}_{eff} = \int d\tau \left(\bar{\psi} \partial_{\tau} \psi + C \bar{C} \bar{\psi}^{q/2} \psi^{q/2} \right), \quad (3.3.4)$$

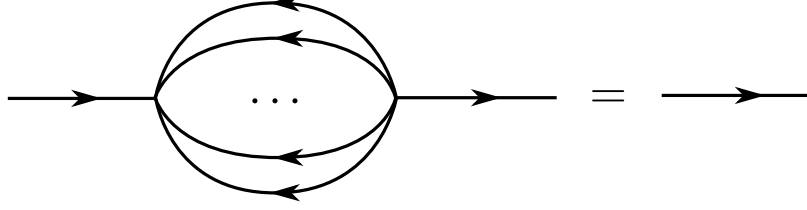


Figure 3.1: Schwinger–Dyson equation for the two-point function. The melonic part contains an even number of propagators.

with $q = 2\hat{q} - 2$. It is very similar to the Lagrangian of the non-supersymmetric SYK model for complex fermions (although the coupling $C\bar{C}$ has different structure), so we can expect the story to be reminiscent of the non-supersymmetric case.

Now we can find the conformal two-point function of the superfield. Keeping in mind (3.2.14), we look for the propagator of the form:

$$\mathcal{G}(1|2) = \mathcal{G}(\langle 12 \rangle) = b \frac{\text{sgn}(\langle 12 \rangle)}{|\langle 12 \rangle|^{2\Delta}}, \quad (3.3.5)$$

where $\langle 12 \rangle$ is the invariant defined in (3.2.12). The propagator has to satisfy the Schwinger–Dyson equation. We can read it off the Lagrangian (3.3.1). Neglecting the $D\mathcal{G}$ term, we find the equation to be (see fig. 3.1):

$$\int d\tau_1 d\theta_1 J \mathcal{G}(\langle 01 \rangle) \mathcal{G}(\langle 21 \rangle)^{\hat{q}-1} = (\bar{\theta}_0 - \bar{\theta}_2) \delta(\langle 02 \rangle). \quad (3.3.6)$$

The delta-function has to be chiral in the first coordinate, hence it depends only on $\langle 02 \rangle$ (and therefore is anti-chiral in the second coordinate). The value of Δ follows from dimensional considerations:

$$2\Delta\hat{q} = 1. \quad (3.3.7)$$

To find b and check the ansatz (3.3.5), we integrate over odd variables in the Schwinger–Dyson equation and then make a one-dimensional Fourier transformation, using the integral:

$$\int d\tau \frac{1}{|\tau|^{2\Delta}} e^{i\omega\tau} = \sqrt{\frac{2}{\pi}} |\omega|^{-1+2\Delta} \Gamma(1-2\Delta) \sin \pi\Delta. \quad (3.3.8)$$

Then the b constant is fixed to:

$$4\pi J b^{\hat{q}} = \tan \pi\Delta. \quad (3.3.9)$$

The four-point function in the model can also be found from an integral equation. To solve it, we

use the fact that the integral kernel commutes with Casimir of the conformal group, and therefore they have a common basis of eigenfunctions. In the next Section, we find eigenfunctions of the Casimir and expand the four-point function in this basis.

3.4 Four-point function

We are looking for a four-point function with two chiral and two anti-chiral fermions:

$$W(1, 2|3, 4) \equiv \langle \bar{\Psi}(1) \Psi(2) \bar{\Psi}(3) \Psi(4) \rangle. \quad (3.4.1)$$

After dividing by propagators, this four-point function becomes invariant under the superconformal group:

$$\mathcal{W} \equiv \frac{W}{\mathcal{G}(\langle 12 \rangle) \mathcal{G}(\langle 34 \rangle)}. \quad (3.4.2)$$

It means that $\mathcal{W}$ can depend only on the cross-ratio of the coordinates. Unlike the non-supersymmetric and $\mathcal{N} = 1$ supersymmetric cases, there is only one cross-ratio consistent with chirality, namely:

$$\chi \equiv \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 14 \rangle \langle 32 \rangle}. \quad (3.4.3)$$

There is no nilpotent invariant as in the $\mathcal{N} = 1$ case either.

We can use the superconformal symmetry to fix the coordinates conveniently. There are four bosonic generators, one of which generates the translation symmetry, and four fermionic ones. We can use the fermionic generators to fix four out of eight odd coordinates. Looking at the structure of the invariant (3.2.12), we see that if we fix $\theta = 0$ for the chiral and $\bar{\theta} = 0$ for the antichiral fields:

$$\theta_2 = \theta_4 = 0, \quad \bar{\theta}_1 = \bar{\theta}_3 = 0, \quad (3.4.4)$$

the cross-ratio reduces to the conventional bosonic cross-ratio:

$$\chi = \frac{\tau_{12} \tau_{34}}{\tau_{14} \tau_{32}}. \quad (3.4.5)$$

Next we can use the bosonic conformal subgroup to fix three out of four coordinates in the standard way:

$$\tau_1 = \chi, \quad \tau_2 = 0, \quad \tau_3 = 1, \quad \tau_4 = \infty. \quad (3.4.6)$$

This implies that the conformal four-point function is a purely bosonic function and does not depend on odd coordinates, unlike the $\mathcal{N} = 1$ four-point function [71]:

$$\mathcal{W} = \mathcal{W}(\chi). \quad (3.4.7)$$

This also means that the Casimir operator as a differential operator acts only on even coordinates. We see in what follows that it is closely related to the Casimir of the non-supersymmetric model.

3.4.1 Casimir of $su(1, 1|1)$

The most general four-point function can be expanded in the basis of eigenfunctions of the two-particle superconformal Casimir. We present our convention for the generators and the Casimir of the $su(1, 1|1)$ algebra in the Appendix 3.8. Conjugating with the two-point functions, we can write the Casimir in terms of the cross-ratio:

$$C_{1+2} \left(\frac{1}{\langle 12 \rangle^{2\Delta}} \frac{1}{\langle 34 \rangle^{2\Delta}} W(1, 2|3, 4) \right) = \frac{1}{\langle 12 \rangle^{2\Delta}} \frac{1}{\langle 34 \rangle^{2\Delta}} \mathcal{C}(\chi) \mathcal{W}(\chi), \quad (3.4.8)$$

where the conformally-invariant Casimir $\mathcal{C}(\chi)$ is a second-order differential operator:

$$\mathcal{C}(\chi) \equiv \chi^2 (1 - \chi) \partial_\chi^2 + \chi (1 - \chi) \partial_\chi. \quad (3.4.9)$$

This operator is diagonalized by functions φ_h :

$$\mathcal{C} \varphi_h(\chi) = h^2 \varphi_h(\chi), \quad (3.4.10)$$

which for $\chi < 1$ can be expressed in terms of a hypergeometric function:

$$\varphi_h(\chi) \equiv \chi^h B(h, h) {}_2F_1(h, h; 1 + 2h; \chi), \quad \chi < 1. \quad (3.4.11)$$

Notice that the equation (3.4.10) is symmetric under $h \leftrightarrow -h$, so the basis of the Casimir is spanned by $\varphi_h(\chi)$ and $\varphi_{-h}(\chi)$.

The Casimir of the $sl(2)$ algebra is very similar to $\mathcal{C}(\chi)$:

$$\mathcal{C}_{\mathcal{N}=0} = \chi^2 (1 - \chi) \partial_\chi^2 - \chi^2 \partial_\chi = \mathcal{C}_{\mathcal{N}=2} - \chi \partial_\chi, \quad (3.4.12)$$

and the eigenfunctions of the $\mathcal{N} = 0$ and $\mathcal{N} = 2$ SYK models are closely related too. If we denote

the eigenfunction of the non-supersymmetric model as $F_h(\chi)$:

$$C_{\mathcal{N}=0}F_h(\chi) = h(h-1)F_h(\chi), \quad F_h(\chi) \equiv B(h, h)\chi^h {}_2F_1(h, h; 2h; \chi) \quad \text{for } \chi < 1, \quad (3.4.13)$$

then the eigenfunction of the $\mathcal{N} = 2$ model φ_h is a linear combination:

$$\varphi_h(\chi) = F_h(\chi) - F_{h+1}(\chi). \quad (3.4.14)$$

For a proof of this relation see Appendix 3.10.

Knowing the basis of the Casimir, we can fix the properties of the four-point function under discrete symmetries (exchange of two fermions) and then find it as a linear combination of φ_h, φ_{-h} . But we find it advantageous to use the shadow formalism to derive an alternative basis of eigenfunctions, which would already have the desired symmetries by construction.

3.4.2 Shadow formalism

Using the shadow prescription, we treat the fields at the points 1 and 2 as living in a different CFT than the fields at the points 3 and 4. Then the four-point function is just a product of independent two-point functions:

$$W = \mathcal{G}(\langle 12 \rangle) \mathcal{G}(\langle 34 \rangle) + O(\varepsilon). \quad (3.4.15)$$

To find the four-point function, we add a fictitious term to the Lagrangian, which introduces a small coupling between these two CFTs:

$$\varepsilon \int d\tau_0 d^2\theta_0 V_h(\tau_0, \theta_0, \bar{\theta}_0) V'_{-h}(\tau_0, \theta_0, \bar{\theta}_0). \quad (3.4.16)$$

Here V_h, V'_{-h} are fictitious bosonic operators with dimensions adding up to zero, so that the whole integral is dimensionless.

To the first order in ε , this interaction adds to the four-point function an integral of a product of two three-point functions:

$$W = \mathcal{G}(\langle 12 \rangle) \mathcal{G}(\langle 34 \rangle) + \sum_h \varepsilon \int d\tau_0 d^2\theta_0 \langle \bar{\Psi}(1) \Psi(2) V_h(0) \rangle \langle \bar{\Psi}(3) \Psi(4) V'_{-h}(0) \rangle + O(\varepsilon^2). \quad (3.4.17)$$

Now we have to fix the form of chiral-antichiral three-point function. In one dimension, a three-

point function with two complex fermions can be either odd or even under exchange of those fermions. Generically it is a linear combination:

$$\langle \bar{\Psi}(1) \Psi(2) V_h(0) \rangle = A f_h^A(1, 2, 0) + S f_h^S(1, 2, 0). \quad (3.4.18)$$

where the form of the three-point functions is fixed by chirality:

$$f_h^A(1, 2, 0) = \frac{\text{sgn}(\langle 12 \rangle)}{|\langle 12 \rangle|^{2\Delta-h} |\langle 10 \rangle|^h |\langle 02 \rangle|^h}, \quad (3.4.19)$$

$$f_h^S(1, 2, 0) = \frac{\text{sgn}(\langle 10 \rangle) \text{sgn}(\langle 20 \rangle)}{|\langle 12 \rangle|^{2\Delta-h} |\langle 10 \rangle|^h |\langle 02 \rangle|^h}. \quad (3.4.20)$$

Here f_h^S , f_h^A are respectively symmetric and antisymmetric under the exchange $(\tau_1, \theta_1, \bar{\theta}_1) \leftrightarrow (\tau_2, \theta_2, \bar{\theta}_2)$.

Dividing the four-point function (3.4.17) over the appropriate propagators to make it conformally invariant, we find:

$$\mathcal{F} = \sum_h \int d\tau_0 d^2\theta_0 \frac{(A + S \text{sgn} \tau_{12} \text{sgn} \tau_{10} \text{sgn} \tau_{20}) (A' + S' \text{sgn} \tau_{34} \text{sgn} \tau_{30} \text{sgn} \tau_{40})}{|\langle 12 \rangle|^{-h} |\langle 10 \rangle|^h |\langle 02 \rangle|^h |\langle 34 \rangle|^h |\langle 30 \rangle|^{-h} |\langle 04 \rangle|^{-h}} + O(\varepsilon^2). \quad (3.4.21)$$

where we denote $\mathcal{W} = 1 + \varepsilon \mathcal{F}$. We call the functions in the sum (3.4.21) Ξ_h . They are eigenfunctions of the Casimir:

$$\mathcal{C} \Xi_h = h^2 \Xi_h. \quad (3.4.22)$$

The shadow representation allows us to find the explicit form of Ξ_h as an integral. In the coordinates chosen as in (3.4.4), (3.4.6), the eigenfunction reads:

$$\Xi_h = \int d\tau_0 d^2\theta_0 \frac{(A - S \text{sgn} \chi \text{sgn} \tau_0 \text{sgn}(\chi - \tau_0)) (A' - S' \text{sgn}(1 - \tau_0))}{|\chi|^{-h} |\tau_0 - \theta_0 \bar{\theta}_0|^h |\chi - \tau_0 - \theta_0 \bar{\theta}_0|^h |1 - \tau_0 - \theta_0 \bar{\theta}_0|^{-h}}. \quad (3.4.23)$$

Now we integrate over Grassmann coordinates and rename $y = \tau_0$, to find the four-point function as an integral over even coordinates:

$$\Xi_h = \int dy (A - S \text{sgn} \chi \text{sgn} y \text{sgn}(\chi - y)) (A' - S' \text{sgn}(1 - y)) \frac{h |\chi|^h |1 - y|^h}{|y|^h |\chi - y|^h} \left(\frac{1}{y} + \frac{1}{\chi - y} - \frac{1}{1 - y} \right). \quad (3.4.24)$$

We break this integral into four parts in a straightforward way:

$$\Xi_h = AA'\Xi_h^{AA} + AS'\Xi_h^{AS} + SA'\Xi_h^{SA} + SS'\Xi_h^{SS}. \quad (3.4.25)$$

Each of the four integrals can be found directly, but we can save the effort if we notice similarities to the non-supersymmetric SYK model with complex fermions. In that case, the four-point function is given by an integral:

$$\Psi_h^{\mathcal{N}=0} = \int dy (a + s \operatorname{sgn} \chi \operatorname{sgn} y \operatorname{sgn} (\chi - y)) (a' + s' \operatorname{sgn} (1 - y)) \frac{|\chi|^h |1 - y|^{h-1}}{|y|^h |\chi - y|^h}. \quad (3.4.26)$$

It is also a sum of four parts:

$$\Psi_h^{\mathcal{N}=0} = aa'\Psi_h^A(\chi) + ss'\Psi_h^S(\chi) + as'\Psi_h^{AS}(\chi) + sa'\Psi_h^{SA}(\chi). \quad (3.4.27)$$

These functions have different parity under exchanges of two fermions. The function Ψ^A is odd under both $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, and it is the same as the eigenfunction in the original SYK model, found in [29]. The function Ψ^S is even under both of these permutations. The functions Ψ^{AS} , Ψ^{SA} have mixed parity. They break the time-reversal symmetry $\mathcal{T}$, whereas Ψ^A and Ψ^S preserve it.

Upon inspection, we see that the $\mathcal{N} = 2$ eigenfunctions are linear combinations of the non-supersymmetric ones, in particular:

$$\Xi_h^{AA} = h (\Psi_{h+1}^{SA}(\chi) - \Psi_h^{AS}(\chi)), \quad (3.4.28)$$

$$\Xi_h^{SS} = h (\Psi_{h+1}^{AS}(\chi) - \Psi_h^{SA}(\chi)), \quad (3.4.29)$$

$$\Xi_h^{AS} = h (-\Psi_{h+1}^S(\chi) + \Psi_h^A(\chi)), \quad (3.4.30)$$

$$\Xi_h^{SA} = h (-\Psi_{h+1}^A(\chi) + \Psi_h^S(\chi)). \quad (3.4.31)$$

We notice that an eigenfunction in the $\mathcal{N} = 2$ model built from three-point functions of the same type (AA or SS) is a sum of “mixed” eigenfunctions in $\mathcal{N} = 0$, and vice versa: a “mixed” $\mathcal{N} = 2$ eigenfunction is a combination of “pure” $\mathcal{N} = 0$ eigenfunctions. As a consequence, “mixed” eigenfunctions in $\mathcal{N} = 2$ preserve time-reversal, and “pure” four-point functions break it. This happens because the $\mathcal{N} = 2$ eigenfunctions are integrals over Grassmann coordinates. The Grassmann measure $d\theta_0 d\bar{\theta}_0$ is an imaginary quantity and therefore is odd under time-reversal. So the functions of mixed parity, which are $\mathcal{T}$ -odd in the $\mathcal{N} = 0$ model, turn out to be $\mathcal{T}$ -even in the $\mathcal{N} = 2$ model.

It is interesting to notice the properties of these eigenfunctions under the transformation $h \leftrightarrow -h$. From (3.4.17), we see that this transformation corresponds to exchange of pairs of fermions: $(1, 2) \leftrightarrow (3, 4)$. We know what happens to the eigenfunctions of the non-supersymmetric SYK when we take $h \leftrightarrow 1 - h$:

$$\Psi_{1-h}^A = \Psi_h^A, \quad (3.4.32)$$

$$\Psi_{1-h}^S = \Psi_h^S, \quad (3.4.33)$$

$$\Psi_{1-h}^{AS} = \Psi_h^{SA}. \quad (3.4.34)$$

From here, we can see that:

$$\Xi_{-h}^{AA} = \Xi_h^{AA}, \quad (3.4.35)$$

$$\Xi_{-h}^{SS} = \Xi_h^{SS}, \quad (3.4.36)$$

$$\Xi_{-h}^{AS} = \Xi_h^{SA}. \quad (3.4.37)$$

The transformation exchanges the $\mathcal{T}$ -even functions and leaves $\mathcal{T}$ -odd functions invariant.

Since the SYK model is $\mathcal{T}$ -invariant, in what follows we are interested in the $\mathcal{T}$ -invariant eigenfunctions, Ξ_h^{AS} and Ξ_h^{SA} . Moreover, because of the relation (3.4.37) we can focus our attention on the Ξ_h^{AS} function only. For brevity, we call it ξ_h :

$$\xi_h(\chi) \equiv \Xi_h^{AS}(\chi) = h(\Psi_h^A(\chi) - \Psi_{h+1}^S(\chi)) = h(\Psi_h^A(\chi) - \Psi_{-h}^S(\chi)). \quad (3.4.38)$$

For $\chi < 1$ we can express the eigenfunctions in terms of φ_h defined in (3.4.11).

$$\xi_h = h \left(1 + \frac{1}{\cos \pi h} \right) \varphi_h(\chi) + h \left(1 - \frac{1}{\cos \pi h} \right) \varphi_{-h}(\chi), \quad \chi < 1. \quad (3.4.39)$$

For $\chi > 1$, we have to do an analytical continuation. Using the results from the $\mathcal{N} = 0$ SYK, we find:

$$\xi_h = \frac{4}{\sqrt{\pi}} \Gamma \left(1 + \frac{h}{2} \right) \Gamma \left(\frac{1-h}{2} \right) \left({}_2F_1 \left(\frac{h}{2}, \frac{1-h}{2}; \frac{1}{2}; \left(\frac{2-\chi}{\chi} \right)^2 \right) + h \frac{2-\chi}{\chi} {}_2F_1 \left(\frac{h}{2}, \frac{1-h}{2}; \frac{3}{2}; \left(\frac{2-\chi}{\chi} \right)^2 \right) \right). \quad (3.4.40)$$

We can expand a supersymmetric conformal four-point function in terms of the ξ_h functions.

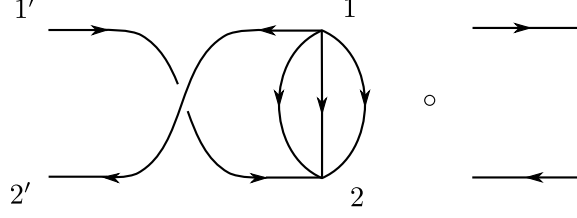


Figure 3.2: $\mathcal{N} = 2$ conformal kernel.

Schematically, the SYK four-point function looks as:

$$\mathcal{F} = \frac{\mathcal{F}_0}{1 - K}. \quad (3.4.41)$$

The SYK kernel K commutes with the $\mathcal{N} = 2$ Casimir and therefore is diagonalized by its eigenfunctions ξ_h . As our next step, we find the eigenvalues of the kernel.

3.4.3 Kernel

Schematically, the $\mathcal{N} = 2$ SYK kernel looks like fig. 3.2. Unlike the non-supersymmetric case, here chirality restricts us to only one form of the kernel operator. The kernel in the integral form is as follows:

$$K = (\hat{q} - 1) b^{\hat{q}} J \frac{\text{sgn } \tau_{12}}{|\langle 12 \rangle|^{2\Delta(\hat{q}-2)}} \frac{\text{sgn } \tau_{1'2}}{|\langle 1'2 \rangle|^{2\Delta}} \frac{\text{sgn } \tau_{12'}}{|\langle 12' \rangle|^{2\Delta}} d\tau_1 d\tau_2 d\bar{\theta}_1 d\theta_2. \quad (3.4.42)$$

The kernel can act either on the 12 or on the 34 channel of the four-point function. In the shadow representation, we construct the four-point point function as an integral of 12y and 34y three-point function, where y is the arbitrary variable we integrate over. This means that to find out how the kernel acts on a four-point function, it suffices to consider how it acts on the three-point functions. We have fixed the form of the possible three-point functions in (3.4.19, 3.4.20). These f_h^A , f_h^S functions diagonalize the kernel:

$$\int K(1', 2'|1, 2) f_h^A(1, 2, 0) = k^A(h) f_h^A(1', 2', 0), \quad \int K(1', 2'|1, 2) f_h^S(1, 2, 0) = k^S(h) f_h^S(1', 2', 0). \quad (3.4.43)$$

To find the eigenvalues k^A and k^S conveniently, we first take τ_0 in the three-point function to infinity, and set:

$$1' \rightarrow (1, \theta), \quad (3.4.44)$$

$$2' \rightarrow (0, \bar{\theta}). \quad (3.4.45)$$

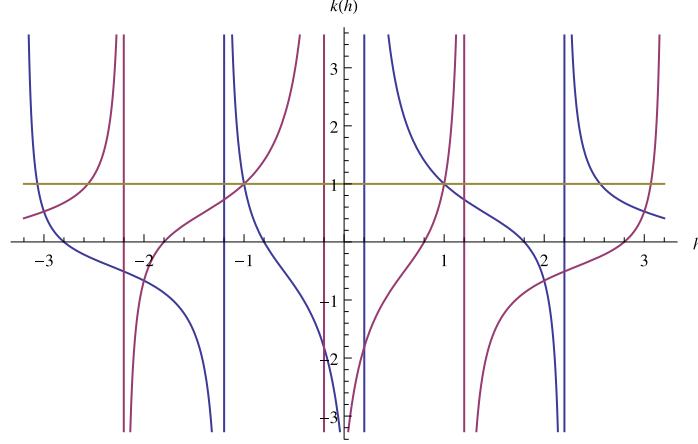


Figure 3.3: Eigenvalues of the antisymmetric (red) and symmetric (blue) kernels at $\hat{q} = 5$.

Then the eigenvalues are given by the integrals,

$$k^A = \frac{\tan \pi \Delta}{4\pi} \int d\tau_1 d\tau_2 d\bar{\theta}_1 d\theta_2 \frac{1}{|\langle 12 \rangle|^{1-2\Delta-h}} \frac{\text{sgn}(1-\tau_2)}{|\langle 1'2 \rangle|^{2\Delta}} \frac{\text{sgn}(\tau_1)}{|\langle 12' \rangle|^{2\Delta}}. \quad (3.4.46)$$

$$k^S = \frac{\tan \pi \Delta}{4\pi} \int d\tau_1 d\tau_2 d\bar{\theta}_1 d\theta_2 \frac{\text{sgn} \tau_{12}}{|\langle 12 \rangle|^{1-2\Delta-h}} \frac{\text{sgn}(1-\tau_2)}{|\langle 1'2 \rangle|^{2\Delta}} \frac{\text{sgn}(\tau_1)}{|\langle 12' \rangle|^{2\Delta}}. \quad (3.4.47)$$

These integrals are of the same type we have encountered in the $\mathcal{N} = 0$ SYK kernel. We can make a change of variables and transform them into products of one-dimensional integrals. The details of the computation can be found in Appendix 2.9. Explicitly, the answer reads:

$$k^A = -\frac{1}{\pi^2} \Gamma(-2\Delta) \Gamma(2-2\Delta) \Gamma(2\Delta-h) \Gamma(2\Delta+h) \sin 2\pi\Delta (\sin 2\pi\Delta - \sin \pi h), \quad (3.4.48)$$

$$k^S = -\frac{1}{\pi^2} \Gamma(-2\Delta) \Gamma(2-2\Delta) \Gamma(2\Delta-h) \Gamma(2\Delta+h) \sin 2\pi\Delta (\sin 2\pi\Delta + \sin \pi h). \quad (3.4.49)$$

These expressions coincide with the results of [78], up to renaming $h \rightarrow h + 1/2$.

We see that the eigenvalues satisfy:

$$k^A(h) = k^S(-h). \quad (3.4.50)$$

This allows for “mixed” four-point functions, i.e. those built from three-point functions with opposite symmetries. The Ξ^{AS} eigenfunctions, which we are going to use to expand the full four-point function, are constructed from three-point function of different types. Acting with the kernel on the Ξ^{AS} eigenfunction from the left (in the 12 channel), we multiply it by the k^A eigenvalue; acting from the right, we multiply it by the k^S eigenvalue. But if we exchange $h \leftrightarrow -h$ (transforming Ξ^{AS} to

Ξ^{SA}), we exchange the two sides in the shadow representation, and therefore exchange two channels. The condition (3.4.50) is needed to allow this transformation.

For consistency, in what follows the kernel always acts on the four-point function from the left, so that the k^A eigenvalue corresponds to the Ξ^{AS} eigenfunction.

The eigenvalues of the $\mathcal{N} = 2$ kernel look very much like the eigenvalues of the non-supersymmetric kernel which we list in Appendix 3.9. The exact relation is:

$$k_{\mathcal{N}=2}^A(h) = \frac{2\Delta + h - 1}{2\Delta - 2} k_{\mathcal{N}=0}^A(h), \quad (3.4.51)$$

$$k_{\mathcal{N}=2}^A(h) = \frac{2\Delta - h - 1}{2\Delta} k_{\mathcal{N}=0}^S(-h). \quad (3.4.52)$$

The symmetry (3.4.50) is a direct consequence of the symmetry $h \leftrightarrow 1 - h$ for the eigenvalues of non-supersymmetric kernel.

The dimensions of the operators in the theory are given by the solutions to the equation $k = 1$ (see fig. 3.3). Generally these dimensions are irrational, given by an asymptotic formula:

$$h^A = 2n + 1 + 2\Delta + O\left(\frac{1}{n}\right), \quad (3.4.53)$$

$$h^S = 2n + 2\Delta + O\left(\frac{1}{n}\right), \quad n > 0. \quad (3.4.54)$$

There is also a mode with $h = 1$ in both channels (which is the same as the $h = 3/2$ mode of [78]. This mode represents the charge multiplet, consisting of the R -charge, the supercharge and the stress tensor:

$$\mathcal{Q} = R + \theta \bar{Q} + \bar{\theta} Q + \theta \bar{\theta} T. \quad (3.4.55)$$

Since the dimension of $\mathcal{Q}$ is one, the dimension of the R -charge operator is also one, and the dimension of the stress tensor is two, just as in the non-supersymmetric complex SYK model [72]. Notice also that like the $U(1)$ charge in the non-supersymmetric model, the R -charge, despite being conserved, has non-zero dimension in the infrared limit.

3.4.4 Inner product

To apply the formula (3.4.41) for the four-point function, we need to project the zero-rung function $\mathcal{F}_0$ to the basis of the Casimir eigenfunctions. To this end, we first find an inner product for the ξ_h

eigenfunctions.

For the non-supersymmetric SYK model, the eigenstates of the Casimir form a Hilbert space [29]. In the supersymmetric case, we should not expect this, since the eigenstates are functions of a superspace and therefore the set of states may contain functions of odd variables. Indeed, it has been found in [71], that the $\mathcal{N} = 1$ eigenfunctions do not form a Hilbert space. Nevertheless, we want to get as close to a Hilbert space as possible.

An invariant inner product of chiral-antichiral four-point functions looks as follows:

$$\langle f, g \rangle = \int \frac{dt_1 dt_2 d\bar{\theta}_1 d\theta_2}{\langle 12 \rangle} \frac{dt_3 dt_4 d\bar{\theta}_3 d\theta_4}{\langle 34 \rangle} f \cdot g \equiv \int d\mu(1, 2) d\mu(3, 4) f \cdot g. \quad (3.4.56)$$

Here we have defined the two-particle integration measure $d\mu(i, j)$, which is conformally invariant but not real: $d\mu(i, j) \neq d\bar{\mu}(i, j)$. Therefore we do not expect the inner product to be real, and this is why we have $f \cdot g$ instead of $\bar{f} \cdot g$ in the inner product. For the same reason, we do not expect the Casimir to be Hermitean with respect to this inner product. Instead, we require it to be bilinear symmetric.

We have shown in the beginning of Section 3.4 that we can fix the coordinates in the four-point function, so that it does not depend on odd coordinates in the superspace. In the same way, we can use the supergroup to make the measure a function of χ only. The details of this calculation can be found in Appendix 3.11, the result being:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \frac{d\chi}{\chi(1-\chi)} f g. \quad (3.4.57)$$

This inner product is clearly not positive-definite, so the $\mathcal{N} = 2$ eigenstates do not form a Hilbert space. It is easy to see that the Casimir 3.4.9 is symmetric with respect to this norm:

$$\langle \mathcal{C}f, g \rangle = \int_{-\infty}^{\infty} d\chi f \partial_{\chi} (\chi \partial_{\chi} g) = (f \chi \partial_{\chi} g - g \chi \partial_{\chi} f)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f \partial_{\chi} (\chi \partial_{\chi} g) = \langle f, \mathcal{C}g \rangle, \quad (3.4.58)$$

provided that a certain boundary condition at infinity is satisfied:

$$(f \chi \partial_{\chi} g - g \chi \partial_{\chi} f)|_{-\infty}^{\infty} = 0. \quad (3.4.59)$$

If the inner product (3.4.57) were positive definite, we would find a complete set of functions by requiring that the eigenvalue of the Casimir h^2 be positive and then looking for normalizable (or

continuum-normalizable) states. We are not in this situation here. Nevertheless we can find a set of functions with non-negative norm. If we require that the Casimir does not bring us out of this set,

$$\langle \xi_h, \xi_h \rangle \geq 0, \quad \langle \mathcal{C}\xi_h, \mathcal{C}\xi_h \rangle \geq 0 \quad \Rightarrow \quad h^4 \geq 0. \quad (3.4.60)$$

then it implies that the eigenvalue of the Casimir has to be real:

$$h^2 \in \mathbb{R}. \quad (3.4.61)$$

In what follows, we see that the condition (3.4.61) is enough to guarantee that the inner product in the ξ_h basis is positive-(semi)definite.

The eigenvalue of the Casimir can be real if h is either purely imaginary or purely real. In the latter case, the eigenstate is normalizable only if we further restrict to integer h :

$$h \in i\mathbb{R} \quad \text{or} \quad h \in \mathbb{Z}. \quad (3.4.62)$$

The first case gives us a continuous series of states, and we expect them to be continuum-normalizable, that is their inner product is proportional to a delta function:

$$\langle \xi_{is}, \xi_{is'} \rangle \sim \delta(s - s'). \quad (3.4.63)$$

This singular contribution comes from the vicinity of $\chi = 0$:

$$\langle \xi_{is}, \xi_{is'} \rangle \sim \int_{-\epsilon}^{\epsilon} \frac{d\chi}{\chi} \xi_{is} \xi_{is'}. \quad (3.4.64)$$

For small positive χ , the Casimir eigenfunctions have a power-like behavior:

$$\chi \rightarrow +0 : \quad \varphi_{is} \sim \chi^{is} B(is, is). \quad (3.4.65)$$

To find the asymptotic of the eigenfunction for negative χ , we once again represent ξ_h via $\mathcal{N} = 0$ eigenfunctions:

$$\xi_h = h(\Psi_h^A - \Psi_{-h}^S). \quad (3.4.66)$$

The function Ψ_h^A is symmetric under $\chi \rightarrow \frac{\chi}{\chi-1}$, and Ψ_h^S is antisymmetric under the same transformation. It means in particular that Ψ_h^A is an even function of χ in the vicinity of zero, and Ψ_h^S is

odd. Since the measure $d\chi/\chi$ is odd, only the terms odd in χ in the integrand of (3.4.64) contribute to the final answer. So in terms of the $\mathcal{N} = 0$ eigenfunctions, the inner product is:

$$\langle \xi_{is}, \xi_{is'} \rangle = \int_{-\epsilon}^{\epsilon} \frac{d\chi}{\chi} is \cdot is' (-\Psi_{is}^A \Psi_{is'+1}^S - \Psi_{is+1}^S \Psi_{is'}^A) = 2ss' \int_0^{\epsilon} \frac{d\chi}{\chi} (\Psi_{is}^A \Psi_{is'+1}^S + \Psi_{is+1}^S \Psi_{is'}^A). \quad (3.4.67)$$

For small positive χ , the Ψ_h^A, Ψ_h^S eigenfunctions behave as follows:

$$\Psi_h^A \sim \left(1 + \frac{1}{\cos \pi h}\right) B(h, h) \chi^h + \left(1 - \frac{1}{\cos \pi h}\right) B(1-h, 1-h) \chi^{1-h}, \quad (3.4.68)$$

$$\Psi_h^S \sim \left(1 - \frac{1}{\cos \pi h}\right) B(h, h) \chi^h + \left(1 + \frac{1}{\cos \pi h}\right) B(1-h, 1-h) \chi^{1-h}, \quad \chi \rightarrow +0 \quad (3.4.69)$$

Bringing (3.4.67, 3.4.69) together, using the integral form of the delta-function:

$$\int_0^{\epsilon} \frac{d\chi}{\chi} \left(\chi^{i(s-s')} + \chi^{-i(s-s')} \right) = 2\pi \delta(s-s'), \quad (3.4.70)$$

and an identity for the Euler's beta function:

$$B(is, is) B(-is, -is) = \frac{4\pi}{s} \coth \pi s, \quad (3.4.71)$$

we can find the norm for the continuous series as:

$$\langle \xi_{is}, \xi_{is'} \rangle = 4\pi s \tanh \pi s \cdot 2\pi \delta(s-s'). \quad (3.4.72)$$

In particular, this norm is real and positive for real non-zero s , as expected.

The reader may be puzzled that the inner product of the basis states ξ_{is} is positive definite, given that the inner product (3.4.57) is not. Indeed, we can easily find a function which has a negative norm, for example one that is close to zero for positive χ and has a bump at negative χ . How can it be expanded in the ξ_{is} basis?

The matter becomes clear if we recall that the ξ_{is} functions are generally complex, as are the expansion coefficients, therefore the condition that the norm be non-negative is not very restrictive. To see this, we can break the eigenfunction into a real and an imaginary parts,

$$\xi_{is} = \zeta_s + i\eta_s. \quad (3.4.73)$$

Its complex conjugate is also in the spectrum and has the same eigenvalue:

$$\bar{\xi}_{is} = \xi_{-is} = \zeta_s - i\eta_s. \quad (3.4.74)$$

From the inner products for ξ_h ,

$$\langle \xi_{is}, \xi_{is'} \rangle = 4\pi s \tanh \pi s \cdot 2\pi \delta(s - s'), \quad \langle \xi_{is}, \bar{\xi}_{is} \rangle = \langle \xi_{is}, \xi_{-is} \rangle = 0, \quad (3.4.75)$$

we can find the inner products for the real and imaginary parts separately:

$$\langle \zeta_s, \zeta_{s'} \rangle = -\langle \eta_s, \eta_{s'} \rangle = 2\pi s \tanh \pi s \cdot 2\pi \delta(s - s'), \quad \langle \zeta_s, \eta_{s'} \rangle = 0. \quad (3.4.76)$$

Hence for each eigenvalue we have two real functions ζ_s and η_s , with positive and negative norm, which are orthogonal to each other. A function that can be expanded in the (ζ_s, η_s) basis, clearly can be expanded in the ξ_{is} basis too, possibly with complex coefficients.

Next we find the inner product of bound states, labeled by integer eigenvalues:

$$h \in \mathbb{Z}. \quad (3.4.77)$$

For a state to be normalizable, we have to further restrict h . For a negative integer h , the eigenfunction φ_h diverges, so we have to make sure that the coefficient in front of it vanishes. In other words, the $\xi_h = \Xi_h^{AS}$ eigenfunction is normalizable at even positive or odd negative h :

$$h^{AS} \in 2\mathbb{Z}_+ \quad \text{or} \quad h^{AS} \in 2\mathbb{Z}_- + 1. \quad (3.4.78)$$

But the spectrum should be symmetric under $h \leftrightarrow -h$. So for the Ξ_h^{SA} eigenfunction, the choice is exactly opposite:

$$h^{SA} \in 2\mathbb{Z}_+ + 1 \quad \text{or} \quad h^{SA} \in 2\mathbb{Z}_-. \quad (3.4.79)$$

To find the norm of a bound state, we take the integral:

$$\langle \xi_h, \xi_h \rangle = \int_{-\infty}^{\infty} \frac{d\chi}{\chi(1-\chi)} \xi_h^2(\chi). \quad (3.4.80)$$

This integral is generally tricky, but we can express it via the norm for the bound state in the non-supersymmetric model (details in Appendix 3.12). The result is:

$$\langle \xi_h, \xi_{h'} \rangle = \delta_{hh'} 4\pi^2 |h|. \quad (3.4.81)$$

Again, we see that the norm is positive, except for the $h = 0$ mode which has a zero norm.

The continuous set ξ_{is} is orthogonal to the discrete series ξ_n since for these two cases the eigenvalues of the Casimir are different.

If we were working in a true Hilbert space, the eigenstates of the Casimir with real eigenvalues would form a complete set. If ξ_{is} formed a complete set, then naively, given the inner products (3.4.72, 3.4.81), the following identity would hold:

$$\int_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{1}{4\pi s \tanh \pi s} \xi_{is}(\chi) \xi_{is}(\chi') + \sum_{h \in \mathbb{Z}_+} \frac{1}{4\pi^2 h} \xi_h(\chi) \xi_h(\chi') \stackrel{?}{=} \chi(1-\chi) \delta(\chi - \chi'). \quad (3.4.82)$$

Then we can integrate both sides of this relation with a function we want to expand in the ξ basis.

However, this expression cannot be correct. The integral over the continuous states has a double pole at $s = 0$ and therefore the left hand side diverges. The root of the problem is that the our functions are not a complete set, because the constant function is orthogonal to all of them. The constant function belongs to both the continuous and the discrete series and is a limit of ξ_{is} at zero s :

$$\xi_0 = \lim_{h \rightarrow 0} \xi_h = 4. \quad (3.4.83)$$

From (3.4.72) and (3.4.81) we see that it is orthogonal to all the eigenstates.

We do not know a general completeness relation for these functions, but for our application it is sufficient to know the expansion of the zero-rung function, that is the relation (3.4.82), convolved with $\mathcal{F}_0$. In Section 3.4.5, we find that the relation (3.4.82) convolved with $\mathcal{F}_0$ is true, provided the integration contour goes to the right of the double pole at $s = 0$.

Another function which is orthogonal to our set is:

$$\left. \frac{d}{ds} \xi_{is} \right|_{s=0} = 4 \log \chi. \quad (3.4.84)$$

We see that a constant and a logarithmic function lie outside of our basis. As we have already mentioned before, we should not a priori expect the eigenfunctions of the Casimir to be a complete set of functions if the inner product is not positive-definite.

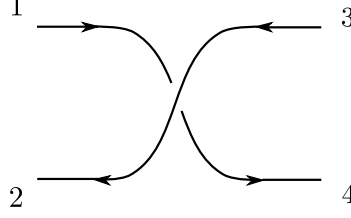


Figure 3.4: Zero-rung four-point function.

3.4.5 Zero-rung four-point function and the $h = 0$ mode

To find the full four-point function, we project the zero-rung function $\mathcal{F}_0$ (see fig. 3.4) to the basis of the Casimir eigenfunctions ξ_h using the completeness relation (3.4.82). Schematically, this expansion is written as:

$$\mathcal{F}_0 = \sum_h \frac{\langle \xi_h, \mathcal{F}_0 \rangle}{\langle \xi_h, \xi_h \rangle} \xi_h. \quad (3.4.85)$$

The “sum” over h includes the discrete sum over the bound states as well as the integral over the continuous series of states. But with the latter, we run into a problem. The integration measure in the completeness relation (3.4.82) has a double pole at $s = 0$. To make the integral meaningful, we have to deform the integration contour away from the origin. The result might depend on this deformation. To see whether the procedure makes sense, we will consider the expansion of the zero-rung four-point function near $\chi = 0$.

The zero-rung four-point function is a (conformally invariant) combination of conformal propagators. Chirality restricts its form to be (see fig. 3.4):

$$\mathcal{F}_0 \equiv \frac{\mathcal{G}(\langle 14 \rangle) \mathcal{G}(\langle 32 \rangle)}{\mathcal{G}(\langle 12 \rangle) \mathcal{G}(\langle 34 \rangle)} = \text{sgn } \chi \cdot |\chi|^{2\Delta}. \quad (3.4.86)$$

The zero-rung function has a finite norm and therefore belongs to our pseudo-Hilbert space:

$$\langle \mathcal{F}_0, \mathcal{F}_0 \rangle = \text{p.v.} \int \frac{d\chi}{\chi(1-\chi)} |\chi|^{4\Delta} < \infty. \quad (3.4.87)$$

The inner product of an eigenfunction with the zero-rung propagator is related to the eigenvalue of the kernel, in full analogy with the non-supersymmetric case:

$$\langle \xi_h, \mathcal{F}_0 \rangle = \frac{1}{2} \alpha k^A(h), \quad (3.4.88)$$

where α is similar to the α_0 coefficient in the non-supersymmetric model:

$$\frac{1}{\alpha} = b^{\hat{q}} J(\hat{q} - 1) = \frac{1 - 2\Delta}{8\pi\Delta} \tan \pi\Delta. \quad (3.4.89)$$

The computation can be found in Appendix 3.14.

To expand the zero-rung four-point function, we have to first determine whether it has the symmetry of AS or SA type. If it has the symmetry of the AS type, it expands in the $\Xi^{AS} = \xi_h$ basis:

$$\begin{aligned} \mathcal{F}_0^{AS}(\chi) = & \alpha \int_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{1}{4\pi h \tan \pi h} k^A(h) \Xi_h^{AS}(\chi) + \\ & \alpha \sum_{h \in 2\mathbb{Z}_+} \frac{1}{4\pi^2|h|} k^A(h) \Xi_h^{AS}(\chi) + \alpha \sum_{h \in 1-2\mathbb{Z}_+} \frac{1}{4\pi^2|h|} k^A(h) \Xi_h^{AS}(\chi). \end{aligned} \quad (3.4.90)$$

Here in the integral we take $h = is$. For integer h , we can use an identity:

$$k^A(h) = k^A(-h), \quad h \in \mathbb{Z}, \quad (3.4.91)$$

and rewrite (3.4.90) as:

$$\mathcal{F}_0^{AS}(\chi) = \alpha \int_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{1}{4\pi h \tan \pi h} k^A(h) \xi_h(\chi) + \alpha \sum_{h \in \mathbb{Z}_+} \frac{1}{4\pi^2 h} k^A(h) \xi_h(\chi). \quad (3.4.92)$$

If however the zero-rung four-point function has the symmetry of the SA type, it expands in terms of Ξ^{SA} functions:

$$\begin{aligned} \mathcal{F}_0^{SA}(\chi) = & \alpha \int_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{1}{4\pi h \tan \pi h} k^S(h) \Xi_h^{SA}(\chi) + \\ & \alpha \sum_{h \in -2\mathbb{Z}_+} \frac{1}{4\pi^2|h|} k^S(h) \Xi_h^{SA}(\chi) + \alpha \sum_{h \in 2\mathbb{Z}_+ - 1} \frac{1}{4\pi^2|h|} k^S(h) \Xi_h^{SA}(\chi). \end{aligned} \quad (3.4.93)$$

However, using the fact that $\Xi_h^{AS} = \Xi_{-h}^{SA}$ and (3.4.91), we can see that these two expansions give exactly the same result:

$$\mathcal{F}_0 = \mathcal{F}_0^{AS} = \mathcal{F}_0^{SA}. \quad (3.4.94)$$

The expression (3.4.92) is a more explicit version of (3.4.85). As we discussed before, the integration measure has a double pole at $h = 0$. To resolve this problem, we deform the contour so that it avoids zero as in fig. 3.5. But this deformation might add to the zero-rung four-point function a

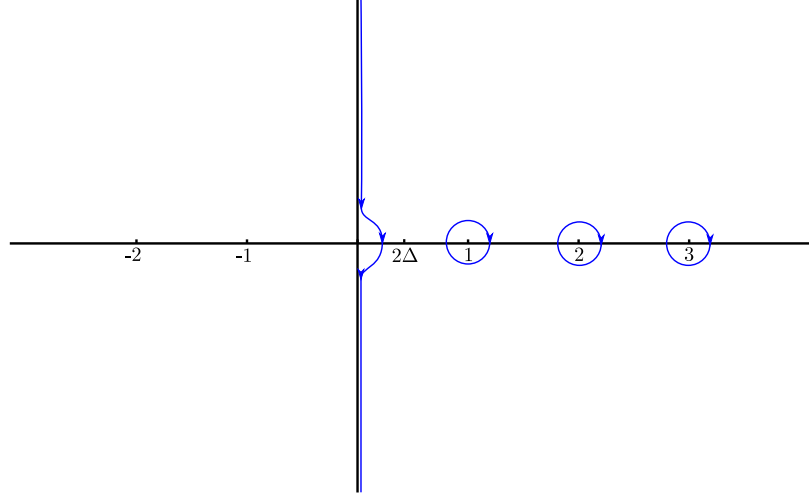


Figure 3.5: The integration contour for the $\mathcal{N} = 2$ SYK model avoids the double pole at zero.

contribution of the form:

$$\text{Res}_{s=0} \frac{1}{s \tanh \pi s} \xi_{is} \sim \left. \frac{d}{ds} \xi_{is}(\chi) \right|_{s=0} \sim \log \chi. \quad (3.4.95)$$

To see if this is the case, we look at the four-point function near $\chi = 0$. In this limit,

$$\xi_h \sim hB(h, h) \left(1 + \frac{1}{\cos \pi h} \right) \chi^h + hB(-h, -h) \left(1 - \frac{1}{\cos \pi h} \right) \chi^{-h}, \quad \chi \sim +0. \quad (3.4.96)$$

Using the simple identity,

$$k^A(h)\xi_h + k^A(-h)\xi_{-h} = \frac{1}{2} (k^A(h) + k^A(-h)) (\xi_h + \xi_{-h}) + \frac{1}{2} (k^A(h) - k^A(-h)) (\xi_h - \xi_{-h}), \quad (3.4.97)$$

we can recast (3.4.92) in the form:

$$\mathcal{F}_0 = \int_C \frac{ds}{2\pi} \frac{1}{8\pi \tan \pi h \tan \pi \Delta} \frac{16\Delta}{\tan \pi \Delta} B(h, h) B(2\Delta - h, 2\Delta + h) B(4\Delta, -2\Delta) \left(\sin \pi h - \frac{\sin 2\pi \Delta}{\cos \pi h} \right) \chi^h + \sum_{h \in \mathbb{Z}_+} (\dots), \quad (3.4.98)$$

where the sum in parentheses is the sum over residues of the integrand at positive integer h , and the contour C goes as in fig. 3.5, crossing the horizontal axis between the origin and 2Δ . Closing the integration contour to the right, we find that $\mathcal{F}_0$ is given by a sum of residues of the integrand

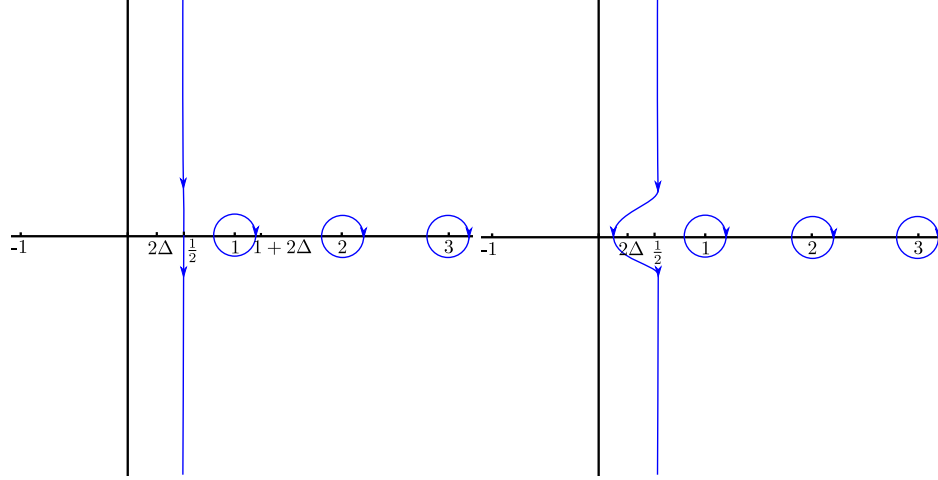


Figure 3.6: Integration contours in the non-supersymmetric SYK model. The anti-symmetric channel is on the left, the symmetric one is on the right.

at the points where the kernel is singular:

$$\mathcal{F}_0 = -\text{Res}_{h \in \mathbb{Z}_+ + 2\Delta} \frac{1}{8\pi \tan \pi h} \frac{16\Delta}{\tan \pi \Delta} \frac{\Gamma^2(h) \Gamma(2\Delta - h) \Gamma(2\Delta + h) \Gamma(-2\Delta)}{\Gamma(2h) \Gamma(2\Delta)} \left(\sin \pi h - \frac{\sin 2\pi \Delta}{\cos \pi h} \right) \chi^h. \quad (3.4.99)$$

In the leading order, this reduces exactly to the zero-rung four-point function:

$$\mathcal{F}_0 = \chi^{2\Delta} + O(\chi^{1+2\Delta}). \quad (3.4.100)$$

If instead we had deformed the contour to lie to the left of the origin, we would have picked up a contribution proportional to $\sim \log \chi$. We have also checked (3.4.99) numerically for any χ .

It is instructive to see how the integration contour is deformed in the non-supersymmetric SYK. Its continuous series is at $h = \frac{1}{2} + is$, so the naive integration contour is parallel to the y axis and intersects the horizontal axis at $h = \frac{1}{2}$. If fermions are complex, there are two distinct channels and two distinct zero-rung four-point functions. In the anti-symmetric channel (where the usual SYK with real fermions lives), the zero-rung four-point function is:

$$\mathcal{F}_0^A(\mathcal{N} = 0) = -\text{sgn}(\chi) |\chi|^{2\Delta} + \text{sgn} \chi \text{sgn}(1 - \chi) \left| \frac{\chi}{\chi - 1} \right|^{2\Delta}. \quad (3.4.101)$$

This function has a finite norm in the $\mathcal{N} = 0$ inner product. Near zero, this reduces to:

$$\mathcal{F}_0^A(\mathcal{N} = 0) \sim -\chi^{2\Delta+1}, \quad \chi \sim +0. \quad (3.4.102)$$

Then, for the expansion in the Casimir eigenfunctions to work, we should make sure that the pole at $h = 2\Delta + 1$ is inside the contour. And for the naive contour at $h = \frac{1}{2} + is$, this is automatically satisfied.

The four-point function in the symmetric channel, however,

$$\mathcal{F}_0^S(\mathcal{N} = 0) = -\text{sgn}(\chi) |\chi|^{2\Delta} - \text{sgn} \chi \text{sgn}(1 - \chi) \left| \frac{\chi}{\chi - 1} \right|^{2\Delta}, \quad (3.4.103)$$

has infinite norm and therefore does not belong to the Hilbert space. Therefore to find a sensible expansion, we have to deform the contour. Near zero, the symmetric zero-rung function behaves as:

$$\mathcal{F}_0^S(\mathcal{N} = 0) \sim -\chi^{2\Delta}, \quad \chi \sim +0. \quad (3.4.104)$$

So to find it in the expansion, we have to make the contour go around the $h = 2\Delta$ pole. We deform it as in fig. (3.6), making it intersect the horizontal axis between zero and 2Δ .

Note that for the $\mathcal{N} = 0$ SYK, 2Δ is always smaller than $\frac{1}{2}$. So in the symmetric channel, we need to shift the contour by a finite distance. This reflects the fact that the symmetric zero-rung function is outside the Hilbert space. In the $\mathcal{N} = 2$ model, the zero-rung function belong to the pseudo-Hilbert space “marginally”, that is the integral (3.4.87) is convergent only in the principal value prescription. Accordingly, the $\mathcal{N} = 2$ integration contour also gets displaced by an infinitesimally small amount, to avoid the origin.

3.4.6 General form of the four-point function

Now we have all the ingredients needed to expand the SYK four-point function. Formally, it is represented as:

$$\mathcal{F}(\chi) = \sum_h \frac{\mathcal{F}_0}{1 - K} = \sum_h \frac{1}{1 - k^A(h)} \frac{\langle \xi_h, \mathcal{F}_0 \rangle}{\langle \xi_h, \xi_h \rangle} \xi_h(\chi). \quad (3.4.105)$$

Using the expansion of the zero-rung function (3.4.92) allows us to write it in the form:

$$\mathcal{F}(\chi) = -\alpha \int_C \frac{dh}{2\pi i} \frac{1}{4\pi h \tan \pi h} \frac{k^A(h)}{1 - k^A(h)} \xi_h(\chi) + \alpha \sum_{h \in \mathbb{Z}_+} \frac{1}{4\pi^2 h} \frac{k^A(h)}{1 - k^A(h)} \xi_h(\chi), \quad (3.4.106)$$

with the integration contour C being deformed as in fig. 3.5 to avoid the double pole at the origin. The integral in this expression is given by the sum of the poles in the integrand. The poles coming from the measure are at the integer values of h , and are cancelled out by the sum in (3.4.106). The

only poles left are the ones coming from the solutions of $k(h) = 1$:

$$\mathcal{F}(\chi) = - \sum_m \text{Res}_{h=h_m > 0} \alpha \frac{1}{4\pi h \tan \pi h} \frac{1}{1 - k^A(h)} \xi_h(\chi), \quad k^A(h_m) = 1. \quad (3.4.107)$$

These solutions correspond to the dimensions of the physical operators in the model. There is also an $h = 1$ subspace which produces a divergence in the four-point function, since $h = 1$ corresponds to the physical operator of supercharge. This subspace should be treated separately by considering the theory outside the conformal limit. We hope to discuss this matter elsewhere.

3.5 Retarded kernel

The next question we address is the Lyapunov exponents of the modes. To find them we introduce the retarded kernel. We make time τ periodic with period $\beta = 2\pi$ and then continue to the complex plane. We take the left rail of the ladder diagram to be at complex time it and the right rail at $(it + \pi)$, so that there is a phase difference of half a period between them.

Generally, the propagator in complex time is:

$$\mathcal{G}_c(1|2) = \frac{b \text{sgn}(\tau_1 - \tau_2)}{|\langle 12 \rangle|^{2\Delta}} \quad \rightarrow \quad \mathcal{G}_c(1|2) = \frac{b (\text{sgn Re}(\tau_1 - \tau_2))^{2\Delta+1}}{\langle 12 \rangle^{2\Delta}}. \quad (3.5.1)$$

The kernel is constructed of the propagators of two types (see fig. 3.7). One is the conventional retarded propagator, which goes along a rail of the ladder:

$$\mathcal{G}_R(1|1') = \Theta(t_1 - t_{1'}) (\mathcal{G}(-\epsilon + it_1, it_{1'}) - \mathcal{G}(\epsilon + it_1, it_{1'})) = \Theta(t_1 - t_{1'}) \frac{2b \cos \pi \Delta}{\langle 11' \rangle_{ll}^{2\Delta}}. \quad (3.5.2)$$

Here $\langle 11' \rangle$ is the supersymmetric invariant distance between two points on the left rail of the ladder. The other goes between the two rails of the ladder:

$$\mathcal{G}_{lr}(1|2) = \frac{b}{\langle 12 \rangle_{lr}^{2\Delta}}, \quad (3.5.3)$$

where $\langle 12 \rangle_{lr}$ is the invariant distance between two points on the left and on the right rail.

To make time periodic, we do a conformal transformation which takes $t \rightarrow \exp(-t)$. Keeping in mind that the odd variables θ have conformal weight $1/2$, we write the new transformed supercoordinates as follows:

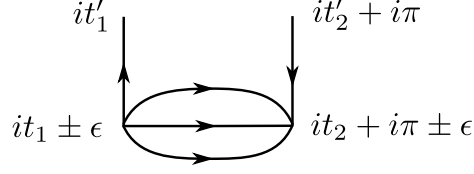


Figure 3.7: The retarded kernel. Retarded propagators go along rails, and the left-right propagator goes between rails.

$$\begin{aligned}
\tau_1 &= e^{-t_1}, & \tau_2 &= e^{-t_2 - i\pi} = -e^{-t_2}, \\
\theta_1 &= e^{-\frac{t_1}{2}} \vartheta_1, \quad (\text{left rail}) & \theta_2 &= e^{-\frac{t_2 + i\pi}{2}} = -ie^{-\frac{t_2}{2}} \vartheta_2, \quad (\text{right rail}) \\
\bar{\theta}_1 &= e^{-\frac{t_1}{2}} \bar{\vartheta}_1, & \bar{\theta}_2 &= e^{-\frac{t_2 - i\pi}{2}} = ie^{-\frac{t_2}{2}} \bar{\vartheta}_2.
\end{aligned} \tag{3.5.4}$$

In these new coordinates, the invariant distances are as follows:

$$\langle 11' \rangle_{ll} = e^{-\frac{t_1 + t'_1}{2}} \left(2 \sinh \frac{t_1 - t'_1}{2} - 2 \bar{\vartheta}_1 \vartheta'_1 - \vartheta_1 \bar{\vartheta}_1 - \vartheta'_1 \bar{\vartheta}'_1 \right), \tag{3.5.5}$$

for the left-left invariant, and:

$$\langle 12 \rangle_{lr} = \tau_1 - \tau_2 - 2 \bar{\theta}_1 \theta_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2 = e^{-\frac{t_1 + t_2}{2}} \left(2 \cosh \frac{t_1 - t_2}{2} + 2i \bar{\vartheta}_1 \vartheta_2 - e^{-\frac{t_1 - t_2}{2}} \vartheta_1 \bar{\vartheta}_1 - e^{\frac{t_1 - t_2}{2}} \vartheta_2 \bar{\vartheta}_2 \right), \tag{3.5.6}$$

for the left-right invariant. The reparameterization invariance of the propagator:

$$\mathcal{G}(t_1, t_2) = \mathcal{G}(\tau_1, \tau_2) \left(\frac{d\tau_1}{dt_1} \frac{d\tau_2}{dt_2} \right)^\Delta, \tag{3.5.7}$$

allows us to write the retarded and the left-right propagators in the following form:

$$\mathcal{G}_R(1|1') = \Theta(t_1 - t_{1'}) \frac{2b \cos \pi \Delta}{\left(2 \sinh \frac{t_1 - t_{1'}}{2} - 2 \bar{\vartheta}_1 \vartheta_{1'} - \vartheta_1 \bar{\vartheta}_{1'} - \vartheta_{1'} \bar{\vartheta}_1 \right)^{2\Delta}}, \tag{3.5.8}$$

$$\mathcal{G}_{lr}(1|2) = \frac{b}{\left(2 \cosh \frac{t_1 - t_2}{2} + 2i \bar{\vartheta}_1 \vartheta_2 - e^{-\frac{t_1 - t_2}{2}} \vartheta_1 \bar{\vartheta}_1 - e^{\frac{t_1 - t_2}{2}} \vartheta_2 \bar{\vartheta}_2 \right)^{2\Delta}}. \tag{3.5.9}$$

The retarded kernel is constructed out of retarded and left-right propagators:

$$K_r(1', 2'|1, 2) = (\hat{q} - 1) J \mathcal{G}_R(1|1') \mathcal{G}_R(2'|2) \mathcal{G}_{lr}^{\hat{q}-2}(1|2) i e^{\frac{1}{2}(t_1 + t_2)} dt_1 dt_2 d\bar{\vartheta}_1 d\vartheta_2. \tag{3.5.10}$$

The factor of $ie^{\frac{1}{2}(t_1+t_2)}$ comes from the transformation of the measure. Using the propagators (3.5.8, 3.5.9), we can write the kernel as follows:

$$K_r(1', 2'|1, 2) = 4 \cos^2 \pi \Delta (\hat{q} - 1) J b^{\hat{q}} i e^{\Delta(t_1+t_2)} e^{-\Delta(t_{1'}+t_{2'})} \frac{\Theta(t_1 - t_{1'}) \Theta(t_2 - t_{2'})}{\langle 11' \rangle^{2\Delta} \langle 2'2 \rangle^{2\Delta} \langle 12 \rangle^{1-4\Delta}}. \quad (3.5.11)$$

Now we diagonalize the retarded kernel, essentially in the same way we did the conformal kernel in Section 3.4.3. The eigenfunctions of the retarded kernel are the same three-point functions of the model (3.4.19, 3.4.20). In complex time, there is no difference between symmetric and antisymmetric eigenfunctions. Taking the third coordinate of the three-point function to infinity, we write the kernel eigenfunction as:

$$f_r^A(1, 2, \infty) = f_r^S(1, 2, \infty) = e^{-\Delta(t_1+t_2)} \frac{1}{\langle 12 \rangle^{2\Delta-h}}. \quad (3.5.12)$$

Integrating over the odd variables in the expression:

$$\int K_r(1', 2'|1, 2) f_r(1, 2, \infty) = k_r f_r(1, 2, \infty), \quad (3.5.13)$$

and fixing $\tau'_1 = 0$, $\tau'_2 = 1$, we find that the eigenvalue is given by the integral of the same kind as for the conformal kernel:

$$k_r = (\hat{q} - 1) J b^{\hat{q}} 2(1 - 2\Delta - h) (2 \cos \pi \Delta)^2 \int d\tau_1 d\tau_2 \frac{\theta(-\tau_1) \theta(\tau_2 - 1)}{|\tau_{12}|^{2-2\Delta-h} |\tau_1|^{2\Delta} |\tau_2|^{2\Delta}}. \quad (3.5.14)$$

Taking the integral, we find:

$$k_r = \frac{\Gamma(-2\Delta)}{\Gamma(2\Delta - 1)} \frac{\Gamma(-h + 2\Delta)}{\Gamma(1 - h - 2\Delta)}. \quad (3.5.15)$$

This eigenvalue is plotted in fig. 3.8. The modes potentially contributing to chaos satisfy $k_r = 1$. The minimal weight h that satisfies this constraint is $h = -1$:

$$k_r|_{h=-1} = 1 \quad \text{for all } \Delta. \quad (3.5.16)$$

At large times, the three-point function $f_r(1, 2, \infty)$ grows (or decays) exponentially:

$$f_r(1, 2, \infty) \sim e^{-ht}, \quad (3.5.17)$$

therefore the $h = -1$ mode shows maximally chaotic behavior. All the other modes have positive h and do not contribute to the exponential growth.

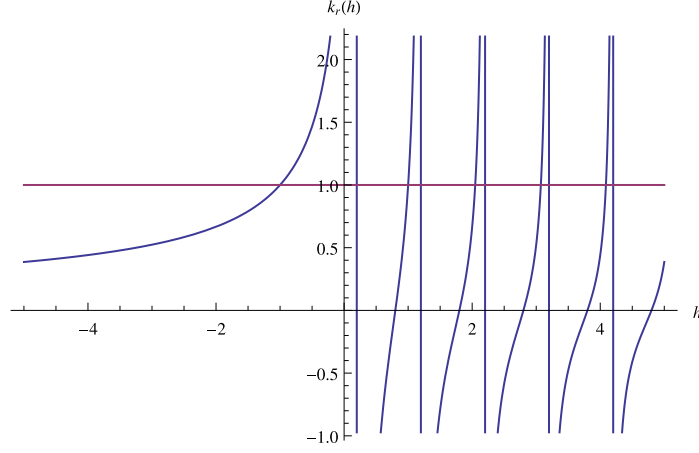


Figure 3.8: Eigenvalues of the retarded kernel at $\hat{q} = 5$.

3.6 Generalization to two dimensions

We can readily generalize our results to two-dimensional spacetime. We work in the $\mathcal{N} = 2$ super-space, parameterized by a set of holomorphic and anti-holomorphic coordinates:

$$(z, \theta, \tilde{\theta}), \quad (\bar{z}, \bar{\theta}, \bar{\tilde{\theta}}). \quad (3.6.1)$$

The two-dimensional superconformal group is a product of two one-dimensional superconformal groups for the left- and right-moving modes. In particular, the $\mathcal{N} = 2$ superconformal symmetry is realized by the $su(1, 1|1) \oplus su(1, 1|1)$ superalgebra. As in one dimension, here we can use the superconformal symmetry to make the correlators depend only on bosonic coordinates.

The superalgebra has two commuting Casimir operators which are complex conjugates of each other. We can write them in terms of bosonic cross-ratios as differential operators:

$$\mathcal{C} = \chi^2 (1 - \chi) \partial_\chi^2 + \chi (1 - \chi) \partial_\chi, \quad \bar{\mathcal{C}} = \bar{\chi}^2 (1 - \bar{\chi}) \partial_{\bar{\chi}}^2 + \bar{\chi} (1 - \bar{\chi}) \partial_{\bar{\chi}}, \quad (3.6.2)$$

where $\chi, \bar{\chi}$ are holomorphic and anti-holomorphic cross-ratios:

$$\chi \equiv \frac{z_{12} z_{34}}{z_{14} z_{32}} = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 14 \rangle \langle 32 \rangle}, \quad \bar{\chi} \equiv \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{14} \bar{z}_{32}} = \frac{\langle \bar{1} \bar{2} \rangle \langle \bar{3} \bar{4} \rangle}{\langle \bar{1} \bar{4} \rangle \langle \bar{3} \bar{2} \rangle}. \quad (3.6.3)$$

Angle brackets $\langle ij \rangle, \langle \bar{i} \bar{j} \rangle$ denote the supersymmetric invariants, completely analogous to the ones we have seen in one dimension:

$$\langle 12 \rangle = z_1 - z_2 - 2\tilde{\theta}_1 \theta_2 - \theta_1 \tilde{\theta}_1 - \theta_2 \tilde{\theta}_2, \quad \langle \bar{1} \bar{2} \rangle = \bar{z}_1 - \bar{z}_2 - 2\bar{\tilde{\theta}}_1 \bar{\theta}_2 - \bar{\theta}_1 \bar{\tilde{\theta}}_1 - \bar{\theta}_2 \bar{\tilde{\theta}}_2. \quad (3.6.4)$$

Knowing the eigenfunctions of the one-dimensional Casimir (3.4.10), we can easily guess the eigenfunctions and eigenvalues in two dimensions:

$$\mathcal{C}(\varphi_h(\chi) \varphi_{\bar{h}}(\bar{\chi})) = h^2 \varphi_h(\chi) \varphi_{\bar{h}}(\bar{\chi}), \quad \bar{\mathcal{C}}(\varphi_h(\chi) \varphi_{\bar{h}}(\bar{\chi})) = \tilde{h}^2 \varphi_h(\chi) \varphi_{\bar{h}}(\bar{\chi}). \quad (3.6.5)$$

The φ_h eigenfunction was defined in (3.4.11). In what follows, we find a more convenient basis of the Casimir eigenfunctions using the shadow formalism.

On physical states, the Casimirs should be Hermitean conjugates, which gives us a condition:

$$\overline{(h^2)} = \tilde{h}^2 \quad \Rightarrow \quad \bar{h} = \tilde{h} \quad \text{or} \quad \bar{h} = -\tilde{h}. \quad (3.6.6)$$

Another restriction comes from the fact that the spin of a bosonic physical state has to be real and in particular integer:

$$l = h - \tilde{h} \in \mathbb{Z}, \quad (3.6.7)$$

which implies that either spin is zero and both the dimensions $h = \tilde{h}$ are purely real, or the dimensions have the following form:

$$h = \frac{l}{2} + is, \quad \tilde{h} = -\frac{l}{2} + is, \quad s \in \mathbb{R}. \quad (3.6.8)$$

To make the discussion more concrete, let's consider the $\mathcal{N} = 2$ SYK model in two dimensions with complex scalar superfield and random superpotential. Our goal is to find the conformal four-point function of the model:

$$\mathcal{W}(\chi, \bar{\chi}) = \frac{\langle \tilde{\Phi}(1, \bar{1}) \Phi(2, \bar{2}) \tilde{\Phi}(3, \bar{3}) \Phi(4, \bar{4}) \rangle}{\langle \tilde{\Phi}(1, \bar{1}) \Phi(2, \bar{2}) \rangle \langle \tilde{\Phi}(3, \bar{3}) \Phi(4, \bar{4}) \rangle}. \quad (3.6.9)$$

Here $\Phi, \tilde{\Phi}$ are chiral superfields with zero spin. In a two-dimensional spacetime, a fermionic field has scaling dimension $\frac{1}{2}$, so a q -fermion interaction is generally irrelevant. To make a q -particle interaction marginal, we consider scalar fields which have zero scaling dimension in the UV. The chiral superfields are annihilated by superderivatives,

$$D\tilde{\Phi} = \bar{D}\Phi = 0, \quad (3.6.10)$$

defined as:

$$D = \frac{\partial}{\partial \theta} + \tilde{\theta} \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\tilde{\theta}} \frac{\partial}{\partial \bar{z}}. \quad (3.6.11)$$

The Lagrangian of the model consists of a kinetic D -term and a superpotential F -term with random coupling:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi \tilde{\Phi} + i \int d^2\theta C_{i_1 i_2 \dots i_{\hat{q}}} \Phi_{i_1} \dots \Phi_{i_{\hat{q}}} + i \int d^2\bar{\theta} \bar{C}_{i_1 i_2 \dots i_{\hat{q}}} \tilde{\Phi}_{i_1} \dots \tilde{\Phi}_{i_{\hat{q}}}, \quad d^2\theta \equiv d\theta d\bar{\theta}. \quad (3.6.12)$$

Here $\hat{q}$ can be any integer, and C is a Gaussian coupling:

$$\langle C_{i_1 \dots i_{\hat{q}}} \bar{C}_{i_1 \dots i_{\hat{q}}} \rangle = (\hat{q} - 1)! \frac{J}{N^{\hat{q}-1}}. \quad (3.6.13)$$

We assume that the F -term is not renormalized, perturbatively or non-perturbatively [83]. As an $\mathcal{N} = 2$ superconformal theory with a holomorphic superpotential, we expect this model to flow to a conformal fixed point in the infrared. The D -term gets renormalized and becomes irrelevant, so the infrared behavior of the model is determined exclusively by the superpotential.

Next we follow the same steps as for the one-dimensional model, finding first the two-point function, then the basis of the four-point functions in the shadow representation and finally eigenvalues of the kernel.

3.6.1 Two-point function in two dimensions

First we look for the chiral–anti-chiral two-point function:

$$\mathcal{G}(1|2) \equiv \langle \tilde{\Phi}(1, \bar{1}) \Phi(2, \bar{2}) \rangle. \quad (3.6.14)$$

The Lagrangian (3.6.12) implies the supersymmetric Schwinger–Dyson equation:

$$D_1 \bar{D}_1 \mathcal{G}(1|3) + J \int d^2 z_2 d^2 \theta_2 \mathcal{G}(1|2) \mathcal{G}^{\hat{q}-1}(3|2) = \left(\tilde{\theta}_1 - \tilde{\theta}_3 \right) \left(\bar{\tilde{\theta}}_1 - \bar{\tilde{\theta}}_3 \right) \delta(\langle 13 \rangle) \delta(\langle \bar{1} \bar{3} \rangle). \quad (3.6.15)$$

The $D^2 \mathcal{G}$ term in the Schwinger–Dyson equation (3.6.15) comes from differentiating the D -term. In the usual non-supersymmetric SYK model, the conformal limit is identified with the large coupling limit, so in the conformal point we can neglect such a term. When considering corrections to the conformal limit however, we have to restore it, and it gives a correction to the two-point function of order $(\beta J)^{-1}$.

In our case, the infrared behavior of the model should be completely determined by the superpotential, therefore the D -term should not affect the Schwinger–Dyson equation. Hence we expect

the integral equation (3.6.15) to be true without the first term in the exact conformal limit.

It is easy to see that the Schwinger–Dyson equation without the first term is satisfied by a conformal propagator of the form:

$$\mathcal{G}(1|2) = \frac{b}{\langle 12 \rangle^\Delta \langle \bar{1}\bar{2} \rangle^\Delta}. \quad (3.6.16)$$

Here Δ is the scaling dimension of the superconformal primary Φ . Dimensional considerations allow us to fix it:

$$\hat{q}\Delta = 1. \quad (3.6.17)$$

The integral in (3.6.15) can be taken in the momentum space. We use the ansatz (3.6.16), integrate over the odd variables, and doing the Fourier transformation of the propagators with the help of an integral:

$$\int \frac{d^2 z}{|z|^{2\Delta}} e^{ip \cdot z} = |p|^{2\Delta-2} \cdot \frac{\pi}{2^{2\Delta-2}} \frac{\Gamma(1-\Delta)}{\Gamma(\Delta)}. \quad (3.6.18)$$

Then the ansatz for the propagator works if we fix the b constant to:

$$b^{\hat{q}} J = \frac{1}{4\pi^2}. \quad (3.6.19)$$

3.6.2 Eigenfunctions of the Casimir operators

Next we proceed to find the basis for the four-point function. Just as in one-dimension, the eigenfunctions of the kernel can be found in the shadow representation. These eigenfunctions are labeled by the eigenvalues of the Casimirs $(h, \tilde{h})$. We formally add an interaction term for fictitious superoperators $\mathcal{V}_{h, \tilde{h}}$:

$$\varepsilon \int d^2 z_0 d^2 \theta_0 d^2 \tilde{\theta}_0 \mathcal{V}_{h, \tilde{h}}(0, \bar{0}) \mathcal{V}'_{-h, -\tilde{h}}(0, \bar{0}). \quad (3.6.20)$$

Note that here we integrate over the full superspace, i.e. this is a D -term. The Casimir eigenfunction is given by an integral:

$$\mathcal{F}_{h, \tilde{h}} \sim \int d^2 z_0 d^2 \theta_0 d^2 \tilde{\theta}_0 \frac{\langle \tilde{\Phi}(1, \bar{1}) \Phi(2, \bar{2}) \mathcal{V}_{h, \tilde{h}}(0, \bar{0}) \rangle \langle \tilde{\Phi}(3, \bar{3}) \Phi(4, \bar{4}) \mathcal{V}'_{-h, -\tilde{h}}(0, \bar{0}) \rangle}{\mathcal{G}(1|2) \mathcal{G}(3|4)}. \quad (3.6.21)$$

The interaction term (3.6.20) makes it clear that eigenfunctions should remain invariant if we

reverse the signs of both holomorphic and anti-holomorphic dimensions $(h, \tilde{h}) \leftrightarrow (-h, -\tilde{h})$:

$$\mathcal{F}_{h, \tilde{h}} = \mathcal{F}_{-h, -\tilde{h}}. \quad (3.6.22)$$

Unlike in one dimension, here we can fix the three-point function uniquely, as a product of a holomorphic and an anti-holomorphic parts:

$$\langle \tilde{\Phi}(1, \bar{1}) \Phi(2, \bar{2}) \mathcal{V}_{h, \tilde{h}}(0, \bar{0}) \rangle = \frac{1}{\langle 12 \rangle^{\Delta-h} \langle 02 \rangle^h \langle 10 \rangle^h} \frac{1}{\langle \bar{1}\bar{2} \rangle^{\Delta-\tilde{h}} \langle \bar{0}\bar{2} \rangle^{\tilde{h}} \langle \bar{1}\bar{0} \rangle^{\tilde{h}}}. \quad (3.6.23)$$

These three-point functions diagonalize both Casimirs $\mathcal{C}, \bar{\mathcal{C}}$, with eigenvalues $h^2, \tilde{h}^2$ correspondingly. Dividing by propagators and integrating over the odd coordinates, we find the conformal block for the four-point function in the integral form, similar to (3.4.24):

$$\Xi_{h, \tilde{h}} = (-1)^{h+\tilde{h}} \int dy d\bar{y} \frac{h \chi^h (1-y)^h}{y^h (\chi-y)^h} \left(\frac{1}{y} + \frac{1}{\chi-y} - \frac{1}{1-y} \right) \frac{\tilde{h} \bar{\chi}^{\tilde{h}} (1-\bar{y})^{\tilde{h}}}{\bar{y}^{\tilde{h}} (\bar{\chi}-\bar{y})^{\tilde{h}}} \left(\frac{1}{\bar{y}} + \frac{1}{\bar{\chi}-\bar{y}} - \frac{1}{1-\bar{y}} \right). \quad (3.6.24)$$

Here we have added a $(-1)^{h+\tilde{h}}$ factor to make our later expressions somewhat simpler. Just as in one dimension, here we see that the $\mathcal{N} = 2$ four-point function does not depend on odd variables, unlike the $\mathcal{N} = 1$ four-point function discussed in [71].

The integral (3.6.24) is tricky, but luckily we can use the results of [71] for a two-dimensional bosonic SYK model. The eigenbasis of the non-supersymmetric conformal Casimirs consists of the $\Psi_{h, \tilde{h}}$ functions, where:

$$\Psi_{h, \tilde{h}}(\chi, \bar{\chi}) \equiv \int dy d\bar{y} \frac{\chi^h (1-y)^{h-1}}{y^h (y-\chi)^h} \frac{\bar{\chi}^{\tilde{h}} (1-\bar{y})^{\tilde{h}-1}}{\bar{y}^{\tilde{h}} (\bar{y}-\bar{\chi})^{\tilde{h}}}. \quad (3.6.25)$$

Explicitly, $\Psi_{h, \tilde{h}}$ is a combination of the eigenfunctions $F_h(\chi)$ (3.9.2) of the non-supersymmetric one-dimensional conformal Casimir:

$$\Psi_{h, \tilde{h}}(\chi, \bar{\chi}) = \frac{1}{2} \frac{\sin \pi h}{\cos \pi \tilde{h}} (F_h(\chi) F_{\tilde{h}}(\bar{\chi}) - F_{1-h}(\chi) F_{1-\tilde{h}}(\bar{\chi})). \quad (3.6.26)$$

Comparing the integral (3.6.24) with the definition of $\Psi_{h, \tilde{h}}$ (3.6.25), we see that the $\mathcal{N} = 2$ eigenfunction is a linear combination of $\mathcal{N} = 0$ eigenfunctions:

$$\Xi_{h, \tilde{h}} = h\tilde{h} \left(\Psi_{h+1, \tilde{h}+1} + \Psi_{h, \tilde{h}} + \Psi_{h+1, \tilde{h}} + \Psi_{h, \tilde{h}+1} \right). \quad (3.6.27)$$

The $\Xi_{h,\tilde{h}}$ eigenfunction is also a linear combination of the Casimir eigenfunctions (3.6.5):

$$\Xi_{h,\tilde{h}}(\chi, \bar{\chi}) = h\tilde{h} \frac{1}{2} \frac{\sin \pi h}{\cos \pi \tilde{h}} (\varphi_h(\chi) \varphi_{\tilde{h}}(\bar{\chi}) - \varphi_{-h}(\chi) \varphi_{-\tilde{h}}(\bar{\chi})). \quad (3.6.28)$$

The eigenvalues of the Casimirs are:

$$\mathcal{C}\Xi_{h,\tilde{h}} = h^2 \Xi_{h,\tilde{h}}, \quad \bar{\mathcal{C}}\Xi_{h,\tilde{h}} = \tilde{h}^2 \Xi_{h,\tilde{h}}. \quad (3.6.29)$$

From this, it is clear that the spectrum of the Casimirs is symmetric under sign reversal:

$$\Xi_{-h,-\tilde{h}} = \Xi_{h,\tilde{h}}. \quad (3.6.30)$$

3.6.3 Two-dimensional kernel

The next step is to diagonalize the two-dimensional SYK kernel. The $\mathcal{N} = 2$ kernel is given by the same diagram (3.2) as before, and it reads as follows:

$$K(1', 2'|1, 2) = (\hat{q} - 1) b^{\hat{q}} J \frac{1}{|\langle 11' \rangle|^{2\Delta} |\langle 2'2 \rangle|^{2\Delta} |\langle 12 \rangle|^{2-4\Delta}} d^2 \tilde{\theta}_1 d^2 \theta_2 d^2 z_1 d^2 z_2. \quad (3.6.31)$$

Note that here, as well as in the one-dimensional case, we integrate only over half of the odd variables.

The kernel acts on the three-point function (3.6.23). To simplify the calculations, we can take the coordinate of the $\mathcal{V}_{h,\tilde{h}}$ field to infinity, so that the three-point function becomes:

$$f(1, 2, \infty; \bar{1}, \bar{2}, \infty) = \frac{1}{\langle 12 \rangle^{\Delta-h} \langle \bar{1}\bar{2} \rangle^{\Delta-\tilde{h}}}. \quad (3.6.32)$$

We can also conveniently fix the coordinates of the 1 and 2 points to be:

$$1 \rightarrow (0, \tilde{\vartheta}_1, \bar{\vartheta}_1), \quad 2 \rightarrow (1, \vartheta_2, \bar{\vartheta}_2), \quad (3.6.33)$$

(the rest of the odd coordinates being zero) so that the corresponding invariants simplify:

$$\langle 11' \rangle = z_1 - \theta_1 \tilde{\theta}_1, \quad \langle 2'2 \rangle = 1 - z_2 - \theta_2 \tilde{\theta}_2, \quad \langle 2'1' \rangle = 1. \quad (3.6.34)$$

Then the eigenvalue of the kernel is:

$$k(h, \tilde{h}) = \int K(1', 2' | 1, 2) f(1, 2, \infty; \bar{1}, \bar{2}, \infty) = \frac{1 - \Delta}{4\pi^2 \Delta} \int \frac{\langle 12 \rangle^h \langle \bar{1}\bar{2} \rangle^{\tilde{h}}}{|\langle 11' \rangle|^{2\Delta} |\langle 2'2 \rangle|^{2\Delta} |\langle 12 \rangle|^{2-2\Delta}} d^2 \tilde{\theta}_1 d^2 \theta_2 d^2 z_1 d^2 z_2. \quad (3.6.35)$$

In the integral over the odd variables, a non-zero contribution comes from the term containing $\tilde{\theta}_1 \bar{\tilde{\theta}}_1 \theta_2 \bar{\theta}_2$. It comes from the expansion of $\langle 12 \rangle^{h+\Delta-1}$ and $\langle \bar{1}\bar{2} \rangle^{\tilde{h}+\Delta-1}$. Then after the integration, the eigenvalue becomes:

$$k(h, \tilde{h}) = -\frac{(1-\Delta)}{\pi^2 \Delta} (-1+h+\Delta) (-1+\tilde{h}+\Delta) \int d^2 z_1 d^2 z_2 \frac{(z_1 - z_2)^h (\bar{z}_1 - \bar{z}_2)^{\tilde{h}}}{|z_1|^{2\Delta} |z_2 - 1|^{2\Delta} |z_1 - z_2|^{4-2\Delta}}. \quad (3.6.36)$$

This expression can be evaluated explicitly with the help of the KLT integral (the calculation is completely analogous to what we did in Appendix 2.9 for the one-dimensional case):

$$\int d^2 x x^a \bar{x}^{\tilde{a}} (1-x)^b (1-\bar{x})^{\tilde{b}} = \frac{\pi}{-1-a-b} \frac{B(1+\tilde{a}, 1+\tilde{b})}{B(-a, -b)}, \quad (3.6.37)$$

the final answer being:

$$k(h, \tilde{h}) = \Delta(1-\Delta) \frac{\Gamma^2(-\Delta)}{\Gamma^2(\Delta)} \frac{\Gamma(-h+\Delta) \Gamma(\tilde{h}+\Delta)}{\Gamma(1-h-\Delta) \Gamma(1+\tilde{h}-\Delta)}. \quad (3.6.38)$$

This is the same as k^{BB} in the $\mathcal{N} = 1$ case [71], up to a sign:

$$k(h, \tilde{h}) = -k^{BB}(h, \tilde{h}). \quad (3.6.39)$$

This eigenvalue has to be symmetric under $h \leftrightarrow \tilde{h}$, and it is if we restrict to physical states with either both dimensions real, or dimensions of the form (3.6.8). Also, for physical states the eigenvalue of the kernel is real. So the condition on the operator spectrum $k(h, \tilde{h}) = 1$ is a single real condition, therefore it is satisfied by a finite number of states for each spin.

As a check to our formula, we notice that there is a solution for $(h, \tilde{h}) = (1, 0)$, which corresponds to the $\mathcal{N} = 2$ multiplet of the holomorphic superconformal current:

$$\mathcal{J} = R + \theta S + \tilde{\theta} \tilde{S} + \theta \tilde{\theta} T, \quad (3.6.40)$$

which contains R -charge, supercurrent and stress tensor. But unlike in one dimension, here the mode

corresponding to the supercurrent is not in the Hilbert space (because neither of the conditions (3.6.6) holds for the supercurrent), so it does not give rise to a divergence in the four-point function.

3.6.4 Normalizable states and the full four-point function

As in the one-dimensional case, the next step towards finding the four-point function is to compute the norm of a state. The inner product has to be invariant under the superconformal group, and the two-dimensional Casimir operators have to be Hermitean with respect to it. Following the same logic as in Section 3.4.4, we define the inner product as:

$$\langle f(\chi, \bar{\chi}), g(\chi, \bar{\chi}) \rangle = \int \frac{d^2\chi}{|\chi|^2|1-\chi|^2} \bar{f}(\chi, \bar{\chi}) g(\chi, \bar{\chi}). \quad (3.6.41)$$

Unlike the one-dimensional inner product (3.4.57), this one is real and the whole inner product is Hermitian. Therefore we expect the eigenfunctions of the Casimir to form a usual Hilbert space, and be a complete set of functions (subject to a boundary condition analogous to (3.4.58)).

We expect the norm of an eigenfunction $\Xi_{h,\tilde{h}}$ to be proportional to δ -function of a combination of $(h, \tilde{h})$. This singular contribution comes from the vicinity of zero. Near $\chi \sim 0$, the eigenfunction behaves as a power of χ :

$$\Xi_{h,\tilde{h}}(\chi) \sim h\tilde{h} \frac{\sin \pi h}{2 \cos \pi \tilde{h}} \left(B(h, h) B(\tilde{h}, \tilde{h}) \chi^h \bar{\chi}^{\tilde{h}} - B(-h, -h) B(-\tilde{h}, -\tilde{h}) \chi^{-h} \bar{\chi}^{-\tilde{h}} \right), \quad \chi \sim 0. \quad (3.6.42)$$

It is convenient to make a change of variables:

$$\chi = e^{\rho+i\varphi}, \quad \bar{\chi} = e^{\rho-i\varphi}. \quad (3.6.43)$$

In these variables and near zero, the integration measure in (3.6.41) becomes:

$$\frac{d^2\chi}{|\chi|^2|1-\chi|^2} \rightarrow d\rho d\varphi, \quad \rho \rightarrow -\infty, \quad (3.6.44)$$

and the eigenfunction is:

$$\Xi_{h,\tilde{h}}(\chi) \sim h\tilde{h} \frac{\sin \pi h}{2 \cos \pi \tilde{h}} \left(B(h, h) B(\tilde{h}, \tilde{h}) e^{\rho(h+\tilde{h})+i\varphi(h-\tilde{h})} - B(-h, -h) B(-\tilde{h}, -\tilde{h}) e^{-\rho(h+\tilde{h})-i\varphi(h-\tilde{h})} \right). \quad (3.6.45)$$

To make this function single-valued, we have to restrict the difference between eigenvalues to be

integer:

$$l \equiv h - \tilde{h} \in \mathbb{Z}. \quad (3.6.46)$$

This is natural since the operator $\mathcal{V}_{h,\tilde{h}}$ in the shadow representation has a bosonic lower component, and l is its spin. In particular, this means that we take the $\mathcal{N} = 0$ eigenfunctions $\Psi_{h,\tilde{h}}$ which can be either even or odd under $\chi \rightarrow \frac{\chi}{\chi-1}$:

$$\Psi_{h,\tilde{h}} \left(\frac{\chi}{\chi-1}, \frac{\bar{\chi}}{\bar{\chi}-1} \right) = (-1)^{h-\tilde{h}} \Psi_{h,\tilde{h}}(\chi, \bar{\chi}). \quad (3.6.47)$$

This is in contrast with the non-supersymmetric case, where $\chi \rightarrow \frac{\chi}{\chi-1}$ is a symmetry of the model and therefore the eigenfunction is even under this transformation. In our case, spin can be odd as well as even. As in the one-dimensional case, the full $\mathcal{N} = 2$ eigenfunction $\Xi_{h,\tilde{h}}$ is neither even nor odd under the $\chi \rightarrow \frac{\chi}{\chi-1}$ transformation, as is clear from (3.6.27).

We have seen in (3.6.6) that the dimensions of the states in the Hilbert space have to either both be real,

$$h = \tilde{h} \in \mathbb{R}, \quad (3.6.48)$$

or be of the form:

$$h = \frac{l}{2} + is, \quad \tilde{h} = -\frac{l}{2} + is, \quad s \in \mathbb{R}. \quad (3.6.49)$$

In the former case, the eigenfunction (3.6.45) always diverges near zero, and the state is not normalizable. In the latter, the product of two states is proportional to a delta function as desired. If we further denote:

$$\mathcal{A}(l, s) \equiv h\tilde{h} \frac{\sin \pi h}{2 \cos \pi \tilde{h}} B(h, h) B(\tilde{h}, \tilde{h}), \quad (3.6.50)$$

then the product of two states is:

$$\begin{aligned} \langle \Xi_{s',l'}, \Xi_{s,l} \rangle &\sim \int_0^{2\pi} d\varphi \int_{-\infty}^0 d\rho \left(\mathcal{A}(l', -s') e^{-i\rho s' - i\varphi l'} + \mathcal{A}(-l', s') e^{i\rho s' + i\varphi l'} \right) \\ &\quad \left(\mathcal{A}(l, s) e^{i\rho s + i\varphi l} + \mathcal{A}(-l, -s) e^{-i\rho s - i\varphi l} \right), \end{aligned} \quad (3.6.51)$$

which gives after integration:

$$\begin{aligned} \langle \Xi_{s',l'}, \Xi_{s,l} \rangle &\sim 2\pi^2 \delta_{l',l} \delta(s - s') (\mathcal{A}(l, -s) \mathcal{A}(l, s) + \mathcal{A}(-l, s) \mathcal{A}(-l, -s)) \\ &\quad + 2\pi^2 \delta_{l',-l} \delta(s + s') (\mathcal{A}(l, -s) \mathcal{A}(l, s) + \mathcal{A}(-l, s) \mathcal{A}(-l, -s)). \end{aligned} \quad (3.6.52)$$

The second line in (3.6.52) reflects the symmetry of the model under $(h, \tilde{h}) \leftrightarrow (-h, -\tilde{h})$. Using once again the Beta function identity (3.4.71) and the fact that $\bar{h} = -\tilde{h}$, we finally arrive at:

$$\langle \Xi_{s', l'}, \Xi_{s, l} \rangle = 4\pi^4 (l^2 + s^2) (\delta_{l, l'} \delta(s - s') + \delta_{l, -l'} \delta(s + s')). \quad (3.6.53)$$

The norm is real and positive-definite for real s and integer l , as expected of a norm in a Hilbert space.

This inner product gives rise to a completeness relation:

$$\sum_{l=-\infty}^{\infty} \int_0^{\infty} \frac{ds}{2\pi} \frac{1}{2\pi^3 (l^2 + s^2)} \Xi_{h, \tilde{h}}(\chi, \bar{\chi}) \Xi_{h, \tilde{h}}(\chi', \bar{\chi}') = |\chi|^2 |1 - \chi|^2 \delta^2(\chi - \chi'). \quad (3.6.54)$$

There is a double pole in this expression, since the norm of a state with $l = s = 0$ vanishes. We avoid this pole by infinitesimally deforming the integration contour to avoid the origin, as in fig. 3.5.

3.6.5 Four-point function in two dimensions

As the $\Xi_{h, \tilde{h}}$ eigenfunctions form a basis, we can find the full four-point function as an expansion:

$$\mathcal{F} = \frac{1}{1 - K} \mathcal{F}_0 = \sum_{h, \tilde{h}} \frac{1}{1 - k(h, \tilde{h})} \frac{\langle \Xi_{h, \tilde{h}}, \mathcal{F}_0 \rangle}{\langle \Xi_{h, \tilde{h}}, \Xi_{h, \tilde{h}} \rangle} \Xi_{h, \tilde{h}}. \quad (3.6.55)$$

Here $\mathcal{F}_0$ is the zero-rung four-point function:

$$\mathcal{F}_0 = \chi^\Delta \bar{\chi}^\Delta. \quad (3.6.56)$$

To make use of the expansion (3.6.55), we have to find the inner product between a Casimir eigenfunction and the zero-rung four-point function $\langle \Xi_{h, \tilde{h}}, \mathcal{F}_0 \rangle$. We can simplify the integral by acting on the eigenfunction with the Casimirs:

$$\langle \mathcal{C}\bar{\mathcal{C}} \Xi_{h, \tilde{h}}, |\chi|^{2\Delta} \rangle = (h\tilde{h})^2 \langle \Xi_{h, \tilde{h}}, |\chi|^{2\Delta} \rangle = \langle \Xi_{h, \tilde{h}}, \mathcal{C}\bar{\mathcal{C}} |\chi|^{2\Delta} \rangle = \Delta^4 \int d^2\chi \Xi_{h, \tilde{h}}(\chi, \bar{\chi}) |\chi|^{2\Delta-2}. \quad (3.6.57)$$

This expression looks similar to the $\mathcal{N} = 0$ inner product:

$$(f, g) \equiv \int \frac{d^2\chi}{|\chi|^4} \bar{f} g. \quad (3.6.58)$$

Since the eigenfunction $\Xi_{h,\tilde{h}}$ is a linear combination of the $\mathcal{N} = 0$ eigenfunctions $\Psi_{h,\tilde{h}}$ (3.6.27), we can express the $\mathcal{N} = 2$ inner product via the non-supersymmetric one:

$$\langle \Xi_{h,\tilde{h}}, |\chi|^{2\Delta} \rangle = \frac{\Delta^4}{h\tilde{h}} \left(\left(\Psi_{h,\tilde{h}}, |\chi|^{2\Delta+2} \right) + \left(\Psi_{h+1,\tilde{h}+1}, |\chi|^{2\Delta+2} \right) + \left(\Psi_{h+1,\tilde{h}}, |\chi|^{2\Delta+2} \right) + \left(\Psi_{h,\tilde{h}+1}, |\chi|^{2\Delta+2} \right) \right). \quad (3.6.59)$$

Now we can apply the results of [71] about the $\mathcal{N} = 0$ inner product:

$$\left(\Psi_{h,\tilde{h}}, |\chi|^{2\Delta} \right) = \frac{\pi^2 \Delta}{(2-\Delta)(1-\Delta)^2} k_{\mathcal{N}=0}(h, \tilde{h}) = \frac{\pi^2 \Delta}{1-\Delta} \frac{k(h, \tilde{h})}{(-1+h+\Delta)(-1+\tilde{h}+\Delta)}, \quad (3.6.60)$$

where $k(h, \tilde{h})$ is the eigenvalue of the $\mathcal{N} = 2$ kernel (3.6.38). Explicitly, it is:

$$\left(\Psi_{h,\tilde{h}}, |\chi|^{2\Delta} \right) = -\pi^2 \frac{\Gamma^2(1-\Delta)}{\Gamma^2(\Delta)} \frac{\Gamma(-h+\Delta)\Gamma(\tilde{h}+\Delta-1)}{\Gamma(2-h-\Delta)\Gamma(\tilde{h}-\Delta+1)}. \quad (3.6.61)$$

Plugging this in the sum (3.6.59), we finally get:

$$\langle \Xi_{h,\tilde{h}}, \mathcal{F}_0 \rangle = \frac{4\pi^2 \Delta}{1-\Delta} k(h, \tilde{h}). \quad (3.6.62)$$

As in all versions of the SYK model we've been discussing so far, the inner product with the zero-rung four-point function is proportional to the eigenvalue of the kernel.

Using this answer in (3.6.55), together with the norm of an eigenfunction (3.6.53), we write the full four-point function as follows:

$$\mathcal{F}(\chi, \bar{\chi}) = -\frac{2}{\pi} \frac{\Delta}{1-\Delta} \sum_{l \in \mathbb{Z}} \int_0^\infty \frac{ds}{2\pi} \frac{1}{l^2 + s^2} \frac{k(h, \tilde{h})}{1 - k(h, \tilde{h})} \Xi_{h,\tilde{h}}(\chi, \bar{\chi}). \quad (3.6.63)$$

The symmetry of the eigenfunctions under $(h, \tilde{h}) \leftrightarrow (-h, -\tilde{h})$ allows us to put it in the form:

$$\mathcal{F}(\chi, \bar{\chi}) = \frac{1}{4\pi} \frac{\Delta}{1-\Delta} \sum_{l \in \mathbb{Z}} \int_{-\infty}^\infty \frac{ds}{2\pi} \frac{k(h, \tilde{h})}{1 - k(h, \tilde{h})} \frac{\sin \pi h}{\cos \pi \tilde{h}} \varphi_h(\chi) \varphi_{\tilde{h}}(\bar{\chi}). \quad (3.6.64)$$

From this, we can find the central charge of the model. On general grounds, the central charge

of an $\mathcal{N} = 2$ two-dimensional CFT of N superfields and with a superpotential of degree q is [83]:

$$c = \sum_{i=1}^N 6 \left(\frac{1}{2} - \frac{1}{q} \right) = 3N(1 - 2\Delta). \quad (3.6.65)$$

Now let us confirm this central charge from the four-point function (3.6.64). As was found in [71], the stress tensor contributes to the χ^2 term of the four-point function, so this term depends on the central charge:

$$\mathcal{F} = \dots + \frac{N\Delta^2}{2c} \chi^2 + O(\chi^2). \quad (3.6.66)$$

The stress tensor lives in the supercurrent multiplet, which is a $(1, 0)$ primary. At $(h, \tilde{h}) = (1, 0)$, or equivalently at $(l, s) = (1, i)$ the integrand in (3.6.63) has a pole. Taking $h = 1 + \tilde{h} = 1 + \varepsilon$ and expanding everything in ε , we find:

$$\varphi_\varepsilon(\bar{\chi}) = \frac{2}{\varepsilon} + O(\varepsilon), \quad (3.6.67)$$

$$\varphi_{1+\varepsilon}(\chi) = \chi + \frac{\chi^2}{3} + O(\varepsilon), \quad (3.6.68)$$

$$k(1 + \varepsilon, \varepsilon) = 1 + \frac{1 - 2\Delta}{\Delta(1 - \Delta)} \varepsilon + O(\varepsilon^2). \quad (3.6.69)$$

(The expressions for φ_h can be derived e.g. from (3.10.5).) Bringing everything together, we find the central charge:

$$c = 3N(1 - 2\Delta). \quad (3.6.70)$$

This is exactly twice the central charge of the $\mathcal{N} = 1$ model found in [71]:

$$c_{\mathcal{N}=2} = 2c_{\mathcal{N}=1}. \quad (3.6.71)$$

3.6.6 Retarded kernel in two dimensions

We can now generalize the analysis of Section 3.5 to the two-dimensional system, to find the chaos exponent and identify the modes contributing to it. To do that, we construct the kernel out of retarded and left-right propagators (see fig. 3.7). We proceed in the same fashion as before, doing an analytical continuation and putting one rail of the ladder diagram at $\tau_l = it$ and the other at $\tau_r = it + \pi$. We also transform the coordinates from $(z, \theta, \bar{\theta})$ to the periodic $(w, \vartheta, \bar{\vartheta})$, where:

$$w = x + i\tau = x - t, \quad \bar{w} = x - i\tau = x + t. \quad (3.6.72)$$

The coordinate transformation differs for the left and the right rails:

$$\begin{aligned}
z_1 &= e^{w_1} & z_2 &= e^{w_2+i\pi} = -e^{w_2}, \\
\theta_1 &= e^{\frac{w_1}{2}} \vartheta_1, \quad (\text{left rail}) & \theta_2 &= e^{\frac{w_2+i\pi}{2}} = ie^{\frac{w_2}{2}} \vartheta_2, \quad (\text{right rail}) \\
\tilde{\theta}_1 &= e^{\frac{w_1}{2}} \tilde{\vartheta}_1, & \tilde{\theta}_2 &= e^{\frac{w_2-i\pi}{2}} = -ie^{\frac{w_2}{2}} \tilde{\vartheta}_2.
\end{aligned} \tag{3.6.73}$$

To make the expressions more symmetrical, we take a different transformation for the anti-holomorphic coordinates:

$$\begin{aligned}
\bar{z}_1 &= e^{-\bar{w}_1} & \bar{z}_2 &= e^{-\bar{w}_2+i\pi} = -e^{-\bar{w}_2}, \\
\bar{\theta}_1 &= e^{-\frac{\bar{w}_1}{2}} \bar{\vartheta}_1, \quad (\text{left rail}) & \bar{\theta}_2 &= e^{-\frac{\bar{w}_2+i\pi}{2}} = ie^{-\frac{\bar{w}_2}{2}} \bar{\vartheta}_2, \quad (\text{right rail}) \\
\bar{\tilde{\theta}}_1 &= e^{-\frac{\bar{w}_1}{2}} \bar{\tilde{\vartheta}}_1, & \bar{\tilde{\theta}}_2 &= e^{-\frac{\bar{w}_2-i\pi}{2}} = -ie^{-\frac{\bar{w}_2}{2}} \bar{\tilde{\vartheta}}_2.
\end{aligned} \tag{3.6.74}$$

Then the supersymmetry-invariant distance between two points belonging to the same rail is:

$$\langle 11' \rangle_u = e^{\frac{w_1+w_{1'}}{2}} \left(2 \sinh \frac{w_1-w_{1'}}{2} - 2\tilde{\vartheta}_1 \vartheta_{1'} - \vartheta_1 \tilde{\vartheta}_{1'} - \vartheta_{1'} \tilde{\vartheta}_1 \right), \tag{3.6.75}$$

and the invariant distance between the rails is:

$$\langle 12 \rangle_{lr} = e^{\frac{w_1+w_2}{2}} \left(2 \cosh \frac{w_1-w_2}{2} - 2i\tilde{\vartheta}_1 \vartheta_2 - e^{\frac{w_1-w_2}{2}} \vartheta_1 \tilde{\vartheta}_1 - e^{\frac{w_2-w_1}{2}} \vartheta_2 \tilde{\vartheta}_2 \right). \tag{3.6.76}$$

For the anti-holomorphic invariants, the exponents in (3.6.75, 3.6.76) are negative:

$$\langle \bar{1}\bar{1}' \rangle_u = e^{-\frac{\bar{w}_1+\bar{w}_{1'}}{2}} \left(2 \sinh \frac{\bar{w}_1-\bar{w}_{1'}}{2} - 2\bar{\tilde{\vartheta}}_1 \bar{\vartheta}_{1'} - \bar{\vartheta}_1 \bar{\tilde{\vartheta}}_{1'} - \bar{\vartheta}_{1'} \bar{\tilde{\vartheta}}_1 \right), \tag{3.6.77}$$

$$\langle \bar{1}\bar{2} \rangle_{lr} = e^{-\frac{\bar{w}_1+\bar{w}_2}{2}} \left(2 \cosh \frac{\bar{w}_1-\bar{w}_2}{2} - 2i\bar{\tilde{\vartheta}}_1 \bar{\vartheta}_2 - e^{\frac{\bar{w}_1-\bar{w}_2}{2}} \bar{\vartheta}_1 \bar{\tilde{\vartheta}}_1 - e^{\frac{\bar{w}_2-\bar{w}_1}{2}} \bar{\vartheta}_2 \bar{\tilde{\vartheta}}_2 \right). \tag{3.6.78}$$

Knowing these supersymmetric invariants, we can construct retarded propagators. To do that, we once again add an infinitesimal imaginary part to t ,

$$t \rightarrow t \pm i\epsilon, \tag{3.6.79}$$

and compute the difference:

$$\mathcal{G}_R(1|1') = \Theta(t_1 - t_{1'}) (\mathcal{G}(w_1 + i\epsilon, \bar{w}_1 - i\epsilon|w_{1'}, \bar{w}_{1'}) - \mathcal{G}(w_1 - i\epsilon, \bar{w}_1 + i\epsilon|w_{1'}, \bar{w}_{1'})), \tag{3.6.80}$$

where we have omitted the Grassmann coordinates for brevity. Taking into account the Jacobian of the transformation, we find:

$$\mathcal{G}_R(1|1') = \Theta(t_{11'} - |x_{11'}|) \frac{-2ib \sin \pi \Delta}{\langle 11' \rangle_{ll}^\Delta \langle \bar{1}\bar{1}' \rangle_{ll}^\Delta} e^{\frac{\Delta}{2}(w_1 - \bar{w}_1)} e^{\frac{\Delta}{2}(w_{1'} - \bar{w}_{1'})}. \quad (3.6.81)$$

The left-right propagator is simply:

$$\mathcal{G}_{lr}(1|2) = \frac{b}{\langle 12 \rangle_{lr}^\Delta \langle \bar{1}\bar{2} \rangle_{lr}^\Delta} e^{\frac{\Delta}{2}(w_1 - \bar{w}_1)} e^{\frac{\Delta}{2}(w_2 - \bar{w}_2)}. \quad (3.6.82)$$

From these propagators, we can build the two-dimensional retarded kernel:

$$K_R(1, 2|1', 2') = \frac{1}{4} J(\hat{q} - 1) \mathcal{G}_R(1'|1) \mathcal{G}_R(2|2') \mathcal{G}_{lr}^{\hat{q}-2}(1'|2') e^{\frac{1}{2}(w_{1'} - \bar{w}_{1'})} e^{\frac{1}{2}(w_{2'} - \bar{w}_{2'})} d^2 w_1 d^2 w_2 d\tilde{\vartheta}_1 d\bar{\vartheta}_1 d\vartheta_2 d\bar{\vartheta}_2. \quad (3.6.83)$$

Using the explicit form of the propagators, we find for the kernel:

$$K_R(1, 2|1', 2') = \sin^2 \pi \Delta (Jb^{\hat{q}}) (\hat{q} - 1) e^{-\Delta(t_1 + t_2)} e^{\Delta(t_{1'} + t_{2'})} \frac{\Theta(t_{11'} - |x_{11'}|) \Theta(t_{22'} - |x_{22'}|)}{\langle 1'1 \rangle_{ll}^\Delta \langle \bar{1}'\bar{1} \rangle_{ll}^\Delta \langle 22' \rangle_{ll}^\Delta \langle \bar{2}\bar{2}' \rangle_{ll}^\Delta \langle 1'2' \rangle_{lr}^{1-2\Delta} \langle \bar{1}'\bar{2}' \rangle_{lr}^{1-2\Delta}} d^2 w_1 d^2 w_2 d\tilde{\vartheta}_1 d\bar{\vartheta}_1 d\vartheta_2 d\bar{\vartheta}_2. \quad (3.6.84)$$

This kernel is diagonalized by three-point functions. We take the coordinate of one of the operator insertions to infinity, and write the eigenfunction of (3.6.84) as:

$$f_R(1, 2, \infty) = e^{-\Delta(w_1 + w_2)} e^{\Delta(\bar{w}_1 + \bar{w}_2)} \frac{1}{\langle 12 \rangle_{lr}^{\Delta-h} \langle \bar{1}\bar{2} \rangle_{lr}^{\Delta-\bar{h}}}. \quad (3.6.85)$$

To see if this three-point function grows with time, consider its bosonic part at $w_1 = w_2$. Then, using (3.6.76), we reduce the retarded three-point function to:

$$f_R \sim e^{(h-\tilde{h})x} e^{-(h+\tilde{h})t}. \quad (3.6.86)$$

We want to find a mode which exhibits exponential growth in time, and no growth in space. Therefore, we restrict:

$$h - \tilde{h} \in i\mathbb{R}, \quad (3.6.87)$$

and look for a mode with negative $h + \tilde{h}$ and the eigenvalue of the retarded kernel equal to one.

Fixing the variables in (3.6.85) and (3.6.84):

$$1 = (0, \vartheta_1 = \bar{\vartheta}_1 = 0), \quad 2 = (0, \tilde{\vartheta}_2 = \bar{\tilde{\vartheta}}_2 = 0), \quad (3.6.88)$$

we write the eigenvalue of the kernel as an integral:

$$k_R(h, \tilde{h}) = \int K(1, 2|1', 2') f_R(1', 2', \infty). \quad (3.6.89)$$

In this expression, the left- and right-moving modes are completely decoupled. We can integrate over odd variables and then use the same integral as for the one-dimensional kernel (3.5.14) to find:

$$k_R(h, \tilde{h}) = -\frac{\Gamma^2(1-\Delta)}{\Gamma(\Delta+1)\Gamma(\Delta-1)} \frac{\Gamma(\Delta-h)\Gamma(\Delta-\tilde{h})}{\Gamma(1-\Delta-h)\Gamma(1-\Delta-\tilde{h})}, \quad (3.6.90)$$

which exactly coincides with the kernel of the $\mathcal{N} = 1$ two-dimensional model.

If the difference $h - \tilde{h}$ is imaginary, this eigenvalue of the kernel is real. Therefore the $k_R = 1$ condition has a continuous family of solutions for different $h, \tilde{h}$. As has already been discussed in [71], the chaos exponents found in this model are below ≈ 0.6 , thus not saturating the maximal chaos bound.

3.7 Conclusion

In this Chapter, we have presented a technical computation of the four-point function of an SYK-inspired model with $\mathcal{N} = 2$ symmetry. We follow the outline of [29], finding first the eigenbasis of the superconformal Casimir and then the action of the SYK kernel on the eigenfunctions. We find the two-particle Casimir of the $\mathcal{N} = 2$ superconformal group as a differential operator (3.4.9) and then compute its eigenfunctions, first directly solving the eigenvalue equation and then using the shadow representation. Then we expand the four-point function of the $\mathcal{N} = 2$ SYK model in this basis, with the result being (3.4.106). We can also write the four-point function as a sum over the positive solutions to the $k(h) = 1$ equation.

We find the $\mathcal{N} = 2$ SYK model very similar to the non-supersymmetric model with complex fermions. The eigenfunctions of the two-particle Casimir are linear combinations of the $\mathcal{N} = 0$ eigenfunctions, and the supergroup can be used to make the four-point function depend only on the bosonic coordinates. The $\mathcal{N} = 2$ eigenfunctions with the conformally invariant inner product do

not form a Hilbert space, and the norm is positive semi-definite in that case. Nevertheless, we can expand the zero-rung four-point function in the eigenfunctions of the Casimir and use this expansion to find the full four-point function. This four-point function has a pole at $h = 1$, which corresponds to the supercharge multiplet, containing the R -charge, the stress tensor and two supercharges. To resolve this pole, we would have to consider the model away from the conformal limit, which is beyond the scope of this paper. A discussion of such a resolution can be found in [70]. We also find that the $h = -1$ mode is maximally chaotic in the out-of-time order four-point function, just as in the non-supersymmetric case.

Since the two-dimensional $\mathcal{N} = 2$ superalgebra is a direct sum of holomorphic and anti-holomorphic copies of one-dimensional $su(1,1|1)$ superalgebras, our results can be easily generalized to the two-dimensional space. We consider a model containing chiral superfields with random holomorphic superpotential and find the expansion of the four-point function in terms of eigenfunctions of the two-dimensional Casimir (3.6.63). We also check that the equation $k(h) = 1$ is satisfied for the supercurrent multiplet with $(h, \tilde{h}) = (1, 0)$. The retarded kernel for this model exactly coincides with the one for the $\mathcal{N} = 1$ two-dimensional SYK model, which has been found in [71] to be non-maximally chaotic. We also find the central charge of the $\mathcal{N} = 2$ model to be twice that of an $\mathcal{N} = 1$ model.

There are numerous broad questions one can ask about the $\mathcal{N} = 2$ SYK model. They include the existence of true RG fixed points outside the large N limit; the realization of this model without random potential in spirit of [59]; a possible holographic dual or further extension to higher dimensions. We hope to address some of these questions elsewhere.

3.8 Appendix: $\mathcal{N} = 2$ Casimir

The generators of the $SU(1,1|1)$ superconformal algebra can be presented in the differential form:

$$L_0 = -\tau\partial_\tau - \frac{1}{2}\theta\partial_\theta - \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}} - \Delta, \quad (3.8.1)$$

$$L_1 = -\partial_\tau, \quad (3.8.2)$$

$$L_{-1} = -\tau^2\partial_\tau - \tau\theta\partial_\theta - \tau\bar{\theta}\partial_{\bar{\theta}} - 2\tau\Delta - \frac{Q}{2}\theta\bar{\theta}, \quad (3.8.3)$$

$$J_0 = -\theta\partial_\theta + \bar{\theta}\partial_{\bar{\theta}} + Q, \quad (3.8.4)$$

$$G_{+1/2} = \tau\partial_{\bar{\theta}} - \tau\theta\partial_\tau - (2\Delta + Q/2)\theta - \theta\bar{\theta}\partial_{\bar{\theta}}, \quad (3.8.5)$$

$$G_{-1/2} = \partial_{\bar{\theta}} - \theta\partial_\tau, \quad (3.8.6)$$

$$\bar{G}_{+1/2} = \tau\partial_\theta - \tau\bar{\theta}\partial_\tau - (2\Delta - Q/2)\bar{\theta} + \theta\bar{\theta}\partial_\theta, \quad (3.8.7)$$

$$\bar{G}_{-1/2} = \partial_\theta - \bar{\theta}\partial_\tau. \quad (3.8.8)$$

A one-particle quadratic Casimir then is:

$$C_2 = L_0L_0 - \frac{1}{4}J_0J_0 - L_1L_{-1} + \frac{1}{2}G_{+1/2}\bar{G}_{-1/2} + \frac{1}{2}\bar{G}_{+1/2}G_{-1/2}. \quad (3.8.9)$$

It commutes with all the other generators of the algebra. It acts on bosonic functions as:

$$C_2f(\tau) = \left(\Delta^2 - \frac{Q^2}{4}\right)f(\tau), \quad (3.8.10)$$

and on fermionic coordinates as:

$$C_2\theta = \left(\Delta^2 - \frac{Q^2}{4} + \frac{Q}{4}\right)\theta, \quad (3.8.11)$$

$$C_2\bar{\theta} = \left(\Delta^2 - \frac{Q^2}{4} - \frac{Q}{4}\right)\bar{\theta}. \quad (3.8.12)$$

A two-particle operator is defined as a sum of one-particle operators:

$$L_0^{2p} = L_0^{(1)} + L_0^{(2)}, \quad (3.8.13)$$

and so on. The two-particle Casimir is the same expression (3.8.9), written in terms of two-particle operators:

$$C^{2p} = L_0^{2p}L_0^{2p} - \frac{1}{4}J_0^{2p}J_0^{2p} - L_1^{2p}L_{-1}^{2p} + \frac{1}{2}G_{+1/2}^{2p}\bar{G}_{-1/2}^{2p} + \frac{1}{2}\bar{G}_{+1/2}^{2p}G_{-1/2}^{2p}. \quad (3.8.14)$$

The Casimir acts on chiral-antichiral correlation functions, so we take the R -charge to be zero:

$$Q = 0. \quad (3.8.15)$$

Then the eigenvalue of one-particle Casimir is Δ^2 . The two-particle Casimir acts on the functions of the cross-ratio χ , conjugated with a two-point function:

$$C^{2p} \left(\frac{\text{sgn } \tau_{12}}{|\langle 12 \rangle|^{2\Delta}} f(\chi) \right) = \frac{\text{sgn } \tau_{12}}{|\langle 12 \rangle|^{2\Delta}} \mathcal{C}(\chi) f(\chi), \quad (3.8.16)$$

where $\mathcal{C}$ is a second-order differential operator:

$$\mathcal{C} = \chi^2 (1 - \chi) \partial_\chi^2 + \chi (1 - \chi) \partial_\chi. \quad (3.8.17)$$

3.9 Appendix: $\mathcal{N} = 0$ SYK with complex fermions

Here we list the eigenfunctions of the $\mathcal{N} = 0$ Casimir we found in Chapter 2. In terms of the cross-ratio, the Casimir reads:

$$\mathcal{C}^{\mathcal{N}=0} = \chi^2 (1 - \chi) \partial_\chi^2 - \chi^2 \partial_\chi. \quad (3.9.1)$$

The eigenvalues of the Casimir are $h(h-1)$ and the eigenfunctions F_h, F_{1-h} :

$$F_h(\chi) \equiv \frac{\Gamma^2(h)}{\Gamma(2h)} \chi^h {}_2F_1(h, h; 2h; \chi), \quad \chi < 1, \quad (3.9.2)$$

$$\mathcal{C}^{\mathcal{N}=0} F_h = h(h-1) F_h. \quad (3.9.3)$$

The eigenfunctions of the Casimir can be $\mathcal{T}$ -even and $\mathcal{T}$ -odd. The $\mathcal{T}$ -even eigenfunctions can be either anti-symmetric or symmetric under exchange of fermions. Explicitly, they are:

$$\Psi_h^A(\chi) = \begin{cases} \frac{2}{\cos \pi h} \left(\cos^2 \frac{\pi h}{2} F_h(\chi) - \sin^2 \frac{\pi h}{2} F_{1-h}(\chi) \right), & \chi < 1, \\ \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{h}{2}\right) \Gamma\left(\frac{1-h}{2}\right) {}_2F_1\left(\frac{h}{2}, \frac{1-h}{2}; \frac{1}{2}; \frac{(2-\chi)^2}{\chi^2}\right), & \chi > 1. \end{cases} \quad (3.9.4)$$

and:

$$\Psi_h^S(\chi) = \begin{cases} \frac{2}{\cos \pi h} \left(-\sin^2 \frac{\pi h}{2} F_h(\chi) + \cos^2 \frac{\pi h}{2} F_{1-h}(\chi) \right), & \chi < 1, \\ -\frac{4}{\sqrt{\pi}} \left(\frac{2-\chi}{\chi} \right) \Gamma\left(1 - \frac{h}{2}\right) \Gamma\left(\frac{1+h}{2}\right) {}_2F_1\left(1 - \frac{h}{2}, \frac{1+h}{2}; \frac{3}{2}; \frac{(2-\chi)^2}{\chi^2}\right), & \chi > 1. \end{cases} \quad (3.9.5)$$

The $\mathcal{T}$ -breaking eigenfunctions have mixed symmetry: they are odd under exchange of one pair of fermions and odd under exchange of the other. They can also be written in terms of F_h :

$$\Psi_h^{AS}(\chi) = \begin{cases} \frac{1}{\pi} \sin^2 \frac{\pi h}{2} \tan \pi h (F_h(\chi) - F_{1-h}(\chi)), & \chi < 1, \\ 0, & \chi > 1. \end{cases} \quad (3.9.6)$$

$$\Psi_h^{SA}(\chi) = \begin{cases} \frac{1}{\pi} \cos^2 \frac{\pi h}{2} \tan \pi h \operatorname{sgn}(\chi) (F_h(\chi) - F_{1-h}(\chi)), & \chi < 1, \\ 0, & \chi > 1. \end{cases} \quad (3.9.7)$$

The $\mathcal{T}$ -even eigenfunctions have bound states. The anti-symmetric eigenfunction is normalizable at even positive h , and the symmetric one is normalizable at odd positive h , with the spectrum of course being symmetric under $h \leftrightarrow 1 - h$.

The eigenvalues of the kernel in non-supersymmetric model are also of two types:

$$k_{\mathcal{N}=0}^A(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(-2\Delta)}{\Gamma(2\Delta-2)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) (\sin \pi h - \sin 2\pi \Delta). \quad (3.9.8)$$

$$k_{\mathcal{N}=0}^S(h, \Delta) = \frac{1}{\pi} \frac{\Gamma(1-2\Delta)}{\Gamma(2\Delta-1)} \Gamma(2\Delta-h) \Gamma(2\Delta+h-1) (\sin \pi h + \sin 2\pi \Delta). \quad (3.9.9)$$

3.10 Appendix: Eigenfunctions of the $\mathcal{N} = 0$ and $\mathcal{N} = 2$ superconformal Casimir operators

Here we show the relation between eigenfunctions:

$$\varphi_h(\chi) = F_h(\chi) - F_{h+1}(\chi). \quad (3.10.1)$$

Given the relation between the Casimir operators:

$$\mathcal{C}_{\mathcal{N}=2} = \mathcal{C}_{\mathcal{N}=0} + \chi \partial_\chi, \quad (3.10.2)$$

we find that the $\mathcal{N} = 2$ Casimir acts on the combination (3.10.1) as:

$$(\mathcal{C}_{\mathcal{N}=0} + \chi \partial_\chi) (F_h - F_{h+1}) = h(h-1) F_h - h(h+1) F_{h+1} + \chi \partial_\chi (F_h - F_{h+1}) = h^2 (F_h - F_{h+1}). \quad (3.10.3)$$

This relies on the following first-order differential relation:

$$\chi \partial_\chi (F_h - F_{h+1}) = h (F_h + F_{h+1}). \quad (3.10.4)$$

Representing $F_h(\chi)$ as a series for $\chi < 1$,

$$F_h(\chi) = \sum_{k=0}^{\infty} \frac{\Gamma^2(h+k)}{\Gamma(2h+k)\Gamma(k+1)} \chi^{h+k}, \quad (3.10.5)$$

we can show that (3.10.4) indeed holds.

3.11 Appendix: $SU(1, 1|1)$ -invariant norm

In this Section we find the $SU(1, 1|1)$ -invariant measure on four-point functions in terms of the χ cross-ratio. We start with the chiral measure:

$$\langle f, g \rangle = \int \frac{d\tau_1 d\bar{\theta}_1 d\tau_2 d\theta_2}{\langle 12 \rangle} \frac{d\tau_3 d\bar{\theta}_3 d\tau_4 d\theta_4}{\langle 34 \rangle} f^* g = \int d\mu f^* g, \quad (3.11.1)$$

for f, g satisfying (anti)chirality conditions:

$$D_{1,3} f = \bar{D}_{2,4} f = D_{1,3} g = \bar{D}_{2,4} g. \quad (3.11.2)$$

With the $SU(1, 1|1)$ group, we can apply a superconformal transformation to all four supercoordinates. The infinitesimal generators of a generic transformation are:

$$V_1 = L_0^{(1)} + L_0^{(2)} + L_0^{(3)} + L_0^{(4)}, \quad (3.11.3)$$

$$V_2 = L_1^{(1)} + L_1^{(2)} + L_1^{(3)} + L_1^{(4)}, \quad (3.11.4)$$

$$\dots \quad (3.11.5)$$

$$V_7 = \bar{G}_{-1/2}^{(1)} + \bar{G}_{-1/2}^{(2)} + \bar{G}_{-1/2}^{(3)} + \bar{G}_{-1/2}^{(4)}, \quad (3.11.6)$$

the generators being listed in the Appendix 3.8. With seven generators, we can fix seven coordinates $\tau_{2,3,4}, \bar{\theta}_{1,3}, \theta_{2,4}$, leaving only $\tau_1 = \chi$. (The final answer won't depend on $\bar{\theta}_{2,4}$ or $\theta_{1,3}$, so we are not fixing those.) We wish to find the invariant measure as a function of χ . In other words, the group action allows us to define a map:

$$\varphi : \mathbb{R}^{4|4} \rightarrow \mathbb{R}, \quad (3.11.7)$$

and we are looking for the invariant measure $d\mu(\chi)$ on $\mathbb{R}$ which is a pushforward of the measure $d\mu$ on $\mathbb{R}^{4|4}$. This measure can be found as a contraction of the infinitesimal generators V_i with the original measure $d\mu$:

$$d\mu(\chi) = \iota_{V_1} \iota_{V_2} \dots \iota_{V_7} \Big|_{\frac{\partial}{\partial \tau_1}=0} d\mu, \quad (3.11.8)$$

with the generator of transformation along τ_1 not acting, so that we can keep the τ_1 coordinate. This contraction is given by a superdeterminant:

$$\iota_{V_1} \iota_{V_2} \dots \iota_{V_7} \Big|_{\frac{\partial}{\partial \tau_1}=0} d\mu \Big|_{\bar{\theta}_{1,3}=\theta_{2,4}=0} = \frac{1}{\tau_1 - \tau_2} \frac{1}{\tau_3 - \tau_4} \text{Ber} \left(\begin{array}{ccc|cccc} -1 & -1 & -1 & & & & \\ -\tau_2 & -\tau_3 & -\tau_4 & & & & \\ -\tau_2^2 & -\tau_3^2 & -\tau_4^2 & & & & \\ \hline 0 & -\tau_3 \theta_3 & 0 & 0 & 0 & \tau_1 & \tau_3 \\ 0 & -\theta_3 & 0 & 0 & 0 & 1 & 1 \\ -\tau_2 \bar{\theta}_2 & 0 & -\tau_4 \bar{\theta}_4 & \tau_2 & \tau_4 & 0 & 0 \\ -\bar{\theta}_2 & 0 & -\bar{\theta}_4 & 1 & 1 & 0 & 0 \end{array} \right), \quad (3.11.9)$$

which gives:

$$d\mu(\tau_1, \tau_2, \tau_3, \tau_4) = \frac{(\tau_2 - \tau_3)(\tau_3 - \tau_4)(\tau_2 - \tau_4)}{(\tau_2 - \tau_4)(\tau_1 - \tau_3)(\tau_1 - \tau_2)(\tau_3 - \tau_4)} d\tau_1 = \frac{\tau_2 - \tau_3}{(\tau_1 - \tau_3)(\tau_1 - \tau_2)} d\tau_1. \quad (3.11.10)$$

Fixing further the even coordinates to be:

$$\tau_1 = \chi, \quad \tau_2 = 0, \quad \tau_3 = 1, \quad \tau_4 = \infty, \quad (3.11.11)$$

we find:

$$d\mu(\chi) = \frac{d\chi}{\chi(1-\chi)}. \quad (3.11.12)$$

3.12 Appendix: Normalization of bound states

In this Appendix we prove the relation (3.4.81). To do that, we first consider the norm of non-supersymmetric SYK model. Let's take the expression:

$$\langle C_{\mathcal{N}=0} \Psi_{h'}^A, \Psi_h^A \rangle_0 - \langle \Psi_{h'}^A, C_{\mathcal{N}=0} \Psi_h^A \rangle_0. \quad (3.12.1)$$

Zero subscript signifies the $\mathcal{N} = 0$ norm:

$$\langle f, g \rangle_0 = \int_{-\infty}^{\infty} \frac{d\chi}{\chi^2} f^* g. \quad (3.12.2)$$

For distinct h, h' this expression should be zero to ensure hermiticity; however if we take h, h' ,

$$h' = h + \epsilon, \quad (3.12.3)$$

it should be proportional to ϵ :

$$\langle C_{\mathcal{N}=0} \Psi_{h'}^A, \Psi_h^A \rangle_0 - \langle \Psi_{h'}^A, C_{\mathcal{N}=0} \Psi_h^A \rangle_0 = \epsilon (2h - 1) \langle \Psi_h^A, \Psi_h^A \rangle_0. \quad (3.12.4)$$

On the other hand, using the explicit form of the Casimir (3.4.12) and the norm (3.12.2), we find:

$$\langle C_{\mathcal{N}=0} \Psi_{h'}^A, \Psi_h^A \rangle_0 - \langle \Psi_{h'}^A, C_{\mathcal{N}=0} \Psi_h^A \rangle_0 = \Psi_{h'}^A (1 - \chi) \partial_\chi \Psi_h^A - \Psi_h^A (1 - \chi) \partial_\chi \Psi_{h'}^A \Big|_{-\infty}^{\infty}. \quad (3.12.5)$$

The eigenfunction $\Psi_h^A(\chi)$ behaves as a logarithm at infinity:

$$\chi \rightarrow \infty : \quad \Psi_h^A \sim a(h) + b(h) \log \chi + O\left(\frac{1}{\chi}\right), \quad (3.12.6)$$

which implies that:

$$\Psi_{h'}^A \partial_\chi \Psi_h^A - \Psi_h^A \partial_\chi \Psi_{h'}^A \Big|_{-\infty}^\infty = 0. \quad (3.12.7)$$

Using formula for the norm of an $\mathcal{N} = 0$ bound state in the right-hand side of (3.12.4),

$$\langle \Psi_h^A, \Psi_h^A \rangle_0 = \frac{4\pi^2}{|2h-1|}, \quad (3.12.8)$$

we find the relation:

$$4\pi^2 \epsilon \cdot \text{sgn} \left(h - \frac{1}{2} \right) = \Psi_h^A \chi \partial_\chi \Psi_{h'}^A - \Psi_{h'}^A \chi \partial_\chi \Psi_h^A \Big|_{-\infty}^\infty, \quad h' = h + \epsilon. \quad (3.12.9)$$

Luckily, this relation allows us to find the norm of the $\mathcal{N} = 2$ eigenstates as well. Indeed, consider two $\mathcal{N} = 2$ eigenfunctions for close values of h . By the same token as before, we have:

$$\langle \mathcal{C}\xi_{h'}, \xi_h \rangle - \langle \xi_{h'}, \mathcal{C}\xi_h \rangle = 2h\epsilon \langle \xi_h, \xi_h \rangle = \xi_{h'} \chi \partial_\chi \xi_h - \xi_h \chi \partial_\chi \xi_{h'} \Big|_{-\infty}^\infty, \quad h' = h + \epsilon. \quad (3.12.10)$$

Since the $\mathcal{N} = 2$ eigenfunction is a linear combination of the non-supersymmetric ones,

$$\xi_h = h (\Psi_h^A - \Psi_{h+1}^S), \quad (3.12.11)$$

and the non-supersymmetric functions of different types are orthogonal,

$$\langle \Psi_h^A, \Psi_{h'}^S \rangle_0 \equiv 0, \quad (3.12.12)$$

we can rewrite (3.12.10) as:

$$2h\epsilon \langle \xi_h, \xi_h \rangle = h^2 (\Psi_{h'}^A \chi \partial_\chi \Psi_h^A - \Psi_h^A \chi \partial_\chi \Psi_{h'}^A + \Psi_{h'+1}^S \chi \partial_\chi \Psi_{h+1}^S - \Psi_{h+1}^S \chi \partial_\chi \Psi_{h'+1}^S) \Big|_{-\infty}^\infty. \quad (3.12.13)$$

Using the relation we have found in the non-supersymmetric model (3.12.9) (and an analogous relation for the Ψ_h^S eigenfunctions), we finally find:

$$\langle \xi_h, \xi_h \rangle = 4\pi^2 |h|. \quad (3.12.14)$$

3.13 Appendix: Eigenvalues of the kernel

Let's compute the integral:

$$\int K f^A(1, 2, \infty) = \frac{\tan \pi \Delta}{4\pi} \int d\tau_1 d\tau_2 d\bar{\theta}_1 d\theta_2 \frac{1}{|\langle 12 \rangle|^{1-2\Delta-h}} \frac{\text{sgn}(\tau'_1 - \tau_2)}{|\langle 1'2 \rangle|^{2\Delta}} \frac{\text{sgn}(\tau_1 - \tau'_2)}{|\langle 12' \rangle|^{2\Delta}}, \quad (3.13.1)$$

where we take three-point function in the form:

$$f^A(1, 2, \infty) = \frac{\text{sgn}(\tau_1 - \tau_2)}{|\langle 12 \rangle|^{2\Delta-h}}. \quad (3.13.2)$$

We can fix the odd coordinates of the points $1', 2'$ to be $(\theta'_1, 0), (0, \bar{\theta}_2)$ and then take the Grassmann integral. The result is:

$$\int K f^A(1, 2, 0) = 2(-1 + h + 2\Delta) \frac{\tan \pi \Delta}{4\pi} \int d\tau_1 d\tau_2 \frac{\text{sgn}(\tau_2 - \tau_1)}{|\tau_1 - \tau_2|^{2-2\Delta-h}} \frac{\text{sgn}(\tau'_1 - \tau_2)}{|\tau'_1 - \tau_2|^{2\Delta}} \frac{\text{sgn}(\tau_1 - \tau'_2)}{|\tau_1 - \tau'_2|^{2\Delta}}. \quad (3.13.3)$$

Changing variables:

$$\tau_1 = (\tau'_2 - \tau'_1)v + \tau'_1, \quad (3.13.4)$$

$$\tau_2 = (\tau'_2 - \tau'_1)u + \tau'_1, \quad (3.13.5)$$

we see that the anti-symmetric three-point function is indeed an eigenvector of the kernel:

$$\int K f^A(1, 2, 0) = \frac{\text{sgn}(\tau'_1 - \tau'_2)}{|\tau'_1 - \tau'_2|^{2\Delta-h}} \cdot k^A, \quad (3.13.6)$$

where the eigenvalue is:

$$k^A = 2(-1 + h + 2\Delta) \frac{\tan \pi \Delta}{4\pi} \int du dv \frac{\text{sgn}(u - v) \text{sgn}(1 - v) \text{sgn} u}{|u - v|^{2-2\Delta-h} |u|^{2\Delta} |v - 1|^{2\Delta}}. \quad (3.13.7)$$

Changing variables further:

$$u = vw, \quad (3.13.8)$$

we see that the integral splits into two of the same type:

$$k^A = -2(-1 + h + 2\Delta) \frac{\tan \pi \Delta}{4\pi} \int dv \frac{\text{sgn } v \text{sgn } (v-1)}{|v|^{1-h}|v-1|^{2\Delta}} \int dw \frac{\text{sgn } w \text{sgn } (w-1)}{|w|^{2\Delta}|w-1|^{2-2\Delta-h}}. \quad (3.13.9)$$

Using the integral definition of the beta-function, we find:

$$\int dt \frac{\text{sgn } t \text{sgn } (t-1)}{|t|^a|t-1|^b} = B(1-a, -1+a+b) - B(1-a, 1-b) + B(1-b, -1+a+b). \quad (3.13.10)$$

Using various identities, we arrive at the answer (3.4.48). The symmetric eigenvalue is recovered from $h \leftrightarrow -h$ symmetry:

$$k^S(h) = k^A(-h). \quad (3.13.11)$$

3.14 Appendix: Zero-rung propagator

In this Appendix, we find the inner product of an eigenfunction with a zero-rung propagator:

$$\langle \xi_h(\chi), \chi^{2\Delta} \rangle. \quad (3.14.1)$$

As before, it is instructive to consider the same problem in the non-supersymmetric model. Let's denote the corresponding product by $n_0(h, \Delta)$:

$$n_0^A(h, \Delta) \equiv \langle \Psi_h^A, \chi^{2\Delta} \rangle_0 = \frac{1}{2} \alpha_0 k_0^A(h), \quad \alpha_0 = \frac{2\pi\Delta}{(1-\Delta)(1-2\Delta)} \cot \pi\Delta. \quad (3.14.2)$$

Applying the Casimir to the functions inside the product and using the hermiticity, we find:

$$\langle \mathcal{C}_{\mathcal{N}=0} \Psi_h^A, \chi^{2\Delta} \rangle_0 = h(h-1) \langle \Psi_h^A, \chi^{2\Delta} \rangle_0 = \langle \Psi_h^A, \mathcal{C}_{\mathcal{N}=0} \chi^{2\Delta} \rangle_0 = 2\Delta(2\Delta-1) \langle \Psi_h^A, \chi^{2\Delta} \rangle_0 - 4\Delta^2 \langle \Psi_h^A, \chi^{2\Delta+1} \rangle_0. \quad (3.14.3)$$

This gives us the following identity:

$$n_0^A\left(h, \Delta + \frac{1}{2}\right) = \frac{(2\Delta-h)(2\Delta+h-1)}{4\Delta^2} n_0^A(h, \Delta). \quad (3.14.4)$$

Now we can follow the same line of reasoning for the $\mathcal{N} = 2$ eigenfunctions. Acting with the Casimir on the inner product 3.14.1, we get:

$$\langle \mathcal{C}\xi_h, \chi^{2\Delta} \rangle = h^2 \langle \xi_h, \chi^{2\Delta} \rangle = \langle \xi_h, \mathcal{C}\chi^{2\Delta} \rangle = 4\Delta^2 \langle \xi_h, \chi^{2\Delta+1} \rangle_0. \quad (3.14.5)$$

Using again the relation (3.4.38) between $\mathcal{N} = 0$ and $\mathcal{N} = 2$ eigenfunctions, we find:

$$\langle \xi_h, \chi^{2\Delta} \rangle = \frac{4\Delta^2}{h} \left(n_0^A \left(h, \Delta + \frac{1}{2} \right) - n_0^S \left(h+1, \Delta + \frac{1}{2} \right) \right). \quad (3.14.6)$$

We need two more identities: the relation between symmetric and antisymmetric eigenvalues (following from (3.9.8, 3.9.9),

$$\frac{k_0^S(h+1, \Delta + \frac{1}{2})}{k_0^A(h, \Delta + \frac{1}{2})} = \frac{2\Delta + h}{2\Delta - h}, \quad (3.14.7)$$

and the relation between $\mathcal{N} = 0$ and $\mathcal{N} = 2$ eigenvalues (3.4.52):

$$k^A(h, \Delta) = \frac{2\Delta + h - 1}{2\Delta - 2} k_0^A(h, \Delta). \quad (3.14.8)$$

Bringing together (3.14.4, 3.14.6, 3.14.7, 3.14.8), we finally find:

$$\langle \xi_h, \chi^{2\Delta} \rangle = \frac{1}{2} \alpha k^A(h). \quad (3.14.9)$$

Chapter 4

Tensor model

4.1 Introduction

In this Chapter, we review an alternative model dominated in large N by melonic diagrams, namely the (uncolored) tensor model in one dimension [59, 85]. In these models we don't have a random coupling constant. Instead, we consider Majorana fermions in a tri-fundamental representation of orthogonal group $O(N)$:

$$\psi^{abc}, \quad a, b, c = 1, \dots, N. \quad (4.1.1)$$

The interaction connects four fermions, forming a “tetrahedron” shape (see fig. 4.1):

$$g\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c}. \quad (4.1.2)$$

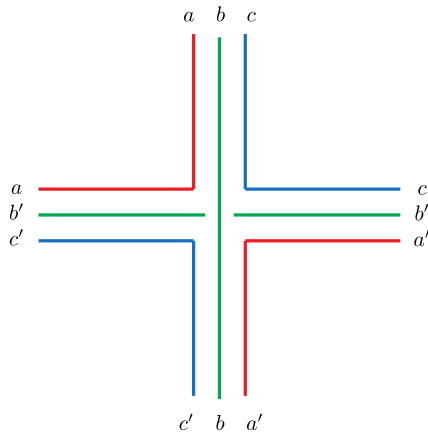


Figure 4.1: Tetrahedral interaction in an “uncolored” tensor model.

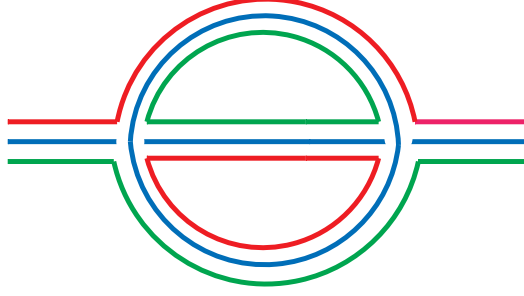


Figure 4.2: A simplest melonic diagram with indices resolved.

This interaction selects melonic diagrams at the leading order. Indeed, we see on fig. 4.2 that if the coupling scales as:

$$g^2 \sim \frac{\lambda^2}{N^3}, \quad \lambda = \text{const}, \quad (4.1.3)$$

the simplest melonic diagram contributing to the two-point function has the same power of N as the free propagator. Every colored loop contributes a factor of N and every intersection a factor of g , so altogether this diagram is multiplied by λ^2 . It is easy to see that every other melonic diagram has the same power of N . It is harder to prove that no other diagram have the same power [86, 87, 88, 58].

Since it is dominated by the same family of diagrams, the large N limit of tensor models shares many properties with that of the *SYK* model. It is also (nearly) conformal in the infrared. Just like SYK, tensor models have a tower of “Regge-like” states of the form:

$$\psi^{abc} \partial^{2k+1} \psi^{abc}, \quad (4.1.4)$$

whose dimensions are dictated by the kernel equation $k(h) = 1$. The lowest operator has the dimension of a graviton $h = 2$ and it corresponds to the Schwarzian term.

In addition to this tower, tensor models have a multitude of other operators which are singlets in $O(N)$. The most straightforward example is tetrahedron (4.1.2), which is identified with the $h = 2$ mode by the equations of motion. In fact, for every operator in on the “Regge trajectory” there is an operator without derivatives which is classically the same. But apart from that, there is an infinite number of other singlet operators present exclusively in the tensor model. These operators are the focus of this Chapter.

This simple fermionic tensor model can be generalized in many directions. The Hamiltonian can include interactions between more than four fermions [89]. The model is readily adapted to include supersymmetric fields [90, 91, 92] or bosons only [93, 94], although it is worth noting that

constructing a one-dimensional supersymmetric model is notably harder than an SYK-like model. Tensor models are also connected to matrix models [95] and as such allow numerical study. Numerical diagonalization of a tensor model Hamiltonian is also harder relative to SYK (see for example [64]), since the dimension of the Hilbert space grows as $\sim 2^{N^3}$ as opposed to $\sim 2^N$ for SYK. Despite this fact, some progress has been made [96, 97, 98]. Finally, we should mention that the original motivation of tensor model was to study random discretization of three-dimensional manifold [99, 100], which potentially provides an interesting connection to quantum gravity.

4.2 Summary

In a model with global symmetry, operators may be classified according to the group representations. In section 4.4 we study the spectra of two-particle operators, which are either symmetric traceless or antisymmetric under two indices belonging to the same $O(N)$ group. We find that the spectrum of symmetric traceless operators (4.4.5) is the same as that in the SYK model with real fermions; in particular it includes the $h = 2$ zero-mode which plays an important role in the dual gravitational dynamics [43, 84, 82]. While in the SYK model there is one $h = 2$ zero-mode, in the $O(N)^3$ tensor model it appears with multiplicity $1 + \frac{3}{2}(N-1)(N+2)$. For the operators anti-symmetric in the two indices, (4.4.6), the spectrum is identical to the additional sector found in the complex tensor and SYK models [60, 70, 11, 71, 78, 72, 101]; it includes the $h = 0$ eigenvalue with multiplicity $\frac{3}{2}N(N-1)$ corresponding to the conserved $O(N)^3$ charges.

An attractive feature of tensor models is that the global symmetry may be gauged [82, 59, 60]; this restricts the operator spectrum to the invariant ones only. The “Regge trajectory” of two-particle operators $\psi^{abc}\partial_t^{2n+1}\psi^{abc}$ is clearly not the full set of $O(N)^3$ invariant operators; there are vastly more operators which may be constructed by multiplying an even number of tensors and contracting all the indices [60]. In section 4.5 we explicitly construct and draw pictorial representations of such operators (these pictures are combinatorially analogous to the Feynman diagrams in the theory of three scalar fields φ_i with interaction vertex $\varphi_1\varphi_2\varphi_3$). Using the techniques developed in [102, 103, 104, 105] (see also [106]), we will calculate the number of $(2k)$ -particle operators and show that it grows asymptotically as $2^k k!$. As a consequence, the theory has a Hagedorn phase transition at the temperature $\sim 1/\log N$, which we discuss in section 4.10. Our work is similar in spirit to the classification of invariants in the $d = 0$ tensor models [107, 108, 109, 110, 111, 112], but some of our specific results appear to be new. Working with the quantum mechanical model of real 3-tensors introduces some subtleties and cancellations: for example, in the $O(N)^3$ fermionic model all the

6-particle operators vanish due to the Fermi statistics, while the number of 10-particle operators is strongly reduced compared to the similar bosonic model. In section 4.9 we also count the invariants in $d = 0$ bosonic models. In addition to the real tensors with $O(N)^3$ symmetry we study the complex tensor theories with $U(N)^3$ and $U(N)^2 \times O(N)$ symmetries, as well as the symmetric traceless and fully antisymmetric rank-3 tensors under a single $O(N)$ group.

Beyond classifying the invariant operators, it is important to determine their infrared scaling dimensions. We begin work on this in section 4.6 and point out that there is a large class of $2k$ -particle operators whose large N scaling dimensions are simply additive, i.e. $k/2$. This is because the melonic ladders contribute only to $1/N$ corrections. However, although less generic, there are operators whose dimensions are not simply quantized. While the Regge trajectory operators studied in [9, 61, 29, 113, 77, 60] receive single ladder contributions, there are operators whose two-point functions have multi-ladder contributions. Since a ladder may contain an $h = 2$ zero-mode, the m -ladder diagram seems to produce a low-temperature enhancement by $(\beta J)^m$. This may be an important physical effect in the melonic tensor models, whose detailed analysis we leave for the future.

Besides our analysis of the spectra of $O(N)^3$ symmetric models, we make some comments about the $O(N)^6$ symmetric Gurau-Witten model [59]. Some features of its spectrum are identical to those in the $q = 4$, $f = 4$ Gross-Rosenhaus flavored generalization [77] of the SYK model. The connections of the Gurau-Witten model with this Gross-Rosenhaus model have been also noted using combinatorial analysis in [114].

The work of [115] has some overlap with the results presented here.

4.3 Comments on the $O(N)^3$ Symmetric Fermionic Tensor Quantum Mechanics

Let us consider the quantum mechanical model of a real anticommuting 3-tensor ψ^{abc} with the action [60]

$$S = \int dt \left(\frac{i}{2} \psi^{abc} \partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \right). \quad (4.3.1)$$

The three indices, each of which runs from 1 to N , are treated as distinguishable, and the Majorana fermions satisfy the anti-commutation relations

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'} \delta^{bb'} \delta^{cc'} . \quad (4.3.2)$$

This model is a somewhat simplified version of the $O(N)^6$ symmetric Gurau-Witten model [59]. Both are in the class of 3-tensor models which possess a “melonic” large N limit where $J = gN^{3/2}$ is held fixed [58, 107, 86, 87, 116, 117, 88, 118, 119, 120, 121]. The large N model is nearly conformal in the IR [4, 9]; for example, the two-point function is

$$\langle T(\psi^{abc}(t_1) \psi^{a'b'c'}(t_2)) \rangle = -\delta^{aa'} \delta^{bb'} \delta^{cc'} \left(\frac{1}{4\pi g^2 N^3} \right)^{1/4} \frac{\text{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}} . \quad (4.3.3)$$

The model (4.3.1) has the $O(N)_1 \times O(N)_2 \times O(N)_3$ symmetry under the replacement¹

$$\psi^{abc} \rightarrow M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'} , \quad (4.3.4)$$

$$M_1 \in O(N)_1, \quad M_2 \in O(N)_2, \quad M_3 \in O(N)_3 . \quad (4.3.5)$$

As far as the group $O(N)_1$ is concerned, we may think of b and c as flavor indices; therefore ψ^{abc} produces N^2 flavors of real fermions in the fundamental of $O(N)_1$. An analogous picture applies to $O(N)_2$ and $O(N)_3$. The three sets of $SO(N)$ symmetry charges are

$$Q_1^{a_1 a_2} = \frac{i}{2} [\psi^{a_1 bc}, \psi^{a_2 bc}] , \quad Q_2^{b_1 b_2} = \frac{i}{2} [\psi^{ab_1 c}, \psi^{ab_2 c}] , \quad Q_3^{c_1 c_2} = \frac{i}{2} [\psi^{abc_1}, \psi^{abc_2}] . \quad (4.3.6)$$

The gauging of $SO(N)_1 \times SO(N)_2 \times SO(N)_3$ sets these charges to zero; this restricts the operators to the invariant ones, where all the indices are contracted. In the ungauged model (4.3.1) a more general class of operators is allowed, and they can be classified according to representations of the $SO(N)_1 \times SO(N)_2 \times SO(N)_3$.

Each $O(N)$ group includes parity transformations (axis reflections) P_{a_0} : for a given a_0 , P_{a_0} sends $\psi^{a_0 bc} \rightarrow -\psi^{a_0 bc}$ for all b, c and leaves all $\psi^{a_1 bc}$, $a_1 \neq a_0$ invariant. In a physical language, these are “big” gauge transformations and operators should be invariant under them. Therefore we can build operators using ψ^{abc} and the delta symbol $\delta^{aa'}$ only. In the case of $SO(N)$ gauge group one can use the fully antisymmetric tensor $\epsilon_{a_1 \dots a_N}$ as well; it is invariant under $SO(N)$, but changes its sign

¹More generally, we could consider a model with $O(N_1) \times O(N_2) \times O(N_3)$ symmetry, where a runs from 1 to N_1 , b from 1 to N_2 , and c from 1 to N_3 . This may be thought of as a model of a large number N_2 of $N_1 \times N_3$ matrices [95].

under the parity transformations. Because of this, there are additional “long” operators containing at least N fields, like

$$O_{\text{long}} = \epsilon_{a_1 \dots a_N} \epsilon_{b_1 \dots b_N} \epsilon_{c_1 \dots c_N} \prod_{j=1}^N \psi^{a_j b_j c_j} . \quad (4.3.7)$$

The difference between gauging $O(N)$ and $SO(N)$ becomes negligible in the large N limit.

Let us define three operations which permute pairs of the $O(N)$ symmetry groups (and thus interchange indices in the tensor field), while also reversing the direction of time,

$$s_{ab} : \psi^{abc} \rightarrow \psi^{bac}, \quad t \rightarrow -t; \quad (4.3.8)$$

$$s_{bc} : \psi^{abc} \rightarrow \psi^{acb}, \quad t \rightarrow -t; \quad (4.3.9)$$

$$s_{ac} : \psi^{abc} \rightarrow \psi^{cba}, \quad t \rightarrow -t. \quad (4.3.10)$$

Each of these transformations preserves the equations of motion for the ψ^{abc} field,

$$\dot{\psi}^{abc} = ig(\psi^3)^{abc}, \quad (\psi^3)^{abc} \equiv \psi^{ab_1 c_1} \psi^{a_1 b c_1} \psi^{a_1 b_1 c}. \quad (4.3.11)$$

The Hamiltonian, including a quantum shift due to (4.3.2),

$$H = -\frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} + \frac{g N^4}{16} = -\frac{1}{4} g [\psi^{a_1 b_1 c_1}, \psi^{a_1 b_2 c_2}] [\psi^{a_2 b_1 c_2}, \psi^{a_2 b_2 c_1}], \quad (4.3.12)$$

changes sign under each of the transformations s_{ab} , s_{bc} , s_{ac} (this is discussed in section 4.5). This means that these transformations are unitary: they preserve e^{iHt} . In contrast, the usual time reversal transformation is anti-unitary because it also requires complex conjugation $i \rightarrow -i$.

The $O(N)^3$ invariant operators form representations under the permutation group S_3 , which acts on the three $O(N)$ symmetry groups (it contains the elements s_{ab} , s_{bc} and s_{ac}). For example, H is in the degree 1 “sign representation” of S_3 : it changes sign under any pair interchange, but preserves its sign under a cyclic permutation.

It is also interesting to study the spectrum of eigenstates of the Hamiltonian for small values of N ; first steps on this were made in [122, 97, 123]. When gauging the $O(N)^3$ symmetry one needs to worry about the Z_2 anomaly, which affects the gauged $O(N)$ quantum mechanics with an odd number of flavors of real fermions in the fundamental representation [124, 125]. Since for each of the three $O(N)$ groups we find N^2 flavors of fundamental fermions, the gauged model is consistent for even N , but is anomalous for odd N . This means that, for odd N , the spectrum does not contain states which are invariant under $O(N)^3$ (for $N = 3$ this can be seen via an explicit diagonalization

of the Hamiltonian (4.3.12) [122]).

4.4 Composite Operators and Schwinger-Dyson Equations

The scaling dimensions of a class of bilinear operators may be extracted from the 4-point function [60]

$$\langle \psi^{a_1 b_1 c_1}(t_1) \psi^{a_1 b_1 c_1}(t_2) \psi^{a_2 b_2 c_2}(t_3) \psi^{a_2 b_2 c_2}(t_4) \rangle, \quad (4.4.1)$$

and factorizing it in the channel where $t_1 \rightarrow t_2$ and $t_3 \rightarrow t_4$. A class of melonic ladder graphs appears in this channel in the large N limit; it may be summed by means of a Schwinger-Dyson equation. The singlet bilinear operators

$$\mathcal{O}_n = \psi^{abc} \partial_t^{2n+1} \psi^{abc}, \quad n = 0, 1, 2, \dots \quad (4.4.2)$$

form a “Regge trajectory.” Their scaling dimensions are the same as in the SYK model [4, 9], and they have been extensively analyzed in the literature [61, 29, 113, 77]. The dimensions are determined by the equation

$$g(h) = -\frac{3 \tan(\frac{\pi}{2}(h - \frac{1}{2}))}{h - 1/2} = 1, \quad (4.4.3)$$

and the first few solutions are $h = 2, 3.77, 5.68, \dots$. As pointed out in [60], the model also contains a multitude of multi-particle singlet operators. As we will see, some special combinations of the multi-particle operators are related by the equations of motion to the operators (4.4.2), but most multi-particle operators are genuinely new.

Interestingly, there are also certain non-singlet operators which are renormalized by the melonic ladder diagrams. This can be seen, for example, from the 4-point function

$$\langle \psi^{a_1 b_1 c_1}(t_1) \psi^{a_2 b_1 c_1}(t_2) \psi^{a_1 b_2 c_2}(t_3) \psi^{a_2 b_2 c_2}(t_4) \rangle \quad (4.4.4)$$

factorized in the channel $t_1 \rightarrow t_2$ and $t_3 \rightarrow t_4$. As shown in figure 4.3, all the melonic ladders again make non-vanishing contributions in the large N limit. Here we find two classes of non-singlet bilinear operators: those symmetric and traceless in a_1 and a_2 , and those anti-symmetric. The

$\frac{1}{2}(N-1)(N+2)$ symmetric traceless operators under $O(N)_1$,

$$\mathcal{O}_n^{(a_1 a_2)} = \psi^{a_1 bc} \partial_t^{2n+1} \psi^{a_2 bc} + \psi^{a_2 bc} \partial_t^{2n+1} \psi^{a_1 bc} - \frac{2}{N} \delta^{a_1 a_2} \psi^{abc} \partial_t^{2n+1} \psi^{abc} , \quad (4.4.5)$$

where $n = 0, 1, 2, \dots$, have the same spectrum as the singlet bilinears (4.4.2) which is determined by (4.4.3). Of course, there are analogous operators $\mathcal{O}_n^{(b_1 b_2)}$ and $\mathcal{O}_n^{(c_1 c_2)}$ that are symmetric traceless under $O(N)_2$ and $O(N)_3$, respectively. Thus, the symmetric traceless operators present in the ungauged model contain the $h = 2$ zero-mode with multiplicity $\frac{3}{2}(N-1)(N+2)$; this appears to imply a significant physical difference between the ungauged $O(N)^3$ model and the SYK model.² While in the gauged model such bilinear operators are projected out, we may form singlet combinations out of their products; such operators have an interesting feature that they are renormalized by multiple ladders. For example, in section 4.6 we will encounter operators related by the equation of motion to $\mathcal{O}_0^{(a_1 a_2)} \mathcal{O}_0^{(a_1 a_2)}$, so they are renormalized by double ladders. The pictorial representations of these operators may be found in column 2 of figure 4.11.

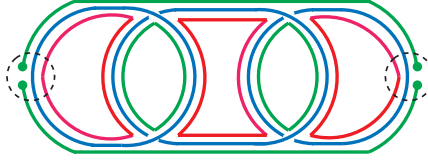


Figure 4.3: A ladder contribution to the two-point function of a bilinear operator with two pairs of indices contracted, $\mathcal{O}^{c_1 c_2}$. It is not suppressed in the large N limit.

There are also the $\frac{1}{2}N(N-1)$ operators in the anti-symmetric two-index representation of $O(N)_1$,

$$\mathcal{O}_n^{[a_1 a_2]} = \psi^{a_1 bc} \partial_t^{2n} \psi^{a_2 bc} - \psi^{a_2 bc} \partial_t^{2n} \psi^{a_1 bc} , \quad (4.4.6)$$

and the analogous anti-symmetric operators under $O(N)_2$ and $O(N)_3$. The Schwinger-Dyson equations for these operators are identical to the "symmetric sector" of the complex tensor model [60, 70, 11, 71, 78, 72, 101]. Their scaling dimensions are determined by

$$\tilde{g}(h) = -\frac{1}{2} \frac{\tan(\frac{\pi}{2}(h + \frac{1}{2}))}{h - 1/2} = 1 . \quad (4.4.7)$$

The first few solutions of this equation are $h = 0, 2.65, 4.58, \dots$, and each one appears with multiplicity $\frac{3}{2}N(N-1)$. The spectrum includes the special $h = 0$ mode corresponding here to the $n = 0$ operators, which are the $O(N)^3$ charges (4.3.6).

²We are grateful to Shiraz Minwalla for very useful discussions on this; see the paper [115].

The 4-point function (4.4.4) may also be factorized in the channel $t_1 \rightarrow t_3$ and $t_2 \rightarrow t_4$. This leads to the spectrum of operators

$$\mathcal{O}_m^{b_1 c_1 b_2 c_2} = \psi^{ab_1 c_1} \partial_t^m \psi^{ab_2 c_2} . \quad (4.4.8)$$

We can see from figure 4.4 that the ladder contribution to this operator are subleading in $1/N$: the rightmost diagram is of ladder type and is $\sim g^2 N^3$, which is suppressed by a power of N relative to the other two diagrams. Therefore the large N scaling dimensions of these operators are $1/2 + m$.

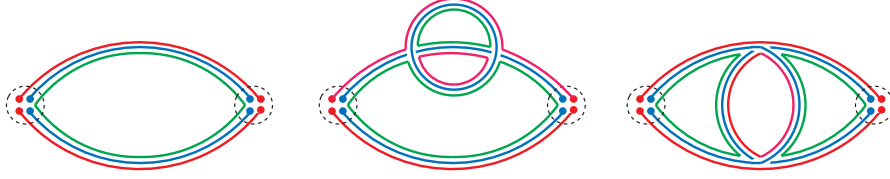


Figure 4.4: Different contributions to the two-point function of a bilinear operator with one pair of indices contracted, $\mathcal{O}_m^{b_1 c_1 b_2 c_2}$. The ladder diagrams, such as the rightmost figure, are suppressed in the large N limit.

We will adopt a pictorial representation of the operators where the ψ^{abc} fields are shown as the vertices. The a -indices which transform under $O(N)_1$ are shown by red lines; the b -indices which transform under $O(N)_2$ are shown by blue lines; and the c -indices which transform under $O(N)_3$ are shown by green lines. For example, the three charges (4.3.6) are shown in figure 4.5.

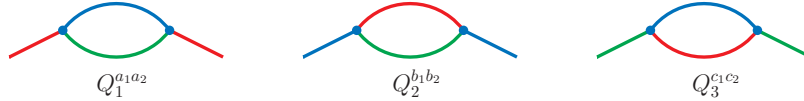


Figure 4.5: The $O(N)_1$, $O(N)_2$ and $O(N)_3$ charges.

4.5 Construction of $O(N)^3$ invariant operators

In this section we study the spectrum of $O(N)^3$ invariant operators. Since a time derivative may be removed using the equations of motion (4.3.11), we may write the operators in a form where no derivatives are present. The bilinear singlet operator, $\psi^{abc} \psi^{abc}$, vanishes classically by the Fermi statistics, while at the quantum level taking into account (4.3.2), it is a C-number. The first non-trivial operators appear at the quartic level and are shown in figure 4.6 (from here on we will not be careful about the quantum corrections to operators).

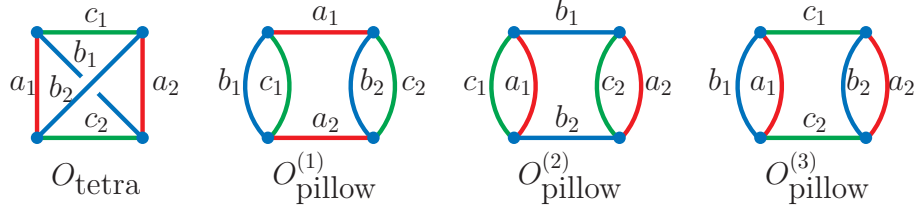


Figure 4.6: All the four-particle operators, the tetrahedron and the three pillows, with the index contractions shown explicitly.

On the left is the “tetrahedron operator” O_{tetra} , which is proportional to the Hamiltonian (4.3.12):

$$O_{\text{tetra}} = \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} . \quad (4.5.1)$$

One can check that

$$\begin{aligned} s_{bc} O_{\text{tetra}} &= \psi^{a_1 c_1 b_1} \psi^{a_1 c_2 b_2} \psi^{a_2 c_2 b_1} \psi^{a_2 c_1 b_2} \\ &= \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_1} \psi^{a_2 b_1 c_2} = -O_{\text{tetra}} , \end{aligned} \quad (4.5.2)$$

and also that $s_{ab} O_{\text{tetra}} = -O_{\text{tetra}}$ and $s_{ac} O_{\text{tetra}} = -O_{\text{tetra}}$. Thus, the tetrahedron operator O_{tetra} is in the degree 1 “sign representation” of S_3 : it changes sign under any pair interchange, but preserves its sign under a cyclic permutation.

The three additional operators in figure 4.6, which we denote as $O_{\text{pillow}}^{(1)}$, $O_{\text{pillow}}^{(2)}$ and $O_{\text{pillow}}^{(3)}$, are the “pillow” operators in the terminology of [117, 120]; they contain double lines between a pair of vertices. For example, for $O_{\text{pillow}}^{(1)}$ we have

$$O_{\text{pillow}}^{(1)} = -\psi^{a_1 b_1 c_1} \psi^{a_2 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_2} = Q_1^{a_1 a_2} Q_1^{a_1 a_2} . \quad (4.5.3)$$

Under the S_3 the three pillow operators decompose into the trivial representation of degree 1 and the standard representation of degree 2. Since the charges (4.3.6) commute with the Hamiltonian (4.3.12), so does each of the three pillow operators. This means that the scaling dimensions of the pillow operators are unaffected by the interactions, i.e. they vanish. In fact, the three pillow operators are simply the quadratic Casimir operators of the three $O(N)$ groups.³ The gauging of $O(N)^3$ symmetry sets the charges (4.3.6) to zero, so the pillow operators do not appear in the gauged model.

³We thank Dan Roberts and Douglas Stanford for discussions on this.

Using the equations of motion (4.3.11) we see that the operator O_{tetra} is related by the equation of motion to the operator $\psi^{abc}\partial_t\psi^{abc}$

$$O_{\text{tetra}} = \psi^{abc}(\psi^3)^{abc} \propto \psi^{abc}\partial_t\psi^{abc}. \quad (4.5.4)$$

If we iterate the use of the equation of motion (4.3.11), then all derivatives in an operator may be traded for extra ψ -fields. Thus, a complete basis of operators may be constructed by multiplying some number $2k$ of ψ -fields and contracting all indices. In this approach, there is a unique operator with $k = 2(n + 1)$ which is equal to the Regge trajectory operator $\psi^{abc}\partial_t^{2n+1}\psi^{abc}$. For $n = 0$ this operator is O_{tetra} , which is proportional to the Hamiltonian; for $n = 1$ it will be constructed explicitly in section 4.5.1.

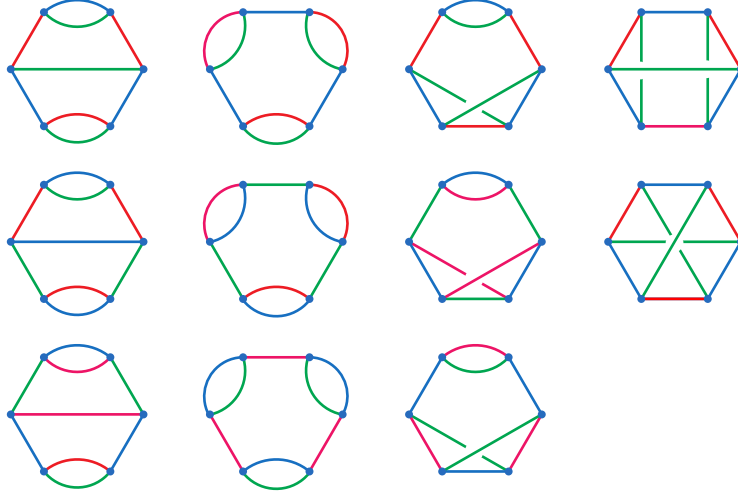


Figure 4.7: All six-particle operators. They are present in the scalar model but vanish in the fermionic model.

All the six-particle operators are represented in figure 4.7, but due to the Fermi statistics all of them vanish. Even if this were not the case, the operators in the first three columns would vanish in the gauged model because they contain insertions of the charges (4.3.6). Let us demonstrate the vanishing of the two operators in the last column in detail. The first operator

$$O_6^{(1)} = \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_3 c_3} \psi^{a_3 b_3 c_1} \psi^{a_3 b_2 c_3}, \quad (4.5.5)$$

may be written as

$$O_6^{(1)} = (\psi^3)^{a_2 b_2 c_1} (\psi^3)^{a_2 b_2 c_1} = 0. \quad (4.5.6)$$

This may be seen by cutting the diagram for this operator in figure 4.7 along the vertical symmetry axis. To show that

$$O_6^{(2)} = \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_3} \psi^{a_2 b_3 c_1} \psi^{a_3 b_3 c_2} \psi^{a_3 b_1 c_3} \quad (4.5.7)$$

also vanishes, we may permute the first two ψ -fields to write it as

$$O_6^{(2)} = -\psi^{a_1 b_2 c_2} \psi^{a_1 b_1 c_1} \psi^{a_2 b_2 c_3} \psi^{a_2 b_3 c_1} \psi^{a_3 b_3 c_2} \psi^{a_3 b_1 c_3} . \quad (4.5.8)$$

After relabeling $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$ and $a_2 \leftrightarrow a_3$, we observe that the RHS equals $-O_6^{(2)}$. Therefore, $O_6^{(2)} = -O_6^{(2)} = 0$.

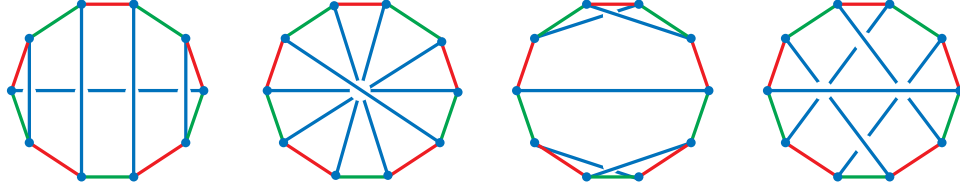


Figure 4.8: Some ten-particle operators which vanish in the fermionic model.

One may wonder if the vanishing extends to the 10-particle operators. We have checked that the operators shown in figure 4.8 all vanish; this is due to the reflection symmetry present for these operators. For example, the left operator in figure 4.8 vanishes because it may be written as $(\psi^5)^{abc}(\psi^5)^{abc}$, which may be seen by cutting the diagram along the vertical symmetry axis. We note that

$$(\psi^5)^{abc} = g^{-2} \partial_t^2 \psi^{abc} . \quad (4.5.9)$$

Similarly, by cutting the third diagram in figure 4.8 along its vertical symmetry axis, we see that the corresponding operator may be written as $(\psi^5)^{ab_1 b_2 b_3 b_4 c}(\psi^5)^{ab_1 b_2 b_3 b_4 c}$ which obviously vanishes as well. This argument extends to all the reflection symmetric $(4n + 2)$ -particle diagrams.

However, not all 10-particle operators vanish. For example, the operators shown in figure 4.9 do not have a reflection symmetry, and we have checked that they do not vanish.

Let us note that each gauge invariant operator, where all the indices are contracted, corresponds to a vacuum Feynman diagram in the theory with three scalar fields and interaction $\lambda \varphi_1 \varphi_2 \varphi_3$ (the three different propagators correspond to the lines of three different colors in our figures). In the theory of bosonic tensors ϕ^{abc} , the number of operators made out of $2k$ fields is precisely the number

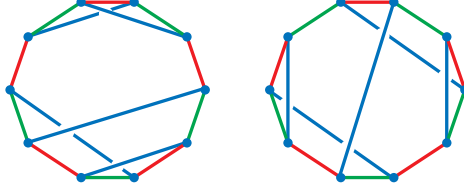


Figure 4.9: Some non-vanishing ten-particle operators.

of distinct Feynman diagrams appearing at order λ^{2k} , which grows as $k!2^k$. In the fermionic model, some of the operators vanish by the Fermi statistics, while others due to the gauge constraint. Nevertheless, we will find that the factorial growth holds also in the fermionic model.

4.5.1 Eight-particle operators

In this section we explicitly construct all the eight-particle operators without bubble (double line) insertions and exhibit their pictorial representations. Having two vertices connected by a double line corresponds to insertion of an $O(N)$ charge which vanishes in the gauged model. For this reason we will omit such operators and list only those where there are no double lines. The possible topologically inequivalent eight-particle operators are shown in figure 4.10; from these we can obtain other admissible operators by interchanging the colors. In this way we find 17 inequivalent operators shown in figure 4.11.

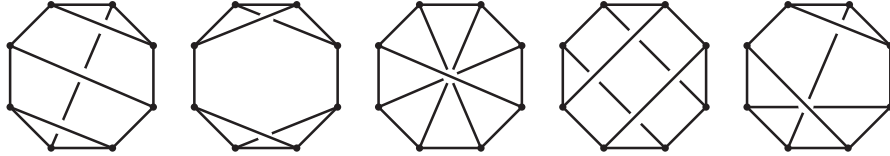


Figure 4.10: Eight-particle operator topologies

Among the eight-particle operators there are three which may be obtained from the tetrahedral vertex

$$\begin{aligned}
 O_1 &= \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_1} \psi^{a_2 b_4 c_4} \psi^{a_3 b_3 c_2} \psi^{a_3 b_1 c_3} \psi^{a_4 b_4 c_3} \psi^{a_4 b_3 c_4} , \\
 O_2 &= \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_1} \psi^{a_2 b_3 c_3} \psi^{a_3 b_3 c_2} \psi^{a_3 b_4 c_4} \psi^{a_4 b_4 c_3} \psi^{a_4 b_1 c_4} , \\
 O_3 &= \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_2 c_1} \psi^{a_2 b_3 c_3} \psi^{a_3 b_1 c_3} \psi^{a_3 b_4 c_4} \psi^{a_4 b_3 c_4} \psi^{a_4 b_4 c_2} .
 \end{aligned}
 \tag{4.5.10}$$

Their pictorial representations are shown in the first column of figure 4.11. Using the equations of

motion, we may write them as

$$\begin{aligned}
O_1 &= \dot{\psi}^{a_1 b_1 c_1} \dot{\psi}^{a_1 b_2 c_2} \dot{\psi}^{a_2 b_1 c_2} \dot{\psi}^{a_2 b_2 c_1} , \\
O_2 &= \dot{\psi}^{a_1 b_1 c_1} \dot{\psi}^{a_1 b_2 c_2} \dot{\psi}^{a_2 b_1 c_2} \dot{\psi}^{a_2 b_2 c_1} , \\
O_3 &= \dot{\psi}^{a_1 b_1 c_1} \dot{\psi}^{a_1 b_2 c_2} \dot{\psi}^{a_2 b_1 c_2} \dot{\psi}^{a_2 b_2 c_1} .
\end{aligned} \tag{4.5.11}$$

It follows that

$$O_1 + O_2 + O_3 \sim \partial_t \psi^{abc} \partial_t^2 \psi^{abc} , \tag{4.5.12}$$

which up to a total derivative equals the Regge trajectory operator $\psi^{abc} \partial_t^3 \psi^{abc}$.

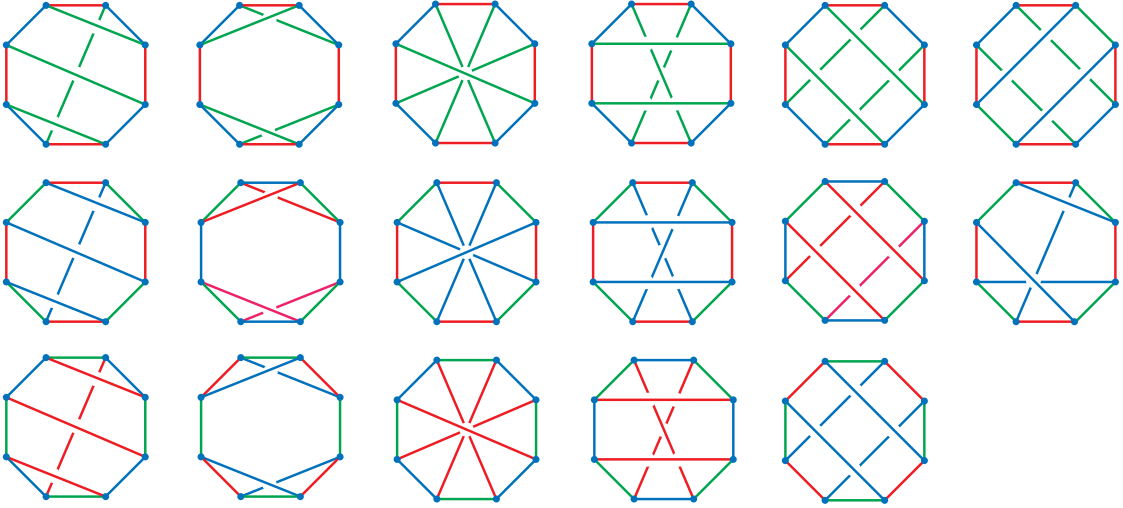


Figure 4.11: All eight-particle operators in the fermionic model.

The transformation properties of operators O_1 , O_2 and O_3 under S_3 are

$$\begin{aligned}
s_{bc} O_3 &= -O_2, & s_{bc} O_2 &= -O_3, & s_{bc} O_1 &= -O_1 , \\
s_{ac} O_3 &= -O_1, & s_{ac} O_2 &= -O_2, & s_{ac} O_1 &= -O_3 , \\
s_{ab} O_3 &= -O_3, & s_{ab} O_2 &= -O_1, & s_{ab} O_1 &= -O_2 .
\end{aligned}$$

It follows that

$$(s_{ab}, s_{ac}, s_{bc}) : (O_1 + O_2 + O_3) \rightarrow -(O_1 + O_2 + O_3) . \tag{4.5.13}$$

Therefore, the operator $\psi^{abc}\partial_t^3\psi^{abc} \sim O_1 + O_2 + O_3$ is in the degree 1 sign representation of S_3 . The other two linear combinations of operators (4.5.10), $O_1 - O_2$ and $O_2 - O_3$, form the standard degree 2 representation of S_3 .

Similarly, we may write down the three operators which correspond to the second column in figure 4.11 (the first of these operators, $\tilde{O}_1$, was written down in [60]):

$$\begin{aligned}\tilde{O}_1 &= \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_3 c_3} \psi^{a_2 b_4 c_4} \psi^{a_3 b_1 c_3} \psi^{a_3 b_3 c_1} \psi^{a_4 b_2 c_4} \psi^{a_4 b_4 c_2}, \\ \tilde{O}_2 &= \psi^{a_1 b_1 c_1} \psi^{a_2 b_1 c_2} \psi^{a_3 b_2 c_3} \psi^{a_4 b_2 c_4} \psi^{a_1 b_3 c_3} \psi^{a_3 b_3 c_1} \psi^{a_2 b_4 c_4} \psi^{a_4 b_4 c_2}, \\ \tilde{O}_3 &= \psi^{a_1 b_1 c_1} \psi^{a_2 b_2 c_1} \psi^{a_3 b_3 c_2} \psi^{a_4 b_4 c_2} \psi^{a_3 b_1 c_3} \psi^{a_1 b_3 c_3} \psi^{a_4 b_2 c_4} \psi^{a_2 b_4 c_4}.\end{aligned}\tag{4.5.14}$$

Via the equations of motion, these operators are related to the bilinear operators defined in (4.4.5):

$$\tilde{O}_1 \sim \mathcal{O}_0^{(a_1 a_2)} \mathcal{O}_0^{(a_1 a_2)}, \quad \tilde{O}_2 \sim \mathcal{O}_0^{(b_1 b_2)} \mathcal{O}_0^{(b_1 b_2)}, \quad \tilde{O}_3 \sim \mathcal{O}_0^{(c_1 c_2)} \mathcal{O}_0^{(c_1 c_2)}.\tag{4.5.15}$$

These relations will be used in the next section.

The action of the discrete symmetries on the operators is

$$\begin{aligned}s_{bc}\tilde{O}_3 &= \tilde{O}_2, & s_{bc}\tilde{O}_2 &= \tilde{O}_3, & s_{bc}\tilde{O}_1 &= \tilde{O}_1, \\ s_{ac}\tilde{O}_3 &= \tilde{O}_1, & s_{ac}\tilde{O}_2 &= \tilde{O}_2, & s_{ac}\tilde{O}_1 &= \tilde{O}_3, \\ s_{ab}\tilde{O}_3 &= \tilde{O}_3, & s_{ab}\tilde{O}_2 &= \tilde{O}_1, & s_{ab}\tilde{O}_1 &= \tilde{O}_2,\end{aligned}\tag{4.5.16}$$

so that

$$(s_{ab}, s_{bc}, s_{ac}) : \tilde{O}_1 + \tilde{O}_2 + \tilde{O}_3 \rightarrow \tilde{O}_1 + \tilde{O}_2 + \tilde{O}_3.\tag{4.5.17}$$

Therefore, this operator is in the trivial representation of S_3 . The other two linear combinations of operators (4.5.14), $\tilde{O}_1 - \tilde{O}_2$ and $\tilde{O}_2 - \tilde{O}_3$, form the standard degree 2 representation of S_3 . The operators corresponding to the other topologies in figure 4.10 may be written down analogously.

4.6 Scaling Dimensions of Multi-Particle Operators

We have seen that the tensor models admit a variety of singlet operators. In this section we discuss their scaling dimensions. Since operators $\mathcal{O}_m^{b_1 c_1 b_2 c_2}$ defined in (4.4.8) do not receive ladder contributions in the large N limit, we expect a large class of m -particle operators to have the quantized

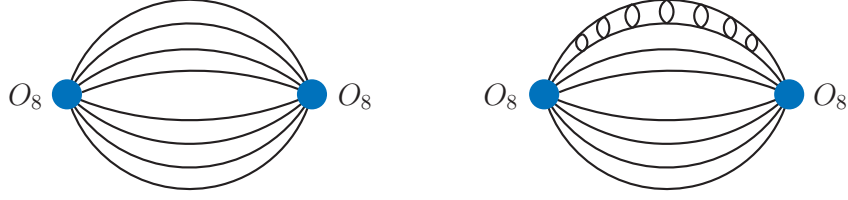


Figure 4.12: Diagrammatics for the “typical” operators whose IR dimensions are quantized. Each line denotes a dressed propagator. a) The melonic diagrams that contribute to the operator two-point functions in the large N limit. b) The ladder diagrams which do not contribute in the large N limit.

dimensions:

$$\Delta_m = \frac{m}{4} + \mathcal{O}(1/N). \quad (4.6.1)$$

This is the dimension of an operator which is not renormalized by ladder diagrams because every pair of tensors have at most one index in common. This situation is illustrated in figure 4.12: the dominant contribution comes from the two operators contracted using the IR two-point function (4.3.3), and the ladder insertions are suppressed by $1/N$. We find that this applies to most of the 17 eight-particle operators shown in figure 4.11. The exceptions are operators O_i and $\tilde{O}_i$, defined in (4.5.10), (4.5.14), and shown in columns 1 and 2. For example, each of the operators $\tilde{O}_i$ in column 2 is renormalized by two ladders, as we discuss below.

Thus, the $m/4$ rule does not apply to all operators: it is violated for the operators whose two-point functions receive the melonic ladder contributions in the large N limit. One class of such singlet operators is the Regge trajectory we have discussed before:

$$\psi^{abc} \partial_t^{2n+1} \psi^{abc}. \quad (4.6.2)$$

After applying the equation of motion (4.3.11), which schematically may be represented as

$$\text{triangle diagram} = \partial_t \text{triangle diagram} = \text{triangle diagram with internal lines} \quad (4.6.3)$$

we may represent the Regge trajectory operators in terms of multi-particle operators without derivatives. For example, the $n = 0$ operator is equivalent to the 4-particle “tetrahedron” operator O_{tetra} , while the $n = 1$ operator is equivalent to $O_1 + O_2 + O_3$, as shown in (4.5.12). The dimensions of

such operators come from solving (4.4.3), so the operator $O_1 + O_2 + O_3$ has $h \approx 3.77$.

Furthermore, using the equation of motion (4.6.3), we can relate many additional singlet operators to operators containing derivatives. Let us denote a vertex with $\partial_t \psi$ by a white circle. By the equations of motion, we can relate the operators whose diagram contains triangles with low-order operators containing derivatives. For example, some of the operators which can be written as lower-order operators with derivatives are shown in figure 4.13.

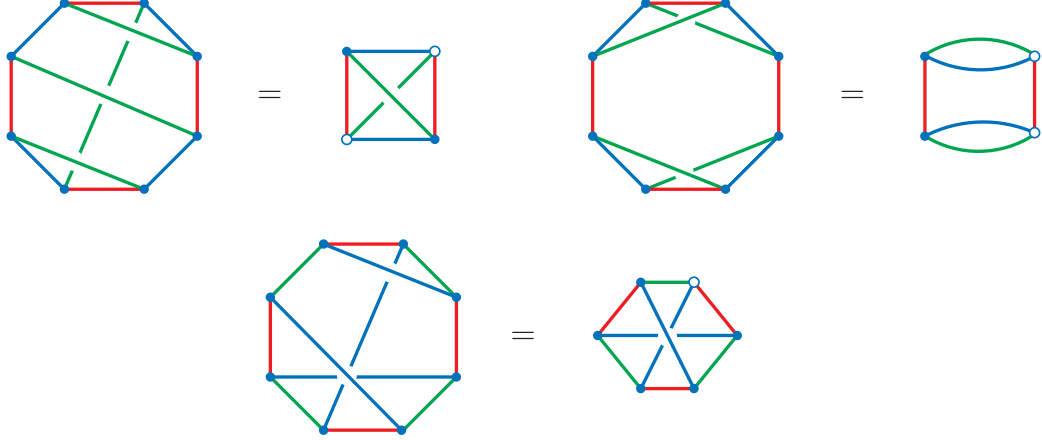


Figure 4.13: The operators which can be represented as lower-order operators with derivative insertions shown by white dots.

As discussed in section 4.4, some of these operators are renormalized by multiple ladder diagrams. For example, the three 4-particle pillow operators, shown in figure 4.6, have dimension $h = 0$ because they are squares of the symmetry charges. Similarly, operators $\mathcal{O}_0^{(a_1 a_2)} \mathcal{O}_0^{(a_1 a_2)}$ related by the equation of motion to column 2 of figure 4.11, are renormalized by double ladders as shown in figure 4.14. One can also see that the correlation function of this operator with four fermionic fields receives a contribution from two ladders as shown in figure 4.15

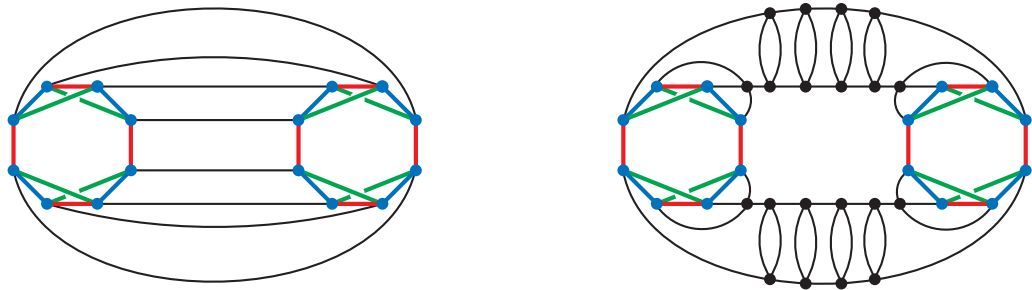


Figure 4.14: An example of an operator renormalized by two ladder diagrams. The diagram with two ladders inserted (right) is of the same order as the diagram with operators connected directly (left). The black dots represent the tetrahedral coupling.

More generally, we may use operators $\mathcal{O}_n^{(a_1 a_2)}$ defined in (4.4.5) to write down the singlet operators

$$\mathcal{O}_{n_1 n_2} = \mathcal{O}_{n_1}^{(a_1 a_2)} \mathcal{O}_{n_2}^{(a_1 a_2)} \quad (4.6.4)$$

renormalized by double ladders,

$$\mathcal{O}_{n_1 n_2 n_3} = \mathcal{O}_{n_1}^{(a_1 a_2)} \mathcal{O}_{n_2}^{(a_2 a_3)} \mathcal{O}_{n_3}^{(a_3 a_1)} \quad (4.6.5)$$

renormalized by triple ladders, and so on. It appears that in the large N limit their scaling dimensions are additive, so that the spectrum of $\mathcal{O}_{n_1 n_2}$ is $h_1 + h_2$, the spectrum of $\mathcal{O}_{n_1 n_2 n_3}$ is $h_1 + h_2 + h_3$, etc., but we postpone a detailed study of the relevant Schwinger-Dyson equations. Here h_i are the eigenvalues which appear in the SYK spectrum; they are the solutions of (4.4.3). The picture of the 12-particle operator which is equivalent by the equation of motion to $\mathcal{O}_0^{(a_1 a_2)} \mathcal{O}_0^{(a_2 a_3)} \mathcal{O}_0^{(a_3 a_1)}$, as well as the analogous operators $\mathcal{O}_0^{(b_1 b_2)} \mathcal{O}_0^{(b_2 b_3)} \mathcal{O}_0^{(b_3 b_1)}$ and $\mathcal{O}_0^{(c_1 c_2)} \mathcal{O}_0^{(c_2 c_3)} \mathcal{O}_0^{(c_3 c_1)}$, are shown in figure 4.17.

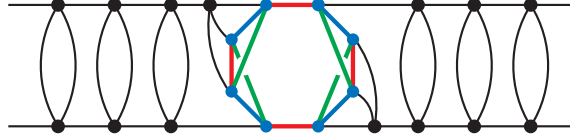


Figure 4.15: A diagram with two ladders contributing to the correlation function $\langle \mathcal{O}_8 \psi \psi \psi \psi \rangle$.

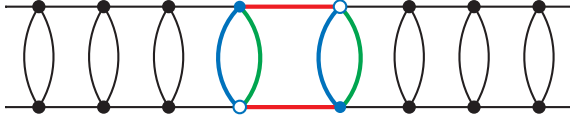


Figure 4.16: Another representation for the same diagram.

We may construct additional operators renormalized by multiple ladders using the operators $\mathcal{O}_n^{[a_1 a_2]}$ (see 4.4.6) in addition to $\mathcal{O}_n^{(a_1 a_2)}$. For example, there is a class of operators $\mathcal{O}_{n_1}^{[a_1 a_2]} \mathcal{O}_{n_2}^{[a_1 a_2]}$ whose scaling dimensions appear to be $h_1 + h_2$, where h_i are the solutions of (4.4.7). Thus, the charges (4.3.6) and their products are not the only exceptions to the $m/4$ rule (since the charges are conserved, we a priori expect their scaling dimension to be zero). In fact, any operator whose diagram contains a bubble subdiagram (i.e. two tensors with a double index contraction) is renormalized by a ladder, and there are as many ladders as there are bubbles. For example, a pillow operator contains two bubbles and is renormalized by two ladders.

Moreover, if we take an operator diagram renormalized by multiple ladders and change one

vertex in the diagram from ψ to $\partial_t \psi$ (blue to white vertex), it will still be renormalized by the same number of ladders. With derivatives we can convert a pillow operator into the second operator in fig. 4.10. It is easy to check that this operator is renormalized by two ladders. Since each of the ladders contains the $h = 2$ zero-mode in its spectrum, and a zero-mode produces a low-temperature enhancement by a factor of βJ [29], we expect the double-ladder to produce an effect of order $(\beta J)^2$. The multi-ladder enhancements by $(\beta J)^n$ seem to be a new effect in the tensor model, which clearly needs to be studied in more detail.

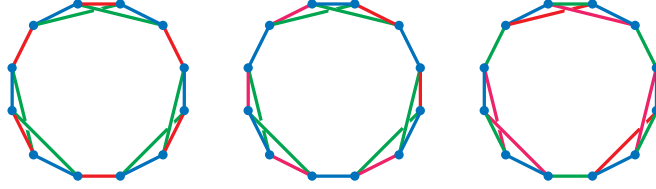


Figure 4.17: Three 12-particle operators of the same topology, which are renormalized by three-ladder diagrams.

To summarize, we find that:

1. The operators containing bubble subgraphs are renormalized with as many ladder diagrams as there are bubble insertions.
2. The operators obtained from operators with bubble subgraphs by inserting derivatives are renormalized by as many ladders as there were bubble insertions in the original diagram.
3. The dimensions of operators which are renormalized with a single ladder are given by the solutions of the conformal kernel equation $g(h) = 1$.
4. The dimensions of the operators which are not renormalized by ladders are multiples of $1/4$.

These results are still far from providing the full information about the singlet spectrum of the $O(N)^3$ tensor quantum mechanics. In particular, we would like to have a more complete understanding of the operators renormalized by multiple ladders and to study their low-temperature contributions. We hope to address these questions elsewhere.

4.7 Some Scaling Dimensions in the Gurau-Witten Model

Let us now consider the $O(N)^6$ symmetric quantum mechanical model [59]. It contains four fermionic rank-3 tensors ψ_A , $A = 0, \dots, 3$, each one transforming in the tri-fundamental representation under a

different subset of the six $O(N)$ groups. The four fermionic tensors and the six $O(N)$ gauge groups may be visualized as the vertices and edges of a tetrahedron [59]. Thus, only two of the fermions transform under a given $O(N)$ symmetry. The Gurau-Witten Hamiltonian is

$$H_{\text{GW}} = -\frac{1}{4}g\psi_0^{abc}\psi_1^{ade}\psi_2^{fde}\psi_3^{fde} . \quad (4.7.1)$$

The model contains bilinear operators of the form $O_A^{c_1c_2} = \psi_A^{abc_1}\psi_A^{abc_2}$. Let us focus on the operators with $A = 0$ and 1 , which transform in the antisymmetric representation of the same $O(N)$ group and can mix with each other:

$$O_+^{c_1c_2} = \psi_0^{abc_1}\psi_0^{abc_2} + \psi_1^{dec_1}\psi_1^{dec_2} , \quad (4.7.2)$$

$$O_-^{c_1c_2} = \psi_0^{abc_1}\psi_0^{abc_2} - \psi_1^{dec_1}\psi_1^{dec_2} . \quad (4.7.3)$$

The operator $O_+^{c_1c_2}$ is the charge of one of the six $O(N)$ symmetries; therefore, its scaling dimension vanishes. The operator $O_-^{c_1c_2}$ has another scaling dimension, h_- . The ladder diagrams contribute to the two-point function $\langle O_-^{c_1c_2}(t_1)O_-^{c_3c_4}(t_2) \rangle$ and we need to derive an appropriate Schwinger-Dyson equation. If we use $\psi_0^{abc_1}\psi_0^{abc_2}$ and $\psi_1^{dec_1}\psi_1^{dec_2}$ as the basis, then the kernel is a 2×2 symmetric matrix with zeros on the diagonal; hence, the two eigenvalues are equal and opposite. To fix the normalization, we note that the two functions $g_{\pm}(h)$ are proportional to $\tilde{g}(h)$, which is given in (4.4.7). Therefore, $g_+(h) = \tilde{g}(h)$ and $g_-(h) = -\tilde{g}(h)$. The spectrum of solutions to $g_+(h) = 1$ indeed includes $h = 0$ corresponding to the conserved charge. The lowest solution to $g_-(h) = 1$ is $h_- \approx 2.33$; this is the scaling dimension of operator $O_-^{c_1c_2}$. Thus, there are three quartic ‘‘pillow operators’’ made out of ψ_0 and ψ_1 : $O_+^{c_1c_2}O_+^{c_1c_2}$ of dimension 0, $O_+^{c_1c_2}O_-^{c_1c_2}$ of dimension h_- , and $O_-^{c_1c_2}O_-^{c_1c_2}$ of dimension $2h_-$. The third operator is the only pillow operator present in the gauged model where $O_+^{c_1c_2}$ is set to zero. Its dimension $2h_- \approx 4.66$ makes it very irrelevant; we find 6 pillow operators with this dimension, corresponding to the presence of 6 different $O(N)$ groups.

We may also study the bilinear singlet operators like

$$O_-^n = \psi_0^{abc}\partial_t^{2n+1}\psi_0^{abc} - \psi_1^{dec}\partial_t^{2n+1}\psi_1^{dec} . \quad (4.7.4)$$

For $n = 0$ this operator vanishes after the use of equations of motion, but it is non-trivial for $n = 1, 2, \dots$. To calculate the scaling dimensions of these operators using the S-D equations we note

that the kernel is the SYK kernel,

$$K_{\text{SYK}}(t_1, t_2; t_3, t_4) = -\frac{3}{4\pi} \frac{\text{sgn}(t_1 - t_3) \text{sgn}(t_2 - t_4)}{|t_1 - t_3|^{1/2} |t_2 - t_4|^{1/2} |t_3 - t_4|}, \quad (4.7.5)$$

times a 4×4 matrix with zeros on the diagonal, and all the off-diagonal elements equal to the same value B . To determine B , we note that the kernel corresponding to the eigenvector $(1, 1, 1, 1)$ with eigenvalue $3B$ should exactly equal the SYK kernel. This means that $B = 1/3$, which gives the spectrum of the SYK model determined by $g(h) = 1$ (see 4.4.3). The three eigenvectors $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 1, -1)$ have eigenvalue $-B = -1/3$; thus, the spectrum of corresponding operators is determined by

$$-\frac{1}{3}g(h) = 1. \quad (4.7.6)$$

The solutions to this equation are shown in figure 4.18.⁴ There is a series of solutions that lie slightly below $2n + \frac{3}{2}$, for $n = 1, 2, 3, \dots$ and approach it at large n . In other words, they lie slightly below the naive dimensions of operators O_-^n . For $n = 1$ the numerical value is 3.39, which is close to 3.5. There is also an exact solution with $h = 1$, whose interpretation is not completely clear.

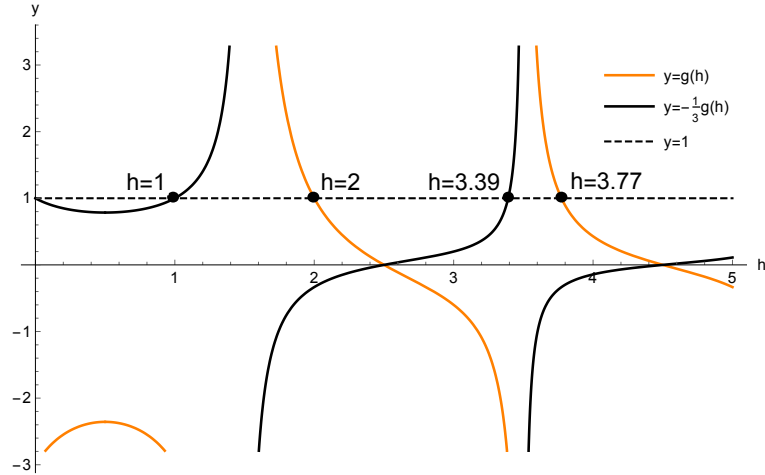


Figure 4.18: Plot of the IR dimensions of the bilinear singlet operators in the GW model.

The dimensions of operators O_-^n that we find are the same as in the Gross–Rosenhaus “generalized SYK model” [77] for $q = 4$. In particular, the $h = 1$ solution is present in that case as well, and the corresponding operator decouples. The Gross–Rosenhaus model that corresponds to the colored

⁴We may decompose the $O(N)^6$ invariant operators into irreducible representations of the symmetry group of the tetrahedron, which is isomorphic to S_4 . Each solution to (4.7.6) corresponds to 3 operators belonging to a degree 3 representation of S_4 .

tensor model has $f = 4$, i.e. it contains four flavors of Majorana fields, χ_a^i , $a = 1, \dots, 4$. Its Hamiltonian may be written as

$$H = J_{ijkl} \chi_i^1 \chi_j^2 \chi_k^3 \chi_l^4, \quad (4.7.7)$$

where J_{ijkl} are random couplings. The operators which are analogous to O_-^n are $\chi_i^1 \partial_t^{n+1} \chi_i^1 - \chi_j^2 \partial_t^{n+1} \chi_j^2$. The $n = 0$ operator vanishes by the equation of motion for any value of J_{ijkl} , which appears to explain the decoupling of the $h = 1$ mode.

4.8 Counting singlet operators in $d = 1$

In this section we proceed to do the singlet operator counting in the $O(N)^3$ quantum mechanics more systematically. We employ the technique used in [104, 105] to find the partition function and free energy of gauge theory. In our case, we will see that the free energy diverges wildly, but nevertheless this procedure allows to count the operators in the gauged or ungauged fermionic and scalar theories.

We work in the one-dimensional spacetime with fields living in the tri-fundamental representation of $O(N)_1 \times O(N)_2 \times O(N)_3$, in the limit of $N \rightarrow \infty$. We will mainly address the case of the free tensor model, which describes the UV fixed point, but also make comments about the IR theory. The partition function may be written in the form:

$$\mathcal{Z} = \sum_{\mathcal{O}_i} x^{h_i}, \quad x \equiv e^{-\beta}, \quad (4.8.1)$$

where $\mathcal{O}_i$ are all operators in the theory which are singlets under $O(N)^3$. Here h_i are the conformal dimensions, so in the UV this partition function is

$$\mathcal{Z} = \sum_k n_k x^{k h_{UV}}, \quad (4.8.2)$$

where k is the number of fields comprising an operator and n_k is the number of admissible operators for each k . In what follows we call k the order of an operator. For the fermionic model $h_{UV} = (d-1)/2$, and for bosonic it is $(d-2)/2$.

The partition function counts all operators including the disconnected ones. To restrict ourselves exclusively to the connected operators, we have to compute the single-sum partition function defined

as:

$$\log \mathcal{Z}(x) = \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}_{\text{s.s.}}(x^m). \quad (4.8.3)$$

To find $\mathcal{Z}_{\text{s.s.}}$ explicitly, we use an elegant formula from [105]:

$$\mathcal{Z}_{\text{s.s.}}(x) = \log \mathcal{Z}(x) + \sum_{m \in \Omega} (-1)^{\nu_m} \frac{1}{m} \log \mathcal{Z}(x^m). \quad (4.8.4)$$

Here m belongs to the set of square-free integers $\Omega = \{2, 3, 5, 6, 7, 10, 11, 13, \dots\}$:

$$m = \prod_{i=1}^{\nu_m} p_i, \quad p_i \text{ prime}. \quad (4.8.5)$$

Our goal in this section is to find the single-sum partition function for the scalar and fermionic tensor models. The partition function for the scalar theory in the UV with one group can be found as [102, 103, 104]:

$$\mathcal{Z}^S = \int dM \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} z_{S,d}(x^m) \chi(M^m) \right), \quad (4.8.6)$$

and for the fermionic theory it is:

$$\mathcal{Z}^F = \int dM \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} z_{F,d}(x^m) \chi(M^m) \right), \quad (4.8.7)$$

with M in the symmetry group and $\chi(M)$ being the character of the desired representation. In our case, we substitute:

$$M \rightarrow M_1 M_2 M_3, \quad \chi(M) \rightarrow \chi(M_1) \chi(M_2) \chi(M_3), \quad M_i \in O(N)_i \quad (4.8.8)$$

and take $\chi(M) = \text{tr } M$.

The single-letter partition functions for scalars and (Majorana) fermions correspondingly are as follows:

$$z_{S,d}(x) = \frac{x^{\frac{d}{2}-1}(1+x)}{(1-x)^{d-1}}, \quad (4.8.9)$$

$$z_{F,d}(x) = \frac{2^{\lfloor \frac{d}{2} \rfloor} x^{\frac{d-1}{2}}}{(1-x)^{d-1}}. \quad (4.8.10)$$

To find $\mathcal{Z}$, we will need the integrals of characters of $O(N)$ [105]:

$$\int dM \prod_l (\text{tr } M^l)^{a_l} = \prod_l \begin{cases} l \text{ odd, } a_l \text{ even} & (2l)^{a_l/2} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{a_l}{2} + \frac{1}{2}\right), \\ l \text{ even} & \sum_{k=0}^{a_l/2} \binom{a_l}{2k} (2l)^k \frac{1}{\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right). \end{cases} \quad (4.8.11)$$

In the next chapter, we first find partition functions for both the fermionic and scalar $d = 1$ models without the constraint that the charges (4.3.6) vanish. Then, to find the partition function for the operators in the gauged model, we subtract the contribution from the operators containing $O(N)$ charge, or a “bubble” subdiagram (4.3.6) (see fig. 4.5). Such operators should vanish in the gauged version of quantum mechanics.

4.8.1 Fermions

The single-letter partition function for real fermions $z_{F,d}$ is not well defined in one dimension. This reflects the divergence of the partition function (and hence free energy). To regularize it, we formally proceed in $(1 + 2\epsilon)$ dimension and neglect all the terms proportional to ϵ in the single-letter partition function; in other words, we simply take:

$$z_{F,1+2\epsilon} = x^\epsilon. \quad (4.8.12)$$

We can justify this choice as follows. The single-letter partition function counts all local operators containing one field ψ^{abc} with any number of derivatives. In our case, the only such operator is ψ^{abc} : since $\partial_t \psi^{abc}$ vanishes by equations of motion in the free theory, all the operators with higher derivatives will vanish too.

In other words, in the fermionic case we are counting only the operators made of fermions without derivatives. We can think of this as operator counting in a $d = 0$ model (for a review see [107]), but with the Fermi statistics imposed.

Computing $\mathcal{Z}$ and using (4.8.7), (4.8.11), we find to first several orders in x :

$$\mathcal{Z}^F = 1 + 4x^{4\epsilon} + 70x^{8\epsilon} + 116x^{10\epsilon} + 3062x^{12\epsilon} + 24788x^{14\epsilon} + 409869x^{16\epsilon} + \dots \quad (4.8.13)$$

From this we can find the single-sum partition function, which counts connected operators:

$$\mathcal{Z}_{\text{s.s.}}^F = 4x^{4\epsilon} + 60x^{8\epsilon} + 116x^{10\epsilon} + 2802x^{12\epsilon} + 24324x^{14\epsilon} + 396196x^{16\epsilon} + \dots \quad (4.8.14)$$

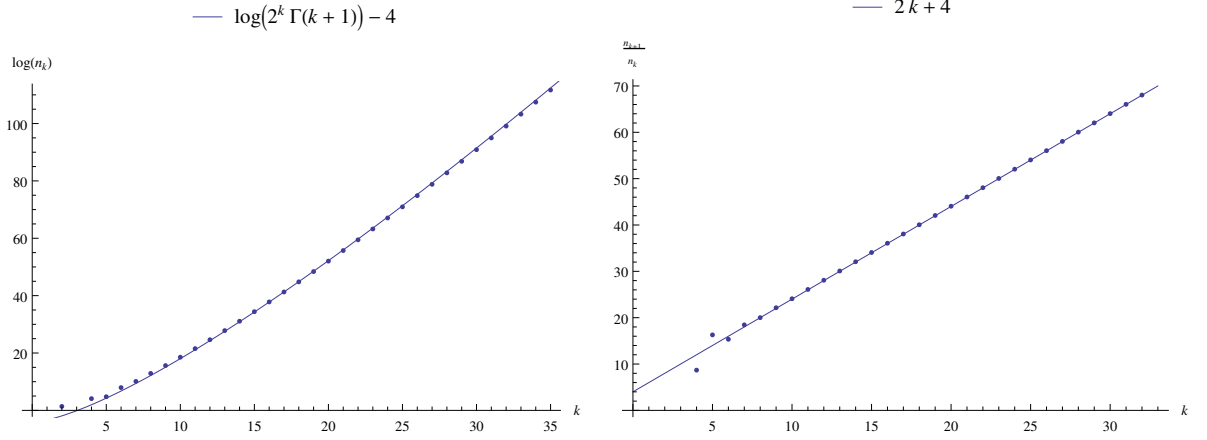


Figure 4.19: Logarithm of the number of allowed $(2k)$ -particle fermionic operators as a function of k . We see that the number of operators grows like $\sim k!2^k$.

The order $2k$ in $x^{2k\epsilon}$ gives the number of fermions in the operator. So we see there are four fermion operators: one tetrahedron and three differently colored pillows (see figure 4.6). Note that, although we employed a gauged theory to count these operators, the pillows and other operators containing $O(N)$ charges are still present. At the sixth order, there are no operators because of the Fermi statistics as we noticed before, but at order 8 there are 60 operators.

The number of $2k$ -particle operators grows roughly as (see fig. 4.19):

$$n_{2k} \sim 2^k k! \quad (4.8.15)$$

To count operators in the gauged model where the vanishing of $O(N)$ charges (4.3.6) is imposed, we have to disregard the operators containing their insertions, i.e. the “bubble” subgraphs. In order to do that, we subtract the operators having the same quantum numbers as a bubble in the exponent of (4.8.7). Each $O(N)$ charge (4.3.6) is antisymmetric in its two indices, which means that it lives in the representation $(N \otimes N)_{\text{antisym}}$ with the character:

$$\chi_A(M) \equiv \chi_{(N \otimes N)_{\text{antisym}}}(M) = \frac{1}{2} \left((\text{tr } M)^2 - \text{tr } M^2 \right). \quad (4.8.16)$$

The bubble is a bosonic operator and its conformal dimension in the UV is 2ϵ . Bringing it all

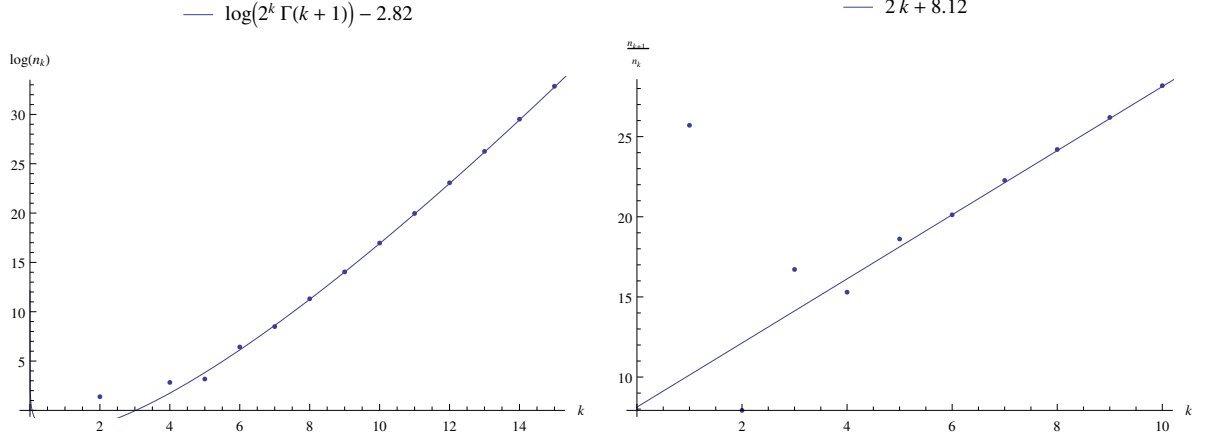


Figure 4.20: Left: The logarithm of the number of $(2k)$ -particle operators n_{2k} in the model where $O(N)^3$ symmetry is gauged. The asymptotic of the number of operators is roughly the same as in the ungauged theory. Right: the ratio n_{2k+2}/n_{2k} plotted against k . The linear behaviour clearly indicates $\sim 2^k k!$ growth.

together, we find that the partition function for operators in the gauge theory is:

$$\mathcal{Z}^{F(\text{gauge})} = \int dM_1 dM_2 dM_3 \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \left((-1)^{m+1} x^{m\epsilon} \chi(M_1) \chi(M_2) \chi(M_3) - x^{2m\epsilon} (\chi_A(M_1) + \chi_A(M_2) + \chi_A(M_3)) \right) \right). \quad (4.8.17)$$

The single-sum partition function for the gauge theory then is as follows:

$$\mathcal{Z}_{\text{s.s.}}^F(\text{gauge}) = x^{4\epsilon} + 17x^{8\epsilon} + 24x^{10\epsilon} + 617x^{12\epsilon} + 4887x^{14\epsilon} + 82466x^{16\epsilon} + \dots \quad (4.8.18)$$

We see that at the fourth order we are left with one operator; namely, the tetrahedron. At the eighth order we see 17 operators, as we already found in section 4.5.1 via explicit construction (see fig. 4.11) We have computed the single-sum partition function up to order 30, and the result matches the same factorial growth as in the model where the $O(N)^3$ symmetry is not gauged (see fig. 4.20).

Finally, let us comment on the IR theory, where we believe there is similarly rapid growth of the number of operators as a function of the conformal dimension. Since for the majority of $2k$ -particle operators the large N IR dimension is $h = k/2$, in view of the result (4.8.15) we expect that the number of operators of dimension h to grow as $\Gamma(2h+1)$, up to an exponential prefactor.

4.8.2 Bosons

We can also count the allowed operators in the scalar theory. Proceeding in the same fashion, we define single-letter partition function in $(1 + 2\epsilon)$ dimensions as follows:

$$z_{S,1+2\epsilon} = x^{-\frac{1}{2}+\epsilon}(1+x), \quad (4.8.19)$$

where $-\frac{1}{2} + \epsilon$ is the dimension of the scalar field. The partition function is:

$$\begin{aligned} \mathcal{Z}^S = & 1 + x^{2\epsilon} (x^{-1} + 1 + x) + x^{4\epsilon} (5x^{-2} + 5x^{-1} + 14 + 5x + 5x^2) \\ & + x^{6\epsilon} (16x^{-3} + 34x^{-2} + 101x^{-1} + 108 + 101x + 34x^2 + 16x^3) + \dots \end{aligned} \quad (4.8.20)$$

The single-sum partition function, which includes the operators with bubble insertions, is:

$$\begin{aligned} \mathcal{Z}_{s.s.}^S = & x^{2\epsilon} (x^{-1} + 1 + x) + x^{4\epsilon} (4x^{-2} + 4x^{-1} + 12 + 4x + 4x^2) \\ & + x^{6\epsilon} (11x^{-3} + 25x^{-2} + 79x^{-1} + 86 + 79x + 25x^2 + 11x^3) + \dots \end{aligned} \quad (4.8.21)$$

In the second order we have operators $\phi^{abc}\phi^{abc}$, $\phi^{abc}\partial_t\phi^{abc}$, and $\partial_t\phi^{abc}\partial_t\phi^{abc}$. In the fourth order, we find the pillows and tetrahedra with various insertions of ∂_t . This partition function also diverges at $\epsilon \rightarrow 0$ and displays the factorial growth of the number of operators with their order.

To count operators in the gauged theory, we once again have to take care of the subgraphs corresponding to the gauge group charge. For a scalar theory, the gauge charge operator is:

$$Q^{a_1 a_2} = \phi^{a_1 bc} \overleftrightarrow{\partial}_t \phi^{a_2 bc}. \quad (4.8.22)$$

This operator lives in the adjoint representation, just like the gauge field. Its dimension is $2\epsilon = (-\frac{1}{2} + \epsilon) + (\frac{1}{2} + \epsilon)$. The character of the adjoint representation is:

$$\chi_{\text{adj}}(M) = \frac{1}{2} \left((\text{tr } M)^2 - \text{tr } M^2 \right). \quad (4.8.23)$$

Taking all this into account, we write the partition function as:

$$\mathcal{Z}^{S(\text{gauge})} = \int dM_1 dM_2 dM_3 \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \left((x^{-\frac{m}{2} + \epsilon m} + x^{\frac{m}{2} + \epsilon m}) \chi(M_1) \chi(M_2) \chi(M_3) \right. \right. \\ \left. \left. - \chi_{\text{adj}}(M_1) x^{2m\epsilon} - \chi_{\text{adj}}(M_2) x^{2m\epsilon} - \chi_{\text{adj}}(M_3) x^{2m\epsilon} \right) \right). \quad (4.8.24)$$

To the first six orders, the partition function reads as:

$$\mathcal{Z}^{S(\text{gauge})} = 1 + x^{2\epsilon} (x^{-1} + 1 + x) + x^{4\epsilon} (5x^{-2} + 5x^{-1} + 11 + 5x + 5x^2) \\ + x^{6\epsilon} (16x^{-3} + 34x^{-2} + 77x^{-1} + 84 + 77x + 34x^2 + 16x^3) + \dots \quad (4.8.25)$$

The single-sum partition function, which counts only the operators with connected diagrams, is as follows:

$$\mathcal{Z}_{\text{s.s.}}^{S(\text{gauge})} = x^{2\epsilon} (x^{-1} + 1 + x) + x^{4\epsilon} (4x^{-2} + 4x^{-1} + 9 + 4x + 4x^2) \\ + x^{6\epsilon} (11x^{-3} + 25x^{-2} + 58x^{-1} + 65 + 58x + 25x^2 + 11x^3) + \dots \quad (4.8.26)$$

The first term in this expression corresponds to the operators $\phi^{abc}\phi^{abc}$, $\phi^{abc}\partial_t\phi^{abc}$, and $\partial_t\phi^{abc}\partial_t\phi^{abc}$ (the second of these operators is a total derivative; such descendant operators are included in the counting). The number 11 in the third term corresponds to all the six-particle graphs discussed in Section 4.5. Now the number of operators containing a string of $2k$ scalars is approximately

$$n_{2k} \sim 2^{2k} \times 2^k k! \quad (4.8.27)$$

Compared to the fermionic case 4.8.15 we have an additional factor of 2^{2k} . As we will see in the next section, for $d = 0$ the leading asymptotic for the number of operators is the same for scalars and fermions. Therefore, the factor 2^{2k} comes from distributing the time derivatives ∂_t among $2k$ fields. Since in the free theory $\partial_t^2\phi^{abc} = 0$, each of the $2k$ fields may be acted on by one or no derivatives. This indeed contributes a factor of 2^{2k} .

4.9 Counting the invariants in $d = 0$

Here we use methods similar to those in the previous section to discuss the counting of invariants in the $d = 0$ model which is simply an integral over the tensor. The construction and counting of such

invariants, which are made out of products of tensors with all indices contracted, has been addressed in [107, 108, 109, 110, 111, 112]. These papers primarily discuss the complex bosonic rank- r tensor models which possess $U(N)^r$ symmetry. We will first consider the bosonic rank-3 tensor model with $O(N)^3$ symmetry and perform the counting using the methods developed in [104, 105]. The model of a real fermionic tensor ψ^{abc} does not work in $d = 0$: since the $O(N)^3$ invariant $\psi^{abc}\psi^{abc}$ vanishes, it is impossible to write down a Gaussian integral. One can write down models of complex fermionic tensors in $d = 0$, but we won't study them here. We will address the bosonic rank-3 symmetric traceless and antisymmetric tensors in subsection 4.9.1, and the bosonic complex tensors with $U(N)^3$ and $U(N)^2 \times O(N)$ symmetries in subsection 4.9.2.

The single-letter partition function counts all the invariants containing one field. In our case the only such operator is ϕ^{abc} , so the single-letter partition function is:

$$z_{S,0}(x) = x. \quad (4.9.1)$$

The invariants in this case are given by the diagrams with $2k$ vertices and three edges of different colors meeting at each vertex. Thus, the invariants are isomorphic to the Feynman diagrams in the theory of three scalar fields with interaction $\varphi_1\varphi_2\varphi_3$. Every edge of the diagram is assigned one of the three colors, and every vertex joins the edges of three different colors. This is a non-trivial condition; for example, one-particle reducible graphs cannot be colored in this way. We consider different colorings of the diagrams as different invariants, so each topology can enter multiple times if there are several distinct ways to color it.

Using (4.8.7), we find the full partition function:

$$\mathcal{Z}^0 = \int dM_1 dM_2 dM_3 \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} x^m \chi(M_1^m) \chi(M_2^m) \chi(M_3^m) \right), \quad (4.9.2)$$

where we have used the character of a tri-fundamental representation (4.8.8). Taking this integral and using (4.8.11), we find in the first several orders:

$$\mathcal{Z}^0 = 1 + x^2 + 5x^4 + 16x^6 + 86x^8 + 448x^{10} + 3580x^{12} + 34981x^{14} + \dots \quad (4.9.3)$$

This partition function counts all the invariants, including the disconnected ones. To remove the latter, we compute the single-sum partition function using (4.8.4):

$$\mathcal{Z}_{\text{s.s.}}^0 = x^2 + 4x^4 + 11x^6 + 60x^8 + 318x^{10} + 2806x^{12} + 29359x^{14} + \dots \quad (4.9.4)$$

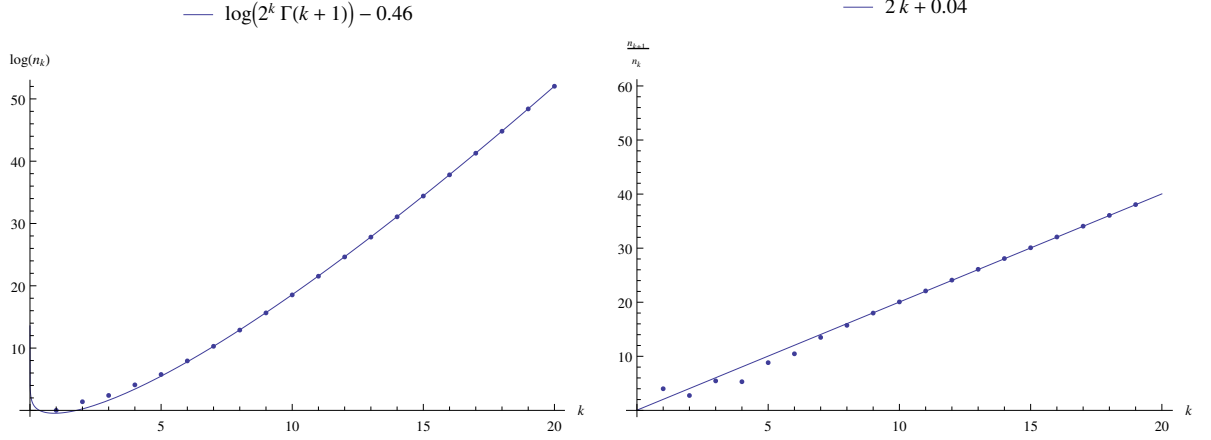


Figure 4.21: Logarithm of the number of invariants with $2k$ scalars as a function of k . The number grows as $\sim k!2^k$.

The only two-scalar invariant is $\phi^{abc}\phi^{abc}$. The four four-scalar invariants are the three inequivalent pillows and the tetrahedron, shown in figure 4.6. The eleven six-scalar invariants are the ones shown in fig. 4.7.

The number of invariants made out of $2k$ fields grows asymptotically as (see fig.4.21):

$$n_{2k} \sim 2^k k! \quad (4.9.5)$$

We can find this asymptotic from an analytic estimate. The key observation is that the integral (4.8.11) grows factorially as $(a_l/2)!$ for large a_l , while only as a power $l^{a_l/2}$ for large l . Besides, for large a_l there is no difference in the leading order between odd and even l . Therefore, the leading contribution to x^{2k} will come simply from the $m = 1$ term:

$$n_{2k} \sim \frac{1}{(2k)!} \int dM_1 dM_2 dM_3 (\chi(M_1)\chi(M_2)\chi(M_3))^{2k} = \frac{1}{(2k)!} (2^k \Gamma(k + 1/2))^3 \sim 2^k k! \quad (4.9.6)$$

Since the dominant term originates only from $m = 1$ term, the same estimate is valid for the fermions.

4.9.1 Symmetric traceless and antisymmetric tensors

Let us also discuss the counting of invariants in models with a single $O(N)$ symmetry, where we will consider the tensors which are either symmetric traceless or fully antisymmetric. Such models with the tetrahedral interactions were recently studied in [85], where evidence was provided that they

have melonic large N limits. The full partition function is

$$\mathcal{Z} = \int dM \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} x^m \chi(M^m) \right), \quad (4.9.7)$$

where for the 3-index symmetric traceless representation the character in the large N limit is ⁵

$$\chi^+(M) = \frac{1}{6}(\text{tr } M)^3 + \frac{1}{2} \text{tr } M \text{tr } M^2 + \frac{1}{3} \text{tr } M^3 - \text{tr } M. \quad (4.9.8)$$

For the fully antisymmetric representation the character is

$$\chi^-(M) = \frac{1}{6}(\text{tr } M)^3 - \frac{1}{2} \text{tr } M \text{tr } M^2 + \frac{1}{3} \text{tr } M^3. \quad (4.9.9)$$

In the symmetric traceless case, the partition function is found to be

$$\mathcal{Z}^+ = 1 + x^2 + 3x^4 + 9x^6 + 32x^8 + 135x^{10} + 709x^{12} + \dots \quad (4.9.10)$$

Extracting the single-sum expression, we find

$$\mathcal{Z}_{\text{s.s.}}^+ = x^2 + 2x^4 + 6x^6 + 20x^8 + 91x^{10} + 509x^{12} + \dots \quad (4.9.11)$$

The numbers of $O(N)$ invariants made of $2k$ fields are the same as the numbers of connected tadpole-free vacuum diagrams in the ϕ^3 theory (here the edges have only one color). They are smaller than the corresponding numbers in (4.9.4) referring to the $O(N)^3$ theory. For example, at order 4 we now have only 2 distinct invariants: in addition to the tetrahedron there is only one pillow, since there are no distinct colorings of it. For large k the number of invariants can be estimated similarly to the tri-fundamental case (4.9.6). Once again, the term with $m = 1$ dominates. Moreover, out of the four terms in (4.9.8), $(\text{tr } M)^3/6$ gives the biggest contribution. Therefore,

$$n_{2k}^{\pm} \sim \frac{1}{(2k)!6^{2k}} \int dM (\text{tr } M)^{6k} \sim \left(\frac{3}{2} \right)^k k! \quad (4.9.12)$$

where we used the integrals (4.8.11).

Since $(\text{tr } M)^3/6$ dominates, the same asymptotic formula is valid for the 3-index antisymmetric

⁵The more complicated expression at finite N may be extracted from eq. (2.4) of [85].

case. Here the partition function is found to be

$$\mathcal{Z}^- = 1 + x^2 + 3x^4 + 7x^6 + 24x^8 + 86x^{10} + 426x^{12} + \dots, \quad (4.9.13)$$

and the single-sum partition function is

$$\mathcal{Z}_{\text{s.s.}}^- = x^2 + 2x^4 + 4x^6 + 14x^8 + 54x^{10} + 298x^{12} + \dots. \quad (4.9.14)$$

4.9.2 Complex 3-Tensors

Let us now consider the complex 3-tensors with $U(N)^3$ or $U(N)^2 \times O(N)$ symmetries. The latter symmetry is particularly interesting because it is preserved by the tetrahedral interaction $\phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \bar{\phi}^{a_2 b_2 c_1}$. This means that there are interacting melonic theories with the $U(N)^2 \times O(N)$ symmetry [117, 119, 60].

In the $U(N)^3$ case we have the fields ϕ^{abc} and $\bar{\phi}^{abc}$, which are in the tri-fundamental representations $N \times N \times N$ and $\bar{N} \times \bar{N} \times \bar{N}$ respectively. The partition function reads:

$$\mathcal{Z}^{U(N)^3} = \int dM_1 dM_2 dM_3 \exp \left(\sum_{m=1}^{\infty} \frac{z(x^m)}{m} (\chi(M_1^m) \chi(M_2^m) \chi(M_3^m) + \bar{\chi}(M_1^m) \bar{\chi}(M_2^m) \bar{\chi}(M_3^m)) \right). \quad (4.9.15)$$

It is straightforward to compute it using the following large N result[105]:

$$\int dM \prod_{l \geq 1} (\text{tr } M^l)^{a_l} (\text{tr } \bar{M}^l)^{b_l} = \prod_{l \geq 1} l^{a_l} a_l! \delta_{a_l, b_l}. \quad (4.9.16)$$

For the scalar we take $z_{S,0}(x) = x$ and find

$$\mathcal{Z}^{U(N)^3} = 1 + x^2 + 4x^4 + 11x^6 + 43x^8 + 161x^{10} + \dots. \quad (4.9.17)$$

This expansion matches the results obtained in [112] using group-theoretic methods. Extracting from $\mathcal{Z}$ the single-sum partition function, we find

$$\mathcal{Z}_{\text{s.s.}}^{U(N)^3} = x^2 + 3x^4 + 7x^6 + 26x^8 + 97x^{10} + \dots. \quad (4.9.18)$$

The coefficient 3 of x^4 is in agreement with the fact that the tetrahedron invariant is not allowed by

the $U(N)^3$ symmetry. Only the 3 pillow invariants are allowed, and they look like:

$$\phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_1 c_2} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_2 b_2 c_1} , \quad \phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_2 c_1} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_2 b_1 c_2} , \quad \phi^{a_1 b_1 c_1} \bar{\phi}^{a_2 b_1 c_1} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_1 b_2 c_2} . \quad (4.9.19)$$

The asymptotic number of operators can be estimated as follows. As in the $O(N)$ case, the integral (4.9.16) grows factorially in a_l and only as a power in l . It means that the term with $m = 1$ again dominates. Besides, to get a non-zero answer we need to extract the term with an equal number of $\chi(M_i)$ and $\bar{\chi}(M_i)$. Therefore,

$$n_{2k}^{U(N)^3} \sim \binom{2k}{k} \frac{1}{(2k)!} \int dM_1 dM_2 dM_3 \prod_{i=1}^3 \chi(M_i)^k \bar{\chi}(M_i)^k \sim k! \quad (4.9.20)$$

In the $U(N)^2 \times O(N)$ case we have representations $N \times N \times N$ and $\bar{N} \times \bar{N} \times N$, so that

$$\mathcal{Z}^{U(N)^2 \times O(N)} = \int dM_1 dM_2 dM_3 \exp \left(\sum_{m=1}^{\infty} \frac{z(x^m)}{m} (\chi(M_1^m) \chi(M_2^m) + \bar{\chi}(M_1^m) \bar{\chi}(M_2^m)) \chi(M_3^m) \right), \quad (4.9.21)$$

where the matrices M_1, M_2 belong to $U(N)$, while M_3 belongs to $O(N)$. The scalar partition function has the following expansion:

$$\mathcal{Z}^{U(N)^2 \times O(N)} = 1 + x^2 + 6x^4 + 21x^6 + 147x^8 + 1043x^{10} + \dots \quad (4.9.22)$$

Extracting the single-sum partition function, we find

$$\mathcal{Z}_{\text{s.s.}}^{U(N)^2 \times O(N)} = x^2 + 5x^4 + 15x^6 + 111x^8 + 821x^{10} + \dots \quad (4.9.23)$$

The coefficient 5 of x^4 is in agreement with the fact that, addition to the tetrahedron invariant, there are 4 pillow invariants allowed by the $U(N)^2 \times O(N)$ symmetry:

$$\begin{aligned} & \phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_1 c_2} \phi^{a_2 b_2 c_1} \bar{\phi}^{a_2 b_2 c_2} , & \phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_1 c_2} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_2 b_2 c_1} , \\ & \phi^{a_1 b_1 c_1} \bar{\phi}^{a_1 b_2 c_1} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_2 b_1 c_2} , & \phi^{a_1 b_1 c_1} \bar{\phi}^{a_2 b_1 c_1} \phi^{a_2 b_2 c_2} \bar{\phi}^{a_1 b_2 c_2} . \end{aligned} \quad (4.9.24)$$

Using the same method as in the $U(N)^3$ case, the asymptotic growth can be found to be

$$n_{2k}^{U(N)^2 \times O(N)} \sim 2^k k! \quad (4.9.25)$$

4.10 The Hagedorn Transition

The special features of the thermodynamics of free theories with fields being tensors of rank $r \geq 3$ under a global symmetry group were recently studied in [105]. It was found that the Hagedorn temperature vanishes in the large N limit as $\sim 1/\log N$. In this section we show it for the models with $O(N)^3$ symmetry.

Schematically, the low temperature expansion of the partition function of a large N tensor model is $\sum_k 2^k k! x^{2k}$, where $-\ln x$ is proportional to β . This power series is divergent and non-Borel summable; therefore, strictly speaking the partition function is not defined for any finite temperature. To illustrate the basic points, we study the large N behavior of the integral (4.9.2) in a standard fashion (it will be convenient to assume that N is even). First of all, for large N there should be no difference between $SO(N)$ and $O(N)$. An $SO(N)$ matrix can always be put in the block-diagonal form with 2×2 blocks corresponding to a rotation by an angle α^i in 2d plane. Including the $SO(N)$ measure [126], the partition function (4.9.2) can be rewritten as:

$$\mathcal{Z} = \int \prod_{r=1}^3 d\alpha_r^i \prod_{i < j}^{N/2} \sin^2 \frac{\alpha_r^i - \alpha_r^j}{2} \sin^2 \frac{\alpha_r^i + \alpha_r^j}{2} \exp \left(8 \sum_{m=1}^{\infty} \frac{z(x^m)}{m} \prod_{r=1}^3 \sum_{i=1}^{N/2} \cos(m\alpha_r^i) \right) = \int [d\alpha] e^{-S_{\text{eff}}}. \quad (4.10.1)$$

Index r labels different $SO(N)_r$ groups and $i, j = 1, \dots, N/2$ go over rotation angles. Also we have introduced a single-letter partition function $z(x)$ to work in more generality. The above equation is valid for scalars, while for fermions we need to include the factor $(-1)^{m+1}$ in front of $z(x^m)$. However, we will see in a moment that for the Hagedorn transition only $m = 1$ term is relevant. Therefore, our main results will be applicable for both cases.

The effective action S_{eff} reads

$$S_{\text{eff}} = -\frac{1}{2} \sum_{r=1}^3 \sum_{i \neq j}^{N/2} \left(\log \sin^2 \frac{\alpha_r^i - \alpha_r^j}{2} + \log \sin^2 \frac{\alpha_r^i + \alpha_r^j}{2} \right) - 8 \sum_{m=1}^{\infty} \frac{z(x^m)}{m} \prod_{r=1}^3 \sum_{i=1}^{N/2} \cos(m\alpha_r^i). \quad (4.10.2)$$

There are three saddle-point equations. One of them is:

$$\sum_{j=1}^{N/2} \left(\cot \frac{\alpha_1^i - \alpha_1^j}{2} + \cot \frac{\alpha_1^i + \alpha_1^j}{2} \right) - 8 \sum_{m=1}^{\infty} z(x^m) \sin(m\alpha_1^i) \sum_{j_2, j_3} \cos(m\alpha_2^{j_2}) \cos(m\alpha_3^{j_3}) = 0. \quad (4.10.3)$$

The other two can be obtained by cyclic permutations of $\alpha_1^i, \alpha_2^i, \alpha_3^i$. Introducing density functions:

$$\rho_r(\alpha) = \frac{2}{N} \sum_{i=1}^{N/2} \delta(\alpha - \alpha_r^i). \quad (4.10.4)$$

The saddle-point equation can be rewritten as:

$$\int_{-\pi}^{\pi} d\alpha'_1 \rho_1(\alpha'_1) \left(\cot \frac{\alpha_1 - \alpha'_1}{2} + \cot \frac{\alpha_1 + \alpha'_1}{2} \right) - 4N \sum_{m=1}^{\infty} z(x^m) \sin(m\alpha_1) \rho_2^m \rho_3^m = 0, \quad (4.10.5)$$

where

$$\rho_r^m = \int_{-\pi}^{\pi} d\alpha \rho_r(\alpha) \cos(m\alpha). \quad (4.10.6)$$

It is natural to assume that because of the cyclic symmetry $\rho_1 = \rho_2 = \rho_3 = \rho(\alpha)$. Moreover, we will assume that ρ is an even function: $\rho(\alpha) = \rho(-\alpha)$. Then the saddle-point equation reads as:

$$2 \int_{-\pi}^{\pi} d\alpha' \rho(\alpha') \cot \frac{\alpha - \alpha'}{2} - 4N \sum_{m=1}^{\infty} z(x^m) \sin(m\alpha) (\rho^m)^2 = 0. \quad (4.10.7)$$

This is the saddle-point equation studied in [105]. There is a Hagedorn transition: for low temperatures when $Nz(x) < 27/16$ the partition function is dominated by the uniform saddle

$$\rho(\alpha) = \frac{1}{2\pi}, \quad \alpha \in [-\pi, \pi]. \quad (4.10.8)$$

And so all ρ^m are zero for $m > 0$. For higher temperatures, the density ρ is not a constant and takes non-zero values only within a smaller interval $[-\alpha_0, \alpha_0]$. Moreover, the transition point itself can be found by assuming that only ρ^1 becomes non-zero. Therefore, the transition happens at $Nz(x) = 27/16$ for both bosons and fermions. For more details see [105, 104].

Consider fermions in $d = 1 + 2\epsilon$. From (4.8.12) we see that in the UV the transition happens at

$$z_{F,1+2\epsilon} = x^\epsilon = \exp(-\beta\epsilon) = \frac{27}{16N}. \quad (4.10.9)$$

In the IR the fermions have dimension $1/4$ for $d = 1$. If most $2k$ -fermion operators have dimension $k/2$, then the transition takes place at:

$$z_{F,IR} = x^{1/4} = \exp(-\beta/4) = \frac{27}{16N}. \quad (4.10.10)$$

Chapter 5

Jackiw–Teitelboim gravity

5.1 Introduction

In this Chapter we go to the gravity side and study correlators in the boundary theory computed in the semiclassical limit of Jackiw–Teitelboim gravity. Although JT gravity is not precisely dual to SYK, it shares many of its interesting features, including exponential growth of out-of-time ordered correlators. As we have seen in Chapter 1, the SYK model has a Schwarzian action term in the infrared limit, and the same term appears in the boundary contribution to dilaton gravity. Considering gravity on a nearly- AdS space, we expect it to be dual to a nearly- CFT , that is a theory in which conformal symmetry is explicitly broken, but holds approximately. This $NAdS/NCFT$ correspondence is the main subject of this Chapter.

In most of the existing literature, one considers operators with fixed dimension in the $NCFT$ which lives on the boundary of the $NAdS$ space. Then in the weak coupling limit, these operators cause negligible back-reaction on the geometry, but they produce interesting quantum fluctuations. Here instead we will consider operators whose dimension scales as the inverse of the coupling parameter. This means that in the weak coupling limit, these operators produce significant back-reaction on the geometry, or more precisely on the dilaton field, since the geometry of the $NAdS$ is really fixed by the equations of motion. For small dimension of the operator, we reproduce the Schwarzian corrections to the two-point function, but for large dimension we find novel behavior. We also analytically continue the two-point function from Euclidean signature in order to compute a real time thermal correlator.

It turns out that the two-point function computed in this way either for large Euclidean distances

or large real time in the case of a thermal correlator has unexpected properties whose origin is not clear. In the Euclidean case, when the length scale in the $NCFT$ goes to infinity, the length of the relevant geodesic that appears in the correlator approaches a finite limit, and the two-point function approaches a limit as well. It suggests that the large dimension operator $\mathcal{O}$ that we consider has a nonzero vacuum expectation value. The interpretation of this fact is obscure.

In the thermal case, the large time limit of a two-point function is small, but finite number. In [127], it was argued that a useful test of the information paradox in black hole physics is to see whether the two-point function decays exponentially for large real time or there are some finite residual correlations even after long time. The size of these fluctuations is expected to be of order $\sim \exp(-cS)$, where S is the entropy and c is a constant. The two-point function after long time is expected to fluctuate wildly, with a characteristic amplitude $\exp(-cS)$. When these fluctuations are averaged, the two-point function becomes an exponentially small constant.

A test of this behavior was performed for the SYK model in [64] and it was shown that the (real part of the) correlator indeed approaches a plateau after a period of exponential descent, reaching a minimum and then rising back to a constant value. Some of these features have been later explored and partly explained in [128]. The plateau is very low, and its height is proportional to $\sim \exp(-N)$.

This effect is of order $O(e^{-1/G_N})$ and is non-perturbative in gravity. However, it may be accessible in a semiclassical treatment [64], [129], and we find a similar phenomenon in our setup. For a two-point function, the exponential decay eventually slows down, and it approaches a plateau at long (real) time. When we identify the parameters of our model with the parameters of the SYK, we find that the long-time limit of the two-point function is also $\sim \exp(-N)$. However, the plateau is reached much sooner than expected on general grounds, which points out that the physics behind it may be different. It is also worth noting that the two-point function found as the exponentiated geodesic length has a shape very similar to what was found in [64] (see fig. 5.17), although this is most likely accidental.

Our setup also allows us to study four-point functions. In particular, we find the out of time ordered thermal four-point function. It is related to a double commutator and serves as a measure of chaotic behavior [69]. We find that the four-point function decays exponentially at first (as can be found from Schwarzian action), then after a relatively short time this decay stops. This can be identified with Ruelle behavior, showing how the system approaches the thermal equilibrium. However, we also find that at long times the four-point function approaches a small but finite value, similarly to the real-time two-point function. To the best of our knowledge, this has not been tested on the SYK side.

We should note that the question of $NAdS$ correlators with back-reaction included has been mentioned in [13] and also in [130]. Also, a similar setup for a Schwarzian two-point function was considered in [33].

5.2 Setup

To set the scene, we introduce the action of two-dimensional gravity in the formulation of Jackiw and Teitelboim [44], [45], [40]. The pure gravity action with the boundary term reads:

$$I = -\frac{1}{16\pi G} \int d^2x \phi \sqrt{g} (R + 2) - \frac{1}{8\pi G} \int_{\partial} \phi_b \sqrt{h} K. \quad (5.2.1)$$

(We omit the term defining the extremal entropy and higher-order terms in ϕ .) The equations of motion for the dilaton set a constant negative curvature:

$$R + 2 = 0. \quad (5.2.2)$$

In a pure anti-de Sitter space this action is topological. However, if we restrict it to a region of AdS_2 , the position of the boundary becomes a non-trivial degree of freedom. Classically, it is set by the boundary condition for the dilaton:

$$\phi|_{\partial} = \phi_b = \text{const}. \quad (5.2.3)$$

We consider massive particles moving in the Jackiw–Teitelboim gravity. With the condition (5.2.2) the action becomes:

$$I = -\frac{\phi_b}{8\pi G} \int_{\partial NAdS} \sqrt{h} K - m \int_{NAdS} ds. \quad (5.2.4)$$

The second integral means that we consider only the part of the worldline lying inside our near- AdS space. Also, from now on we absorb the gravitational constant G in the definition of ϕ_b .

As has been discussed in [43], we can think of this action as a low-energy limit of some unknown theory. The position of the boundary is then a UV cutoff, and we choose to make this cutoff consistent with the equations of motion following from the JT action (5.2.1).

The boundary of the $NAdS$ space is at finite distance from the “center” of the true AdS_2 , and has a finite length. This allows us to study the boundary theory using the gravitational action only. The symmetries of the boundary theory are generated by isometries of AdS_2 . Therefore the theory

does not possess the full conformal symmetry, but, keeping the boundary “close” to the boundary of the true AdS_2 space, we can hope to see a nearly-conformal theory. The “closeness” is measured by the value of the boundary dilaton. In particular, the true conformal theory corresponds to the boundary dilaton being infinitely large. To be more specific and following [43], we define:

$$\phi_b = \frac{\phi_r}{\epsilon}, \quad (5.2.5)$$

where ϕ_r is the renormalized value of the dilaton, and ϵ is a small number measuring how close the $NAdS$ boundary is to the true boundary of AdS_2 . We want the boundary lengths to be finite as $\epsilon \rightarrow 0$, and therefore we rescale quantum mechanical distances as:

$$du_{QM} \equiv \epsilon \cdot du_{AdS}. \quad (5.2.6)$$

When ϵ is small, the extrinsic curvature term in (5.2.4) reduces to a Schwarzian derivative:

$$\phi_b \int \sqrt{h} K \rightarrow \phi_r \int \text{Sch}(t, u) du. \quad (5.2.7)$$

The same Schwarzian term appears in the effective action of the SYK model as the first correction to the conformal answer. This allows us to tentatively identify the parameters of the two theories as:

$$\phi_r^{(JT)} \sim \left(\frac{N}{J} \right)^{(\text{SYK})}. \quad (5.2.8)$$

The factor between the parameters is the function of q in SYK which we are not discussing here. So, we expect the boundary theory to be close to conformal when ϕ_r is large, and the $1/N$ corrections to SYK to correspond to $1/\phi_r$ corrections on the boundary of JT gravity.

We want to study this $NAdS/NCFT$ correspondence in a semiclassical regime with $G \ll 1$, or in our notation $\phi_b \gg 1$. In particular, we use the conventional holographic prescription [8] to find the two-point function of boundary operators.

The two-point function G satisfies the Laplace equation written in terms of the geodesic length (we ignore the angular part of the Laplacian):

$$\left(-\frac{1}{\sinh \ell} \frac{d}{d\ell} \sinh \ell \frac{d}{d\ell} + m^2 \right) G = 0, \quad G \equiv \langle \mathcal{O}_1 \mathcal{O}_2 \rangle. \quad (5.2.9)$$

Here ℓ is the length of the geodesic connecting the two operators (see fig. 5.1). This equation typically

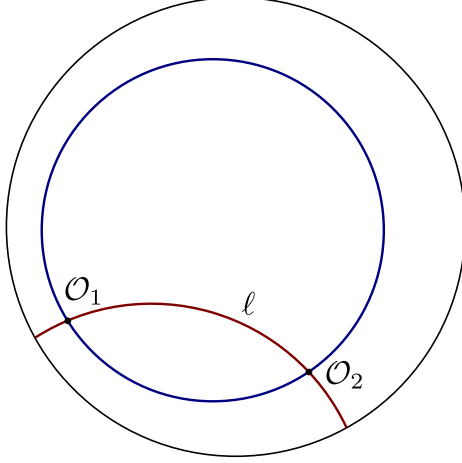


Figure 5.1: Poincaré disc (black) with near- AdS space inside (blue), crossed by a massive particle (red). Two operators $\mathcal{O}_1, \mathcal{O}_2$ belong to a near-conformal theory on the boundary of $NAdS$ space. The mass is small, and the back-reaction absent, therefore the $NAdS$ boundary looks like a full circle.

has an exponentially growing and an exponentially decaying solution, with only the latter making physical sense. For large ℓ , the two-point function is:

$$G(\ell) \sim e^{-\Delta \cdot \ell} \sim \exp\left(-m \int ds\right), \quad \Delta(\Delta - 1) = m^2. \quad (5.2.10)$$

We are interested in particles with large mass, and will assume:

$$\Delta \sim m. \quad (5.2.11)$$

In this prescription takes into account only the second part of the action (5.2.4). The action of extrinsic curvature provides a correction to this result. In Section 5.4.5, we find that this correction is (numerically) small in Euclidean signature, but is significant for real-time correlation functions. But first, we find the two-point function as the exponentiated geodesic length.

In the absence of back-reaction (see fig. 5.1), or with an extremely small mass, this prescription gives usual conformal answer:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle \sim \frac{1}{\sin^{2\Delta} x}. \quad (5.2.12)$$

However, the massive particle creates a jump in the dilaton and distorts the boundary, therefore introducing corrections to this result. This has to be taken into account to compute the length of the geodesic; one cannot compute the length of the geodesic as if one were in an undistorted $NAdS_2$. Our goal is to find the full semiclassical answer for the two-point function, taking this back-reaction

into account.

5.3 Near- AdS boundary

In this Section, we set up the geometry we are working in. We find the boundary of the $NAdS$ space, consistent with the Dirichlet condition for the dilaton:

$$\phi|_{\text{bdry}} = \phi_b. \quad (5.3.1)$$

The classical solution for the dilaton field is found from the pure Jackiw-Teitelboim action (5.2.1). The equations of motion for the metric define the energy-momentum tensor for the dilaton:

$$T_{\mu\nu}^\phi = \frac{1}{8\pi G} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi + g_{\mu\nu} \phi). \quad (5.3.2)$$

In the absence of matter, this energy-momentum tensor vanishes:

$$T_{\mu\nu}^\phi = 0. \quad (5.3.3)$$

This condition is conveniently solved in the embedding coordinates Y_i , which define the AdS_2 inside flat three-dimensional space as:

$$Y = (Y_0, Y_1, Y_2), \quad Y^2 = Y_0^2 - Y_1^2 - Y_2^2 = 1. \quad (5.3.4)$$

In these coordinates, the solution to (5.3.3) is linear:

$$\phi = Z \cdot Y = Z_0 Y_0 - Z_1 Y_1 - Z_2 Y_2. \quad (5.3.5)$$

The Z_i constants are the $SO(1, 2)$ charges of the solution. In what follows, we will sometimes use the word “dilaton” to mean this vector of charges.

We could work in the embedding coordinates, but find it more convenient to switch to the two-dimensional space. There are several conventional representation of the (Euclidean) AdS_2 space. One (perhaps more intuitive) is a Poincaré disc, as on fig. 5.1. We will use a hyperbolic half-plane instead (see fig. 5.2).

We choose the coordinates on the half-plane to be (t, z) , with the boundary at $z = 0$ and the t

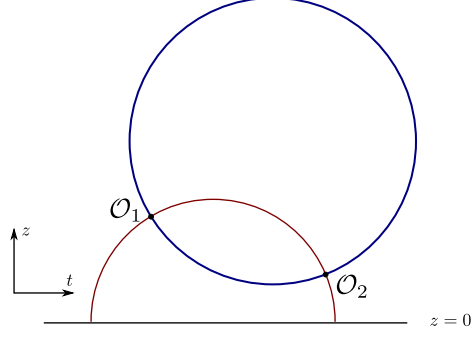


Figure 5.2: When mass is small, the $NAdS$ boundary is a perfect circle.

axis along the boundary. The Euclidean hyperbolic metric is:

$$g = \frac{dt^2 + dz^2}{z^2}. \quad (5.3.6)$$

The embedding coordinates map to (t, z) as:

$$Y_0 = \frac{1 + t^2 + z^2}{2z}, \quad Y_1 = -\frac{t}{z}, \quad Y_2 = \frac{1 - t^2 - z^2}{2z}. \quad (5.3.7)$$

In coordinates (t, z) on the half-plane, the classical solution for the dilaton is:

$$\phi = \frac{1}{2z} \left((Z_0 - Z_2) (t^2 + z^2) - 2Z_1 t + (Z_0 + Z_2) \right). \quad (5.3.8)$$

The boundary of the near- AdS_2 space is fixed by the Dirichlet boundary condition on the dilaton (5.3.1), which looks a circle on the half-plane (see fig. 5.2):

$$\phi = \phi_b \quad \Rightarrow \quad \left(t - \frac{Z_1}{Z_0 - Z_2} \right)^2 + \left(z - \frac{\phi_b}{Z_0 - Z_2} \right)^2 = \frac{\phi_b^2 - Z^2}{(Z_0 - Z_2)^2}. \quad (5.3.9)$$

There are some restrictions on the parameters of the dilaton. First, to make the Schwarzian action positive, the boundary dilaton has to be positive:

$$\phi_b > 0. \quad (5.3.10)$$

The $NAdS_2$ space makes sense only if it lies completely inside the hyperbolic plane, that is, if its boundary does not intersect the true boundary at $z = 0$. This amounts to the requirement that the

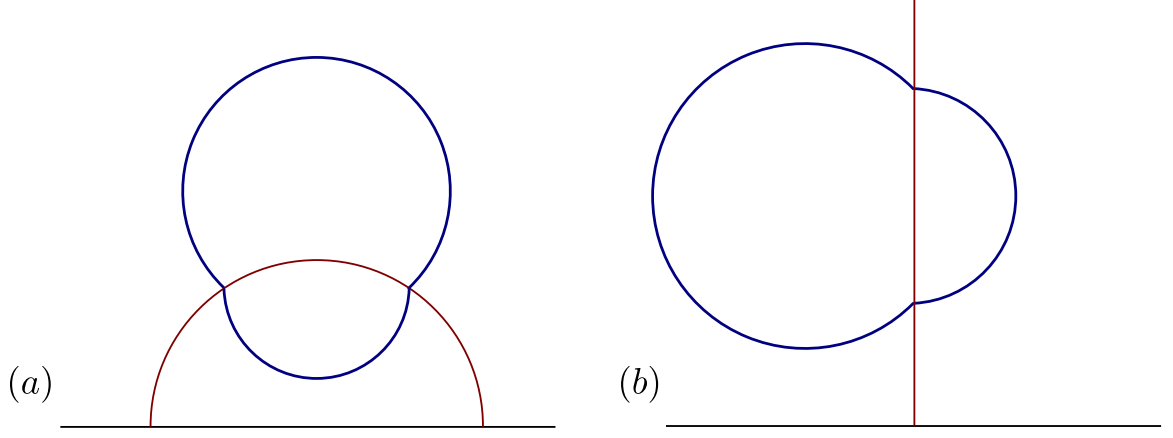


Figure 5.3: Two different gauge choices.

square of the charge vector for the dilaton is positive:

$$Z^2 > 0, \quad (5.3.11)$$

and that the center of the circle (5.3.9) is above the $z = 0$ boundary:

$$Z_0 - Z_2 > 0. \quad (5.3.12)$$

To find correlators in the semi-classical approximation, we introduce a massive particle. On a half plane, its trajectory is generally a half-circle, intersecting the boundary at a right angle (see fig. 5.2). The parameters of the trajectory form another vector of $SO(1, 2)$ charges A . In the embedding coordinates, the trajectory is also given by a linear condition:

$$A \cdot Y = 0. \quad (5.3.13)$$

In the coordinates on the half-plane, this condition reads:

$$(A_0 - A_2)(t^2 + z^2) - 2A_1t + (A_0 + A_2) = 0. \quad (5.3.14)$$

Generally, the radius of the circle is:

$$r^2 = \frac{-A^2}{(A_0 - A_2)^2}. \quad (5.3.15)$$

From this we see that $A^2 < 0$. The $SO(1, 2)$ transformations allow us to rotate the A vector, keeping

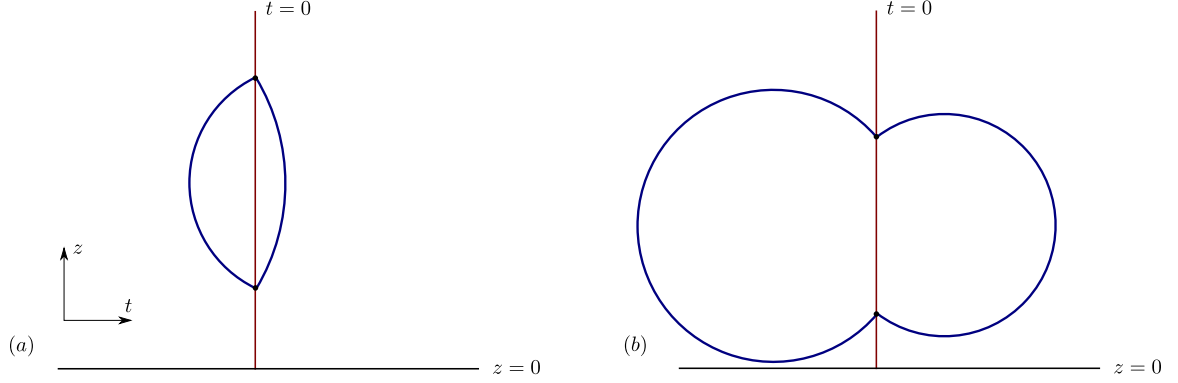


Figure 5.4: Positive mass makes the cusps turn inward (b), negative mass turns them outward (a).

its square invariant. This invariant fixes the mass of the particle:

$$A^2 = -m^2. \quad (5.3.16)$$

The massive particle creates a jump in the parameters of the dilaton. That is, on the other side of the geodesic the dilaton changes to:

$$\phi = Z \cdot Y \quad \Rightarrow \quad \phi = (Z + A) \cdot Y. \quad (5.3.17)$$

This means that the boundary of the $NAdS_2$ space with massive particle inside consists of two arcs meeting at an angle (see fig. 5.3). This angle is conformally invariant, and is zero for massless particles, corresponding to a fully conformal theory on the boundary. For a finite mass, there is cusp. A particle with positive mass draws together pieces of the boundary, creating an inward cusp as on part (b) of fig. 5.4. A particle with negative mass would pull the boundary apart, and creates an outward cusp, as can be seen on part (a). We discuss negative mass in Appendix 5.8.

We can use an $SO(1,2)$ transformation to choose a convenient gauge. With it, we can either make a picture look symmetric, or choose the degenerate case with the geodesic being a straight line, as on fig. 5.3:

$$(a) : A_1 = 0, \quad (b) : A_0 = A_2 = 0. \quad (5.3.18)$$

We will find it easier to use the (b) choice.

Since the particle creates a jump in the parameters of the dilaton, we have to make sure once again that the whole boundary of the near- AdS lies above the true boundary at $z = 0$. To do that,

we can require:

$$Z^2 > 0, \quad (Z + A)^2 > 0, \quad (5.3.19)$$

together with the condition that the centers of the circle segments lie above the boundary:

$$Z_0 - Z_2 > 0, \quad (Z + A)_0 - (Z + A)_2 > 0. \quad (5.3.20)$$

5.4 Euclidean two-point function

From the bulk point of view and in the semi-classical picture, the two-point function depends on the length of the trajectory of the massive particle. Since we work in a part of the anti-de Sitter space, we are interested in the part of the geodesic cut out by the requirement $\phi = \phi_b$ (the part of the dark-red line in fig. 5.5 inside the blue boundary of the $NAdS$ space). This geodesic length is:

$$\ell = \ln \frac{z_+}{z_-}. \quad (5.4.1)$$

However, the boundary quantum mechanics does not know about the trajectory of a particle in the bulk. So from the boundary point of view, the two-point function should be expressed in terms of the distance *along the boundary*, which by definition must be longer than the geodesic distance. In fig. 5.5 it is the length of the blue segment of the circle:

$$u_{12} \sim \int_{-}^{+} \frac{\sqrt{dz^2 + dt^2}}{z} \quad (\text{along the } NAdS \text{ boundary}). \quad (5.4.2)$$

The metric in the boundary theory can differ from the metric inherited from the bulk of AdS_2 by a constant factor. We choose this factor so that when we come close to the boundary of the true AdS , the quantum mechanical length remains finite. Using the ϵ parameter defined in (5.2.5), we define the distance in the boundary theory as:

$$u_{12} \equiv \epsilon \int_{-}^{+} \frac{\sqrt{dz^2 + dt^2}}{z} \quad (\text{along the } NAdS \text{ boundary}). \quad (5.4.3)$$

Accordingly, we rescale the two-point function so that the limit of large dilaton is meaningful. We can do it since the Laplace equation (5.2.9) defines the two-point function up to a constant factor.

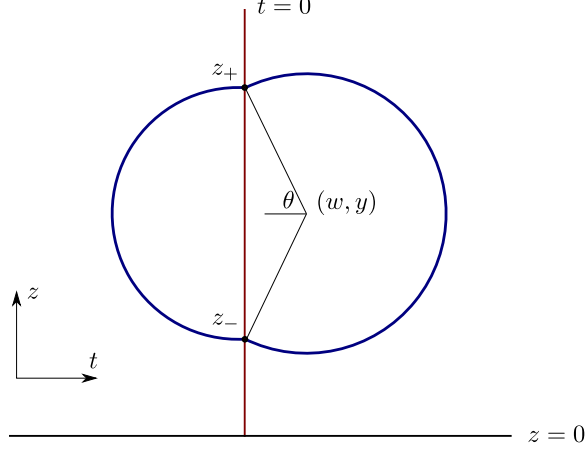


Figure 5.5: Massive particle distorting the boundary of a near- AdS_2 space. Red line is the trajectory of the particle, blue line is two arcs of the $NAdS_2$ boundary, and solid black line is the boundary of the true AdS space.

We want the two-point function to be consistent with OPE at small distances:

$$G = \frac{1}{|u_{12}|^{2\Delta}}, \quad u_{12} \rightarrow 0. \quad (5.4.4)$$

In what follows, we see that it is so if we rescale G as:

$$G = \frac{1}{\epsilon^{2\Delta}} \exp(-\Delta \cdot \ell). \quad (5.4.5)$$

In what follows, it will sometimes be convenient to define the exponentiated distance $\gamma \equiv \epsilon \cdot \exp(\ell/2)$, so that the two-point function is:

$$G = \frac{1}{\gamma^{2\Delta}}. \quad (5.4.6)$$

Our goal is to express the two-point function (5.4.6) in terms of the boundary length (5.4.3). In our choice of the gauge, if the center of this circle is at $(t, z) = (w, y)$ and the radius of the circle is R , this length is:

$$u_{12} = \epsilon \int_{\theta_-}^{\theta_+} \frac{R d\theta}{y - R \sin \theta} = \epsilon \cdot \frac{2R}{\sqrt{1-w^2}} \arccos\left(\frac{wy}{R}\right), \quad (5.4.7)$$

with a similar formula for the length of the other segment:

$$u_{21} = \epsilon \cdot \frac{2R}{\sqrt{1-w^2}} \left(\pi - \arccos\left(\frac{wy}{R}\right) \right). \quad (5.4.8)$$

Here and in what follows u_{12} is the length of the segment on the left-hand side of the geodesic on

fig. 5.5, and u_{21} is the length on the right-hand side.

The sum of the lengths of these two segments is the length of the boundary:

$$L \equiv u_{12} + u_{21}. \quad (5.4.9)$$

The geodesic length in terms of the same variables is given by:

$$\ell = \ln \frac{z_+}{z_-} = \ln \frac{y + \sqrt{R^2 - w^2}}{y - \sqrt{R^2 - w^2}}. \quad (5.4.10)$$

The parameters (y, w, R) of the circle are defined by the $SO(1, 2)$ charges of the dilaton. From (5.3.9), we see that:

$$w = \frac{Z_1}{Z_0 - Z_2}, \quad y = \frac{\phi_b}{Z_0 - Z_2}, \quad R = \frac{\sqrt{\phi_b^2 - Z^2}}{Z_0 - Z_2}. \quad (5.4.11)$$

The charge vectors are different on the different sides of the particle's worldline. The worldline is specified by the vector of charges A . We take the worldline to be vertical, therefore A is fixed to be:

$$A = \begin{pmatrix} 0 \\ m \\ 0 \end{pmatrix}. \quad (5.4.12)$$

The dilaton charges get shifted by this vector when crossing the worldline:

$$Z \mapsto Z + A. \quad (5.4.13)$$

This means that the Z_1 component of the dilaton is fixed. However, we still have the freedom of boosting Z_0 and Z_2 . The boost acts as a rescaling of the boundary:

$$w \mapsto e^\rho w, \quad y \mapsto e^\rho y, \quad R \mapsto e^\rho R. \quad (5.4.14)$$

We choose the boost in such a way that the exchange of the two operator insertions acts as an inversion:

$$t \rightarrow \frac{-t}{t^2 + z^2}, \quad z \rightarrow \frac{z}{t^2 + z^2}, \quad (5.4.15)$$

that is, such that:

$$z_+ z_- = 1. \quad (5.4.16)$$

This choice corresponds to $Z_2 = 0$. It allows us to connect the radius of the boundary to the coordinates of the center:

$$R^2 = y^2 + w^2 - 1. \quad (5.4.17)$$

In the semiclassical approximation to gravity, the boundary value of the dilaton is a large parameter. This allows us to treat y as a large parameter as well:

$$\phi_b \gg 1 \quad \Rightarrow \quad y = \frac{\phi_b}{Z_0} \gg 1, \quad (5.4.18)$$

In this choice of gauge and this approximation, the geodesic length is:

$$\ell = 2 \cosh^{-1} y \sim 2 \ln(2y), \quad (5.4.19)$$

and the two-point function becomes:

$$G = \frac{1}{\gamma^{2\Delta}}, \quad \gamma = 2\epsilon y. \quad (5.4.20)$$

The $y \gg 1$ approximation simplifies our expression for the length of the segment u_{12} :

$$u_{12} = \epsilon \frac{2y}{\sqrt{1-w^2}} \arccos w, \quad (5.4.21)$$

which asks for the following parameterization:

$$w \equiv \cos \alpha. \quad (5.4.22)$$

We express everything in terms of the angle α , and in what follows call (5.4.21) the small ϵ approximation.

There is an important difference between working in the small ϵ approximation (5.4.21) and using the precise answer (5.4.7). For the integral in (5.4.7) to make sense when $w < 0$, we need to require that $y > 1$. This means that very small distances are not available to us. The boundary dilaton ϕ_b , or rather the ϵ parameter, plays the role of a UV cutoff. This makes sense since we are working in the semiclassical approximation to gravity, and the underlying ultraviolet theory is beyond our

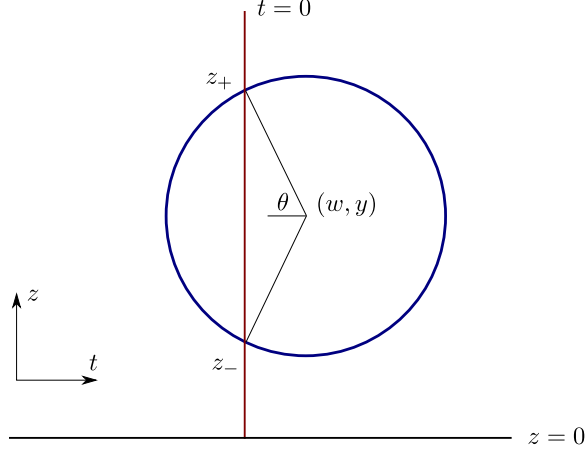


Figure 5.6: Near- AdS space without backreaction.

scope. However, the small ϵ approximation formally continues to small distances and, in particular, it gives the correct OPE for the operators. We interpret the small ϵ approximation as a certain limit where ϵ is extremely small; however, we need to keep in mind that at any given value of ϵ at some point in the UV this approximation, together with OPE, is bound to break down.

We discuss the implications of the small ϵ approximation in more detail in Section 5.4.4. Before that, we find the two-point function in terms of the boundary length. First we do it with backreaction absent, then consider the symmetric case where the distance between the operators is exactly half the length of the boundary, and finally we look how the two-point function depends on the separation between the operators when the boundary length is fixed. In doing so, we work in the small ϵ approximation and rely on numerical methods.

5.4.1 No back-reaction

Without the back-reaction, the boundary of the $NAdS$ remains a perfect circle (see fig. 5.6). In particular, it is invariant under a part of the conformal group, namely rescaling of the boundary (5.4.14). Therefore, in the expression for the two-point function we should recover the familiar conformal answer. Without back-reaction, there is no jump in the parameters of the dilaton, and it remains the same on both side of the particle's trajectory:

$$\phi = Z \cdot Y. \quad (5.4.23)$$

The parameters (y, w, R) are the same on both sides of the circles, and the boundary distances

between the operators in the small ϵ approximation are:

$$\begin{aligned} u_{12} &= 2y\epsilon \frac{\alpha}{\sin \alpha}, \\ u_{21} &= 2y\epsilon \frac{\pi - \alpha}{\sin \alpha}. \end{aligned} \tag{5.4.24}$$

Then the length of the boundary is:

$$L = y \frac{2\pi\epsilon}{\sin \alpha}. \tag{5.4.25}$$

From here, we see that 2α is the arc angles between the two operators on fig. 5.6:

$$u_{12} = \frac{L}{\pi} \alpha. \tag{5.4.26}$$

The conformal (or no-backreaction) limit is when these arc angles add up to precisely a full circle. In Section 5.4.6 we consider small deviations from this, recovering the perturbative answer for the two-point function.

The two-point function depends only on the coordinate of the center of the circle y . From (5.4.25), we find:

$$y = \frac{L}{2\pi\epsilon} \sin \alpha. \tag{5.4.27}$$

Then the two-point function has the form we would expect in a conformal theory:

$$G = \left(\frac{L}{\pi} \sin \frac{\pi u_{12}}{L} \right)^{-2\Delta}. \tag{5.4.28}$$

Notice that because of rescaling G in (5.4.5), we in particular recover the usual OPE expected in a conformal theory:

$$G \sim \frac{1}{u_{12}^{2\Delta}}, \quad u_{12} \ll L. \tag{5.4.29}$$

5.4.2 Two-point function with back-reaction: symmetric case

Having reproduced the conformal result, let us turn to discussing the two-point function with back-reaction. To simplify, we first consider the symmetric case, when $u_{12} = u_{21} = L/2$ (see fig. 5.7). The conformal answer gives:

$$G = \left(\frac{L}{\pi} \sin \frac{\pi}{2} \right)^{-2\Delta} = \left(\frac{L}{\pi} \right)^{-2\Delta}. \tag{5.4.30}$$

The parameters of the dilaton jump by $(0, m, 0)$ when crossing the trajectory of the particle.

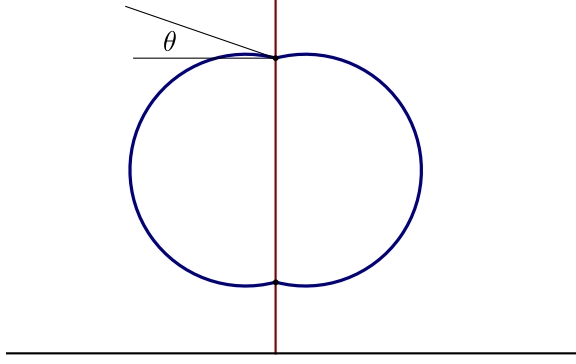


Figure 5.7: The near- AdS boundary for the symmetric case.

Left-right symmetry of the picture corresponds to $Z_1 \leftrightarrow -Z_1$. Since Z_1 has to jump by m when crossing the worldline of the particle, in the symmetric case we can choose the parameters of the dilaton to be:

$$Z = \left(Z_0, \pm \frac{m}{2}, 0 \right). \quad (5.4.31)$$

The angle θ in fig. 5.7 is conformally invariant, and it shows how large the back-reaction is:

$$\sin \theta = \frac{m}{2\phi_b}. \quad (5.4.32)$$

In the massless, or conformal, case, $\theta = 0$ and the cusp on the boundary is absent. The combination ϕ_b/m governs how close our theory is to conformal. It roughly tells the AdS distance at which the conformal symmetry breaks down in the infrared. We want this distance to be small:

$$\frac{m}{\phi_b} \ll 1. \quad (5.4.33)$$

In the symmetric case, the w parameters have opposite signs at the opposite sides of the worldline:

$$w_{\text{left}} = -w_{\text{right}} < 0. \quad (5.4.34)$$

The boundary of $NAdS$ touches the true AdS boundary when $w = -1$. It is convenient to use the α angles, as before:

$$w = \cos \alpha, \quad \pi/2 \leq \alpha \leq \pi. \quad (5.4.35)$$

These angles add up to π on the opposite sides:

$$\alpha_{\text{left}} = \pi - \alpha_{\text{right}}. \quad (5.4.36)$$

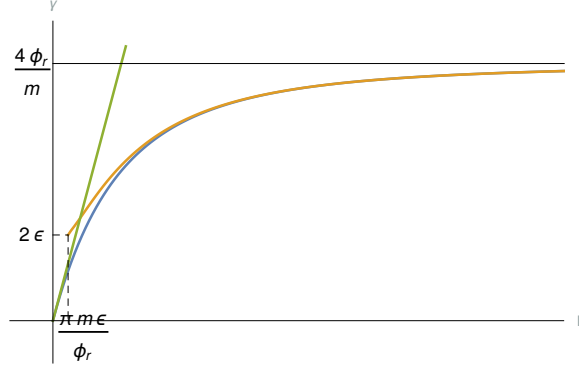


Figure 5.8: Numerical solution to (5.4.40) and (5.4.39) when mass is positive. The blue line is the conformal result, the orange line is the full solution, and the green line is the solution with $R \sim y$, or ϕ_b/m taken to be large.

From the jump in the parameters of the dilaton, we find:

$$w_{\text{left}} = -\frac{m}{Z_0}. \quad (5.4.37)$$

We are still working in the gauge $Z_2 = 0$ where inversion acts as the exchange of the two operators, so the radius is given by (5.4.17):

$$R^2 = y^2 - \sin^2 \alpha. \quad (5.4.38)$$

This time, y is also expressed using α :

$$y = \frac{\phi_b}{Z_0} = -\frac{2\phi_b}{m} \cos \alpha, \quad \gamma = -\frac{4\phi_r}{m} \cos \alpha. \quad (5.4.39)$$

The length of the boundary is found from (5.4.24):

$$L = 2u_{12} = 4\epsilon R \frac{\alpha}{\sin \alpha}, \quad (5.4.40)$$

Using (5.4.40) and (5.4.39), we cannot find the two-point function of the boundary theory in a closed form, as we did in Section 5.4.1. However, we can solve these two conditions numerically, the result shown in fig. 5.8.

The limits of small and large L can also be studied numerically. If α is close to $\pi/2$,

$$\alpha = \frac{\pi}{2} + \delta, \quad \delta \gg \frac{m}{\phi_b}, \quad (5.4.41)$$

then

$$\gamma = \frac{4\phi_r}{m}\delta, \quad L \sim \pi\gamma. \quad (5.4.42)$$

This is the conformal limit. Note that in this limit, only the approximate formulas (5.4.40, 5.4.39) make sense. The precise distance (5.4.7) stops working when $L \sim \pi m \epsilon^2 / \phi_r$. This is when y goes to 1, so the circles forming the boundary barely touch the worldline. $y = 1$ is the physical limit, restricting our insight into the UV physics. This reminds us once again that our theory is cut off in the ultraviolet, much like the SYK model. Our α parameterization allows us to continue past the cutoff and in fact make theory nearly conformal in the UV. But this continuation is formal, and we should keep in mind that the true ultraviolet theory is not accessible for us.

Also, from fig. 5.8 we see that there is a lower limit on ϕ_b parameter:

$$\phi_b > m/2. \quad (5.4.43)$$

In the opposite limit, when α is near π ,

$$\alpha = \pi - \delta, \quad (5.4.44)$$

the length of the boundary becomes infinite, and γ saturates:

$$\gamma = \frac{4\phi_r}{m}, \quad L \sim \frac{1}{\delta}. \quad (5.4.45)$$

The two-point function depends on the y parameter and also becomes constant in this limit:

$$G = \gamma^{-2\Delta} \sim \left(\frac{4m}{\phi_r} \right)^{2m}. \quad (5.4.46)$$

So we see that the two-point function approaches a non-zero constant when boundary length is large.

Although we have found it in the small ϵ approximation, the full answer for the two-point function gives the same result. Geometrically, this happens because we can easily bring the length of the boundary to diverge, just making it approach the boundary of the true AdS space. If ϕ_b is finite, the bulk distance between the two operators can be kept finite in this limit, as on fig. 5.9.

There is a comment to be made about the exchange symmetry of the operators. So far, we treated the operator insertions as identical, and naively the picture on fig. 5.9 is symmetric under inversion. However, a closer look shows that this is not so. The parameters of the dilaton jump by $A_1 > 0$

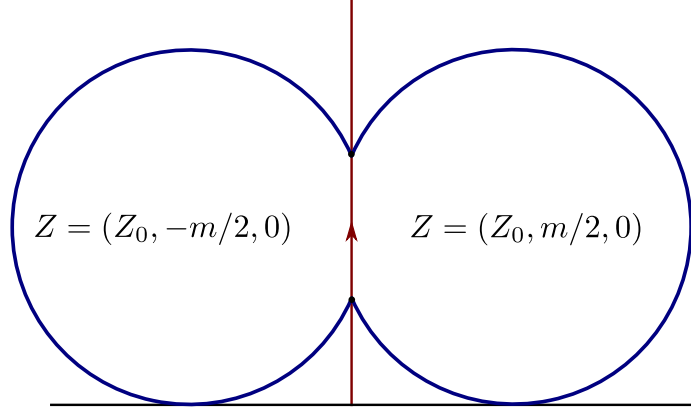


Figure 5.9: The two-point function stays finite as the $NAdS$ boundary touches the boundary of the true AdS , making the boundary length diverge.

when we cross the worldline of the particle from left to right. (We need A_1 to be positive, so that the cusps at the operator insertions point inward.) This is a matter of convention, but this convention breaks the exchange symmetry between the operators. To make “left” and “right” well-defined, we draw an arrow pointing from one operator insertion to the other on fig. 5.9. Inversion reverses the direction of this arrow and at the same time exchanges left and right sides of the picture. In what follows, we implicitly assume that all the trajectories are directional.

5.4.3 Two-point function: generic case

Having discussed the symmetric case, we move on to consider $u_{12} \neq u_{21}$. From what we have seen above, we expect the two-point function to be approximately conformal at short distances, and saturate at a constant when the operators are far away. We also expect to approach the conformal result when mass goes to zero.

The parameters of the dilaton are:

$$Z = \left(Z_0, Z_1 \pm \frac{m}{2}, 0 \right), \quad (5.4.47)$$

where we have again fixed the gauge so that the inversion exchanges the two operators. The horizontal displacements for the centers of the circles are:

$$w_{1,2} = \frac{Z_1 \pm \frac{m}{2}}{Z_0}. \quad (5.4.48)$$

The radii of the circular segments are now different:

$$R_{1,2}^2 = y^2 + w_{1,2}^2 - 1. \quad (5.4.49)$$

Here y is the vertical coordinate of the center, and it is the same for both segments. We can express it via $w_{1,2}$:

$$y = \frac{\phi_b}{Z_0} = \frac{\phi_b}{m} (w_2 - w_1). \quad (5.4.50)$$

The lengths of the boundary segments are given by:

$$\begin{aligned} u_{12} &= \epsilon \frac{2R_1}{\sqrt{1-w_1^2}} \arccos\left(\frac{w_1 y}{R_1}\right), \\ u_{21} &= \epsilon \frac{2R_2}{\sqrt{1-w_2^2}} \left(\pi - \arccos\left(\frac{w_2 y}{R_2}\right)\right). \end{aligned} \quad (5.4.51)$$

For these lengths to be real, the argument of the arccos has to be greater than -1. It implies that:

$$y > 1. \quad (5.4.52)$$

This is our condition for the UV cutoff. For notational simplicity, we switch signs in (5.4.51), so that the boundary distances are positive when mass is positive.

If we take $y \gg 1$, we can assume $y \sim R$ and use the angular ansatz, as above,

$$w_{1,2} = \cos \alpha_{1,2}, \quad (5.4.53)$$

and find a relatively simple expression for the boundary distances:

$$\begin{aligned} u_{12} &= \frac{2\phi_r}{m} \frac{\alpha_1}{\sin \alpha_1} (\cos \alpha_2 - \cos \alpha_1), \\ u_{21} &= \frac{2\phi_r}{m} \frac{\pi - \alpha_2}{\sin \alpha_2} (\cos \alpha_2 - \cos \alpha_1). \end{aligned} \quad (5.4.54)$$

The γ parameter then is given by:

$$\gamma = \frac{2\phi_r}{m} (\cos \alpha_2 - \cos \alpha_1). \quad (5.4.55)$$

An important property of the small ϵ approximation is that neither distances nor the two-point function depends on ϵ . In a way, we can treat ϵ as a parameter saying how close the small ϵ approximation is to the true answer. We will see how the two-point function depends on this

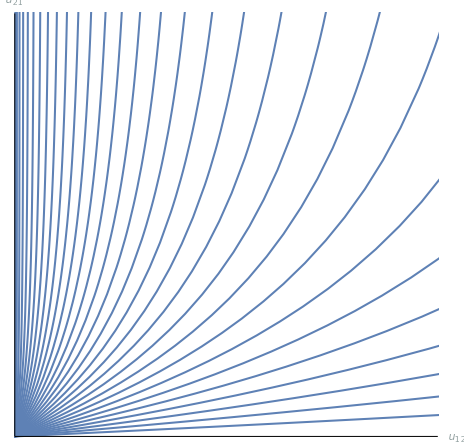


Figure 5.10: Every line on the u plane corresponds to fixed α_1 and varying α_2 .

parameter in Section 5.4.4.

A natural question is whether the angular parameters $\alpha_{1,2}$ are in one-to-one correspondence with the boundary distances. On fig. 5.10 we vary the angles and find that the quarter of the u plane with both positive distance gets covered exactly once. From here, we conclude that we can indeed change variables to $\alpha_{1,2}$ without introducing any additional singularities.

In the small ϵ approximation, the UV theory is nearly conformal. Indeed, if we take:

$$\alpha_1 = \delta_1, \quad \alpha_2 = \delta_2, \quad \delta_1^2 - \delta_2^2 = \frac{mL}{\pi\phi_r} \cdot \delta_2, \quad \delta_{1,2} \ll 1, \quad (5.4.56)$$

we get for the distances:

$$u_{21} \sim L, \quad u_{12} \sim \frac{L}{\pi} \delta_2, \quad (5.4.57)$$

and the two-point function is:

$$G = \frac{1}{\gamma^{2\Delta}} = \frac{1}{|u_{12}|^{2\Delta}}. \quad (5.4.58)$$

This is the behavior one expects from the OPE. We found in the small ϵ approximation, and cannot take it literally. However, for every u_{12} we can find ϵ small enough so that at that distance, the two-point function looks like (5.4.58). For that same ϵ , the OPE will break down as we go to shorter distances.

Let us also look in more detail how the two-point function depends on mass. To do that, we fix the full length of the boundary L and find γ as a function of u_{12} . The two-point function, as before, is $G = \gamma^{-2\Delta}$. There is no analytical solution, however the equations can be solved numerically with good convergence.

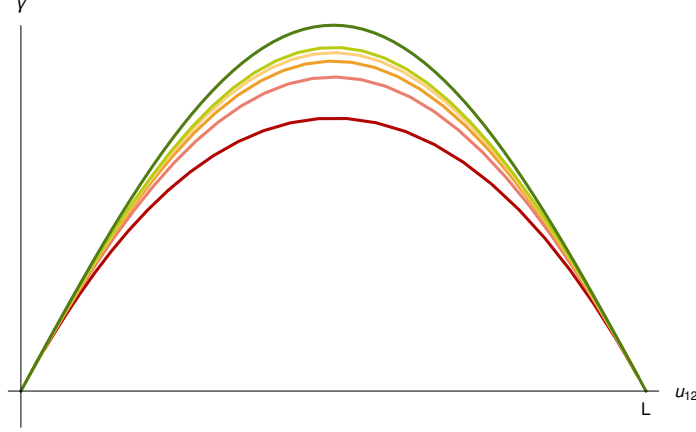


Figure 5.11: $\gamma(u_{12})$ for various masses, the curves becoming greener as mass decreases. The dark green line is the conformal result $\gamma = L/\pi \cdot \sin(\pi u_{12}/L)$.

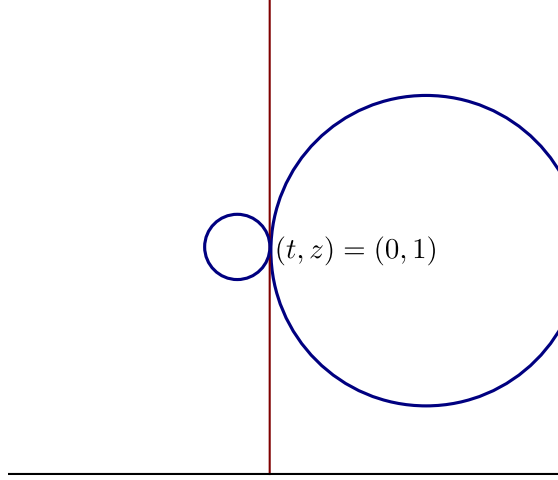


Figure 5.12: The geodesic length can go to zero when one of the distances u_{12} becomes small. This dictates the maximal value of the two-point function $G \sim 1/\gamma^{2m}$. In this situation, invariance under inversion fixes $y = 1$. Thus we can treat $y = 1$ as an ultraviolet cutoff.

The solutions are plotted on fig. 5.11. We see that $\gamma(u_{12})$ starts linear for small distances. For every mass, when distance is small enough, the two-point function is indistinguishable from the conformal one, as we have seen in (5.4.58). As distance grows, γ deviates from the conformal answer, however having a similar general shape. As mass decreases, γ comes closer to conformal, as we expect on general grounds.

5.4.4 Small ϵ approximation

In this Section, we look at how the precise answer for the two-point function coming from (5.4.7) differs from the small ϵ approximation we discussed in Section 5.4.3. The small ϵ approximation

relies on the boundary dilaton ϕ_b being large compared to all the components of the dilaton charge vector. The “distance” to the boundary ϵ also tells how far we are from the pure AdS setup. We expect the precise two-point function to converge to the approximation when $\epsilon \rightarrow 0$.

However, generally ϵ is finite, and the boundary dilaton is also finitely large. We see that it makes a difference in the ultraviolet, and in particular that the ultraviolet limit is not conformal. It should be emphasized that the semiclassical description eventually breaks down in the UV and our analysis no longer works there. We also show that at finite distances the small ϵ approximation is very close to the full answer.

As we mentioned before, the integral in (5.4.7) makes sense only when $y > 1$. Thus we can think of $y \sim 1$ as a condition on the UV cutoff, coming from our semiclassical approximation of the gravitational theory. On fig. 5.12, we have schematically drawn a setup with $y = 1$. The vertical coordinate of one of the operators is always greater than 1, thus it happens when both the operators are at the same point. This is our cutoff and at the same time the smallest value of γ possible.

In the small ϵ approximation, $y \sim 1$ belongs to the conformal regime, with $\gamma \sim u_{12}$. From this, we find that the small ϵ approximation breaks down roughly at distances:

$$u_{12} \sim \epsilon. \tag{5.4.59}$$

This is where our semiclassical approximation stops being reliable. Together with it, the OPE also breaks down. Therefore we should consider OPE carefully, taking into account the order of limits we are taking. For every distance between the two operators, we can find ϵ small enough that the OPE holds. However, after we fix ϵ , we can bring operators close enough to ensure that OPE no longer works.

We can think of the small ϵ approximation as a way to continue our theory in the UV consistently with conformal symmetry. We should keep in mind that this way may not be physical, and this behavior is not typical for one-dimensional quantum mechanics. In the well-studied example of the SYK model, the two-point function looks conformal only at large distances, and in the ultraviolet the theory is essentially free. Thus our full answer (5.4.7) behaves more like a conventional SYK model, and the approximation (5.4.21) is more like the conformal part of the SYK, or the cSYK of [131]. On fig. 5.13, we draw the precise result for the two-point function and the small ϵ approximation to it, and it reminds us of the way the precise two-point function of the SYK compares to the conformal approximation (see for example [29]).

For large distances the precise answer and the approximation get very close. On fig. 5.14, we fix

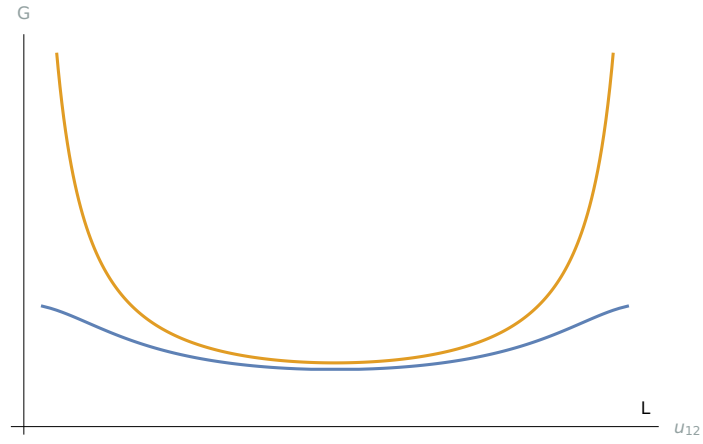


Figure 5.13: The two-point function in the small ϵ approximation (yellow) versus the precise one (blue).

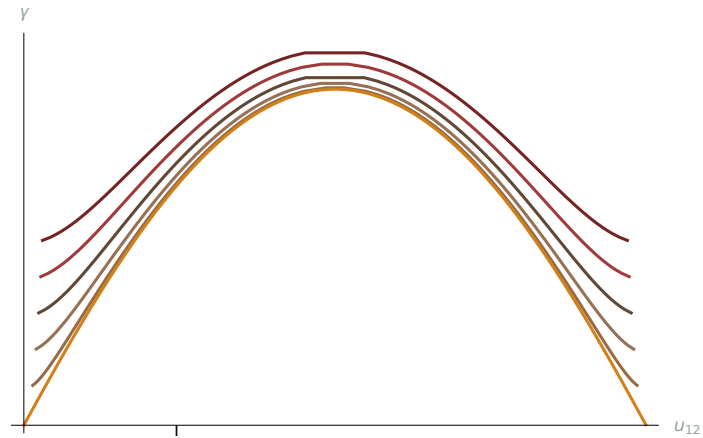


Figure 5.14: The precise answer for $\gamma(u_{12})$ (from red to orange) gets closer to the small ϵ approximation (light orange) as ϵ decreases.

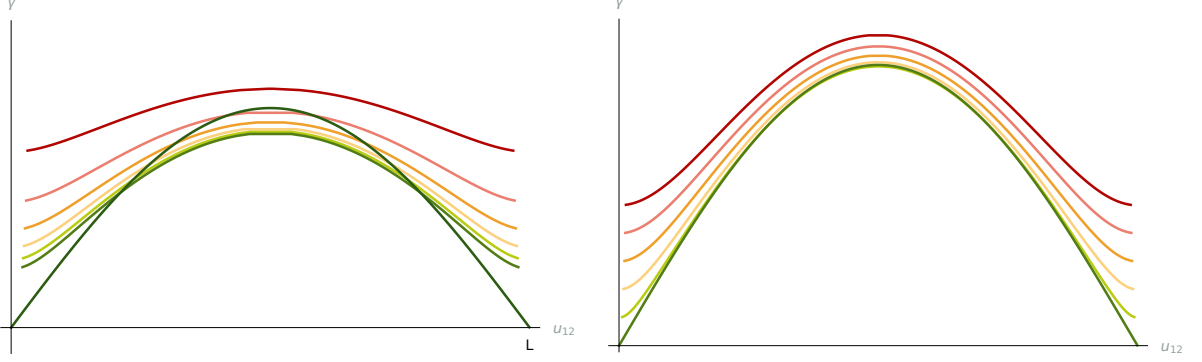


Figure 5.15: $\gamma(u_{12})$ for various values of the boundary dilaton compared to the conformal answer. The curves become greener as ϵ decreases. The renormalized dilaton ϕ_r on the right is larger than on the left, and the result is accordingly close to the conformal.

the length of the boundary and solve for $\gamma(u_{12})$ numerically for various values of ϵ . (It should be said that the approximation is much friendlier to numerical methods.) We see that as ϵ decreases, the answer gets closer to the small ϵ approximation, as expected.

Thus we see that the precise solution has two parameters: ϵ , which says how far in the ultraviolet we can extend the conformal symmetry and therefore how close it is to the small ϵ approximation, and the ratio ϕ_r/m , which governs how close we are to the conformal answer, including in the infrared. It may be instructive to draw a further analogy with the SYK model. In the large N limit, the SYK model is approximately conformal at large distances. Since $\phi_r \sim N$, we expect ϕ_r to be large, and as it grows the two-point function gets closer to the conformal one. In the UV, the SYK model is effectively free and the two-point function approaches a limit. To get the same result in our theory, we keep ϵ fixed (but small). Since our results depend only on the ratio $\phi_r/m \sim N/\Delta$, we expect the two-point functions of heavier operators to get farther away from conformal in the SYK as well.

It should be said that this analogy relies on our description of the UV region, where the semi-classical approximation we are working in stops being applicable. So we think of the SYK as an (approximation to an) effective theory, rather than the precise holographic dual of the gravitational theory in the bulk.

Finally, we can look at how the precise answer for $\gamma(u_{12})$ approaches the conformal answer when ϵ changes. The numerical results are on fig. 5.15. When mass is large (or ϕ_r small), there is little resemblance of the conformal answer both in the ultraviolet and the infrared. When mass becomes smaller (or ϕ_r larger), the $\gamma(u_{12})$ function approaches the conformal answer from above, unlike in the small ϵ approximation (see fig. 5.11).

5.4.5 Extrinsic curvature

Up to this point, when finding the two-point function, we have considered only the action of massive particle. The full gravitational action (5.2.1) also contains a term with extrinsic curvature:

$$I = -\phi_b \int_{\partial} \sqrt{h} K. \quad (5.4.60)$$

Since the massive particle shifts the boundaries of the $NAdS$ space, this term creates a non-zero correction to the two-point function. To find this correction, we subtract from (5.4.60) the action for $NAdS$ without the massive particle:

$$I - I_0 = \phi_b \left(\int \sqrt{h} K_0 - \int \sqrt{h} K \right). \quad (5.4.61)$$

Here the second term denotes the curvature of the empty $NAdS$ space with the boundary of length L_{AdS} , and the first term is the space with the same boundary length, but with a massive particle inside. ϕ_b in (5.4.61) is a large number, so this difference is potentially large. The corrected two-point function becomes:

$$G = \frac{1}{\gamma^{2m}} e^{-(I-I_0)}. \quad (5.4.62)$$

In this Section, we discuss this correction and find it small for small distance, and finite constant when the distance gets large.

In the hyperbolic half-plane, the extrinsic curvature is:

$$K = \frac{t' (t'^2 + z'^2 + z z'') - t'' z z'}{(t'^2 + z'^2)^{\frac{3}{2}}}. \quad (5.4.63)$$

For a circle of radius R and with a center at vertical coordinate $z = y$ the extrinsic curvature is:

$$K = -\frac{y}{R}. \quad (5.4.64)$$

In Section 5.4.1, we have found the boundary length of the empty $NAdS$ (in AdS units) as:

$$L_{AdS} = \frac{2\pi R}{\sqrt{y^2 - R^2}}. \quad (5.4.65)$$

Therefore the extrinsic curvature action of the empty $NAdS$ space is:

$$I_0 = \phi_b \frac{y}{R} \cdot L_{AdS} = \phi_b \sqrt{L_{AdS}^2 + (2\pi)^2}. \quad (5.4.66)$$

The extrinsic curvature action for the space with a massive particle is a sum of two parts. One comes from the finite segments of the boundary, and the other from the cusps where the particle meets the boundary:

$$I = I_{\text{seg}} + 2I_{\text{cusp}}. \quad (5.4.67)$$

The first part depends on the boundary length and have roughly the same structure as (5.4.66). More precisely, it is:

$$I_{\text{seg}} = \phi_b \frac{y}{\epsilon} \left(\frac{u_{12}}{R_1} + \frac{u_{21}}{R_2} \right). \quad (5.4.68)$$

Here we cannot use the small ϵ approximation (since it requires $y \rightarrow R$), and the distances are as in (5.4.7, 5.4.8). They are chosen so as to match the distance in (5.4.66):

$$u_{12} + u_{21} = \epsilon \cdot L_{AdS}. \quad (5.4.69)$$

The second term in (5.4.67) is largely universal and independent of distances. To find it, we replace a cusp with a segment of a small circle of radius r . Since the circle is small, we can think that the metric is constant, $g \sim \frac{1}{y^2}$. If the cusp angle is θ , the extrinsic curvature action is:

$$I_{\text{cusp}} = \phi_b \lim_{r \rightarrow 0} \left(\frac{y}{r} \frac{r\theta}{y} \right) = \phi_b \cdot \theta. \quad (5.4.70)$$

The angle is found as:

$$\theta = \arccos \frac{w_1 y}{R_1} - \arccos \frac{w_2 y}{R_2} = \arcsin \left(\frac{m}{\phi_b} \frac{y \sqrt{y^2 - 1}}{R_1 R_2} \right). \quad (5.4.71)$$

If we take ϕ_b to be large, the full cusp action becomes:

$$I_{\text{cusp}} = m \frac{y \sqrt{y^2 - 1}}{R_1 R_2}. \quad (5.4.72)$$

Away from the UV cutoff at $y \sim 1$, this action changes little with distance. When distance is large, $y \sim R$ and the cusp action becomes:

$$I_{\text{cusp}} \sim m. \quad (5.4.73)$$

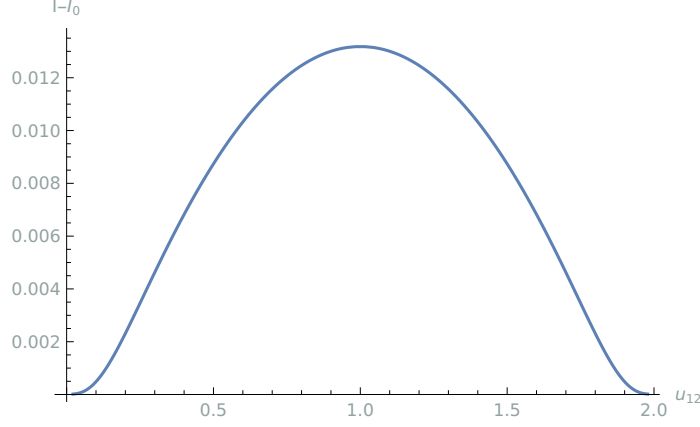


Figure 5.16: $(I - I_0)$ as a function of u_{12} . The parameters are: $\phi_r = 8, m = 1, \epsilon = .1$.

The precise answer for the correction is hard to find analytically. In the symmetric case, $u_{12} = u_{21}$, the numerical method gives us the answer very close to:

$$I - I_0 \sim \frac{2m}{1 + \frac{3\pi^2\phi_b}{mL}}. \quad (5.4.74)$$

When L is small, this action is negligible. When L is large, it comes mostly from the contribution of the cusps, $I \sim 2m$. In particular, it changes the value of the two-point at large distances (5.4.46) to:

$$G \rightarrow \left(\frac{4m}{\phi_r \cdot e} \right)^{2m}. \quad (5.4.75)$$

We can also fix L and find the action as a function of u_{12} . The numerical result for some fixed values of parameters is plotted on fig. 5.16. When the u_{12} distance is small and $y \rightarrow 1$, the angle θ goes to zero, and the action from the cusps and the u_{12} segment also vanishes. The contribution of the u_{21} segment exactly coincides with the action of the empty space, and the whole difference in action $(I - I_0)$ becomes zero, as can be seen on fig. 5.16. Overall, the action the extrinsic curvature is numerically small for finite distances. However, we see later that this holds only for the Euclidean time, and the real-time correlators receive a correction from the extrinsic curvature action at long time. This corrections makes the correlators exponentially small.

5.4.6 Schwarzian limit

A limit we can use for a reality check is taking ϕ_r/m to be large. Zero mass corresponds to the conformal two-point function, and expansion in m/ϕ_r gives a correction to the conformal answer. The same correction can be found from Schwarzian theory (see [2]), corresponding to a limit of large

Δ .

In the conformal case, the cusp at the insertion of the boundary operator is absent, and the α_1, α_2 angles are equal. We relax this condition and take the angles to be:

$$\begin{aligned}\alpha_1 &= \alpha - \delta, \\ \alpha_2 &= \alpha + \delta.\end{aligned}\tag{5.4.76}$$

As δ goes to zero, we want to come back to the conformal case discussed in Section 5.4.1. The same happens as mass goes to zero. Therefore we take:

$$\delta = c \cdot \frac{m}{\phi_r}.\tag{5.4.77}$$

with c being some constant to be fixed later.

Our goal is to find the two-point function in this limit, up to the second order in δ . Since δ is proportional to m , it means that we find γ in terms of u_{12}, u_{21} to the first order in δ :

$$\gamma = \frac{2\phi_r}{m} (\cos \alpha_2 - \cos \alpha_1) = \frac{4\phi_b}{m} \sin \delta \sin \alpha.\tag{5.4.78}$$

Expanding u_{12} in δ , we find:

$$u_{12} = \frac{\phi_r}{m} \cdot 4\delta (\alpha - \delta \cdot \eta(\alpha)) + O(\delta^2),\tag{5.4.79}$$

where we have defined:

$$\eta(\alpha) \equiv 1 - \alpha \cot \alpha.\tag{5.4.80}$$

The full length of the boundary then is:

$$L = u_{12} + u_{21} = \frac{\phi_r}{m} \cdot 4\pi\delta \left(1 - \frac{\delta}{\pi} (1 + (\pi - 2\alpha) \eta(\alpha))\right) + O(\delta^2).\tag{5.4.81}$$

Since we expand to the first order in δ , this equation allows us to fix the constant in (5.4.77):

$$\delta \sim \frac{mL}{4\pi\phi_r} + O(\delta^2).\tag{5.4.82}$$

The fraction of the boundary belonging to the first segment is:

$$\frac{\pi u_{12}}{L} = \alpha + \delta \left(1 + \left(\frac{2\alpha}{\pi} - 1 \right) \eta(\alpha) \right) + O(\delta^2). \quad (5.4.83)$$

From here, we see that in this near-conformal case, α is roughly the arc angle for u_{12} , with a correction of order ϵ .

Putting everything together, from (5.4.81) and (5.4.83) we find:

$$\gamma = \frac{L}{\pi} \sin \frac{\pi u_{12}}{L} \left(1 - \frac{2\delta}{\pi} \eta(\alpha) \eta(\pi - \alpha) \right) + O(\delta^2). \quad (5.4.84)$$

Raising this to a power, we find a correction for the two-point function:

$$G = \frac{1}{\left(\frac{L}{\pi} \sin \frac{\pi u_{12}}{L} \right)^{2m}} \left(1 + \frac{m^2}{\phi_r} \frac{L}{\pi^2} \eta\left(\frac{\pi u_{12}}{L}\right) \eta\left(\frac{\pi u_{21}}{L}\right) \right). \quad (5.4.85)$$

This is the correction that was found in [2] from the Schwarzian propagator. Since $\phi_r \sim N$, it corresponds to a $1/N$ correction in the SYK model.

5.5 Two-point function in real time

Our analysis allows us to extend our discussion to real time and consider a thermal correlator:

$$G_\beta(t) = \left\langle \mathcal{O}\left(\frac{\beta}{2} + it\right) \mathcal{O}\left(\frac{\beta}{2} - it\right) \right\rangle. \quad (5.5.1)$$

In quantum mechanics, this correlator looks like a sum over energy eigenstates:

$$G_\beta(t) = \frac{1}{Z(\beta)} \sum_{m,n} e^{-\beta(E_m + E_n)/2} e^{it(E_m - E_n)} |\langle m | \mathcal{O} | n \rangle|^2. \quad (5.5.2)$$

In a chaotic system, it is believed that for very large t , the off-diagonal terms have large and essentially random phases. In this case, after some averaging over t , the off-diagonal terms do not contribute. If so, one expects that at least in an averaged sense, the large real time behavior can be approximated by:

$$G_\beta|_{t \rightarrow \infty} \sim \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} |\langle n | \mathcal{O} | n \rangle|^2. \quad (5.5.3)$$

If the non-diagonal elements can be neglected in some sense, and the diagonal elements are of

order one, this correlator is a close cousin to the spectral form factor:

$$\text{SFF}(t) = \frac{\left| Z\left(\frac{\beta}{2} + it\right) \right|^2}{\left| Z\left(\frac{\beta}{2}\right) \right|^2}, \quad (5.5.4)$$

which has been discussed at length in the context of SYK in [64] and other studies. At large time, and if the spectrum of the system has no degeneracies, the averaged spectral form factor becomes:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{SFF}(t) dt = \frac{Z(2\beta)}{Z(\beta)^2}. \quad (5.5.5)$$

The partition function generally scales as $Z \sim \exp(-cS)$, therefore the long-time value of the spectral form factor (and of the real-time two-point function) is exponential in S [132]. On the gravitational side, the entropy is $S \sim 1/G_N$, and in SYK, the entropy is $S \sim N$. In our setup, it means that $S \sim \phi_r$, and the two-point function should have an exponentially small limit at long times.

The existence of this limit for the two-point function is a non-perturbative effect, both in the SYK model and in gravity. We find that our two-point function also approaches an exponentially small number, when the extrinsic curvature term is taken into account. However, in our case the two-point function is averaged, that is insensitive to small oscillations which have been observed for SYK in [64] and are anticipated on general grounds. Also, we find that in our case the long time limit is approached much faster than expected for a spectral form factor in SYK.

We first find the two-point function as the exponentiated geodesic length and then take into account extrinsic curvature. We once again use the small ϵ approximation and translate the distances between operators to (now complex) angles:

$$\begin{aligned} \frac{\beta}{2} + it &= \frac{2\phi_r}{m} \frac{\alpha}{\sin \alpha} (\cos \alpha + \cos \bar{\alpha}), \\ \frac{\beta}{2} - it &= \frac{2\phi_r}{m} \frac{\bar{\alpha}}{\sin \bar{\alpha}} (\cos \alpha + \cos \bar{\alpha}). \end{aligned} \quad (5.5.6)$$

Here we have defined:

$$\alpha = \alpha_1, \quad \bar{\alpha} = \pi - \alpha_2. \quad (5.5.7)$$

The two-point function depends, as before, on the γ parameter:

$$G(\beta, t) = \frac{1}{\gamma^{2\Delta}}, \quad \gamma = \frac{2\phi_r}{m} (\cos \alpha + \cos \bar{\alpha}). \quad (5.5.8)$$

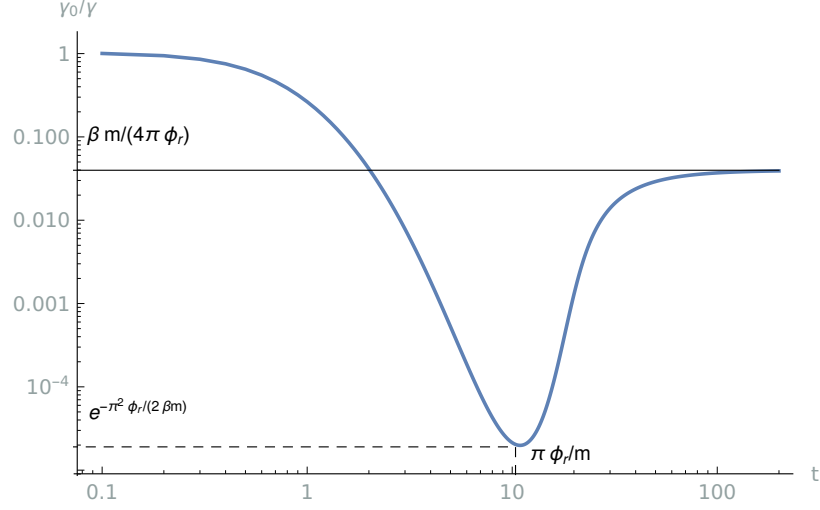


Figure 5.17: Two-point function in real time without the extrinsic curvature correction.

To simplify the discussion, we take mass to be relatively small, in particular:

$$\beta m / \phi_r \ll 1. \quad (5.5.9)$$

In this limit, we can find the two-point function at $t = 0$:

$$\alpha_0 = \frac{\pi}{2} + \frac{\beta m}{4\pi\phi_r} \quad \Rightarrow \quad \gamma_0 = \frac{\beta}{\pi}. \quad (5.5.10)$$

We normalize the thermal two-point function (5.5.1) by the two-point function at $t = 0$, imitating the spectral form factor (5.5.4):

$$\frac{G_\beta(t)}{G_\beta(0)} = \left(\frac{\gamma_0}{\gamma} \right)^{2\Delta}. \quad (5.5.11)$$

On fig. 5.17, we plot the γ_0/γ ratio as a function of time. We cannot solve (5.5.6) analytically, but can see how the correlator behaves in various limits. Let the angle α be complex:

$$\alpha = \xi + ix. \quad (5.5.12)$$

Then the two-point function is:

$$G = \gamma^{-2\Delta} = \left(-\frac{4\phi_r}{m} \cos \xi \cosh x \right)^{-2\Delta}. \quad (5.5.13)$$

With time, the real part ξ grows from α_0 to π . When ξ is small, (5.5.6) gives:

$$t = \frac{\beta}{\pi}x, \quad (\text{small } t), \quad (5.5.14)$$

and y goes as:

$$\gamma_0/\gamma \sim 1/\cosh\left(\frac{\pi t}{\beta}\right). \quad (5.5.15)$$

The two-point function then decays exponentially, as can be expected from a thermal correlator. This can also be seen from the Schwarzian theory. However, this decay eventually stops. The imaginary part x grows faster than ξ , and the minimum on fig. 5.17 occurs when x is large and ξ still close to $\pi/2$. Plugging this information as an approximation into (5.5.6), we find the minimum at:

$$\xi_{\min} = \frac{\pi}{2} + \frac{\beta m}{2\pi}, \quad t_{\min} = \frac{\pi\phi_r}{m}. \quad (5.5.16)$$

The minimum of the correlator, consistently with the initial exponential decay, occurs at:

$$\gamma_0/\gamma_{\max} \sim \exp\left(-\frac{\pi^2\phi_r}{2\beta m}\right). \quad (5.5.17)$$

The minimal value of the normalized two-point function then is:

$$\frac{G_{\min}}{G_0} \sim \exp\left(-\frac{\pi^2\phi_r}{\beta}\right). \quad (5.5.18)$$

After the minimum, the imaginary part x decreases, and at the same time ξ covers most of the distance to π . The two-point function grows exponentially with roughly the same speed as it decreased. Finally, the correlator approaches a plateau, where both x and ξ are small.

The similarity of the real-time correlator to the form-factor of the SYK [128] is striking, but likely accidental. The finite value at which the two-point function saturates is also exponential in N in the SYK, and is $\sim 1/N$ in our problem. It is also worth noting that the large real time limit is the same as the large Euclidean time limit (5.4.46). From (5.5.3), we expect the large t limit of the thermal two-point function to be the square of the thermal one-point function of the operator. This is a consequence of the eigenstate thermalization hypothesis [133], [134]. Without action of the extrinsic curvature, the thermal one-point function appears to be the same as the one-point function in Euclidean time.

However, when we take into account extrinsic curvature, the picture changes drastically. In our discussion so far, we worked in the small ϵ approximation when $y = R$, and the extrinsic curvature

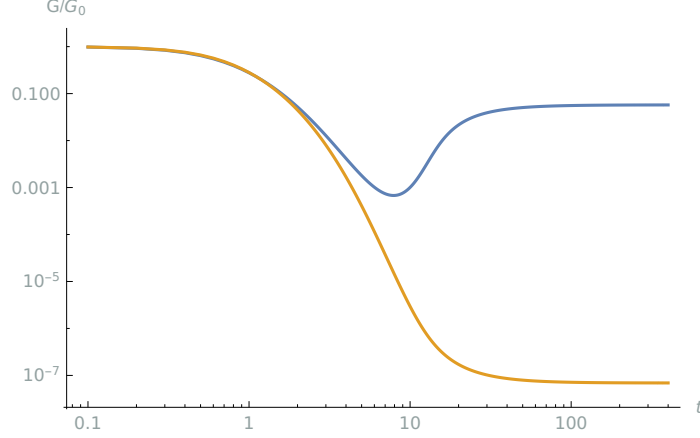


Figure 5.18: The two-point function including the extrinsic curvature term (yellow) compared to the two-point function as exponentiated geodesic (blue). The long-time value of the two-point function is $\sim \exp(-N)$.

is equal to one. The full answer for u_{12} , (5.4.7), is hard to continue to the complex plane, but for us it is sufficient to find the extrinsic curvature to the first order in $1/y^2$. When ϵ is small, y is a large parameter, so all other corrections will be subleading in ϵ .

The extrinsic curvature is:

$$K = -\frac{y}{R} = -\frac{y}{\sqrt{y^2 - \sin^2 \alpha}} = -1 - \frac{1}{2y^2} \sin^2 \alpha + O(y^{-4}) \quad (5.5.19)$$

In particular, we see that at large time $y = R$ with good accuracy, so the “segment” part of action of the extrinsic curvature is:

$$I_{\text{seg}}|_{t \rightarrow \infty} = -\phi_b L / \epsilon = -\phi_r \beta. \quad (5.5.20)$$

This is a large number. When we normalize to the action of the empty AdS space, it becomes:

$$(I - I_0)|_{t \rightarrow \infty} = \phi_b \left(\sqrt{\left(\frac{\beta}{\epsilon}\right)^2 + (2\pi)^2} - \frac{\beta}{\epsilon} \right) = \frac{2\pi^2 \phi_r}{\beta} + O(\epsilon^2). \quad (5.5.21)$$

This creates a large correction to the two-point function at long times:

$$\frac{G}{G_0} \Big|_{t \rightarrow \infty} \sim \exp\left(-\frac{2\pi^2 \phi_r}{\beta}\right). \quad (5.5.22)$$

This correction is similar to the minimal value of the two-point function (5.5.18). Therefore, when the extrinsic curvature is taken into account, the two-point function decays to a value $\sim \exp(-\phi_r)$, and therefore exponential in N . This can be expected in quantum mechanics on general grounds, as

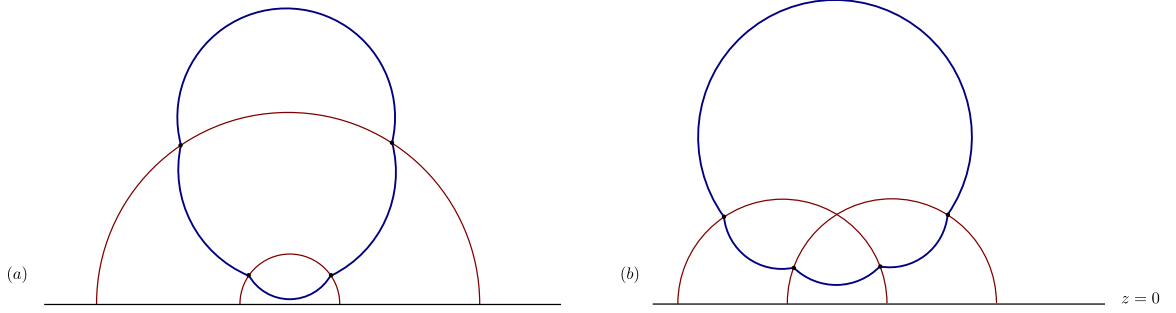


Figure 5.19: Four-point function for (a) time-ordered and (b) out of time ordered operators.

an average of oscillations with a large phase [128]. We plot the numerical result (the first correction to the small ϵ approximation) on fig. 5.18. We see that the “ramp” is gone, and the two-point function decays monotonically. Roughly at $t \sim \phi_r/m \sim N$ this decay slows down, and the final value is $\sim \exp(-N)$.

It would seem that our calculation reproduces the plateau in the SYK two-point function on the $NAdS$ side. However, the time at which this plateau is reached seems much shorter in our case. In [64], the “plateau time” has been found to be exponential in N from random matrix considerations. In our case the plateau starts at roughly the time when the two-point function (5.5.11) reaches its minimum:

$$t_{\min} \sim \phi_r \sim N. \quad (5.5.23)$$

This time is linear in N . The reason for this behavior is not clear, but it points out that the plateau we find in JT gravity may be governed by different physics than the plateau in SYK.

5.6 Euclidean four-point function

Following the same logic, we can find the four-point function in the semi-classical approximation. This time, we consider trajectories of two massive particles, both intersecting the boundary of the $NAdS$ space. In a stable theory, when particles have positive masses, they create inward cusps (see fig. 5.19).

The four-point function is defined by the lengths of the trajectories, lying inside the $NAdS$ space. As before, we are going to express this four-point function in terms of the lengths of the boundary segments. For simplicity, we take the segments to be pairwise equal. This means that the picture on fig. 5.19 is both left-right symmetric, and invariant under inversion.

In this picture, there are two significantly different cases. If the trajectories of the particles do

not intersect, it corresponds to the time-ordered four-point function. However if the trajectories do intersect, we get an out of time ordered correlator, and expect to see exponential growth after analytical continuation. The presence or absence of an intersection is conformally invariant, and can be defined from the cross product of the charge vectors for the massive particles. Let one trajectory be defined by the three-vector A and the other by B . We will be working in the convention where:

$$\begin{aligned}(A \times B)^2 > 0 &\Rightarrow \text{intersection,} \\ (A \times B)^2 < 0 &\Rightarrow \text{no intersection.}\end{aligned}\tag{5.6.1}$$

There is a boundary case when both trajectories are straight vertical lines and the cross product is exactly zero, but we will not be considering it.

In this Section, we work in the small ϵ approximation. We do not discuss the precise answer with a UV cutoff, as we did for the two-point function. We expect the approximation to work in a similar way for a four-point function, removing a cutoff with an approximately conformal region in the ultraviolet. This approximately conformal region allows us to find a small mass correction to the four-point function in Section 5.6.3.

We impose a significant amount of symmetry, making the pictures on fig. 5.19 both left-right and inversion symmetric. This allows us to parameterize the boundary distance in a relatively simple way, using the angular variables. In particular, the answer for the time-ordered four-point function is strikingly similar to the answer for the two-point function, compare (5.6.23, 5.6.24) to (5.4.54, 5.4.55). For the out of time ordered four-point function, we can also use a similar parameterization, with the result being (5.6.42, 5.6.46), however in this case there is an extra condition (5.6.43).

Our parameterization helps us to analytically continue the four-point function to real time. However, here the imposed symmetry appears to be restrictive and does not allow to find the time-ordered correlator. For the out of time ordered four-point function, we find that it first decays exponentially and then stabilizes at a small ($\sim \exp(-N)$) value. We find this after taking into account the action of extrinsic curvature, which is small in Euclidean signature but is significant for real-time correlators.

5.6.1 Four-point function: time-ordered

First, we start with the picture of the type (a) on fig. 5.19 with non-intersecting trajectories. We want to make it left-right symmetric and inversion invariant. These two symmetries act on the embedding coordinates Y as reflections:

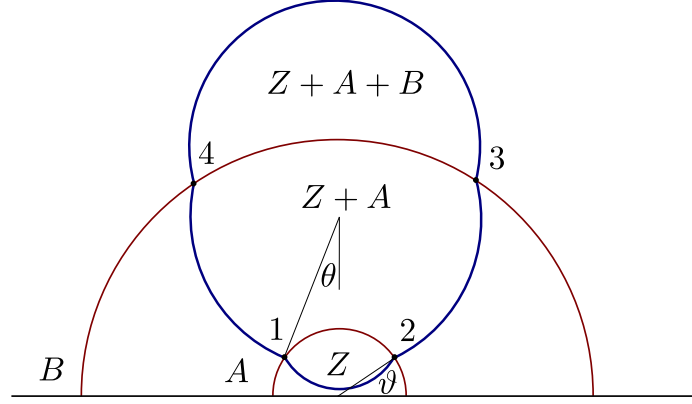


Figure 5.20: Bulk diagram for the time ordered four-point function.

$$\text{Left-right : } Y_1 \rightarrow -Y_1, \quad (5.6.2)$$

$$\text{Inversion : } Y_2 \rightarrow -Y_2.$$

Let us denote Z the charge vector of the dilaton inside the “smaller” circle, A the vector for the trajectory on the “smaller” circle, and B the vector for the trajectory of the “larger” circle. To make the picture invariant under the left-right reflection, we make the first component of each of these vectors vanish:

$$Z_1 = A_1 = B_1 = 0. \quad (5.6.3)$$

To make them inversion-invariant, we take:

$$A_2 = B_2 = -Z_2. \quad (5.6.4)$$

The individual components of the vectors prove not to be the convenient parameters for our calculations. Instead, we introduce two angle parameters, as we did previously for the two-point function. Using the condition on masses,

$$A^2 = B^2 = -m^2, \quad (5.6.5)$$

together with (5.6.3), we parameterize A and B vectors as:

$$\begin{aligned} A &= (-m \sinh \alpha, 0, m \cosh \alpha), \\ B &= (m \sinh \alpha, 0, m \cosh \alpha), \quad \alpha > 0. \end{aligned} \quad (5.6.6)$$

In these variables, the radius of the “larger” circle is:

$$r \equiv e^\alpha, \quad (5.6.7)$$

and the radius of the “smaller” circle is $1/r$, since they are exchanged by inversion.

By the same logic, we parameterize the dilaton vector Z as:

$$Z = (Z \cosh \zeta, 0, -Z \sinh \zeta). \quad (5.6.8)$$

From inversion invariance (5.6.4), we find the Z constant:

$$Z = \frac{m \cosh \alpha}{\sinh \zeta}, \quad \zeta > 0. \quad (5.6.9)$$

Thus we are left with two angles α, ζ , which we are going to determine from two boundary lengths, which we call u_{12}, u_{23} .

The boundary length of the lower segment is given by an integral:

$$u_{12} = 2\epsilon \cdot \int_0^{\theta_*} \frac{R d\theta}{y - R \sin \theta} \quad (5.6.10)$$

We have rescaled from the AdS units to the quantum mechanicals ones, so that the distance stays finite as ϵ becomes small. Here R is the radius of the boundary segment and y is the vertical position of its center. Both are taken to be large, and are connected by:

$$y^2 - R^2 = e^{-2\zeta}. \quad (5.6.11)$$

Hence ζ is the measure of how “close” the $NAdS$ is to the real boundary (coordinate-wise, since the real distance to the boundary is infinite). y is given by:

$$y = \frac{\phi_b}{m} \cdot \frac{1 - e^{-2\zeta}}{2 \cosh \alpha} \sim \phi_b/m, \quad (5.6.12)$$

We take ϕ_b/m to be large, and find the answer in the leading order in $1/y$. This is the condition of the small ϵ approximation.

In this limit, the θ_* angle in (5.6.10) is small and is found to be:

$$\theta_* = \frac{r^{-1}}{y}. \quad (5.6.13)$$

Expressing everything in terms of the (α, ζ) angles, we take the integral (5.6.10) and find the boundary distance:

$$u_{12} = \frac{2\phi_r}{m} \cdot \frac{\sinh \zeta}{\cosh \alpha} \arctan \left(\frac{1}{\sinh(\alpha - \zeta)} \right), \quad (5.6.14)$$

A quick check shows that $u_{34} = u_{12}$.

In the same way, we can find the u_{23} distance. We use the same formula (5.6.10) for the integral, except with different parameters of the circle. The angles between which we integrate are found from (5.6.13) and inversion invariance,

$$\theta_1 = \frac{r^{-1}}{y}, \quad \theta_2 = \frac{r}{y}. \quad (5.6.15)$$

Here y is different from before and is equal to:

$$y = \frac{\phi_b}{m} \cdot \frac{\sinh \zeta}{\cosh(\alpha - \zeta)}. \quad (5.6.16)$$

Bringing everything together, we get for the second distance:

$$u_{23} = \frac{2\phi_r}{m} \cdot \frac{\sinh \zeta}{\cosh(\alpha - \zeta)} \arctan(\sinh \alpha). \quad (5.6.17)$$

The four-point function in the semiclassical approximation is determined by the geodesic distances between the operators. We rescale the four-point function, so that it is consistent with our definition of the two-point function (5.4.5):

$$W = \epsilon^{-4\Delta} \exp(-2\ell \cdot 2\Delta) = \frac{1}{\gamma^{4m}}, \quad \gamma \equiv \epsilon \cdot e^\ell. \quad (5.6.18)$$

The geodesic length of the trajectory of one particle is given by an integral:

$$\ell = \int_{\vartheta_*}^{\frac{\pi}{2}} \frac{r d\vartheta}{r \sin \vartheta} = -\ln \tan \frac{\vartheta}{2}, \quad (5.6.19)$$

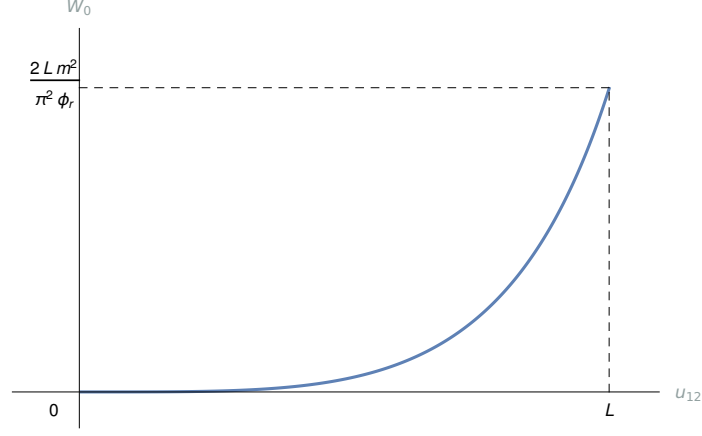


Figure 5.21: The connected part of the in-order four-point function.

with ϑ defined on fig. 5.20. ϑ_* is a small angle, determined by the geometry of fig. 5.20 to be:

$$\vartheta_* = \frac{1 + r^2}{2yr}. \quad (5.6.20)$$

Plugging it in the integral (5.6.19) and using the definition of the (α, ζ) angles, we find the exponentiated geodesic length γ :

$$\gamma = \frac{2\phi_r}{m} \cdot \frac{\sinh \zeta}{\cosh \alpha \cosh (\alpha - \zeta)}. \quad (5.6.21)$$

Another quick check shows that the lengths of the both segments of geodesics inside $NAdS$ are the same.

We can find the four-point function using directly (5.6.14) and (5.6.17), but find it convenient to change variables once again. Let us denote:

$$\begin{aligned} \sigma &\equiv \frac{\pi}{2} - \arctan (\sinh (\alpha - \zeta)), & 0 \leq \sigma \leq \pi, \\ \psi &\equiv \frac{\pi}{2} - \arctan (\sinh \alpha), & 0 \leq \psi \leq \pi/2. \end{aligned} \quad (5.6.22)$$

Then using various identities for hyperbolic and trigonometric functions, we find:

$$\begin{aligned} u_{12} &= \frac{2\phi_r}{m} \cdot \frac{\sigma}{\sin \sigma} (\cos \psi - \cos \sigma), \\ u_{23} &= \frac{2\phi_r}{m} \cdot \frac{\pi/2 - \psi}{\sin \psi} (\cos \psi - \cos \sigma), \end{aligned} \quad (5.6.23)$$

and the exponentiated geodesic length is now:

$$\gamma = \frac{2\phi_r}{m} \cdot (\cos \psi - \cos \sigma). \quad (5.6.24)$$

These expressions are much easier to analyze. They are also very similar to the answer for the two-point function (5.4.54, 5.4.55). As was the case for the two-point function, we cannot find an analytical solution for $\gamma(u_{12}, u_{23})$ in a closed form. Nevertheless, we can find a numerical solution relatively easily.

It is convenient to focus on the connected part of the four-point function:

$$W_0 \equiv \frac{\langle \mathcal{O}_1(u_1) \mathcal{O}_1(u_2) \mathcal{O}_2(u_3) \mathcal{O}_2(u_4) \rangle}{\langle \mathcal{O}_1(u_1) \mathcal{O}_1(u_2) \rangle \langle \mathcal{O}_2(u_3) \mathcal{O}_2(u_4) \rangle} - 1 = \frac{W(u_{12}, u_{23})}{G^2(u_{12})} - 1. \quad (5.6.25)$$

In terms of γ , this is:

$$W_0 = \left(\frac{\gamma_{2\text{pt}}}{\gamma_{4\text{pt}}} \right)^{4m} - 1. \quad (5.6.26)$$

We plot the numerical solution for W_0 on fig. 5.21. When the distance between operators u_{12} is small, the connected part is close to zero. It grows monotonically and reaches a maximum when $u_{12} = L/2$. If the boundary length L is relatively small, $L \ll \phi_r/m$, the maximum value of the four-point function is:

$$W_0(u_{12} = L/2) \sim \frac{2m^2 L}{\pi^2 \phi_r}. \quad (5.6.27)$$

We see that in general, the in-order four-point function is relatively close to zero. In the next Section, we find the out-of-order four-point function numerically and see that it is also closer to zero for lighter particles.

5.6.2 Four-point function: out-of-time ordered

Next we turn to the out-of-time ordered four-point function. We consider the worldlines of the particles intersecting in the $NAdS$ space. As before, we use the symmetries of the problem to simplify the discussion. We pick the parameters in such a way that the picture is both left-right and inversion symmetric.

The left-right symmetry requires that the first component of the dilaton on the left-hand side of the picture was the opposite of the one on the right-hand side:

$$Z_1|_{\text{left}} = -Z_1|_{\text{right}}. \quad (5.6.28)$$

In particular, it means that the first component of the dilaton in the 12 and 34 segments is zero (see fig. 5.22).

The invariance under inversion states that the trajectories of the particles intersect at the point

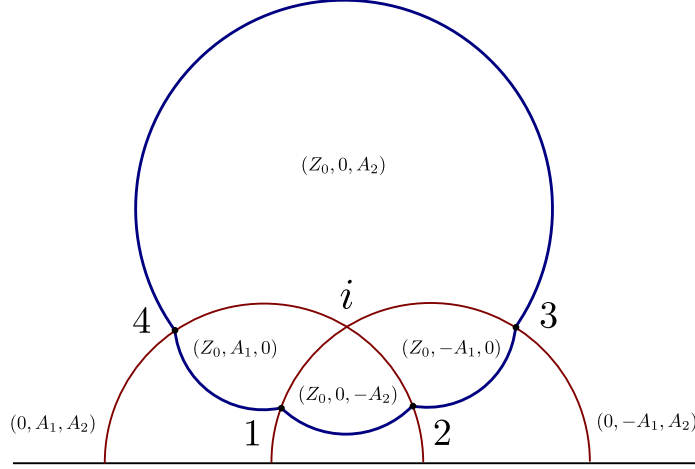


Figure 5.22: Out-of-time ordered four-point function

$(t, z) = (0, 1)$, or i in the complex plane. It implies that the zeroth component of the particles' charge vector vanishes:

$$A_0 = 0. \quad (5.6.29)$$

In addition, it requires that the second component of the dilaton vector be the opposite on the top and bottom parts of the picture:

$$Z_2|_{\text{left}} = -Z_2|_{\text{right}}. \quad (5.6.30)$$

It implies that the second component of the dilaton vanishes in the 23 and 14 segments. It also implies that the second component of the dilaton in the 12 segment is:

$$Z_2 = -A_2. \quad (5.6.31)$$

Bringing all this together, we arrive at the setup on fig. 5.22.

These symmetries guarantee that the lengths of the boundary segments are pairwise equal:

$$\begin{aligned} u_{12} &= u_{34}, \\ u_{23} &= u_{14}. \end{aligned} \quad (5.6.32)$$

Also, from fig. 5.22 we notice that there is yet another symmetry. Unlike in the time-ordered case, here there is no topological difference between segments 12 and 23, so after an exchange:

$$\begin{aligned} A_1 &\leftrightarrow A_2, \\ u_{12} &\leftrightarrow u_{23}, \end{aligned} \quad (5.6.33)$$

the picture goes back to itself. This means that when determining how the four-point function depends on the distance, we only have to find u_{12} and the other distance can be recovered from this symmetry.

The square of the vector A is fixed by the mass of the particle, and for convenience we introduce a parameter α such that:

$$\begin{aligned} A_1 &= m \sin \alpha, \\ A_2 &= m \cos \alpha. \end{aligned} \tag{5.6.34}$$

Therefore we have two parameters, Z_0 and α . Our goal is to find the two boundary distances, u_{12} and u_{23} , and the exponentiated geodesic length γ in terms of these parameters.

We focus on the 12 segment. The boundary distance is, as before, given by an integral:

$$u_{12} = 2\epsilon \cdot \int_0^{\theta_*} \frac{R d\theta}{y - R \cos \theta}. \tag{5.6.35}$$

Here we have rescaled from AdS to quantum mechanical length. As before, y and R are the vertical coordinate and the radius of the circle describing the $NAdS$ boundary, and they are given by:

$$\begin{aligned} y &= \frac{\phi_b}{Z_0 + m \cos \alpha}, \\ R^2 &= y^2 - \frac{Z_0 - m \cos \alpha}{Z_0 + m \cos \alpha}. \end{aligned} \tag{5.6.36}$$

We treat y as a large parameter. By the same logic as in Section 5.4, this allows us to simplify the expressions for distances and at the same time gives a conformal limit in the ultraviolet.

The radius of the trajectory of a particle is:

$$r = \frac{m}{A_2} = \frac{1}{\cos \alpha}, \tag{5.6.37}$$

and the center of the right half-circle has the horizontal coordinate of:

$$v = \frac{A_1}{A_2} = \tan \alpha. \tag{5.6.38}$$

The angle θ_* in 5.6.35 is found from the intersection of the two circles as:

$$\theta_* = \frac{r - v}{y} + O(y^{-2}). \tag{5.6.39}$$

Bringing all the parameters together and taking the integral, we find (in the $y \gg 1$ approximation):

$$u_{12} = \frac{4\phi_r}{\sqrt{Z_0^2 - m^2 \cos^2 \alpha}} \arctan \left(\sqrt{\frac{Z_0 + m \cos \alpha}{Z_0 - m \cos \alpha}} \cdot \frac{1 - \sin \alpha}{\cos \alpha} \right). \quad (5.6.40)$$

Using the symmetry (5.6.33), we immediately find the u_{23} distance as well:

$$u_{23} = \frac{4\phi_r}{\sqrt{Z_0^2 - m^2 \sin^2 \alpha}} \arctan \left(\sqrt{\frac{Z_0 + m \sin \alpha}{Z_0 - m \sin \alpha}} \cdot \frac{1 - \cos \alpha}{\sin \alpha} \right). \quad (5.6.41)$$

To make these expressions more manageable, we take the inverse tangents to be the new angular variables $\psi/2$, $\sigma/2$. In terms of these variables, the boundary distances become:

$$\begin{aligned} u_{12} &= \frac{2\phi_r}{m} \frac{\cos \alpha - \cos \psi}{\sin \psi \sin^2 \alpha} \cdot \psi, \\ u_{23} &= \frac{2\phi_r}{m} \frac{\sin \alpha - \cos \sigma}{\sin \sigma \cos^2 \alpha} \cdot \sigma, \end{aligned} \quad (5.6.42)$$

together with a constraint:

$$(\cos \alpha - \cos \psi) \cos^3 \alpha = (\sin \alpha - \cos \sigma) \sin^3 \alpha. \quad (5.6.43)$$

The conformal limit is reached when:

$$\begin{aligned} \psi &\rightarrow \alpha, \\ \sigma &\rightarrow \pi/2 - \alpha. \end{aligned} \quad (5.6.44)$$

The exponentiated length γ of a geodesic is:

$$\gamma = \epsilon \cdot e^\ell = \frac{2\phi_r}{Z_0 - m \sin \alpha \cos \alpha}. \quad (5.6.45)$$

In terms of the angles, it becomes:

$$\gamma = \frac{2\phi_r}{m} \frac{\cos \alpha - \cos \psi}{\sin^3 \alpha}. \quad (5.6.46)$$

This answer might appear not symmetrical in (ψ, σ) , which is an effect of the constraint (5.6.43).

We can find γ as a function of u_{12} numerically. As before, we are interested in the connected part of the four-point function:

$$W_0 = \left(\frac{\gamma_{2\text{pt}}(L/2)}{\gamma_{4\text{pt}}} \right)^{4m} - 1. \quad (5.6.47)$$

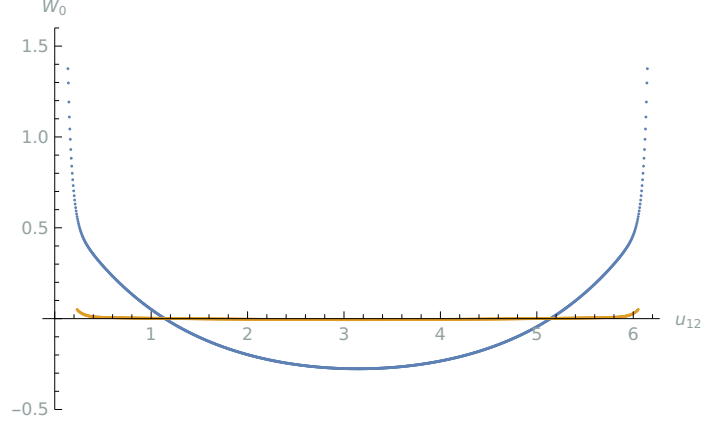


Figure 5.23: The connected part of the out-of-time order four-point function. The yellow graph has a larger ϕ_r and is closer to zero.

The solution for the connected part of the four-point function is plotted on fig. 5.23.

The UV limit, when $u_{12} \sim 0$ of the four-point function is reached when $\psi \sim 0$, $\sigma \sim \pi$, $\alpha \sim 0$. In this limit γ grows linearly with distance. When mass is relatively small, $mL \ll 1$, the four-point function is:

$$W_0|_{u_{12} \rightarrow 0} \sim \left(\frac{\gamma_{2\text{pt}}}{\gamma_{4\text{pt}}} \right)^{4m} - 1 \sim 0. \quad (5.6.48)$$

When mass is relatively large, $mL \gg 1$, the four-point function becomes:

$$W_0|_{u_{12} \rightarrow 0} \sim \left(\frac{\gamma_{2\text{pt}}}{\gamma_{4\text{pt}}} \right)^{4m} - 1 \sim 2^{4m} - 1. \quad (5.6.49)$$

In the symmetric case, when $u_{12} = u_{23} = L/4$, the angles are $\sigma = \psi$, $\alpha = \pi/4$. When $mL \ll 1$, then $\gamma \sim L/\pi$ and the four-point function is once again close to zero. When $mL \gg 1$, the exponentiated length is constant:

$$\gamma \sim \frac{4\phi_r}{m} (1 + \sqrt{2}), \quad (5.6.50)$$

and the four-point function becomes:

$$W_0 \sim \left(\frac{1}{1 + \sqrt{2}} \right)^{4m} - 1. \quad (5.6.51)$$

So we see that the connected four-point function for the particles of small mass always stays close to zero. The four-point function for heavier particles grows to a potentially large value in the ultraviolet, and becomes negative in the “infrared” when both distances are macroscopic. This is exactly what we see on fig. 5.23.

5.6.3 Schwarzian limit

As a reality check, we find the correction to the conformal four-point function when mass is small.

This should reproduce the result in [2, 43]. We proceed in the same way as in Section 5.4.6.

The time-ordered four-point function is conformal when $\psi = \sigma$. We relax the condition and take:

$$\begin{aligned}\sigma &= \alpha + \delta, \\ \psi &= \alpha - \delta.\end{aligned}\tag{5.6.52}$$

The δ parameter measures how far we are from the conformal limit and is proportional to mass:

$$\delta = c \cdot m.\tag{5.6.53}$$

Then we find for the segment lengths:

$$\begin{aligned}u_{12} &= \frac{4\phi_r}{m} \delta (\alpha + \delta (1 - \alpha \cot \alpha)) + O(\delta^2), \\ u_{23} &= \frac{4\phi_r}{m} \delta (\pi/2 - \alpha + \delta (1 + (\pi/2 - \alpha) \cot \alpha)) + O(\delta^2).\end{aligned}\tag{5.6.54}$$

We see that σ, ψ are roughly the segment angles for u_{12}, u_{23} .

The full boundary length then is:

$$L = 2(u_{12} + u_{23}) = \frac{4\phi_r}{m} \delta (\pi + 4\delta (1 + (\pi/4 - \alpha) \cot \alpha)) + O(\delta^2),\tag{5.6.55}$$

which allows us to fix c :

$$c = \frac{L}{4\pi\phi_r}.\tag{5.6.56}$$

The exponentiated geodesic length is:

$$\gamma = \frac{4\phi_r}{m} \delta \sin \alpha + O(\delta^2).\tag{5.6.57}$$

Bringing everything together, we can find γ in terms of the segment length:

$$\gamma = \frac{L}{\pi} \sin \frac{\pi u_{12}}{L} \left(1 - \frac{4\delta}{\pi} (1 - \alpha \cot \alpha) (1 + (\pi/2 - \alpha) \cot \alpha) \right) + O(\delta^2).\tag{5.6.58}$$

The four-point function $W = 1/\gamma^{4m}$. To study the corrections, we extract the connected part:

$$W_0 = \frac{W(u_{12}, u_{23})}{G^2(u_{12})} - 1. \quad (5.6.59)$$

In the Schwarzian limit, the two-point function is given by (5.4.85), and the connected part of the four-point function becomes:

$$W_0^{(\text{in-order})} = \left(\frac{\gamma_{2\text{pt}}}{\gamma_{4\text{pt}}} \right)^{4m} - 1 = \frac{2m^2 L}{\pi^2 \phi_r} \cdot \eta \left(\frac{\pi u_{12}}{L} \right)^2 + O(\delta^3). \quad (5.6.60)$$

where as before $\eta(\alpha) \equiv 1 - \alpha \cot \alpha$. This is the same answer as in [43].

In the same way, we find the first correction to the out-of-time ordered four-point function. To do that, we relax the condition (5.6.44). In doing so, we need to ensure that the constraint (5.6.43) is satisfied. Then the angles become:

$$\begin{aligned} \psi &= \alpha + \delta \sin^2 \alpha, \\ \sigma &= \pi/2 - \alpha + \delta \cos^2 \alpha. \end{aligned} \quad (5.6.61)$$

The small parameter δ is again proportional to the mass:

$$\delta = c \cdot m. \quad (5.6.62)$$

Expanding (5.6.42) in δ , we find:

$$\begin{aligned} u_{12} &= \frac{2\phi_r}{m} \delta \left(\alpha + \delta \sin^2 \alpha \left(1 - \frac{\alpha}{2} \cot \alpha \right) \right) + O(\delta^2), \\ u_{23} &= \frac{2\phi_r}{m} \delta \left(\pi/2 - \alpha + \delta \cos^2 \alpha \left(1 - \left(\pi/4 - \frac{\alpha}{2} \right) \tan \alpha \right) \right) + O(\delta^2). \end{aligned} \quad (5.6.63)$$

We see that here as well, (σ, ψ) are approximately the segment angles. The full boundary length is:

$$L = \frac{2\pi\phi_r}{m} \cdot \delta \left(1 + \delta \left(\frac{2}{\pi} - \frac{1}{4} \sin 2\alpha \right) \right) + O(\delta^2). \quad (5.6.64)$$

From here we fix the coefficient in (5.6.62):

$$\delta \sim \frac{mL}{2\pi\phi_r}. \quad (5.6.65)$$

The exponentiated geodesic length becomes:

$$\gamma = \frac{2\phi_r}{m} \delta \left(1 + \frac{1}{4} \delta \sin 2\alpha \right) + O(\delta^2). \quad (5.6.66)$$

Plugging in the boundary length (5.6.64), we find:

$$\gamma = \frac{L}{\pi} \left(1 + \frac{mL}{2\pi\phi_r} \left(\frac{1}{2} \sin 2\alpha - \frac{2}{\pi} \right) \right). \quad (5.6.67)$$

The connected part of the four-point function then becomes:

$$W_0^{(\text{out-of-order})} = -\frac{2m^2L}{\pi^2\phi_r} \left(\frac{\pi}{2} \sin \frac{2\pi u_{23}}{L} + 1 \right) + O(\delta^3). \quad (5.6.68)$$

After analytic continuation, the sine in (5.6.68) becomes exponentially decaying. The real-time correlation function corresponds to $u_{23} = \beta/4 + it$. Then the connected four-point function found from the Schwarzian limit becomes:

$$W_0^{(\text{out-of-order})} \sim -\frac{1}{N} \cosh \left(\frac{2\pi}{\beta} t \right). \quad (5.6.69)$$

This demonstrates the chaotic behavior of the four-point function. In the next Section, we see that the Schwarzian limit describes well the out-of-order four-point function at early times.

5.7 Four-point function in real time

A useful measure of chaotic behavior of theory is the out-of-time ordered four-point function [81], [69]. To construct it, we place our operators equidistantly on the thermal circle as follows:

$$W^{(\text{out-of-order})} = \langle \mathcal{O}_1(-it/2) \mathcal{O}_2(\beta/4 + it/2) \mathcal{O}_1(\beta/2 - it/2) \mathcal{O}_2(3\beta/4 + it/2) \rangle. \quad (5.7.1)$$

It is convenient to divide this four-point function by the product of the two-point functions $\langle \mathcal{O}_1 \mathcal{O}_1 \rangle$, $\langle \mathcal{O}_2 \mathcal{O}_2 \rangle$:

$$W_0^{(\text{out-of-order})} = \frac{W^{(\text{out-of-order})}}{G(\beta/2)^2}. \quad (5.7.2)$$

In terms of the geodesic lengths, the normalized four-point function becomes:

$$W_0^{(\text{out-of-order})} = \left| \frac{\gamma_{2\text{pt}}(\beta/2)}{\gamma_{4\text{pt}}(\beta/2 - it)} \right|^{4m}. \quad (5.7.3)$$

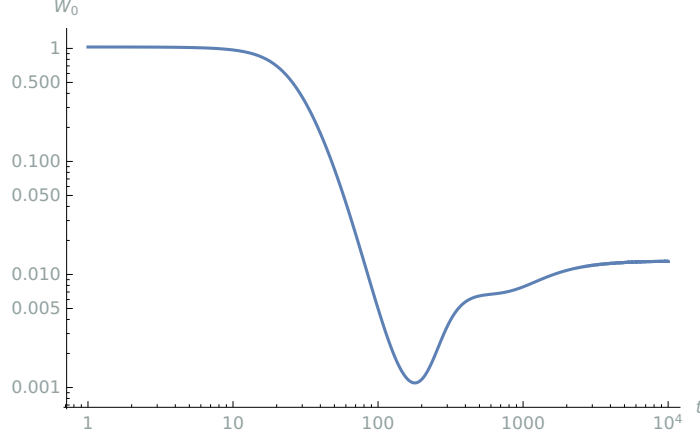


Figure 5.24: The out-of-time ordered four-point function in the small ϵ approximation.

The out of time ordered four-point function is related to thermal average of a double commutator:

$$C(t) = \left\langle -[\mathcal{O}_2(t), \mathcal{O}_1(0)]^2 \right\rangle_\beta = 2 \left(1 - W_0^{(\text{out-of-order})} \right). \quad (5.7.4)$$

In a theory dual to a black hole [68], one expects the double commutator to grow exponentially at first, $C(t) \sim \exp(\lambda_L t)$. This growth is referred to as Lyapunov behavior. The exponent of the growth is bounded above, $\lambda_L \leq 2\pi/\beta$. This growth does not continue indefinitely, and at about scrambling time $t_* \sim \beta$ the double commutator saturates. It approaches a constant value exponentially slowly. This is similar to what is expected at the beginning of the Ruelle region.

In our notation, the arrangement of operators in (5.7.1) corresponds to the distances being complex conjugates:

$$\begin{aligned} u_{12} &= \beta/4 + it, \\ u_{23} &= \beta/4 - it. \end{aligned} \quad (5.7.5)$$

In the small ϵ approximation, the boundary distances u_{12}, u_{23} and the exponentiated geodesic distance γ are analytic functions of the angles and therefore can be relatively easily continued to complex plane. The angles become subject to the conditions:

$$\begin{aligned} \sigma &= \bar{\psi}, \\ \alpha + \bar{\alpha} &= \pi/2. \end{aligned} \quad (5.7.6)$$

Using these two conditions (which improve the convergence of the numerical method), the definitions of the boundary distances (5.6.42) and the exponentiated geodesic distance (5.6.46), we can find the out-of-time ordered four-point function numerically. The result is on fig. 5.24. The overall structure

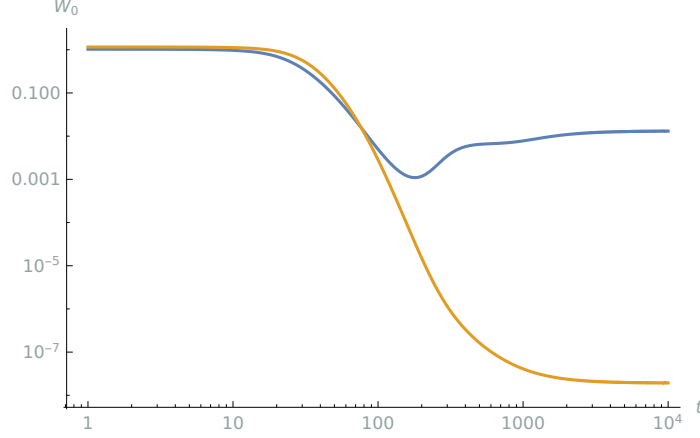


Figure 5.25: The out-of-time ordered four-point function with the extrinsic curvature correction (blue) compared to the two-point function found as the exponentiated geodesic distance (yellow).

of this four-point function is very similar to the real-time two-point function on fig. 5.17. We see that the normalized four-point function starts close to 1, and exponentially decays to a minimum at $t \sim \phi_r/m$. After that, it grows to a plateau. At long times, the four-point function is:

$$W_0^{(\text{out-of-order})} \Big|_{t \rightarrow \infty} \sim \frac{m}{\phi_r} \sim 1/N. \quad (5.7.7)$$

As we have seen for a real-time two-point function, addition of the extrinsic curvature term changes the picture quite a bit. The two-point function in (5.7.2) is in Euclidean time, and the extrinsic curvature correction for it is small. Therefore, to find the full correction, we consider only the four-point function. We do it numerically, but first let us consider the long time limit.

At long times, the curvature is very close to 1:

$$K_{t \rightarrow \infty} = -\frac{y}{R} = -1. \quad (5.7.8)$$

The curvature of the empty AdS space is never very close to 1. Hence the action of extrinsic curvature (normalized by the action of the empty AdS space) is:

$$(I - I_0)_{t \rightarrow \infty} = (4I_{\text{seg}} + 4I_{\text{cusp}} - I_0)_{t \rightarrow \infty} = -\phi_b \left(\frac{\beta}{\epsilon} - \sqrt{\left(\frac{\beta}{\epsilon} \right)^2 + (2\pi)^2} \right) - 4m = \frac{2\pi^2 \phi_r}{\beta} - 4m. \quad (5.7.9)$$

Therefore, the plateau on fig. 5.24 becomes exponentially lower:

$$W_0^{(\text{out-of-order})} \Big|_{t \rightarrow \infty} \sim \exp \left(-\frac{2\pi^2 \phi_r}{\beta} + 4m \right) \sim \exp(-N). \quad (5.7.10)$$

Expanding the extrinsic curvature to first order in ϵ^2 , we can find the correction numerically. (We do not use the full answer for the four-point function, since it has branch points and is hard to continue to complex plane.) The result is on fig. 5.25. At small time, the correction is small, and the four-point function demonstrates exponential decay. However, at time $t \sim \beta$ the decay slows down, and the four-point function approaches a small but nonzero value. This is the beginning of the Ruelle region, describing thermalization of a black hole [68]. However, the onset of the Ruelle behavior is expected to be at roughly the scrambling time $t_* \sim \log N$, and in our case the exponentially decaying four-point function reaches the value (5.7.10) at time $t \sim \phi_r \sim N$. This seems puzzling to us.

Note that the four-point function never approaches zero, and its value at long times is exponentially small in N . If we look at the four-point function in the energy basis, we find a very similar structure to what we have seen in Section 5.5. Disregarding the off-diagonal terms in the $\mathcal{O}_1, \mathcal{O}_2$ operators, we can write the four-point function as:

$$W \sim \sum_{n,m} |\langle n | \mathcal{O}_1 | n \rangle|^2 |\langle m | \mathcal{O}_2 | m \rangle|^2 e^{-\beta/2 \cdot E_n - \beta \cdot E_m} e^{i(E_n - E_m)t}. \quad (5.7.11)$$

If the diagonal terms in the operators are close to 1, then by the same logic as before the four-point function approaches a finite value at long times. Since we normalize by the two-point functions which are independent of real time t , this should hold after normalization as well. As was the case for the two-point function, our four-point function does not capture the rapid fluctuations in (5.7.11) and therefore represents the quantum mechanical four-point function only in the averaged sense.

To our knowledge, this effect has not been tested in the SYK model. Although it is non-perturbative in N , it should be visible in a numerical simulation.

5.8 Appendix: Two-point function for particles with negative mass

Our setup allows us to also study the two-point function for particles with negative mass. It corresponds to a picture like fig. 5.4(a), with cusps in the $NAdS$ boundary pointing outward. We use the same angular parameterization as before:

$$w_{1,2} = \cos \alpha_{1,2}, \quad (5.8.1)$$

with the $w_1 > w_2$ to make the cusps point outward.

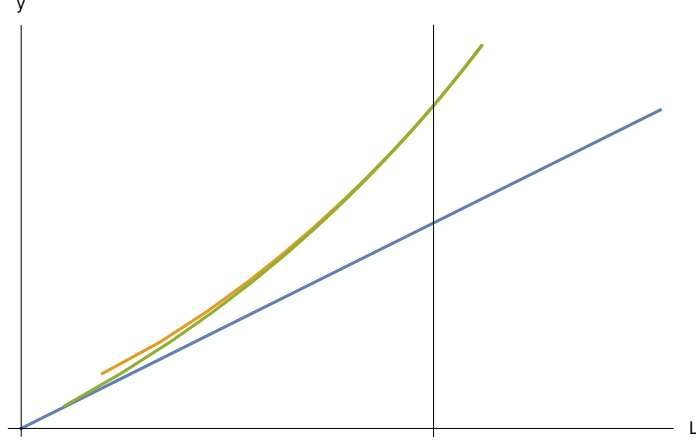


Figure 5.26: y as a function of boundary length L for a theory with negative mass. The blue line is the conformal result, the blue line is the result with $y \sim R$, or very large l_0 compared to all the other parameters. The vertical line divides the region with real α from the region with imaginary α , and one can see that there is no cusp or discontinuity at this point.

For a positive w , there is no upper bound on its value. It can be greater than one, making the formal parameter α imaginary. We will see that the point where α becomes imaginary does not introduce any singularities, and it is special only for our choice of parameters.

We start with finding the symmetric correlation function, that is, consider $\alpha_1 = \pi - \alpha_2$. The length of the boundary L and the exponentiated geodesic length γ are as before, see (5.4.40) and (5.4.39). $\gamma(L)$ cannot be in general solved analytically, but large and small distances are accessible to us. First, let us take α close to $\pi/2$:

$$\alpha = \frac{\pi}{2} - \delta, \quad \delta \gg \frac{m}{2\phi_b}. \quad (5.8.2)$$

We need the second condition in (5.8.2) to keep y large. Then:

$$\gamma = \frac{4}{m}\delta, \quad L = \pi\gamma. \quad (5.8.3)$$

From this, we see that the two-point function has the conformal form:

$$G \sim \left(\frac{L}{\pi}\right)^{-2\Delta}. \quad (5.8.4)$$

Like before, it is a consequence of the approximation we are taking. The two-point function has to be cut off at $L \sim m\epsilon^2/\phi_r$. We can see it on fig. 5.26.

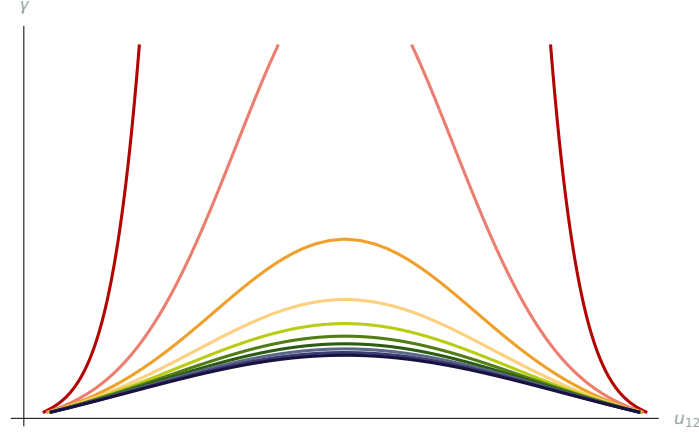


Figure 5.27: $\gamma(u_{12})$ for negative mass. The conformal result $\gamma \sim \sin u_{12}$ is the dark blue line on the bottom. The red line corresponds to the largest mass.

Next, let us see what happens when α goes to the complex plane:

$$\alpha = \delta. \quad (5.8.5)$$

The two-point function is finite at that point. The length parameters are:

$$\gamma = \frac{4\phi_r}{m}, \quad L = 2\gamma. \quad (5.8.6)$$

The two-point function is still close to the conformal one (5.4.30), although it starts to deviate:

$$G \sim \left(\frac{L}{2}\right)^{-2\Delta}. \quad (5.8.7)$$

We can also look at the expressions for L and γ , (5.4.40, 5.4.39), and see that they are regular at $\alpha \rightarrow 0$. Therefore we do not encounter any cusp or discontinuity when α becomes imaginary.

Considering large imaginary α , we recover the region of large L :

$$\alpha = i\Lambda \quad \Rightarrow \quad y = \frac{\phi_b}{m} \cdot e^\Lambda. \quad (5.8.8)$$

Then the radius of the boundary is:

$$R^2 = y^2 + \sinh^2 \alpha \sim y^2, \quad \phi_b/m \gg 1, \quad (5.8.9)$$

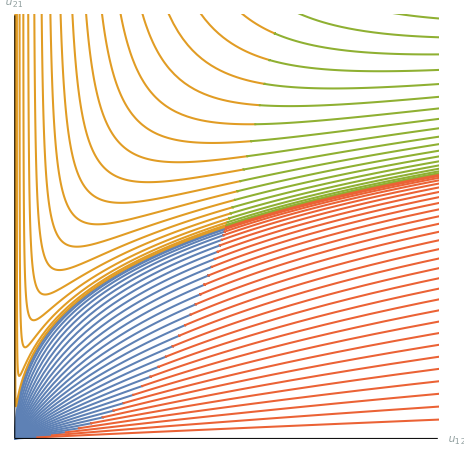


Figure 5.28: Boundary distances u_{12} and u_{21} as functions of angles α_1, α_2 . Blue region corresponds to real angles, green to imaginary ones, and yellow and red to one angle being real and one imaginary.

and the length of the boundary is:

$$L = 4y\epsilon \frac{\Lambda}{\sinh \Lambda} \sim \frac{4\phi_r}{m} \Lambda. \quad (5.8.10)$$

Then the two-point function is exponential, as expected in a theory with particles of negative mass:

$$G \sim e^{m^2 L / (2\phi_r)}. \quad (5.8.11)$$

On fig. 5.26 we see the beginning of this exponential growth.

We can also solve for the two-point function with fixed boundary length L , while changing mass (see fig. 5.27). In the small ϵ approximation, the two-point function approaches the conformal $\sin u_{12}$ as mass decreases. For larger mass, the two-point function grows exponentially for small distances.

Using numerical methods, we can also answer the question of whether the angular parameters are in one-to-one correspondence with the length parameters. To answer that, we vary $\alpha_{1,2}$ and plot the length on the u plane. The angles are artificial parameters and can be real or imaginary:

$$0 \leq \alpha \leq \pi \quad \text{or} \quad \alpha \in i\mathbb{R}. \quad (5.8.12)$$

We can easily check that the distances are either both positive or both negative, depending on the sign of the mass. So varying angles, we cover two quarters of the u plane. Right now we are interested in the case when both distances in (5.4.51) are negative. The results are on fig. 5.28. Varying the α parameters, we cover a quarter of the u plane exactly once.

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