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Matching Curved Lattices to Anisotropic Tangent Planes

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History - Near-Conformal Field Theories

I last gave a seminar at KEK in 2014, titled “Non-perturbative quantum field theory on curved manifolds”, so I am here to give a once-per-decade update.

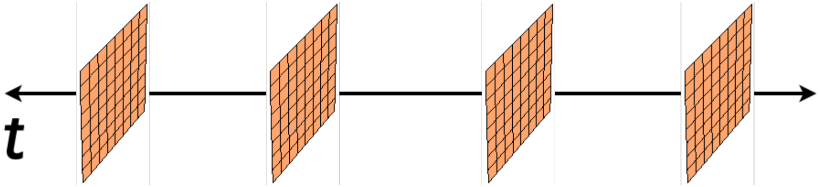
The motivation today is still the same as back then: the microscopic physics explaining the Higgs mechanism remains unclear. It could be due to new strong dynamics, which would make the Higgs boson a composite particle.

It was long realized that a composite Higgs mechanism is a problem for the Yukawa mechanism of fermion mass generation since it would naturally lead FCNC through four-fermi operators. Yukawa $\frac{\bar{q}q\,\bar{Q}Q}{\Lambda^{\Delta-1}}$ FCNC $\frac{\bar{q}q\,\bar{q}q}{\Lambda^{\Delta-1}}$ where q are SM fermions and Q are Higgs constituents. Δ is scaling dimension of $\bar{Q}Q$ mass operator. Suppressing FCNC requires $\Lambda > 1000$ TeV.

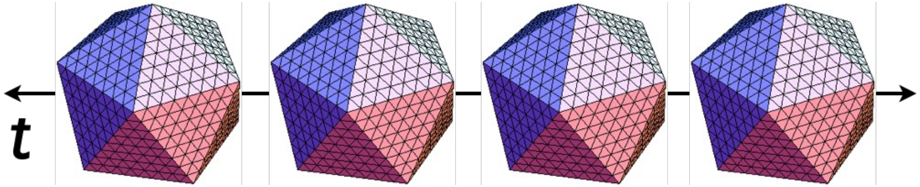
The apparent solution, also long known, is a near-conformal (walking) gauge theory.

Studying near-conformal gauge theories on the lattice using traditional methods is challenging due to the very long correlation lengths and very slow approach to the continuum limit.

Hamiltonian vs. Radial quantization



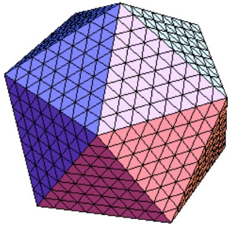
- Eigenstates of **Hamiltonian** defined on surfaces of constant Euclidean time.
- Eigenstates labeled by spatial momenta $\vec{p}$ due to translation invariance.
- Dynamical dispersion relation:
$$E_n^2 = |\vec{p}|^2 + m_n^2$$
- Correlations:
$$C(\vec{p}, t, t') = \sum_n A_n(\vec{p})\, e^{-E_n|t-t'|}$$
- Conformal correlations are power-law.



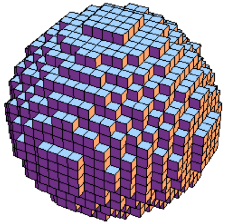
- Eigenstates of **Dilatation operator** defined on surfaces of constant radius.
- Eigenstates labeled by angular momenta (ℓ, m_ℓ) due to rotational invariance.
- Dynamical dispersion relation (conformal):
$$\Delta_{\mathcal{O},\ell} = \Delta_{\mathcal{O},0} + \ell$$
- Correlations (conformal):
$$C(\ell, t, t') = \sum_{\mathcal{O}} B(\Delta_{\mathcal{O}}, \ell)\, e^{-\Delta_{\mathcal{O},\ell}|t-t'|}$$
- Near-conformal would modify integer spacing and t-dependence.

History - Attempts at Radial Quantization of D=3 critical Ising model

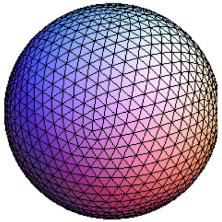
- $\mathbb{R}\times$Icosahedron with Ising action.
  - Scale Invariance 👍 but no rotational invariance: not conformal 👎.



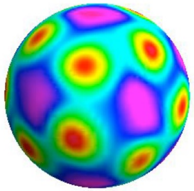
- $\mathbb{R}\times$Cubic sphere with Ising action.
  - Frustrated scale invariance 👎.



- $\mathbb{R}\times\mathbb{S}^2$ with unequal areas and lengths and classically perfect ϕ^4 action using finite element method (FEM).
  - Frustrated scale invariance 👎.



- $\mathbb{R}\times\mathbb{S}^2$ with equal area prescription of [Tegmark 1996] and FEM action.
  - Better but still frustrated scale invariance 👎.



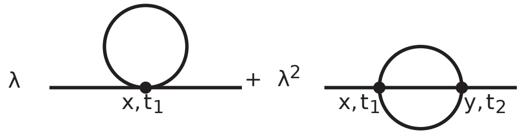
- Frustration means can't make theory critical everywhere.

Quantum Finite Elements – A method that works

- I concluded my 2014 KEK seminar by mentioning we would explore perturbative one- and two-loop quantum counterterms to the classical FEM action: Quantum Finite Elements (QFE)
  - Works on discretization with unequal lengths and areas.
  - One limitation: the lattice coupling must be small enough for lattice perturbation theory to be valid.
  - Divergences not a problem on lattice, bare mass is fine-tuned to critical surface.
  - Finite parts of relevant diagrams include contributions from non-uniform UV cutoff.
  - Also, discovered a novel irrelevant contribution $Ric\ \phi^2$ due to non-uniform curvature density $\sim\mathcal{O}(a^{0.41})$.

$$S = \frac{1}{2}\sum_{y\in\langle x,y\rangle}\frac{l^*_{xy}}{l_{xy}}\left(\tilde\phi_{t,x}-\tilde\phi_{t,y}\right)^2 + \frac{a^2}{4R^2}\sqrt{\tilde g_x}\,\tilde\phi^2_{t,x} + \sqrt{\tilde g_x}\left[\frac{a^2}{a_t^2}\left(\tilde\phi_{t,x}-\tilde\phi_{t+1,x}\right)^2 + m_0^2\tilde\phi^2_{t,x} + \lambda_0\tilde\phi^4_{t,x}\right]$$

$$S_{QFE} = S - \sum_{t,x}\sqrt{\tilde g_x}\left[6\lambda_0\delta G_x - 24\lambda_0^2\delta G_x^{(3)}\right]\tilde\phi^2_{t,x}.$$



Demonstration that QFE works in three plots: (1) Binder Cumulant

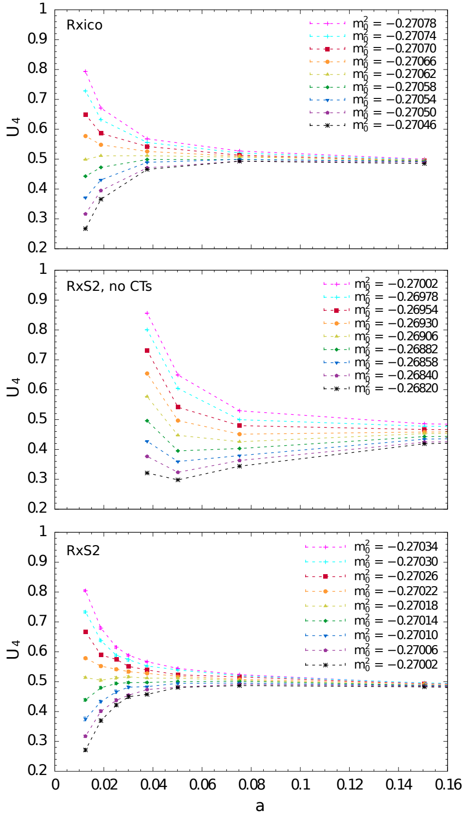
- 4th-order Binder cumulant is a common observable used to locate the critical surface:

$$U_4 = \frac{3}{2}\left[1 - \frac{\langle M^4 \rangle}{3\,\langle M^2 \rangle^2}\right], \qquad M = \sum_x \sqrt{g_x}\,\phi_x$$

- The figure shows the Ising model on $\mathbb{R}\times$Icosahedron is critical, FEM ϕ^4 theory is not, and QFE ϕ^4 theory restores criticality.
- Finite size scaling:

$$U_4(m_0^2, \lambda_0) = U_{4,crit} + A\left(m_0^2 - m_{0,crit}^2\right)a^{-1.6} + B\,a^{0.8} + \cdots$$

- Brower, GTF, Neuberger, PLB 271 (2013) 299.
- A. Gasbarro et al, PRD 104 (2021) 094502.



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Demonstration that QFE works in three plots: (2) Two-Point Function

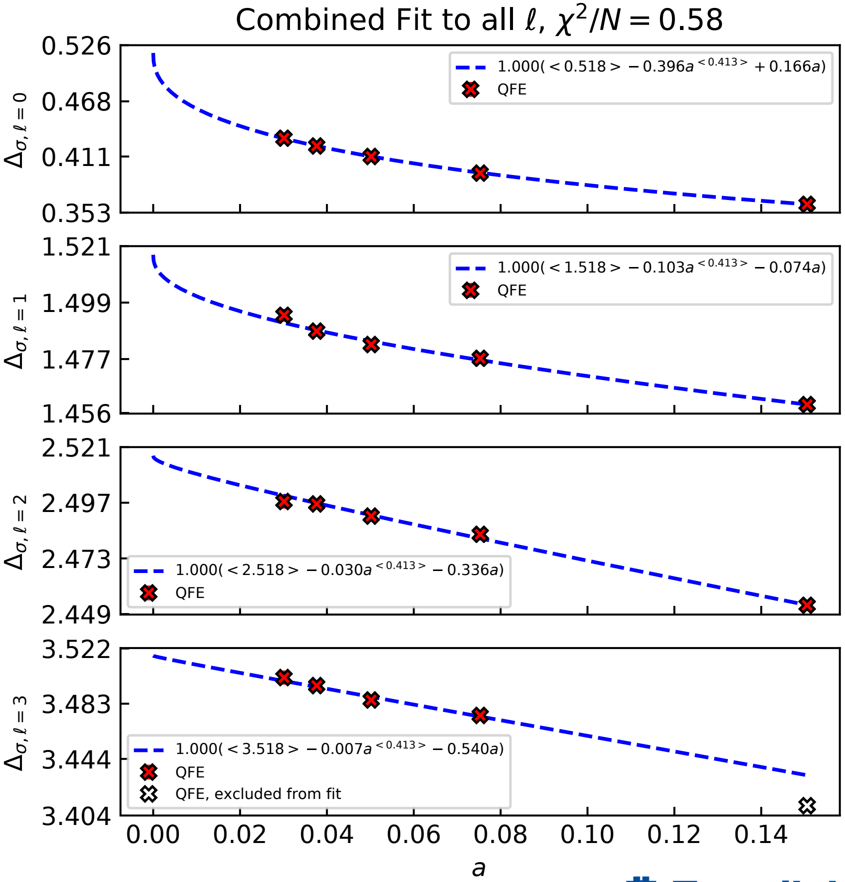
- Extract scaling dimensions for lowest Z_2-odd scalar with different angular momenta ℓ.

$$C(\ell, t, t') = \sum_{\mathcal{O}} B(\Delta_{\mathcal{O}}, \ell)\, e^{-\Delta_{\mathcal{O},\ell}|t-t'|}$$

- Compare to conformal dispersion relation:

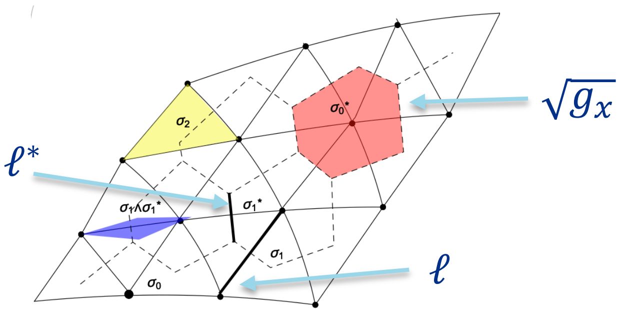
$$\Delta_{\mathcal{O},\ell} = \Delta_{\mathcal{O},0} + \ell$$

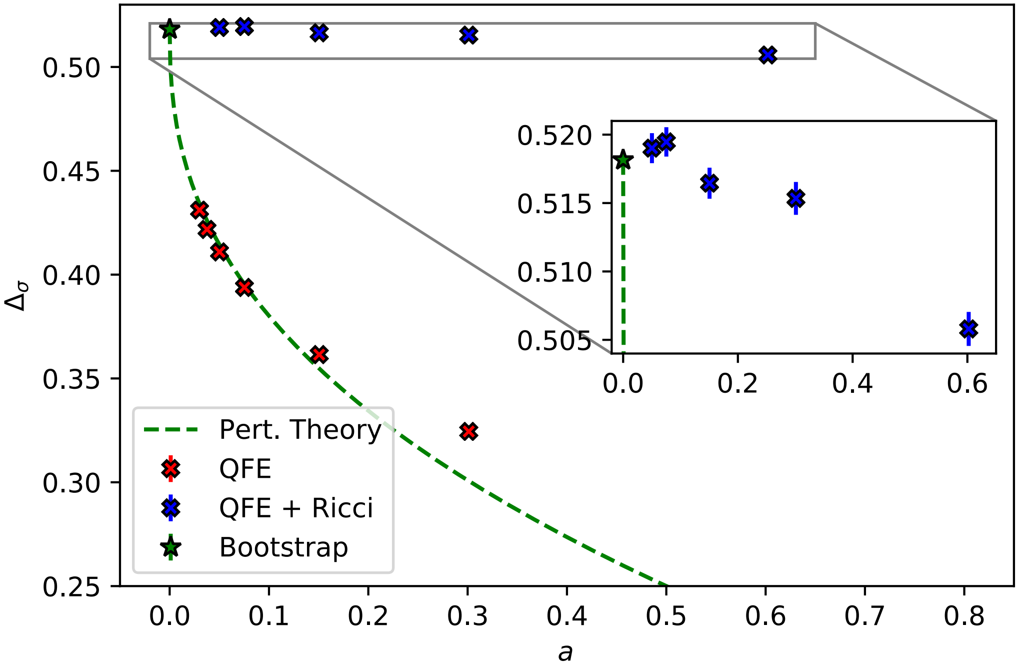
- Dispersion relation indicates full Poincare invariance recovered in continuum limit.
- Conformal invariance requires scale invariance and Poincare invariance.
- $\mathbb{R}\times$Icosahedron had scale invariance but small breaking of rotational invariance, not conformal.
- Note Ricci term artifact.



Demonstration that QFE works in three plots: (3) Ricci-Improved Action

- Ricci term on a sphere is normally just a constant that is absorbed by a shift in the bare mass: $m_0^2 \to m_0^2 + R^{-2}$
- On our lattice, uniform curvature was achieved in continuum limit, but curvature density was not uniform.
- Effect computable in PT and can be cancelled with a counterterm.





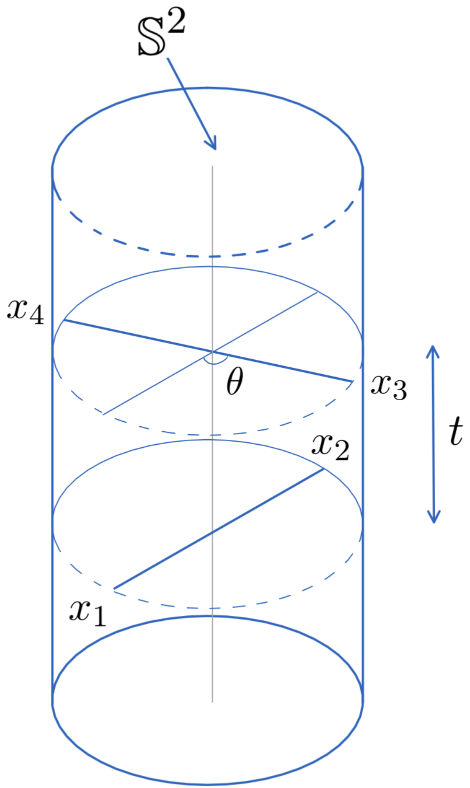
Four-point Function – Essential for Studying CFTs

- Full data of CFT is scaling dimensions $\Delta_{\mathcal{O}}$ and OPE coeffs $f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$.
- OPE coeffs computed easily in radial quantization using partial wave expansion. Lüscher's method for a cylinder.
- Conformal blocks $B_{n,j}(\Delta_{\mathcal{O}})$ are known functions with no free parameters, like $Y_{\ell m}$ for conformal symmetry. Possible question with best choice of normalization for lattice studies.

$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{j\in 2\mathbb{N}_0} c_j(t) P_j(\cos\theta)$$

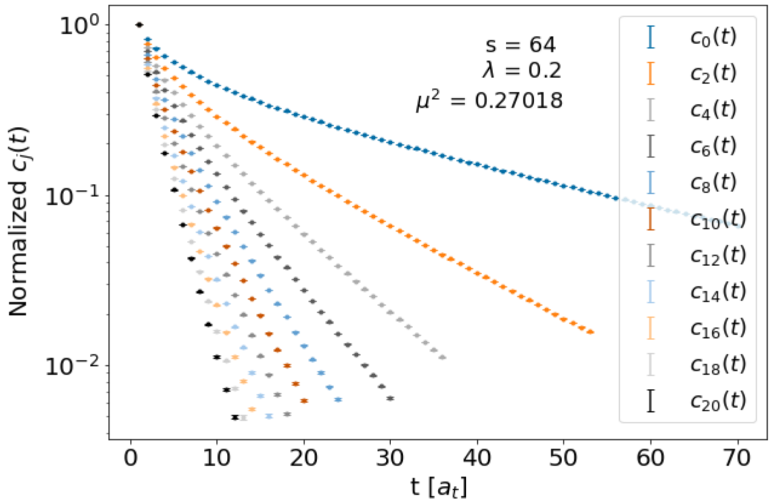
$$c_j(t) = \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{\substack{n\in 2\mathbb{N}_0 \\ n\geq|j-l|}}^{\infty} B_{n,j}(\Delta_{\mathcal{O}}) e^{-(\Delta_{\mathcal{O}}+n)t}$$

even spin & parity



Four-point Function – Data Analysis

- Simultaneous fits to $c_0(t)$ and $c_2(t)$ using four primaries $\mathcal{O} = \epsilon, \epsilon', T, T'$ and $n \in [0,20]$ and various $[t_{min}, t_{max}]$ combinations.
- Final results using Bayesian model averaging [Jay, Neil, 2020].
- Conformal Blocks: Hogervorst and Rychkov 2013, M. S. Costa et al. 2016.
- QFE: Anna-Maria Glück et al. 2023.



$$\frac{\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle\sigma(x_1)\sigma(x_2)\rangle\langle\sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{j\in 2\mathbb{N}_0} c_j(t) P_j(\cos\theta)$$

$$c_j(t) = \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{\substack{n\in 2\mathbb{N}_0 \\ n\geq|j-l|}}^{\infty} B_{n,j}(\Delta_{\mathcal{O}}) e^{-(\Delta_{\mathcal{O}}+n)t}$$

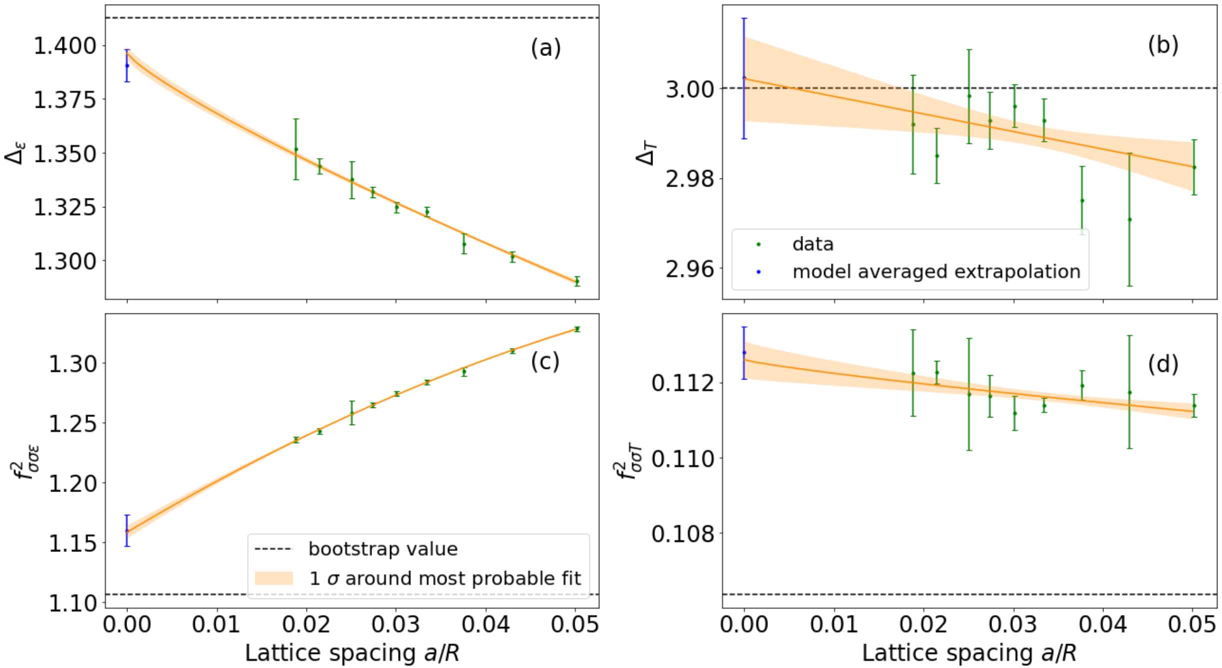
Four-point Function – Continuum extrapolation at small fixed bare m_0^2, λ_0

- Extrapolation to continuum limit and again Bayesian model averaging to estimate extrapolated values.
- Note Δ_T consistent with 3, which is required for energy-momentum tensor.

$$f_{\text{FSS}} = c_1\left(\frac{a}{R}\right)^{0.83} + c_0$$

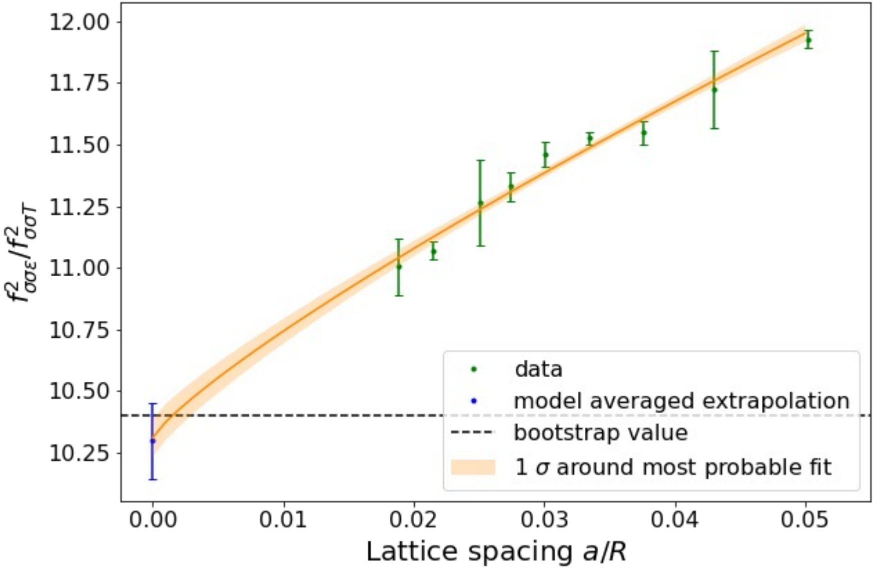
$$f_{\text{linear}} = c_1\left(\frac{a}{R}\right) + c_0$$

$$f_{\text{quadratic}} = c_2\left(\frac{a}{R}\right)^2 + c_1\left(\frac{a}{R}\right) + c_0$$



Four-point Function – Ratios of OPE coefficients

- Ratios of OPE coefficients agree with conformal bootstrap. Not sure why:
  - Maybe a normalization issue with conformal blocks?
  - Maybe m_0^2 not tuned accurately enough to $m_{0,crit}^2$ at fixed λ_0?
  - Maybe m_0^2 and λ_0 must also be tuned simultaneously to zero when using perturbative counterterms?

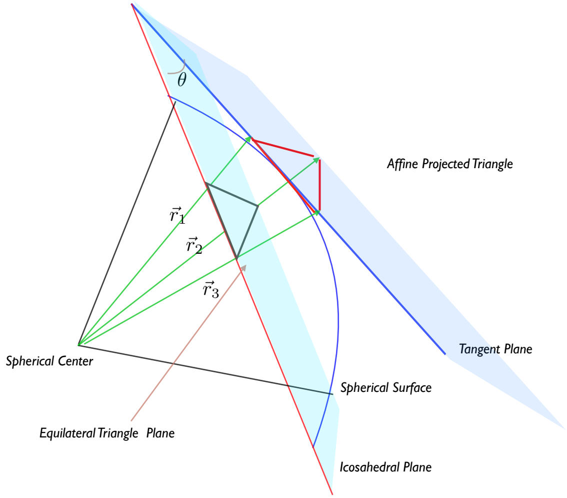


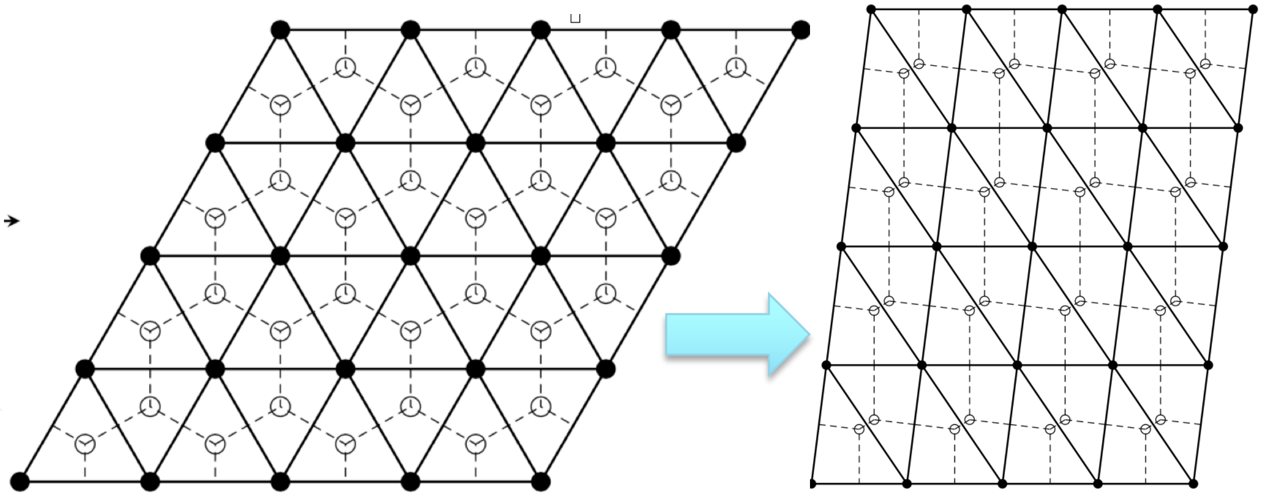
$$\left(f^2_{\sigma\sigma\epsilon}/f^2_{\sigma\sigma T}\right)^{fit} = 10.30(16)$$

$$\left(f^2_{\sigma\sigma\epsilon}/f^2_{\sigma\sigma T}\right)^{bootstrap} = 10.4017(24)$$

Going Beyond Perturbative QFE: Affine Conjecture

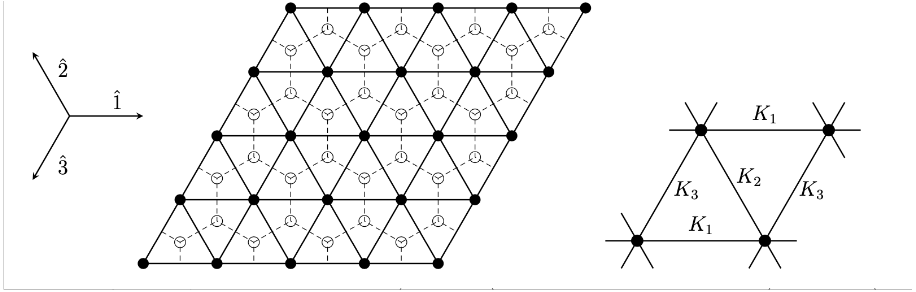
- We can keep the icosahedral graph the same and move the points on the surface of the sphere to make the deficit angles and dual areas as identical as possible (curvature density).
- But, the triangles will never be equilateral. So, when projected onto the tangent plane it will look like an affine transformation. Nearby lattice points will have slightly different tangent planes with different affine transformations: smooth affine connection describes how the affine transformation varies from point to point.





Going Beyond Perturbative QFE: Affine Conjecture Problem

- Geometry tells us that at each lattice point on the sphere, there will be a tangent plane that is an affine transformation of a regular lattice with a known set of lengths.
- Lattice QFT challenge: can we find the anisotropic lattice action which non-perturbatively has the same set of lengths at the critical point?
- Yes! In one case at least, the D=2 critical Ising model on an affine transformed triangular lattice can be solve analytically [Evan Owen and Brower, 2023].



$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}} \, ,$$

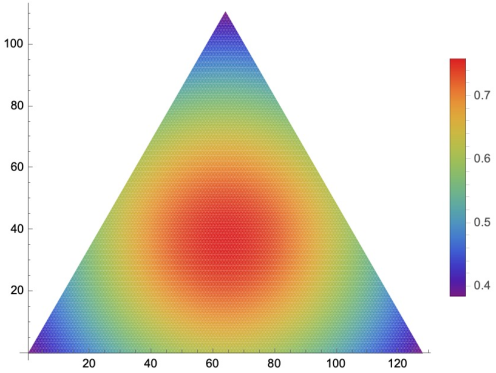
$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

Going Beyond Perturbative QFE: Affine Conjecture Problem

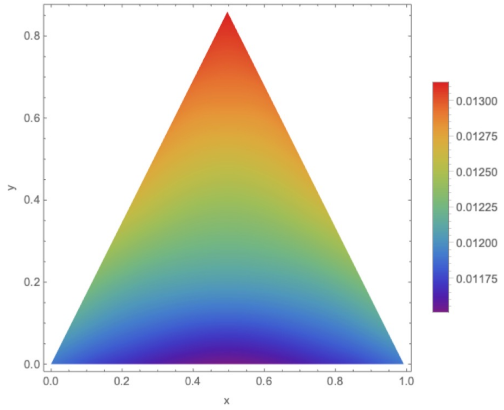
- Next, move points to minimize deviation in scalar curvature density and find target lengths.
- Using given lengths, assign couplings everywhere. Note, only ratios of couplings important. There will be an overall inverse temperature β that needs to be tuned to find the critical surface for the sphere which will be different from the plane.
- Note perturbative counterterms are not needed.

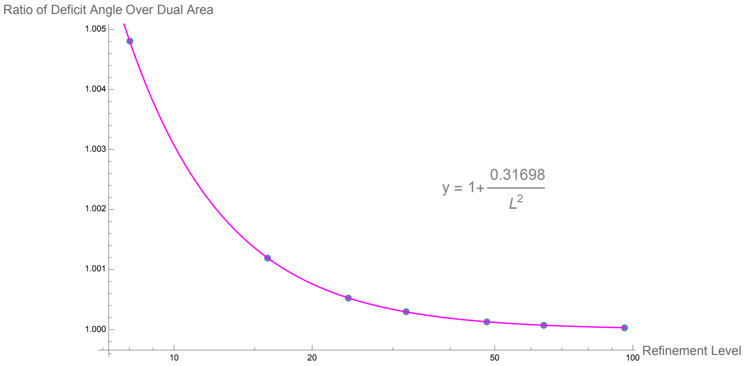
$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

Scalar curvature before smoothing



Scalar curvature after smoothing





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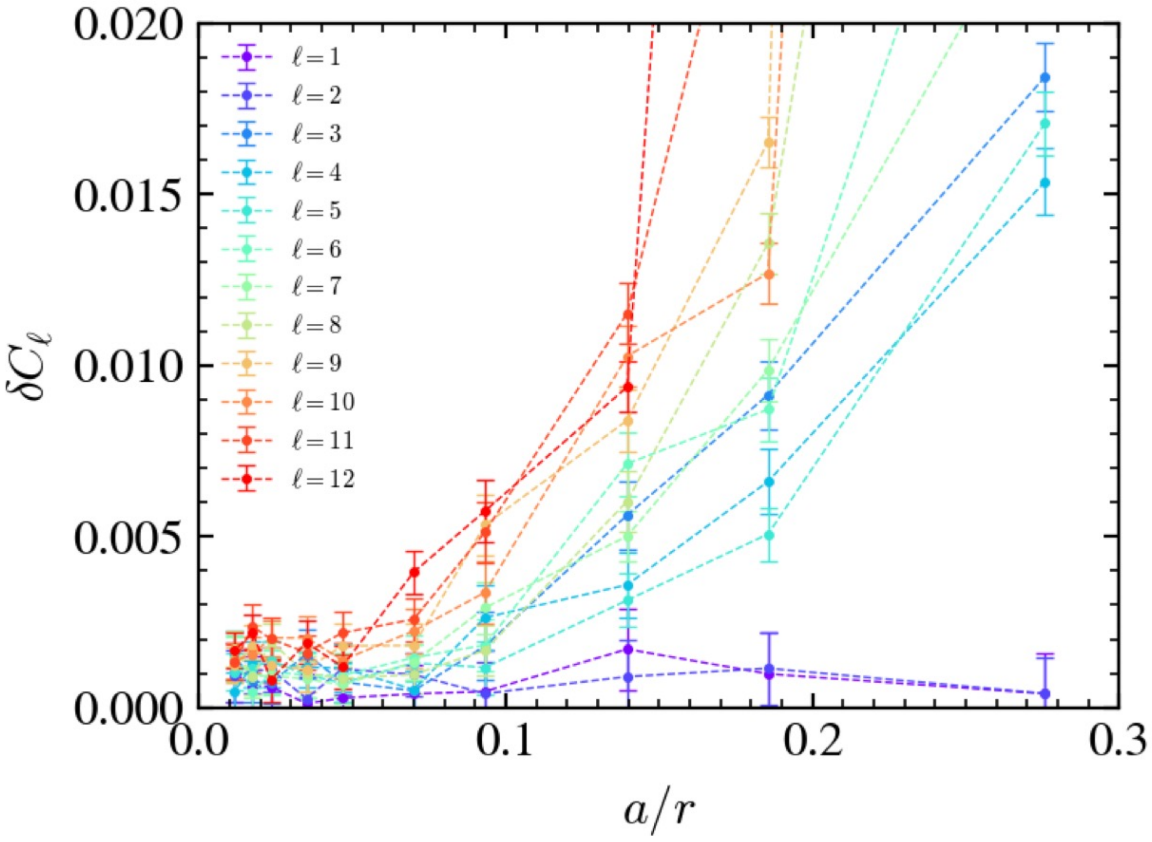
Results for Ising Model on $\mathbb{S}^2$: Rotational Invariance

- Evan Owen Ph.D. Thesis 2023
- Plot shows measure of breaking rotational invariance:

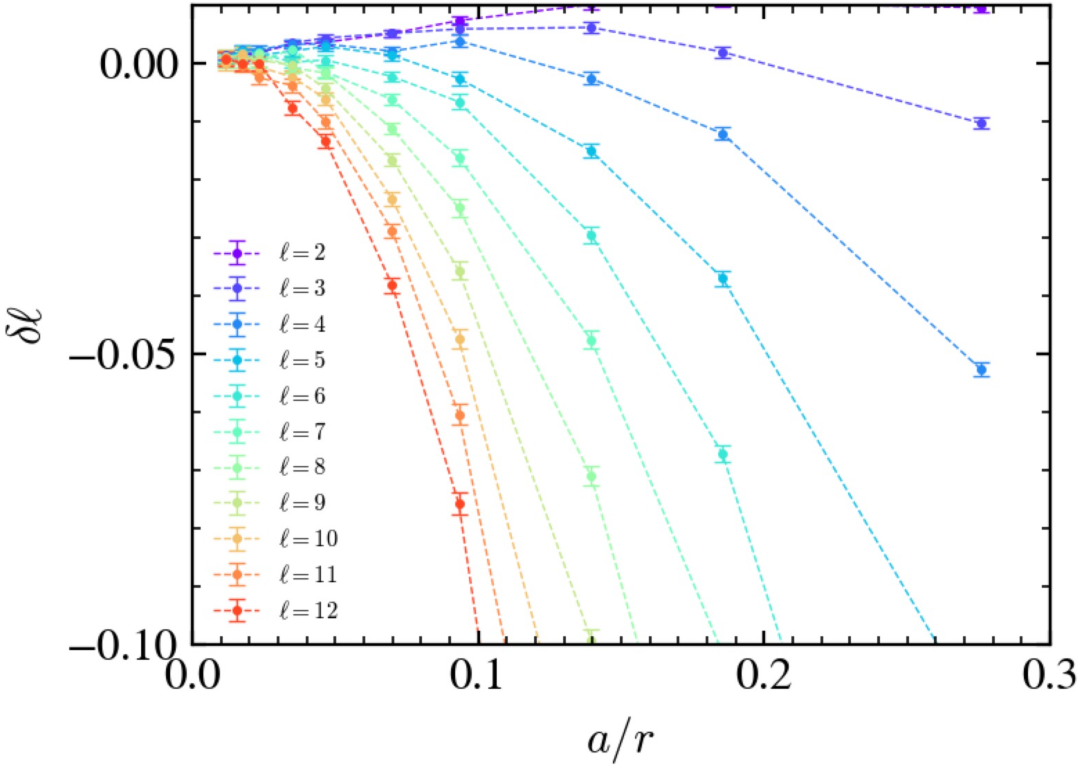
$$\delta C_\ell \propto \sqrt{\sum_m (\Delta_{\ell,m} - \overline{\Delta_\ell})^2}$$



Results for Ising Model on $\mathbb{S}^2$: Rotational Invariance

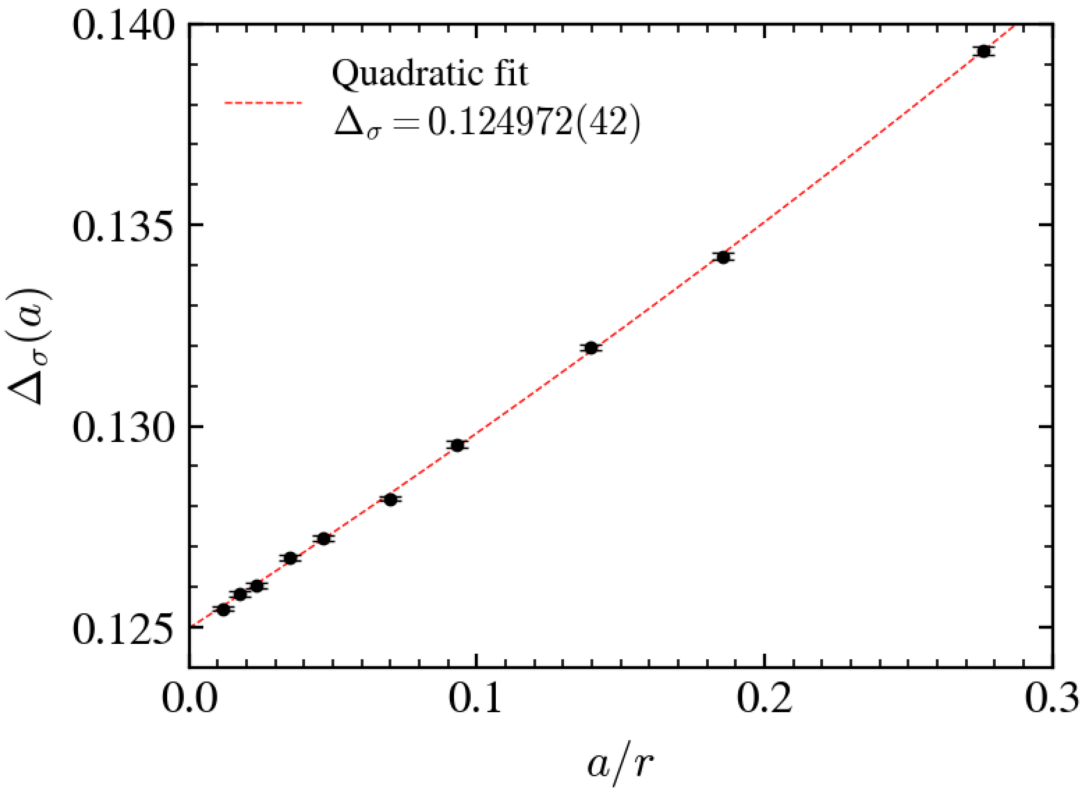
- Evan Owen Ph.D. Thesis 2023
- Plot shows measure of breaking of conformal invariance:

$$\delta\ell \propto (\bar{\Delta}_\ell - \Delta_0) - \ell$$



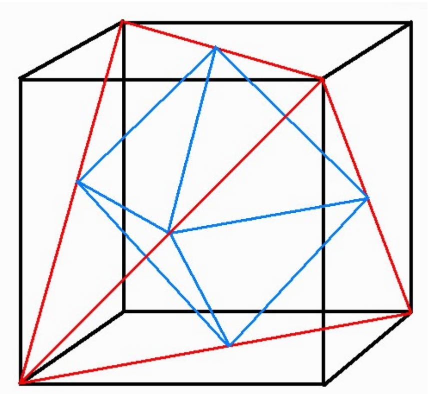
Results for Ising Model on $\mathbb{S}^2$: Scaling Dimension of Z_2-odd scalar σ

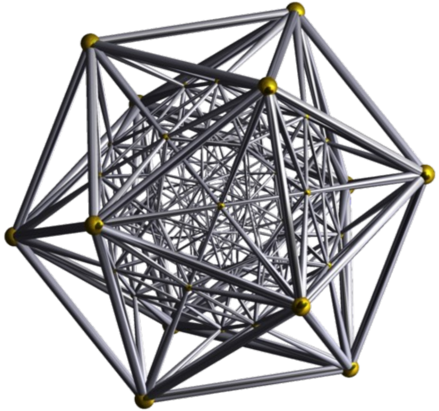
- Evan Owen Ph.D. Thesis 2023
- Exact value is known:
$$\Delta_\sigma = {}^1/_8$$
- Expected power of lattice artifact known from finite-size scaling:
$$\left(\frac{a}{R}\right)^{\Delta_{\epsilon'}-d} = \left(\frac{a}{R}\right)^{4-2}$$
- Work in progress to repeat calculation on $\mathbb{R}\times\mathbb{S}^2$ by GS Jin-Yun Lin (CMU).
- Work in progress by Nobuyuki Matsumoto (Boston U) on extraction of Energy-Momentum Tensor.



Affine Conjecture in Higher Dimensions

- Testing the affine conjecture in higher dimensions ($\mathbb{R}\times\mathbb{S}^3$) requires finding anisotropic actions for three-dimensional affine transformed lattices.
- Choice is face-centered cubic (FCC) lattices, since the number of unique links (6) matches the number of parameters of the affine transformation.
- FCC lattices can tile the interiors of the 600 tetrahedral faces of the 600-cell, the higher-dimensional analog of the icosahedron.
- One problem is FCC lattice has alternating cells of tetrahedrons and octahedrons, not simplicial! Need to figure out how to break octahedrons into tetrahedrons without introducing three new independent parameters.
- Think of FCC as red-black cubic lattice. FCC links connect nearest red-red sites and black sites sit at center of octahedra. Adding red-black links makes the lattice simplicial.





Multihistogram Reweighting [Ferrenberg, Swendsen 1989]

- Affine FCC partition function

$$Z(K_1, \cdots, K_6) = \sum_{s_n = \pm 1} e^{K_1 s_n s_{n+\hat{1}} + \cdots + K_6 s_n s_{n+\hat{6}}}, \qquad E_{n,\hat{e}} = -s_n s_{n+\hat{e}}$$

- Multihistogram master equation (solved iteratively)

$$Z_k = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j \, Z_j^{-1} \, e^{(\vec{K}_k - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$

where R is number of runs, N_r is length of run r, $\vec{E}_{r,i}$ are energies on i-th configuration of run r.

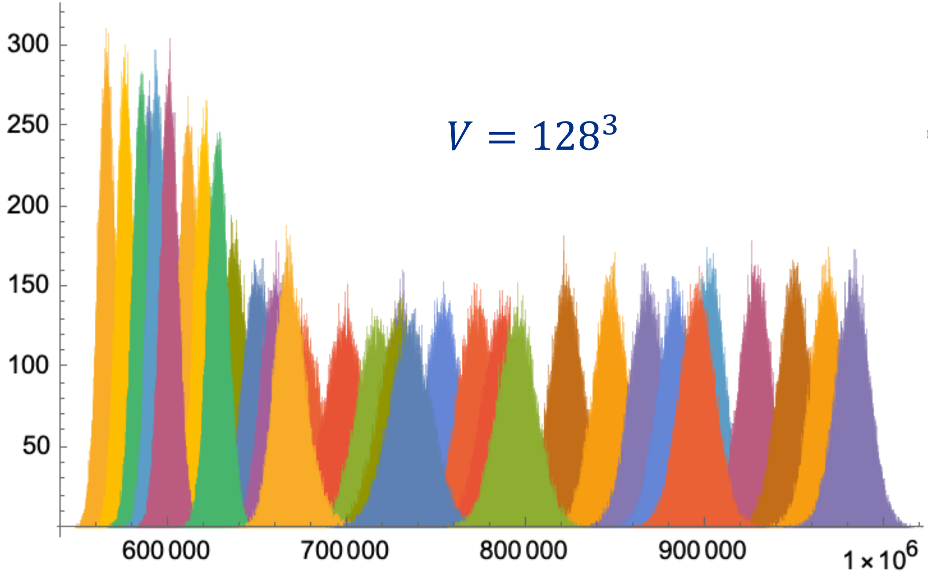
- Observables for any other nearby $\vec{K}$:

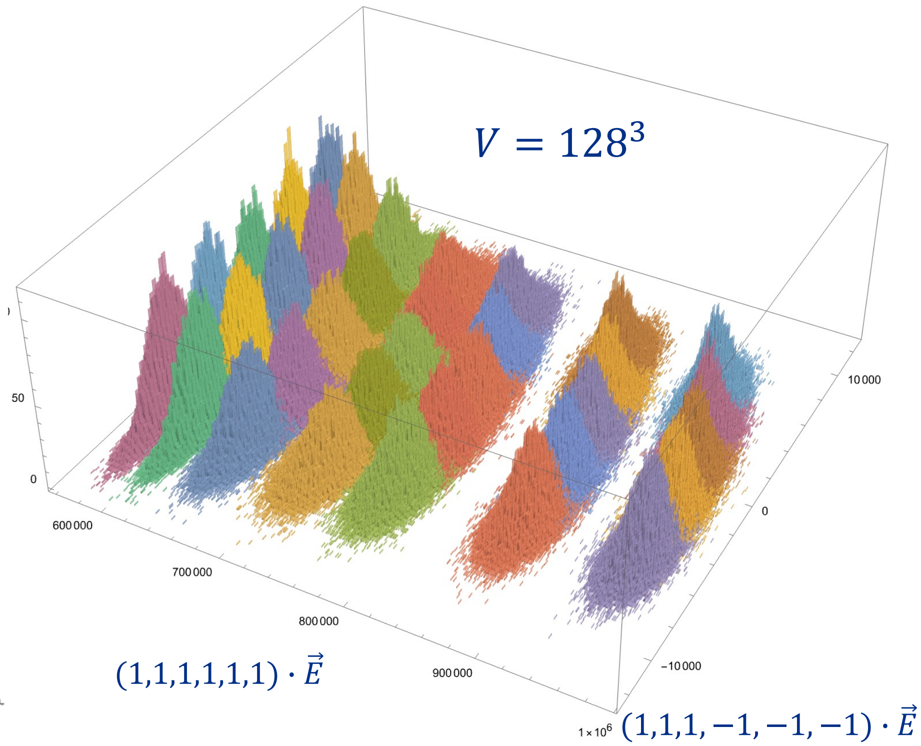
$$\langle \mathcal{O}(\vec{K}) \rangle = \frac{1}{Z(\vec{K})} \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{\mathcal{O}_{r,i}}{\sum_{j=1}^{R} N_j \, Z_j^{-1} \, e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$

$$Z(\vec{K}) = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j \, Z_j^{-1} \, e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$$

Affine Conjecture in Higher Dimensions

- First test: $\frac{K_2}{K_1} = \frac{K_3}{K_1} = 1,\ \frac{K_4}{K_1} = \frac{K_5}{K_1} = \frac{K_6}{K_1} \in \{0.94, 0.97, 1.00, 1.03, 1.06\}$
- K_1 is tuned close to critical point.
- Solve multihistogram consistency condition for all 35 runs, each run $N_r = 50{,}000$ configs.





Next Steps for FCC lattice

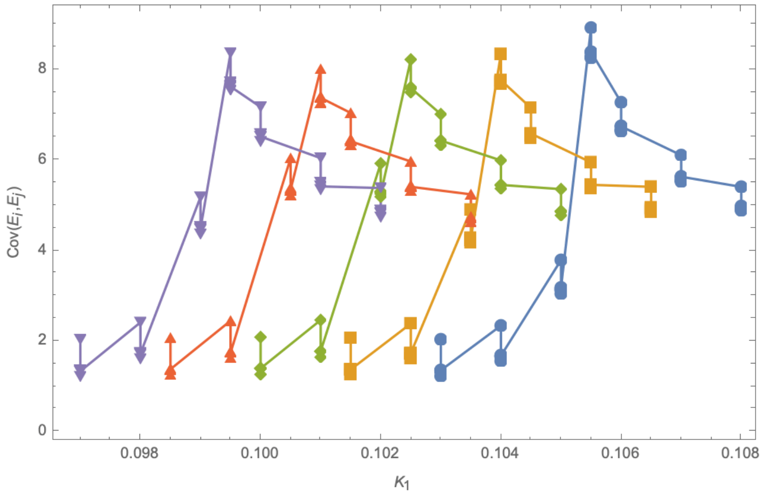
- Using multihistogram reweighting, find critical surface $\vec{K}_{crit}$ by identifying peak in $Cov(\vec{E}, \vec{E})$. In general, it is a 5-d surface with permutation symmetry.

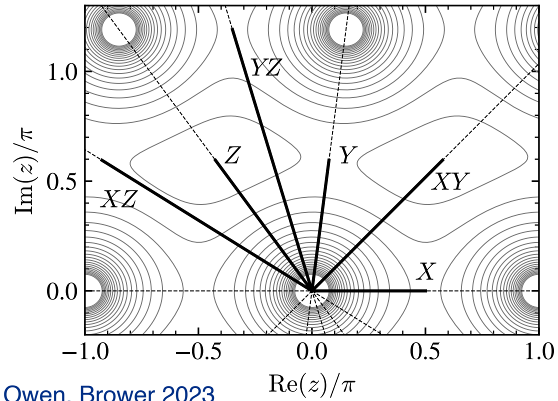
- Then, along critical surface compute two-point function:

$$\langle s(\vec{x}) s(0) \rangle = \frac{1}{\left(x_i\, G_{ij}(\vec{K}_{crit})\, x_j\right)^{\Delta_\sigma}}$$

- Numerically, try to find the matrix $G_{ij}(\vec{K}_{crit})$ that best fits the two-point function data in elliptical region (curve collapse).

- Caution, avoid $\vec{x}$ too large due to finite volume effects of affine transformation on torus. This is often called the modular torus.





Owen, Brower 2023

Conclusions

- Radial quantization would be an ideal tool to study conformal and near-conformal theories.
- We have used radial quantization to solve the critical 3-D Ising model in two limits:
  - Perturbative $\lambda\phi^4$ quantum field theory as $\lambda \to 0$ along critical surface with counter terms.
  - Ising spin model ($\lambda \to \infty$ limit of $\lambda\phi^4$ theory) using affine conjecture without counterterms.
- In near future, we should be able to solve critical 3-D Ising model $\lambda\phi^4$ theory at fixed λ without counterterms using affine conjecture.
- Starting work on radial quantization on $\mathbb{R}\times\mathbb{S}^3$ by first attempting to solve the critical 3-D Ising model on $\mathbb{S}^3$ using 600-cell discretization and affine-transformed FCC lattice tangent planes.
- Multihistogram reweighting plays an essential role in finding the 5-d critical surface

$$K_{1,crit}\left({}^{K_2}/_{K_1}, \cdots, {}^{K_6}/_{K_1}\right)$$

- As an aside, this method can help us define a local energy-momentum tensor operator since insertion of this operator induces the affine transformation.

Thanks to My Collaborators

- Current
  - Prof. Rich Brower (Boston U)
  - Anna-Maria Glück (Heidelburg/Yale → Princeton)
  - Jin-Yun Margit Lin (Carnegie Mellon U)
  - Dr. Nobuyuki Matsumoto (Boston U)
- Past
  - Dr. Venkitesh Ayyar (Boston U)
  - Prof. Casey Berger (Smith College)
  - Dr. Daniel Berkowitz (Yale U)
  - Dr. Michael Cheng (Boston U)
  - Dr. Cameron Cogburn (Rensselaer Poly)
  - Dr. Andy Gasbarro (Bern U)
  - Dr. Dean Howarth (LBNL)
  - Prof. Herbert Neuberger (Rutgers U)
  - Dr. Evan Owen (Boston U)
  - Dr. Tim Raben (Michigan State U)
  - Prof. Chung-I Tan (Brown U)