

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FÍSICAS



TESIS DOCTORAL

Aspectos de gravedad modificada en el universo temprano y tardío

Aspects of modified gravity in the early and late universe

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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ACKNOWLEDGEMENTS

“Si el suelo se inclina (las cosas se complican), uno aprieta los puños y los músculos y tira hacia arriba porque la gran mayoría de las veces nos espera una cumbre que conquistar y disfrutar mirando desde ella sabiendo que entendemos y comprendemos más y mejor que antes de llegar a ella.”

Miguel Ángel Ramos Osorio
(1961-2019)

I am profoundly grateful to my PhD supervisor, Mindaugas Karčiauskas, for his incredible support and insightful guidance over these five years of hard work. He gave me the chance to meet and collaborate with true experts in the field, while I became involved in many projects that enriched my knowledge greatly. His willingness to discuss any question in a precise and respectful way was always encouraging and truly vital. He has been an *extraordinary* supervisor and I feel extremely fortunate to have learnt from him even the most basic aspects of the field of cosmology which I believed to understand. On the other hand, little did I know I would have to struggle so much to write a scientific paper properly! With the aim of making this a well-established tradition, I want to say as well *labai ačiū, Mindaugas!*

I am very much obliged to Antonio López Maroto for kindly agreeing to be my tutor in the last year of my PhD, and to every member of the Department of Theoretical Physics, especially my colleagues Adrián, Andrea, Clara, Dani, Fernando, Rita and Valentin for the always nice and friendly atmosphere. Part of the work developed in this thesis was partially supported by the ‘Atracción de Talento Investigador (Comunidad de Madrid)’ Grant No. 2017-T1/TIC-5305. Also, I am thankful to the Institute of Particle and Cosmos Physics (IPARCOS) for the grant ‘Ayudas de doctorado IPARCOS-UCM/2021’.

The time I spent in Lancaster in order to fulfil part of the requirements to receive the International PhD Mention was inspiring and lovely. I want to thank Konstantinos Dimopoulos for his hospitality and for the interesting and challenging project we undertook along with Dr. Antonio Racioppi and my supervisor. In fact, the results of that investigation constitute almost half of this dissertation. I should not forget to mention the members of the Physics Department at Lancaster University as they were extraordinarily kind and amicable. My

colleagues Samuel, Lucy, Mark and Josh. Thank you so much for such a warm welcome!

Additionally, I would like to thank the two external reviewers, Prof. Tomi Koivisto and Prof. Sami Nurmi, and every member of the thesis panel, for taking the time to read and assess this dissertation.

Finally, I express my gratefulness to my family and, in particular, to my beloved parents, José Terente-Terente and María Pilar Díaz Álvarez. *Por todo.*

J.J.

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ACRONYMS

GR	General Relativity	1
HBB	Hot Big Bang	3
BBN	Big Bang Nucleosynthesis	3
CP	Cosmological Principle	3
FLRW	Friedmann-Lemaître-Robertson-Walker	3
DE	Dark Energy	5
WEC	Weak Energy Condition	6
EoS	Equation of State	7
CMB	Cosmic Microwave Background	8
SEC	Strong Energy Condition	20
CC	Cosmological Constant	22
NEC	Null Energy Condition	23
SR	Slow-Roll	25
GWs	Gravitational Waves	63
TT	Transverse-Traceless	64
GB	Gauss-Bonnet	141
SGB	Scalar-Gauss-Bonnet	145
GBDE	Gauss-Bonnet Dark Energy	153
LC	Levi-Civita	183
EH	Einstein-Hilbert	191
EHP	Einstein-Hilbert-Palatini	194
EL	Euler-Lagrange	195
WGB	Weyl-Gauss-Bonnet	207

RESUMEN

Mientras que las modificaciones a la gravedad einsteniana continúan esquivas observacionalmente, sus aspectos más relevantes juegan papeles significativos en la comprensión teórica del Universo temprano y tardío. Esta tesis, titulada “Aspectos de Gravedad Modificada en el Universo Temprano y Tardío”, se centra en algunos de esos aspectos durante dos periodos clave de la evolución del Universo tal y como lo conocemos: la etapa temprana de inflación y la transición tardía de expansión decelerada a acelerada. Esta investigación ha resultado en tres publicaciones, refs. [1–3], y un artículo en evaluación (véase ref. [4]).

Los objetivos del primer proyecto son formular las condiciones de inflación *slow-roll* en la representación de Jordan, determinar las ecuaciones necesarias para calcular los observables de inflación en términos de variables en la representación de Jordan únicamente, y aplicar los resultados al análisis concreto de modelos de inflación de gravedad inducida generalizada. Consideramos gravedad tipo $F(\phi)R$. El espaciotiempo se configura en hipersuperficies de tipo espacio y mantenemos dicha configuración. Las condiciones de inflación *slow-roll* (claramente definidas en la representación de Einstein) se traducen en la representación de Jordan bajo la misma configuración con una transformación conforme. Para obtener predicciones de observables, sólo es necesario traducir las relaciones homogéneas. Empleando las variables en la representación de Jordan, examinamos dos tipos de modelos de inflación de gravedad inducida generalizada: *hilltop* y caóticos. Concluimos que inflación *slow-roll* sólo puede realizarse en ciertas regiones del espacio de parámetros, y algunos casos del régimen de tipo caótico caen en la región 2σ de los resultados recientes de BICEP/Keck. Para validar el método empleado, resolvemos numéricamente ecuaciones de perturbaciones en la representación de Einstein y comparamos éstas con los resultados usando las relaciones *slow-roll* en la representación de Jordan. Ambas expresiones coinciden durante los últimos 65 *e-folds* de inflación, aproximadamente.

En el segundo proyecto, consideramos la perturbación lineal de curvatura en la representación de Jordan definida en *slicings* comóviles, $\mathcal{R}$, en gravedad $F(\phi)R$. Es sabido que $\mathcal{R}$ es constante después de que los modos de perturbación abandonen el horizonte durante inflación si la presión es adiabática. Esto lo convierte en esencial para conectar las predicciones de inflación con cantidades que podamos medir actualmente. El objetivo es comprobar cómo se transforma $\mathcal{R}$ bajo una transformación conforme. Tomamos hipersuperficies comóviles con el fluido efectivo con momento y energía conservados, y relacionamos las expresiones de $\mathcal{R}$ en las representaciones de Jordan y Einstein. Se demuestra que, en el régimen más allá del horizonte durante inflación *slow-roll*, aquéllas son equivalentes salvo por un término que es

cuadrático en los gradientes, multiplicado por un cociente, $\omega \equiv -\frac{d \ln \sqrt{F}}{d \ln H}$, entre las variaciones de las dos escalas homogéneas dependientes del tiempo en gravedad $F(\phi)R$. En el caso de los modelos de gravedad inducida generalizada, ω es de orden 1 para aquellos que concuerdan con las observaciones, lo que significa que la equivalencia conforme se logra poco después del momento en que las perturbaciones abandonan el horizonte. En el régimen dentro del horizonte, la invariancia conforme no se satisface a no ser que ω sea lo suficientemente pequeña. Como demostración de la equivalencia conforme, encontramos que es posible formular los observables de inflación empleando $\mathcal{R}$ en ambas representaciones obteniendo las expresiones en ref. [1].

Considerando el Universo tardío, en el tercer proyecto, comprobamos la consistencia de los modelos de energía oscura de tipo Gauss-Bonnet con las últimas observaciones de la velocidad de propagación de las ondas gravitacionales, c_{GW} . Estos modelos predicen $c_{\text{GW}} \neq c$, siendo c la velocidad de la luz en el vacío, mientras que la diferencia relativa en velocidades se determina observacionalmente menor que 10^{-15} aproximadamente. Analizamos el sistema dinámico asumiendo un potencial exponencial, con factor λ . Consideramos principalmente los regímenes *scaling* y de Sitter, que son puntos fijos del sistema autónomo. Ejecutamos simulaciones numéricas y abarcamos una amplia región del espacio de parámetros que conduce a trayectorias en el espacio de fases consistentes con los valores actuales de los parámetros de densidad de los campos de materia Ω_m y de la ecuación de estado de energía oscura w_f . Empleamos algunas de las simulaciones para examinar la evolución del parámetro $|\alpha_T| \equiv |c_{\text{GW}}^2 - 1|$ (donde $c = 1$) y encontramos que, o el límite observacional sobre c_{GW} se viola por muchos órdenes de magnitud, o podría ser muy complicado encontrar modelos que sean consistentes con los valores de Ω_m y w_f . Esto se aplica al caso de acoplo Gauss-Bonnet exponencial con factor $\kappa \neq \lambda$, y al acoplo lineal y $\alpha_T = 0$ dado que de Sitter es inestable y el régimen *scaling* es un atractor.

Aunque los modelos de energía oscura de tipo Horndeski (y los de tipo Gauss-Bonnet porque éstos son una subclase de los primeros) están severamente constreñidos por el límite observacional sobre $|\alpha_T|$, esto es estrictamente cierto en el formalismo métrico. En el cuarto proyecto, motivados por los resultados recientes relativos a modelos de Horndeski específicos en el formalismo de Palatini, consideramos la acción de los modelos de energía oscura de tipo Gauss-Bonnet pero asumiendo la geometría de Weyl (llamamos a este modelo “Weyl-Gauss-Bonnet”). Los puntos fijos se calculan bajo ciertas suposiciones, y su estabilidad se determina a través del análisis lineal. Obtenemos los regímenes *scaling* (**ScI**) y de Sitter (**dS**) con estabilidad similar a la de ref. [3]. Además, tenemos un segundo régimen *scaling* con dos

puntos fijos, **ScII** $\pm$. Realizamos simulaciones numéricas y examinamos las trayectorias que cruzan ambos regímenes *scaling*. Analizamos la evolución de $|\alpha_T|$ para aquellas que satisfacen las condiciones observacionales sobre Ω_m y w_f . Encontramos que la condición sobre $|\alpha_T|$ se viola considerablemente y atribuimos esto al hecho de que el parámetro densidad efectivo del acoplo de Gauss-Bonnet crece y domina actualmente. No encontramos mejoras significativas con respecto a las predicciones del formalismo métrico en los casos de acoplo lineal y $\alpha_T = 0$. Mostramos que nuestros resultados son aplicables a una conexión con torsión pero compatible con la métrica debido a las denominadas “transformaciones proyectivas” de la conexión.

SUMMARY

While gravity modifications remain elusive observationally, their most relevant aspects are found to play significant roles in the theoretical understanding of the early and late Universe. This dissertation, *Aspects of Modified Gravity in the Early and Late Universe*, focusses on some of such aspects over two key periods of the evolution of the Universe as we know it: the early inflation and the late transition from decelerating to current accelerating expansion. The investigation has resulted in three published papers, Refs. [1–3], and a fourth one under review (see Ref. [4]).

The goals in the first project are to formulate the slow-roll inflation conditions in the Jordan frame, determine the equations needed to compute inflation observables in terms of Jordan frame quantities alone, and apply the results to the concrete analysis of generalised induced gravity inflation models. Our study considers $F(\phi)R$ gravity. We slice the space-time into spacelike hypersurfaces and keep that slicing fixed. Thus the slow-roll inflation conditions (clearly defined in the Einstein frame) are mapped to the Jordan frame on the same slice by performing the conformal transformation. We only need to map the homogeneous relations in order to derive observable model predictions. Making use of Jordan frame quantities, two categories of generalised induced gravity inflation models are examined: the hilltop- and chaotic-like ones. We conclude that slow-roll inflation can be realised in certain regions of the parameters space and only some cases in the chaotic-like regime fall within the 2σ region of the recent BICEP/Keck results. To validate the method developed in this work, some numerical simulations are carried out where exact perturbation equations are solved in the Einstein frame and are compared with the results using the Jordan frame slow-roll relations. These agree well over the last 65 e-folds of inflation, approximately.

In the second project, we consider the Jordan frame linear curvature perturbation on comoving slicings, $\mathcal{R}$, in $F(\phi)R$ gravity. This quantity is known to become constant after horizon exit of the perturbation modes during inflation if the non-adiabatic pressure vanishes. This makes it essential to connect inflation predictions with quantities that can be measured today. The goal is to check how it transforms under the conformal rescaling. We take hypersurfaces that comove with the effective fluid with conserved energy and momentum, and relate the expressions of $\mathcal{R}$ in Jordan and Einstein frames. It is shown that, on the superhorizon regime during slow-roll, those are equivalent up to a term which is quadratic in gradients, multiplied by a ratio, $\omega \equiv -\frac{d \ln \sqrt{F}}{d \ln H}$, of the variation of the two time dependent background scales of $F(\phi)R$ gravity. In the case of the generalised induced gravity inflation models, ω is shown to be of order 1 for those fitting the data, meaning that the conformal

equivalence is achieved shortly after horizon exit. On the subhorizon regime however, the conformal invariance does not hold unless ω is sufficiently small. As a demonstration of the conformal equivalence, we find that it is possible to formulate the inflation observables using $\mathcal{R}$ in both frames by arriving at the expressions in Ref. [1].

Turning our attention to the late Universe, in the third project, we seek to check the consistency of Gauss-Bonnet Dark Energy models with the latest constraints on the speed of gravitational waves, c_{GW} . These models quite generically predict $c_{\text{GW}} \neq c$, c being the speed of light in vacuum, while the relative difference between the two speeds is constrained to be smaller than 10^{-15} or so. We do the dynamical systems analysis assuming an exponential potential, with strength λ . The scaling and de Sitter regimes are considered chiefly, which are fixed points of the autonomous system. We perform numerical simulations and scan a wide region of the space of parameters that leads to trajectories in the phase space consistent with present values of the density parameter of matter fields Ω_m and of the equation of state parameter of the dark energy fluid w_f . We use some of those simulations to examine the evolution of the parameter $|\alpha_T| \equiv |c_{\text{GW}}^2 - 1|$ (where $c = 1$) and find that, either the observational bound on c_{GW} is violated by many orders of magnitude, or it could be very hard to find models that are consistent with the constraints on Ω_m and w_f . This conclusion applies to the exponential Gauss-Bonnet coupling, with strength $\kappa \neq \lambda$, and to the cases of linear coupling and $\alpha_T = 0$ given that de Sitter becomes unstable and the scaling fixed point is an attractor.

While Horndeski type dark energy models (and hence the Gauss-Bonnet ones because they are a subclass of Horndeski's) are severely constrained by the bound on $|\alpha_T|$, this is strictly true in the metric formalism of gravity. In our fourth project, motivated by recent results regarding specific Horndeski models in the Palatini formalism, we consider the action of the Gauss-Bonnet Dark Energy models but in Weyl geometry. We call this model 'Weyl-Gauss-Bonnet'. The fixed points are calculated under some assumptions, and their stability is determined via linear analysis. We obtain the scaling (**ScI**) and de Sitter (**dS**) fixed points with stability conditions similar to those in Ref. [3]. On top of these, we have a second scaling regime which is represented by a pair of fixed points, **ScII** $\pm$. We then carry out numerical simulations and examine trajectories entering both scaling regimes. The evolution of $|\alpha_T|$ is analysed for those trajectories satisfying the constraints on Ω_m and w_f . We find the constraint on $|\alpha_T|$ to be grossly violated and this is attributed to the fact that the effective density parameter associated with the Gauss-Bonnet coupling function grows and dominates at present. No significant improvement is found in the cases of linear coupling and $\alpha_T = 0$.

when compared to the predictions of the metric formalism. All of our results are shown to be applicable to a connection with torsion but metric compatible by virtue of the so-called ‘projective transformations’ of the connection.

CHAPTER 1

COSMIC INFLATION AND COSMOLOGICAL PERTURBATION THEORY

The General Relativity (GR) theory [5,6] is an essential step towards a deep understanding of the origin of the Universe and its entire evolution. The theory provides the physical principles and the mathematical formulation needed to address the dynamics of the Universe to a spectacular accuracy. Not only has it promoted the field of Cosmology, but it has showed the intimate connection between this and other branches of Physics which were being explored at the time it was conceived. As is the case with any good theory, it soon displayed its limitations, as well as the need for a completion that was consistent with the other two major discoveries of the twentieth century: the Quantum Mechanics and the Quantum Field Theory.

In GR, the gravitational interaction is described in the vein of the classical electromagnetic theory; that is, as a classical field theory. The symmetric metric tensor $g_{\mu\nu}(x)$ represents the gravitational field and the field equations governing its dynamics are the Einstein field equations [5], which can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} , \quad (1.1)$$

where c is the speed of light in vacuum and G is Newton's gravitational constant, which is present in Newton's theory of gravity. $R_{\mu\nu}(x)$ and $R(x)$ are tensor and scalar quantities respectively, containing second-order derivatives of the metric tensor. Although Eq. (1.1) looks simple, these field equations have few exact solutions known to date. One of such solutions will be crucial in this dissertation and shall be introduced later.

On the RHS of Eq. (1.1), the tensor quantity $T_{\mu\nu}(x)$ holds information about the matter content. Mass densities are responsible for the gravitational interaction in Newton's gravity already, but in Einstein's theory mass and energy are equivalent and any form of energy produces gravitational fields. Therefore, *pressures* can also generate gravitational fields. This well established fact will be of special importance when presenting the inflationary universe and how the mechanism of cosmic inflation works.

The symmetries of GR allow for a slight modification of Eq. (1.1) [7]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} , \quad (1.2)$$

where Λ is another constant of the theory in addition to c and G . This term introduced by hand on the LHS is not forbidden by the theory, but there is no reason *a priori* to include it. It can be removed from the equation if it is interpreted as a redefinition of $T_{\mu\nu}$

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \frac{\Lambda c^4}{8\pi G}g_{\mu\nu} . \quad (1.3)$$

However, its role since around 1998 has become essential and Λ is currently seen as a true conundrum which demands new physics [8–10]. We will comment on its relevance and how it became a decisive incentive to take the paradigm of cosmic inflation seriously.

The three constants of GR have the following values [11, 12]:

$$c = 299\,792\,458 \text{ m s}^{-1}, \quad (1.4)$$

$$G = 6.67430(15) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}, \quad (1.5)$$

$$\Lambda = 1.088(30) \times 10^{-52} \text{ m}^{-2}. \quad (1.6)$$

We shall however adopt the units where $c = \hbar = 1$, $\hbar$ being the reduced Planck constant. The reduced Planck mass defined as

$$M_{\text{Pl}} \equiv \sqrt{\frac{\hbar c}{8\pi G}} , \quad (1.7)$$

equals the inverse of $\sqrt{8\pi G}$ in these units. The Einstein field equations (1.1) then read

$$G_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2}T_{\mu\nu} , \quad (1.8)$$

where we have introduced the symmetric Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R . \quad (1.9)$$

The metric signature employed in this thesis is the mostly positive one: $(-+++)$. x , which has been used already, encodes both temporal and spatial coordinates.

1.1 Standard Big Bang Theory: Achievements and Issues

The conventional theory of the Big Bang, usually called ‘Hot Big Bang (HBB) model’ [13], provides a thorough description of how the Universe evolved since it was 1 second old up until today (the latest estimate of the age of the Universe being 13.80 ± 0.02 Gyr [11, 12]). It primarily predicts with remarkable accuracy the abundances of light elements synthesised at the end of the first three minutes [14]. This important process is known as ‘Big Bang Nucleosynthesis (BBN)’ [15] and we will comment on it to a greater extent later.

1.1.1 Equations of a Homogeneous and Isotropic Universe

The big number of successes of the theory is rather impressive if one considers the assumptions on which it relies. The theory postulates that the Universe was initially a hot and highly dense gas of particles in thermal equilibrium, which was expanding and being pulled back by the gravitational interaction. It then describes how the Universe expanded and cooled. One can calculate how fast it expanded as well as its temperature and the density of matter at any given instant of time. The equations that govern the dynamics of the expansion are built upon a major assumption: the Cosmological Principle (CP), which states that the Universe is homogeneous and isotropic on large scales. An exact solution to the Einstein field equations describing an expanding, homogeneous and isotropic universe is the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric* [16–18]

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (1.10)$$

written in spherical coordinates, where $c = 1$ as was stated earlier. $ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$ is the line element, and the constant term K describes the uniform curvature of space. $a(t)$ denotes the scale factor, which accounts for the change of size of the Universe over time (that is, for the change in distance among objects in the Universe due solely to the expansion of the Universe). One may define the expansion rate as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}, \quad (1.11)$$

called the *Hubble rate*. Overdots are used to denote derivatives with respect to the coordinate time t . The coordinates (r, θ, ϕ) are called ‘comoving coordinates’ [19], given that the effect of the expansion has been subtracted out. In these coordinates, the ‘grid’ or coordinate chart

expands along with the Universe.

Also, one can define the *conformal* time as

$$d\eta \equiv \frac{dt}{a(t)} , \quad (1.12)$$

and the *conformal* Hubble rate

$$\mathcal{H}(\eta) \equiv \frac{a'(\eta)}{a(\eta)} , \quad (1.13)$$

meaning that $\mathcal{H} \equiv aH$, where primes denote derivatives with respect to the conformal time (although we will often use this very same notation to refer to derivatives with respect to other quantities, and this will be neatly indicated).

From the Einstein field equations in the tensor form (1.2), one can write out the homogeneous differential equations dictating the evolution of the Universe. One must specify the matter content however, which is encoded in the symmetric *stress-energy tensor* $T_{\mu\nu}$, or *energy-momentum tensor*. In line with the homogeneity and isotropy assumption, the energy-momentum tensor takes on the form of an ideal (perfect) fluid whose covariant components are [17, 20]

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu} , \quad (1.14)$$

where ρ and P are the *energy density* and *isotropic pressure* of the fluid respectively, as measured by a comoving observer. u_μ are the covariant components of the four-velocity of a fluid element, and $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ are the covariant components of the projection tensor onto the constant time (spacelike) hypersurfaces. The 00 component of the Einstein field equations is the so-called *Friedmann equation* [19], given by

$$3 \left(H^2 + \frac{K}{a^2} \right) = \frac{\rho + \rho_\Lambda}{M_{\text{Pl}}^2} , \quad (1.15)$$

$\rho_\Lambda \equiv \Lambda M_{\text{Pl}}^2$ being the energy density that corresponds to the constant term Λ . If we restore the units, we can calculate the *mass* density to be [11, 12]

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} = 5.83(16) \times 10^{-30} \text{ g cm}^{-3}. \quad (1.16)$$

This density is of the same order of magnitude as the mass density of a single proton in a cubic metre [11]. Thus this Λ is very low-dense when compared to other macroscopic objects in the Universe acting as sources of gravity.

In view of Eq. (1.15), we may introduce the density parameter in Einstein gravity [17]:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\text{cr}}(t)} , \quad (1.17)$$

which measures the deviation of the energy density from the critical one

$$\rho_{\text{cr}}(t) \equiv 3H^2(t)M_{\text{Pl}}^2 , \quad (1.18)$$

defined as the time dependent energy density of a universe without Λ and with zero uniform curvature K . In this respect, it is worth mentioning that three possible curvatures are considered in Cosmology, all of them taken to be uniform. The $K = 0$ is the so-called *flat* case [17], which means that, given the coordinate system, on the constant time hypersurfaces, the axioms of the Euclidean geometry are fulfilled. Therefore, ‘flat’ does not refer to the dimensionality but it alludes to the *spatial* curvature in the spacetime picture. Also, given that K/a^2 in the Friedmann equation represents the spatial curvature, we may interpret H^2 as the ‘temporal’ curvature of the Universe.

Plugging Eq. (1.17) into Eq. (1.15), we arrive at

$$\Omega - 1 = \frac{K}{a^2 H^2} , \quad (1.19)$$

where ρ_Λ has been absorbed into ρ as part of the full energy density content of the Universe. We can actually interpret this Λ as an ideal fluid, satisfying the equation

$$P_\Lambda = -\rho_\Lambda . \quad (1.20)$$

This implies that its energy-momentum tensor is proportional to the metric tensor (see Eqs. (1.3) and (1.14)) [21]. This fluid is an example of what is called *Dark Energy* (DE) [22].

The second homogeneous field equation that can be obtained from the Einstein field equations (ij components of the field equations) is [19]

$$2\dot{H} + 3H^2 + \frac{K}{a^2} = -\frac{P + P_\Lambda}{M_{\text{Pl}}^2} , \quad (1.21)$$

where

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} . \quad (1.22)$$

Using Eq. (1.15), Eq. (1.21) becomes

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M_{\text{Pl}}^2} + \frac{\rho\Lambda}{3M_{\text{Pl}}^2} . \quad (1.23)$$

The *Weak Energy Condition* (*WEC*) for a perfect fluid [20] stipulates that $\rho \geq 0$. If this DE obeys the WEC, it shall contribute to the *increase* in the acceleration of the expansion of the Universe ($\ddot{a} > 0$), irrespective of what the other matter content does. Note that, due to Eq. (1.20), the isotropic pressure of this fluid is *negative*, which is an essential feature of DE. This repulsive effect that accelerates the expansion of the Universe is in sharp contrast to the usual behaviour of gravity which tends to attract massive objects and pull them together. As was pointed out earlier, pressures can produce gravitational fields in GR and, in this particular case, the gravity interaction generated by this fluid drives the objects in the Universe away from one another.

The conservation of the non-gravitational energy and momentum in GR is the covariant generalization of the usual conservation law when gravity is not taken into account. Thus the energy-momentum tensor of the matter content in the Universe satisfies

$$\nabla_\mu T^{\mu\nu} = 0 . \quad (1.24)$$

∇_μ denotes the *covariant derivative* which, as opposed to the usual partial derivative ∂_μ , when acting on a tensor, transforms as a tensor under coordinate transformations [23]. If we elaborate more on Eq. (1.24), we find (assuming the energy-momentum tensor in Eq. (1.14))

$$u^\mu u^\nu \partial_\nu \rho + h^{\mu\nu} \partial_\nu P + (\theta u^\mu + a^\mu) (\rho + P) = 0 , \quad (1.25)$$

where we introduced the scalar expansion rate $\theta \equiv \nabla_\mu u^\mu$ and the four-acceleration vector $a^\mu \equiv u^\nu \nabla_\nu u^\mu$. In GR, the metric is covariantly conserved, meaning that

$$\nabla_\alpha g_{\mu\nu} = 0 , \quad (1.26)$$

and no such term appears in Eq. (1.25). This fact is what makes possible to introduce by hand the Λ term in the Einstein field equations, as it does not alter the result after taking the covariant derivative on both sides of the equation. Consequently, Eq. (1.24) is not affected either.

Since the four-velocity has components $u^\mu = (1, 0, 0, 0)$ in a comoving coordinate system

[17], there is only one non-trivial equation that we can derive from Eq. (1.25) which reads

$$\dot{\rho} + 3H(\rho + P) = 0, \quad (1.27)$$

where $\theta = 3H$ [17]. This equation can be rewritten in terms of the derivative with respect to the natural logarithm of the scale factor a

$$\frac{d\rho}{d \ln a} = -3(\rho + P). \quad (1.28)$$

In line with the DE fluid, we can propose the following relation between the energy density and the isotropic pressure:

$$P = w\rho, \quad (1.29)$$

where the constant term w is known as the *Equation of State (EoS) parameter*, and Eq. (1.29) is the EoS [17] of a *barotropic fluid* in particular, as the pressure depends on the energy density only. The EoS parameter in the case of the DE is $w = -1$ (see Eq. (1.20)). If we solve Eq. (1.28) for a constant EoS parameter, we find

$$\rho(a) = \rho_0 \left(\frac{a}{a_0} \right)^{-3(1+w)}, \quad (1.30)$$

where $\rho_0 = \rho(a_0)$. We readily see that the energy density of the DE fluid is not affected whatsoever by the expansion of the Universe as $\rho_\Lambda(a) = \rho_\Lambda(a_0) \equiv \text{constant}$.

The EoS parameter can take other known values. For example, if we consider a fluid made of non-interacting particles, this fluid shall be *pressureless*: $P = 0$. On the other hand, if the energy-momentum tensor of the fluid is said to be *traceless* in 1+3 spacetime dimensions, the fluid is barotropic and the pressure is $P = \rho/3$. In a FLRW universe, the first case corresponds to *pressureless matter* ($w = w_m = 0$), sometimes simply referred to as ‘matter’, and the second one is applicable to *radiation* ($w = w_r = 1/3$) [17]. Plugging these values of the EoS parameter into Eq. (1.30), we obtain

$$\rho_m(a) \propto a^{-3}, \quad (1.31)$$

$$\rho_r(a) \propto a^{-4}. \quad (1.32)$$

In accordance with the HBB model, if we trace back the predicted evolution, we find that ρ_r must dominate over ρ_m in the early stages of the evolution of the Universe. Notice that $\rho_m \propto a^{-3}$ shows that the mass (or the energy associated with the mass) is conserved while

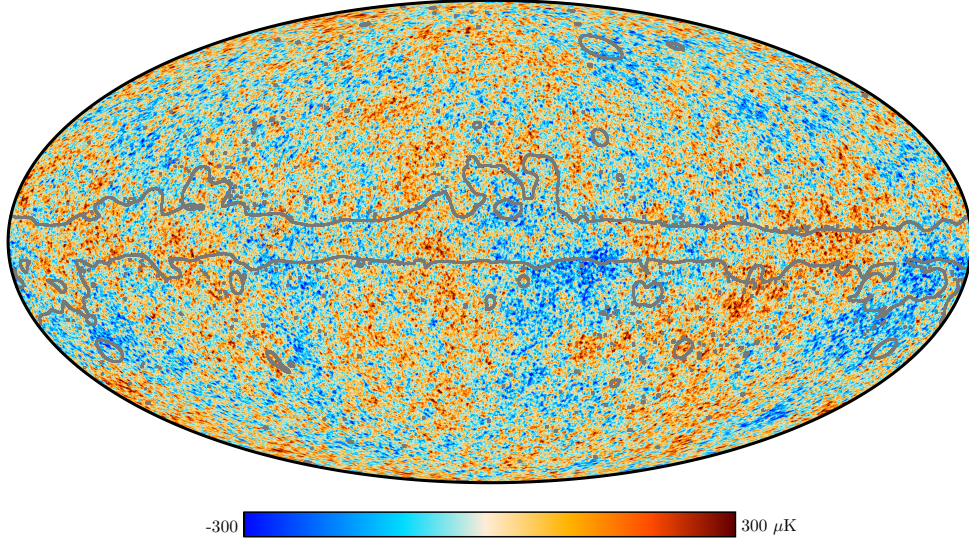


Figure 1.1: 2018 *Planck* map of the anisotropies in temperature of the CMB. Figure taken from the *Planck* website (see Ref. [35], ESA and the *Planck* Collaboration).

the Universe expands (the proper volume is proportional to a^3), whereas the extra factor a^{-1} in the predicted evolution of the energy density of radiation stems from the *redshift* it undergoes. Indeed, in the FLRW universe, the frequency of the radiation is proportional to a^{-1} , meaning that it loses energy (its wavelength gets stretched) due to the expansion of the Universe [24]. This genuine prediction of the HBB model in regard to the energy loss of radiation while the Universe expands has been confirmed on numerous occasions (the first known report dates back to 1912; see Ref. [25]). In contrast to this, the DE fluid gains energy because of the expansion [26]. Given that the energy density ρ_Λ remains constant while the volume increases, this means that the total energy inside this expanding volume must grow accordingly.

1.1.2 Isotropy of the Cosmic Microwave Background

The isotropy of the Universe on large scales, which the HBB model is based on, is supported by numerous observations, although there is one that will be of interest due to its relevance in the development of the inflation paradigm. This involves the *Cosmic Microwave Background (CMB) radiation*, discovered in 1965 by A. A. Penzias and R. W. Wilson [27]. This colossal laboratory, where many theories of the early Universe are tested, has become our biggest clue for what the Universe looked like in its early stages. The radiation dates back to the epoch of *recombination* [24], where electrons and protons combined to form electrically neutral hydrogen atoms. Before that, photons and electrons were strongly interacting in the

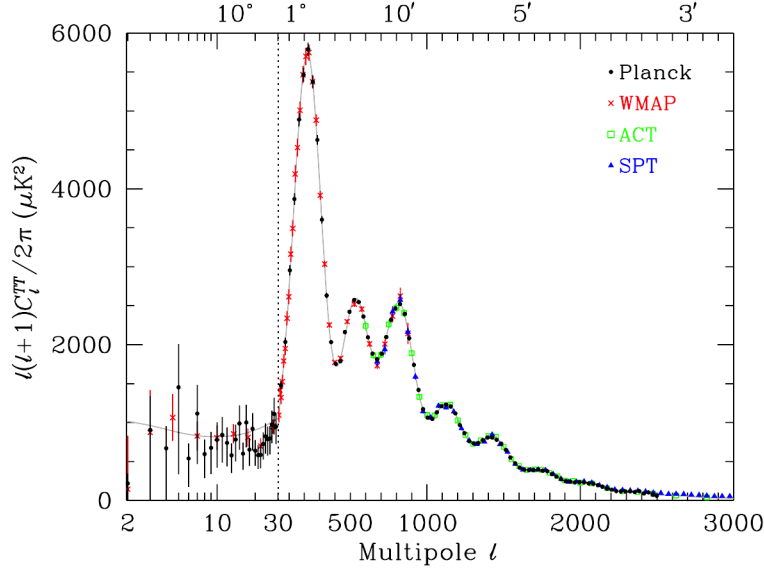


Figure 1.2: CMB temperature anisotropy power spectrum. Figure taken from Ref. [11], where only a set of the most recent experimental results has been selected. Some points with large error bars were removed, as indicated in the caption of that figure (see Sec. 29.5, p. 9 of Ref. [11]).

primordial plasma, with a very large cross section, and these electrons were strongly coupled to the protons. Temperatures were greater than¹ 0.1 eV (i.e. greater than 1.16×10^3 K). Only when they dropped below 0.1 eV did the photons decouple and formed the free-streaming radiation we observe today. Since photons would no longer scatter off the electrons, the surface of the sphere that the CMB formed is known as the *last-scattering surface* [24, 28].

Those photons have been travelling to us on a straight line since about 380 000 yrs after the gravitational singularity, and we now observe a picture of what the Universe looked like at that time [28]. It is worth mentioning that this singularity is the result of a mere extrapolation of the expanding evolution of the Universe backwards in time [20] and it is often referred to as ‘Big Bang’ as well [29]. The photons from the recombination epoch are a source of essential information on the early Universe, but they are not expected to be the only one that we shall be able to register in the coming future, as we might also detect ‘primordial gravitational waves’ or ‘B-modes’ [30–33], those providing much more details about that epoch and the preceding ones.

1. When referring to the temperatures in those units we are setting the Boltzmann constant k_B equal to one. The value of this important constant when units are restored is [11]

$$k_B = 8.617\,333\,262\dots \times 10^{-5} \text{ eV K}^{-1}. \quad (1.33)$$

Despite the relevance of the discovery in 1965, the relevant detection for our purposes entails the anisotropies measured by the Cosmic Background Explorer (COBE) satellite in 1992 [34]. These are temperature and polarization anisotropies, the latter being of special interest in current experiments. In particular, when one maps the measured CMB temperature fluctuations (see map of anisotropies in temperature in Fig. 1.1), one can decompose the map into spherical harmonics [36]

$$\frac{\delta T(x, \mathbf{n})}{T_\gamma} = \sum_{lm} a_{lm}(x) Y_l^m(\mathbf{n}) , \quad (1.34)$$

$T_\gamma = 2.7255 \pm 0.0006$ K being the mean temperature of the CMB [11, 37]. $\mathbf{n}$ denotes the direction in the sky in which the photon travelling to us at spacetime position $x = (t, \mathbf{x})$ from the CMB is seen. Invariance of the stochastic properties under rotations (statistical isotropy) demands that the complex coefficients $a_{lm}(x)$ are not correlated for different values of l and m , such that [36]

$$\langle a_{lm}(x) a_{l'm'}^*(x) \rangle = \delta_{ll'} \delta_{mm'} C_l(t) , \quad (1.35)$$

$C_l(t)$ being the angular spectrum of the CMB (see power spectrum in Fig. 1.2), which is evaluated at the present time, $C_l(t = t_0)$, and where a_{lm}^* is the complex conjugate of a_{lm} . ‘ $\langle \rangle$ ’ denotes the average over many realisations of the Universe [24, 36]. Also, the statistical homogeneity erases the spatial dependence of the result.

The largest anisotropy in the CMB comes from the dipole term ($l = 1$) [36]. The dipole anisotropy was first reported by E. Conklin in 1969 [38] and by P. Henry in 1971 [39]. The fluctuation amplitude of the anisotropy on a scale of 180° is 3.3621 ± 0.0010 mK (latest measurement made by the *Planck* team [40]), meaning that

$$\left. \frac{\delta T}{T_\gamma} \right|_{\text{dipole}} \simeq 10^{-3} . \quad (1.36)$$

This anisotropy is attributed to the Solar System motion relative to the last-scattering surface, which leads to one hot pole and a cold one. The dipole then indicates that the Solar System is moving at 369.82 ± 0.11 km s $^{-1}$ [40]. Once this Doppler shift is taken into account and one sets the CMB frame in which the dipole from the CMB would be zero, the

contribution from all higher multipoles yields [34, 36]

$$\frac{\delta T}{T_\gamma} \simeq 10^{-5}. \quad (1.37)$$

These are anisotropies in temperature to one part in a hundred thousand. These anisotropies in temperature are therefore associated, not with our proper motion with respect to the CMB, but with the Universe and are of primordial origin. When this high isotropy in temperatures of the CMB is combined with the Copernican Principle, we find that the evidence for isotropy amounts to evidence for *spatial* homogeneity (we shall always refer to homogeneity in space).

The strong evidence for homogeneity and isotropy on large scales brings in an immediate issue. As we saw at the beginning of this section, the major conclusion that one can extract from the HBB theory is that the age of the Universe is finite [24]. One can actually make estimations of the age of the Universe based on observations and approximate relations [11]. If the CMB displays such a striking homogeneity and isotropy in temperature, this means that the regions encompassing the last-scattering surface should have been in causal contact before the epoch of recombination, and this process of homogenisation should have taken place in a finite interval of time. To see the troubling aspects of this, we assume the Universe is either dominated by radiation or by pressureless matter, and the density is comparable to the critical density at Hubble rate $H = \dot{a}/a$. Then the Friedmann equation (1.15) with negligible spatial curvature and ρ_Λ yields in both cases

$$a_r(t) \propto t^{1/2}, \quad (1.38)$$

$$a_m(t) \propto t^{2/3}, \quad (1.39)$$

respectively. This gives $H(t) \propto 1/t$, where the constant of proportionality is $1/2$ for radiation domination and $2/3$ for matter domination [17]. Before recombination, the Universe was already matter dominated and the corresponding energy density was indeed very close to the critical one at that time [24]. If one assumes adiabatic expansion of the Universe since then and that the matter domination phase held until very recently, then, at the recombination epoch, the Universe we now observe was formed by 10^5 causally disconnected regions and still was uniform to one part in a hundred thousand [41]. This means that matter and energy should have travelled at a hundred times the speed of light, which is not permitted by the laws of physics as we know them. Therefore the initial conditions inferred from the HBB model do not allow the nearly homogeneity and isotropy we do observe in the CMB.

One could argue that some of the assumptions we made to derive this approximate result

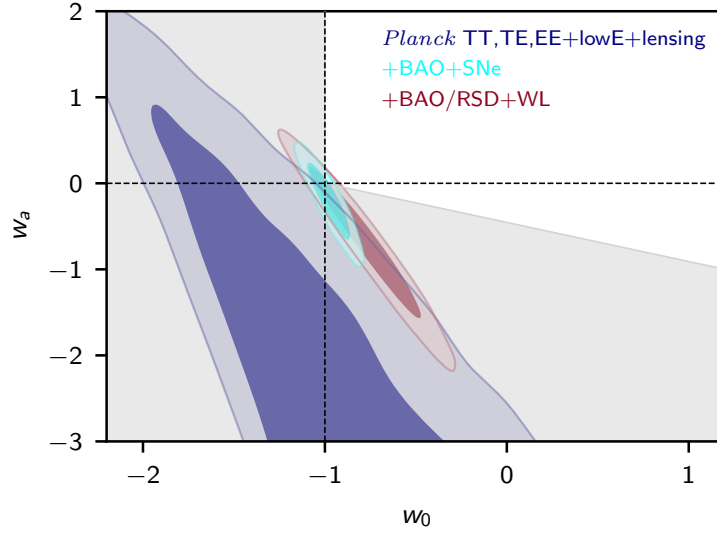


Figure 1.3: Marginalised posterior distributions of the (w_0, w_a) parameters for various data combinations which are listed above on the right. To test the time-varying EoS parameter w , the following functional form has been adopted: $w(a) = w_0 + (1 - a)w_a$. Figure taken from Ref. [12].

might not be true, but these are well tested by observations. The matter domination epoch is known to have covered the longest period in the evolution of the observable universe since the Big Bang until today [11]. Such a period of matter domination no longer applies to our current observable universe, whose dynamics is mainly driven by the DE fluid we introduced earlier. Recent measurements find that its EoS parameter is consistent with $w = -1$ (see Refs. [11, 12] and Fig. 1.3), corresponding to the ideal fluid associated with the constant Λ in Sec. 1.1.1. This DE fluid, although not clearly ascribed to a specific fluid, is known to have taken over when the observable universe was 9.8 yrs old [41]. When $w = -1$, we notice that H is constant and

$$a(t) \propto e^{Ht}, \quad (1.40)$$

meaning that the scale factor grows (or decreases for $H < 0$) exponentially in time.

Regarding the adiabatic expansion assumption, there is no heat transfer in an isotropic Universe [24], and therefore it is equally well-supported by observations. In particular, this yields a basic evolution law for the mean temperature of the CMB. The spectrum of the CMB is the best observed example in Cosmology of the spectrum of black-body's radiation [11]. The sum of all the energy densities of all the species that follow the black-body's distribution

is [24]

$$\rho = \frac{\pi^2}{30} g_*(T) T^4, \quad (1.41)$$

where the $g_*(T)$ denotes the effective number of degrees of freedom under consideration at a given temperature

$$g_*(T) \equiv \sum_{\text{bosons}} g_f + \frac{7}{8} \sum_{\text{fermions}} g_f. \quad (1.42)$$

At different temperatures, the number of relativistic degrees of freedom is different due to distinct particle masses and various *phase transitions*. The second law of thermodynamics states that

$$d\rho = (sT - \rho - P) \frac{dV}{V} + Tds, \quad (1.43)$$

where $V \propto a^3$ is the proper volume with total energy $E = \rho V$ and total entropy $S = sV$. Since the energy density for the black-body's radiation depends only on the temperature and the EoS parameter is $1/3$

$$\frac{S}{a^3} = \frac{2\pi^2}{45} g_*(T) T^3. \quad (1.44)$$

Hence the adiabaticity condition implies $T \propto [g_*(T)]^{-1/3} / a$. g_* is constant for a range of T values and changes almost like a step function. For such a range, $T(a) \propto 1/a$.² We can see that $\rho \propto a^{-4}$ from Eq. (1.41) is the expected behavior for radiation, where the extra $1/a$ was owing to the redshift of the radiation due to the expansion of the Universe. The relation between the redshift ' z ' and the scale factor (taking $a_{\text{today}} = 1$ for convenience) is

$$1 + z = \frac{1}{a}, \quad (1.45)$$

and the scale factor a is usually replaced by the redshift to mark down different important stages in the evolution of the observable universe since the epoch of matter-radiation equality (when the matter contribution overcame the radiation one). For example, the matter-radiation equality and recombination epochs occurred at redshifts $z = 3402$ and $z = 1100$, respectively. The starting moment at which DE becomes the dominant component corresponds to $z \sim 2$ [41].

Thus, the HBB theory cannot explain how the Universe became so uniform and isotropic but assume that it was so from the beginning. We shall address this issue with the initial conditions soon, but the big success of the HBB model that was mentioned earlier, the

2. Also, g_* comes with a power $-1/3$, hence T is not very sensitive to small changes in g_* . For an example of how g_* changes with T , see Fig. 2.1 in Ref. [42].

BBN, is worth of a brief summary due to its vital importance in the development of modern cosmology.

1.1.3 Primordial Nucleosynthesis

The BBN theory describes the processes involved in the formation of the light chemical elements in the early Universe [24, 43]. It must be pointed out however that most of the elements in the Universe were not formed in the Big Bang, but much later, in the interior of stars [44]. To be precise, the formation of mostly all of the chemical elements which are heavier than lithium was the result of supernovae explosions, and those were then collected into later generations of stars, the Sun being an example of them.

About five isotopes of hydrogen H, helium He and lithium Li were primarily formed in the Big Bang and one can calculate their abundances [43] (see predictions in Fig. 1.4). To that end, we first introduce the temperature dependent *rate of interactions per particle* $\Gamma(T)$. This quantity parametrises the efficiency of a given process as a function of the temperature. In an expanding universe with Hubble rate given by $H(T)$, $\Gamma(T)/H(T) > 1$ means that the particles of a given species χ can be considered to be in thermal equilibrium due to the high rate of interactions (remember that, during radiation or matter domination, $1/H$ characterises the age of the Universe; see Eqs. (1.38) and (1.39), respectively). If the temperature drops below the mass of the particles (we shall consider non-relativistic particles), their abundance in thermal equilibrium is given by [43]

$$n_\chi = g_\chi \left(\frac{m_\chi T}{2\pi} \right)^{3/2} \exp \left(-\frac{m_\chi}{T} \right). \quad (1.46)$$

This expression indicates that, if the thermal equilibrium persists, the comoving number density $N_\chi \equiv a^3 n_\chi$ will exponentially decrease towards zero ($a \propto 1/T$). However, when the temperature becomes smaller than a certain value T_D , then $\Gamma(T)/H(T) < 1$, and *particles fall out of equilibrium*. T_D is called ‘decoupling temperature’. Hence, the change in the comoving number density is determined by the ratio Γ/H [43]

$$\frac{dN_\chi}{d \ln a} \simeq -\frac{\Gamma}{H}, \quad (1.47)$$

and therefore the abundance of the particles χ becomes constant within a comoving volume as the Universe expands, *freezing out*. In the case of relativistic particles, $n \propto T^3$, and the comoving number density does not change over time. If this comoving number density

corresponds to that of thermal equilibrium, it will then remain so regardless of the ratio Γ/H .

The rate of interactions when processes of particle-antiparticle annihilation dominate is given by [43]

$$\Gamma = n\langle\sigma v\rangle , \quad (1.48)$$

where $\langle\sigma v\rangle$ is the velocity-averaged product of the cross section σ and the velocity v . At temperatures higher than 1 MeV (around 10 MeV), the following processes at thermal equilibrium are dictated by the weak interactions:

$$n \longleftrightarrow p + e^- + \bar{\nu}_e , \quad (1.49)$$

$$n + \nu_e \longleftrightarrow p + e^- , \quad (1.50)$$

$$n + e^+ \longleftrightarrow p + \bar{\nu}_e , \quad (1.51)$$

ν_e and $\bar{\nu}_e$ being the electron neutrino and antineutrino, respectively. Then, the ratio of the neutron and proton abundances is (see Eq. (1.46))

$$\frac{n_n}{n_p} \simeq \exp\left(-\frac{m_n - m_p}{T}\right) , \quad (1.52)$$

m_n and m_p being the masses of the neutron and the proton, respectively. Such a difference is $m_n - m_p = 1.293$ MeV [11]. Since $T \gg m_n - m_p$, the rate of interactions of these processes is given by $\Gamma(T) \sim G_F^2 T^5$, G_F being Fermi's constant [47]. Using the Friedmann equation in a spatially-flat universe dominated by radiation (with ρ given by Eq. (1.41)), the Hubble parameter is $H(T) \propto T^2$ (see Eq. (1.15)), meaning that the rate of interactions will fall faster than the Hubble rate as the temperature drops. The decoupling temperature can be easily estimated to be $T_D \simeq 1$ MeV [24], and the previous ratio at decoupling time takes on the approximated value $n_n/n_p \simeq 1/6$. This ratio decreases slowly and, by the time of formation of the first nuclei, it reaches the value $n_n/n_p \simeq 1/7$ due to the β -decay of the neutrons [11].

Before the nucleons (neutrons and protons) freeze out, photons are energetic enough to dissociate deuterium nuclei D whose formation is favoured by the strong interaction [43]

$$n + p \longleftrightarrow D + \gamma . \quad (1.53)$$

The binding energy of the deuterium is $B_D = 2.23$ MeV. Given that $m_p \simeq m_n \simeq \frac{1}{2}m_D$, we

have

$$\frac{n_D}{n_n n_p} \simeq \frac{3}{4} \left(\frac{4\pi}{m_p T} \right)^{3/2} \exp \left(\frac{B_D}{T} \right). \quad (1.54)$$

When $T \ll B_D$, the abundance in equilibrium of deuterium starts dominating, this temperature being $T = 0.1$ MeV (which marks the start of the BBN). The formation of stable deuterium nuclei triggers the generation of ${}^3\text{He}$ [11]



formed by the collision of two deuterium nuclei too. Once the temperatures are low enough, the production of ${}^4\text{He}$, with binding energy greater than that of the deuterium [43], becomes the main process and all the neutrons end up incorporating into the ${}^4\text{He}$ atoms. The mass fraction of helium-4, Y , is then [43]

$$Y = 1 - \frac{n_p - n_n}{n_p + n_n} = \frac{2n_n}{n_p + n_n}, \quad (1.56)$$

which, for the previous value of the ratio $n_n/n_p \simeq 1/7$, yields $Y \simeq 0.25$. Finally, ${}^7\text{Li}$ is produced directly by the collision between tritium and helium-4 atoms. It can also be produced indirectly by ${}^3\text{He} + {}^4\text{He}$, due to the subsequent β -decay of ${}^7\text{Be}$ into ${}^7\text{Li}$ [11, 48].

It is not possible to go beyond helium-4 through processes involving two-body collisions in the early Universe. The density is too low and the timescales are small when compared to the interior of stars, for example, where such reactions do occur. There are then no stable nuclei with mass number 5 or 8 and the production of elements heavier than helium-4 is not significant during BBN [11].

1.1.4 Flatness Problem

The density parameter introduced in Eq. (1.17) can be written as

$$\Omega(t) = \frac{1}{1 - \frac{3K}{[a(t)]^2 \rho(t)}}, \quad (1.57)$$

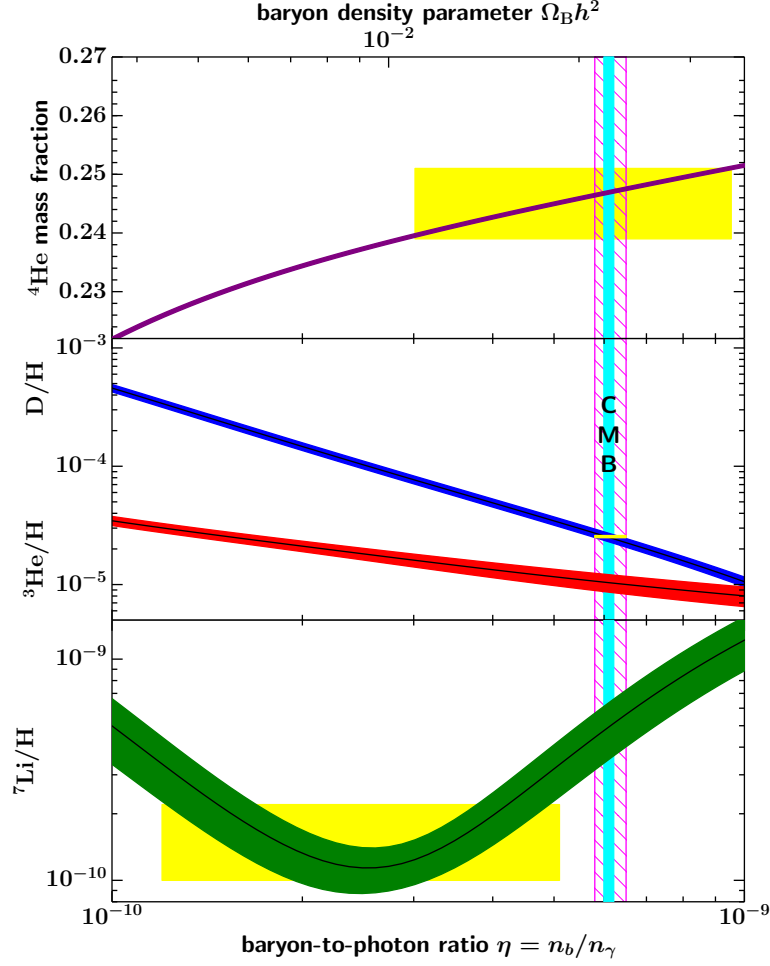


Figure 1.4: Primordial abundances of ^4He , D , ^3He , and ^7Li as predicted by the BBN model against the *baryon-to-photon* ratio defined as $\eta \equiv n_b/n_\gamma$, and the baryon density parameter $\Omega_B h^2$, where $h = 0.674$ is the scaling factor for the Hubble parameter today [11]. The bands represent 95% CL and the boxes indicate the observed light element abundances, while narrow and wider vertical bands represent the value of η inferred from CMB measurements and the BBN concordance range (both at 95% CL). Figure taken from Ref. [11]. Notice that, while the boxes overlap with the vertical bands in the case of ^4He and D , this is not the case of ^7Li . This is known as the *lithium problem* [11, 45, 46].

using Eq. (1.15). Now, Ω can be expressed as a function of the scale factor if one assumes a perfect fluid and makes use of Eq. (1.30)

$$\Omega(a) = \frac{\Omega_0}{\Omega_0 - (\Omega_0 - 1) \left(\frac{a}{a_0}\right)^{1+3w}}, \quad (1.58)$$

where $\Omega_0 = \Omega(a_0)$. Since all the fluids we have considered satisfy $w > -1/3$ (excepting DE), it is clear that any initial departure from $\Omega = 1$ (flat universe) will grow as time passes. In this sense, $\Omega = 1$ is an *unstable equilibrium point*, very much like a pencil that balances on its tip. Had the density parameter been slightly smaller than 1 in the early Universe, it would have rapidly fallen to zero meaning that the Universe would have ended up being devoid of any trace of matter. On the other hand, an initial value of Ω slightly larger than 1 would lead to a quick increase in Ω , which would rise to infinity. This implies that the Universe would recollapse (see Fig. 1.5a). Neither of both scenarios leads to the formation of galaxies that we do observe.

The current observed value for Ω is [11, 12]

$$\Omega = 1.011 \pm 0.006, \quad (1.59)$$

with 68% CL, where *Planck* CMB data and *Planck* lensing measurements have been put together, along with the assumption that DE corresponds to a perfect fluid with $w = -1$. Then, we know that the Universe is spatially-flat at present to about 0.6% or, perhaps, 1%. For the Universe to have such a level of flatness currently, it should have been flat to 1 part in 10^{57} within 10^{-43} seconds after the Big Bang [41]. One might argue that at that time the equations governing the dynamics of the Universe might not be those provided by Einstein's theory. However, the same estimation during BBN gives a level of flatness of 1 part in 10^{16} , and 1 part in 10^5 at the time of formation of the last scattering surface of the CMB [41]. This extraordinary coincidence is usually referred to as the *flatness problem* and there is nothing in the HBB theory that gives a dynamical explanation of how the Universe was so close to being flat. This problem is sometimes rephrased as an 'age problem' since, as was explained above, the tiny initial departures from 1 would soon drive Ω to zero or infinity, impeding the formation of any structure that we observe today in the Universe.

1.1.5 Horizon Problem

As we commented in Sec. 1.1.2, the high level of isotropy of the CMB is in strong conflict with the laws of physics as we know them and the HBB theory cannot provide an explanation of how the Universe became so uniform and isotropic if it does not assume that it was uniform and isotropic from the start. The fact that the problem lies in explaining the uniformity reached in causally disconnected regions is what gives it the name ‘*horizon problem*’.

As was mentioned earlier, the Universe was dominated by matter at the time of recombination. Then, the Hubble parameter can be written as (remember the relation between the redshift and the scale factor in Eq. (1.45))

$$\frac{H^2(z)}{H_0^2} \simeq \Omega_{m,0} (1+z)^3, \quad (1.60)$$

$\Omega_{m,0}$ being the density parameter of matter today ($z=0$) and H_0 the Hubble parameter at present. We introduce the particle horizon d_p , which gives the maximum distance light from particles could have travelled from the Big Bang ($z \rightarrow \infty$) to the observer, in this case, at the time of recombination:

$$d_p(z) = \int_z^\infty \frac{d\tilde{z}}{H(\tilde{z})}, \quad (1.61)$$

and solve the integral assuming matter domination

$$d_p(z) \simeq \frac{2}{\sqrt{\Omega_{m,0}} H_0} \frac{1}{\sqrt{1+z}}. \quad (1.62)$$

Hence the comoving particle horizon at the time of recombination ($z=1100$; see the end of Sec. 1.1.2) is

$$d_p(z_{\text{rec}}) \simeq 180.8 \Omega_{m,0}^{-1/2} h^{-1} \text{ Mpc}, \quad (1.63)$$

where the current Hubble parameter is measured to be [11, 12]

$$H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (1.64)$$

or $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$, with $h = 0.674$.³ The horizon at the time of recombination

3. There is a striking discrepancy subject to current research between the value of the present-day Hubble parameter determined by the Planck Collaboration from measurements of the CMB, and that found by the SHOES team that uses the traditional cosmic distance ladder methods (a *Hubble tension*). The latter is found to be slightly higher than the former [49]. Such a deviation reaches the 4.2σ [11] which constitutes a potential motivation for models alternative to the Λ CDM model, the standard model of Cosmology at present. In our calculations, we choose the value determined by *Planck* (see Eq. (1.64)).

subtends an angle of about 1.7° across the sky

$$\theta = \frac{d_p(z_{\text{rec}})}{d_p(0)} \simeq (1 + z_{\text{rec}})^{-1/2} \simeq 0.03 = 1.7^\circ. \quad (1.65)$$

Angular scales larger than 1.7° in the sky would have not been in causal contact at the time of recombination (or at the time of formation of the last scattering surface), which contradicts what the CMB small anisotropies in temperature seem to imply.

1.2 Inflationary Cosmology

The issues we have explained in earlier sections are tackled by an addition to the HBB model called *cosmological inflation* [26]. This paradigm exploits the new feature ascribed by Einstein's theory of gravity to the gravitational interaction, which was the fact that pressures, along with energy densities, can produce gravitational fields (see the beginning of this chapter, before Eq. (1.2)).

1.2.1 Particle Horizon and Comoving Hubble Horizon

To see how cosmic inflation addresses the flatness and horizon problems, we first notice that the particle horizon can be written as

$$d_p(a) = \int_{\ln(a_i)}^{\ln(a)} \frac{d \ln \tilde{a}}{\tilde{a} H}. \quad (1.66)$$

If we assume a perfect fluid, with EoS parameter $w = P/\rho$, the *comoving Hubble horizon* reads (see Eq. (1.30))

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}, \quad (1.67)$$

where $a_0 = 1$, corresponding to today's scale factor. This implies that, as long as we consider either radiation or pressureless matter, or any matter source of the form of a perfect fluid that fulfills the *Strong Energy Condition (SEC)* [41] ($w \geq -1/3$), the comoving Hubble horizon will increase as the Universe expands. Or, equivalently, the particle horizon will continue growing as a increases

$$d_p(a) = \frac{2H_0^{-1}}{1+3w} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)} \right], \quad (1.68)$$

as, under these assumptions involving matter, the particle horizon and the comoving Hubble horizon are virtually equivalent at late times ($a \gg a_i$).

The inflation paradigm assumes that, at early times, the Universe was dominated by a perfect fluid that violated the SEC [41]. Consequently, the EoS parameter of such a fluid satisfied $w < -1/3$ and the comoving Hubble horizon shrank as the Universe expanded. This means that, as the scale factor grew, the particle horizon became much greater than the comoving Hubble horizon, and therefore, as opposed to the usual evolution prescribed by the SEC and the HBB model, they both would end up being very different in magnitude at late times.

This result is crucial for the inflation paradigm. The particle horizon, as was explained already (see Eq. (1.61)), determines whether two particles could have communicated or not throughout the history of the Universe.⁴ If both particles are separated by a comoving distance greater than the particle horizon, they never could have communicated with each other. However, the comoving Hubble horizon determines if these particles are in causal contact *over a given period* of the evolution of the Universe, so they can communicate with each other [41]. If they are separated by a comoving distance larger than the comoving Hubble horizon, they cannot communicate with each other *now*, which however does not imply that they had not been in causal contact at an earlier moment in time. This then goes to the heart of the homogeneity and isotropy of the CMB and the horizon problem.

Then, the inflation paradigm provides a way of overcoming the horizon problem. Moreover, this decreasing comoving Hubble horizon also addresses the flatness problem. Eq. (1.19) shows that if $(aH)^{-1}$ decreases, Ω is driven towards 1; i.e. the Universe becomes spatially-flat. If $w < -1/3$ in Eq. (1.58), then we see that, regardless of how far from 1 the density parameter started out, it will tend to 1 rather than deviating from it as in the usual HBB model (see Fig. 1.5b). The farther away Ω starts from 1, the more inflation is needed to drive it to 1 sufficiently. So the inflation paradigm favours a spatially-flat universe which, as we discussed already, is widely accepted today.

When the inflation paradigm was proposed, Ω was constrained to lie in a range between 0.2 or 0.3, which constituted an inconvenience to the idea of inflation itself [50]. After 1998, when the Universe was found to be accelerating in its expansion [51, 52], the new energy component included in the equations, the DE fluid, changed the inferred value of Ω , this now being the value most favoured by the inflationary cosmology. Furthermore, the DE fluid is the type of matter component one would need to realise this period of shrinking Hubble

4. By ‘communicate’ we mean that they both can receive light signals from each other.

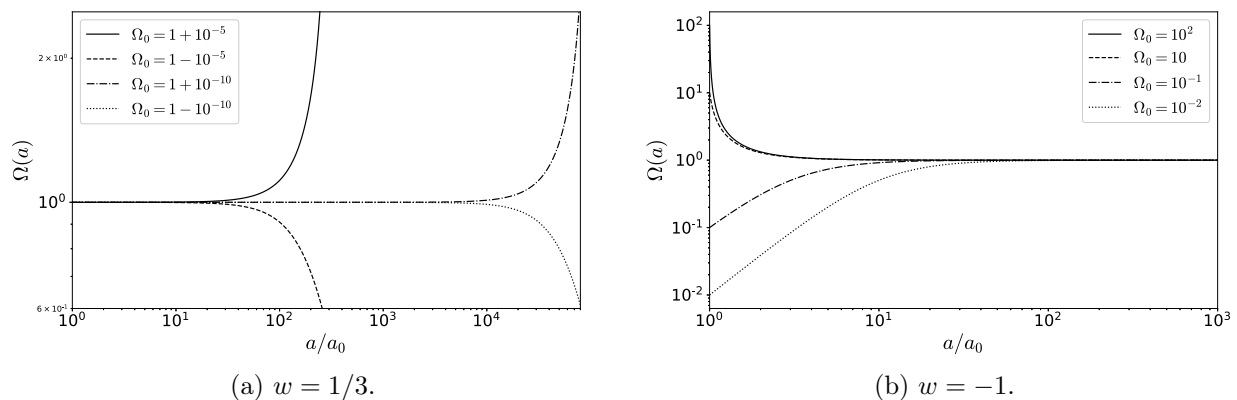


Figure 1.5: Evolution of Ω in a universe dominated by radiation (left) and by a CC (right). During radiation domination, initial deviations from 1 of the order of 10^{-10} take $10^9 t_{\text{Pl}}$ to start altering the value of Ω drastically, so it does not match the inferred value from observations. Since radiation domination in the evolution of the Universe lasted for $10^{44} t_{\text{Pl}}$, it is clear that the Universe as we know it could have not evolved until the present time. However, a universe dominated by a CC quickly drives way different initial values of Ω towards the flat value.

radius as its EoS parameter violates the SEC (see Fig. 1.3).

The possibility of adding such a fluid to the field equations dates back to the birth of GR, two years after the definite field equations were presented by A. Einstein (1917) [7]. Although at that time it was not conceived as a *fluid* but as a constant of the theory (a *Cosmological Constant* (CC), as Einstein called it), this modification to his equations has now, more than one hundred years after, become an unquestionable relevant addition with striking theoretical and observational consequences [53, 54].

The original motivation of Einstein was to describe a static universe, which was how the Universe was believed to be because of the best data available at that time, before the evidence for expansion arose (prior to this, the relative velocities of the stars were observed to be small and this is referred to as a *fact* in the paper by Einstein to justify the CC) [55]. The work by Friedmann in 1922 [16] and 1924 [56], and by Lemaître in 1927 [57], led to the eventual retraction of Einstein (1931) and the common acceptance of an expanding universe [55]. Then, the CC was considered to be zero by most physicists until 1998, when two independent teams that were determining the redshift of distant Type Ia supernovae found evidence for an *accelerating* expansion of the Universe. This acceleration can be caused by a *positive* non-zero CC (see Eq. (1.23)). This breakthrough led S. Perlmutter [58], B. Schmidt and A. Riess to jointly receive the Nobel Prize in Physics in 2011.

1.2.2 Quasi-de Sitter Phase

The inflation period must conclude at some point to recover the successful predictions of the HBB model, and hence one must ensure that the value of the EoS parameter of the fluid responsible for inflation does not remain close to -1 for long. That is, in the early stages of inflation, $w \simeq -1$, but when inflation has successfully circumvented the horizon and flatness problems, w deviates significantly from -1 and the DE-like fluid must become subdominant in the energy budget of the Universe. To see the implications of this strong deviation from -1 and understand how this could be realised, we use the Friedmann equation (1.15) and Eq. (1.21) for a spatially-flat universe (because enough inflation has occurred so spatial curvature is negligible) to obtain

$$\dot{H} = -\frac{\rho + P}{2M_{\text{Pl}}^2}, \quad (1.69)$$

or

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}(1 + w), \quad (1.70)$$

where ϵ is a small parameter in the early stages of inflation, but grows as inflation ends. We notice from Eq. (1.23) that an accelerating expanding universe ($\ddot{a} > 0$) demands a small ϵ

$$\frac{\ddot{a}}{aH^2} = -\frac{1}{2}(1 + 3w) = 1 - \epsilon. \quad (1.71)$$

This however assumes that the *Null Energy Condition* (*NEC*) is satisfied by the perfect fluid under consideration [20, 41], such that $\rho + P \geq 0$, and then $\epsilon \geq 0$. Consequently, $\epsilon < 1$ in order for $\ddot{a} > 0$. Since w must be close to -1 , but not be equal to -1 exactly, so inflation ends at some point, inflation amounts to a *quasi-de Sitter* phase of acceleration of the expansion of the Universe rather than an exact one, which would correspond to a CC with $w = -1$ [24, 41]. From the definition of ϵ (1.70), we clearly see that this quasi-de Sitter phase entails a nearly constant Hubble parameter since $-\dot{H} \ll H^2$.

1.2.3 Scalar Field Dynamics

In the *fluid flow approach* [59, 60], a single scalar field may be employed to realise a quasi-de Sitter phase of inflation. The energy-momentum tensor of the scalar field $\phi(t, \mathbf{x})$ reads [24]

$$T_{\mu\nu}^{\phi} = \partial_{\mu}\phi\partial_{\nu}\phi - g_{\mu\nu} \left[\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + V(\phi) \right], \quad (1.72)$$

where $V(\phi)$ is the potential energy of the field. The action of this field in curved spacetime is

$$S_\phi = \int d^4x \sqrt{-g} \mathcal{L}_\phi = - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right], \quad (1.73)$$

$g \equiv \det(g_{\mu\nu})$ being the determinant of the metric tensor and $\mathcal{L}_\phi$ the Lagrangian. We indeed see that we obtain the energy-momentum tensor via the usual definition [24]

$$T_{\mu\nu}^\phi \equiv - \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_\phi)}{\delta g^{\mu\nu}}, \quad (1.74)$$

which can be applied to any matter field action. It is important to write down *Jacobi's formula* which shall be used later on, in different sections and on many occasions [23]:

$$g_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}}. \quad (1.75)$$

In line with the fluid flow approach, we introduce the fluid four-velocity for the scalar field [60]:

$$u_\mu = - \frac{\partial_\mu \phi}{\sqrt{-g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi}}, \quad (1.76)$$

which clearly satisfies the normalisation condition $g^{\mu\nu} u_\mu u_\nu = -1$. The energy-momentum tensor then reads

$$T_{\mu\nu}^\phi = \rho_\phi u_\mu u_\nu + P_\phi h_{\mu\nu}, \quad (1.77)$$

where the energy density and pressure are

$$\rho_\phi = \frac{1}{2} (u^\mu \partial_\mu \phi)^2 + V(\phi), \quad (1.78)$$

$$P_\phi = \frac{1}{2} (u^\mu \partial_\mu \phi)^2 - V(\phi), \quad (1.79)$$

respectively. Notice that no assumption of homogeneity and isotropy has been made so far; i.e. the metric is completely general. We then interpret the scalar field as a perfect fluid (see Eq. (1.14)) with energy density and pressure given by ρ_ϕ and P_ϕ respectively, as measured by a comoving observer (i.e. an observer in the rest frame of u^μ). Assuming the conservation of energy and momentum, Eq. (1.25) yields

$$u^\mu \partial_\mu \rho_\phi + \theta (\rho_\phi + P_\phi) = 0, \quad (1.80)$$

$$h^{\mu\nu} \partial_\nu P_\phi + a^\mu (\rho_\phi + P_\phi) = 0, \quad (1.81)$$

where the first equation was obtained by multiplying the terms of Eq. (1.25) by u^μ , bearing in mind the normalisation condition of u^μ . The second equation is obtained from Eq. (1.25) as a consequence of the first one.

We then consider a homogeneous scalar field $\phi(t)$ and a spatially-flat FLRW metric, such that $u^\mu = (1, \mathbf{0})$. The energy density and isotropic pressure read

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) , \quad (1.82)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) , \quad (1.83)$$

respectively. From Eq. (1.80) we obtain

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 , \quad (1.84)$$

whereas Eq. (1.81) yields a trivial equation as was stated before Eq. (1.27).

Eq. (1.84) could have been obtained by extremising the action (1.73) with respect to the scalar field ϕ . We would have the following scalar field equation [24]:

$$\square\phi - V_{,\phi} = 0 , \quad (1.85)$$

$\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ being the d'Alembert operator. This equation yields Eq. (1.84) when we assume a homogeneous scalar field and the spatially-flat FLRW metric [24].

The EoS parameter of this ‘fluid’ reads

$$w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} . \quad (1.86)$$

If the potential energy dominates over the kinetic energy $\dot{\phi}^2/2$, then w_ϕ approaches the value of the EoS parameter of the CC ($w_\phi \simeq -1$). Indeed, if the kinetic energy is virtually zero or very small, the potential energy becomes constant and can be interpreted as a CC term in the action. If one thinks of a scalar field potential sufficiently flat so the field moves slowly on it and $w_\phi \simeq -1$, that may be a good candidate to realise inflation.

1.2.4 Slow-Roll Inflation

The simplest inflationary scenario turns out to be the one that fits the current observations surprisingly well, as we shall see later. This scenario is called *Slow-Roll (SR) inflation* and

assumes, in the simplest model, that a single scalar field rolls over a nearly flat potential such that $\dot{\phi}^2 \ll 2V$ and $w_\phi \simeq -1$ [41]. In this single-field SR model, the Friedmann equation (for a spatially-flat universe) and Eq. (1.69) take on the following exact forms:

$$3H^2 = \frac{1}{M_{\text{Pl}}^2} \left(\frac{1}{2} \dot{\phi}^2 + V \right), \quad (1.87)$$

$$\dot{H} = -\frac{1}{2} \frac{\dot{\phi}^2}{M_{\text{Pl}}^2}, \quad (1.88)$$

respectively. Using these two equations, we can write the ϵ parameter defined in Eq. (1.70) as

$$\epsilon = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V}, \quad (1.89)$$

such that $\epsilon \ll 1$ implies $\dot{\phi}^2 \ll 2V$ and therefore $w_\phi \simeq -1$, as expected.⁵ This implies that $3H^2 \simeq V/M_{\text{Pl}}^2$ (see Eq. (1.87)).

Consequently, a slow-rolling scalar field violates the SEC and leads to a universe that expands with acceleration, which is what inflation prescribes to solve the flatness and horizon problems. However, we must ensure that inflation lasts long enough, and hence, that the ϵ parameter remains small over a certain period during the accelerating expansion of the Universe. To that end, we introduce a second parameter ϵ_2 (while the previous ϵ parameter is now dubbed as ϵ_1) such that [61, 62]

$$\epsilon_2 \equiv \frac{\dot{\epsilon}_1}{H\epsilon_1}, \quad (1.90)$$

where $|\epsilon_2| \ll 1$. Notice that ϵ_2 can be negative as opposed to ϵ_1 (as ϵ_1 may decrease during the early stages of inflation despite having to grow as we approach the end of inflation). Using these two parameters we can introduce a third one which relates the acceleration (or deceleration) term in the scalar field equation and $H\dot{\phi}$ [24]:⁶

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon_1 - \frac{1}{2}\epsilon_2. \quad (1.91)$$

The conditions for inflation to occur ($\epsilon_1 \ll 1$) and for inflation not to end prematurely

5. We make it clear that Eq. (1.89) is a consequence of Eq. (1.70) applied to a scalar field driven inflation. The general definition of ϵ is given by Eq. (1.70).

6. Do not confuse the η parameter with the conformal time defined in Eq. (1.12). We follow the usual convention in the literature.

($|\epsilon_2| \ll 1$) imply that one can neglect the acceleration (deceleration) term in the scalar field equation (1.84) against the Hubble friction term $3H\dot{\phi}$, such that the latter is locked to the slope of the potential $V_{,\phi}$

$$3H\dot{\phi} \simeq -V_{,\phi} . \quad (1.92)$$

This is the *SR equation of motion* [24, 41].

Under the conditions $\epsilon_1 \ll 1$ and $|\epsilon_2| \ll 1$ (hence $|\eta| \ll 1$), which eliminate the second time derivative $\ddot{\phi}$ from Eq. (1.84), the first time derivative $\dot{\phi}$ becomes a single function of the scalar field value (remember that $3H^2(\phi) \simeq V(\phi)/M_{\text{Pl}}^2$ because $\epsilon_1 \ll 1$, so H is a single function of the field ϕ). This means we can write the coordinate time t as a single function of the field too and replace the time derivatives by derivatives with respect to the scalar field.

If we assume that the sign of $\dot{\phi}$ does not change ($\dot{\phi} \neq 0$) over a given trajectory in the phase space $(\phi, \dot{\phi})$, the Friedmann equation can be written as [24]

$$3H^2(\phi) - 2M_{\text{Pl}}^2 \left(\frac{dH}{d\phi} \right)^2 = \frac{V(\phi)}{M_{\text{Pl}}^2} , \quad (1.93)$$

where Eq. (1.88) yields

$$\dot{\phi} = -2M_{\text{Pl}}^2 \frac{dH}{d\phi} , \quad (1.94)$$

and where we assume $\dot{\phi} > 0$ (the choice of sign of $\dot{\phi}$ will not be altered during the evolution of the field as was assumed already). We have replaced the time dependence of the Hubble parameter, $H = H(t)$, by a dependence on the scalar field, such that $H = H(\phi)$, which is not problematic as long as, again, $\dot{\phi}$ does not change its sign. This procedure is known as the ‘Hamilton-Jacobi formalism’ for inflation [24, 61, 63].

Now, consider a small deviation from a solution to the previous field equations, $H \rightarrow H + \delta H$, such that Eq. (1.93) becomes [24, 41, 61]

$$\frac{1}{\delta H} \frac{d\delta H}{d\phi} = \frac{3H}{2M_{\text{Pl}}^2} \frac{1}{\frac{dH}{d\phi}} \Rightarrow \delta H(\phi) = \delta H_i \exp \left(\frac{3}{2M_{\text{Pl}}^2} \int_{\phi_i}^{\phi} \frac{H}{\frac{dH}{d\phi}} d\tilde{\phi} \right) , \quad (1.95)$$

where $\delta H_i = \delta H(\phi_i)$. If we assume that the solution $H(\phi)$ corresponds to the SR solution, then

$$\left(\frac{1}{H} \frac{dH}{d\phi} \right)^2 \ll \frac{1}{2M_{\text{Pl}}^2} , \quad (1.96)$$

which means that any deviation δH shall be suppressed exponentially fast *at least*, as the

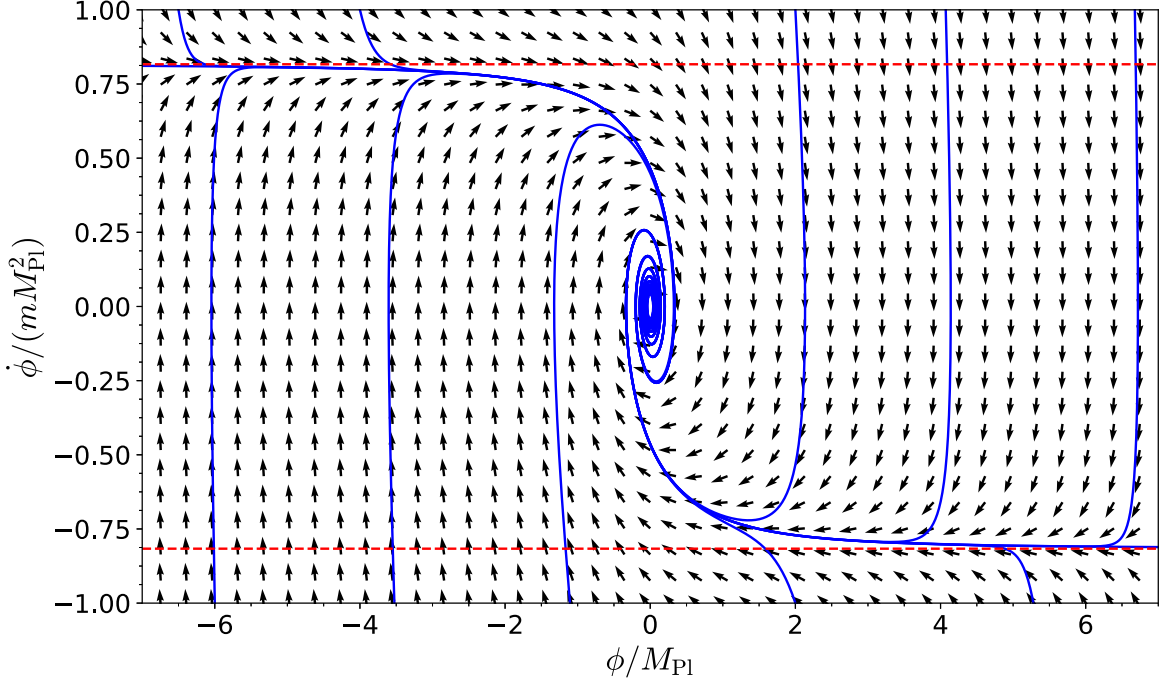


Figure 1.6: Attractor in the phase space for the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. Red dashed lines are the SR solution and blue lines denote trajectories in the phase space that start from different initial conditions.

field rolls

$$\delta H < \delta H_i \exp \left(- \int_{\phi_i}^{\phi} \frac{3}{\sqrt{2}M_{\text{Pl}}} d\tilde{\phi} \right) = \delta H_i \exp \left(- \frac{3|\phi - \phi_i|}{\sqrt{2}M_{\text{Pl}}} \right), \quad (1.97)$$

where remember that $dH/d\phi < 0$.

The SR solution is then a stable *attractor* solution. For some potentials like the quadratic potential, which we shall consider soon, this SR trajectory is attained from a quite ample set of different initial conditions as we can see in Fig. 1.6.

Eqs. (1.93) and (1.94) are widely known in the literature. Eq. (1.93) is usually referred to as the Hamilton-Jacobi equation [24, 61]. These are nothing but the usual field equations written in terms of derivatives with respect to the scalar field, which is possible to do on the proviso that t is a single-valued function of the scalar field; i.e. $\dot{\phi}$ does not change sign as the scalar field rolls. This approach also assumes that, as we said already but it is worth stressing, H can be expressed as a single function of ϕ .

The Friedmann equation in inflation

$$3H^2(\phi) \simeq \frac{V(\phi)}{M_{\text{Pl}}^2}, \quad (1.98)$$

and the SR equation (1.92), allow to write the ϵ_1 parameter in terms of the potential and the slope of the potential:

$$\epsilon_1 \simeq \epsilon_V \equiv \frac{M_{\text{Pl}}^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2. \quad (1.99)$$

As long as the SR approximation is valid during inflation, this ϵ_V parameter can be used in place of ϵ_1 , but its smallness is a necessary but not sufficient condition for SR [41]. One can also write a parameter analogous to η in terms of the potential and its derivatives with respect to the scalar field. To that end, we notice that the SR approximation includes a further assumption which is to consider that the derivative of Eq. (1.92) holds [24], so η is given by

$$\eta \simeq M_{\text{Pl}}^2 \frac{V_{,\phi\phi}}{V} - \epsilon_V. \quad (1.100)$$

This means that SR translates into

$$|\eta_V| \equiv M_{\text{Pl}}^2 \frac{|V_{,\phi\phi}|}{V} \ll 1. \quad (1.101)$$

Usually, the *SR conditions* [24, 41] refer to the fact that the relative potential value is much larger than the first and second derivatives of the potential ($\epsilon_V \ll 1$ and $|\eta_V| \ll 1$, respectively). In SR one can indeed choose a potential, a single function of the scalar field value, and develop an inflationary model imposing the SR conditions. These conditions restrict the form of the potential during inflation although one can certainly choose among a wide variety of potentials. In order to discriminate among them, these potentials must be confronted with observations and, as we shall see, SR allows to do this in a simple manner.

The ϵ_1 and ϵ_2 parameters belong to a set known as the ‘Hubble-flow’ parameters, defined as [61, 62]

$$\epsilon_{i+1} \equiv \frac{\dot{\epsilon}_i}{H\epsilon_i} ; \quad (i \geq 1), \quad (1.102)$$

where $i \in \mathbb{N}$. The SR approximation demands all the parameters of the above set be negligible. On the other hand, the parameters ϵ_V and η_V are referred to as the *SR parameters* in the literature, or ‘potential-slow-roll’ (PSR) parameters when the Hamilton-Jacobi formalism is considered [41, 61]. When these parameters are regarded, one works at first-order in a *SR expansion* that encompasses higher derivatives of the potential with respect to the scalar field. Such an expansion reads [61]

$${}^n\beta_V \equiv M_{\text{Pl}}^2 \left(\frac{(V_{,\phi})^{n-1}}{V^n} \frac{\partial^{n+1} V}{\partial \phi^{n+1}} \right)^{1/n} ; \quad (n \geq 1), \quad (1.103)$$

where ${}^1\beta_V = \eta_V$ and $n \in \mathbb{Z}^+$. Note that any n -th order of this expansion involves the knowledge of a $(n+1)$ -th derivative of the potential, and none of the terms involved diverges if the potential is constant.

As was noticed earlier, although $\epsilon_1 = 1$ signals the end of inflation, $\epsilon_V = 1$ cannot be interpreted in that sense. This can be readily seen if we write the *exact* relation between these two parameters [61]

$$\epsilon_V = \epsilon_1 \left(\frac{3 - \eta}{3 - \epsilon_1} \right)^2, \quad (1.104)$$

where the scalar field equation was used to write the η parameter as

$$\eta = 3 + \frac{V_{,\phi}}{H\dot{\phi}}, \quad (1.105)$$

and the Friedmann equation was written as

$$H^2 = \frac{V/M_{\text{Pl}}^2}{3 - \epsilon_1}. \quad (1.106)$$

This relation cannot be inverted in an exact manner; i.e. one cannot express ϵ_1 as a function of the potential parameters unless the SR approximation is assumed [61]. However, $\epsilon_V = 1$ is in most cases a good proxy for the true end of inflation.

We need to have a value for the minimum duration of inflation so horizon and flatness problems are resolved. To that end, a very suitable time variable is the *number of e-folds* [24, 41]

$$N(t) \equiv \int_{t_i}^t H d\tilde{t}, \quad (1.107)$$

although a different convention is broadly regarded in the literature where [64]

$$N(t) \equiv \int_t^{t_f} H d\tilde{t}, \quad (1.108)$$

and $N(t_f) = 0$ as opposed to the former case in which $N(t_i) = 0$. We shall work mainly with Eq. (1.107), but on several occasions, mostly those involving numerical simulations, we shall make use of the case that considers the number of e-folds as counting backwards from the end of inflation.

The approximate linear dependence of the number of e-folds on the coordinate time during inflation (given that $H \simeq \text{constant}$) makes N very useful (although it becomes a rather inadequate time variable right after the end of inflation). In single-field SR models of

inflation, we can rewrite the number of e-folds as [24]

$$N(\phi) = \int_{\phi_i}^{\phi} \frac{H}{\dot{\phi}} d\tilde{\phi} \simeq \frac{1}{M_{\text{Pl}}^2} \int_{\phi}^{\phi_i} \frac{V}{V_{,\tilde{\phi}}} d\tilde{\phi}, \quad (1.109)$$

where we used the SR equation in the last step (Eq. (1.92)). Then, choosing a potential, we can estimate the number of e-folds prescribed by the SR assumption that are left before the end of inflation for a given field value. A simple case is the quadratic potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2, \quad (1.110)$$

$m^2 \equiv V_{,\phi\phi}$ being the mass squared of the field. Eq. (1.109) yields

$$N(\phi) = \frac{1}{4M_{\text{Pl}}^2} (\phi_i^2 - \phi^2). \quad (1.111)$$

To find the final field value ϕ_e we can resort to the SR parameter ϵ_V and set it equal to one. We have stressed that this does not represent the true end of inflation but that usually, and more often than not, $\epsilon_V = 1$ is a good estimate.⁷ Then

$$\epsilon_V(\phi) = 2 \frac{M_{\text{Pl}}^2}{\phi^2}, \quad (1.112)$$

such that $\epsilon_V(\phi_e) = 1$ yields $\phi_e/M_{\text{Pl}} = \sqrt{2}$. Therefore, for a given ϕ_i , we can determine the number of e-folds before the end of inflation according to the SR approximation. In the same way that $\epsilon_V(\phi_e) = 1$ is a good estimate for the end of inflation in most cases, so is the SR formula of $N(\phi)$ (1.109) for the actual duration of inflation. It should be noted that this quadratic potential has an unknown parameter m which must be fixed somehow. We shall see in one of the coming sections how this parameter can be determined via observations.

An example of more involved potential that can be used to realise SR is the following hyperbolic tangent potential [65]:

$$V(\phi) = \frac{V_0}{2} \left[1 - \tanh \left(p \frac{\phi}{M_{\text{Pl}}} \right) \right], \quad (1.113)$$

7. The differences between $\epsilon_V = 1$ and $\epsilon_1 = 1$ are usually less than an e-fold as we were able to check ourselves while performing numerical simulations like the ones displayed in Fig. 1.6.

where $p > 0$ and $V(0) = V_0/2$. The number of e-folds in the SR approximation is

$$N(\phi) = -\frac{1}{2p^2} \left[p \frac{\phi_i - \phi}{M_{\text{Pl}}} + \frac{1}{2} e^{-2p\phi/M_{\text{Pl}}} \left(1 - e^{2p(\phi - \phi_i)/M_{\text{Pl}}} \right) \right], \quad (1.114)$$

and the SR parameter reads

$$\epsilon_V(\phi) = \frac{p^2}{2} \left[1 + \tanh \left(p \frac{\phi}{M_{\text{Pl}}} \right) \right]^2, \quad (1.115)$$

such that $\epsilon_V(\phi_e) = 1$ yields the final field value, which is a function of the parameter p :

$$\phi_e(p) = -\frac{M_{\text{Pl}}}{2p} \ln \left(\sqrt{2p} - 1 \right). \quad (1.116)$$

The SR scenario is not new and dates back to 1982 when A. Linde put forward a model of inflation where the *inflaton* field, ϕ , slowly rolled down to the minimum of its effective potential [66]. His study was motivated by the previous work of A. Guth (commonly regarded as one of the main initial proponents of the theory of inflation) in 1981 [50]. In this work, Guth discussed the implausibility of the assumptions made by the HBB model on the initial conditions and how the problems that emerged from these disappeared if the Universe underwent a supercooling phase transition below a certain critical temperature. However, in that very same paper, Guth acknowledged the formation of gross inhomogeneities this process led to and therefore the latent difficulty in achieving a smooth ending to this supercooling period (*a graceful exit*). Since Linde's paper dealt with this issue, Guth's inflation model is known as *old inflation* while Linde's proposal is referred to as *new inflation* [67].

1.2.5 Preheating and Reheating

In this section we discuss the exit from inflation and, in particular, from the slow-rolling inflation phase. As was commented earlier, inflation must end in order for the Universe to go through the processes described by the HBB model. Since the Universe expanded quasi-exponentially in this inflation era (that is, $a(t) \propto e^{Ht}$ approximately), the temperature dropped considerably (see below Eq. (1.44)), meaning that once inflation concluded, the Universe was far from radiation domination. We know with extreme precision that at BBN, the Universe was dominated by radiation (see Sec. 1.1.3), so a phase of *reheating* is needed after inflation [41, 68–71].

In the simplest models, one can envisage the mechanism of reheating as the field oscil-

lating about the minimum of its potential. Near this minimum ϕ_0 , the leading term of the potential is the quadratic one $\frac{1}{2}m^2(\phi - \phi_0)^2$. Let us introduce the angular variable θ via the following change of variables [72]:

$$m\phi \equiv \sqrt{6}HM_{\text{Pl}} \cos \theta , \quad (1.117)$$

$$\dot{\phi} \equiv \sqrt{6}HM_{\text{Pl}} \sin \theta . \quad (1.118)$$

Eq. (1.88) and the scalar field equation (1.84) yield (let us assume that $\phi_0 = 0$)

$$\dot{H} = -3H^2 \sin^2 \theta , \quad (1.119)$$

$$\dot{\theta} = -m - \frac{3}{2}H \sin(2\theta) . \quad (1.120)$$

As H decreases, the angular variable θ approaches the linear relation $\theta(t) \simeq -mt + \alpha$, and the scalar field oscillates with frequency m (the constant phase α can be removed). If we replace $\theta \simeq -mt$ in Eq. (1.119), we obtain [71]

$$H(t) \simeq \frac{2}{3t} \left(1 - \frac{\sin(2mt)}{2mt} \right)^{-1} , \quad (1.121)$$

where the constant term $\alpha_1 m$ in the denominator, α_1 being a constant of integration, was removed by a time shift. When $t \gg 1/m$, $H(t) \simeq 2/(3t)$, which is the Hubble parameter corresponding to a universe dominated by matter (see after Eq. (1.39)). If we plug Eq. (1.121) and $\theta \simeq -mt$ into Eq. (1.117), we obtain the field value in this oscillatory regime

$$\phi(t) \simeq 2\sqrt{\frac{2}{3}}M_{\text{Pl}} \frac{\cos(mt)}{mt} . \quad (1.122)$$

Therefore, the aftermath of SR consists of a regime of *damped oscillations* of the field about $\phi = \phi_0 = 0$. The *time averaged* EoS parameter, after we take the period of oscillations to be much smaller than the cosmic time, corresponds to that of a *pressureless ideal fluid* [72]. During this phase of oscillations, the energy stored in the inflaton field ϕ may be released leading to the production of many elementary particles which interact with one another and eventually reach a state of thermal equilibrium with certain temperature T_R [67]. The period of reheating is of extreme importance as, depending on the specifics of it, the time during inflation at which one evaluates key observable inflation parameters shall differ. We will examine those inflation observables when we introduce linear perturbations in Cosmology and, in particular, during inflation.

Historically, the first treatment of reheating was *perturbative* [73]. The inflaton in the oscillating regime can be viewed as a condensate of massive scalar particles, each one decaying independently [71, 72]. The inflaton interactions are of the form [72]

$$S_M \supset \int d^4x \sqrt{-g} \left(-g\phi\chi^2 - h\phi\bar{\psi}\psi \right), \quad (1.123)$$

χ and ψ being the scalar and fermion decay products, respectively. g and h are two coupling constants, the former with the dimension of mass and the latter being dimensionless. During inflation, such couplings are not considered as the field is not oscillating, but they become crucial once inflation ends.

The corresponding decay rates at the tree level order are [68, 69, 72]

$$\Gamma_{\phi \rightarrow \chi\chi} = \frac{g^2}{8\pi m}, \quad (1.124)$$

$$\Gamma_{\phi \rightarrow \bar{\psi}\psi} = \frac{h^2 m}{8\pi}. \quad (1.125)$$

The total width $\Gamma \equiv \Gamma_{\phi \rightarrow \chi\chi} + \Gamma_{\phi \rightarrow \bar{\psi}\psi}$ can be inserted in the homogeneous field equation (1.84) as a friction term to account for the energy loss of the inflaton due to particle production [41, 70]

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V_{,\phi} \simeq 0. \quad (1.126)$$

When H has decreased to a value comparable to Γ , particle production becomes an effective process against the energy loss due to the expansion of the Universe (as we explained in Sec. 1.1.3). Thus, when $H \sim \Gamma$, we can find a maximum value of the reheating temperature if the decay products are assumed to be light relativistic particles mainly (see the Friedmann equation (1.15) for a spatially-flat universe and Eq. (1.41))

$$T_R \simeq \left(\frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{M_{\text{Pl}} \Gamma}. \quad (1.127)$$

There are many issues in relation to this approach [70]. One of them is what we remarked in regard to the independent decay of each particle of the inflaton condensate. Although, due to the large occupation number, the condensate can be treated classically, the decay products must be accounted for *quantum mechanically* [74]. Analysis based on the quantum production of matter particles in a classical background inflaton field showed that reheating might involve a *parametric resonance instability* [68, 69, 74, 75]. To see this, we consider the

following interaction Lagrangian [70]:

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}g^2\phi^2\chi^2 . \quad (1.128)$$

Following the recipe of the quantum theory, we expand the quantum field operator $\hat{\chi}(x)$ into annihilation and creation operators, $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ respectively,

$$\hat{\chi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\hat{a}_{\mathbf{k}}\chi_k(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger\chi_k^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}} \right] , \quad (1.129)$$

where k denotes the momentum and χ_k^* is the complex conjugate of χ_k . The time dependent mode function $\chi_k(t)$ satisfies the following equation in an expanding background [70]:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left[\frac{k^2}{a^2} + m_\chi^2 + g^2\Phi^2(t)\sin^2(mt) \right] \chi_k = 0 , \quad (1.130)$$

m_χ being the bare mass of χ and $\Phi(t)$ the amplitude of oscillations of the inflaton field, such that

$$\phi(t) = \Phi(t)\sin(mt) . \quad (1.131)$$

Eq. (1.130) is that of a harmonic oscillator with variable frequency damped by the expansion of the Universe. We will see how to obtain that equation in the next part of this chapter, when we analyse linear perturbations in Cosmology. Depending on the momentum k , some of the modes with mode functions χ_k are expected to become *parametrically* excited. If we assume that the time period is shorter than the Hubble expansion time given by H^{-1} , we can perform the analysis in a flat background spacetime (with $a = 1$), such that we obtain the *Mathieu equation* [70]

$$\chi_k'' + [A_k - 2q\cos(2mt)]\chi_k = 0 , \quad (1.132)$$

where

$$A_k \equiv \frac{k^2 + m_\chi^2}{m^2} + 2q , \quad (1.133)$$

$$q \equiv \frac{g^2\Phi^2}{4m^2} , \quad (1.134)$$

and primes denote derivatives with respect to mt . For values $q \ll 1$, we find that a resonance occurs in a narrow instability band (*narrow resonance*), whilst the resonance is much more

efficient if $q \gg 1$ and occurs in broad bands (*broad resonance*) [72]. In the latter case, particles are produced in bursts in contrast to the smooth production that takes place in the narrow resonance. The bursts occur when the adiabaticity condition at some time t_0 [70]

$$\frac{|\dot{\omega}_k(t_0)|}{\omega_k^2(t_0)} < 1 , \quad (1.135)$$

is violated, where the frequency $\omega_k(t)$ in the case under consideration is given by

$$\omega_k^2(t) = k^2 + m_\chi^2 + g^2 \Phi^2(t) \sin^2(mt) . \quad (1.136)$$

In the case of an expanding universe, the broad resonance is very efficient as more k modes are encompassed by the instability band due to the cosmological redshift. Since these resonances are due to the choice of parameters, they are known as ‘*parametric resonances*’.

This early period of non-perturbative decay of the inflaton due to the effect of broad resonance was called *preheating* [68], in order to differentiate it from the stage of particle decay and thermalisation described using perturbation theory, which shall be employed eventually to describe the energy transfer of the oscillating inflaton field when the parametric resonance becomes inefficient. The process of preheating was shown to lead to nonthermal cosmological phase transitions [76–78] and production of large amount of superheavy particles [67–69, 75]. The latter is of great interest as the interactions and decay of these superheavy particles may lead to the production of baryons at the scale of Grand Unification (the so-called ‘GUT baryogenesis’) [79].

We considered a hyperbolic tangent potential in Eq. (1.113) that displayed a period of SR. In this potential, in contrast to the quadratic one (1.110), the field *does not* oscillate at the bottom [65]. In this case then, the standard perturbative techniques do not work, and one must resort to other ways to reheat the Universe. An example put forward in 1998 that can be used in this case is *instant preheating* [80], which is based on the same mechanism as the parametric resonance. The reason why ‘instant’ was coined is that, in those models where oscillations occur, this process of preheating takes place within less than one oscillation of the inflaton, immediately after inflation concludes. Due to this, it works in those models of inflation with potentials with no oscillatory regime.

In this model, the decay products of the inflaton field, the χ particles, interact with fermions ψ with the interaction Lagrangian [80]

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}g^2\phi^2\chi^2 - h\chi\bar{\psi}\psi . \quad (1.137)$$

The interesting aspect of instant preheating is that the energy density of the χ particles at the moment of their decay may become much greater than their energy density at the moment of their creation. This is due to the fact that the effective mass of each of these χ particles depends on ϕ such that $m_\chi = g\phi$ and, when the field rolls back from the minimum, these are ‘fattened’ and tend to decay when the field reaches its maximal value. The energy density of these particles is [80]

$$\rho_\chi \simeq \frac{g^{5/2} |\dot{\phi}_{\text{ip}}|^{3/2} \phi_{\text{ip}}}{8\pi^3} , \quad (1.138)$$

‘ip’ denoting the time of instant preheating. For the quadratic potential, $\phi_{\text{ip}} \simeq 10^{-1} M_{\text{Pl}}$ and $|\dot{\phi}_{\text{ip}}| \simeq m\phi_{\text{ip}}$, and the ratio between ρ_χ and the energy density of the inflaton field, which is dominated by the kinetic energy density $\dot{\phi}_{\text{ip}}^2/2$, is [80]

$$\frac{\rho_\chi}{\rho_\phi} \simeq 2g^{5/2} , \quad (1.139)$$

where $m \simeq 10^{-6} M_{\text{Pl}}$ for reasons that will be clear in a later section (as we commented before Eq. (1.113)). This ratio is indeed larger than that if each χ particle decays at the moment of their production [80]

$$\frac{\rho_\chi}{\rho_\phi} \simeq 5 \times 10^{-3} g^2 , \quad (1.140)$$

where the energy density of the χ particles is $\rho_\chi \propto g^2 \dot{\phi}_{\text{ip}}^2$, and $g \gtrsim 10^{-4}$ for efficient preheating.

The fermions produced by the decay of the χ particles can be as massive as two orders of magnitude greater than the particles that may be produced by the parametric resonance. These superheavy fermions (or the products of their decay) may eventually dominate the energy budget of the Universe even if their energy density was small in the beginning, and reheating will be completed [80], meaning that the Universe will be dominated by radiation as prescribed by the successful BBN.

Another reheating mechanism, which makes use of the change of sign of the Ricci scalar after inflation, is the *Ricci reheating* [81, 82]. The Ricci scalar is defined as the trace of the Ricci tensor $R_{\mu\nu}$ [23]

$$R \equiv g^{\mu\nu} R_{\mu\nu} , \quad (1.141)$$

and it was shown to play a role in the Einstein field equations (1.1). For the spatially-flat

FLRW metric, the Ricci scalar reads [82]

$$R = 12H^2 + 6\dot{H} = 3H^2(1 - 3w). \quad (1.142)$$

During inflation, $w \simeq -1$ and $R > 0$. However, when inflation ends and the kinetic energy dominates over the potential, the EoS parameter becomes $w \simeq 1$ (assuming the inflaton field energy density still dominates), and $R < 0$. This period of kinetic energy domination is called *kination* [83]. If, in addition to the inflaton, we consider a further scalar field (*spectator field*) χ , with the following action [82]:

$$S_\chi = -\frac{1}{2} \int d^4x \sqrt{-g} \left[\xi \chi^2 R + g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + 2V(\chi) \right], \quad (1.143)$$

this field will have the following effective potential:

$$V_{\text{eff}}(\chi) \equiv V(\chi) + \frac{1}{2} \xi \chi^2 R, \quad (1.144)$$

ξ being a coupling constant. Assuming the term $\frac{1}{2} \xi \chi^2 R$ dominates over the potential $V(\chi)$ at the end of inflation, the change of sign of R shall correspond to a change of sign of the effective mass squared of χ , $m_\chi^2 \equiv V_{\text{eff},\chi\chi}(0)$, so it becomes *tachyonic* ($m_\chi^2 < 0$). This field, which was at the minimum of its effective potential during inflation, rolls towards the new minimum and begins to oscillate. The field χ oscillating about this minimum has the particle interpretation that was previously discussed, and may decay into radiation, reheating the Universe eventually. One may argue that, since the inflaton dominates, this stage of reheating would not occur. However, during kination, the inflaton field equation reads [83]

$$\ddot{\phi} \simeq -3H\dot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} \Rightarrow \dot{\phi} \propto a^{-3}, \quad (1.145)$$

such that the energy density of the inflaton decays as a^{-6} , much faster than that of the resulting radiation (see Eq. (1.32)).

The coupling between the spectator field χ (that we might call the *reheaton* [82]) and the Ricci scalar in Eq. (1.143) is very well motivated by the results of applying quantum field theory in curved spacetimes [84–86], and we shall address it to a greater extent in the next chapter of this thesis.

1.2.6 Alternatives to the Inflationary Cosmology

Since the late 70's, the inflation paradigm has gained increasing ascendancy over many other alternatives which still attempt to address the issues concerning the initial conditions in the HBB model. Some notorious examples are the ekpyrotic model [87], the cyclic universe [88], bouncing [89] and string gas [90] cosmologies, the pre-Big Bang scenario [91] and the conformal cyclic cosmology [92]. Most of these scenarios, although they do reproduce some features predicted by the inflation paradigm, they do not entirely address the flatness and horizon problems [67].

Despite being commonly accepted and included in many cosmology textbooks along with the HBB model, cosmic inflation *has not* been directly confirmed by observations yet. A detection reported in March 2014 from the BICEP2 experiment [93] seemed to have confirmed gravitational waves caused by inflation in the form of one type of the so-called B-modes (we mentioned them in Sec. 1.1.2 when we discussed the CMB), which are a result of the polarization of the CMB. However, in September 2014, new results of the Planck experiment indicated that much of the BICEP2 signal was attributed to interstellar dust (see Ref. [94] for the joint analysis).

Due in part to the lack of direct observational evidence, the inflation paradigm lacks a definite model of particle physics too [95]. Indeed, it can be realised by many scalar field potentials [64], by more than one single scalar field [96–98], and even by vector fields [99–104]. There is even a version of cosmic inflation which accounts for the couplings between the inflaton and other matter fields, which were regarded in the post-inflationary phase, but were neglected during inflation. This version is called *warm inflation* [105–108]. In the warm inflation paradigm both, dissipative effects and radiation production, occur concurrently with the inflationary expansion. The inflaton dissipates its energy and transfers it to the subdominant radiation bath. This radiation bath is then sustained by the energy transfer and is not diluted by the expansion. When dissipation becomes stronger, the reheating period may take place during inflation, without the need for a separate period after inflation [107].

Also, the warm inflation paradigm deals with one problematic aspect of SR: the so-called η -problem [109, 110]. If we remember, the η parameter in SR gave the ratio between $m_\phi^2 \equiv V_{,\phi\phi}$ and the Hubble parameter squared $3H^2 \simeq V/M_{\text{Pl}}^2$ (Eq. (1.101)). Since during SR, $|\eta_V| \ll 1$, this means that the inflaton mass must be smaller than the Hubble parameter; i.e., smaller than *the energy scale of inflation*. However, a small inflaton mass is *radiatively unstable*, meaning that, in the absence of symmetries preventing it, the η parameter will receive quantum corrections which shall spoil a prolonged phase of SR. In the warm version

however, the η_V parameter is modified and no longer required to be smaller than unity but smaller than $1 + Q$, Q being the dissipative ratio between the dissipation coefficient Υ and the Hubble parameter [95, 111, 112]. If the dissipative effects are strong, the smallness of the inflaton mass is not a requirement for successful SR.

Despite the theoretical and observational challenges the inflation paradigm must face, there is one further advantage which we have not discussed yet and will be an essential part of the next section: the inflation paradigm is capable of explaining, in a simple manner, the origin of all the large scale structures we observe in the Universe nowadays. It can also provide details about the properties of the primordial density perturbations which were the seed of those structures [24].

1.3 Linear Perturbations in Cosmology

So far we have regarded an isotropic and homogeneous universe and used the FLRW metric under the guidance of the CP. However, we observe inhomogeneities on small scales which are the result of gravitational instabilities developed in slightly denser regions of the early Universe [41], and we need to understand where these regions stemmed from in the first place. For this purpose, the *relativistic cosmological perturbation theory* has proven to be essential [24, 113, 114].

1.3.1 Introduction to Cosmological Perturbation Theory

To describe those tiny inhomogeneities on cosmological scales, we differentiate between the perturbed spacetime and a *background* spacetime by writing the metric as [24]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} , \quad (1.146)$$

where $\bar{g}_{\mu\nu}$ is the metric of the background spacetime and $g_{\mu\nu}$ is the full metric of the perturbed spacetime. $\delta g_{\mu\nu}$ is a small perturbation (a *linear* perturbation).

In GR (or in any covariant theory of gravity) the physics is independent of the coordinate system under consideration [23]. We assume then that there exists a coordinate system on the perturbed spacetime that allows to write the metric as Eq. (1.146) [115]. However such a separation is not unique as, given a point $\bar{P}$ on the background spacetime, we have many ways to associate it with points on the perturbed spacetime. Let x^μ be the coordinates of a coordinate system on the background, and $\hat{x}^\mu$ and $\tilde{x}^\mu$ the coordinates of two different

coordinate systems on the perturbed spacetime. We assume that the coordinate system $\{\tilde{x}^\mu\}$ associates the point $\bar{P}$ in the background spacetime with a point $\tilde{P}$ in the perturbed spacetime, whilst $\{\hat{x}^\mu\}$ does the same with $\bar{P}$ and another point $\hat{P}$ in the perturbed spacetime [115]. These coordinate systems correspond to two ‘gauges’ in the context of relativistic cosmological perturbation theory [24]. The perturbation is then *gauge dependent* and we have

$$\delta\hat{g}_{\mu\nu} \equiv g_{\mu\nu}(\hat{P}) - \bar{g}_{\mu\nu}(\bar{P}) , \quad (1.147)$$

$$\delta\tilde{g}_{\mu\nu} \equiv g_{\mu\nu}(\tilde{P}) - \bar{g}_{\mu\nu}(\bar{P}) . \quad (1.148)$$

To relate both perturbations in different gauges, we assume that both gauges are related by an infinitesimal coordinate transformation (a *gauge transformation*) [116]

$$\tilde{x}^\mu = \hat{x}^\mu + \xi^\mu , \quad (1.149)$$

where ξ^μ and its first derivatives are first-order small. Consequently, the difference $\xi^\mu(\tilde{P}) - \xi^\mu(\hat{P})$ is second-order small, and we can just simply assume that ξ^μ lives on the background spacetime [115]

$$\xi^\mu = \xi^\mu(\bar{P}) = \xi^\mu(x^\alpha) . \quad (1.150)$$

Hence, we see that the perturbations live on the background spacetime as well, such that $\delta\hat{g}_{\mu\nu} = \delta\hat{g}_{\mu\nu}(x^\alpha)$ and $\delta\tilde{g}_{\mu\nu} = \delta\tilde{g}_{\mu\nu}(x^\alpha)$. Taking all this into account, the metric in different coordinate systems transforms as [23]

$$g_{\tilde{\mu}\tilde{\nu}}(\tilde{P}) = \frac{\partial\hat{x}^\alpha}{\partial\tilde{x}^\mu} \frac{\partial\hat{x}^\beta}{\partial\tilde{x}^\nu} g_{\hat{\alpha}\hat{\beta}}(\hat{P}) \simeq (\delta_\mu^\alpha - \partial_\mu\xi^\alpha) (\delta_\nu^\beta - \partial_\nu\xi^\beta) g_{\hat{\alpha}\hat{\beta}}(\hat{P}) , \quad (1.151)$$

at first order in ξ^μ . In different coordinate systems *and* on different points in the perturbed spacetime [115]:

$$g_{\tilde{\mu}\tilde{\nu}}(\tilde{P}) \simeq (\delta_\mu^\alpha - \partial_\mu\xi^\alpha) (\delta_\nu^\beta - \partial_\nu\xi^\beta) \left[g_{\hat{\alpha}\hat{\beta}}(\hat{P}) - \partial_\sigma\bar{g}_{\alpha\beta}\xi^\sigma \right] , \quad (1.152)$$

at first order in ξ^μ and at linear order in perturbation theory. Therefore, the infinitesimal transformation in Eq. (1.149) induces a gauge transformation of the perturbations, this being (see Eqs. (1.147) and (1.148))

$$\delta\tilde{g}_{\mu\nu} \simeq \delta\hat{g}_{\mu\nu} - \partial_\mu\xi^\sigma\bar{g}_{\sigma\nu} - \partial_\nu\xi^\sigma\bar{g}_{\mu\sigma} - \xi^\sigma\partial_\sigma\bar{g}_{\mu\nu} , \quad (1.153)$$

at linear order. Or, remembering that in GR the metric is covariantly conserved (see Eq. (1.26)), we can rewrite it as

$$\delta\tilde{g}_{\mu\nu} \simeq \delta\hat{g}_{\mu\nu} - \bar{\nabla}_\mu \xi^\sigma \bar{g}_{\sigma\nu} - \bar{\nabla}_\nu \xi^\sigma \bar{g}_{\mu\sigma} . \quad (1.154)$$

From now on, we shall remove the bars when referring to background variables except when we write a full spacetime dependent quantity explicitly. We will also replace ‘ $\simeq$ ’ by ‘ $=$ ’ as we have stressed multiple times that the relations hold at linear order.

We analyse the *scalar* part of the linear metric perturbations.⁸ The line element reads [24]

$$ds^2 = -(1 + 2A) dt^2 - 2\partial_i B dx^i dt + a^2(t) [(1 - 2\psi) \delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j, \quad (1.155)$$

where we assume that the background spacetime is endowed with the spatially-flat FLRW metric. $A(x)$ denotes the linear perturbation of the lapse function, which describes how *time slices* are embedded in the spacetime, and $\partial_i B(x)$ is the shift vector perturbation, describing how the spatial coordinates are *threaded* through the spacetime [116]. $\psi(x)$ and $\partial_i \partial_j E(x)$ are the intrinsic curvature perturbation and the anisotropy in the spatial metric, respectively. These perturbations are gauge dependent, and therefore, subject to the gauge transformations. The gauge transformation vector ξ^μ , when the scalar perturbations are regarded, can be parametrised as [116]

$$\xi^\mu \equiv (\alpha, \partial^i \beta), \quad (1.156)$$

where $\alpha(x)$ and $\beta(x)$ are arbitrary spacetime functions. Then, from Eq. (1.154), we obtain the following transformation rules of the scalar perturbations in Eq. (1.155) [41]:⁹

$$A \rightarrow A - \dot{\alpha}, \quad (1.157)$$

$$B \rightarrow B + a^2 \dot{\beta} - \alpha, \quad (1.158)$$

$$E \rightarrow E - \beta, \quad (1.159)$$

$$\psi \rightarrow \psi + H\alpha. \quad (1.160)$$

8. ‘Scalar’ in the sense of how they transform under rotations in the background space [41, 115]. The same applies to the ‘vector’ and ‘tensor’ perturbations.

9. Notice that the transformation rule of some variables like B is true up to a constant of space. However, the homogeneous values of such variables vanish by construction. The same will happen with another variables like V and Ψ (see Eqs. (1.173) and (1.191)).

The lapse function and the curvature perturbations are spatially *gauge-invariant*; that is, they are not affected by the gauge mode $\beta(x)$ from the spatial components of the gauge transformation vector. This does not apply to B and E , as can be seen. We may however define the *shear potential* of worldlines orthogonal to the spacelike hypersurfaces, combining the shift vector perturbation and the anisotropy terms [115]:

$$\chi \equiv B + a^2 \dot{E} . \quad (1.161)$$

The shear potential then transforms as

$$\chi \rightarrow \chi - \alpha , \quad (1.162)$$

and we see that it is not affected by β .

Motivated by this, we can define quantities which are *fully* gauge-invariant. The following quantities are two examples of them:

$$A^\psi \equiv A + \left(\frac{\dot{\psi}}{H} \right) , \quad (1.163)$$

$$\psi^\chi \equiv \psi + H\chi . \quad (1.164)$$

These can be interpreted as the lapse function in the gauge where $\psi = 0$ (called *flat gauge* [41]), and the intrinsic curvature perturbation in the gauge where $\chi = 0$ (called *shear-free gauge* or *Newtonian gauge* [41]), respectively. Every time we write a perturbation with a superscript referring to another perturbation quantity, this will mean that the latter must be set equal to zero to obtain the former. These perturbations are then written in a *gauge-invariant way*.

Another important tensor, besides the metric, is the energy-momentum one $T_{\mu\nu}$. We introduced the energy-momentum tensor of a perfect fluid in Eq. (1.14), which shall correspond to the background energy-momentum tensor (the one compatible with the assumption of homogeneity and isotropy [115, 116]). However, the most general symmetric $T_{\mu\nu}$ that we may write is [117–119]

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \Pi_{\mu\nu} , \quad (1.165)$$

where

$$\rho \equiv u_\mu u_\nu T^{\mu\nu}, \quad (1.166)$$

$$P \equiv \frac{1}{3} h_{\mu\nu} T^{\mu\nu}, \quad (1.167)$$

are the energy density and the pressure respectively, and

$$q^\mu \equiv -h^{\mu\nu} u^\sigma T_{\nu\sigma}, \quad (1.168)$$

$$\Pi^{\mu\nu} \equiv h^{\mu\alpha} h^{\nu\beta} T_{\alpha\beta} - P h^{\mu\nu}, \quad (1.169)$$

are the *energy flux* and the *anisotropic pressure*, respectively. The normalisation condition, $u_\mu u^\mu = -1$, implies $u_\mu q^\mu = 0$ and $u_\mu \Pi^{\mu\nu} = 0$. Besides, $\Pi^{\mu\nu} = \Pi^{\nu\mu}$ and $\Pi^\mu{}_\mu = 0$ [119].

In line with the gauge transformation vector, the four-velocity is parametrised as [116]

$$u_\mu = (-1 - A, \partial_i V), \quad (1.170)$$

where at linear order we see that it satisfies the normalisation condition

$$g_{\mu\nu} u^\mu u^\nu \simeq g_{00} (u^0)^2 \simeq -1. \quad (1.171)$$

Knowing how the covariant components of the perturbation of a vector transform under gauge transformations [115]

$$\delta \tilde{u}_\mu = \delta \hat{u}_\mu - \xi^\alpha \partial_\alpha \bar{u}_\mu - \bar{u}_\alpha \partial_\mu \xi^\alpha, \quad (1.172)$$

we can easily see how V transforms (and confirm the transformation rule of A ; see footnote 9 too):

$$V \rightarrow V + \alpha. \quad (1.173)$$

We can calculate the scalar expansion rate and the four-acceleration vector, both introduced above Eq. (1.26). Before that however, let us calculate the determinant of the full metric up to linear order:

$$\sqrt{-g} = a^3 \left(1 + A - 3\psi + \partial_i \partial^i E \right), \quad (1.174)$$

where $\partial_i \partial^i \equiv \delta^{ij} \partial_i \partial_j$, and Jacobi's formula (1.75) was used. The scalar expansion rate

reads [116]

$$\theta = \nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma_{\mu\nu}^\mu u^\nu = 3H(1-A) - 3\dot{\psi} + a^{-2}\partial_i\partial^i(V+\chi). \quad (1.175)$$

The four-acceleration has vanishing background components and is given by (see App. A, where the Christoffel symbols perturbations and perturbations of other geometric quantities are shown in detail)

$$a_\mu = \left(0, \partial_i(\dot{V} + A)\right). \quad (1.176)$$

Let us now determine the components of the energy flux at linear order. Using the definition in Eq. (1.168), we obtain

$$q^0 = 0, \quad (1.177)$$

$$q^i = -(\rho + P)a^{-2}\partial^i V + a^{-2}\delta^{ij}\delta T_j^0, \quad (1.178)$$

where we used

$$\delta T_0^i = -(\rho + P)a^{-2}\partial^i B - a^{-2}\delta^{ij}\delta T_j^0. \quad (1.179)$$

Notice that we can make the energy flux vanish if we choose $\partial_i V$ such that

$$\partial_i V = \frac{1}{\rho + P}\delta T_i^0. \quad (1.180)$$

This choice corresponds to the *energy frame four-vector* [120]. Another choice, called the ‘normal frame four-vector’ corresponds to $u_i = 0$.

Regarding the anisotropic pressure, it has vanishing background components and its only non-zero components at linear order are the perturbations of the spatial components [115]. Since the components at background level are zero, we shall simply write $\Pi^{\mu\nu}$ rather than $\delta\Pi^{\mu\nu}$ to denote the perturbation components.

We can now see how the linear perturbations of the mixed components of the energy-momentum tensor T^μ_ν transform [116]:

$$\delta\tilde{T}^\mu_\nu = \delta\hat{T}^\mu_\nu + \partial_\sigma\xi^\mu\bar{T}^\sigma_\nu - \partial_\nu\xi^\sigma\bar{T}^\mu_\sigma - \xi^\sigma\partial_\sigma\bar{T}^\mu_\nu. \quad (1.181)$$

The scalar components are

$$\delta T_0^0 = -\delta\rho, \quad (1.182)$$

$$\delta T_i^0 = (\rho + P)\partial_i V, \quad (1.183)$$

$$\delta T_j^i = \delta P \delta_j^i + \Pi_j^i . \quad (1.184)$$

Consequently, the perturbations transform as (we verify the transformation rule of V as well)

$$\delta \rho \rightarrow \delta \rho - \dot{\rho} \alpha , \quad (1.185)$$

$$\delta P \rightarrow \delta P - \dot{P} \alpha , \quad (1.186)$$

$$\Pi_j^i \rightarrow \Pi_j^i . \quad (1.187)$$

We see that the anisotropic pressure is gauge-invariant. We observe the same feature for the four-acceleration

$$\partial_i(\dot{V} + A) \rightarrow \partial_i(\dot{V} + A) . \quad (1.188)$$

This is in line with the fact that both quantities vanish in a FLRW universe (this being one necessary and sufficient condition of the three prescribed by the Stewart-Walker lemma [121] for gauge-invariant quantities).

From the transformation rules of $\delta \rho$ and V , we can write two important gauge-invariant quantities involving matter perturbations

$$\psi^\Psi \equiv \psi - \frac{H}{\rho + P} \Psi , \quad (1.189)$$

$$\psi^{\delta \rho} \equiv \psi - \frac{\delta \rho}{3(\rho + P)} , \quad (1.190)$$

where we defined the *scalar momentum perturbation*

$$\Psi \equiv (\rho + P) V , \quad (1.191)$$

such that $\delta T_i^0 = \partial_i \Psi$, and where we applied the background continuity equation (1.27). ψ^Ψ is usually referred to as ‘ $\mathcal{R}$ ’ in the literature [41], whereas $\zeta \equiv -\psi^{\delta \rho}$. We shall use this notation from now on. $\mathcal{R}$ and $-\zeta$ are the *comoving curvature perturbation* and the *curvature perturbation on uniform-density slicings*, respectively. In the former case, the time slicing is chosen such that $\delta T_i^0 = 0$; i.e. $V = 0$, meaning that the vector field u^μ is orthogonal to the constant time hypersurfaces [116]. Therefore, the hypersurface may be regarded as being ‘comoving’ with the fluid. In the latter case, the slicing is chosen such that $\delta \rho = 0$, so the energy density is uniform on those slicings.

One may define another important gauge-invariant quantity in that sense, which is the

pressure perturbation on uniform-density slicings

$$\delta P^{\delta\rho} \equiv \delta P - \frac{\dot{P}}{\dot{\rho}} \delta\rho . \quad (1.192)$$

This is called the *non-adiabatic pressure perturbation* [115], and it is usually denoted by δP_{nad} [116]. It is the remaining contribution to the pressure perturbation when $\delta\rho = 0$ (hence the name ‘non-adiabatic’).

The perturbed Einstein field equations at linear order give the *energy constraint equation* ($\delta G^0_0 = M_{\text{Pl}}^{-2} \delta T^0_0$; see the components of the perturbed Einstein tensor in App. A) [122]

$$H\kappa + a^{-2} \partial_i \partial^i \psi = \frac{1}{2M_{\text{Pl}}^2} \delta\rho , \quad (1.193)$$

where κ is defined in App. A (see Eq. (A.33)). We also have the *momentum constraint equation* ($\delta G^0_i = M_{\text{Pl}}^{-2} \delta T^0_i$) [122]

$$HA + \dot{\psi} = -\frac{1}{2M_{\text{Pl}}^2} \Psi , \quad (1.194)$$

up to a constant of space, which is zero by construction (see footnote 9).

The trace-free part of the Einstein field equations, $\delta G^i_j - \frac{1}{3} \delta^i_j \delta G^k_k = M_{\text{Pl}}^{-2} \left(\delta T^i_j - \frac{1}{3} \delta^i_j \delta T^k_k \right)$, yields [122]

$$a^{-2} \left(\partial^i \partial_j - \frac{1}{3} \delta^i_j \partial_k \partial^k \right) (\dot{\chi} + H\chi - A + \psi) = \frac{1}{M_{\text{Pl}}^2} \Pi^i_j , \quad (1.195)$$

where we remind the reader that the anisotropic pressure is traceless. This equation is usually called the *ADM propagation equation*. Finally, we can obtain the *Raychaudhuri equation*, $\delta G^i_i - \delta G^0_0 = M_{\text{Pl}}^{-2} (\delta T^i_i - \delta T^0_0)$ [122]

$$\dot{\kappa} + 2H\kappa - \left(a^{-2} \partial_i \partial^i + 3\dot{H} \right) A = -\frac{1}{2M_{\text{Pl}}^2} (\delta\rho + 3\delta P) . \quad (1.196)$$

Notice that, using the energy constraint equation, we obtain

$$\frac{1}{2M_{\text{Pl}}^2} (\delta\rho - 3\delta P) = \dot{\kappa} + 4H\kappa - \left(a^{-2} \partial_i \partial^i + 3\dot{H} \right) A + 2a^{-2} \partial_i \partial^i \psi , \quad (1.197)$$

from the Raychaudhuri equation, which is $\delta R/2$ (see Eq. (A.32)). Indeed, the trace of the Ricci tensor is the trace-reversed of the energy-momentum tensor in GR, $R = -T/M_{\text{Pl}}^2$ [23], so $M_{\text{Pl}}^2 \delta R = -\delta T = -\delta T^0_0 - \delta T^i_i = \delta\rho - 3\delta P$.

From the conservation of energy and momentum, Eq. (1.24), we can obtain the *energy density perturbation conservation equation* ($\nu = 0$) [122]

$$\dot{\delta\rho} + 3H(\delta\rho + \delta P) = -(\rho + P) \left[a^{-2} \partial_i \partial^i \left(\frac{\Psi}{\rho + P} + \chi \right) - 3\dot{\psi} \right], \quad (1.198)$$

and the *momentum perturbation conservation equation* ($\nu = i$) [122]

$$\partial_i \left[\dot{\Psi} + 3H\Psi + \delta P + (\rho + P) A \right] = -\partial_j \Pi_i^j. \quad (1.199)$$

Now, consider that there is no contribution from the anisotropic pressure for simplicity (we will see an example of this soon). The momentum perturbation conservation equation can be plugged into the energy density perturbation conservation equation, such that

$$a^{-2} \partial_i \partial^i \left(\chi + \frac{\Psi}{\rho + P} \right) = -\frac{1}{a^3(\rho + P)} (a^3 \dot{\delta\rho}^\Psi), \quad (1.200)$$

where we have defined

$$\delta\rho^\Psi \equiv \delta\rho - 3H\Psi, \quad (1.201)$$

which is the comoving energy density perturbation. We also employed the momentum constraint equation to arrive at this equation. This gauge-invariant quantity can be related to $\mathcal{R}$ and $-\zeta$ such that

$$\delta\rho^\Psi = 3(\rho + P)(\mathcal{R} + \zeta). \quad (1.202)$$

Therefore, Eq. (1.200) reads

$$\frac{1}{3} a^{-2} \partial_i \partial^i \left(\chi + \frac{\Psi}{\rho + P} \right) = -\frac{[a^3(\rho + P)(\mathcal{R} + \zeta)]}{a^3(\rho + P)} = -\dot{\mathcal{R}} - \dot{\zeta} + 3H(\mathcal{R} + \zeta) \frac{\dot{P}}{\dot{\rho}}, \quad (1.203)$$

where the background continuity equation was used again.

Finally, from Eqs. (1.193) and (1.194) (and the definition of κ), we obtain

$$a^{-2} \partial_i \partial^i \psi^\chi = \frac{1}{2M_{\text{Pl}}^2} \delta\rho^\Psi, \quad (1.204)$$

where ψ^χ is defined in Eq. (1.164).¹⁰ Plugging these into Eq. (1.202)

$$\mathcal{R} + \zeta = -\frac{1}{3\dot{H}}a^{-2}\partial_i\partial^i\psi^\chi = \frac{1}{3H^2}a^{-2}\partial_i\partial^i\left(\frac{\psi^\chi}{\epsilon_1}\right), \quad (1.206)$$

where we brought back the first Hubble-flow parameter ϵ_1 introduced in Eq. (1.102).

Since it will be useful later, we derive $\mathcal{R}$ with respect to time

$$\dot{\mathcal{R}} = \frac{H}{\rho + P}\delta P - \frac{3H^2}{\rho + P}\frac{\dot{P}}{\dot{\rho}}\Psi, \quad (1.207)$$

where we have used the momentum constraint equation and the momentum perturbation conservation equation (again, ignoring the anisotropic pressure). This equation can be written as

$$\dot{\mathcal{R}} = \frac{H}{\rho + P}\left(\delta P_{\text{nad}} + \frac{\dot{P}}{\dot{\rho}}\delta\rho^\Psi\right). \quad (1.208)$$

We shall see in the next section that, for some fluids like a scalar field (in its effective fluid description; see Sec. 1.2.3), we can proceed further and derive very interesting results.

We have analysed the scalar perturbations separately and ignore other perturbations. The reason for this is twofold: firstly, scalar perturbations are extremely important in Cosmology [24, 41, 72, 115]. They couple to density and pressure perturbations and are responsible for the formation of the large scale structures in the Universe. Secondly, scalar, vector and tensor perturbations *do not* couple to one another at linear order in perturbation theory in the FLRW model, so we can safely treat them separately [115].

Before moving onto the next section, we list some important gauges that apply to scalar perturbations, some of which were mentioned before already [41]:

- Synchronous gauge: this gauge corresponds to the proper time slicing ($A = 0$) and the orthogonal threading ($B = 0$).
- Newtonian gauge: this gauge is also called ‘longitudinal gauge’ [124], and refers to the shear-free slicing ($\chi = 0$) and the isotropic threading ($E = 0$). Notice that these slicing and threading lead to $B = 0$.
- Uniform-curvature gauge: also called ‘spatially-flat’ gauge [41], it corresponds to the flat slicing ($\psi = 0$) and the isotropic threading ($E = 0$).

10. ψ^χ and A^χ defined as

$$A^\chi \equiv A - \dot{\chi}, \quad (1.205)$$

are known as *Bardeen potentials* in the literature [41, 115, 123].

- Uniform-density gauge: this corresponds to the uniform-density slicing ($\delta\rho = 0$) and the isotropic threading ($E = 0$).
- Comoving gauge: this gauge is sometimes known as ‘total matter gauge’ in the literature [124], and corresponds to the comoving slicing ($\delta T^0_i = 0$) and the isotropic threading ($E = 0$).

We note that all the gauges listed above fix the temporal gauge mode (α) completely, except for the synchronous one [115].

1.3.2 Mukhanov-Sasaki Equation

Eq. (1.200) can be rewritten as

$$\delta\dot{\rho}^\Psi + 3H\delta\rho^\Psi = -a^{-2}\partial_i\partial^i\Psi^\chi, \quad (1.209)$$

where

$$\Psi^\chi \equiv \Psi + (\rho + P)\chi, \quad (1.210)$$

is the scalar momentum perturbation on shear-free slicings. Notice that this gauge-invariant perturbation is related to ψ^χ and $\mathcal{R}$ by

$$\Psi^\chi = \frac{\rho + P}{H}(\psi^\chi - \mathcal{R}). \quad (1.211)$$

Using Eq. (1.204), we obtain a differential equation for ψ^χ (up to a constant of space, which vanishes by construction; see footnote 9)

$$\frac{\dot{\psi}^\chi}{H} + (1 + \epsilon_1)\psi^\chi = \epsilon_1\mathcal{R}, \quad (1.212)$$

or

$$\frac{2}{3}\frac{\dot{\psi}^\chi}{H} + \left(\frac{5}{3} + w\right)\psi^\chi = (1 + w)\mathcal{R}, \quad (1.213)$$

where we used Eq. (1.70).

Now, let us assume a fluid that satisfies the following equation:

$$\delta P_{\text{nad}} = \delta\rho^\Psi \left(1 - \frac{\dot{P}}{\dot{\rho}}\right). \quad (1.214)$$

Then, Eq. (1.208) reads

$$\dot{\mathcal{R}} = \frac{H}{\rho + P} \delta\rho^\Psi = \frac{1}{H} a^{-2} \partial_i \partial^i \left(\frac{\psi^\chi}{\epsilon_1} \right). \quad (1.215)$$

Eq. (1.206) yields

$$\dot{\mathcal{R}} = 3H (\mathcal{R} + \zeta), \quad (1.216)$$

which indicates that if $\mathcal{R}$ is a constant of time, then $\mathcal{R} = -\zeta$; in other words, for this type of fluid satisfying Eq. (1.214), both, the comoving curvature perturbation and the curvature perturbation on uniform-density slicings, coincide ($\psi^\Psi = \psi^{\delta\rho}$) if the former is conserved. On the other hand, Eqs. (1.212) and (1.215) allow us to get a second-order differential equation for $\mathcal{R}$:

$$\ddot{\mathcal{R}} + 3H \left(1 + \frac{1}{3} \epsilon_2 \right) \dot{\mathcal{R}} - a^{-2} \partial_i \partial^i \mathcal{R} = 0. \quad (1.217)$$

An example of fluid that satisfies Eq. (1.214) is a single scalar field in its effective fluid description [125], as was commented earlier. The linear perturbations of the energy density and the pressure, and the scalar momentum perturbation of the scalar field read (see the corresponding energy-momentum tensor in Eq. (1.72))

$$\delta\rho_\phi = \dot{\phi} \left(\delta\dot{\phi} - A\dot{\phi} \right) + V_{,\phi} \delta\phi, \quad (1.218)$$

$$\delta P_\phi = \dot{\phi} \left(\delta\dot{\phi} - A\dot{\phi} \right) - V_{,\phi} \delta\phi, \quad (1.219)$$

$$\Psi_\phi = -\dot{\phi} \delta\phi, \quad (1.220)$$

respectively. We can readily see that the comoving slicing coincides with the *uniform- ϕ* slicing (as $\Psi_\phi = 0$ amounts to $\delta\phi = 0$) [125]. Then, $\delta\rho_\phi^\Psi = \delta P_\phi^\Psi$, and that leads to Eq. (1.214).¹¹ Using the expressions of the energy density and pressure perturbations, their background values, and Eq. (1.192), the non-adiabatic pressure can be easily calculated to be

$$\delta P_\phi^{\text{nad}} = -\frac{2V_{,\phi}}{3H\dot{\phi}} \left(\dot{\phi} \delta\dot{\phi} - A\dot{\phi}^2 - \ddot{\phi} \delta\phi \right), \quad (1.222)$$

where the background scalar field equation (1.84) was used as well.

11. If the fluid does not satisfy $\delta\rho^\Psi = \delta P^\Psi$, then Eq. (1.214) naively reads

$$\delta P_{\text{nad}} = \delta\rho^\Psi \left(1 - \frac{\dot{P}}{\dot{\rho}} \right) + \delta P^\Psi - \delta\rho^\Psi. \quad (1.221)$$

We now notice that the comoving curvature perturbation for the scalar field ‘fluid’ reads (replacing the scalar momentum perturbation in Eq. (1.189) by Eq. (1.220))

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}} \delta\phi \equiv \frac{H}{\dot{\phi}} \delta\phi^\psi, \quad (1.223)$$

where $\rho + P = \dot{\phi}^2$. We introduced another gauge-invariant quantity¹²

$$\delta\phi^\psi \equiv \delta\phi + \frac{\dot{\phi}}{H} \psi, \quad (1.224)$$

which is the *scalar field perturbation on flat slicings*. Note that the field perturbation transforms as

$$\delta\phi \rightarrow \delta\phi - \dot{\phi} \alpha, \quad (1.225)$$

which can be easily seen from the transformation rule of Ψ (remember that $\Psi \equiv (\rho + P) V$ and the transformation rule of V (1.173))

$$\Psi \rightarrow \Psi + (\rho + P) \alpha. \quad (1.226)$$

Then, replacing Ψ by Ψ_ϕ (1.220), we arrive at Eq. (1.225).

Let us define the following gauge-invariant field [41]:

$$u \equiv a \delta\phi^\psi = -z \mathcal{R}, \quad (1.227)$$

where $z \equiv -a\dot{\phi}/H$ is the *Mukhanov variable* [41], such that

$$\frac{\dot{z}}{Hz} = 1 + \frac{1}{2} \epsilon_2. \quad (1.228)$$

Then, Eq. (1.217) can be written as

$$\frac{1}{az} \left[az^2 \left(\frac{\dot{u}}{z} \right) \right]^\bullet = a^{-2} \partial_i \partial^i u. \quad (1.229)$$

12. If we use Eq. (1.223) to replace $\mathcal{R}$ by $\delta\phi^\psi$ in Eq. (1.217), we will obtain an equation for $\delta\phi^\psi$ which is said to be written in the *gauge-invariant* form [126, 127]. This equation is similar to Eq. (1.130). If Eq. (1.224) is used to rewrite it in terms of $\delta\phi$, ψ and other perturbations, it will be written in the *gauge-ready* form, in which the spatial gauge is fixed but *not* the temporal one [127]. Note that, in the gauge-ready form, if the flat slicing is chosen, such that $\psi = 0$, one recovers the same equation as in the former gauge-invariant form because $\delta\phi^\psi$ is the scalar field fluctuation on flat slicings.

At this point, the conformal time η is brought back, which was introduced in Sec. 1.1 (Eq. (1.12)). Primes are used to denote derivatives with respect to it. Thus, Eq. (1.229) becomes

$$\frac{1}{a^2 z} \left[z^2 \left(\frac{u}{z} \right)' \right]' = a^{-2} \partial_i \partial^i u , \quad (1.230)$$

or

$$u'' - \left(\partial_i \partial^i + \frac{z''}{z} \right) u = 0 . \quad (1.231)$$

We now Fourier expand the perturbation u

$$u(\eta, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^{3/2}} u_k(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} . \quad (1.232)$$

At first order, each Fourier component evolves independently and we can analyse the evolution of a single Fourier component with momentum k [115]. It must be pointed out that, since $\mathbf{x}$ are comoving coordinates, $\mathbf{k}$ is the *comoving* wavevector. In Fourier space then, Eq. (1.231) reads

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0 , \quad (1.233)$$

where z''/z is a (effective) mass term given by [128]

$$\frac{z''}{z} = \mathcal{H}^2 \left[\left(1 + \frac{1}{2} \epsilon_2 \right) \left(2 - \epsilon_1 + \frac{1}{2} \epsilon_2 \right) + \frac{\epsilon_2'}{2\mathcal{H}} \right] . \quad (1.234)$$

$\mathcal{H}$ is the conformal Hubble rate defined in Eq. (1.13).

Eq. (1.233) is the *Mukhanov-Sasaki equation* [41], and it can be solved analytically under certain assumptions.

1.3.3 Bunch-Davies Vacuum

In order to solve the Mukhanov-Sasaki equation (1.233), we need to specify some initial conditions. *Quantum fluctuations* set the initial conditions for the small classical inhomogeneities that seeded the formation and growth of galaxies and galactic clusters [24, 41, 72].

We assume that the perturbations start inside the comoving Hubble horizon, such that their comoving wavenumbers k are larger than the conformal Hubble parameter, $k \gg \mathcal{H}$. The modes are then *subhorizon* [41, 129]. This amounts to stating that the physical wavelengths of those modes are much smaller than the Hubble horizon, $1/H$. In regard to these subhorizon

scales, we make the following two assumptions [129]:

- At the time when initial conditions are set, the Universe approximates an exact de Sitter spacetime, where $\dot{H} = 0$. This means that to a good approximation we can set $z''/z \simeq 2\mathcal{H}^2$ in Eq. (1.234).
- The initial quantum state corresponds to the vacuum state, such that the average occupation number of particles with momentum k is virtually zero. This makes sense by virtue of the first assumption, as de Sitter tends to wipe out any trace of matter content.

We now follow the usual quantisation procedure in field theory [129–131]. We promote complex, commuting numbers to complex, non-commuting operators and impose the equal-time canonical commutation relations, which are

$$\left[\hat{u}(\eta, \mathbf{x}), \hat{\Pi}_u(\eta, \tilde{\mathbf{x}}) \right] = i\hat{I}\delta(\mathbf{x} - \tilde{\mathbf{x}}) , \quad (1.235)$$

$$\left[\hat{u}(\eta, \mathbf{x}), \hat{u}(\eta, \tilde{\mathbf{x}}) \right] = \left[\hat{\Pi}_u(\eta, \mathbf{x}), \hat{\Pi}_u(\eta, \tilde{\mathbf{x}}) \right] = \hat{0} , \quad (1.236)$$

where $\hat{\Pi}_u$ is the conjugate momentum of the field operator $\hat{u}$. Hats are used to stress that we are dealing with field operators. δ is the Dirac's delta, and $\hat{I}$ and $\hat{0}$ are the identity and null operators, respectively. The general solution to the field equation is expressed in terms of the sum of the complete sets of positive and negative frequency solutions, denoted by $\{u_n\}$ and $\{u_n^*\}$ respectively

$$\hat{u}(x) = \sum_n \left[\hat{a}_n u_n(x) + \hat{a}^\dagger u_n^*(x) \right] , \quad (1.237)$$

where $u_n^*(x)$ is the complex conjugate of $u_n(x)$. The coefficients of the expansion are operators since the general solution is an operator too. These operators, $\hat{a}$ and $\hat{a}^\dagger$, are the *lowering and rising operators in Fock space*, respectively (they are also commonly called the annihilation and creation operators, respectively [41, 130]). The vacuum state (defined by a generic complete set of orthonormal solution $\{v_n\}$) is built upon the annihilation operator such that [129]

$$\hat{a}_n |0\rangle_v = 0 . \quad (1.238)$$

These operators satisfy the following commutation relations (deduced from the previous ones):

$$\left[\hat{a}_n, \hat{a}_m^\dagger \right] = \hat{I}\delta_{nm} ; \quad \left[\hat{a}_n, \hat{a}_m \right] = \left[\hat{a}_n^\dagger, \hat{a}_m^\dagger \right] = \hat{0} . \quad (1.239)$$

The solutions $\{u_n\}$ and $\{u_n^*\}$ must satisfy the orthonormality conditions

$$(u_n, u_m) = \delta_{nm} ; \quad (u_n^*, u_m^*) = -\delta_{nm} ; \quad (u_n, u_m^*) = (u_n^*, u_m) = 0 , \quad (1.240)$$

where δ_{nm} is the Kronecker's delta in case n is a discrete label. If n is continuous, the Kronecker's delta must be replaced by the Dirac's delta mentioned above. The scalar product $(.,.)$ reads [41, 131]

$$(u_n, u_m) \equiv i \int \left[u_m^* u_n' - u_n (u_m^*)' \right] d^3 \mathbf{x} . \quad (1.241)$$

We now make the following *Ansatz*:

$$u_n(x) \equiv \frac{1}{(2\pi)^{3/2}} u_{\mathbf{k}}(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} , \quad (1.242)$$

where the index becomes continuous (denoted by $\mathbf{k}$). We then obtain¹³

$$(u_n, u_m) = i \left[u_{\tilde{\mathbf{k}}}^* u_{\mathbf{k}}' - u_{\mathbf{k}} (u_{\tilde{\mathbf{k}}}^*)' \right] \delta^3(\mathbf{k} - \tilde{\mathbf{k}}) . \quad (1.244)$$

Note that the Wronskian [130]

$$\mathbf{W} \left(u_{\tilde{\mathbf{k}}}^*, u_{\mathbf{k}} \right) \equiv u_{\tilde{\mathbf{k}}}^* u_{\mathbf{k}}' - u_{\mathbf{k}} (u_{\tilde{\mathbf{k}}}^*)' , \quad (1.245)$$

must equal $-i$ in order for the orthonormality conditions above to be satisfied; i.e.

$$u_{\mathbf{k}} (u_{\tilde{\mathbf{k}}}^*)' - u_{\tilde{\mathbf{k}}}^* u_{\mathbf{k}}' = i . \quad (1.246)$$

This is a condition that every set of mode functions $\{u_{\mathbf{k}}\}$ that is a part of the general solution must satisfy. With that *Ansatz*, the field operator now reads (in terms of the continuous index $\mathbf{k}$)

$$\hat{u}(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \left[\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(\eta) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] , \quad (1.247)$$

13. Dirac's delta is usually expressed as [130]

$$\delta^3(\mathbf{k} - \tilde{\mathbf{k}}) = \frac{1}{(2\pi)^3} \int e^{i(\mathbf{k} - \tilde{\mathbf{k}}) \cdot \mathbf{x}} d^3 \mathbf{x} . \quad (1.243)$$

where $u_{\mathbf{k}}$ satisfies a harmonic oscillator equation [130]

$$u_{\mathbf{k}}'' + \omega_{\mathbf{k}}^2 u_{\mathbf{k}} = 0 . \quad (1.248)$$

Notice that in our case, $\omega_{\mathbf{k}}^2 \simeq k^2 - 2\mathcal{H}^2$, and the equations only involve the norm of the wavenumber k , so the mode functions shall only depend on k . We then use the subscript ‘ k ’ rather than ‘ $\mathbf{k}$ ’ (compare Eq. (1.247) with Eq. (1.129)).

It is clear from Eq. (1.246) that there is certain ambiguity when fixing the quantum initial state as any function satisfying it is a good choice [41]. In flat spacetime (i.e., $H = 0$), that is not an ambiguity anymore, since one can take a global timelike Killing vector of the Poincaré group that leads to positive frequency solutions of the form of plane waves, $u_k \propto e^{-i\omega_k \eta}$ [130, 131]. The Poincaré group in curved spacetime is not a symmetry group and therefore one cannot follow the same strategy. However, notice that, according to the *equivalence principle* of GR [23], for sufficiently small spacetime volume, one should recover flat spacetime. Thus, on subhorizon scales during inflation, assuming the equivalence principle, the quantum initial state ambiguity can be resolved.

The first of the two assumptions made at the beginning of this section leads to $\mathcal{H}^2 \simeq 1/\eta^2$, which is a good proxy for the relation between $\mathcal{H}$ and η during SR inflation. In general, we would have [128]

$$\int d\eta = \int \frac{da}{a^2 H} = - \int d \left(\frac{1}{aH} \right) - \int \frac{dH}{aH^2} = - \int d \left(\frac{1 + \epsilon_1}{aH} \right) + \int \left(\frac{\dot{\epsilon}_1}{H\epsilon_1} + \epsilon_1 \right) \frac{\epsilon_1}{a^2 H} da , \quad (1.249)$$

or

$$\eta = - \frac{1 + \epsilon_1}{aH} + \int \epsilon_1 (\epsilon_1 + \epsilon_2) \frac{da}{a^2 H} , \quad (1.250)$$

where the constants of integration have been ignored. In SR ($\epsilon_1, |\epsilon_2| \ll 1$) we have, at first order [128]

$$\eta_{\text{SR}} \simeq - \frac{1}{aH} (1 + \epsilon_1) , \quad (1.251)$$

which, indeed, yields $\eta \simeq -1/\mathcal{H}$ if ϵ_1 is neglected.

Consequently, the Mukhanov-Sasaki equation (1.248) reads [41, 129]

$$u_k'' + \left(k^2 - \frac{2}{\eta^2} \right) u_k = 0 . \quad (1.252)$$

The exact solution to this equation is

$$u_k(\eta) = \alpha \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) + \beta \frac{e^{ik\eta}}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right), \quad (1.253)$$

where α and β are time independent complex coefficients that characterise the non-uniqueness of the solutions (of the mode functions).

Now, on subhorizon scales $k^2 \gg \mathcal{H}^2 \simeq 1/\eta^2$ (the latter approximate relation is due to the SR conditions as shown above), the oscillation frequency is much larger than the expansion rate H . One has $u_k'' \simeq -k^2 u_k$ (so $\omega_k^2 \simeq k^2$ satisfies the adiabaticity condition; see Eqs. (1.135) and (1.248)). Solving this equation yields [129]

$$u_{k \gg aH}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \quad (1.254)$$

where the prefactor $1/\sqrt{2k}$ was set so Eq. (1.254) is almost identical to the mode function for which the vacuum state of the harmonic oscillator with constant frequency is the state of minimum energy [41]. Eq. (1.254) corresponds to the mode functions describing the *Bunch-Davies vacuum state* [84, 132]. The latter prescribes that the corresponding light¹⁴ particles, on scales much smaller than the inverse of the Hubble parameter, do not feel the interaction of the gravitational field in accordance with the equivalence principle.

We can now fix the coefficients α and β of the solution $u_k(\eta)$ in Eq. (1.253):

- The orthonormality conditions on the mode functions u_k (1.246) imply that [41]

$$|\alpha|^2 - |\beta|^2 = 1. \quad (1.255)$$

- The initial vacuum state corresponding to the Bunch-Davies vacuum (1.254) yields

$$\alpha = 1; \quad \beta = 0. \quad (1.256)$$

Therefore we arrive at the solution

$$u_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right), \quad (1.257)$$

14. The SR parameters have been neglected and a small η_V implies that we assume a light scalar field (we mentioned this in Sec. 1.2.6 when we presented the η problem; see footnote 6 too).

or (remember that $\delta\phi^\psi = u/a$)

$$\delta\phi_k^\psi(\eta) = \frac{a^{-1}(\eta)e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right). \quad (1.258)$$

1.3.4 Inflation Observables

The physical wavelengths of the modes are *stretched* by the inflationary expansion (because the comoving Hubble horizon shrinks), and some of these modes become *superhorizon* ($k \ll \mathcal{H}$) [41]. Eq. (1.215) implies that, on superhorizon scales, the comoving curvature perturbation is *conserved* [133] (as long as ϵ_1 does not become too small).

In Fourier space, Eq. (1.215) reads

$$\frac{\dot{\mathcal{R}}_k}{H} = -\frac{k^2}{a^2 H^2} \frac{\psi_k^\chi}{\epsilon_1}. \quad (1.259)$$

Then, $\dot{\mathcal{R}}_k \rightarrow 0$ as $k/(aH) \rightarrow 0$. This also means that $\psi^\Psi = \psi^{\delta\rho}$ as we argued in Sec. 1.3.2 (see Eq. (1.206)), and therefore ζ_k becomes constant as well. The *horizon-crossing* value of $\mathcal{R}_k$ (i.e. when $k = aH$) shall be indicated with asterisks. These will be used with any time dependent quantity, be it perturbation or homogeneous. In the latter case, the asterisk will denote the value of the homogeneous quantity at the time when $k = aH$ (then, we can write $k = a_* H_*$ for example).

The scalar field perturbations $\delta\phi$, we have been dealing with, can be viewed as random fields with zero mean [24]. The properties of these random fields are specified by the form of the *probability distribution function* (PDF); i.e. the probability of realising some field configuration from the whole ensemble of possible configurations. In Sec. 1.1.2, we mentioned the statistical isotropy and homogeneity of the coefficients of the decomposition into spherical harmonics of the CMB temperature fluctuations. Similarly we can assume the same about the scalar fluctuations which amounts to the invariance of the PDF under rotations and space translations, respectively. These two assumptions imply that the two-point correlation function $\langle\delta\phi(\mathbf{x})\delta\phi(\tilde{\mathbf{x}})\rangle$ depends solely on the distance between the two points $\mathbf{x}$ and $\tilde{\mathbf{x}}$ [24]. Statistical homogeneity and isotropy in Fourier space implies that the two-point correlation function of the comoving curvature perturbation can be written as [128, 129]

$$\langle\mathcal{R}(\mathbf{k})\mathcal{R}^*(\tilde{\mathbf{k}})\rangle = \frac{2\pi^2}{k^3} \Delta_{\mathcal{R}}^2(k) \delta(\mathbf{k} - \tilde{\mathbf{k}}), \quad (1.260)$$

where $\Delta_{\mathcal{R}}^2$ is the dimensionless *power spectrum* of $\mathcal{R}$ [41]. Statistical isotropy in particular

implies that $\Delta_{\mathcal{R}}^2$ is a function only of the length of the wavevector, $k \equiv |\mathbf{k}|$: $\Delta_{\mathcal{R}}^2 = \Delta_{\mathcal{R}}^2(k)$.

Although $\langle \mathcal{R} \rangle = 0$, the mean of the absolute square of $\mathcal{R}$, $|\mathcal{R}|^2$, is not zero (the so-called *variance* $\sigma_{\mathcal{R}}^2$), and it is related to the power spectrum by [24]

$$\sigma_{\mathcal{R}}^2 = \int_0^\infty \Delta_{\mathcal{R}}^2(k) \, d \ln k . \quad (1.261)$$

Therefore, the power spectrum $\Delta_{\mathcal{R}}^2(k)$ determines the contribution from the mode with wavenumber k to the total variance.

The curvature perturbation on uniform-density slicings can be used in lieu of $\mathcal{R}$, as it may be more interesting due to its direct relation with the *density contrast* $\delta \equiv \delta\rho/\rho$ [24]

$$\zeta = -\psi + \frac{\delta\rho}{3(\rho + P)} = -\frac{H}{\dot{\rho}} \delta\rho^\psi = \frac{\delta^\psi}{1+w} , \quad (1.262)$$

where $\delta\rho^\psi$ is defined as

$$\delta\rho^\psi \equiv \delta\rho + \frac{\dot{\rho}}{H} \psi = \delta\rho - 3(\rho + P) \psi , \quad (1.263)$$

and we used the continuity equation (1.27).

However, the interest of these two gauge-invariant curvature perturbations during inflation lies in the evolution near horizon-crossing and on superhorizon scales [41, 129]. In the latter case, we have seen that both quantities are constant and coincide, so we can opt for either of them to perform the analysis of the power spectrum. We point out that, since the power spectrum is related to the amplitude of ‘scalar’ perturbations, we shall use ‘ A_s ’ [64] to denote it instead of ‘ $\Delta_{\mathcal{R}}^2$ ’.

We then determine the power spectrum at horizon exit. In SR inflation, this is *approximately* equal to the value of A_s on superhorizon scales (i.e. $k\eta \rightarrow 0$). From Eqs. (1.258) and (1.223), we obtain [128]

$$A_s \equiv \frac{k^3}{2\pi^2} |\mathcal{R}_k|^2 = \frac{H^2}{8\pi^2 M_{\text{Pl}}^2 \epsilon_1} , \quad (1.264)$$

where we have used $k = a_* H_*$ and therefore H and ϵ_1 are evaluated at horizon exit (we do not use asterisks however for simplicity). Remember that this expression has been derived by assuming an almost exact de Sitter expansion, and then first-order SR parameters were neglected. Since ϵ_2 is highly suppressed, ϵ_1 is indeed constant. Since ϵ_1 is small too, H is constant and so is A_s for the subsequent superhorizon evolution, as expected.

We may replace the Hubble parameter and the first Hubble-flow parameter by the po-

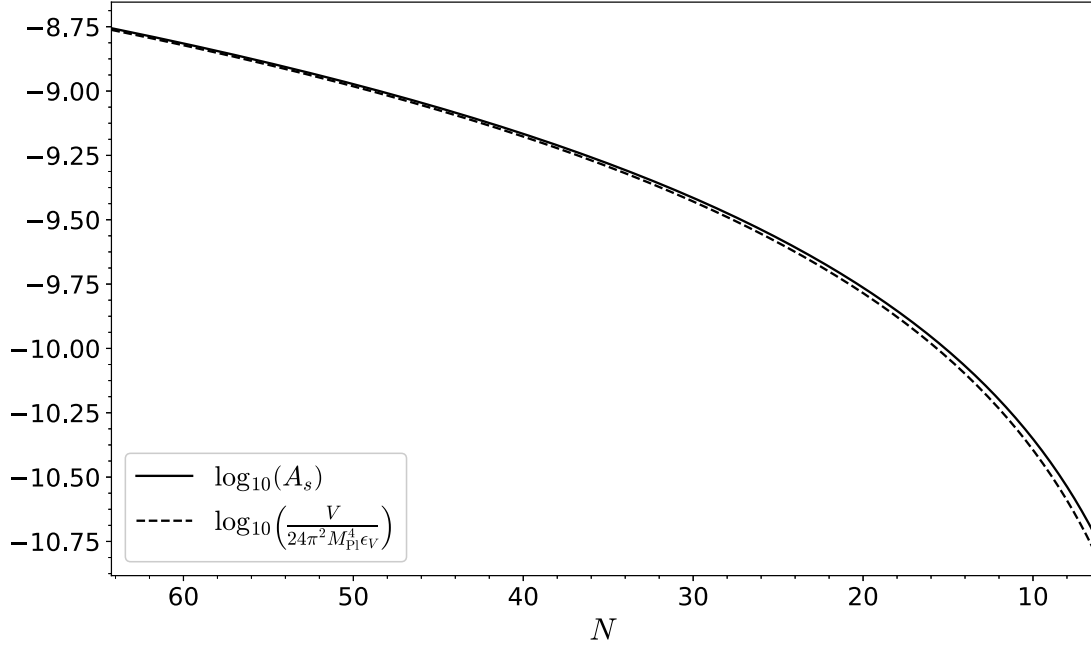


Figure 1.7: Power spectrum of scalar perturbations $\log_{10}(A_s)$ and the SR formula (1.265) for the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. The exact power spectrum has been evaluated numerically at 5 e-folds after horizon-crossing to make sure $\mathcal{R}$ has become constant, while the background values used in the SR formula are evaluated at horizon-crossing.

tential and the SR parameter ϵ_V , respectively [128], and arrive at

$$A_s = \frac{V}{24\pi^2 M_{\text{Pl}}^4 \epsilon_V} , \quad (1.265)$$

so we obtain the power spectrum as a function of the field value at horizon exit, $A_s = A_s(\phi)$. In Sec. 1.2.4, we used the quadratic potential as an example to analyse SR, and this had an unknown parameter m to be fixed via observations. If we know the value of A_s at a certain scale k (or for some field value ϕ), we can obtain m from Eq. (1.265).

In line with the *Planck* team [64], we consider a power-law form to parametrise the scale dependence of the power spectrum of primordial curvature perturbations on comoving slicings, such that

$$A_s(k) = A_s(k_*) \left(\frac{k}{k_*} \right)^{n_s - 1} , \quad (1.266)$$

where k_* is a reference scale (called ‘pivot’ scale) which is chosen to be $k_* = 0.05 \text{ Mpc}^{-1}$, and $n_s = n_s(k_*)$ is the *scalar spectral index*, which gives a measure of the deviation of the spectrum from the *scale invariance*. Such a scale invariant spectrum is referred to

as *Harrison-Zeldovich spectrum* in the literature [134–136]. The power spectrum is then expanded about the pivot scale

$$\ln A_s(k) = \ln A_s(k_*) + (n_s - 1) \ln \left(\frac{k}{k_*} \right) + \frac{1}{2!} \alpha_s \ln^2 \left(\frac{k}{k_*} \right) + \frac{1}{3!} \beta_s \ln^3 \left(\frac{k}{k_*} \right) + \dots \quad (1.267)$$

We exclude higher derivatives as there are no strong observational constraints on the corresponding parameters [64]. The scalar spectral index is given by

$$n_s(k_*) - 1 \equiv \frac{d \ln A_s(k_*)}{d \ln k} \quad (1.268)$$

We define the *running of the spectral index* to be

$$\alpha_s(k_*) \equiv \frac{dn_s(k_*)}{d \ln k} = \frac{d^2 \ln A_s(k_*)}{(d \ln k)^2} \quad (1.269)$$

and the *running of the running* of the spectral index is

$$\beta_s(k_*) \equiv \frac{d^2 n_s(k_*)}{(d \ln k)^2} = \frac{d^3 \ln A_s(k_*)}{(d \ln k)^3} \quad (1.270)$$

To calculate the spectral index in terms of the Hubble flow parameters, we can use the chain rule [41]

$$\frac{d}{d \ln k} = \frac{dN_*}{d \ln k} \frac{d}{dN_*} = \frac{1}{1 - \epsilon_{1*}} \frac{d}{dN_*} \quad (1.271)$$

where we have also used the horizon-crossing relation, $k = a_* H_*$, in the last step, and the number of e-folds is $dN = H dt = d \ln a$. Dropping the asterisks again for simplicity and plugging Eq. (1.264) into Eq. (1.268), we obtain the first-order expression in the SR expansion [128]¹⁵

$$n_s(k_*) - 1 = -4\epsilon_1 + 2\eta = -2\epsilon_1 - \epsilon_2 \quad (1.272)$$

The running of the scalar spectral index is simply

$$\alpha_s(k_*) = -2\epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_3 \quad (1.273)$$

which includes the third Hubble-flow parameter ϵ_3 and is second-order in SR. The running

15. See Ref. [137] for a calculation of the power spectrum and the spectral index of scalar perturbations up to second-order corrections in SR.

of the running reads

$$\beta_s(k_*) = -2\epsilon_1\epsilon_2^2 - 2\epsilon_1\epsilon_2\epsilon_3 - \epsilon_2\epsilon_3^2 - \epsilon_2\epsilon_3\epsilon_4 , \quad (1.274)$$

which is third-order in the SR expansion.

In order to derive the expressions above, we used the definition of the Hubble-flow parameters (1.102). However, we can still write these *inflation observables* as functions of the field value by virtue of the SR regime, as we did with the amplitude of scalar perturbations (1.265) [128, 138]

$$n_s(k_*) - 1 = -6\epsilon_V + 2\eta_V , \quad (1.275)$$

$$\alpha_s(k_*) = -2\xi_V^2 - 24\epsilon_V^2 + 16\eta_V\epsilon_V , \quad (1.276)$$

$$\beta_s(k_*) = 2\omega_V^3 - 24\xi_V^2\epsilon_V + 2\xi_V^2\eta_V + 192\eta_V\epsilon_V^2 - 32\eta_V^2\epsilon_V - 192\epsilon_V^3 , \quad (1.277)$$

where ϵ_V and η_V were defined in Eqs. (1.99) and (1.101), respectively. ξ_V^2 and ω_V^3 are defined as [138]

$$\xi_V^2 \equiv M_{\text{Pl}}^4 V_{,\phi\phi\phi} \frac{V_{,\phi}}{V^2} , \quad (1.278)$$

$$\omega_V^3 \equiv M_{\text{Pl}}^6 V_{,\phi\phi\phi\phi} \frac{V_{,\phi}^2}{V^3} , \quad (1.279)$$

and are the second and third terms of Eq. (1.103) (i.e. $\xi_V \equiv {}^2\beta_V$ and $\omega_V \equiv {}^3\beta_V$).

$\{A_s, n_s, \alpha_s, \beta_s\}$ are inflation observables as they allow us to make predictions and connect them with observations [64]. The procedure is clear: we use the SR formula of the number of e-folds, Eq. (1.109), to compute the field value ϕ_* that corresponds to a given number of e-folds N_* before the end of inflation, when SR is a good approximation. Once we know the field value, we calculate the potential and its derivatives and find the SR parameters. With these parameters, we can compute the values of the inflation observables listed above and compare them with observational constraints. In line with the choice made by the *Planck* team [64], the corresponding number of e-folds to make the comparisons is taken to be 50 to 60 e-folds before the end of inflation, although the largest observable scales exit the horizon somewhere between 45 and 70 e-folds before the end of inflation [139]. The latest constraints on the Hubble-flow functions at horizon exit are [64]

$$\epsilon_{1*} < 0.0052 \text{ (95\% CL)} , \quad (1.280)$$

$$\epsilon_{2*} = 0.034 \pm 0.008 \text{ (68\% CL)} , \quad (1.281)$$

$$\epsilon_{3*} = 0.13^{+0.40}_{-0.45} \text{ (95\% CL)} . \quad (1.282)$$

However, how do we relate the primordial perturbations at the time of horizon exit with a *cosmological observable*, something that we can measure? To that end we need a *transfer function* between the comoving curvature perturbation at horizon exit and the perturbations at some later time [24, 41]. Once the perturbations reenter the horizon, after inflation ends and the comoving Hubble horizon increases again, these will vary over time until today. An example of *fluctuation* quantity that can be measured is the anisotropies in temperature of the CMB that we saw in Sec. 1.1.2, or the density contrast of some galaxy.

When we began our analysis of perturbations at linear order, we focussed on the scalar ones and ignored other perturbations. At this point however, the *tensor perturbations* $h_{ij}(x)$ about the spatially-flat FLRW metric should be tackled briefly. These read [41, 114, 115, 140, 141]

$$ds^2 = -dt^2 + a^2(t) [\delta_{ij} + h_{ij}(t, \mathbf{x})] dx^i dx^j , \quad (1.283)$$

and, as explained below, are quite significant in the inflationary cosmology. h_{ij} is *transverse*, $\partial_i h^i_j = 0$, and *traceless*, $h \equiv h^i_i = 0$ [115, 141]. Also, from Eq. (1.153), one sees that h_{ij} is gauge-invariant, as *no new tensor part* is introduced due to the gauge transformation [115]

$$\delta \tilde{g}_{ij} = \delta \hat{g}_{ij} - \partial_i \xi_j - \partial_j \xi_i - 2a^2 H \xi^0 \delta_{ij} . \quad (1.284)$$

The linear tensor perturbations (*Gravitational Waves (GWs)*) satisfy the following equation, derived from Einstein field equations:

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2}\partial_k\partial^k h_{ij} = \frac{2}{M_{\text{Pl}}^2}\Pi_{ij} , \quad (1.285)$$

where the perturbations of the Christoffel symbols $\Gamma_{\mu\nu}^\alpha$, the Ricci tensor $R_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu}$ are shown in App. A. Indeed, given the Einstein field equations, Eq. (1.8), one may perturb them linearly and consider the tensor perturbations solely. Using Eq. (A.38) in App. A, we obtain

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2}\partial_k\partial^k h_{ij} - 2\left(2\dot{H} + 3H^2\right)h_{ij} = \frac{2}{a^2}\frac{1}{M_{\text{Pl}}^2}\delta T_{ij} . \quad (1.286)$$

δT_{ij} is (see Eq. (1.184))

$$\delta T_{ij} = a^2 (Ph_{ij} + \Pi_{ij}) , \quad (1.287)$$

where only the tensor perturbations were regarded. The background equation (1.21) (with $K = 0$) gives Eq. (1.285).

We assume a vanishing anisotropic pressure (as is the case for the single scalar field model) and write this equation in Fourier space. For this purpose, we decompose h_{ij} into Fourier modes [41]

$$h_{ij}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_s \varepsilon_{ij}^s(k) h_k^s(t) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (1.288)$$

where s denotes the *polarisation* and ε_{ij}^s is the polarisation tensor. The two polarisation states of the Transverse-Traceless (TT) tensor perturbation [141] are $s = \{+, \times\}$, and the polarisation tensor satisfies the following properties [115]:

$$\varepsilon_{ij}(k) = \varepsilon_{ji}(k) , \quad (1.289)$$

$$k^i \varepsilon_{ij}(k) = \varepsilon^i_i(k) = 0 , \quad (1.290)$$

which are clear by looking at those of h_{ij} (see below Eq. (1.283)), and

$$\varepsilon_{ij}^s(k) \varepsilon_{ij}^{\tilde{s}}(k) = 2\delta_{s\tilde{s}} . \quad (1.291)$$

Then, in Fourier space, Eq. (1.285) reads

$$\ddot{h}_k^s + 3H\dot{h}_k^s + \frac{k^2}{a^2} h_k^s = 0 . \quad (1.292)$$

Or, in terms of derivatives with respect to the conformal time

$$h_k^{s''} + 2\mathcal{H}h_k^{s'} + k^2 h_k^s = 0 . \quad (1.293)$$

If we define the field [41, 128]

$$v_k^s \equiv \frac{1}{2} a M_{\text{Pl}} h_k^s , \quad (1.294)$$

Eq. (1.293) reads

$$v_k^{s''} + \left(k^2 - \frac{a''}{a} \right) v_k^s = 0 , \quad (1.295)$$

where [128]

$$\frac{a''}{a} = \mathcal{H}^2 + \mathcal{H}' = \mathcal{H}^2 (2 - \epsilon_1) . \quad (1.296)$$

Then, Eq. (1.295) is an equation similar to Eq. (1.233), but simpler. We can then follow the same quantisation procedure as in Sec. 1.3.3 and make the same assumptions to determine the initial conditions, so the solution to this equation for a given polarisation reads (compare with Eq. (1.257)) [41, 128]

$$v_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) . \quad (1.297)$$

On superhorizon scales ($k\eta \rightarrow 0$), the dimensionless *tensor power spectrum* can be written as (see Eq. (1.264)) [41, 128]

$$A_t \equiv \frac{k^3}{2\pi^2} \sum_{s=+,\times} |h_k^s|^2 = \frac{2H^2}{\pi^2 M_{\text{Pl}}^2} . \quad (1.298)$$

In contrast to Eq. (1.264), we see that the tensor power spectrum only depends on H . In a way similar to the scalar power spectrum however, we can expand A_t about the pivot scale

$$\ln A_t(k) = \ln A_t(k_*) + n_t \ln \left(\frac{k}{k_*} \right) + \frac{1}{2!} \alpha_t \ln^2 \left(\frac{k}{k_*} \right) + \dots , \quad (1.299)$$

where, again, we ignore those parameters that are not sufficiently constrained by observations yet. The *tensor spectral index* $n_t = n_t(k_*)$ reads

$$n_t(k_*) \equiv \frac{d \ln A_t(k_*)}{d \ln k} , \quad (1.300)$$

and the *running of the tensor spectral index* is given by

$$\alpha_t(k_*) \equiv \frac{dn_t(k_*)}{d \ln k} = \frac{d^2 \ln A_t(k_*)}{(d \ln k)^2} . \quad (1.301)$$

Plugging Eq. (1.298) into Eq. (1.300) and using Eq. (1.271), we obtain the tensor spectral index in terms of the first Hubble-flow parameter [128]

$$n_t(k_*) = -2\epsilon_1 . \quad (1.302)$$

The running of the tensor spectral index is second-order, like the scalar one, and reads

$$\alpha_t(k_*) = -2\epsilon_1\epsilon_2 . \quad (1.303)$$

In terms of the potential and its derivatives, these two inflation observables can be written as [138]

$$n_t(k_*) = -2\epsilon_V , \quad (1.304)$$

$$\alpha_t(k_*) = 4\epsilon_V\eta_V - 8\epsilon_V^2 . \quad (1.305)$$

We introduce the *tensor-to-scalar ratio* [64]

$$r \equiv \frac{A_t}{A_s} , \quad (1.306)$$

which gives the value of the amplitude of tensor perturbations in terms of that of the scalar ones. Inserting Eqs. (1.298) and (1.264), we find

$$r(k_*) = 16\epsilon_1 , \quad (1.307)$$

or

$$r(k_*) = 16\epsilon_V . \quad (1.308)$$

Notice that one can write a relation of proportionality between the tensor-to-scalar ratio and the tensor spectral index

$$r(k_*) = -8n_t(k_*) , \quad (1.309)$$

meaning that if the spectrum approaches the flat one ($n_t(k_*) \rightarrow 0$), then the tensor perturbations amplitude must be highly suppressed compared to that of the scalar ones. This *consistency relation* is satisfied by any single-field theory of SR inflation. In addition to this, since the tensor-to-scalar ratio is proportional to ϵ_1 , it is related to the field velocity by (see Eqs. (1.70) and (1.88))

$$r = \frac{8}{M_{\text{Pl}}^2} \left(\frac{d\phi}{dN} \right)^2 . \quad (1.310)$$

Integrating, we find

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \int_{N_e}^{N_*} \sqrt{\frac{r}{8}} dN , \quad (1.311)$$

where we have assumed that $\dot{\phi} < 0$. Assuming that SR holds until the very end of inflation,

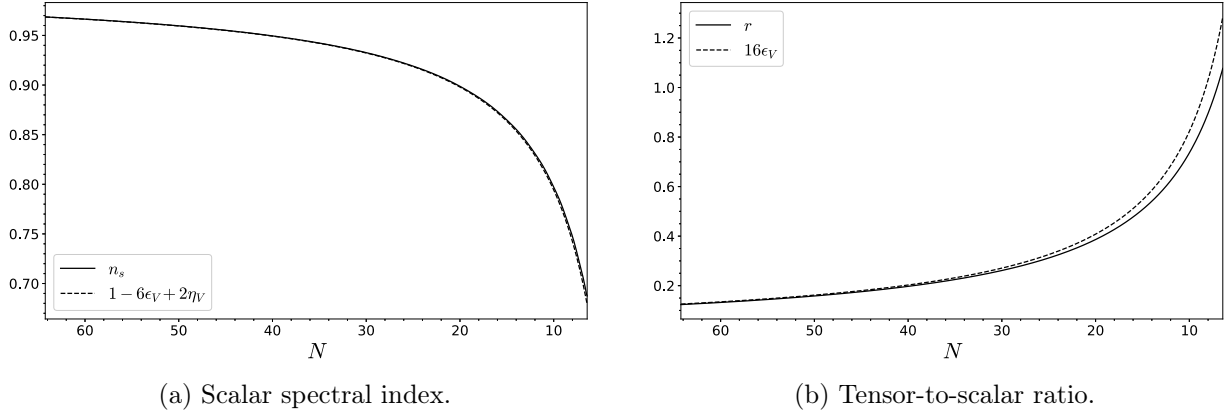


Figure 1.8: Scalar spectral index n_s (left) and the tensor-to-scalar ratio r (right) and their SR formulas, Eqs. (1.275) and (1.308), for the quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. The exact n_s and r have been evaluated numerically at 5 e-folds after horizon-crossing to make sure $\mathcal{R}$ and h have become constant, while the background values used in the SR formula are evaluated at horizon-crossing.

r can be taken out of the integral such that

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \sqrt{\frac{r(k_*)}{8}} \Delta N. \quad (1.312)$$

Typically, ΔN is taken to be 45 to 70 e-folds as we said earlier, and $1/\sqrt{8} = 0.35$, so we have approximately

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \mathcal{O}(1) \times \sqrt{\frac{r(k_*)}{0.01}}. \quad (1.313)$$

This result implies that values of r larger than 10^{-2} (i.e. tensor perturbations amplitudes larger than $10^{-2}A_s$) correspond to *super-Planckian* displacements of the inflaton field, $\Delta\phi > M_{\text{Pl}}$. This is known as the *Lyth bound* [142] in the literature.

The tensor-to-scalar ratio, and therefore the tensor perturbations, are extremely important as they can help us determine the *energy scale of inflation* [142, 143]. Given that A_t depends on H only and A_s can be determined observationally, we only have to use the Friedmann equation in Eq. (1.306) such that

$$\frac{V^{1/4}}{M_{\text{Pl}}} = \left(\frac{3\pi^2}{2} r(k_*) A_s(k_*) \right)^{1/4}. \quad (1.314)$$

Since, as we know, the energy budget was dominated by the potential energy of the inflaton during inflation, determining $V^{1/4}/M_{\text{Pl}}$ amounts to determining the energy scale of inflation

in Planck units.

The latest measurements of some of the inflation observables presented in this section are [64]

$$\ln(10^{10} A_s) = 3.044 \pm 0.014 \text{ (68\% CL)} , \quad (1.315)$$

$$n_s = 0.9649 \pm 0.0042 \text{ (68\% CL)} , \quad (1.316)$$

$$\alpha_s = -0.005 \pm 0.013 \text{ (95\% CL)} , \quad (1.317)$$

$$r < 0.036 \text{ (95\% CL)} , \quad (1.318)$$

so that an inflation model that predicts an *almost scale invariant spectrum* of scalar perturbations, with a slightly *redshifted* scalar spectral index ($n_s < 1$) with almost vanishing running, and a highly suppressed amplitude of tensor perturbations, is consistent with observations (it must satisfy those tight constraints). The value of the running set by *Planck* assumes $\beta_s = 0$. No hints of running of the running were found with the *Planck* 2018 data [64].

If we consider the quadratic potential (see Eq. (1.110)), we can estimate the parameter m^2 using the value of the amplitude of scalar perturbations. Using Eq. (1.265) and the value in Eq. (1.315), we find

$$m \simeq 10^{-6} M_{\text{Pl}} , \quad (1.319)$$

where we have used Eqs. (1.111) and (1.112), and assumed that $\phi_*^2 \gg \phi_e^2 = 2M_{\text{Pl}}^2$, such that $\phi_*^2/M_{\text{Pl}}^2 \simeq 4\Delta N$. We also took $\Delta N = 60$ e-folds, which is within the range of e-folds that was said to be used to make comparisons with the results from the *Planck* team. This value of m was indeed used in Sec. 1.2.5, when the instant preheating was being discussed (see below Eq. (1.139)).

Note also that the bound on r inferred from observations (1.318) favours single-field SR models with $\Delta\phi/M_{\text{Pl}} \simeq \mathcal{O}(1)$ at most, according to the Lyth bound (see Eq. (1.313)). Combining this with the value of A_s , we obtain the maximum approximate energy scale of inflation (see Eq. (1.314)), which is

$$\frac{V^{1/4}}{M_{\text{Pl}}} < 10^{-2} , \quad (1.320)$$

i.e. near the GUT scale ($\sim 10^{16}$ GeV) [24].

Despite its simplicity, the quadratic potential pertains to a set of the so-called *monomial* models of inflation with $V(\phi) \propto \phi^p$ (where $p > 1$) which are *strongly disfavoured* by the

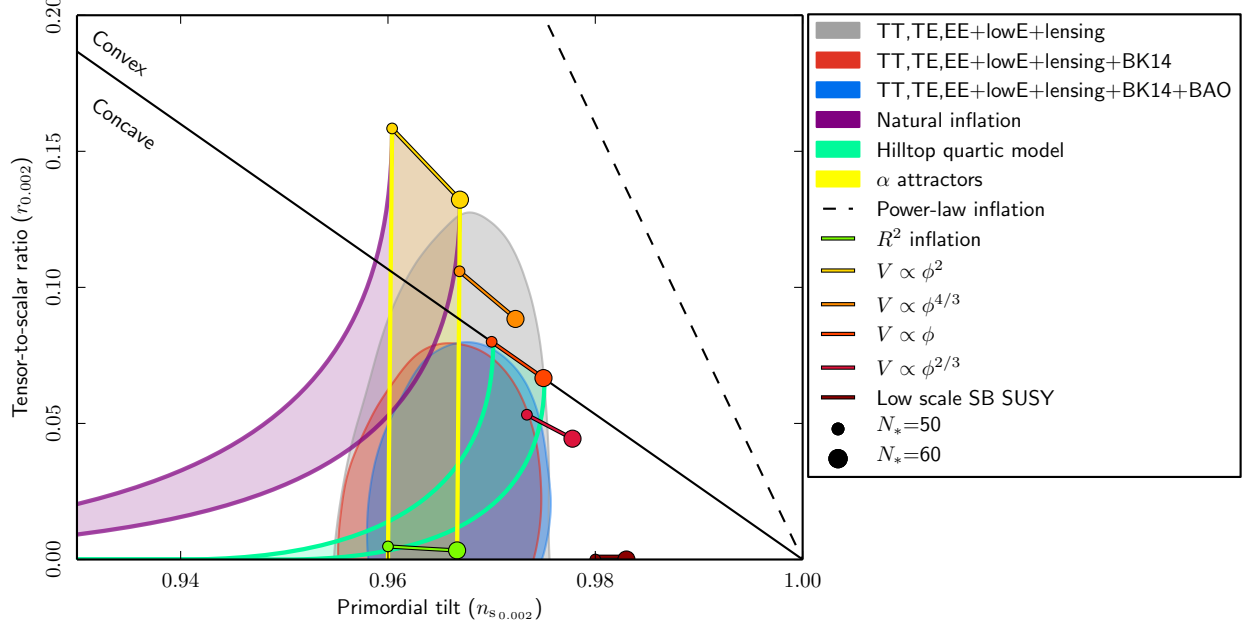


Figure 1.9: Some inflation models are confronted with the inferred scalar spectral index (primordial tilt) and the tensor-to-scalar ratio from *Planck* 2018 and lensing data, and their combination with BICEP2/Keck Array (BK14) and BK14 + BAO data. Figure taken from the *Planck* website ([64], ESA and the *Planck* Collaboration). Note that ‘ R^2 inflation’ gives the best prediction.

most recent data released by *Planck* 2018, in combination with the BK15 data [64, 144] (see Fig. 1.9). Another example of inflation model that may be discarded in view of the observational constraints is *natural inflation* [145], which leads to predictions similar to the quadratic inflation potential [11]. The simplest potential in natural inflation is given by

$$V(\phi) = V_0 [1 + \cos(\phi/f)], \quad (1.321)$$

f being a parameter with dimensions of mass, which must be larger than M_{Pl} .

According to Eq. (1.215), a constant $\mathcal{R}$ on superhorizon scales implies that the density perturbation on the comoving slicing vanishes. Eq. (1.214) then implies that there is no non-adiabatic contribution to the pressure and therefore the perturbations must be *adiabatic* [146]. Although there is still no clear evidence for non-adiabaticities [64] (also called ‘isocurvature’ perturbations [41]), if there was such an evidence in the future, this might single out the need for *multi-field* inflation models, where more than one scalar field is considered [41, 97, 116, 147, 148].

Cosmological observations also seem to suggest that the primordial perturbations are

highly *Gaussian* [64, 116]. This implies that the two-point correlation function (Eq. (1.260)) contains the main statistical information (it specifies the PDF completely) [24], whilst higher-order correlations are virtually zero. In Fourier space, these are called the *bispectrum*, the *trispectrum*, and so on. Any deviation from Gaussianities (Gaussian perturbations) is called *non-Gaussianity* [116, 149]. Multi-field inflation models, for example, generate non-Gaussianities as well [24, 116, 150, 151].

Consequently, inflation models leading to an almost scale invariant spectrum of scalar perturbations, which are nearly Gaussian and adiabatic, are the ones favoured by observations up to date.

Having reviewed the HBB model, its achievements, the flaws that motivate the need for inflation, the inflationary cosmology itself, the background dynamics and the role of linear perturbations and inflation observables, we turn our attention to the *scalar-tensor theories of gravity* and their application to the inflation paradigm in particular. Theories of modified gravity like R^2 are known to be in accordance with observations regarding inflation (see Fig. 1.9), and we expect this given the need for a high energy completion of GR. The generalisation of Einstein gravity will lead us into the notion of conformal frames, which provides an interesting framework in the context of cosmic inflation.

CHAPTER 2

SCALAR-TENSOR GRAVITY AND CONFORMAL FRAMES

The Einstein field equations plus the CC term, Eq. (1.2), can be derived from the following action [152]:

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_{\text{M}} [g_{\mu\nu}, \Psi], \quad (2.1)$$

where ‘ Ψ ’ denotes collectively the matter fields that belong to the *matter sector*. The energy-momentum tensor of matter fields, $T_{\mu\nu}^{\text{M}}$, is given by (see Eq. (1.74))

$$T_{\mu\nu}^{\text{M}} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_{\text{M}})}{\delta g^{\mu\nu}}, \quad (2.2)$$

where $\mathcal{L}_{\text{M}}$ is the Lagrangian of matter fields such that

$$S_{\text{M}} [g_{\mu\nu}, \Psi] = \int d^4x \sqrt{-g} \mathcal{L}_{\text{M}} (g_{\mu\nu}, \Psi). \quad (2.3)$$

However, the *gravity sector* can include an arbitrary function of the scalar curvature R and further degrees of freedom. The goal to achieve with these modifications of the GR action (2.1) is to tackle some observations regarding high energy (early time) and low energy (late time) physics, such as the inflationary and DE periods of evolution of the Universe, respectively [152–160] (see updated version of Ref. [161] on the arxiv). Since we are interested in the early phase of inflationary expansion, we shall analyse the role certain modifications of Einstein’s gravity may play during inflation.

2.1 Introduction to Scalar-Tensor Theories of Gravity

2.1.1 Effective Energy-Momentum Tensor in $f(\phi, R)$ Gravity

One reason why modified gravity is of enormous relevance in the inflation paradigm is the undeniable success of the *Starobinsky model* (‘ R^2 inflation’) [162, 163] in fitting the *Planck* 2018 data, combined with some other experiments (see Fig. 1.9, where the predictions of the Starobinsky model in regard to the scalar spectral index and the tensor-to-scalar ratio are displayed). This model falls within the so-called ‘ $f(R)$ theories of gravity’ [164].

Besides the arbitrary function of R , one can consider a scalar field ϕ *non-minimally coupled* to gravity (see Eq. (1.143)). A vanishing coupling is not stable under radiative corrections when quantum effects become relevant in curved spacetime [85, 165, 166] and, therefore, we should consider a generic coupling function as well. Taking both modifications into account, we write the full action (which corresponds to that of a *scalar-tensor theory of gravity* [152, 167])

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} \mathcal{K}(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_M[g_{\mu\nu}, \Psi], \quad (2.4)$$

where the matter action S_M refers to fields *minimally coupled* to gravity (i.e. ‘ Ψ ’), and hence the scalar field ϕ pertains to the gravity sector. We assume that the scalar field is *non-canonical*, which implies that the kinetic term includes an arbitrary function $\mathcal{K}$ of the field ϕ .

The variation of the function $f(\phi, R)$ yields

$$\sqrt{-g} \delta f(\phi, R) = \sqrt{-g} (F R_{\mu\nu} + g_{\mu\nu} \square F - \nabla_\mu \partial_\nu F) \delta g^{\mu\nu} + \sqrt{-g} f_{,\phi} \delta \phi, \quad (2.5)$$

where $F(\phi, R) \equiv f_{,R}(\phi, R)$, and where we have ignored the resulting total derivatives. The d’Alembert operator ‘ $\square$ ’ was introduced after Eq. (1.85). Varying the total action with respect to the inverse metric and making it vanish for any $\delta g^{\mu\nu}$ gives

$$F R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f - 2V - \mathcal{K} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2\square F) - \mathcal{K} \partial_\mu \phi \partial_\nu \phi - \nabla_\mu \partial_\nu F = T_{\mu\nu}^M, \quad (2.6)$$

where Jacobi’s formula, Eq. (1.75), has been used. This equation can be written in terms of the Einstein tensor (see Eq. (1.9))

$$F G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f - F R - 2V - \mathcal{K} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2\square F) - \mathcal{K} \partial_\mu \phi \partial_\nu \phi - \nabla_\mu \partial_\nu F = T_{\mu\nu}^M. \quad (2.7)$$

We first notice that $f(\phi, R) = M_{\text{Pl}}^2 R$ (and then $F(\phi, R) = M_{\text{Pl}}^2$) leads to the Einstein field equations (1.1), as expected. We also observe that all the contributions to the Einstein tensor on the LHS of the equation (2.7) can be moved to the RHS, so we can introduce an *effective* energy-momentum tensor $\tilde{T}_{\mu\nu}$

$$\tilde{T}_{\mu\nu} \equiv \frac{1}{F} T_{\mu\nu} = \frac{1}{F} (T_{\mu\nu}^\phi + T_{\mu\nu}^f + T_{\mu\nu}^M), \quad (2.8)$$

where, aside from the energy-momentum tensor of matter fields, we have an energy-momentum

tensor that would be associated with a non-canonical scalar field ϕ minimally coupled to gravity (see Eq. (1.72))

$$T_{\mu\nu}^\phi = \mathcal{K} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(\mathcal{K} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 2V \right), \quad (2.9)$$

and another tensor accounting for the gravity modifications

$$T_{\mu\nu}^f = \nabla_\mu \partial_\nu F - g_{\mu\nu} \left[\square F + \frac{1}{2} (RF - f) \right]. \quad (2.10)$$

We then have the field equations resembling the Einstein field equations (1.8)

$$G_{\mu\nu} = \tilde{T}_{\mu\nu}. \quad (2.11)$$

The scalar field equation reads (compare with Eq. (1.85))

$$\mathcal{K} \square \phi + \frac{1}{2} \mathcal{K}_{,\phi} \partial_\mu \phi \partial^\mu \phi - V_{,\phi} = -\frac{1}{2} f_{,\phi}. \quad (2.12)$$

We notice that, despite the modified gravity terms, the energy-momentum tensor of matter fields is still covariantly conserved [168]; i.e. $\nabla_\mu T_M^{\mu\nu} = 0$ (see Eq. (1.24)). Let us show this: the *Bianchi identity* [23]

$$\nabla_\mu G^{\mu\nu} = 0, \quad (2.13)$$

leads to the equation

$$G^{\mu\nu} \partial_\mu F = \nabla_\mu T_\phi^{\mu\nu} + \nabla_\mu T_f^{\mu\nu} + \nabla_\mu T_M^{\mu\nu}. \quad (2.14)$$

The covariant divergence of $T_\phi^{\mu\nu}$ yields

$$\nabla_\mu T_\phi^{\mu\nu} = \left(\mathcal{K} \square \phi + \frac{1}{2} \mathcal{K}_{,\phi} \partial_\mu \phi \partial^\mu \phi - V_{,\phi} \right) \partial^\nu \phi = -\frac{1}{2} f_{,\phi} \partial^\nu \phi, \quad (2.15)$$

where Eq. (2.12) has been used. On the other hand, the covariant divergence of $T_f^{\mu\nu}$ is

$$\nabla_\mu T_f^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} (\nabla_\mu \nabla_\rho - \nabla_\rho \nabla_\mu) \partial_\sigma F - \frac{1}{2} R \partial^\nu F + \frac{1}{2} f_{,\phi} \partial^\nu \phi. \quad (2.16)$$

But the Riemann tensor in GR can be introduced in this equation as it is defined as [23]

$$R^\rho{}_{\sigma\mu\nu}V^\sigma \equiv [\nabla_\mu, \nabla_\nu] V^\rho = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho, \quad (2.17)$$

where, in our case, $V^\mu = \partial^\mu F$. Therefore

$$\nabla_\mu T_f^{\mu\nu} = R^{\mu\nu} \partial_\mu F - \frac{1}{2} R \partial^\nu F + \frac{1}{2} f_{,\phi} \partial^\nu \phi = G^{\mu\nu} \partial_\mu F + \frac{1}{2} f_{,\phi} \partial^\nu \phi, \quad (2.18)$$

where the Ricci tensor $R_{\mu\nu}$ is obtained from the Riemann tensor by taking the trace as follows [23]

$$R_{\mu\nu} \equiv R^\sigma{}_{\mu\sigma\nu}, \quad (2.19)$$

$R_{\mu\nu}$ being symmetric. We readily see that Eqs. (2.14), (2.15) and (2.18) lead to

$$\nabla_\mu T_M^{\mu\nu} = 0, \quad (2.20)$$

as promised. We could have guessed this in view of the fact that the matter fields are neither non-minimally coupled to gravity nor coupled to the scalar field ϕ in the action (2.4).

2.1.2 Brans-Dicke Theory and the Conformal Transformation

One of the earliest theories of modified gravity that has been put forward after GR is the *Brans-Dicke theory* (1961) [169], which can be recovered if we choose the function $f(\phi, R)$, the non-canonical function $\mathcal{K}(\phi)$ and the potential such that¹

$$f(\phi, R) = \phi R; \quad \mathcal{K}(\phi) = \frac{\omega}{\phi}; \quad V(\phi) = 0, \quad (2.21)$$

where ω is a dimensionless constant. Then, the metric and scalar field equations read

$$\phi G_{\mu\nu} = T_{\mu\nu}^M + \nabla_\mu \partial_\nu \phi - g_{\mu\nu} \square \phi + \frac{\omega}{\phi} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right), \quad (2.22)$$

$$\frac{\omega}{\phi} \square \phi = \frac{1}{2} \left(\frac{\omega}{\phi^2} \partial_\mu \phi \partial^\mu \phi - R \right), \quad (2.23)$$

respectively. The last equation can be written in terms of the trace of the energy-momentum tensor T_M by taking the trace of Eq. (2.22), and then replacing the RHS in Eq. (2.23) such

1. Since ϕ shall replace G^{-1} (i.e. M_{Pl}^2), then $[\phi] = +2$ and not $[\phi] = +1$.

that

$$\Box\phi = \frac{T_M}{3 + 2\omega} . \quad (2.24)$$

The original motivation for this theory was to develop a relativistic theory of gravitation compatible with *Mach's principle* and, partially, the formulation of boundary (and initial) conditions for the theory determined solely by the *matter distribution* of the Universe [169]. As Brans and Dicke pointed out in their paper, the idea of matter determining the physical properties of the space it occupies is not fully realised in Einstein's theory, despite the attempts of Einstein himself to incorporate Mach's principle into his theory, and despite the latter having been initially a central inspiration for him. However, in this model, we see that Newton's gravitational constant G has been replaced by the scalar field ϕ , which obeys a dynamical equation that involves the trace of the energy-momentum tensor. Consequently, matter fields, through the trace of $T_{\mu\nu}^M$, determine the dynamics of this changing Newton's gravitational 'constant', which ultimately affects the inertial masses of elementary particles, which are determined by measuring the gravitational accelerations [170].

The action of the Brans-Dicke theory reads (see Eq. (2.4) and (2.21))

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) + S_M [g_{\mu\nu}, \Psi] . \quad (2.25)$$

The gravity sector of this action can be shown to be invariant under a specific *conformal rescaling* of the metric [171, 172]

$$\hat{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) , \quad (2.26)$$

and a field redefinition, where the conformal factor $\Omega(x)$ is a smooth, non-zero function of the spacetime coordinates. Notice that the conformal transformation does not change coordinates, but it modifies the metric. To see the invariance of the gravity action, we first show how R changes under the conformal rescaling of the metric [172]:

$$R = \Omega^2 \left[\hat{R} + \frac{6}{\Omega} \left(\hat{\Box} \Omega - 2 \hat{g}^{\mu\nu} \frac{\partial_\mu \Omega \partial_\nu \Omega}{\Omega} \right) \right] , \quad (2.27)$$

this transformation rule being valid only in 4 dimensions, and where $\hat{\Box} \equiv \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu$. Since $\sqrt{-g} = \Omega^{-4} \sqrt{-\hat{g}}$, Eq. (2.25) reads

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} \left[\Omega^{-2} \phi \hat{R} + 6 \phi \frac{\hat{\Box} \Omega}{\Omega^3} - \frac{\phi}{\Omega^2} \hat{g}^{\mu\nu} \left(\frac{\omega}{\phi^2} \partial_\mu \phi \partial_\nu \phi + 12 \frac{\partial_\mu \Omega \partial_\nu \Omega}{\Omega^2} \right) \right] +$$

$$+ S_{\text{M}} \left[\Omega^{-2} \hat{g}_{\mu\nu}, \Psi \right]. \quad (2.28)$$

Now, let us assume that $\Omega = \Omega(\phi)$ and define the new field [173] such that

$$\hat{\phi}(\phi) \equiv \frac{\phi}{\Omega^2(\phi)}. \quad (2.29)$$

Action (2.28) becomes (again, total derivatives are integrated out and not considered)

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} \left\{ \hat{\phi} \hat{R} - \left[\frac{\hat{\phi}}{\phi^2} \left(\omega + \frac{3}{2} \right) \left(\frac{\partial \phi}{\partial \hat{\phi}} \right)^2 - \frac{3}{2\hat{\phi}} \right] \hat{g}^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} \right\} + S_{\text{M}} \left[\Omega^{-2} \hat{g}_{\mu\nu}, \Psi \right]. \quad (2.30)$$

In order to recover the gravitational part of the action in Eq. (2.25), we define

$$\frac{\hat{\omega}}{\hat{\phi}} \equiv \frac{\hat{\phi}}{\phi^2} \left(\omega + \frac{3}{2} \right) \left(\frac{\partial \phi}{\partial \hat{\phi}} \right)^2 - \frac{3}{2\hat{\phi}}, \quad (2.31)$$

where $\hat{\omega}$ is a constant, and solve for $\hat{\phi} = \hat{\phi}(\phi)$ to obtain

$$\hat{\phi}(\phi) \propto \phi^\alpha, \quad (2.32)$$

where α is given by

$$\alpha \equiv \sqrt{\frac{\omega + 3/2}{\hat{\omega} + 3/2}}. \quad (2.33)$$

Consequently, if we choose the conformal factor such that $\Omega^2(\phi) \propto \phi^a$, a being a constant, and perform the field redefinition $\phi \rightarrow \hat{\phi} \propto \phi^{1-a}$, the gravitational sector of the action is invariant, with the constant parameter ω being replaced by $\hat{\omega}$ given by [173]

$$\hat{\omega} = \frac{\omega + \frac{3}{2}a(2-a)}{(1-a)^2}. \quad (2.34)$$

Notice in Eq. (2.33) that $\alpha = 0$ ($a = 1$) if $\omega = -3/2$, and that the action (2.28) becomes instead

$$S_{\text{BD}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-\hat{g}} \hat{R} + S_{\text{M}} \left[\Omega^{-2} \hat{g}_{\mu\nu}, \Psi \right], \quad (2.35)$$

where M_{Pl}^2 is the constant of proportionality in the conformal factor that we fix to match

the (reduced) Planck mass squared,² and (remember that $[\phi] = +2$)

$$\Omega^2(\phi) = \frac{\phi}{M_{\text{Pl}}^2} . \quad (2.36)$$

The gravitational sector resembles Einstein gravity action without a CC (see Eq. (2.1)), but the matter sector is now coupled to the field ϕ . However, since $\omega = -3/2$, this implies that $T_{\text{M}} = 0$ (see Eq. (2.24)), and this means that the conservation of energy-momentum is *not* affected by the conformal transformation.

To check this, let us see how the covariant divergence of the energy-momentum tensor of matter fields, $\nabla_\mu T_{\text{M}}^{\mu\nu} = 0$, transforms when we perform the conformal rescaling of the metric. Firstly, the energy-momentum tensor transforms as (see Eq. (2.2)) [172]

$$\hat{T}_{\text{M}}^{\mu\nu} = \Omega^{-6} T_{\text{M}}^{\mu\nu} , \quad (2.37)$$

where $\sqrt{-\hat{g}}\hat{\mathcal{L}}_{\text{M}} = \sqrt{-g}\mathcal{L}_{\text{M}}$. The trace transforms as [172]

$$\hat{T}_{\text{M}} = \hat{g}_{\mu\nu} \hat{T}_{\text{M}}^{\mu\nu} = \Omega^{-4} T_{\text{M}} . \quad (2.38)$$

Then

$$\hat{\nabla}_\mu \hat{T}_{\text{M}}^{\mu\nu} = -\frac{\partial^\nu \Omega}{\Omega^5} T_{\text{M}} . \quad (2.39)$$

If $T_{\text{M}} = 0$, then the conformal transformation does not alter the conservation of energy and momentum, and the dynamics of matter dictated by the action (2.35) is the same as the one prescribed by GR. So we have effectively recovered Einstein gravity (without a CC) [173].

However, the choice $\omega = -3/2$ implies a change of sign of the kinetic term in Eq. (2.25). This means that ϕ violates the NEC [20] and becomes a ‘phantom’ field. The common lore is that the NEC cannot be violated in any ‘healthy’ field theory, with some possible exceptions that we shall not discuss here.

2.1.3 Einstein Frame

In line with Brans-Dicke theory, we can seek a conformal rescaling of the metric (and the subsequent field redefinition) that modifies the action (2.4) so the gravity sector of the

2. If not, we can perform a rescaling of the metric such that $\hat{g}_{\mu\nu} \rightarrow e^{-\lambda} \hat{g}_{\mu\nu}$, λ being a constant. Then $\hat{R} \rightarrow e^\lambda \hat{R}$ and $\sqrt{-\hat{g}} \rightarrow e^{-2\lambda} \sqrt{-\hat{g}}$, and hence $\beta^{-1} \rightarrow \beta^{-1} e^{-\lambda}$, where β is the constant of proportionality in the conformal factor: $\Omega^2(\phi) = \beta\phi$ (remember that $[\phi] = +2$; see footnote 1). Thus, we can choose $e^{-\lambda}$ such that $e^{-\lambda} \equiv M_{\text{Pl}}^2 \beta$.

resulting one resembles Eq. (2.1) (without Λ). In few words: we aim to find the so-called *Einstein frame* [174–176].

Let us then see how the Einstein tensor changes under the conformal rescaling of the metric in 4 dimensions [172]:

$$\hat{G}_{\mu\nu} = G_{\mu\nu} + \frac{1}{\Omega^2} (4\partial_\mu\Omega\partial_\nu\Omega - g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\Omega\partial_\rho\Omega) - \frac{2}{\Omega} (\nabla_\mu\partial_\nu\Omega - g_{\mu\nu}\square\Omega), \quad (2.40)$$

and rewrite Eq. (2.7) in terms of ‘hatted’ quantities [177]:

$$\begin{aligned} \hat{G}_{\mu\nu} = & \frac{\mathcal{K}}{F}\partial_\mu\phi\partial_\nu\phi + \frac{1}{F^2}\partial_\mu F\partial_\nu F + \frac{2}{\Omega^2}\partial_\mu\Omega\partial_\nu\Omega + \nabla_\mu \left[\frac{1}{F}\partial_\nu F - \frac{1}{\Omega^2}\partial_\nu(\Omega^2) \right] - \\ & - \frac{1}{2}g_{\mu\nu} \left\{ \frac{\mathcal{K}}{F}\partial_\sigma\phi\partial^\sigma\phi + \frac{2}{F^2}\partial_\sigma F\partial^\sigma F - \frac{2}{\Omega^2}\partial_\sigma\Omega\partial^\sigma\Omega + \frac{2V + RF - f}{F} + \right. \\ & \left. + 2\nabla_\sigma \left[\frac{1}{F}\partial^\sigma F - \frac{1}{\Omega^2}\partial^\sigma(\Omega^2) \right] \right\} + \frac{\Omega^2}{F}\hat{T}_{\mu\nu}^{\text{M}}. \end{aligned} \quad (2.41)$$

Indices are contracted, raised and lowered with respect to $g_{\mu\nu}$. It is clear that, in order to get rid of the terms involving higher derivatives, the conformal factor must be chosen such that $\Omega^2 = \alpha F(\phi, R)$ (α being a non-zero constant with units of the inverse of mass squared, $[\alpha] = -2$). Then

$$\begin{aligned} \hat{G}_{\mu\nu} = & \frac{\mathcal{K}}{F}\partial_\mu\phi\partial_\nu\phi + \frac{3}{2F^2}\partial_\mu F\partial_\nu F - \frac{1}{2}\hat{g}_{\mu\nu} \left\{ \frac{\mathcal{K}}{F}\partial_\sigma\phi\partial^\sigma\phi + \frac{3}{2F^2}\partial_\sigma F\partial^\sigma F + \frac{2V + RF - f}{\alpha F^2} \right\} + \\ & + \alpha\hat{T}_{\mu\nu}^{\text{M}}, \end{aligned} \quad (2.42)$$

where now indices are contracted, raised and lowered with respect to the metric in the new representation $\hat{g}_{\mu\nu}$.

We then have the Einstein field equations with the Einstein tensor $\hat{G}_{\mu\nu}$, the energy-momentum tensor of matter fields $\hat{T}_{\mu\nu}^{\text{M}}$ and two non-canonical scalar fields ϕ and F with a potential term $U(\phi, F)$ [178]

$$U(\phi, F) \equiv \frac{2V(\phi) + RF - f(\phi, R)}{2\alpha^2 F^2}. \quad (2.43)$$

Eq. (2.42) can be obtained by setting equal to zero the variation of the following action with respect to the inverse of the metric $\hat{g}^{\mu\nu}$:

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2\alpha}\hat{R} - \frac{\mathcal{K}}{2\alpha F}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{3}{4\alpha F^2}\hat{g}^{\mu\nu}\partial_\mu F\partial_\nu F - U(\phi, F) \right] +$$

$$+ S_M \left[(\alpha F)^{-1} \hat{g}_{\mu\nu}, \Psi \right], \quad (2.44)$$

where the gravity sector is now described by Eq. (2.1) with $\Lambda = 0$ if we identify $\alpha = M_{\text{Pl}}^{-2}$ (if not, see footnote 2), and every field is minimally coupled to gravity. The matter sector however contains a coupling to one of the scalar fields (and since both fields are coupled due to the interacting potential $U(\phi, F)$, matter fields also affect the evolution of ϕ).

Regarding the scalar field equation, the d'Alembert operators acting on the scalar field in both conformal frames are related by [172]

$$\square\phi = \Omega^2 \left(\hat{\square}\phi - \frac{2}{\Omega} \hat{g}^{\mu\nu} \partial_\mu \Omega \partial_\nu \phi \right) = \alpha F \left(\hat{\square}\phi - \frac{1}{F} \hat{g}^{\mu\nu} \partial_\mu F \partial_\nu \phi \right). \quad (2.45)$$

Then, Eq. (2.12) becomes

$$\mathcal{K} \left(\hat{\square}\phi - \frac{1}{F} \hat{g}^{\mu\nu} \partial_\mu F \partial_\nu \phi \right) + \frac{1}{2} \mathcal{K}_{,\phi} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{1}{\alpha F} \left(V_{,\phi} - \frac{1}{2} f_{,\phi} \right). \quad (2.46)$$

It can be seen that one arrives at this equation by setting equal to zero the variation of the action (2.44) with respect to ϕ .

Finally, we vary S with respect to F and make the variation vanish. Since now S_M is a functional of F too, we must take it into account while we do the calculations. We notice that

$$\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta F} = \alpha \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \hat{g}^{\mu\nu} = -\frac{1}{2F} \sqrt{-g} \hat{T}_M, \quad (2.47)$$

and therefore, the field equation of F reads

$$\frac{1}{F} \left[\mathcal{K} \partial_\mu \phi \partial^\mu \phi + 3 \left(\hat{\square}F - \frac{1}{F} \partial_\mu F \partial^\mu F \right) \right] + 4\alpha U - \frac{R}{\alpha F} = \alpha \hat{T}_M. \quad (2.48)$$

Using Eq. (2.27), such that

$$R = \alpha F \left[\hat{R} + \frac{3}{F} \left(\hat{\square}F - \frac{3}{2F} \partial_\mu F \partial^\mu F \right) \right], \quad (2.49)$$

Eq. (2.48) can be rewritten as

$$\frac{1}{F} \left(\mathcal{K} \partial_\mu \phi \partial^\mu \phi + \frac{3}{2F} \partial_\mu F \partial^\mu F \right) + 4\alpha U - \hat{R} = \alpha \hat{T}_M. \quad (2.50)$$

This equation is actually the trace equation of Eq. (2.42). Therefore, F adds no further

information and can be regarded as an *arbitrary function* rather than a dynamical field.

2.1.4 Starobinsky Model and Higgs Inflation

We have mentioned the Starobinsky model (1979 and 1980) [162, 163], which corresponds to the following choices of $f(\phi, R)$, $\mathcal{K}(\phi)$ and $V(\phi)$:

$$f(\phi, R) = f(R) = M_{\text{Pl}}^2 \left(R + \frac{1}{6M^2} R^2 \right) ; \quad \mathcal{K}(\phi) = 0 ; \quad V(\phi) = 0 , \quad (2.51)$$

M being a constant with units of mass. Since this model is applied to the inflationary cosmology, the matter sector is not considered, and neither is the interaction that emerges in the Einstein frame between χ , to be defined now, and the matter fields (see the matter action in Eq. (2.44)). We shall comment on this coupling with matter fields later on.

The metric field equations read (see Eq. (2.7))

$$G_{\mu\nu} = \frac{1}{3M^2} \left[\nabla_\mu \partial_\nu R - R R_{\mu\nu} - g_{\mu\nu} \left(\square R - \frac{1}{4} R^2 \right) \right] . \quad (2.52)$$

In the Einstein frame (see Eq. (2.42), where we directly set $\alpha \equiv M_{\text{Pl}}^{-2}$), we have

$$\hat{G}_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} \left\{ \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} \hat{g}_{\mu\nu} \left[\hat{g}^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi + 2U(\chi) \right] \right\} , \quad (2.53)$$

where the field χ (often called ‘scalaron’ [152, 164]; compare the energy-momentum tensor of the scalar field in Eq. (1.72) with the χ terms in Eq. (2.53)) has been defined as

$$\chi \equiv \sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \left(\frac{F}{M_{\text{Pl}}^2} \right) , \quad (2.54)$$

such that the potential $U(\chi)$ is given by (see Eq. (2.43))

$$U(\chi) = \frac{3M^2 M_{\text{Pl}}^2}{4} \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right)^2 \equiv U_0 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right)^2 . \quad (2.55)$$

From Eq. (2.48), we can derive the equation for the scalaron. It can be noticed that

$$4 \frac{U}{M_{\text{Pl}}^2} - M_{\text{Pl}}^2 \frac{R}{F} = 3M^2 e^{-\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \left(e^{-\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} - 1 \right) = -\frac{\sqrt{6}}{M_{\text{Pl}}} U_{,\chi} , \quad (2.56)$$

and hence (compare with Eq. (1.85) and remember that we have neglected matter fields)

$$\hat{\square}\chi - U_{,\chi} = 0 . \quad (2.57)$$

Both equations, (2.53) and (2.57), may be obtained from the action

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{M_{\text{Pl}}^2}{2} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right] , \quad (2.58)$$

which is the action of a canonical scalar field χ in curved spacetime, minimally coupled to gravity.

Starobinsky inflation is the result of studying the interaction between quantum free, massless, conformally-invariant matter fields and a classical gravitational background field. Starobinsky did calculations involving the one-loop corrections to see if non-singular solutions could be found which would avoid the Big Bang singularity in the early Universe [162, 163]. The original field equations had more second-order curvature terms than those shown in Eq. (2.52), but it was shown that the inflationary expansion was essentially determined by the Ricci-squared term in the action [179, 180]. We indeed see that the scalaron potential $U(\chi)$ in Eq. (2.56) is asymptotically flat for large positive field values, meaning that, in this region of the field space, a de Sitter phase can be reached. Starobinsky was then among the first to show the possibility that the Universe went through an early de Sitter era, and that this period would be unstable, all this as a consequence of quantum effects taking place in a classic FLRW background metric.

SR inflation can be readily realised in the Starobinsky model. The SR parameters, ϵ_U and η_U (see Eqs. (1.99) and (1.101)), for the Starobinsky model, are given by

$$\epsilon_U(\chi) = \frac{4}{3} \left(1 - e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right)^{-2} , \quad (2.59)$$

$$\eta_U(\chi) = \frac{4}{3} \left(2 - e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right) \left(1 - e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right)^{-2} , \quad (2.60)$$

where, for $\chi > M_{\text{Pl}}$, $\epsilon_U(\chi)$ and $\eta_U(\chi)$ become exponentially smaller (although η_U decreases slower than ϵ_U). Regarding the number of e-folds, we can derive the SR formula

(see Eq. (1.109))

$$N(\chi) = \frac{1}{2} \left[\sqrt{\frac{3}{2}} \frac{\chi - \chi_i}{M_{\text{Pl}}} + \frac{3}{2} \left(e^{\sqrt{\frac{2}{3}} \frac{\chi_i}{M_{\text{Pl}}}} - e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{\text{Pl}}}} \right) \right]. \quad (2.61)$$

And, like in the quadratic case (below Eq. (1.319)), we can assume that $\chi_* \gg \chi_e$, where χ_e can be estimated to be

$$\chi_e = \sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \left(1 + \frac{2}{\sqrt{3}} \right) \simeq 0.4 M_{\text{Pl}}, \quad (2.62)$$

which is obtained by setting $\epsilon_U(\chi_e) = 1$. Since $\chi_* \gg \chi_e$ implies $\chi_* > M_{\text{Pl}}$, we can write the number of e-folds SR formula as

$$\Delta N \simeq \frac{3}{4} e^{\sqrt{\frac{2}{3}} \frac{\chi_*}{M_{\text{Pl}}}}, \quad (2.63)$$

and then, the SR parameters as functions of the number of e-folds read

$$\epsilon_U(N) = \frac{3}{4N^2}, \quad (2.64)$$

$$\eta_U(N) = -\frac{1}{N}, \quad (2.65)$$

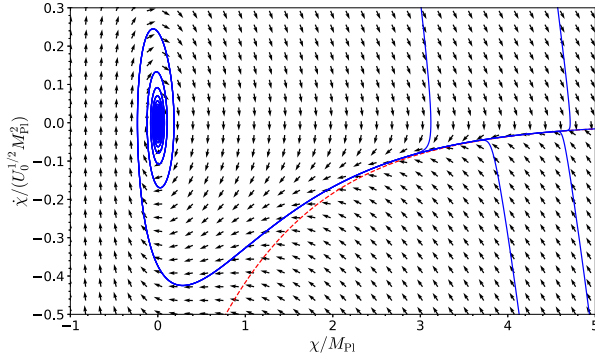
where N goes from somewhere between 70 and 45 to 0 (end of inflation; see discussion before Eqs. (1.280), (1.281) and (1.282)).

Regarding the inflation observables n_s and r , we can easily determined how they evolve with the number of e-folds during SR using the SR formulae (1.275) and (1.308)

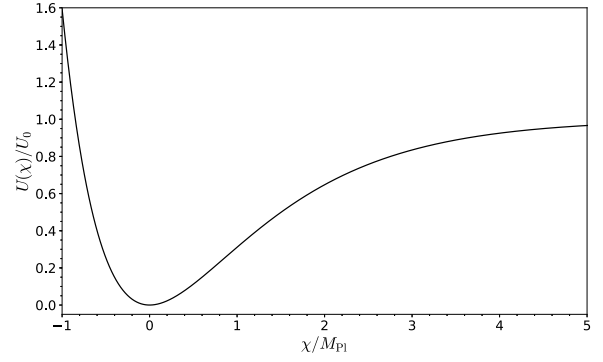
$$n_s(N) - 1 = -\frac{2}{N}, \quad (2.66)$$

$$r(N) = \frac{12}{N^2}. \quad (2.67)$$

Notice that if we take the shortest period of inflation since the largest observable scales left the horizon, $N = 45$, then $r = 0.0059$, which means that Starobinsky inflation predicts a tiny spectrum of primordial tensor perturbations compared to the scalar one, with $r \simeq 10^{-2}$ at most. This prediction is in very good agreement with the bound given in Eq. (1.318). As to the scalar spectral index, the model clearly predicts a redshifted one with $n_s = 0.9600$ for $N = 50$ and $n_s = 0.9667$ for $N = 60$, both falling within the range in Eq. (1.316) (see Fig. 1.9).



(a) Phase space of the Starobinsky model.



(b) Potential of the Starobinsky model.

Figure 2.1: Phase space of the Starobinsky model on the left shows the SR trajectory (dashed red line) which, in contrast to the quadratic potential (see Fig. 1.6), is not a straight line (the SR formula for the field velocity is not constant). On the right, we have the potential of the Starobinsky model, Eq. (2.55), which displays the flat region for large positive field values.

We can determine U_0 in the Starobinsky model using the measured value of A_s (Eq. (1.315)) and the SR formula (1.265)

$$A_s = \frac{U_0}{2\pi^2 M_{\text{Pl}}^4} \sinh^4 \left(\frac{\chi}{\sqrt{6} M_{\text{Pl}}} \right), \quad (2.68)$$

where $2 \sinh(x) \equiv e^x - e^{-x}$. Then

$$\frac{U_0}{M_{\text{Pl}}^4} = \frac{3.71 \times 10^{-7}}{N^2}. \quad (2.69)$$

Choosing the shortest period of inflation again, we find $U_0^{1/4} = 3.679 \times 10^{-3} M_{\text{Pl}}$. We have $U_0^{1/4} \simeq 3 \times 10^{-3} M_{\text{Pl}}$ in any case. If we write it in terms of M

$$M \simeq 10^{-5} M_{\text{Pl}}, \quad (2.70)$$

which is well below the Planck scale (given by M_{Pl}).

The action of the Starobinsky model

$$S_S = \frac{M_{\text{Pl}}^2}{2} \int d^4 x \sqrt{-g} \left(R + \frac{1}{6M^2} R^2 \right), \quad (2.71)$$

or the action of any $f(R)$ theory of gravity, is *equivalent* to the following action (using what

is called the ‘Legendre transformation’) [152, 164]:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [f(\phi) + f_{,\phi}(\phi) (R - \phi)] , \quad (2.72)$$

as long as $f_{,\phi\phi} \neq 0$ (e.g. the choice $f(R) = M_{\text{Pl}}^2 R$ does not satisfy this). This equivalence is established in the sense that if one derives the field equations of Eq. (2.72), one obtains $R = \phi$ and this goes over to the action of $f(R)$. Notice that $[\phi] = +2$, like in Brans-Dicke theory.

The equivalence holds in the case of $f(\phi, R)$ theories as well. In this case, Eq. (2.4) would be equivalent to

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [f(\phi, \chi) + f_{,\chi} (R - \chi)] - \frac{1}{2} \mathcal{K}(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\} + S_{\text{M}} [g_{\mu\nu}, \Psi] , \quad (2.73)$$

as long as $f_{,\chi\chi} \neq 0$ (e.g. the choice $f(\phi, R) = (M_{\text{Pl}}^2 + \xi \phi^2) R$ does not satisfy this, ξ being a dimensionless constant).

In the case of Starobinsky inflation, Eq. (2.72) reads

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\left(M_{\text{Pl}}^2 + \xi h^2 \right) R - \frac{\lambda}{2} h^4 \right] , \quad (2.74)$$

where h is a scalar field such that $\phi \equiv h^2$, with $[h] = +1$, and where we defined

$$\xi = \lambda \equiv \frac{M_{\text{Pl}}^2}{3M^2} . \quad (2.75)$$

This is the action of a scalar field non-minimally coupled to gravity with no kinetic term and with a *quartic* potential. The potential can be turned into a *symmetry breaking potential* if we dismiss the limit where $h^2 \gg v^2$ (v being the vacuum expectation value), and the kinetic term can be added if no SR assumption is made, which would make it negligible when compared to the potential. Then the action becomes

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} (M_{\text{Pl}}^2 + \xi h^2) R - \frac{1}{2} g^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{\lambda}{4} (h^2 - v^2)^2 \right] , \quad (2.76)$$

ξ being a coupling constant and λ the self-coupling constant. If the field h is set to be the *Higgs field* (which is the only scalar field that has been found experimentally up to date), then the model with the action (2.76) is usually called *Higgs inflation* [181–184]. It is then claimed in the literature that both, Starobinsky inflation and Higgs inflation in the large field

limit ($h^2 \gg v^2$) during SR (when the kinetic term can be neglected against the potential), lead to equivalent predictions [185, 186].

One can check the equivalence during SR by writing the action (2.76) in the Einstein frame (see Eq. (2.44))

$$S = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} \left[M_{\text{Pl}}^2 \hat{R} - \frac{M_{\text{Pl}}^2}{F} \left(1 + \frac{3F_{,h}^2}{2F} \right) \hat{g}^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{\lambda M_{\text{Pl}}^4}{2F^2} (h^2 - v^2)^2 \right], \quad (2.77)$$

where $F(h) \equiv M_{\text{Pl}}^2 + \xi h^2$, and the conformal factor $\Omega^2(h)$ has been chosen such that

$$\Omega^2(h) \equiv \frac{F(h)}{M_{\text{Pl}}^2}. \quad (2.78)$$

Notice that, in the Einstein frame, h is minimally coupled to gravity at the expense of becoming non-canonical, with $\mathcal{K}(h)$ given by (see Eq. (2.4))

$$\mathcal{K}(h) \equiv \frac{M_{\text{Pl}}^2}{F} \left(1 + \frac{3F_{,h}^2}{2F} \right) = \frac{M_{\text{Pl}}^2 [M_{\text{Pl}}^2 + (1 + 6\xi) \xi h^2]}{(M_{\text{Pl}}^2 + \xi h^2)^2}. \quad (2.79)$$

We can perform a field redefinition, $h \rightarrow \varphi(h)$, such that the new field is canonical

$$\frac{d\varphi}{dh} \equiv \sqrt{\mathcal{K}(h)}, \quad (2.80)$$

and integrate Eq. (2.80) to find $\varphi = \varphi(h)$. In the large field limit, $h \gg M_{\text{Pl}}/\sqrt{\xi}$, Eq. (2.79) reads

$$\mathcal{K}(h) \simeq \frac{1 + 6\xi}{\xi} \frac{M_{\text{Pl}}^2}{h^2}. \quad (2.81)$$

Plugging these into Eq. (2.80) and integrating yields

$$\varphi(h) = \sqrt{\frac{1 + 6\xi}{\xi}} M_{\text{Pl}} \ln \left(\frac{\sqrt{\xi} h}{M_{\text{Pl}}} \right) + \varphi_0, \quad (2.82)$$

φ_0 being a constant of integration such that $\varphi_0 = \varphi(h = h_0)$, with $h_0 = M_{\text{Pl}}/\sqrt{\xi}$. Then, $h(\varphi)$ reads

$$h(\varphi) = \frac{M_{\text{Pl}}}{\sqrt{\xi}} e^{\sqrt{\frac{\xi}{1+6\xi}} \frac{\varphi}{M_{\text{Pl}}}}, \quad (2.83)$$

where we took $\varphi_0 = 0$. Therefore, $h \gg M_{\text{Pl}}/\sqrt{\xi}$ implies $\varphi \gg M_{\text{Pl}} \sqrt{(1 + 6\xi)/\xi}$. If $\xi \gg 1$, we have $\xi \gg \sqrt{6} M_{\text{Pl}} \simeq 0.4 M_{\text{Pl}}$, which is approximately the lower bound on χ (the final field

value prescribed by the SR approximation) that we obtained in the Einstein frame in the Starobinsky model (see below Eq. (2.62)). Inserting Eq. (2.83) in the potential

$$U(h) = \frac{\lambda M_{\text{Pl}}^4}{4F^2(h)} \left(h^2 - v^2 \right)^2, \quad (2.84)$$

and assuming $h^2 \gg v^2$, yields

$$U(\varphi) = \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left(1 + e^{-2\sqrt{\frac{\xi}{1+6\xi}} \frac{\varphi}{M_{\text{Pl}}}} \right)^{-2} \simeq \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left(1 - e^{-2\sqrt{\frac{\xi}{1+6\xi}} \frac{\varphi}{M_{\text{Pl}}}} \right)^2. \quad (2.85)$$

This exponentially flat potential in the limit where $h^2 \gg v^2$ and $h^2 \gg M_{\text{Pl}}^2/\xi$ resembles that of the Starobinsky model (Eq. (2.55)), where

$$3M^2 = \frac{\lambda M_{\text{Pl}}^2}{\xi^2}. \quad (2.86)$$

If we choose $\lambda = \xi$, then from Eq. (2.86) we obtain Eq. (2.75). In the opposite limit, where $h^2 \ll M_{\text{Pl}}^2/\xi$, Eq. (2.79) becomes $\mathcal{K}(h) \simeq 1$. Then $\varphi = h$, and the potential in this limit is the symmetry breaking potential or the quartic potential if $h^2 \gg v^2$. Notice that we assumed that $(1 + 6\xi)\xi h^2 \ll M_{\text{Pl}}^2$ held as well. This does not have to be the case if $\xi \gg 1$. In that case, $\mathcal{K}(h)$ becomes

$$\mathcal{K}(h) \simeq \frac{6\xi^2 h^2}{M_{\text{Pl}}^2}. \quad (2.87)$$

Therefore, integrating Eq. (2.80) yields

$$\varphi(h) = \sqrt{\frac{3}{2}} \frac{\xi h^2}{M_{\text{Pl}}} + \varphi_0, \quad (2.88)$$

where $\varphi_0 = \varphi(0)$. Taking $\varphi_0 = 0$, writing $h(\varphi)$ and replacing it in the potential (2.84), we obtain either the symmetry breaking potential again or the *quadratic* potential if $h^2 \gg v^2$ as $h^2 \propto \varphi$.

Despite both models leading to the same predictions in the large field limit during SR, it is more realistic and suitable to consider *both* modifications to gravity simultaneously. This model is called ‘Higgs- R^2 ’ inflation or ‘Higgs Scalon Mixed’ inflation [187, 188], and it is

of the type of Eq. (2.4) with

$$f(h, R) = M_{\text{Pl}}^2 \left(R + \frac{\xi h^2}{M_{\text{Pl}}^2} R + \frac{1}{6M^2} R^2 \right). \quad (2.89)$$

One reason that motivates this approach is the large coupling constant ξ that Higgs inflation demands. If we consider $\lambda = \xi$ and Eq. (2.86), and the value of M predicted by Starobinsky inflation (2.70), we find $\xi \simeq 3 \times 10^9$. In terms of λ , this would be [184]

$$\frac{\xi}{\sqrt{\lambda}} \simeq 6 \times 10^4. \quad (2.90)$$

The experimentally measured Higgs boson mass, $m_H = 125.25 \pm 0.17$ GeV [11], and the expectation value of the Higgs field, $v \simeq 246$ GeV [11], allow to determine the self-coupling constant $\lambda = m_H^2/(2v^2)$, such that $\lambda \simeq 0.13$. Considering this as an example and plugging this value into Eq. (2.90), yields $\xi \simeq 10^4$ [181]. Since $\xi \gg \sqrt{\lambda}$ and $\xi \gg 1$, this constitutes a serious issue during reheating which demands a sensible UV (‘ultraviolet’; that is, high-energy) extension of the model.

During reheating, the momentum of the produced particles is $\mathcal{O}(\sqrt{\lambda} M_{\text{Pl}})$, which is much larger than the cutoff scale of the theory, comparable to that in vacuum, M_{Pl}/ξ [188, 189]. The Higgs- R^2 model however lifts up the cutoff scale in vacuum for masses of the scalaron $M \lesssim M_{\text{Pl}}/\xi$, such that the cutoff is now above or near the Planck scale [190], guaranteeing the perturbative analysis during inflation and in the post-inflationary phase, during preheating [191]. This model is, as we pointed out, realistic too since the Ricci-squared term is expected to be radiatively-induced in the large coupling ξ limit [188].

2.1.5 Einstein and Jordan Frames

We have reviewed briefly examples of $F(\phi)R$ (Brans-Dicke model and Higgs inflation), $f(R)$ (Starobinsky model) and $f(\phi, R)$ (Higgs- R^2 inflation) theories of gravity. However, the analysis of the SR parameters and inflation observables was done in the Einstein frame mainly, which is the frame where the gravity sector of the action resembles that of Einstein theory. Switching to this frame implied that one had to consider a coupling in the matter sector with the field that was non-minimally coupled to gravity in the starting frame. The frame in which the scalar field is non-minimally coupled to gravity but *does not* enter the matter action like in the Einstein frame is widely known in the literature as the *Jordan frame* [175], named after P. Jordan, whose work on the variation of Newton’s gravitational

constant [192, 193] preceded that of Brans and Dicke, and was inspired by Dirac’s large numbers hypothesis [194].

As was pointed out already, such a coupling with matter fields is ignored when the theory is applied to inflation, but when that is not the case, this causes isolated test particles to *not* move on geodesics with respect to the Einstein frame metric, which constitutes a violation of the equivalence principle [159, 175]. On the other hand, it is sometimes the case that the dynamical analysis is performed in an easier manner in the Jordan frame, and therefore, the conformal rescaling of the metric that converts the action into the Einstein frame action is not always *desirable* [195]. Furthermore, we mentioned *warm inflation*, which is considered as an alternative to the usual, ‘cold’ inflation scenario (see Sec. 1.2.6) that accounts for the couplings between matter and the inflaton during inflation. Such a scheme, when embedded in the framework of conformal frames, might result in interesting new aspects which would require an analysis in the Jordan frame to avoid complications derived from the direct coupling between matter fields and the field dependent conformal factor.

Thus, it is clear that, besides the analysis performed in the Einstein frame, one needs to tackle the Jordan frame despite its seeming difficulty. We shall do this in the next section by considering $F(\phi)R$ theories of gravity in the Jordan frame and, in line with our interests, we shall apply this study to the inflationary cosmology.³

2.2 Slow-Roll Inflation in the Jordan Frame

2.2.1 Spacelike Hypersurfaces in Jordan and Einstein Frames

We begin by adopting a coordinate time t that is used to slice the 4-dimensional spacetime into the constant time, spacelike hypersurfaces (the so-called *foliation of spacetime*) [196], such that they coincide with the constant energy density ones. The choice of t is arbitrary but, once it is chosen, we keep t fixed.⁴ The slices are assumed to be homogeneous and isotropic, as we are interested in FLRW spacetimes.

Now, notice that we have *not* used the metric to introduce this coordinate time. The *proper time* τ however does depend on the metric, such that [196]

$$\delta\tau = \mathcal{N}\delta t , \tag{2.91}$$

3. The work developed in Secs. 2.2 and 2.3 has resulted in a publication; see Ref. [1].

4. We shall choose the slicing that simplifies the expressions in the Jordan frame.

where $\mathcal{N}$ is the *lapse function*, given by

$$\mathcal{N} \equiv \sqrt{-g^{\mu\nu} \nabla_\mu t \nabla_\nu t} . \quad (2.92)$$

$\nabla_\mu t$ is the gradient of t .⁵ We also fix the spatial coordinate such that the shift vector vanishes. Once the slicing and threading are fixed, the spatially-flat FLRW metric can be written as

$$ds^2 = -\mathcal{N}^2 dt^2 + a^2(t) \delta_{ij} dx^i dx^j . \quad (2.93)$$

If we now perform a conformal transformation (see Eq. (2.26)), this yields a rescaling of the lapse function and the scale factor

$$\hat{\mathcal{N}} = \Omega \mathcal{N} , \quad (2.94)$$

$$\hat{a} = \Omega a . \quad (2.95)$$

Therefore, the usual procedure of rescaling the time coordinate that is often followed in the literature is inconvenient as t does not depend on the metric, and makes the computation more difficult. Normally, when dealing with the FLRW metric, the lapse function is set equal to one as, in GR, $\mathcal{N}$ is a Lagrange multiplier and plays no dynamical role [197]. Here however we consider different conformal frames and the lapse function is retained *a priori* and then chosen in a consistent manner.

With every constant time hypersurface one can associate the *intrinsic* and *extrinsic* curvature tensors [196]. These are denoted by ${}^3R_{\mu\nu}$ and $K_{\mu\nu}$, respectively. The trace of the latter determines the volume expansion rate [198]

$$K = -\frac{1}{\delta\mathcal{V}} \frac{d\delta\mathcal{V}}{d\tau} , \quad (2.96)$$

where $\delta\mathcal{V} \propto a^3$ is the proper volume element (see below Eqs. (1.31) and (1.32)), and $K \equiv K^\mu{}_\mu$. In place of K , it is conventional to use the Hubble parameter (see Eq. (1.11))

$$H \equiv -\frac{1}{3}K = \frac{1}{a} \frac{da}{d\tau} . \quad (2.97)$$

If one chooses the lapse function to be 1, then H is defined as in Eq. (1.11). The relation

5. More precisely, $\nabla_\mu t$ are the coordinate components of the timelike vector field which is the *metric dual* to the gradient of t , this being a 1-form quantity.

between Hubble parameters in the two conformally-related frames is

$$\hat{H} = \frac{1}{\hat{a}} \frac{d\hat{a}}{d\hat{\tau}} = \frac{1}{\Omega} \left(H + \frac{1}{\Omega} \frac{d\Omega}{d\tau} \right). \quad (2.98)$$

We now introduce the Hubble-flow parameters, defining them as the relative change of the Hubble parameter above as measured by the comoving observer

$$\epsilon_{i+1} \equiv \frac{1}{H\epsilon_i} \frac{d\epsilon_i}{d\tau}; \quad (i \geq 1), \quad (2.99)$$

where $i \in \mathbb{N}$ and

$$\epsilon_1 \equiv -\frac{1}{H^2} \frac{dH}{d\tau}. \quad (2.100)$$

It is clear again that if the lapse function is set equal to 1, then these parameters are defined as in Eq. (1.102).

As we did in the previous section, we consider the ‘hatted’ frame to be the Einstein frame, whereas we do not use carets for geometric quantities in the Jordan frame. Now, inflation occurs when the expansion rate of the spacelike slices is accelerating which, in accordance with the formalism developed here, translates into (compare with Eq. (1.71))

$$\frac{d^2\hat{a}}{d\hat{\tau}^2} > 0. \quad (2.101)$$

If the lapse function in the Einstein frame is set equal to 1, then we recover $\ddot{\hat{a}} > 0$. If the scale factor is assumed to be accelerating in the Einstein frame, the same does not necessarily apply to the Jordan frame in order to have inflation [199–201], as can be easily seen by rewriting this in terms of ‘unhatted’ quantities

$$\frac{1}{a} \frac{d^2a}{d\tau^2} > -\frac{1}{\Omega} \frac{d^2\Omega}{d\tau^2} + \left(\frac{1}{\Omega} \frac{d\Omega}{d\tau} \right)^2 - \frac{H}{\Omega} \frac{d\Omega}{d\tau}, \quad (2.102)$$

where $\Omega > 0$. In order to go from the Jordan frame to the Einstein one, the conformal factor must be chosen such that $\Omega^2 \equiv F/M_{\text{Pl}}^2 = F(\phi)/M_{\text{Pl}}^2$ (see above Eq. (2.42) and remember that we chose $\alpha \equiv M_{\text{Pl}}^{-2}$). Besides, for simplicity, we set $M_{\text{Pl}} = 1$ and work with those units.

As was commented in footnote 4, for convenience we fix the constant time hypersurfaces such that the coordinate time equals the proper time *in the Jordan frame* (without the loss of generality). This means that $\mathcal{N} = 1$ and (see Eqs. (2.94) and (2.95))

$$\hat{\mathcal{N}} = \sqrt{F}, \quad (2.103)$$

$$\hat{a} = \sqrt{F}a . \quad (2.104)$$

Consequently, the relation between Hubble parameters (2.98) becomes

$$\hat{H} = \frac{H}{\sqrt{F}} (1 + \theta_1) , \quad (2.105)$$

and the condition for the second time derivative of the scale factor (2.102) reads

$$\frac{\ddot{a}}{aH^2} > -\frac{\theta_1\theta_2}{1+\theta_1} , \quad (2.106)$$

where we introduced the following set of parameters (compare with Eq. (1.102)):

$$\theta_{i+1} \equiv \frac{\dot{\theta}_i}{H\theta_i} ; \quad (i \geq 1) , \quad (2.107)$$

where

$$\theta_1 \equiv \frac{\dot{F}}{2HF} , \quad (2.108)$$

and $i \in \mathbb{N}$.

2.2.2 Field Equations in the Jordan Frame

The motivation for those parameters is rather clear if we write the action in the Jordan frame (see Eq. (2.4))

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} F(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] . \quad (2.109)$$

We see that the Planck mass has been replaced by a time dependent function. This means that we no longer have one time dependent background scale like in GR ($H(t)$) but an additional one given by $\sqrt{F(t)}$. In passing, we should also mention that we consider a canonical field in the Jordan frame for simplicity. However, in case one wants to consider a non-canonical field, we can always perform a field redefinition and introduce a canonical one.

From Eq. (2.109) we can derive the homogeneous field equations in the Jordan frame. The metric field equations are

$$3H^2 = \frac{1}{F} \left(\frac{1}{2} \dot{\phi}^2 + V - 3H\dot{F} \right) , \quad (2.110)$$

$$\dot{H} = -\frac{1}{2F} \left(\dot{\phi}^2 - H\dot{F} + \ddot{F} \right). \quad (2.111)$$

The scalar field equation reads

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 3F_{,\phi} \left(2H^2 + \dot{H} \right). \quad (2.112)$$

Eq. (2.112) can be written in a more suggestive manner. Using Eqs. (2.110) and (2.111), Eq. (2.112) becomes

$$\left(1 + \frac{3F_{,\phi}^2}{2F} \right) \left(\ddot{\phi} + 3H\dot{\phi} \right) + V_{,\phi} - 2\frac{F_{,\phi}}{F}V = -\frac{F_{,\phi}\dot{\phi}^2}{2F} \left(1 + 3F_{,\phi\phi} \right), \quad (2.113)$$

or

$$\ddot{\phi} + \left[3 + \frac{\theta_1}{F\mathcal{K}} \left(1 + 3F_{,\phi\phi} \right) \right] H\dot{\phi} + \frac{F}{\mathcal{K}} \left(\frac{V}{F^2} \right)_{,\phi} = 0, \quad (2.114)$$

where (see Eq. (2.79))

$$\mathcal{K} \equiv \frac{1}{F} \left(1 + \frac{3F_{,\phi}^2}{2F} \right). \quad (2.115)$$

We then define

$$\theta_{\mathcal{K}} \equiv \frac{\dot{\mathcal{K}}}{2H\mathcal{K}} = -\frac{\theta_1}{F\mathcal{K}} \left(1 + \frac{3F_{,\phi}^2}{F} - 3F_{,\phi\phi} \right), \quad (2.116)$$

so Eq. (2.114) becomes

$$\ddot{\phi} + (3 + 2\theta_1 + \theta_{\mathcal{K}}) H\dot{\phi} + \frac{F}{\mathcal{K}} \left(\frac{V}{F^2} \right)_{,\phi} = 0. \quad (2.117)$$

This equation resembles the homogeneous scalar field equation in GR, Eq. (1.84), but with a modified Hubble friction term and a new slope term.

We find convenient to define the following set of parameters:

$$\gamma_{i+1} \equiv \frac{\dot{\gamma}_i}{H\gamma_i}; \quad (i \geq 1), \quad (2.118)$$

where

$$\gamma_1 \equiv \gamma \equiv \sqrt{\frac{\mathcal{K}}{2}} \frac{\dot{\phi}}{H}, \quad (2.119)$$

and $i \in \mathbb{N}$. Equations are more compact and the SR approximations are made clearer if these parameters are used in lieu of the Hubble-flow parameters ϵ_i of the Jordan frame (see

Eq. (1.102) to remember the definition). γ_2 contains θ_K as defined above

$$\gamma_2 = \frac{\ddot{\phi}}{H\dot{\phi}} + \epsilon_1 + \theta_K . \quad (2.120)$$

Also, from Eq. (2.111), we can obtain γ^2 in terms of the θ parameters and the first Hubble-flow parameter, such that

$$\gamma^2 = (\epsilon_1 + \theta_1)(1 + \theta_1) - \theta_1\theta_2 . \quad (2.121)$$

2.2.3 Slow-Roll Approximation

The physical equivalence of conformal frames (that is, the issue regarding the frame that should be used to make the physical predictions [175]) is an old debate. In Refs. [202–204], the equivalence between frames when the classical homogeneous mode dominates over small quantum fluctuations is well established (see Ref. [205] for a differing view on this matter). For frame independent formulations of some late universe processes see, e.g., Refs. [206–209]. Intended analogous computations regarding inflation and the corresponding observables are developed in Refs. [208, 210–213].

To address this question of the *physical frame*, we must note that the metric *defines* the units of measure. Therefore, a conformal rescaling of the metric implies a transformation of the units [174]. As we saw, this gives a rescaled lapse function and a rescaled scale factor, and does not imply any change in *physical observables* (only in their *interpretation*). This was illustrated by N. Deruelle and M. Sasaki in Ref. [214]. They dealt with this issue by showing a simple example that observational predictions made from a frame where the metric couples minimally to the matter fields (GR) and those made from a conformal frame where the new metric is no longer minimally coupled to the matter fields (a so-called ‘veiled’ version of GR) are the same. The difference stems ultimately from the interpretation of the physical phenomena under consideration. Applying the conformal transformation to an FLRW metric for instance, we can map all the dynamical equations into a universe which is *static*. Observed phenomena like the redshift is still predicted by the model but, in a static frame, this is ascribed to the time dependence of particle masses. The relation between the emitted and observed wavelengths of photons is the same in both frames.

Although that paper in particular does not refer to scalar-tensor theories (as we said, the starting theory is GR and the conformal frame is the veiled version of GR), the same idea can be applied to inflation (see Ref. [215] for an application in this context and Ref. [216]

for a comprehensive analysis in line with the one carried out in Ref. [214] regarding a Brans-Dicke-like action with a potential function and matter fields). Regardless of the different frames when conformal transformations are considered, inflation is a physical phenomenon that happens in all frames, but the way we interpret it *depends on the frame*. Particularly, we have seen how to interpret SR inflation in the Einstein frame (see Sec. 1.2.4) and, following on from this, one can infer the SR dynamics in the Jordan frame. To that end, we consider the SR conditions as we know them in the Einstein frame

$$\frac{1}{2}\mathcal{K}\dot{\phi}^2 \ll \hat{\mathcal{N}}^2 U , \quad (2.122)$$

$$\frac{1}{2\hat{H}\mathcal{K}\dot{\phi}^2} \left| \frac{d}{dt} \left(\frac{\mathcal{K}\dot{\phi}^2}{\hat{\mathcal{N}}^2} \right) \right| \ll \frac{3}{\hat{\mathcal{N}}} . \quad (2.123)$$

$\hat{H}$ is given in Eq. (2.105) and $U(\phi)$ is (see Eq. (2.43) and remember that $M_{\text{Pl}} = 1$; i.e. $\alpha = 1$ in that equation)

$$U(\phi) = \frac{V(\phi)}{F^2(\phi)} . \quad (2.124)$$

As we saw, given the choice of the spacetime slicing, $\hat{\mathcal{N}} = \sqrt{F}$ in the Einstein frame, and the SR equation reads (compare with Eq. (1.92))

$$3\hat{H}\dot{\phi} \simeq -\frac{\hat{\mathcal{N}}U_{,\phi}}{\mathcal{K}} , \quad (2.125)$$

where

$$U_{,\phi} = U \left(\frac{V_{,\phi}}{V} - 2\frac{F_{,\phi}}{F} \right) . \quad (2.126)$$

The Friedmann equation during inflation is given by (compare with Eq. (1.98))

$$3\hat{H}^2 \simeq U . \quad (2.127)$$

Using the SR equation, Eq. (2.123) becomes

$$\frac{1}{2\hat{H}\mathcal{K}\dot{\phi}^2} \left| \frac{d}{dt} \left(\frac{\mathcal{K}\dot{\phi}^2}{\hat{\mathcal{N}}^2} \right) \right| \ll \frac{3}{\hat{\mathcal{N}}} \simeq \left| \frac{U_{,\phi}}{\hat{H}\mathcal{K}\dot{\phi}} \right| . \quad (2.128)$$

The first SR condition (2.122), when written in terms of Jordan frame quantities, yields

$$\frac{1}{2}F\mathcal{K}\dot{\phi}^2 \ll V , \quad (2.129)$$

i.e.

$$\gamma^2 \ll \frac{V}{H^2 F} . \quad (2.130)$$

Notice that Eq. (2.129) gives (see Eq. (2.115))

$$\frac{1}{2}\dot{\phi}^2 \ll V , \quad (2.131)$$

$$\theta_1^2 \ll \frac{V}{3H^2 F} . \quad (2.132)$$

Writing Eq. (2.110) such that

$$1 + 2\theta_1 = \frac{1}{3H^2 F} \left(\frac{1}{2}\dot{\phi}^2 + V \right) \simeq \frac{V}{3H^2 F} , \quad (2.133)$$

means that

$$\theta_1^2 \ll 1 + 2\theta_1 . \quad (2.134)$$

This however implies that

$$|\theta_1| \ll 1 . \quad (2.135)$$

Therefore, the Friedmann equation in the Jordan frame during inflation reads

$$3H^2(\phi) \simeq \frac{V(\phi)}{F(\phi)} . \quad (2.136)$$

Knowing that $\hat{\mathcal{N}} = \sqrt{F}$, the relation between $\hat{H}$ and H (Eq. (2.105)), and Eq. (2.126), we may write Eq. (2.128) as

$$\frac{1}{2} \left| \frac{\left(\mathcal{K}\dot{\phi}^2 \right)^\bullet}{\mathcal{K}\dot{\phi}^2} - \frac{\dot{F}}{F} \right| \ll 3H(1 + \theta_1) \simeq \frac{V}{F\mathcal{K}\dot{\phi}^2} \left| \frac{\dot{V}}{V} - 2\frac{\dot{F}}{F} \right| . \quad (2.137)$$

In terms of γ_2 (Eq. (2.120)), θ_1 and γ^2 , this condition becomes

$$|\gamma_2 - \epsilon_1 - \theta_1| \ll 3(1 + \theta_1) \simeq \frac{V}{F\mathcal{K}\dot{\phi}^2} \left| \frac{\dot{V}}{HV} - 4\theta_1 \right| . \quad (2.138)$$

Since $|\theta_1| \ll 1$, Eq. (2.138) yields

$$|\gamma_2 - \epsilon_1| = \left| \frac{\ddot{\phi}}{H\dot{\phi}} + \theta_{\mathcal{K}} \right| \ll 1 . \quad (2.139)$$

Condition (2.139) has been derived starting from the SR condition in the Einstein frame, which we know how to interpret. Using this SR condition, which now involves Jordan frame quantities, we can obtain the following approximate equation from Eq. (2.117):

$$3H\dot{\phi} \simeq -\frac{F}{\mathcal{K}} \left(\frac{V}{F^2} \right)_{,\phi}. \quad (2.140)$$

This is the SR equation *in the Jordan frame*. We can indeed check this by taking the SR equation in the Einstein frame (Eq. (2.125) with $\hat{\mathcal{N}} = \sqrt{F}$) and rewrite it in terms of Jordan frame variables, knowing that $|\theta_1| \ll 1$. In that case we obtain Eq. (2.140).

There is a version of this equation that is usually found in the literature (see Refs. [217–220]) which reads

$$3H\dot{\phi} \simeq -F^2 \left(\frac{V}{F^2} \right)_{,\phi}. \quad (2.141)$$

This equation can be obtained if one assumes that $|\epsilon_1| \ll 1$ and $|\ddot{\phi}/(H\dot{\phi})| \ll 1$ in Eq. (2.112), and then uses the Friedmann equation. These conditions, which are imposed and not derived from well-known SR conditions in the Einstein frame as we did, are sometimes referred to as ‘generalised slow-roll approximation’ [217, 221, 222]. We notice that the very same equation is obtained by assuming that $F_{,\phi}^2 \ll F$ in Eq. (2.140) (such that $\mathcal{K} \simeq 1/F$). Generally, neglecting this term is *not* justified and we shall see why soon. Moreover, by performing numerical simulations, L. Järv and A. Toporensky demonstrated that Eq. (2.140) *is* the SR attractor [223]. Some implications resulting from the two different expressions can be found in the work by K. Akin, A. S. Arapoglu and A. E. Yükselci in Ref. [222].

We can find a condition on $\dot{V}/(HV)$ in Eq. (2.138). The RHS reads

$$1 \simeq \frac{V}{3H^2 F \gamma^2} \left| \frac{\dot{V}}{2HV} - 2\theta_1 \right|. \quad (2.142)$$

From Eq. (2.130), $V/(3H^2 F \gamma^2) \gg 1$. Since $|\theta_1| \ll 1$, then

$$\left| \frac{\dot{V}}{HV} \right| \ll 1, \quad (2.143)$$

in order for Eq. (2.142) to be approximately satisfied.

When justifying Eq. (2.141), we said that $|\epsilon_1| \ll 1$ is imposed. We take the absolute value because ϵ_1 , as opposed to the first Hubble-flow parameter in the Einstein frame, $\hat{\epsilon}_1$, may be negative. The relation between ϵ_1 and $\hat{\epsilon}_1$ can be easily derived by first writing $\hat{\epsilon}_1$ as

(see Eq. (1.88))

$$\hat{\epsilon}_1 = -\frac{1}{\hat{H}^2} \frac{d\hat{H}}{d\hat{\tau}} = \frac{1}{2} \frac{\mathcal{K}\dot{\phi}^2}{\hat{H}^2 F} = \frac{\gamma^2}{(1+\theta_1)^2}, \quad (2.144)$$

where $d\hat{\tau} = \sqrt{F}dt$ (see Eq. (2.91)). Then, using Eq. (2.121), one obtains

$$\hat{\epsilon}_1 = \frac{1}{1+\theta_1} \left(\epsilon_1 + \theta_1 - \frac{\theta_1\theta_2}{1+\theta_1} \right). \quad (2.145)$$

It can be seen that the condition $\hat{\epsilon}_1 \ll 1$ for quasi-de Sitter inflation amounts to

$$\epsilon_1 \ll 1 + \frac{\theta_1\theta_2}{1+\theta_1}, \quad (2.146)$$

which could have obtained from Eq. (2.106) using Eq. (1.71), which holds in the Jordan frame too. $|\theta_1| \ll 1$ implies

$$\hat{\epsilon}_1 \simeq \gamma^2 \simeq \epsilon_1 + \theta_1(1 - \theta_2). \quad (2.147)$$

We can compute higher-order Hubble-flow parameters in the Einstein frame and write them in terms of Jordan frame variables as in Eq. (2.144). The second Hubble-flow parameter as an example is

$$\hat{\epsilon}_2 = \frac{2}{1+\theta_1} \left(\gamma_2 - \frac{\theta_1\theta_2}{1+\theta_1} \right). \quad (2.148)$$

Up to the lowest order in θ_1 , this expression can be simplified such that

$$\hat{\epsilon}_2 \simeq 2(\gamma_2 - \theta_1\theta_2). \quad (2.149)$$

If $|\gamma_2|$ and $|\theta_1\theta_2|$ happen to cancel out approximately, then they can be large enough not to be negligible separately and still lead to SR inflation ($|\hat{\epsilon}_2| \ll 1$). If this is combined with Eq. (2.139), then $|\gamma_2| \simeq |\epsilon_1| \simeq |\theta_1\theta_2|$ (and $\hat{\epsilon}_1 \simeq |\theta_1| \simeq \gamma^2$ if the cancellation $\epsilon_1 - \theta_1\theta_2$ is smaller than θ_1 in Eq. (2.145)). We however do not consider this case any further and generically write the condition as

$$|\gamma_2| \ll 1, \quad (2.150)$$

$$|\epsilon_1| \ll 1, \quad (2.151)$$

$$|\theta_1\theta_2| \ll 1. \quad (2.152)$$

The latter inequality still permits $|\theta_2| \sim 1$. Since θ_2 is given by

$$\theta_2 = \frac{\ddot{F}}{H\dot{F}} + \epsilon_1 - 2\theta_1 , \quad (2.153)$$

values of $|\theta_2|$ of order one imply $|\ddot{F}/(H\dot{F})|$ of order $\mathcal{O}(1)$ too, which might provide a counterexample to the approximation $|\ddot{F}| \ll H|\dot{F}|$, usually regarded in the literature, and part of the ‘generalised slow-roll approximation’ mentioned earlier (below Eq. (2.141)).

The first and second Hubble-flow parameters in the Einstein frame can be expressed in terms of the SR parameters ϵ_U and η_U (see Eqs. (1.99), (1.100) and (1.91)) such that

$$\epsilon_U \simeq \hat{\epsilon}_1 \simeq \gamma^2 , \quad (2.154)$$

$$2(2\epsilon_U - \eta_U) \simeq \hat{\epsilon}_2 \simeq 2(\gamma_2 - \theta_1\theta_2) . \quad (2.155)$$

$\epsilon_U = \epsilon_U(\phi)$ reads

$$\epsilon_U = \frac{1}{2\mathcal{K}} \left(\frac{V_{,\phi}}{V} - 2\frac{F_{,\phi}}{F} \right)^2 , \quad (2.156)$$

which can be obtained either by replacing U by V and F in the definition of ϵ_U (see Eq. (1.99) and remember that $M_{\text{Pl}} = 1$)

$$\epsilon_U = \frac{1}{2\mathcal{K}} \left(\frac{U_{,\phi}}{U} \right)^2 , \quad (2.157)$$

or by using the definition of γ^2 in Eq. (2.119), the SR equation (2.140), the Friedmann equation during inflation (2.136), and Eq. (2.154).

The second SR equation (2.155) yields

$$\gamma_2 - \theta_1\theta_2 \simeq \eta_{FV} , \quad (2.158)$$

during SR, where the parameter η_{FV} is defined as

$$\begin{aligned} \eta_{FV} \equiv 2\epsilon_U - \eta_U &= \frac{1}{\mathcal{K}} \left(\frac{U_{,\phi}}{U} \right)^2 - \frac{1}{\mathcal{K}} \left(\frac{U_{,\phi\phi}}{U} - \frac{\mathcal{K}_{,\phi}}{2\mathcal{K}} \frac{U_{,\phi}}{U} \right) = \frac{1}{\mathcal{K}} \left[2\frac{F_{,\phi\phi}}{F} - \frac{V_{,\phi\phi}}{V} - 2\left(\frac{F_{,\phi}}{F} \right)^2 + \right. \\ &\quad \left. + \left(\frac{V_{,\phi}}{V} \right)^2 + \frac{\mathcal{K}_{,\phi}}{2\mathcal{K}} \left(\frac{V_{,\phi}}{V} - 2\frac{F_{,\phi}}{F} \right) \right] . \end{aligned} \quad (2.159)$$

2.2.4 Slow-Roll Parameters

So far, we have considered a generic expression for $\mathcal{K}$, without making any assumption on both, F and the first field derivative $F_{,\phi}$. These assumptions may serve to simplify the SR parameters further and should be considered carefully as was commented already.

We first consider the case $F_{,\phi}^2 \ll F$. Then (see Eq. (2.115))

$$\mathcal{K} \simeq \frac{1}{F} . \quad (2.160)$$

The SR parameter ϵ_U reads

$$\epsilon_U \simeq \frac{F_{,\phi}^2}{2F} \left(2 - \frac{F}{F_{,\phi}} \frac{V_{,\phi}}{V} \right)^2 . \quad (2.161)$$

The smallness of ϵ_U implies

$$\frac{1}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \ll \frac{1}{F} \ll \frac{1}{F_{,\phi}^2} . \quad (2.162)$$

The second SR parameter, η_{FV} , is simplified as

$$\eta_{FV} \simeq 2F_{,\phi\phi} + F \left[\left(\frac{V_{,\phi}}{V} \right)^2 - \frac{V_{,\phi\phi}}{V} - \left(\frac{F_{,\phi}}{F} \right)^2 - \frac{F_{,\phi}}{2F} \frac{V_{,\phi}}{V} \right] , \quad (2.163)$$

where

$$\frac{\mathcal{K}_{,\phi}}{2\mathcal{K}} = -\frac{F_{,\phi}}{2F} + \frac{3F_{,\phi}}{2\mathcal{K}F^2} \left(F_{,\phi\phi} - \frac{F_{,\phi}^2}{2F} \right) , \quad (2.164)$$

and therefore we have assumed that $|F_{,\phi\phi}| \ll 1$ as well, so $\mathcal{K}_{,\phi}/(2\mathcal{K}) \simeq -F_{,\phi}/(2F)$. From Eq. (2.162), $|\eta_{FV}| \ll 1$ and (2.163), we can then infer that

$$\left| \frac{V_{,\phi\phi}}{V} \right| \ll \frac{1}{F} \ll \frac{1}{F_{,\phi}^2} . \quad (2.165)$$

On the other hand, the case $F_{,\phi}^2 \gg F$ yields

$$\mathcal{K} \simeq \frac{3F_{,\phi}^2}{2F^2} . \quad (2.166)$$

The SR parameter ϵ_U can then be approximated as

$$\epsilon_U \simeq \frac{1}{3} \left(2 - \frac{F}{F_{,\phi}} \frac{V_{,\phi}}{V} \right)^2. \quad (2.167)$$

We see that the SR condition then requires the cancellation

$$\frac{V_{,\phi}}{V} \simeq 2 \frac{F_{,\phi}}{F}, \quad (2.168)$$

up to SR precision. η_{FV} becomes

$$\eta_{FV} \simeq \frac{2F^2}{3F_{,\phi}^2} \left[\left(\frac{V_{,\phi}}{V} \right)^2 - \frac{V_{,\phi\phi}}{V} + \left(F_{,\phi\phi} - \frac{F_{,\phi}^2}{F} \right) \frac{V_{,\phi}}{F_{,\phi}V} \right]. \quad (2.169)$$

There is also the possibility that $F_{,\phi}^2$ and F are of the same order of magnitude. This yields a specific form of F :

$$F(\phi) = (\alpha\phi + \beta)^2, \quad (2.170)$$

with $\alpha = \mathcal{O}(1)$. If $\alpha = 1/\sqrt{6}$, then $\mathcal{K} = 2/F$ and the terms are exactly equal for that F ($3F_{,\phi}^2 = 2F$). The SR parameter ϵ_U is then

$$\epsilon_U \simeq \mathcal{O}(1) \left(2 - \frac{F}{F_{,\phi}} \frac{V_{,\phi}}{V} \right)^2. \quad (2.171)$$

The same condition as in Eq. (2.168) must be satisfied for SR inflation to be realised.

2.2.5 Inflation Observables in the Jordan Frame

Inflation observables during SR in the Einstein frame were discussed already in the previous chapter (see Sec. 1.3.4). We now seek to express those very same observables in terms of the Jordan frame variables. This would avoid the need for a conformal rescaling of the metric which, as we said, is sometimes not desirable. Besides, scalar-tensor theories of gravity are formulated in the Jordan frame in the first place and one must know how to work out the inflation observables with variables linked with that frame in a generic way.

The scalar perturbations amplitude in the Einstein frame reads (see Eqs. (1.265) and (2.124))

$$A_s = \frac{U}{24\pi^2\epsilon_U} = \frac{V}{24\pi^2 F^2 \epsilon_U}, \quad (2.172)$$

during SR. In terms of the Jordan frame variables, we use Eq. (2.154) to write

$$A_s \simeq \frac{V}{24\pi^2 F^2 \gamma^2} . \quad (2.173)$$

On the other hand, the scalar spectral index and the tensor-to-scalar ratio are (see Eq. (1.275) and (1.308))

$$n_s - 1 = -2(\epsilon_U + \eta_{FV}) , \quad (2.174)$$

$$r = 16\epsilon_U , \quad (2.175)$$

in the Einstein frame, respectively. In terms of the Jordan frame variables

$$n_s - 1 \simeq -2(\gamma^2 + \gamma_2 - \theta_1 \theta_2) , \quad (2.176)$$

$$r \simeq 16\gamma^2 . \quad (2.177)$$

Besides those relations, we need to specify the moment at which they must be evaluated. For that purpose, as was explained in the previous chapter, the number of e-folds is a suitable variable. However the number of e-folds is subject to the ambiguities of the conformal rescaling of the metric as can be readily seen if we write the expression

$$d\hat{N} = d \ln \hat{a} = \hat{H} d\hat{\tau} = \left(1 + \frac{1}{H\Omega} \frac{d\Omega}{d\tau} \right) dN . \quad (2.178)$$

In accordance with our previous choices, Einstein and Jordan frame number of e-folds are related by

$$d\hat{N} = (1 + \theta_1) dN . \quad (2.179)$$

Since we keep the spacetime slicing fixed, the initial slice at t coincides in the Einstein and Jordan frames, and the same can be applied to the t_i slice (or t_f if you consider the convention given in Eq. (1.108)). Integrating Eq. (2.179) yields the exact relation

$$\hat{N}(t) = N(t) + \frac{1}{2} \ln \left(\frac{F(t)}{F(t_i)} \right) , \quad (2.180)$$

or

$$\hat{N}(t) = N(t) + \frac{1}{2} \ln \left(\frac{F(t_f)}{F(t)} \right) , \quad (2.181)$$

if you opt to start counting e-folds of inflation from the end of inflation. As we see, the

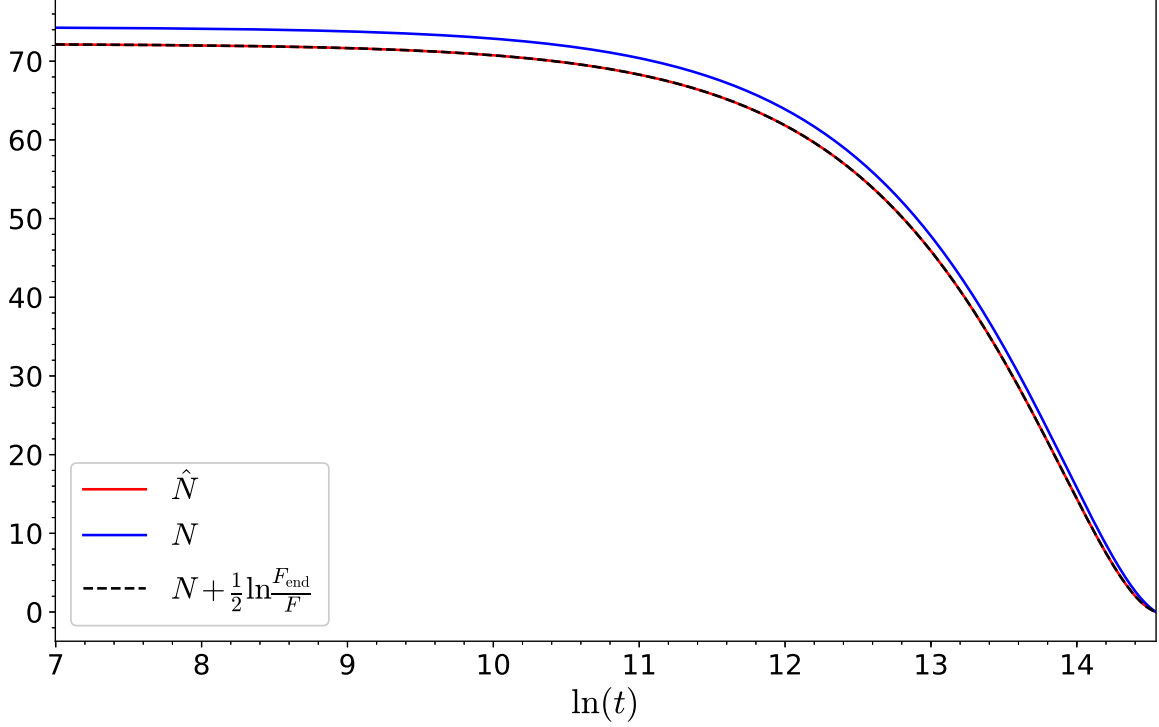


Figure 2.2: The e-fold number in each frame after solving numerically the exact field equations. We show the effect of the e-fold shift number as in Eq. (2.181), where $F_{\text{end}} = F(t_f)$.

Jordan frame number of e-folds N does not necessarily correspond to the same numerical value as $\hat{N}$ [209, 224, 225] (see Fig. 2.2 for the mismatch and how it is corrected by adding the *e-fold shift number*, $\frac{1}{2} \ln(F(t_f)/F(t))$).

Although we regard the homogeneous quantities only, we must also address the question about the *horizon-crossing time* (see also Ref. [209]). In the Einstein frame, as was commented already in an earlier section, Sec. 1.3.4, a mode of the perturbation is said to leave the horizon when its wavenumber is $k = \hat{a}\hat{H}$ (see, e.g., Ref. [24] for details). This means that the same does not hold in the Jordan frame, as

$$\hat{a}\hat{H} = aH(1 + \theta_1), \quad (2.182)$$

so the equalities $k = aH$ and $k = \hat{a}\hat{H}$ are satisfied on two different slices. The error of the often used choice $k = aH$ is of order SR however as, for cosmological scales, those SR conditions are meant to be applicable and $|\theta_1| \ll 1$ as was shown above, in Sec. 2.2.3. Hence the difference between the slices is immaterial. It would be interesting to check how this changes in *non-slow-roll regimes*; i.e. those in which SR inflation no longer holds, and the approximations made so far may become substantially different.

Finally, the inflation observables, Eqs. (2.173), (2.176) and (2.177), can be written in terms of the Jordan frame Hubble-flow parameters. To that end, we use Eq. (2.147) and derive γ^2 with respect to time to calculate γ_2 (see Eq. (2.118)). We then have

$$A_s \simeq \frac{V}{24\pi^2 F^2 [\epsilon_1 + \theta_1 (1 - \theta_2)]} , \quad (2.183)$$

$$n_s - 1 \simeq -2 \left[\epsilon_1 + \theta_1 (1 - 2\theta_2) + \frac{1}{2} \frac{\epsilon_1 \epsilon_2 + \theta_1 \theta_2 (1 - \theta_2 - \theta_3)}{\epsilon_1 + \theta_1 (1 - \theta_2)} \right] , \quad (2.184)$$

$$r \simeq 16 [\epsilon_1 + \theta_1 (1 - \theta_2)] . \quad (2.185)$$

2.2.6 Concluding Remarks of Section 2.2

The central idea of the method developed in Sec. 2.2 was to use the fact that the conformal transformation corresponds to the change of units of measure. Keeping the slicing fixed, we mapped the relations defined in the Einstein frame to the Jordan frame on the same slice by changing the units of measure (i.e. by performing the conformal transformation). We took the conditions required for inflation and more stringent ones for SR, which were clearly defined in the Einstein frame, and mapped them to the Jordan frame. To find a compact parametrisation for those conditions, we had to introduce a number of flow parameters, in analogy to the Hubble-flow parameters (see Eq. (1.103)). These were ϵ_i , θ_i and γ^2 , defined in Eqs. (2.99), (2.108) and (2.118) respectively. We have shown that $\hat{\epsilon}_i \ll 1$ translates into the Jordan frame as

$$|\epsilon_1|, \gamma^2, |\gamma_2|, |\theta_1|, |\theta_1 \theta_2| \ll 1 . \quad (2.186)$$

Only two of the three sets of parameters ϵ_i , θ_i and γ_i are needed to describe the system and we found that θ_i and γ_i give the most compact equations. We derived the SR equations in the Jordan frame (see Eqs. (2.136) and (2.140)). These approximate relations hold as long as SR conditions, $\epsilon_U, |\eta_{FV}| \ll 1$ are satisfied (see Eqs. (2.156) and (2.159), respectively).

Given that the inflation observables can be conveniently written in terms of homogeneous quantities, we only needed to map the homogeneous relations in order to derive observable model predictions. In Eqs. (2.173), (2.176) and (2.177), the amplitude of scalar perturbations, the scalar spectral index and the tensor-to-scalar ratio, respectively, were written in terms of γ_i and θ_i , whilst in Eqs. (2.183)-(2.185) were expressed as functions of ϵ_i and θ_i . In Eqs. (2.172), (2.174) and (2.175) we wrote A_s , n_s and r respectively in terms of ϵ_U and η_{FV} .

When computing numerical values of the homogeneous variables, one must take into

account the difference between the number of e-folds of inflation as defined with respect to a and $\hat{a}$; that is, the scale factor in Jordan and Einstein frames, respectively. The difference is proportional to the logarithm of F as shown in Eq. (2.181) (we opt to count e-folds backwards, starting from the end of inflation; see Fig. (2.2)).

To conclude this section, let us show how to determine the range of possible values of θ_1 at the end of inflation (when $\hat{\epsilon}_1 = 1$). By setting $\hat{\epsilon}_1 = 1$ in Eq. (2.144) we obtain

$$\frac{1}{3H^2} \frac{V}{F} = \frac{1}{3} \mathcal{K} \frac{\dot{\phi}^2}{H^2} = \frac{2}{3} (1 + \theta_1)^2; \quad (\text{end of inflation}), \quad (2.187)$$

where we make it clear that these equalities hold only at the end of inflation. The first equality is simply $\dot{\phi}^2/F(\phi) = U(\phi)$ with Einstein frame variables (see Eq. (1.89)). Now, regarding the second equality, the definition of $\mathcal{K}$ (see Eq. (2.115)) yields

$$\frac{\dot{\phi}^2}{2H^2 F} = 1 + 2\theta_1 - 2\theta_1^2; \quad (\text{end of inflation}), \quad (2.188)$$

so $1 + 2\theta_1 - 2\theta_1^2$ at the end of inflation must be positive. We then arrive at the following range for allowed values of θ_1 at the end of inflation:

$$\frac{1}{2} (1 + \sqrt{3}) > \theta_1 > \frac{1}{2} (1 - \sqrt{3}); \quad (\text{end of inflation}), \quad (2.189)$$

implying that $|\theta_1|$ is not much larger than 1 at the end of inflation.

Additionally, the equality $V/(3H^2 F) = (2/3)(1 + \theta_1)^2$ in Eq. (2.187) and Eq. (2.189) allow to find a range of values for the background energy density ρ in the Jordan frame at the end of inflation relative to the potential energy ($\rho = 3H^2 F$)

$$\frac{2}{(\sqrt{3} - 1)^2} > \frac{\rho}{V} > \frac{2}{(\sqrt{3} + 1)^2}; \quad (\text{end of inflation}). \quad (2.190)$$

As opposed to what happens in the Einstein frame, in the Jordan frame we do *not* have a single value assigned to the background energy density when inflation ends. In the Einstein frame, when inflation finishes, $\hat{\rho}/U = 3/2$ (see Eq. (2.144) or Eq. (1.89) and impose $\hat{\epsilon}_1 = 1$). However, the ratio between the energy density and the potential energy in the Jordan frame at the end of inflation can be more than two times larger than that of the Einstein frame (the upper bound is ~ 3.73), or almost one fifth smaller (the lower bound is ~ 0.27).

This range of possible values at the end of inflation is due to the varying Newton's gravitational constant in the Jordan frame, $F = F(\phi)$. Another consequence of this changing

F is that we can no longer assume that, by starting from initial conditions at rest, the field always *rolls down* towards the minimum of the potential V (see the forcing term on the RHS of Eq. (2.112)). In contrast to the Einstein frame intuition, the field can also climb up the potential. If we look at Eq. (2.126), we see that the gravitational force is not uniquely determined by the gradient of V . It depends on the gradient of F as well. Therefore, the gravitational force can be directed in the uphill direction of V .

2.3 Generalised Induced Gravity Inflation

We shall now address the generalised version of an example of $F(\phi)R$ theory of gravity known as *induced gravity* [226–228], where gravity is generated by a spontaneous symmetry breaking [229–232], and apply the SR inflation formalism we have developed.⁶ The action for the induced gravity is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \xi \phi^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right], \quad (2.191)$$

such that $F(\phi) = \xi \phi^2$ and $V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2$ (see Eq. (2.109)), where v is the vacuum expectation value (see below Eq. (2.75)). We generalise the induced gravity model applied to inflation by allowing for arbitrary powers of ϕ in the non-minimal function $F(\phi)$ and $\phi^2 - v^2$ in the spontaneous symmetry breaking potential $V(\phi)$. We are going to search for inflation models which are consistent with the current observational constraints on the scalar spectral index and the tensor-to-scalar ratio.

2.3.1 Classification of Generalised Induced Gravity Inflation Models

The generalisation of the induced gravity model that we study can be written as

$$F(\phi) = \xi \phi^p, \quad (2.192)$$

$$V(\phi) = \frac{\lambda}{2q} (\phi^2 - v^2)^q, \quad (2.193)$$

6. Recently some works (see for example Refs. [233, 234]) have been published which analyse models that overlap with the ones considered in this section in some parameter space. However, our main goal is to derive analytical expressions of the constraints that can be compared with observations.

where $|p|$ and q are of order 1. Additionally, q must be an even, positive number in order for the potential to be bounded from below.⁷ F must be positive to avoid instabilities. Therefore, we fix $\xi > 0$ and, in case we have odd values of p , we shall only be interested in the $\phi > 0$ region.

The essence of the induced gravity inflation scenario is that, when the field ϕ eventually settles down at the minimum of the potential V (with $\phi = v$), then $F(v) = 1$ and we recover the action of GR. This requirement leads to the normalisation $v = \xi^{-1/p}$. We make use of this relation to simplify the expressions by normalising the field such that

$$x \equiv \frac{\phi}{v} . \quad (2.194)$$

Using this definition, F and V read

$$F(x) = x^p, \quad (2.195)$$

$$V(x) = \frac{\lambda \xi^{-2q/p}}{2q} (x^2 - 1)^q . \quad (2.196)$$

Since the potential is symmetric under the transformation $\phi \rightarrow -\phi$, x is positive (the field ϕ either runs towards the $\phi \rightarrow v$ minimum if $\phi > 0$ initially, or towards $\phi \rightarrow -v$ otherwise). In the context of inflation, the regime $x < 1$ corresponds to the *hilltop-like* inflation models (i.e. $\phi < v$), whereas the regime $x > 1$ leads to the *chaotic-like* models (i.e. $\phi > v$). We shall use this terminology below to distinguish the two regimes.

As was discussed earlier, at the end of the previous section, the field does not necessarily roll down the potential V in the Jordan frame. It can climb it up as well. We need to impose additional conditions in order to guarantee that the field reaches the minimum (i.e. $x = 1$). Writing $U_{,x}$ (see Eq. (2.126))

$$U_{,x} = 2 \frac{U}{x} \frac{(q-p)x^2 + p}{x^2 - 1} , \quad (2.197)$$

we notice that, for very small x , the force is directed towards the minimum of U only for $p > 0$. While for large x , the same is true either for $p > 0$ in the case of $p = q$, or for $q > p$ in the case of $p \neq q$. We can summarise this as follows

$$p > 0 \quad \text{if } x \ll 1 ; \quad (\text{hilltop-like models}) , \quad (2.198)$$

7. An unbounded potential V also results in an unbounded U (see Eq. (2.124)).

$$p \leq q \quad \text{if } x \gg 1 ; \quad (\text{chaotic-like models}) . \quad (2.199)$$

The SR equation (2.140) in terms of x is given by

$$\frac{\dot{x}}{H} \simeq -\frac{4\xi^{2/p}x^{p-1}}{2+3p^2\xi^{2/p}x^{p-2}} \frac{(q-p)x^2+p}{x^2-1} . \quad (2.200)$$

If we take the initial conditions to be $x \ll 1$ (hilltop-like setup) then, in order for x to slow-roll towards $x = 1$, $\dot{x} > 0$, which means that $p > 0$. In the opposite regime, where $x \gg 1$ (chaotic-like setup), the field velocity must be negative $\dot{x} < 0$, which leads to the condition $p > 0$ if $p = q$, or $q > p$ if $p \neq q$. We see that this is consistent with the former analysis in terms of the gradient of U . Notice also that, in the latter case, p can be negative.

Finally, we write the SR parameter ϵ_U in terms of x

$$\epsilon_U = \frac{4p^2\xi^{2/p}x^{p-2}}{2+3p^2\xi^{2/p}x^{p-2}} \left[\frac{(q/p-1)x^2+1}{x^2-1} \right]^2 , \quad (2.201)$$

which shall serve as a good proxy to indicate the regime where $\hat{\epsilon}_1 \ll 1$ as, during SR inflation, $\hat{\epsilon}_1 \simeq \epsilon_U$ (see discussion below Eq. (1.99)).

The $p = q = 2$ case with $\phi \gg v$ has been studied extensively in the literature (see Refs. [226–228, 235–242] for some early work on such models). In this section however, our goal is to investigate if inflation is possible in a broader range of parameters, where q and p are not necessarily equal and the field ϕ reaches the minimum from both sides.

2.3.2 Hilltop Type Models

In this regime, we take the approximation $x_* \ll 1$. x_* is the field value in v -units when cosmological scales exit the horizon (see the discussion after Eq. (1.259)).

Within this approximation, we can simplify the expression for ϵ_U in Eq. (2.201) as

$$\epsilon_U \simeq \frac{4p^2\xi^{2/p}x^{p-2}}{2+3p^2\xi^{2/p}x^{p-2}} . \quad (2.202)$$

Let us consider first $p > 2$. This implies that ϵ_U monotonically decreases as x decreases. In order for $\epsilon_U \ll 1$, we need to impose

$$p^2\xi^{2/p}x^{p-2} \ll 1 . \quad (2.203)$$

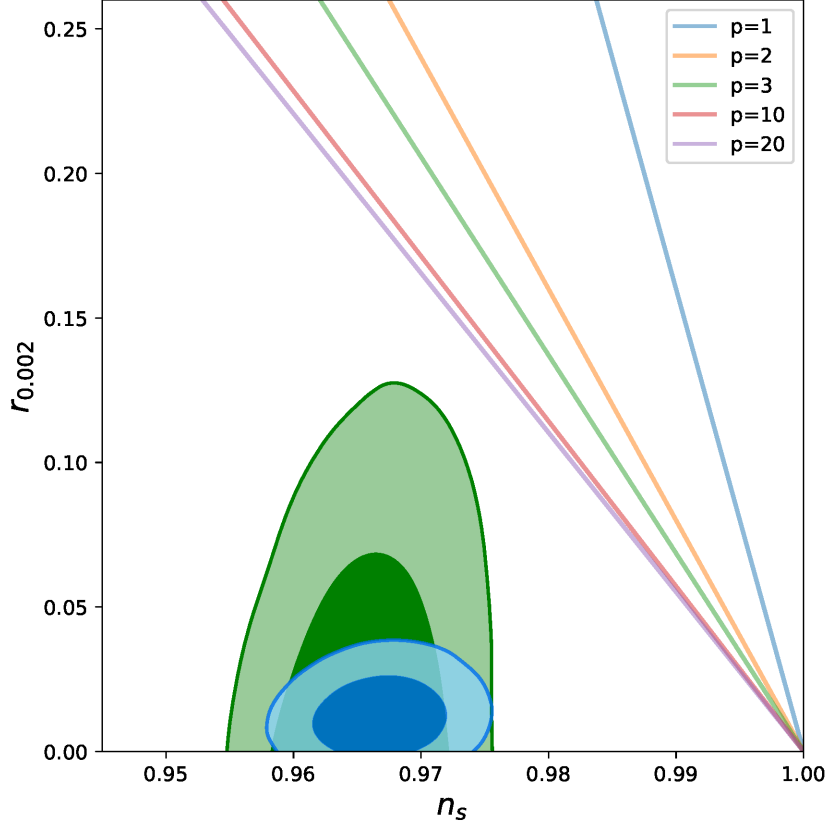


Figure 2.3: Comparison of observational constraints on scalar spectral index n_s and the tensor-to-scalar ratio r with predictions of the generalised induced gravity models in the *hilltop-like regime*. The green contours represent 1σ and 2σ constraints from Ref. [64] and the blue contours add the new BICEP/Keck observations (see Ref. [243] for details). $p > 0$ is required for hilltop-like models. Note also that the presence of a line for some values of r v n_s does not necessarily imply that those values can be actually realised within a given model (which also holds for the next plots). To establish this, one needs to compute the number of e-folds $\hat{N}_*$. As the lines lie outside the observationally allowed region anyway, we did not perform numerical simulations involving the hilltop-like setup.

We can then approximate ϵ_U as

$$\epsilon_U \simeq 2p^2 \xi^{2/p} x^{p-2}. \quad (2.204)$$

We notice that Eq. (2.203) is nothing but $F_{,\phi}^2 \ll F$ as

$$\frac{F_{,\phi}^2}{F} = \xi^{2/p} \frac{F_{,\phi}^2}{F} = p^2 \xi^{2/p} x^{p-2}. \quad (2.205)$$

We can then use Eq. (2.163) to compute η_{FV} , such that

$$\eta_{FV} \simeq -p(2-p) \xi^{2/p} x^{p-2}. \quad (2.206)$$

Using those and Eqs. (2.174) and (2.175), we find

$$n_s - 1 \simeq \frac{2-3p}{16p} r. \quad (2.207)$$

As can be seen in Fig. 2.3, these models fall outside the observationally allowed region.

We shall also consider the cases $p = 2$ and $p = 1$. In the former one:

$$\epsilon_U|_{p=2} \simeq \frac{8\xi}{1+6\xi}. \quad (2.208)$$

Therefore SR inflation ($\epsilon_U \ll 1$) is only possible for $\xi \ll 1$. In the case of $p = 1$:

$$\epsilon_U|_{p=1} \simeq \frac{4\xi^2 x^{-1}}{2+3\xi^2 x^{-1}}, \quad (2.209)$$

and the $\xi^2 x^{-1}$ term is a decreasing function of x . This leads to

$$\lim_{x \rightarrow 0} \epsilon_U|_{p=1} = \frac{4}{3} > 1, \quad (2.210)$$

which is a *non-inflationary* regime. Thus, SR inflation can be realised for field values such that $x \gg \xi^2$. Since for $p = 2$ and $p = 1$, $F_{,\phi}^2/F$ is given by

$$\left. \frac{F_{,\phi}^2}{F} \right|_{p=2} = 4\xi, \quad (2.211)$$

$$\left. \frac{F_{,\phi}^2}{F} \right|_{p=1} = \xi^2 x^{-1}, \quad (2.212)$$

the previous conditions on ξ imply that $F_{,\phi}^2 \ll F$ in both cases. Therefore, Eq. (2.207) relating n_s and r is valid in those cases too. This means however that the cases $p = 2$ and $p = 1$ also lie outside the observationally allowed region (as shown in Fig. 2.3).

2.3.3 Chaotic Type Models

In this regime we assume that $x_* \gg 1$. We then have to deal carefully with Eq. (2.201), as the lowest order approximation in terms of x^{p-2} depends on the relation between p and q .

Let us consider first the case where $p \neq q$. ϵ_U is then approximately given by

$$\epsilon_U \simeq \frac{4p^2 \xi^{2/p} x^{p-2}}{2 + 3p^2 \xi^{2/p} x^{p-2}} \left(\frac{q}{p} - 1 \right)^2. \quad (2.213)$$

As we said already (see below Eq. (2.200)), $p \neq q$ permits negative values of p . In the limit $p^2 \xi^{2/p} x^{p-2} \gg 1$ (i.e. $F_{,\phi}^2 \gg F$):

$$\epsilon_U \simeq \frac{4}{3} \left(\frac{q}{p} - 1 \right)^2, \quad (2.214)$$

but since $p \neq q$, SR inflation can only be realised for models with

$$p^2 \xi^{2/p} x^{p-2} \ll 1, \quad (2.215)$$

i.e. $F_{,\phi}^2 \ll F$. We can then approximate η_{FV} as in Eq. (2.163) again, which results in

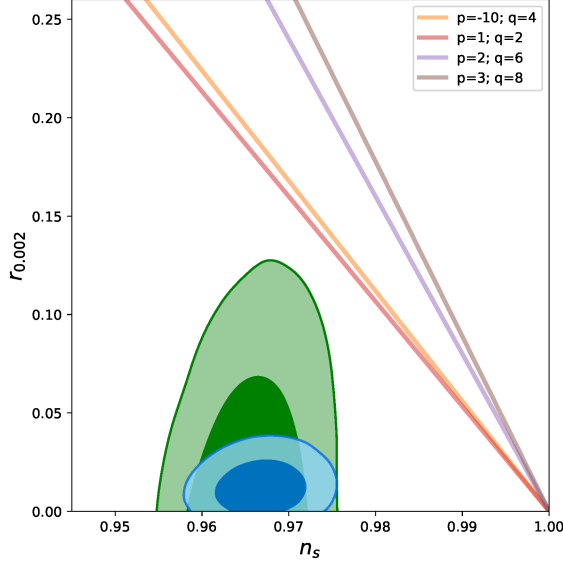
$$\eta_{FV} \simeq \begin{cases} qp^2 \xi^{2/p} x^{p-4} & \text{for } p = 2 \\ (q-p)(2-p) \xi^{2/p} x^{p-2} & \text{for } p \neq 2 \end{cases}, \quad (2.216)$$

to the lowest orders in x^{-2} and $p^2 \xi^{2/p} x^{p-2}$.

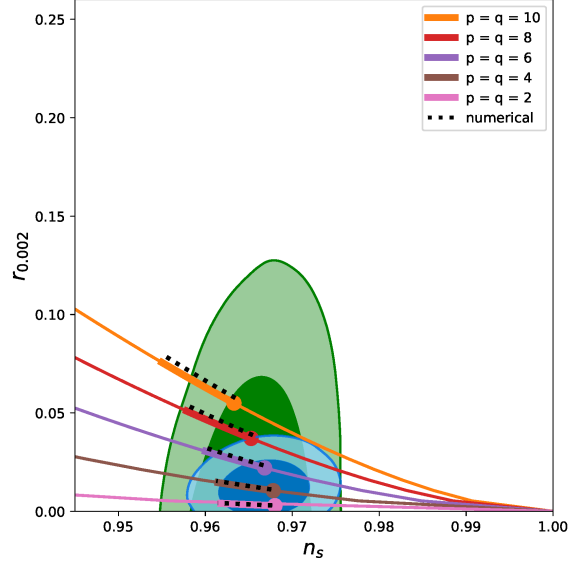
In the case of $p = 2$, Eqs. (2.174) and (2.175) yield

$$n_s - 1 \simeq -\frac{r}{8} \left[1 + \frac{q}{2x^2} \left(\frac{q}{2} - 1 \right)^{-2} \right] \simeq -\frac{r}{8}, \quad (2.217)$$

where $x \gg 1$ as we are dealing with chaotic type models.



(a) Models with $p < q$ and $F_{,\phi}^2 \ll F$.



(b) Models with $p = q$ and $F_{,\phi}^2 \gg F$. Thick sections of curves show $\hat{N}_* \in [50; 60]$ range, where large points correspond to the $\hat{N}_* = 60$ value. The black dotted curves represent the results from exact numerical simulations in the same $\hat{N}_*$ range.

Figure 2.4: Comparison of observational constraints of inflationary parameters n_s and r with chaotic type ($x_* \gg 1$) models of generalised induced gravity theories. The contours are the same as in Fig. 2.3.

In the case of $p \neq 2$:

$$n_s - 1 \simeq -\frac{r}{16} \left(3 - \frac{q-2}{q-p} \right). \quad (2.218)$$

Notice then that we can write a unified expression for all values of p as

$$n_s - 1 \simeq \frac{3(p-2) - 2(q-2)}{16(q-p)} r. \quad (2.219)$$

If $q = 2$, then

$$n_s - 1|_{q=2} \simeq -\frac{3}{16} r, \quad (2.220)$$

which is the same for any $p < q$. This is the curve which gives the smallest r value for a given n_s . However, the parameter range in this case is excluded by observations (see Fig. 2.4a).

In the case of $p = q$, ϵ_U is shown in Fig. 2.5 for several values of p and ξ . To the lowest

order in x^{-2} , we can approximate it as

$$\epsilon_U \simeq \frac{4p^2 \xi^{2/p} x^{p-2}}{2 + 3p^2 \xi^{2/p} x^{p-2}} \frac{1}{x^4} . \quad (2.221)$$

In this case, SR inflation can be realised in both regimes, for $F_{,\phi}^2 \ll F$ and for $F_{,\phi}^2 \gg F$. The expression for ϵ_U can be simplified in both cases such that

$$\epsilon_U \simeq \begin{cases} 2p^2 \xi^{2/p} x^{p-6} & \text{for } F_{,\phi}^2 \ll F \\ \frac{4}{3} x^{-4} & \text{for } F_{,\phi}^2 \gg F \end{cases} . \quad (2.222)$$

If we consider that the cosmological scales exit the horizon in the $F_{,\phi}^2 \ll F$ regime, then we can use Eq. (2.163) again to calculate η_{FV} and obtain

$$\eta_{FV} \simeq \begin{cases} \frac{6-p}{2p} x^2 \epsilon_U & \text{for } p \neq 6 \\ \frac{1}{3} \epsilon_U & \text{for } p = 6 \end{cases} , \quad (2.223)$$

where we used Eq. (2.222) too. In the case of $p = 6$, we find the relation (see Eqs. (2.174) and (2.175))

$$n_s - 1 \simeq -\frac{r}{6} . \quad (2.224)$$

These values lie outside the observationally allowed region.

In the case of $p \neq 6$:

$$n_s - 1 \simeq -\frac{r}{8} \left(1 + \frac{6-p}{2p} x^2 \right) \simeq \frac{p-6}{16p} x^2 r , \quad (2.225)$$

where $p < 6$ in order to get a redshifted scalar spectral index (remember that $p > 0$). Since $x_* \gg 1$, then the constraint $1 - n_s \simeq 0.01$ is violated unless $r \ll 1$. We are only left with the region in which $F_{,\phi}^2 \gg F$. In this limit, ϵ_U becomes (see Eq. (2.222))

$$\epsilon_U \simeq \frac{4}{3x^4} , \quad (2.226)$$

which does not depend on p . To determine η_{FV} we use Eq. (2.169) such that

$$\eta_{FV} \simeq \frac{4}{p} \left(\sqrt{\frac{\epsilon_U}{3}} + \epsilon_U \right) , \quad (2.227)$$

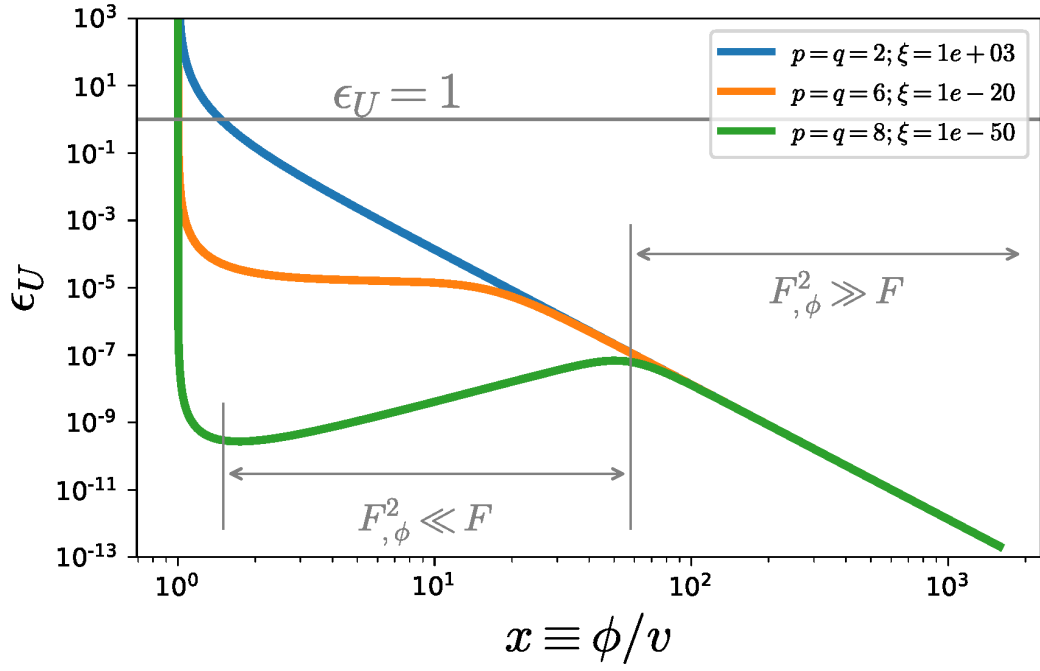


Figure 2.5: An illustration of the SR parameter ϵ_U for chaotic type models with $p = q$. The regions that correspond to $F^2_{,\phi} \ll F$ and $F^2_{,\phi} \gg F$ are shown for the $p = 8$ model. ϵ_U can be approximated as in Eq. (2.222) in those regions.

which leads to

$$n_s - 1 \simeq -\frac{2}{p}\sqrt{\frac{r}{3}} - \left(\frac{4}{p} + 1\right)\frac{r}{8}. \quad (2.228)$$

As can be seen in Fig. 2.4b, these models cross the BICEP/Keck region with $p = q$ values up to 10.

However, we need to determine if this happens at the right number of e-folds before the end of inflation. We then write the SR formula of the number of e-folds in the Jordan frame (taking $dN \equiv -Hdt$; see Eq. (1.108)), which reads

$$N(\phi) \simeq \int_{\phi}^{\phi_f} \left(2\frac{F_{,\tilde{\phi}}}{F} - \frac{V_{,\tilde{\phi}}}{V}\right)^{-1} \mathcal{K} d\tilde{\phi}, \quad (2.229)$$

where the SR equation (2.140) was used. In terms of x , the equation above becomes

$$N(x_*) \simeq \frac{1}{8\xi^{2/p}p} \int_{x_f}^{x_*} \left(2 + 3p^2\xi^{2/p}x^{p-2}\right) (x^2 - 1) \frac{d(x^2)}{x^p}, \quad (2.230)$$

which can be readily integrated, leading to

$$N(x_*) \simeq \begin{cases} \frac{3}{4} \left(\frac{1}{6\xi} + 1\right) \left[x_*^2 - x_f^2 + \ln\left(\frac{x_f^2}{x_*^2}\right)\right] & \text{for } p = 2 \\ \frac{1}{16\xi^2} \left[x_*^{-2} - x_f^{-2} - \ln\left(\frac{x_f^2}{x_*^2}\right)\right] + \frac{3}{2} \left[x_*^2 - x_f^2 + \ln\left(\frac{x_f^2}{x_*^2}\right)\right] & \text{for } p = 4 \\ \frac{1}{2p\xi^{2/p}} \left(\frac{x_*^{4-p} - x_f^{4-p}}{4-p} - \frac{x_*^{2-p} - x_f^{2-p}}{2-p}\right) + \frac{3p}{8} \left[x_*^2 - x_f^2 + \ln\left(\frac{x_f^2}{x_*^2}\right)\right] & \text{otherwise} \end{cases} \quad (2.231)$$

Since we assume that $F_{,\phi}^2 \gg F$ when cosmological scales exit the horizon, this translates into $4\xi \gg 1$ in the case of $p = 2$ and $p^2\xi^{2/p}x_*^{p-2} \gg 1$ for other values of p . Thus, we can safely neglect the $\left(p^2\xi^{2/p}x_*^{p-2}\right)^{-1}$ terms from the expression above (as compared to the $x_*^2 \gg 1$ term or 1). This, however, *does not* imply that we can neglect the $\left(p^2\xi^{2/p}x_f^{p-2}\right)^{-1}$ terms.⁸ As we shall see soon, the observational constraints force ξ to be large and we can neglect the $\left(p^2\xi^{2/p}x_f^{p-2}\right)^{-1}$ terms as well.

Making use of $x_* \gg x_f$ (see after Eq. (1.319)), we can approximate N_* by

$$N_* \simeq \frac{3p}{8} \left[x_*^2 + \ln\left(\frac{x_f^2}{x_*^2}\right)\right]. \quad (2.232)$$

8. In the models with $p > 2$, ξ is allowed to be small and still satisfy the $p^2\xi^{2/p}x_*^{p-2} \gg 1$ condition, in principle.

Then, the number of e-folds in the Einstein frame reads (see Eq. (2.181))

$$\hat{N}_* \simeq \frac{p}{8} \left[3x_*^2 + 5 \ln \left(\frac{x_f^2}{x_*^2} \right) \right]. \quad (2.233)$$

As we can see, the logarithmic term in Eq. (2.232) is of the same order of magnitude as the $\int \theta dN$ term, which we called the *e-fold shift number* after Eq. (2.181). The logarithmic term cannot be neglected at first, as it can introduce a sizeable contribution to $\hat{N}_*$. Notice that one might have been tempted to remove θ_1 from Eq. (2.179) and then $d\hat{N} \simeq dN$ during SR as $|\theta| \ll 1$. But this does not hold throughout inflation and $\hat{N}$ is obtained by integration. We must then solve a transcendental equation for x_* .

Before that however, we must find x_f . With the assumption $p^2 \xi^{2/p} \gg 1$ (or $\xi \gg p^{-p}$), x_f can be easily computed from Eq. (2.201). Taking $\epsilon_U(x_f) = 1$ and $q = p$, we find

$$x_f \simeq \sqrt{1 + \frac{2}{\sqrt{3}}} \simeq 1.35. \quad (2.234)$$

Now, x_* is determined from Eq. (2.233), which shall correspond to $\hat{N}_* = 50$ and $\hat{N}_* = 60$ e-folds before the end of inflation (see above Eq. (1.280)). Then, plugging these values of x_* into Eqs. (2.226), (2.227) and (2.228), one obtains $n_s(r)$ and we are able to compare the results with observations. This is shown in Fig. 2.4b. As can be seen, only the cases of $p = q = 2$, $p = q = 4$ and $p = q = 6$ are allowed.

2.3.4 Slow-Roll Inflation for $x \simeq 1$

We now look for observationally acceptable models with $x_* \simeq 1$; i.e. a regime which is neither hilltop-like nor chaotic-like. To that end, we define

$$\delta \equiv 1 - x^2, \quad (2.235)$$

where $|\delta| \ll 1$. Eq. (2.197) reads

$$U_{,x} \simeq -2U \frac{q - (q - p)\delta}{\left(1 - \frac{1}{2}\delta\right)\delta}. \quad (2.236)$$

If $x \lesssim 1$, δ is positive and $U_{,x}$ is negative. On the other hand, $x \gtrsim 1$ means that δ is negative and $U_{,x}$ is positive. Therefore, the gravitational force is *always* directed towards

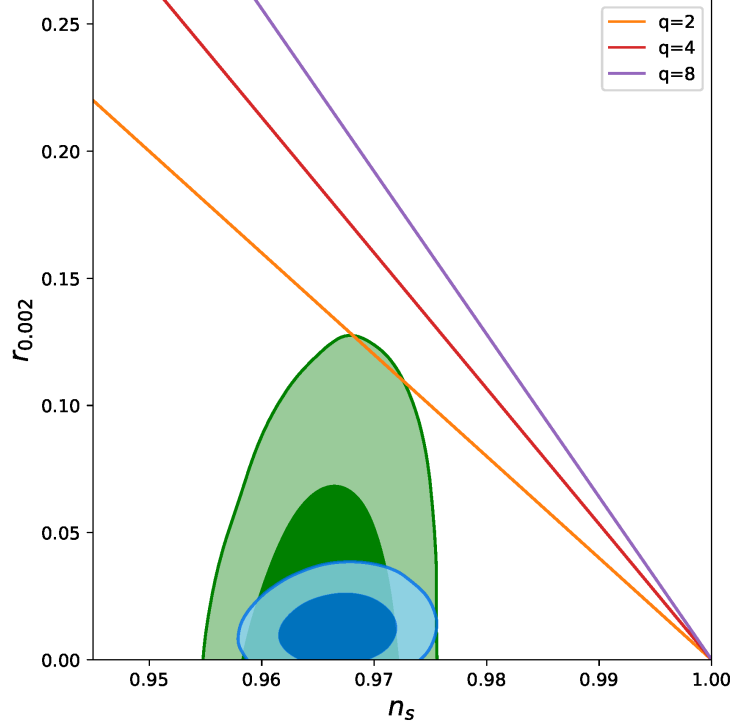


Figure 2.6: Comparison of observational constraints with $x_* \simeq 1$ models. The contours are the same as in Fig. 2.3.

the minimum of the potential and the constraints in Eqs. (2.198) and (2.199) do not apply.

Taking the first two terms in the expansion of ϵ_U in terms of δ^{-1} (see Eq. (2.201)), we have

$$\epsilon_U \simeq \frac{4p^2\xi^{2/p}}{2 + 3p^2\xi^{2/p}} \left(\frac{q}{p}\right)^2 \frac{1}{\delta^2} . \quad (2.237)$$

Imposing $\epsilon_U \ll 1$, we find

$$p^2\xi^{2/p} \ll 1 , \quad (2.238)$$

i.e. $F_{,\phi}^2 \ll F$. Then, Eq. (2.237) becomes

$$\epsilon_U \simeq \frac{2q^2\xi^{2/p}}{\delta^2} . \quad (2.239)$$

Since $F_{,\phi}^2 \ll F$, Eq. (2.163) can be used once more, and one obtains

$$\eta_{FV} \simeq \frac{4q\xi^{2/p}}{\delta^2} . \quad (2.240)$$

		$F_{,\phi_*}^2 \ll F_*$	$F_{,\phi_*}^2 \gg F_*$
Hilltop-like ($x_* \ll 1$)		$n_s - 1 \simeq \frac{2-3p}{16p}r$	No slow-roll
Chaotic-like ($x_* \gg 1$)	$q > p$	$n_s - 1 \simeq \frac{3(p-2)-2(q-2)}{16(q-p)}r$	No slow-roll
	$q = p = 6$	$n_s - 1 \simeq -\frac{r}{6}$	$n_s - 1 \simeq -\frac{2}{p}\sqrt{\frac{r}{3}} - \left(\frac{4}{p} + 1\right)\frac{r}{8}$
	$q = p \neq 6$	$n_s - 1 \simeq \frac{p-6}{16p}x_*^2r$	
$x_* \simeq 1$		$n_s - 1 \simeq -\frac{q+2}{8q}r$	No slow-roll

Table 2.1: Summary of observational predictions of the generalised induced gravity inflation models in different regimes. In this table, ‘no slow-roll’ signifies the absence of the $\epsilon_U \ll 1$ region. The only models that are compatible with observations are chaotic type models with $p = q \leq 6$ in the regime $F_{,\phi_*}^2 \gg F_*$ (see Fig. 2.4b).

Using Eqs. (2.174) and (2.175), one arrives at the following relation between n_s and r :

$$n_s - 1 \simeq -\frac{q+2}{8q}r, \quad (2.241)$$

which is independent of p . Notice that the smallest value of r for a given n_s is achieved when $q = 2$. As we see in Fig. 2.6, the predictions lie outside the 2σ contour of the BICEP/Keck results even for the $q = 2$ value.

2.3.5 Concluding Remarks of Section 2.3

In Sec. 2.3, we have studied the generalisation of the induced gravity models of inflation and learnt how to compute the main spectral properties in terms of Jordan frame variables applying the formalism developed previously. We have outlined the results of this section in Table 2.1. We found that, among the many regions where SR inflation could be realised, only the case of $p = q = 2, 4$, and 6 in the chaotic type regime (where $|\phi_*| \gg |v|$) falls within the 2σ region of the newest BICEP/Keck results.

We carried out this analysis because the Einstein frame provides a way to interpret inflation with which we are familiar, and we know the essential observables needed to test predictions from the inflation model under consideration. The results from the Jordan and Einstein frames have been compared on the same time slice in Fig. 2.7. In Figs. 2.7a and 2.7b, it can be seen the excellent agreement between the first two Einstein frame Hubble-flow

parameters, $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$, and their expressions in terms of Jordan frame SR parameters before the end of inflation. We conclude the same about the exact simulations of inflation observables and their approximate expressions in terms of Jordan frame parameters by looking at Figs. 2.7c, 2.7d and 2.7e. The time variable t in both frames has been computed by simply integrating (see Eq. (1.108))

$$t_{\text{end}} - t = \int_0^N \frac{dN}{H} . \quad (2.242)$$

As was stated earlier, the modified gravity models are essentially formulated in the Jordan frame and, similarly to the homogeneous quantities, we should analyse the dynamics of the linear perturbations *in the Jordan frame* too, which shall be the goal of the next section. There have been works that studied cosmological perturbations in the Jordan frame already (see Refs. [120, 244–250]), as well as the equivalence between the curvature perturbation in the two frames [202, 251–255]. We will derive the corresponding perturbation equations in the Jordan frame and show how they transform under conformal transformations. Additionally, the conformal invariance of the comoving curvature perturbation in particular will be examined given the notable relevance of this perturbation (discussed in Sec. 1.3.4) in the predictions extracted from the inflationary paradigm.

2.4 Linear Perturbations in Scalar-Tensor Gravity

2.4.1 Perturbed Effective Energy-Momentum Tensor

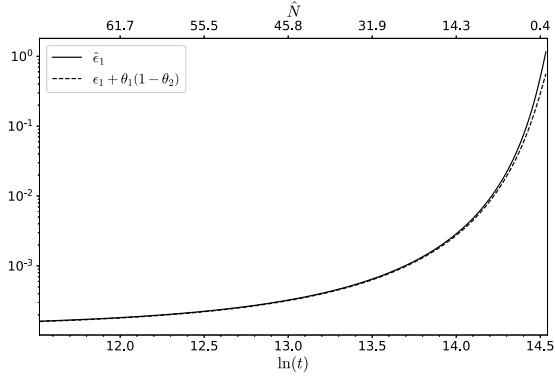
In $f(\phi, R)$ theories of gravity, we saw that one can move the modified gravity terms to the RHS of the metric field equations such that an effective energy-momentum tensor $\tilde{T}_{\mu\nu}$ is defined (see Eq. (2.8))

$$\tilde{T}_{\mu\nu} = \frac{1}{F} \left\{ \partial_\mu \phi \partial_\nu \phi + \nabla_\mu \partial_\nu F - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V + \square F + \frac{1}{2} (RF - f) \right] \right\}, \quad (2.243)$$

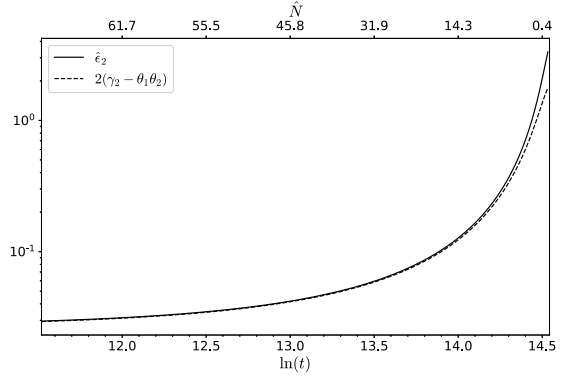
where we have neglected matter fields and we have considered a canonical scalar field such that $\mathcal{K} = 1$.

Assuming the perturbed metric in Eq. (1.155), the scalar components of the linear perturbation of the effective energy-momentum tensor, $\delta \tilde{T}^\mu_\nu$, are

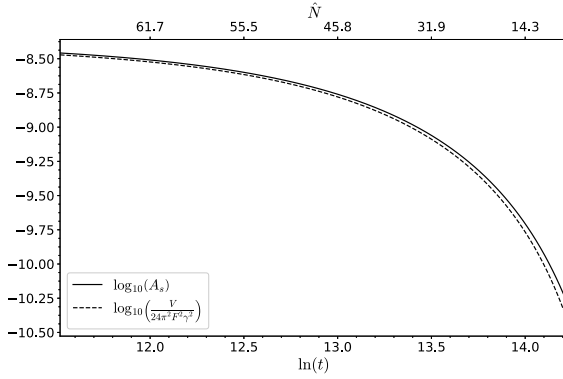
$$\delta \tilde{T}^0_0 = \frac{1}{F} \left[-\dot{\phi} \delta \dot{\phi} + A \dot{\phi}^2 - V_{,\phi} \delta \phi + \frac{1}{2} f_{,\phi} \delta \phi - a^{-2} \partial_i \partial^i \left(\delta F - \dot{F} \chi \right) + 3H \delta \dot{F} - \right.$$



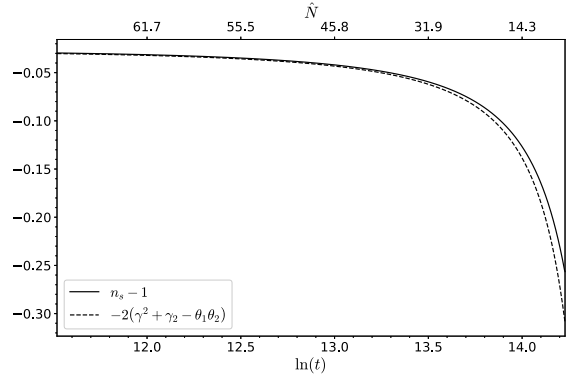
(a) Numerical check of Eq. (2.154).



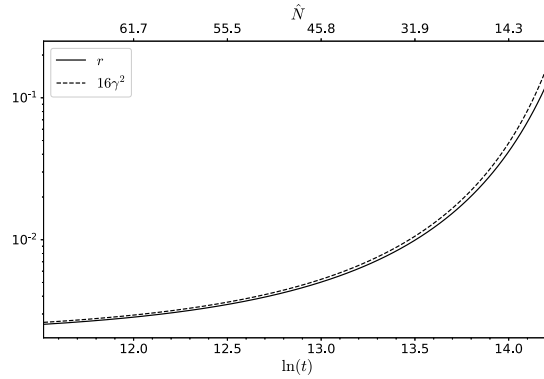
(b) Numerical check of Eq. (2.155).



(c) Numerical check of Eq. (2.173).



(d) Numerical check of Eq. (2.176).



(e) Numerical check of Eq. (2.177).

Figure 2.7: Comparison of exact, numerical Einstein frame results and approximate relations derived in this section. We used the model defined in Eqs. (2.192) and (2.193) with $p = q = 2$. These functions are plotted against the time t , which enumerates spatial slices and are shown on the lower horizontal axis. The upper horizontal axis shows the corresponding e-fold number $\hat{N}(t)$ in the Einstein frame for the reference.

$$-3\dot{F} \left(\dot{\psi} + 2HA \right) + \frac{1}{2} \left(\tilde{\rho} + 3\tilde{P} \right) \delta F \Big], \quad (2.244)$$

$$\delta \tilde{T}^0_i = \partial_i \left[\frac{1}{F} \left(-\dot{\phi} \delta \phi - \delta \dot{F} + H \delta F + \dot{F} A \right) \right], \quad (2.245)$$

$$\begin{aligned} \delta \tilde{T}^i_j = \frac{1}{F} \left[\dot{\phi} \delta \phi - A \dot{\phi}^2 - V_{,\phi} \delta \phi + \frac{1}{2} f_{,\phi} \delta \phi + \delta \ddot{F} + 2H \delta \dot{F} - \left(4HA + 2\dot{\psi} + \dot{A} \right) \dot{F} - 2A \ddot{F} - \right. \\ \left. - a^{-2} \partial_k \partial^k \left(\delta F - \dot{F} \chi \right) - \frac{1}{2} \left(\tilde{\rho} - \tilde{P} \right) \delta F \right] \delta^i_j + a^{-2} \partial^i \partial_j \left(\frac{\delta F - \dot{F} \chi}{F} \right), \end{aligned} \quad (2.246)$$

where $\tilde{\rho}$ and $\tilde{P}$ are the homogeneous effective energy density and pressure

$$\tilde{\rho} = \frac{1}{F} \left[\frac{1}{2} \dot{\phi}^2 + V + \frac{1}{2} (RF - f) - 3H \dot{F} \right], \quad (2.247)$$

$$\tilde{P} = \frac{1}{F} \left[\frac{1}{2} \dot{\phi}^2 - V - \frac{1}{2} (RF - f) + \ddot{F} + 2H \dot{F} \right], \quad (2.248)$$

respectively. According to Eq. (1.184), we identify the non-vanishing spatial components of the anisotropic pressure

$$\Pi^i_j = a^{-2} \left(\partial^i \partial_j - \frac{1}{3} \delta^i_j \partial_k \partial^k \right) \left(\delta F - \dot{F} \chi \right), \quad (2.249)$$

Π^i_j being symmetric and traceless, as was commented already (see Eq. (1.165)). It is also gauge-invariant since

$$\delta F \rightarrow \delta F - \dot{F} \alpha, \quad (2.250)$$

and χ transforms as Eq. (1.162) (this had been mentioned already; see Eq. (1.187)).

The effective energy density, pressure and scalar momentum perturbations are given by (see Eqs. (1.182), (1.183) and (1.184))

$$\begin{aligned} \delta \tilde{\rho} = \frac{1}{F} \left[\dot{\phi} \delta \phi - A \dot{\phi}^2 + V_{,\phi} \delta \phi - \frac{1}{2} f_{,\phi} \delta \phi + a^{-2} \partial_i \partial^i \left(\delta F - \dot{F} \chi \right) - 3H \delta \dot{F} + 3\dot{F} \left(\dot{\psi} + 2HA \right) - \right. \\ \left. - \frac{1}{2} \left(\tilde{\rho} + 3\tilde{P} \right) \delta F \right], \end{aligned} \quad (2.251)$$

$$\begin{aligned} \delta \tilde{P} = \frac{1}{F} \left[\dot{\phi} \delta \phi - A \dot{\phi}^2 - V_{,\phi} \delta \phi + \frac{1}{2} f_{,\phi} \delta \phi + \delta \ddot{F} + 2H \delta \dot{F} - \left(4HA + 2\dot{\psi} + \dot{A} \right) \dot{F} - 2A \ddot{F} - \right. \\ \left. - \frac{2}{3} a^{-2} \partial_i \partial^i \left(\delta F - \dot{F} \chi \right) - \frac{1}{2} \left(\tilde{\rho} - \tilde{P} \right) \delta F \right], \end{aligned} \quad (2.252)$$

$$\tilde{\Psi} = \frac{1}{F} \left(-\dot{\phi} \delta \phi - \delta \dot{F} + H \delta F + \dot{F} A \right), \quad (2.253)$$

respectively. Notice that $\delta\tilde{\rho} \neq \delta\rho/F$ and $\delta\tilde{P} \neq \delta P/F$. Rather, we have

$$\delta\tilde{\rho} = \frac{1}{F} \left(\delta\rho - \frac{\rho}{F} \delta F \right), \quad (2.254)$$

$$\delta\tilde{P} = \frac{1}{F} \left(\delta P - \frac{P}{F} \delta F \right), \quad (2.255)$$

$$\tilde{\Psi} = \frac{1}{F} \Psi. \quad (2.256)$$

The latter equality is reasonable as the scalar momentum perturbation has no background analogue.

2.4.2 Perturbation Equations in $f(\phi, R)$ Gravity

Since the conservation of energy-momentum is now formulated in terms of the *effective* energy-momentum tensor (due to Bianchi's identity; see Eq. (2.13)), we have

$$\nabla_\mu \tilde{T}^{\mu\nu} = \frac{1}{F} \nabla_\mu T^{\mu\nu} - \frac{1}{F^2} T^{\mu\nu} \partial_\mu F = 0, \quad (2.257)$$

and the energy density and momentum perturbation conservation equations are the same as Eqs. (1.198) and (1.199), but with perturbation quantities of the energy-momentum tensor wearing tildes [120]

$$\delta\dot{\tilde{\rho}} + 3H(\delta\tilde{\rho} + \delta\tilde{P}) = -(\tilde{\rho} + \tilde{P}) \left[a^{-2} \partial_i \partial^i \left(\frac{\tilde{\Psi}}{\tilde{\rho} + \tilde{P}} + \chi \right) - 3\dot{\psi} \right], \quad (2.258)$$

$$\dot{\tilde{\Psi}} + 3H\tilde{\Psi} + \delta\tilde{P} + (\tilde{\rho} + \tilde{P}) A = -\frac{2}{3} a^{-2} \partial_i \partial^i \sigma, \quad (2.259)$$

where the momentum perturbation conservation equation includes the contribution from the anisotropic pressure, σ being defined as

$$\sigma \equiv \frac{\delta F - \dot{F}\chi}{F} \equiv \frac{\delta F^\chi}{F}. \quad (2.260)$$

Hence $F\sigma$ is the perturbation δF on shear-free slicings and, in consequence, σ is gauge-invariant.

Due to Eq. (2.257), one must also insert quantities with tildes in the background conti-

nuity equation (1.27), such that

$$\dot{\rho} - \frac{\dot{F}}{F}\rho = -3H(\rho + P). \quad (2.261)$$

This means that $\delta\tilde{\rho}$ and $\tilde{\Psi}$ transform as

$$\delta\tilde{\rho} \rightarrow \delta\tilde{\rho} - \frac{1}{F} \left(\dot{\rho} - \frac{\dot{F}}{F}\rho \right) \alpha = \delta\tilde{\rho} + 3H(\tilde{\rho} + \tilde{P}) \alpha, \quad (2.262)$$

$$\tilde{\Psi} \rightarrow \tilde{\Psi} + (\tilde{\rho} + \tilde{P}) \alpha, \quad (2.263)$$

under gauge transformations. Therefore

$$\delta\tilde{\rho}^\Psi \equiv \delta\tilde{\rho} - 3H\tilde{\Psi}, \quad (2.264)$$

is gauge-invariant (we write ‘ Ψ ’ in $\delta\tilde{\rho}^\Psi$ because $\tilde{\Psi} = 0$ implies $\Psi = 0$ and vice versa). Notice that $\delta\rho^\Psi$, defined in Eq. (1.201), is *no longer* gauge-invariant:

$$\delta\rho^\Psi = \delta\rho - 3H\Psi \rightarrow \delta\rho^\Psi - \frac{\dot{F}}{F}\rho\alpha. \quad (2.265)$$

Since

$$\frac{\tilde{\Psi}}{\tilde{\rho} + \tilde{P}} = \frac{\Psi}{\rho + P}, \quad (2.266)$$

the comoving curvature perturbation is defined in the same way as in Eq. (1.189) (i.e. $\tilde{\mathcal{R}} = \mathcal{R}$). On the other hand, when we introduced ζ (or $-\psi^{\delta\rho}$) in Eq. (1.190) we made use of the continuity equation; otherwise the definition would be

$$\zeta \equiv -\psi - \frac{H}{\dot{\rho}}\delta\rho. \quad (2.267)$$

We see that both, ζ and $\tilde{\zeta}$ (which is defined in the same way as ζ in Eq. (2.267) but with tildes) are gauge-invariant. However, they are not equal and one must resort to $\tilde{\zeta}$ to write the analogue of Eq. (1.202), which is

$$\delta\tilde{\rho}^\Psi = 3(\tilde{\rho} + \tilde{P})(\mathcal{R} + \tilde{\zeta}). \quad (2.268)$$

Regarding the energy and momentum constraint equations (see Eqs. (1.193) and (1.194), respectively), we only have to replace the perturbation quantities of the energy-momentum

tensor by the ones with tildes (and also remember that M_{Pl}^2 is time dependent and such a role is played by F) [120]. Then

$$H\kappa + a^{-2}\partial_i\partial^i\psi = \frac{1}{2}\delta\tilde{\rho} \ , \quad (2.269)$$

$$HA + \dot{\psi} = -\frac{1}{2}\tilde{\Psi} \ . \quad (2.270)$$

Using Eqs. (2.269) and (2.270) (and the definition of κ in App. A), one arrives at

$$a^{-2}\partial_i\partial^i\psi^\chi = \frac{1}{2}\delta\tilde{\rho}^\Psi \ , \quad (2.271)$$

and plugging this into Eq. (2.268), we obtain (compare with Eq. (1.206))

$$\mathcal{R} + \tilde{\zeta} = -\frac{1}{3\dot{H}}a^{-2}\partial_i\partial^i\psi^\chi = \frac{1}{3H^2}a^{-2}\partial_i\partial^i\left(\frac{\psi^\chi}{\epsilon_1}\right) \ , \quad (2.272)$$

ϵ_1 being the first Hubble-flow parameter *in the Jordan frame*.

Eqs. (2.258) and (2.259), along with the momentum constraint equation, yield (compare with Eq. (1.200))

$$a^{-2}\partial_i\partial^i\left(\chi + \frac{\Psi}{\rho + P}\right) = -\frac{1}{a^3(\tilde{\rho} + \tilde{P})}\left(a^3\delta\tilde{\rho}^\Psi\right)^\bullet + \frac{2H}{\tilde{\rho} + \tilde{P}}a^{-2}\partial_i\partial^i\sigma \ . \quad (2.273)$$

Using Eq. (2.268), Eq. (2.273) becomes

$$\frac{1}{3}a^{-2}\partial_i\partial^i\left(\frac{\Psi^\chi}{\rho + P}\right) = -\dot{\mathcal{R}} - \dot{\tilde{\zeta}} + 3H\left(\mathcal{R} + \tilde{\zeta}\right)\frac{\dot{\tilde{P}}}{\tilde{\rho}} + \frac{2H}{3(\tilde{\rho} + \tilde{P})}a^{-2}\partial_i\partial^i\sigma \ , \quad (2.274)$$

where Ψ^χ was defined in Eq. (1.210). We notice that

$$\frac{\Psi^\chi}{\rho + P} = \frac{\tilde{\Psi}^\chi}{\tilde{\rho} + \tilde{P}} = \frac{1}{H}(\psi^\chi - \mathcal{R}) \ , \quad (2.275)$$

and using Eq. (2.272), we obtain

$$\frac{\dot{\psi}^\chi}{H} + (1 + \epsilon_1)\psi^\chi = \sigma + \epsilon_1\mathcal{R} \ , \quad (2.276)$$

up to a constant of space which vanishes by construction (compare with Eq. (1.212) and see

footnote 9); or (see Eq. (1.70))

$$\frac{2}{3} \frac{\dot{\psi}^\chi}{H} + \left(\frac{5}{3} + w \right) \psi^\chi = \frac{2}{3} \sigma + (1 + w) \mathcal{R} . \quad (2.277)$$

Eq. (2.276) is very similar to Eq. (1.212), although we have the contribution from the anisotropic pressure which, in this case, might have important effects on the dynamics of ψ^χ , irrespective of the scales. It is also clear that ϵ_1 refers to the first Hubble-flow parameter in the Jordan frame, as was pointed out earlier below Eq. (2.272). On the other hand, one could have obtained Eq. (2.276) by noticing that Eq. (2.273) can be written as (compare with Eq. (1.209))

$$\left(\delta \tilde{\rho}^\Psi \right)^\bullet + 3H \delta \tilde{\rho}^\Psi = -a^{-2} \partial_i \partial^i \left(\tilde{\Psi}^\chi - 2H\sigma \right) , \quad (2.278)$$

and then by using Eqs. (2.271) and (2.275) to replace $\delta \tilde{\rho}^\Psi$ and $\tilde{\Psi}^\chi$, respectively.

The presence of a non-vanishing anisotropic pressure yields the following ADM propagation equation (see Eq. (1.195)) [120]:

$$\dot{\chi} + H\chi - A + \psi = \sigma . \quad (2.279)$$

Bearing in mind the definitions of the Bardeen potentials (Eqs. (1.164) and (1.205)), the equation above becomes

$$\psi^\chi - A^\chi = \sigma . \quad (2.280)$$

The Bardeen potentials are therefore equal in GR but differ when gravity modifications of the $f(\phi, R)$ type are regarded. We can define however the following perturbation variables:

$$\hat{\psi}^\chi \equiv \psi^\chi - \frac{1}{2} \sigma , \quad (2.281)$$

$$\hat{A}^\chi \equiv A^\chi + \frac{1}{2} \sigma , \quad (2.282)$$

such that one recovers Einstein's gravity equation. We shall go over this transformation of ψ^χ and A^χ (and of other perturbation variables) later on. Also, we use ' $\hat{\chi}$ ' rather than ' χ ' to denote these gauge-invariant quantities because we refer to the Einstein frame. More details will be discussed later.

As in Eq. (1.208), we may calculate $\dot{\mathcal{R}}$ as well (using the momentum constraint and the

momentum perturbation conservation equations, as we know):

$$\dot{\mathcal{R}} = \frac{H}{\tilde{\rho} + \tilde{P}} \left(\delta\tilde{P}_{\text{nad}} + \frac{\dot{\tilde{P}}}{\tilde{\rho}} \delta\tilde{\rho}^\Psi + \frac{2}{3} a^{-2} \partial_i \partial^i \sigma \right), \quad (2.283)$$

where $\delta\tilde{P}_{\text{nad}}$ is defined as (see Eq. (1.189))

$$\delta\tilde{P}_{\text{nad}} \equiv \delta\tilde{P} - \frac{\dot{\tilde{P}}}{\tilde{\rho}} \delta\tilde{\rho}. \quad (2.284)$$

Leaving the tildes aside, the difference between this equation and Eq. (1.208) comes from the anisotropic pressure contribution. On superhorizon scales however, this contribution is expected not to be important. Thus, $\mathcal{R}$ becomes constant on the superhorizon regime if the non-adiabatic pressure vanishes, same as in GR. This was proven to hold non-perturbatively in Ref. [133] (assuming that the *separate universes approach* holds, which we shall not discuss in this dissertation).

δF obeys an equation that we can easily find by first taking the trace of Eq. (2.7) (remember that we neglect matter fields and take $\mathcal{K} = 1$), such that

$$\square F = \frac{1}{3} (2f - FR - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4V). \quad (2.285)$$

The background equation is

$$\ddot{F} + 3H\dot{F} = -\frac{1}{3} (2f - FR + \dot{\phi}^2 - 4V), \quad (2.286)$$

and the perturbation equation at linear order (see App. A) [120]

$$\begin{aligned} \delta\ddot{F} + 3H\delta\dot{F} - \left(a^{-2} \partial_i \partial^i + \frac{1}{3} R \right) \delta F + \frac{2}{3} \dot{\phi} \delta\dot{\phi} - \frac{4}{3} \left(V_{,\phi} - \frac{1}{2} f_{,\phi} \right) \delta\phi = \\ = \left(2\ddot{F} + 3H\dot{F} + \frac{2}{3} \dot{\phi}^2 \right) A + \dot{F} (\dot{A} - \kappa) - \frac{1}{3} F \delta R. \end{aligned} \quad (2.287)$$

If we plug this equation into Eq. (2.252) to replace $\delta\ddot{F}$, we obtain $\delta R = \delta\tilde{\rho} - 3\delta\tilde{P}$, as expected since $R = -\tilde{T} = -\tilde{T}_0^0 - \tilde{T}_i^i$ (see below Eq. (1.197)). Using this equation of δR , the one given in App. A, Eq. (A.32), which holds in $f(\phi, R)$ gravity, and the energy constraint equation, Eq. (2.269), we can obtain the analogue of Raychaudhuri's equation in GR, Eq. (1.196),

which reads [120]

$$\dot{\kappa} + 2H\kappa - \left(a^{-2}\partial_i\partial^i + 3\dot{H}\right)A = -\frac{1}{2}\left(\delta\tilde{\rho} + 3\delta\tilde{P}\right). \quad (2.288)$$

This equation will be an essential part of the derivation of the equation analogous to the Mukhanov-Sasaki equation in GR, Eq. (1.233). We will derive such an equation in the next section by considering $F(\phi)R$ theories of gravity, which have been the object of our discussion in the context of SR inflation in the Jordan frame in the preceding sections.

2.4.3 Mukhanov-Sasaki Equation in $F(\phi)R$ Gravity

Let us assume that $F = F(\phi)$. In that case, the uniform- ϕ slicing ($\delta\phi = 0$) coincides with the uniform- F slicing ($\delta F = 0$). Applying this to $\delta\tilde{P}$ and $\delta\tilde{\rho}$ we obtain⁹

$$\delta\tilde{\rho} + 3\delta\tilde{P} = \frac{1}{F} \left[3\dot{F} \left(\frac{1}{3}\kappa - HA - \dot{A} \right) - 2A \left(3\ddot{F} + 2\dot{\phi}^2 \right) \right]. \quad (2.289)$$

Inserting this in Eq. (2.288) and using Eq. (2.111), we get

$$\dot{\kappa} + (2 + \theta_1)H\kappa - a^{-2}\partial_i\partial^i A - 3H\theta_1\dot{A} + H^2 \left(\frac{\dot{\phi}^2}{H^2 F} - 3\epsilon_1 - 9\theta_1 \right) A = 0, \quad (2.290)$$

where θ_1 was defined in Eq. (2.107). On the other hand, the momentum constrain equation (2.270) yields

$$A = -\frac{1}{1 + \theta_1} \frac{\dot{\psi}}{H}, \quad (2.291)$$

and the energy constraint equation (2.269)

$$\kappa = \frac{1}{H(1 + \theta_1)} \left[\frac{1}{2F} \left(3H\dot{F} - \dot{\phi}^2 \right) A - a^{-2}\partial_i\partial^i \psi \right]. \quad (2.292)$$

In terms of ψ and its time derivatives, we have

$$\kappa = -\frac{1}{H(1 + \theta_1)} \left[\left(3H\theta_1 - \frac{\dot{\phi}^2}{2HF} \right) \frac{\dot{\psi}}{1 + \theta_1} + a^{-2}\partial_i\partial^i \psi \right]. \quad (2.293)$$

9. Since we are fixing $\delta\phi = \delta F = 0$, we should write the perturbation quantities as $A^{\delta\phi}$, $\psi^{\delta\phi}$, etc. However, we do not do that for the sake of simplicity.

Plugging this and Eq. (2.291) into Eq. (2.290), we obtain

$$\frac{\gamma^2}{(1+\theta_1)^2} \left(\ddot{\psi} - a^{-2} \partial_i \partial^i \psi \right) + \left\{ \frac{(\dot{\gamma}^2)}{(1+\theta_1)^2} + \frac{H\gamma^2}{(1+\theta_1)^3} [3(1+\theta_1) + 2\theta_1(1+\theta_1-\theta_2)] \right\} \dot{\psi} = 0 , \quad (2.294)$$

where γ^2 was defined in Eq. (2.119). The equation takes on the form [120]

$$\ddot{\psi} + \left[3 + 2 \left(\theta_1 + \gamma_2 - \frac{\theta_1 \theta_2}{1+\theta_1} \right) \right] H \dot{\psi} - a^{-2} \partial_i \partial^i \psi = 0 , \quad (2.295)$$

γ_2 being defined in Eq. (2.118), and (see footnote 9)

$$\psi = \psi^{\delta\phi} . \quad (2.296)$$

Eq. (2.295) can be written as

$$\frac{(1+\theta_1)^2}{a^3 F \gamma^2} \left[\frac{a^3 F \gamma^2}{(1+\theta_1)^2} \dot{\psi} \right]^{\bullet} - a^{-2} \partial_i \partial^i \psi = 0 . \quad (2.297)$$

We now introduce the gauge-invariant field (see Eq. (1.227)) [120]

$$u \equiv -z \psi^{\delta\phi} = \frac{\sqrt{F\mathcal{K}}}{1+\theta_1} a \delta\phi^{\psi} , \quad (2.298)$$

where z is given by

$$z \equiv -a \frac{\dot{\phi}}{H} \frac{\sqrt{F\mathcal{K}}}{1+\theta_1} = -a \frac{\sqrt{2F\gamma^2}}{1+\theta_1} , \quad (2.299)$$

and $\mathcal{K}$ was defined in Eq. (2.115). z is the Mukhanov variable (see Eq. (1.227)) in this $F(\phi)R$ gravity theory. The relation between $\psi^{\delta\phi}$ and $\delta\phi^{\psi}$ is (see Eq. (1.224))

$$\psi^{\delta\phi} = \frac{H}{\dot{\phi}} \delta\phi^{\psi} . \quad (2.300)$$

Eq. (2.297) then becomes Eq. (1.229) in coordinate time, and Eq. (1.231) in conformal time. In Fourier space, we obtain Eq. (1.233), which is the Mukhanov-Sasaki equation, but with z''/z now given by (compare with Eq. (1.234) and remember that primes denote derivatives

with respect to the conformal time introduced in Eq. (1.12)) [120]

$$\frac{z''}{z} = \mathcal{H}^2 \left[(1 + \theta_1 + \gamma_2) \left(2 + \theta_1 + \gamma_2 - \epsilon_1 - 2 \frac{\theta_1 \theta_2}{1 + \theta_1} \right) + \gamma_2 \gamma_3 + \frac{\theta_1 \theta_2}{1 + \theta_1} (\theta_1 + \epsilon_1 - \theta_2 - \theta_3 + 2 \frac{\theta_1 \theta_2}{1 + \theta_1}) \right], \quad (2.301)$$

where γ_2 is given in Eq. (2.120). In GR, $\theta_i = 0$ where $i = 1, 2, \dots$, and $\gamma_2 = \frac{1}{2}\epsilon_2$, so we recover the mass term of Eq. (1.234).

2.5 Comoving Curvature Perturbation in Jordan and Einstein Frames

Although the Mukhanov-Sasaki equations in $F(\phi)R$ gravity and in GR are similar, we see that, in the former case, u is no longer directly related to the *comoving* curvature perturbation $\mathcal{R}$ as in Eq. (1.227). If we set $\delta\phi = 0$ (and hence $\delta F = 0$ because $F = F(\phi)$) in Eq. (2.253), we get

$$\Psi^{\delta\phi} = \dot{F} A^{\delta\phi}, \quad (2.302)$$

meaning that, in $F(\phi)R$ gravity, the comoving slicing *does not* generally coincide with the uniform-field slicing,¹⁰ and therefore $\mathcal{R} \neq \psi^{\delta\phi}$.

Here, $\mathcal{R}$ is the curvature perturbation defined on hypersurfaces that are comoving with an effective fluid, which is the one with covariantly conserved energy and momentum (see Eq. (2.257)), and whose energy-momentum tensor does not correspond to that of the canonical scalar field (see Eq. (2.8)) in the Jordan frame. Also, it is to this perturbation, $\mathcal{R}$, that the results of Ref. [133] apply directly, as mentioned below Eq. (2.284).

We shall examine the *conformal invariance* of this gauge-invariant curvature perturbation $\mathcal{R}$ which, in contrast to that of the curvature perturbation on the uniform- ϕ slicing, $\psi^{\delta\phi}$, is an unexplored issue in the literature to the best of our knowledge.¹¹

10. In the case of $f(R)$ gravity [164], an analogous mismatch is mentioned in Ref. [202], where the uniform- F slicing does not necessarily coincide with the comoving one, F being defined as $F \equiv f_{,R}$.

11. The work developed in Sec. 2.5 has resulted in a publication; see Ref. [2].

2.5.1 Conformal Frames and Linear Perturbations

The curvature perturbation *on uniform-field slicings* or, equivalently, on those hypersurfaces that comove with the scalar field fluid, can be shown to be *invariant* under a conformal transformation and a field redefinition (see Refs. [202, 220, 253]). In order to prove this, let us see how the scalar components of the perturbed metric transform under a conformal transformation (this being introduced in Eq. (2.26)):

$$\begin{aligned} d\hat{s}^2 = & - \left(1 + 2A + 2\frac{\delta\Omega}{\Omega} \right) \Omega^2 \mathcal{N}^2 dt^2 - 2\Omega^2 \partial_i B dx^i dt + \Omega^2 a^2 \left[\left(1 - 2\psi + 2\frac{\delta\Omega}{\Omega} \right) \delta_{ij} + \right. \\ & \left. + 2\partial_i \partial_j E \right] dx^i dx^j, \end{aligned} \quad (2.303)$$

where we have considered an arbitrary lapse function $\mathcal{N}$ in contrast to Eq. (1.155), and

$$d\hat{s}^2 = (\hat{g}_{\mu\nu} + \delta\hat{g}_{\mu\nu}) dx^\mu dx^\nu \simeq \Omega^2 \left(g_{\mu\nu} + \delta g_{\mu\nu} + 2\frac{\delta\Omega}{\Omega} g_{\mu\nu} \right) dx^\mu dx^\nu, \quad (2.304)$$

$\Omega = \Omega(t)$ being the homogeneous conformal factor and $\delta\Omega = \delta\Omega(t, \mathbf{x})$ the linear perturbation. Therefore

$$A \rightarrow A + \frac{\delta\Omega}{\Omega}, \quad (2.305)$$

$$B \rightarrow \Omega^2 B, \quad (2.306)$$

$$\psi \rightarrow \psi - \frac{\delta\Omega}{\Omega}, \quad (2.307)$$

$$E \rightarrow E, \quad (2.308)$$

given how $\mathcal{N}$ and a transform (see Eqs. (2.94) and (2.95), respectively).

The linear lapse function and curvature perturbations in the Einstein frame are (remember that $\Omega^2 = F$ because we set $M_{\text{Pl}}^2 = 1$)

$$\hat{A} \equiv A + \frac{\delta F}{2F}, \quad (2.309)$$

$$\hat{\psi} \equiv \psi - \frac{\delta F}{2F}, \quad (2.310)$$

respectively. These were partially introduced in Eqs. (2.281) and (2.282) to bring Eq. (2.280) into the Einstein frame form. Considering the conformal transformation of ψ , we turn the

quantities on the RHS of Eq. (2.310) into gauge-invariant ones such that

$$\hat{\psi} = \psi^\chi - \frac{1}{2}\sigma - H(1 + \theta_1)\chi , \quad (2.311)$$

where we remind the reader the definition of σ (2.260), and that θ_1 was defined in Eq. (2.108). Hence, we have (see Eq. (2.281))

$$\hat{\psi}^{\hat{\chi}} \equiv \hat{\psi} + H(1 + \theta_1)\chi . \quad (2.312)$$

One might wonder whether this quantity, $\hat{\psi}^{\hat{\chi}}$, is gauge-invariant (it should be because the RHS is gauge-invariant). Since we know that $\chi \rightarrow \chi - \alpha$ (see Eq. (1.162)), then $\hat{\psi}$ should transform as

$$\hat{\psi} \rightarrow \hat{\psi} + H(1 + \theta_1)\alpha . \quad (2.313)$$

Let us show that this is so.

The gauge transformations that we derived after Eq. (1.156) assumed that the lapse function was equal to 1. This is no longer true in the Einstein frame given the choice of the spacetime slicing (which for convenience was $\mathcal{N} = 1$ in the Jordan frame, so the proper time and coordinate time coincide in that frame; see Sec. 2.2), and we must then derive them again.

Considering an arbitrary lapse function, as we did in the ‘hatted’ frame already (see Eq. (2.303)), we have

$$ds^2 = -\mathcal{N}^2 (1 + 2A) dt^2 - 2\partial_i B dx^i dt + a^2 [(1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j, \quad (2.314)$$

rather than Eq. (1.155). In App. B, we calculate the homogeneous geometric quantities given the FLRW background metric with the lapse function. Bearing the perturbed metric in mind, the gauge transformations now read (consider the same parametrisation of the gauge transformation vector as in Eq. (1.156))

$$A \rightarrow A - \dot{\alpha} - \frac{\dot{\mathcal{N}}}{\mathcal{N}}\alpha , \quad (2.315)$$

$$B \rightarrow B - \mathcal{N}^2\alpha + a^2\dot{\beta} , \quad (2.316)$$

$$\psi \rightarrow \psi + H\mathcal{N}\alpha , \quad (2.317)$$

$$E \rightarrow E - \beta , \quad (2.318)$$

where $H \equiv \dot{a}/(\mathcal{N}a)$ and overdots denote derivatives with respect to the cosmic time t , as usual. Regarding the transformation of ψ , and using hats to refer to the Einstein frame, we have

$$\hat{\psi} \rightarrow \hat{\psi} + \hat{H}\hat{\mathcal{N}}\alpha = \hat{\psi} + \hat{H}\sqrt{F}\alpha = \hat{\psi} + H(1 + \theta_1)\alpha , \quad (2.319)$$

where we used the fact that $\hat{\mathcal{N}} = \sqrt{F}$ (see Eq. (2.103)) and the conformal transformation of H (see Eq. (2.105)).

We then recover Eq. (2.313) as promised. From Eq. (2.312), the curvature perturbation on shear-free slicings in the Einstein frame is (compare with Eq. (1.164))

$$\hat{\psi}^{\hat{\chi}} = \hat{\psi} + \hat{H}\hat{\chi} , \quad (2.320)$$

where $\hat{\chi} \equiv \sqrt{F}\chi$. Notice that, rather than $\hat{\psi}^{\hat{\chi}}$, we could write $\hat{\psi}^\chi$ as well because $\chi = 0$ implies $\hat{\chi} = 0$.

Finally, regarding A , considering how it transforms under conformal transformations (see Eq. (2.305)) so it is converted into the Einstein frame perturbation (see Eq. (2.309)), we have

$$\hat{A} = A^\chi + \frac{1}{2}\sigma + \dot{\chi} + H\theta_1\chi , \quad (2.321)$$

if we turn the perturbations on the RHS of Eq. (2.309) into gauge-invariant ones. Then

$$\hat{A}^{\hat{\chi}} \equiv \hat{A} - \dot{\chi} - H\theta_1\chi . \quad (2.322)$$

In line with the previous reasoning involving $\hat{\psi}$, $\hat{A}$ must transform as

$$\hat{A} \rightarrow \hat{A} - \dot{\alpha} - H\theta_1\alpha = \hat{A} - \dot{\alpha} - \frac{\dot{F}}{2F}\alpha = \hat{A} - \dot{\alpha} - \frac{\dot{\mathcal{N}}}{\mathcal{N}}\alpha , \quad (2.323)$$

in order for $\hat{A}^{\hat{\chi}}$ to be gauge-invariant. This is precisely the gauge transformation we derived earlier in Eq. (2.315). Again, defining $\hat{\chi} \equiv \sqrt{F}\chi$, we obtain (compare with Eq. (1.205) and remember that $\mathcal{N} = 1$ in the Jordan frame, so $d\tau = dt$)

$$\hat{A}^{\hat{\chi}} = \hat{A} - \frac{d\hat{\chi}}{d\hat{\tau}} . \quad (2.324)$$

Now that the conformal transformations of perturbations are clear, we can show the

invariance of $\psi^{\delta\phi}$ under the conformal transformation and a field redefinition [220]:

$$\hat{\psi}^{\delta\varphi} = \hat{\psi} + \frac{\hat{H}}{d\varphi/d\hat{\tau}} \delta\varphi = \psi - \frac{\delta F}{2F} + \frac{H}{\dot{\phi}} (1 + \theta_1) \delta\phi = \psi + \frac{H}{\dot{\phi}} \delta\phi = \psi^{\delta\phi}, \quad (2.325)$$

where $\delta F = F_{,\phi} \delta\phi$ and φ is the canonical field in the Einstein frame (see Eq. (2.80), which is the same field redefinition we use here but with ϕ instead of h and $\mathcal{K}$ given in Eq. (2.115)).

Notice that $\hat{\mathcal{R}} = \hat{\psi}^{\delta\varphi}$ because, in the Einstein frame, the effective fluid corresponds to the canonical scalar field (see Eq. (2.42) and remember the field redefinition with $\mathcal{K}$ given by Eq. (2.115)), that we do not consider matter field and $F = F(\phi)$. But, since $\psi^{\delta\phi} \neq \mathcal{R}$ as was explained below Eq. (2.302), the *comoving* curvature perturbation is *not* invariant under a conformal transformation and a field redefinition generally; i.e.

$$\hat{\mathcal{R}} = \hat{\psi}^{\delta\varphi} = \psi^{\delta\phi} \neq \mathcal{R}. \quad (2.326)$$

Eq. (2.326) is generic, valid at all scales and irrespective of how the background spacetime behaves, same as the transformations equations derived above in this section so far. This then shows that the curvature perturbation is conformally equivalent on uniform- ϕ slicings (or on those that comove with the scalar field fluid but not with the effective one) for single scalar field models on all length scales and at any expansion rate, but the same does not appear to hold for $\mathcal{R}$. This issue will be tackled in the next sections.

2.5.2 Conformal Equivalence of $\mathcal{R}$

We now proceed to derive the corresponding exact relation between $\mathcal{R}$ and $\hat{\mathcal{R}}$ for a generic $F(\phi)R$ gravity theory at linear order in perturbation theory. We will check if the invariance is possible and perform the analysis on subhorizon and superhorizon scales during SR inflation (this will allow us to resort to the results derived in previous sections).

The linear comoving curvature perturbation in the Jordan frame can be written as (using Eq. (2.253) and the momentum constraint equation, Eq. (2.270), to replace A)

$$\mathcal{R} = \frac{H}{\rho + P} \frac{\dot{\phi} \delta\phi^\psi + \delta\dot{F}^\psi - H \delta F^\psi}{1 + \theta_1}, \quad (2.327)$$

δF^ψ being defined as

$$\delta F^\psi \equiv \delta F + \frac{\dot{F}}{H} \psi. \quad (2.328)$$

In $F(\phi)R$ gravity, $\delta F = F_{,\phi}\delta\phi$ as was pointed out earlier. Then

$$\mathcal{R} = \frac{H}{\rho + P} \frac{\dot{\phi} (1 + F_{,\phi\phi}) \delta\phi^\psi + F_{,\phi} (\delta\dot{\phi}^\psi - H\delta\phi^\psi)}{1 + \theta_1}, \quad (2.329)$$

or

$$\mathcal{R} = \frac{H}{\dot{\phi}} \frac{1}{1 + \theta_1} \frac{1}{\epsilon_1} \left\{ [\epsilon_1 (1 + \theta_1) + \theta_1 (\theta_{\mathcal{K}} - \gamma_2)] \delta\phi^\psi + \theta_1 \frac{\delta\dot{\phi}^\psi}{H} \right\}. \quad (2.330)$$

$\theta_{\mathcal{K}}$ was introduced in Eq. (2.116). Since $\hat{\mathcal{R}} = \hat{\psi}^{\delta\varphi} = \psi^{\delta\phi}$, we have (see Eq. (2.300))

$$\mathcal{R} = \hat{\mathcal{R}} \left[1 + \frac{\theta_1}{\epsilon_1 (1 + \theta_1)} \left(\theta_{\mathcal{K}} - \gamma_2 + \frac{\delta\dot{\phi}^\psi}{H\delta\phi^\psi} \right) \right]. \quad (2.331)$$

Now, given that

$$\delta\phi^\psi = \frac{\dot{\phi}}{H} \psi^{\delta\phi} = \frac{\dot{\phi}}{H} \hat{\mathcal{R}} = -\frac{\dot{\phi}}{H\hat{z}} \hat{u} = \frac{1 + \theta_1}{a\sqrt{F\mathcal{K}}} \hat{u}, \quad (2.332)$$

where $\hat{u} \equiv -\hat{z}\hat{\mathcal{R}}$ (compare with Eq. (1.227)) and $\hat{z}$ is the Mukhanov variable in the Einstein frame, defined as

$$\hat{z} \equiv -\frac{\hat{a}}{\hat{H}} \frac{d\varphi}{d\hat{\tau}}, \quad (2.333)$$

then

$$\frac{\delta\dot{\phi}^\psi}{H\delta\phi^\psi} = -1 - \theta_1 - \theta_{\mathcal{K}} + \frac{\theta_1\theta_2}{1 + \theta_1} + \frac{\dot{\hat{u}}}{H\hat{u}}. \quad (2.334)$$

On the other hand, we remember that $\hat{\epsilon}_2$ (the second Hubble-flow parameter in the Einstein frame) is related to γ_2 by Eq. (2.148), so we can write

$$\mathcal{R} = \hat{\mathcal{R}} \left[1 - \frac{\theta_1}{\epsilon_1} \left(1 + \frac{1}{2}\hat{\epsilon}_2 - \frac{1}{\hat{H}\hat{u}} \frac{d\hat{u}}{d\hat{\tau}} \right) \right], \quad (2.335)$$

where we used Eq. (2.105) too.

SR inflation demands that $\hat{\epsilon}_1, |\hat{\epsilon}_2| \ll 1$ and, as was learnt from Sec. 2.2, one obtains that $|\theta_1| \ll 1$. Hence, during SR , Eq. (2.335) can be written as

$$\mathcal{R} \simeq \hat{\mathcal{R}} \left[1 - \frac{\theta_1}{\epsilon_1} \left(1 - \frac{1}{\hat{H}\hat{u}} \frac{d\hat{u}}{d\hat{\tau}} \right) \right], \quad (2.336)$$

and $H \simeq \sqrt{F}\hat{H}$ as $|\theta_1| \ll 1$ (see Eq. (2.105)). Since $\hat{u}$ obeys Eq. (1.233) (simply replace the variables by the hatted ones given that the Einstein frame is that of GR), and we assume

SR inflation, we have the solution in Eq. (1.257). Therefore

$$1 - \frac{1}{\hat{H}\hat{u}_k} \frac{d\hat{u}_k}{d\hat{\tau}} = 1 - \frac{\hat{u}_k'}{\hat{\mathcal{H}}\hat{u}_k} = \frac{k^2}{\hat{\mathcal{H}}^2} \frac{1}{1 - ik/\hat{\mathcal{H}}} , \quad (2.337)$$

where the prime refers to the derivative with respect to the conformal time which, in the presence of an arbitrary lapse function, is defined as

$$d\eta \equiv \frac{\mathcal{N}}{a} dt , \quad (2.338)$$

and hence is invariant under conformal transformations given Eqs. (2.94) and (2.95). Also, $\hat{\mathcal{H}} \equiv \hat{a}\hat{H}$. Notice that $\mathcal{H}$ is invariant under conformal transformations during SR because

$$\mathcal{H} = aH = \hat{a} \frac{H}{\sqrt{F}} \simeq \hat{a}\hat{H} = \hat{\mathcal{H}} . \quad (2.339)$$

The comoving wavenumber k is invariant as well because the spatial coordinates are not affected by the conformal rescaling of the metric.

Eq. (2.336) becomes

$$\mathcal{R}_k \simeq \frac{1 - ik/\hat{\mathcal{H}} - \omega(k/\hat{\mathcal{H}})^2}{1 - ik/\hat{\mathcal{H}}} \hat{\mathcal{R}}_k , \quad (2.340)$$

in Fourier space, where

$$\omega \equiv \frac{\theta_1}{\epsilon_1} = - \frac{d \ln \sqrt{F}}{d \ln H} . \quad (2.341)$$

ω gives an indication of how large the variations of $\ln H$ and $\ln \sqrt{F}$ are relative to each other. Remember that H and $\sqrt{F}$ are the two time dependent background scales of the theory in the Jordan frame (whereas, in GR, we have H only; see discussion below Eq. (2.109)). Notice that if $\omega = 0$ (because $\theta_1 = 0$, as in GR), then $\mathcal{R}_k = \hat{\mathcal{R}}_k$ for all k , as expected. If $|\omega| < 1$, $\ln H$ varies faster than $\ln \sqrt{F}$, while $|\omega| > 1$ means that the rate of change of $\ln H$ is slower than that of $\ln \sqrt{F}$.

On superhorizon scales ($k/\hat{\mathcal{H}} \ll 1$), we have

$$\mathcal{R}_{k \ll \hat{\mathcal{H}}} \simeq \left[1 - \omega \left(\frac{k}{\hat{\mathcal{H}}} \right)^2 \right] \hat{\mathcal{R}}_{k \ll \hat{\mathcal{H}}} , \quad (2.342)$$

and the equivalence of the two linear comoving curvature perturbations in Jordan and Einstein frames is valid up to a term quadratic in gradients. This equivalence implies that the

comoving and uniform- ϕ slicings coincide whenever $|\omega|k^2/\hat{\mathcal{H}}^2 \ll 1$ in SR (see Eq. (2.326))

$$\psi_{|\omega|k^2 \ll \hat{\mathcal{H}}^2}^{\delta\phi} = \hat{\psi}_{|\omega|k^2 \ll \hat{\mathcal{H}}^2}^{\delta\varphi} = \hat{\mathcal{R}}_{|\omega|k^2 \ll \hat{\mathcal{H}}^2} \simeq \mathcal{R}_{|\omega|k^2 \ll \hat{\mathcal{H}}^2} . \quad (2.343)$$

For subhorizon modes ($k/\hat{\mathcal{H}} \gg 1$):

$$\mathcal{R}_{k \gg \hat{\mathcal{H}}} \simeq \left(1 - i\omega \frac{k}{\hat{\mathcal{H}}}\right) \hat{\mathcal{R}}_{k \gg \hat{\mathcal{H}}} , \quad (2.344)$$

meaning that the equivalence between $\mathcal{R}$ and $\hat{\mathcal{R}}$ in the two frames does not hold.

However, due to the factor ω , the length scales on which $\mathcal{R}$ can be considered to be conformally invariant depend on the time dependence of F and H . For models with very small $|\omega|$ that invariance holds on subhorizon scales already. For very large $|\omega|$ instead, it holds only long after such scales have exited the horizon.

2.5.3 Conformal Invariance in Generalised Induced Gravity Inflation Models

As we mentioned in Sec. 2.2 (see after Eq. (2.119)) the γ parameters make approximations clearer if they replace the Hubble-flow parameters of the Jordan frame. Assuming the SR approximation, Eq. (2.147) may be written as

$$\gamma^2 \simeq \epsilon_1 + \theta_1 , \quad (2.345)$$

assuming $|\theta_2| \ll 1$ for simplicity, although it was pointed out in Ref. [1] (and in this dissertation; see discussion following Eq. (2.152)) that the SR assumption still permits $|\theta_2| \sim 1$. Given the SR relation between γ^2 and ϵ_1 , ω can be written as

$$\omega \simeq \frac{\theta_1}{\gamma^2 - \theta_1} = \left(\frac{\gamma^2}{\theta_1} - 1\right)^{-1} = \left(\frac{\mathcal{K}\dot{\phi}}{H} \frac{F}{F_{,\phi}} - 1\right)^{-1} . \quad (2.346)$$

Now, two different regimes are in order: either $\mathcal{K} \simeq 1/F$ because $F_{,\phi}^2 \ll F$; or $\mathcal{K} \simeq 3F_{,\phi}^2/(2F^2)$ as $F_{,\phi}^2 \gg F$. We explored both possibilities in Ref. [1] and in Sec. 2.2.4. The latter yields

$$\omega \simeq (3\theta_1 - 1)^{-1} \simeq -1 , \quad (2.347)$$

so ω is of order $\mathcal{O}(1)$ (i.e. the evolution rate of both scales, H and $\sqrt{F}$, is similar; see

Eq. (2.341)). On the other hand, $F_{,\phi}^2 \ll F$ gives

$$\omega \simeq \left(\frac{\dot{\phi}}{HF_{,\phi}} - 1 \right)^{-1} \simeq \left(1 - \frac{FV_{,\phi}}{F_{,\phi}V} \right)^{-1}, \quad (2.348)$$

where the SR equations, Eqs. (2.136) and (2.140), were used. ω would be larger than unity if

$$\frac{F_{,\phi}}{F} \simeq \frac{V_{,\phi}}{V}. \quad (2.349)$$

This is viable given that

$$\frac{1}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \ll \frac{1}{F}, \quad (2.350)$$

must be satisfied if $\hat{\epsilon}_1 \ll 1$ (see Eq. (2.162)), and if we insert Eq. (2.349) we obtain $F_{,\phi}^2 \ll F$, which is precisely the condition we had imposed in this case. For generalised models of induced gravity inflation however, it was shown that those cases, in which $F_{,\phi}^2 \ll F$, do not fall within the 2σ region of the latest BICEP/Keck results [243] (see table 2.1), in contrast to those satisfying $F_{,\phi}^2 \gg F$. Therefore, for this class of models, ω is of order $\mathcal{O}(1)$, and the comoving curvature perturbation is invariant under the conformal transformation for a few e-folds after horizon-crossing (see Eq. (2.342)).

2.5.4 Inflation Observables in Jordan and Einstein Frames

In Sec. 2.5.2, we have shown the conformal invariance of the linear perturbation $\mathcal{R}$ on sufficiently large scales during SR using the conformal transformation. Now, we check the consistency of that conformal invariance by doing all the computations in the Jordan frame. We shall calculate the inflation observables in that frame and compare them with those obtained in Ref. [1] (see Sec. 2.2.5) where we regarded the Einstein frame expressions firstly and then performed the conformal rescaling of the metric.

We begin with the exact relation between $\mathcal{R}$ and $\psi^{\delta\phi}$

$$\mathcal{R} = \psi^{\delta\phi} + \frac{\theta_1}{\epsilon_1(1+\theta_1)} \frac{\dot{\psi}^{\delta\phi}}{H}, \quad (2.351)$$

where we used Eqs. (1.189), (2.253) and (2.270). $\psi^{\delta\phi}$ can be written as (see Eq. (2.298))

$$\psi^{\delta\phi} = \frac{H}{\dot{\phi}} \delta\phi^\psi \equiv -\frac{u}{z}, \quad (2.352)$$

where we defined u and z as in Eqs. (2.298) and (2.299), respectively. The second term in Eq. (2.351) is the direct consequence of the fact that uniform-field slices do not coincide with the comoving ones, as argued below Eq. (2.325). Since the perturbed quantities defined on the uniform-field slices are conformally invariant (because $\delta\phi = 0$ implies $\delta\Omega(\phi) = 0$; see Ref. [202]), it is not surprising to find that $\mathcal{R} \neq \hat{\mathcal{R}}$.

Eq. (2.351) may be rewritten as

$$\mathcal{R} = -\frac{u}{z} \left[1 - \frac{\omega}{1 + \theta_1} \left(1 - \frac{u'}{\mathcal{H}u} + \theta_1 + \gamma_2 - \frac{\theta_1\theta_2}{1 + \theta_1} \right) \right], \quad (2.353)$$

where ω was defined in Eq. (2.341) and z'/z is

$$\frac{z'}{z} = \mathcal{H} \left(1 + \theta_1 + \gamma_2 - \frac{\theta_1\theta_2}{1 + \theta_1} \right). \quad (2.354)$$

u satisfies the Mukhanov-Sasaki equation (1.233) [120], as was pointed out at the end of Sec. 2.4.3, where z''/z is given by Eq. (2.301). During SR, assuming that all the parameters are negligible, we have $z''/z \simeq 2/\eta^2$ and we can use the same solution for u_k as that in Eq. (1.257). Then

$$1 - \frac{u'_k}{\mathcal{H}u_k} = \frac{k^2}{\mathcal{H}^2} \frac{1}{1 - ik/\mathcal{H}}. \quad (2.355)$$

Hence, whenever $|\omega|k^2/\mathcal{H}^2 \ll 1$, $\mathcal{R}_{|\omega|k^2 \ll \mathcal{H}^2} \simeq -u_{|\omega|k^2 \ll \mathcal{H}^2}/z = \psi_{|\omega|k^2 \ll \mathcal{H}^2}^{\delta\phi}$ during SR.

Given that result, we may write the power spectrum of the comoving curvature perturbation, A_s , as [41] (we drop ‘ $|\omega|k^2 \ll \mathcal{H}^2$ ’ for simplicity)

$$A_s = \frac{k^3}{2\pi^2} |\mathcal{R}_k|^2 \simeq \frac{k^3}{2\pi^2} \frac{|u_k|^2}{z^2} \simeq \frac{H^2}{8\pi^2 F \gamma^2} \simeq \frac{V}{24\pi^2 F^2 \gamma^2}. \quad (2.356)$$

In the last equality, one of the two SR equations, the Friedmann equation during SR (2.136), was used.

Another important inflation observable is the scalar spectral index n_s (see definition in Eq. (1.268)). Using the chain rule as in Eq. (1.271), and dropping the asterisks for simplicity, we have (given that $|\epsilon_1| \ll 1$ during SR)

$$n_s - 1 \simeq \frac{d \ln A_s}{dN} \simeq 2 \frac{d \ln H}{dN} - \frac{d \ln F}{dN} - \frac{d \ln \gamma^2}{dN} \simeq -2(\epsilon_1 + \theta_1 + \gamma_2), \quad (2.357)$$

where we inserted A_s from Eq. (2.356) and in the last equality we used the definition of

γ_2 that can be inferred from Eq. (2.118). We may replace $\epsilon_1 + \theta_1$ by γ^2 given Eq. (2.345). Hence

$$n_s - 1 \simeq -2(\gamma^2 + \gamma_2) . \quad (2.358)$$

Finally, the tensor-to-scalar ratio r is defined in Eq. (1.306). In $F(\phi)R$ gravity, we have non-vanishing anisotropic pressure, as we know (see Eq. (2.249)). Hence, the equation of GWs will be given by Eq. (1.285) but with $\Pi_{ij} \neq 0$ (also, remember that M_{Pl}^2 is replaced by F ; see Eq. (2.8)). The tensor perturbation of Π_{ij} yields [247]

$$h_{ij}'' + (3 + 2\theta_1) H \dot{h}_{ij} - a^{-2} \partial_k \partial^k h_{ij} = 0 , \quad (2.359)$$

given that (see Eq. (2.243) and App. A, where we calculate the tensor perturbations of the geometric quantities)

$$\delta \tilde{T}^i_j = -\frac{1}{2} \frac{\dot{F}}{F} \dot{h}^i_j , \quad (2.360)$$

and hence $\tilde{\Pi}_{ij} = -\frac{\dot{F}}{2F} \dot{h}_{ij}$ (see Eq. (1.287)).

Eq. (2.359) can be written as (compare with Eq. (1.295))

$$v_k^{s''} + \left(k^2 - \frac{z''}{z} \right) v_k^s = 0 , \quad (2.361)$$

where z is defined as

$$z \equiv \frac{a}{2} \sqrt{F} , \quad (2.362)$$

and (compare with Eq. (1.294))

$$v_k^s \equiv \frac{1}{2} a \sqrt{F} h_k^s . \quad (2.363)$$

We have decomposed h_{ij} into Fourier modes, as in Eq. (1.288). z''/z is then

$$\frac{z''}{z} = \mathcal{H}^2 (1 + \theta_1) \left(2 + \theta_1 - \epsilon_1 + \frac{\theta_1 \theta_2}{1 + \theta_1} \right) . \quad (2.364)$$

In GR, $\theta_i = 0$ where $i = 1, 2, \dots$ (see below Eq. (2.301)). We then recover Eq. (1.295), i.e. $z''/z = a''/a$ (see Eq. (1.296)).

During SR, $z''/z \simeq 2/\eta^2$, and we can resort to the solution in Eq. (1.297). The tensor

power spectrum becomes (see Eq. (1.298))

$$A_t = \frac{k^3}{2\pi^2} \sum_{s=+,\times} |h_k^s|^2 = \frac{k^3}{\pi^2} \frac{2}{a^2 F} \sum_{s=+,\times} |v_k^s|^2 \simeq \frac{2H^2}{\pi^2 F}. \quad (2.365)$$

Plugging Eqs. (2.356) and (2.365) into Eq. (1.306), we obtain

$$r \simeq 16\gamma^2. \quad (2.366)$$

Notice that

$$A_t \simeq \frac{2}{\pi^2} \frac{H^2}{F} \simeq \frac{2}{\pi^2} \hat{H}^2, \quad (2.367)$$

where we used Eq. (2.105) and the fact that, during SR, $|\theta_1| \ll 1$. This is the amplitude of tensor perturbations in GR during SR (see Eq. (1.298) and remember that $M_{\text{Pl}}^2 = 1$; see below Eq. (2.102)). Moreover, it can be shown that the tensor perturbations are conformally invariant in general to fully non-linear order [202].

The expressions in Eqs. (2.356), (2.358) and (2.366) are the same in Eqs. (2.173), (2.176) and (2.177) (Eqs. (96)-(98) in Ref. [1]) respectively (except for the $\theta_1\theta_2$ term in Eq. (2.176) because we took $|\theta_2| \ll 1$ here). Those equations were obtained by using the Einstein frame expression of A_s , n_s and r , applying the conformal transformation and using the SR approximation. In this section, on the other hand, we have arrived at the same results by starting with the analogue of the Mukhanov-Sasaki equation in the Jordan frame and then by using the SR approximation, without resorting to the conformal transformation. This confirms the conformal invariance of the linear comoving curvature perturbation and therefore the equivalence between the comoving and the uniform-field slicings on scales sufficiently large such that $|\omega|k^2/\mathcal{H}^2 \ll 1$ (or $|\omega|k^2\hat{\mathcal{H}}^2 \ll 1$ given that $\hat{\mathcal{H}} \simeq \mathcal{H}$ during SR as shown in Eq. (2.339)).

2.5.5 Concluding Remarks of Section 2.5

We have reviewed linear cosmological perturbations in $f(\phi, R)$ modified gravity in the Jordan frame and shown the conformal relation between the comoving curvature perturbation in Jordan and Einstein frames in $F(\phi)R$ gravity.

As was stated, this curvature perturbation is defined here on hypersurfaces that comove with the effective fluid with conserved energy and momentum in the Jordan frame. While the conformal equivalence does not hold on the subhorizon regime during SR inflation (see Eq. (2.344)), we have shown that, on superhorizon scales, $\mathcal{R} \simeq \hat{\mathcal{R}}$ up to a term which is

quadratic in gradients (2.342). The conformal relation includes a parameter θ_1/ϵ_1 which is of order $\mathcal{O}(1)$ in the generalised induced gravity inflation models allowed by observations (see Sec. 2.3), ensuring the conformal invariance on superhorizon scales shortly after horizon-crossing. That ratio reflects the relative variation of the two time dependent background scales of the theory: H and $\sqrt{F}$ (see Eq. (2.341)). The equivalence on superhorizon scales in the generalised induced gravity inflation models agrees with the result shown in Ref. [202] non-perturbatively given that the uniform- ϕ and comoving slicings coincide on superhorizon scales during SR inflation, as has been clarified in this section. Such an equivalence however, cannot be asserted on the subhorizon regime as argued already (see Eq. (2.344)).

The same expressions for the inflation observables of Ref. [1] (see Eqs. (2.356), (2.358) and (2.366)) were obtained using the comoving curvature perturbation of the Jordan frame and the analogue of Mukhanov-Sasaki equation (see Eq. (1.233) and the mass term in Eq. (2.301) for $F(\phi)R$ gravity). Without using the conformal transformation in Sec. 2.5.4, we have shown that it is possible to formulate those inflation observables using the comoving curvature perturbation at linear order in perturbation theory in Jordan and Einstein frames. This has served as a demonstration of the conformal invariance of the linear comoving curvature perturbation on the *superhorizon regime* in SR inflation. It would be interesting to explore the implications of a *non-slow-roll phase*, which we know how to interpret in the Einstein frame very well [60, 256–261]. We would need first to find out how it translates into the Jordan frame and how it affects the parameters involved in it. Once the dynamics is examined, since the expression in Eq. (2.331) is valid regardless of the dynamics of the background, we might make the corresponding approximations starting from that expression directly and see how $\mathcal{R}$ and $\hat{\mathcal{R}}$ relate in this case. We leave this non-slow-roll analysis for a future project.

In the next chapter, we turn our attention to the late Universe and, in particular, to a class of field models widely regarded in the literature. The use of modified gravity theories has been restricted to the early inflationary phase in the preceding sections, although, as was pointed out at the beginning of this chapter, these are also considered with the aim of addressing questions that involve late time physics. We shall consider one of these modified gravity theories that stems from low energy corrections in string theory and explore its applicability in a *formalism* of gravity which is a relatively old alternative to the one that has been assumed so far. That formalism has been gradually attracting the current interest in the field of Cosmology. We shall explain the motivations for this framework in the context of late time cosmology and present our model and its implications.

CHAPTER 3

GAUSS-BONNET QUINTESSENCE AND PALATINI FORMALISM

In the preceding chapter, we dealt with modified gravity theories and their applications to the inflationary cosmology. Models such as $f(R)$ gravity gave rise to higher-order curvature terms in the gravitational action, leading to additional degrees of freedom. However, there is a certain combination of quadratic curvature scalars which does not contribute to the metric field equations in four spacetime dimensions, and its inclusion cannot be regarded as a modified gravity theory. Such a combination is known as the *Gauss-Bonnet (GB) term* $\mathcal{G}$ and is given by (see Ref. [262] for a recent review of this subject)

$$\mathcal{G} \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} . \quad (3.1)$$

3.1 Scalar-Gauss-Bonnet Theory of Gravity

3.1.1 Gauss-Bonnet Term and Lovelock Gravity

Considering the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_{\text{Pl}}^2 R - \xi \mathcal{G} \right) , \quad (3.2)$$

where ξ is a dimensionless constant, and varying each term with respect to the inverse metric (see the details in App. C), we arrive at the following metric field equations (see Eqs. (C.9) and (C.34)):

$$M_{\text{Pl}}^2 G_{\mu\nu} - 2\xi \mathcal{G}_{\mu\nu} = 0 , \quad (3.3)$$

where we have introduced the *GB tensor*

$$\mathcal{G}_{\mu\nu} \equiv RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha_{\nu} - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} + R_{\alpha\beta\sigma\mu}R^{\alpha\beta\sigma}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{G} . \quad (3.4)$$

Although we mentioned ‘four spacetime dimensions’, $d = 4$, when writing the action (3.2) we did not make use of this fact actually. Let us then consider that $d \geq 3$ instead, and introduce a traceless tensor called *Weyl tensor* $C^\alpha_{\beta\mu\nu}$. This tensor carries the independent

Riemann tensor components that are not accounted for by the Ricci tensor. In $d \geq 3$ spacetime dimensions, the covariant Weyl tensor can be written as [23]

$$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - \frac{2}{d-2} \left(g_{\mu[\alpha} R_{\beta]\nu} - g_{\nu[\alpha} R_{\beta]\mu} \right) + \frac{2R}{(d-1)(d-2)} g_{\mu[\alpha} g_{\beta]\nu} . \quad (3.5)$$

In $d = 3$, the Riemann and Ricci tensors have the *same* number of independent components,¹ and therefore $C_{\mu\nu\alpha\beta} = 0$.

The Weyl tensor has the same symmetries as the Riemann tensor and, on top of these, it includes the vanishing of *all* of its traces (i.e. by contracting *any* pair of indices) [23]. As a result of this, an important relation involving the Weyl tensor in four spacetime dimensions is the so-called *Lanczos identity* (see the appendix of Ref. [263], where a proof can be found), which reads

$$C_{\mu\alpha\beta\sigma} C_{\nu}{}^{\alpha\beta\sigma} = \frac{1}{4} C_{\alpha\beta\sigma\rho} C^{\alpha\beta\sigma\rho} g_{\mu\nu} . \quad (3.7)$$

The Weyl tensor is also *conformally invariant*; i.e. invariant under the conformal rescaling of the metric [23, 172] (see Eq. (2.26)).

Using Eq. (3.5), we may write the GB tensor in terms of the Weyl tensor. This will serve to extract an important result. We begin with

$$\begin{aligned} R^{\alpha\beta} R_{\alpha\mu\beta\nu} &= R^{\alpha\beta} C_{\alpha\mu\beta\nu} + \frac{d}{(d-1)(d-2)} R R_{\mu\nu} - \frac{1}{(d-1)(d-2)} R^2 g_{\mu\nu} - \frac{2}{d-2} R_{\mu\alpha} R^{\alpha}{}_{\nu} + \\ &+ \frac{1}{d-2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} . \end{aligned} \quad (3.8)$$

The $R_{\alpha\beta\sigma\mu} R^{\alpha\beta\sigma}{}_{\nu}$ term is slightly more intricate:

$$\begin{aligned} R_{\alpha\beta\sigma\mu} R^{\alpha\beta\sigma}{}_{\nu} &= C_{\alpha\beta\sigma\mu} C^{\alpha\beta\sigma}{}_{\nu} + \frac{4}{d-2} R^{\alpha\beta} C_{\alpha\mu\beta\nu} + \frac{4}{(d-1)(d-2)^2} R R_{\mu\nu} - \\ &- \frac{2}{(d-1)(d-2)^2} R^2 g_{\mu\nu} + \frac{2(d-4)}{(d-2)^2} R_{\mu\alpha} R^{\alpha}{}_{\nu} + \frac{2}{(d-2)^2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} . \end{aligned} \quad (3.9)$$

1. The number of independent components of the Riemann tensor in d dimensions is [23]

$$\frac{d^2(d^2-1)}{12} . \quad (3.6)$$

In $d = 3$, there are 6 independent components. This figure coincides with the number of independent components of the second-rank, symmetric Ricci tensor. In $d = 1$, the Riemann tensor vanishes everywhere, meaning that there is no *intrinsic* curvature in one dimension.

Contracting μ and ν , we obtain $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, so the GB term reads

$$\mathcal{G} = \frac{d(d-3)}{(d-1)(d-2)}R^2 - 4\frac{d-3}{d-2}R^{\mu\nu}R_{\mu\nu} + C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} . \quad (3.10)$$

Putting all together, we see that the GB tensor can be written as

$$\begin{aligned} \mathcal{G}_{\mu\nu} = & C_{\alpha\beta\sigma\mu}C^{\alpha\beta\sigma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}C^{\alpha\beta\sigma\rho}C_{\alpha\beta\sigma\rho} + \frac{d-4}{d-2} \left[\frac{d(d-3)}{(d-1)(d-2)}RR_{\mu\nu} - 2R^{\alpha\beta}C_{\alpha\mu\beta\nu} - \right. \\ & \left. - 2\frac{d-3}{d-2}R_{\mu\alpha}R^{\alpha}{}_{\nu} + \frac{d-3}{d-2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} - \frac{(d+2)(d-3)}{4(d-1)(d-2)}g_{\mu\nu}R^2 \right] . \end{aligned} \quad (3.11)$$

$d = 4$ yields

$$\mathcal{G}_{\mu\nu} = C_{\alpha\beta\sigma\mu}C^{\alpha\beta\sigma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}C^{\alpha\beta\sigma\rho}C_{\alpha\beta\sigma\rho} = C_{\mu\alpha\beta\sigma}C^{\alpha\beta\sigma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}C^{\alpha\beta\sigma\rho}C_{\alpha\beta\sigma\rho} . \quad (3.12)$$

Using Lanczos identity (Eq. (3.7)), the GB tensor vanishes: $\mathcal{G}_{\mu\nu} = 0$. Consequently, the metric field equations (3.3) turn out to be the same as those in GR (see Eq. (1.8)). This is true in four spacetime dimensions, although one may point out that the same holds in three dimensions (remember that $C_{\alpha\beta\mu\nu} = 0$ in $d = 3$). However, the GB term itself is zero in three dimensions (see Eq. (3.10)), whereas it is non-vanishing when $d = 4$. In higher dimensions, the GB term *does* affect the field equations.

The reason for the result above (the fact that $\mathcal{G}_{\mu\nu} = 0$ in four dimensions) can be understood better when we notice that this combination of quadratic curvature terms is included in a generalisation of GR to higher spacetime dimensions known as *Lovelock's theory of gravity*, introduced by D. Lovelock in 1971 [264–267]. This theory is the most general one leading to second-order metric field equations in an arbitrary number of spacetime dimensions. The Lagrangian of the theory reads

$$\mathcal{L}_n = c_0 + \sum_{m=1}^n c_m \mathcal{R}^m, \quad (3.13)$$

where $\mathcal{R}^m$ is given by

$$\mathcal{R}^m = \frac{1}{2^m} \delta^{\mu_1\nu_1\ldots\mu_m\nu_m}_{\alpha_1\beta_1\ldots\alpha_m\beta_m} \prod_{i=1}^m R^{\alpha_i\beta_i}_{\mu_i\nu_i} . \quad (3.14)$$

The coefficients c_m have dimensions of mass to the power of $d - 2m$: $[c_m] = d - 2m$. The coefficient c_0 would correspond to the pressure associated with the CC Λ (see Eq. (1.20) and below Eq. (1.15)). c_1 has dimensions of mass to $d - 2$ (i.e. $[c_1] = 2$ in $d = 4$, which is a

mass squared that may be ascribed to the Planck mass squared or the inverse of Newton's gravitational constant G).

The generalised Kronecker delta $\delta_{\alpha_1\beta_1\dots\alpha_m\beta_m}^{\mu_1\nu_1\dots\mu_m\nu_m}$ is defined as [262]

$$\delta_{\alpha_1\beta_1\dots\alpha_m\beta_m}^{\mu_1\nu_1\dots\mu_m\nu_m} \equiv \begin{vmatrix} \delta_{\alpha_1}^{\mu_1} & \delta_{\beta_1}^{\mu_1} & \dots & \delta_{\alpha_m}^{\mu_1} & \delta_{\beta_m}^{\mu_1} \\ \delta_{\alpha_1}^{\nu_1} & \delta_{\beta_1}^{\nu_1} & \dots & \delta_{\alpha_m}^{\nu_1} & \delta_{\beta_m}^{\nu_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{\alpha_1}^{\mu_m} & \delta_{\beta_1}^{\mu_m} & \dots & \delta_{\alpha_m}^{\mu_m} & \delta_{\beta_m}^{\mu_m} \\ \delta_{\alpha_1}^{\nu_m} & \delta_{\beta_1}^{\nu_m} & \dots & \delta_{\alpha_m}^{\nu_m} & \delta_{\beta_m}^{\nu_m} \end{vmatrix}. \quad (3.15)$$

Then, $n = 1$ yields

$$\mathcal{L}_1 = c_0 + c_1 \mathcal{R}^1 = c_0 + \frac{1}{2} c_1 \delta_{\alpha_1\beta_1}^{\mu_1\nu_1} R^{\alpha_1\beta_1}_{\mu_1\nu_1} = c_0 + c_1 R, \quad (3.16)$$

which is the Lagrangian of the gravitational sector of the action (2.1) (identifying $c_0 = -M_{\text{Pl}}^2 \Lambda$ and $c_1 = M_{\text{Pl}}^2/2$). The GB term is the second-order term of the theory:

$$\mathcal{R}^2 = \frac{1}{4} \delta_{\alpha_1\beta_1\alpha_2\beta_2}^{\mu_1\nu_1\mu_2\nu_2} R^{\alpha_1\beta_1}_{\mu_1\nu_1} R^{\alpha_2\beta_2}_{\mu_2\nu_2} = R^2 - 4R^\mu{}_\nu R^\nu{}_\mu + R^{\alpha\beta}_{\mu\nu} R^{\mu\nu}_{\alpha\beta}, \quad (3.17)$$

where the symmetries of the Riemann and Ricci tensors were used. Then

$$\mathcal{L}_2 = c_0 + c_1 R + c_2 \mathcal{G}, \quad (3.18)$$

where $[c_2] = d - 4$. Therefore, c_2 is dimensionless in four spacetime dimensions and, consequently, we may identify $c_2 = -\xi/2$ in view of Eq. (3.2).

3.1.2 Field Equations of the Scalar-Gauss-Bonnet Model

The terms that contribute to the metric field equations in Lovelock gravity are those for which $m < d/2$ [264, 266]. Therefore, in four spacetime dimensions, the GB term ($m = 2$) *does not* affect the metric field equations, as was shown above, and it is a total derivative

(although it can still be important for other aspects of the theory under consideration, such as regularisation [268, 269]). However, if the constant ξ is promoted to a field dependent function such that the action becomes

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[M_{\text{Pl}}^2 R - \xi(\phi) \mathcal{G} \right] - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\}, \quad (3.19)$$

then the GB term will have an effect on the field equations in four spacetime dimensions. This scalar-tensor theory, that we shall refer to as ‘Scalar-Gauss-Bonnet (SGB) theory’ [270, 271], and the GB term in higher dimensions, are well-motivated extensions of GR, emerging in UV theories such as string theory [272–288].

To derive the field equations, we can follow the same procedure as in App. C, but taking the field dependence of ξ into account. Eq. (C.10) now reads

$$\xi(\phi) \delta(\sqrt{-g} R^2) = 2\sqrt{-g} \xi R S_{\mu\nu} \delta g^{\mu\nu} + 2\sqrt{-g} \xi R \nabla_\mu \gamma^\mu, \quad (3.20)$$

where $S_{\mu\nu}$ is defined in Eq. (C.13). The second term leads to

$$\begin{aligned} 2\sqrt{-g} \xi R \nabla_\mu \gamma^\mu &= -2\sqrt{-g} \left[\xi (\nabla_\mu \partial_\nu R - g_{\mu\nu} \square R) + R (\nabla_\mu \partial_\nu \xi - g_{\mu\nu} \square \xi) + \right. \\ &\quad \left. + 2 (\partial_\mu \xi \partial_\nu R - g_{\mu\nu} \partial^\sigma \xi \partial_\sigma R) \right] \delta g^{\mu\nu}, \end{aligned} \quad (3.21)$$

where we have neglected the total derivatives as we did in App. C. Therefore

$$\begin{aligned} \xi(\phi) \delta(\sqrt{-g} R^2) &= 2\sqrt{-g} \left[\xi (R S_{\mu\nu} - \nabla_\mu \partial_\nu R + g_{\mu\nu} \square R) - R (\nabla_\mu \partial_\nu \xi - g_{\mu\nu} \square \xi) - \right. \\ &\quad \left. - 2 (\partial_\mu \xi \partial_\nu R - g_{\mu\nu} \partial_\sigma \xi \partial^\sigma R) \right] \delta g^{\mu\nu}. \end{aligned} \quad (3.22)$$

The second term in Eq. (C.14), multiplied by $\xi(\phi)$, becomes

$$\begin{aligned} 2\sqrt{-g} \xi R^{\mu\nu} \delta R_{\mu\nu} &= \sqrt{-g} \left[\xi \left(\square R_{\mu\nu} + g_{\mu\nu} \nabla_\alpha \nabla_\beta R^{\alpha\beta} - 2 \nabla_\alpha \nabla_\nu R^\alpha_\mu \right) + \square \xi R_{\mu\nu} + \right. \\ &\quad + g_{\mu\nu} \left(R^{\alpha\beta} \nabla_\alpha \partial_\beta \xi + 2 \partial_\alpha \xi \nabla_\beta R^{\alpha\beta} \right) - 2 (\nabla_\alpha \partial_\mu \xi R^\alpha_\nu + \partial_\mu \xi \nabla_\alpha R^\alpha_\nu + \partial_\alpha \xi \nabla_\mu R^\alpha_\nu - \\ &\quad \left. - \partial^\alpha \xi \nabla_\alpha R_{\mu\nu}) \right] \delta g^{\mu\nu}, \end{aligned} \quad (3.23)$$

up to total derivatives. Using Bianchi’s identity (see Eq. (2.13)) and the symmetries of the Riemann tensor, we may write the previous equation as

$$2\sqrt{-g} \xi R^{\mu\nu} \delta R_{\mu\nu} = \sqrt{-g} \left[\xi \left(\square R_{\mu\nu} - \nabla_\mu \partial_\nu R + \frac{1}{2} g_{\mu\nu} \square R - 2 R_{\mu\alpha} R^\alpha_\nu + 2 R^{\alpha\beta} R_{\alpha\mu\beta\nu} \right) + \right.$$

$$+\square\xi R_{\mu\nu} + g_{\mu\nu} \left(R^{\alpha\beta} \nabla_\alpha \partial_\beta \xi + \partial_\alpha \xi \partial^\alpha R \right) - 2 \left(\nabla_\alpha \partial_\mu \xi R^\alpha{}_\nu + \frac{1}{2} \partial_\mu \xi \partial_\nu R + \partial_\alpha \xi \nabla_\mu R^\alpha{}_\nu - \partial^\alpha \xi \nabla_\alpha R_{\mu\nu} \right) \delta g^{\mu\nu}, \quad (3.24)$$

and hence

$$\begin{aligned} \xi(\phi) \delta(\sqrt{-g} R^{\mu\nu} R_{\mu\nu}) = \sqrt{-g} \left[\xi \left(2R^{\alpha\beta} R_{\alpha\mu\beta\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} + \square R_{\mu\nu} - \nabla_\mu \partial_\nu R + \right. \right. \\ \left. \left. + \frac{1}{2} g_{\mu\nu} \square R \right) + \square \xi R_{\mu\nu} + g_{\mu\nu} \left(R^{\alpha\beta} \nabla_\alpha \partial_\beta \xi + \partial_\alpha \xi \partial^\alpha R \right) - 2 \left(\nabla_\alpha \partial_\mu \xi R^\alpha{}_\nu + \frac{1}{2} \partial_\mu \xi \partial_\nu R + \right. \right. \\ \left. \left. + \partial_\alpha \xi \nabla_\mu R^\alpha{}_\nu - \partial^\alpha \xi \nabla_\alpha R_{\mu\nu} \right) \right] \delta g^{\mu\nu}. \end{aligned} \quad (3.25)$$

Regarding Eq. (C.19), the second term, multiplied by $\xi(\phi)$, yields

$$2\sqrt{-g} \xi R_\alpha{}^{\beta\mu\nu} \delta R^\alpha{}_{\beta\mu\nu} = -4\sqrt{-g} \left(\xi \nabla_\alpha \nabla_\beta R_\mu{}^{\alpha\beta}{}_\nu + 2\partial_\alpha \xi \nabla_\beta R_\mu{}^{\alpha\beta}{}_\nu + \nabla_\alpha \partial_\beta \xi R_\mu{}^{\alpha\beta}{}_\nu \right) \delta g^{\mu\nu}, \quad (3.26)$$

where

$$\partial_\alpha \xi \nabla_\beta R_\mu{}^{\alpha\beta}{}_\nu \delta g^{\mu\nu} = \partial_\alpha \xi \nabla_\beta R_\nu{}^{\beta\alpha}{}_\mu \delta g^{\mu\nu} = \partial_\beta \xi \nabla_\alpha R_\mu{}^{\alpha\beta}{}_\nu \delta g^{\mu\nu}. \quad (3.27)$$

Using Bianchi's identity again and the symmetries of the Riemann tensor, we obtain

$$\begin{aligned} 2\sqrt{-g} \xi R_\alpha{}^{\beta\mu\nu} \delta R^\alpha{}_{\beta\mu\nu} = -4\sqrt{-g} \left[\xi \left(R_{\mu\alpha} R^\alpha{}_\nu - R^{\alpha\beta} R_{\alpha\mu\beta\nu} - \square R_{\mu\nu} + \frac{1}{2} \nabla_\mu \partial_\nu R \right) - \right. \\ \left. - 2\partial_\alpha \xi \nabla^\alpha R_{\mu\nu} + 2\partial_\alpha \xi \nabla_\mu R^\alpha{}_\nu + \nabla_\alpha \partial_\beta \xi R_\mu{}^{\alpha\beta}{}_\nu \right] \delta g^{\mu\nu}. \end{aligned} \quad (3.28)$$

Therefore

$$\begin{aligned} \xi(\phi) \delta(\sqrt{-g} R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}) = 2\sqrt{-g} \left[\xi \left(R_{\alpha\beta\sigma\mu} R^{\alpha\beta\sigma}{}_\nu - \frac{1}{4} g_{\mu\nu} R^{\alpha\beta\sigma\rho} R_{\alpha\beta\sigma\rho} - 2R_{\mu\alpha} R^\alpha{}_\nu + \right. \right. \\ \left. \left. + 2R^{\alpha\beta} R_{\alpha\mu\beta\nu} + 2\square R_{\mu\nu} - \nabla_\mu \partial_\nu R \right) + 4\partial_\alpha \xi \nabla^\alpha R_{\mu\nu} - 4\partial_\alpha \xi \nabla_\mu R^\alpha{}_\nu - 2\nabla_\alpha \partial_\beta \xi R_\mu{}^{\alpha\beta}{}_\nu \right] \delta g^{\mu\nu}. \end{aligned} \quad (3.29)$$

Consequently, the variation of the GB term yields

$$\xi(\phi) \delta(\sqrt{-g} \mathcal{G}) = 2\sqrt{-g} \left[\xi \mathcal{G}_{\mu\nu} + 2 \left(R_\mu{}^\alpha{}_\nu{}^\beta + 2S^\alpha{}_\mu \delta_\nu^\beta - g_{\mu\nu} G^{\alpha\beta} - R_{\mu\nu} g^{\alpha\beta} \right) \nabla_\alpha \partial_\beta \xi \right] \delta g^{\mu\nu}. \quad (3.30)$$

The metric field equations in four spacetime dimensions then read

$$G_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} \left[4 \left(R_{\mu}^{\alpha}{}_{\nu}{}^{\beta} + 2S_{(\mu}^{\alpha} \delta_{|\nu)}^{\beta} - g_{\mu\nu} G^{\alpha\beta} - R_{\mu\nu} g^{\alpha\beta} \right) \nabla_{\alpha} \partial_{\beta} \xi + \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi + 2V \right) \right], \quad (3.31)$$

where

$$R_{\mu}^{\alpha}{}_{\nu}{}^{\beta} \nabla_{\alpha} \partial_{\beta} \xi = R_{\nu}^{\beta}{}_{\mu}{}^{\alpha} \nabla_{\alpha} \partial_{\beta} \xi = R_{\nu}^{\alpha}{}_{\mu}{}^{\beta} \nabla_{\alpha} \partial_{\beta} \xi = R_{(\mu}^{\alpha}{}_{|\nu)}{}^{\beta} \nabla_{\alpha} \partial_{\beta} \xi. \quad (3.32)$$

The scalar field equation is (compare with Eq. (1.85))

$$\square \phi - V_{,\phi} = \frac{1}{2} \xi_{,\phi} \mathcal{G}. \quad (3.33)$$

Written in terms of a homogeneous scalar field $\phi(t)$ in the spatially-flat FLRW background metric, the scalar field equation becomes

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = -\frac{1}{2} \xi_{,\phi} \mathcal{G}, \quad (3.34)$$

where the GB term is given by (see App. A, where the homogeneous components of the Riemann and Ricci tensors are shown, as well as the background Ricci scalar)

$$\mathcal{G} = 24H^2 \left(\dot{H} + H^2 \right). \quad (3.35)$$

We notice that $\mathcal{G}$ is always positive if the Universe accelerates in its expansion (see Eq. (1.22)). Also, Eq. (3.34) bears resemblance with Eq. (2.112), both equations showing the presence of a forcing term on the RHS which affects the dynamics of the homogeneous scalar field ϕ . However, since $\xi(\phi)$ is dimensionless and $F(\phi)$ has dimension 2, the factor depending on H and $\dot{H}$ is quadratic in the SGB model, and therefore more suppressed over the late time evolution of the Universe (when the curvature is lower).

The homogeneous metric field equations are

$$3H^2 M_{\text{Pl}}^2 = \frac{1}{2} \dot{\phi}^2 + V + 12\xi \dot{H}^3, \quad (3.36)$$

$$- \left(2\dot{H} + 3H^2 \right) M_{\text{Pl}}^2 = \frac{1}{2} \dot{\phi}^2 - V - 4\xi \dot{H}^2 - 8\xi H \left(\dot{H} + H^2 \right), \quad (3.37)$$

where the homogeneous components of the tensor $S_{\mu\nu}$ read

$$S_{00} = -\frac{3}{2}\dot{H}, \quad (3.38)$$

$$S_{ij} = -\frac{1}{2}a^2\dot{H}\delta_{ij}. \quad (3.39)$$

Combining Eqs. (3.36) and (3.37), we obtain

$$\dot{H}M_{\text{Pl}}^2 = -\frac{1}{2}\dot{\phi}^2 + 2H^2(\ddot{\xi} - H\dot{\xi}) + 4\dot{\xi}H\dot{H}. \quad (3.40)$$

We notice that the energy density and isotropic pressure associated with the effective GB-induced energy-momentum tensor are

$$\rho_{\text{GB}} \equiv 12\dot{\xi}H^3, \quad (3.41)$$

$$P_{\text{GB}} \equiv -4\ddot{\xi}H^2 - 8\dot{\xi}H(\dot{H} + H^2), \quad (3.42)$$

respectively, which can be read off from the homogeneous equations (3.36) and (3.37). The density parameter is then (see Eq. (1.17))

$$\Omega_{\text{GB}} \equiv 4H\frac{\dot{\xi}}{M_{\text{Pl}}^2}. \quad (3.43)$$

Hence, we write the homogeneous equations as (ρ_ϕ and P_ϕ were defined in Eqs. (1.82) and (1.83), respectively) [270]

$$3H^2M_{\text{Pl}}^2(1 - \Omega_{\text{GB}}) = \rho_\phi + \rho_{\text{M}}, \quad (3.44)$$

$$-2\dot{H}M_{\text{Pl}}^2(1 - \Omega_{\text{GB}}) = \rho_\phi + P_\phi + \rho_{\text{M}} + P_{\text{M}} - 4H^2(\ddot{\xi} - H\dot{\xi}), \quad (3.45)$$

where we have included the energy density and isotropic pressure of matter fields (which are not coupled with the scalar field and the GB coupling function). If $\dot{\phi} \neq 0$, we can see that the scalar field equation (3.34) can be written as (see Eq. (1.27))

$$(\rho_\phi + \rho_{\text{GB}}) = -3H(\rho_\phi + P_\phi + \rho_{\text{GB}} + P_{\text{GB}}), \quad (3.46)$$

as expected since ϕ and ξ are obviously coupled with each other. Also

$$\dot{\rho}_{\text{M}} = -3H(\rho_{\text{M}} + P_{\text{M}}). \quad (3.47)$$

We may plug Eq. (3.34) into Eq. (3.45) to get rid of the second derivative $\ddot{\phi}$:

$$\dot{H} = -\frac{1}{2\left(M_{\text{Pl}}^2 - 4\xi H + 24\xi_{,\phi}^2 H^4\right)} \left\{ \dot{\phi}^2 + \rho_{\text{M}}(1 + w_{\text{M}}) + 4H^2 \left[4\xi H + \xi_{,\phi} \left(V_{,\phi} + 12\xi_{,\phi} H^4 \right) - \xi_{,\phi\phi} \dot{\phi}^2 \right] \right\}, \quad (3.48)$$

where the denominator is supposed not to vanish. We have $\dot{H}$ as a function of H , ϕ and $\dot{\phi}$, so we can integrate it numerically and obtain $H(t)$. Inserting $\dot{H}$ in Eq. (3.34), we may also solve $\ddot{\phi}$ numerically and obtain $\phi(t)$. Regarding the matter fields, we shall assume that $w_{\text{M}} \equiv P_{\text{M}}/\rho_{\text{M}}$ is constant and falls within the range $0 \leq w_{\text{M}} < 1$, which includes pressureless matter and radiation (see Eqs. (1.31) and (1.32), respectively).

3.1.3 Gravitational Waves

In Sec. 1.3, we derived the perturbation equations and focussed on the scalar components chiefly. Here however, our interest lies in the GWs equation that may be obtained from Eq. (3.19), and the speed of propagation of the tensor modes which were introduced in Eq. (1.283). This was shown to coincide with the speed of light in GR (see Eq. (1.285)) and in $F(\phi)R$ gravity (see Eq. (2.359)).

We perturb the metric field equations given in Eq. (3.31) about the spatially-flat FLRW metric such that

$$\begin{aligned} \delta G_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} \left[4 \left(\delta R_{\mu}^{\alpha}{}_{\nu}{}^{\beta} + 2\delta S^{\alpha}_{(\mu} \delta^{\beta}_{|\nu)} - \delta g_{\mu\nu} G^{\alpha\beta} - g_{\mu\nu} \delta G^{\alpha\beta} - \delta R_{\mu\nu} g^{\alpha\beta} - \right. \right. \\ \left. \left. - R_{\mu\nu} \delta g^{\alpha\beta} \right) \nabla_{\alpha} \partial_{\beta} \xi - 4 \left(R_{\mu}^{\alpha}{}_{\nu}{}^{\beta} + 2S^{\alpha}_{(\mu} \delta^{\beta}_{|\nu)} - g_{\mu\nu} G^{\alpha\beta} - R_{\mu\nu} g^{\alpha\beta} \right) \delta \Gamma_{\alpha\beta}^0 \dot{\xi} + \delta g_{\mu\nu} P_{\phi} \right], \end{aligned} \quad (3.49)$$

where we have only considered tensor perturbations and P_{ϕ} was defined in Eq. (1.83). Those quantities which are not preceded by the δ symbol are considered to be homogeneous (we then avoid the use of bars to denote homogeneous quantities, for simplicity).

The only non-trivial equations are $(\mu, \nu) = (i, j)$. By resorting to the perturbations derived in App. A that carry the TT tensor perturbations h_{ij} , we see that the field equations above become

$$\delta G_{ij} = \frac{1}{M_{\text{Pl}}^2} \left[4\delta R_{i0j0} \ddot{\xi} - 4H\dot{\xi} \delta R_i^k{}_j g_{kl} - 4H\dot{\xi} \left(\delta S^k{}_i g_{kj} + \delta S^k{}_j g_{ki} \right) + 4 \left(\ddot{\xi} + 3H\dot{\xi} \right) \delta R_{ij} - \right.$$

$$-12H \left(H\ddot{\xi} + 2\dot{H}\dot{\xi} + 3H^2\dot{\xi} \right) \delta g_{ij} + 4\dot{\xi} \left(H^2 + \dot{H} \right) \delta \Gamma_{ij}^0 + P_\phi \delta g_{ij} \Big]. \quad (3.50)$$

The linear tensor perturbation of $S_{\mu\nu}$ is

$$\delta S_{ij} = \frac{1}{2}a^2 \left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2}\partial_k\partial^k h_{ij} - \dot{H}h_{ij} \right). \quad (3.51)$$

Hence

$$\delta S^k{}_i g_{kj} + \delta S^k{}_j g_{ki} = a^2 \left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - \partial_k\partial^k h_{ij} \right). \quad (3.52)$$

Also

$$\delta R_{i0j0} = -a^2 \left[\frac{1}{2}\ddot{h}_{ij} + H\dot{h}_{ij} + \left(\dot{H} + H^2 \right) h_{ij} \right], \quad (3.53)$$

$$\delta R_i{}^k{}_j{}^l g_{kl} = \frac{1}{2}a^2 \left(H\dot{h}_{ij} - a^{-2}\partial_k\partial^k h_{ij} + 6H^2 h_{ij} \right). \quad (3.54)$$

Therefore, the GWs equation takes on the form

$$\begin{aligned} \delta G_{ij} = \frac{a^2}{M_{\text{Pl}}^2} \Big[& 2H\dot{\xi}\ddot{h}_{ij} + \left(2\ddot{\xi}H + 6\dot{\xi}H^2 + 2\dot{\xi}\dot{H} \right) \dot{h}_{ij} - 2\ddot{\xi}a^{-2}\partial_k\partial^k h_{ij} + \left(P_\phi - 4\ddot{\xi}H^2 - \right. \\ & \left. - 8\dot{\xi}H^3 - 8\dot{\xi}\dot{H}H \right) h_{ij} \Big], \end{aligned} \quad (3.55)$$

and using the tensor perturbation part of Eq. (A.38) and the background equation (3.37), the term accompanying h_{ij} gets removed. The equation then becomes

$$\begin{aligned} & \left(1 - 4H\frac{\dot{\xi}}{M_{\text{Pl}}^2} \right) \ddot{h}_{ij} + \left[3H \left(1 - 4H\frac{\dot{\xi}}{M_{\text{Pl}}^2} \right) - \frac{4}{M_{\text{Pl}}^2} \left(\ddot{\xi}H + \dot{\xi}\dot{H} \right) \right] \dot{h}_{ij} - \\ & - \left(1 - 4\frac{\ddot{\xi}}{M_{\text{Pl}}^2} \right) a^{-2}\partial_k\partial^k h_{ij} = 0. \end{aligned} \quad (3.56)$$

Assuming that $4H\dot{\xi} \neq M_{\text{Pl}}^2$ (i.e. $\Omega_{\text{GB}} \neq 1$; see Eq. (3.43)),² this equation can be rewritten

2. Therefore the density parameter associated with the GB term is assumed not to dominate the energy budget of the Universe. This is a reasonable assumption, because then the potential energy, $V(\phi)$, would have to be negative and cancel the contribution from the kinetic energy of the scalar field, $\dot{\phi}^2/2$, exactly. Otherwise, $\dot{\phi} = 0$, which implies $\Omega_{\text{GB}} = 0$ and makes the dominance of the density parameter of ξ impossible.

as

$$h_{ij}'' + (3 + \alpha_M) H \dot{h}_{ij} - (1 + \alpha_T) a^{-2} \partial_k \partial^k h_{ij} = 0 , \quad (3.57)$$

where we defined

$$\alpha_M \equiv -\frac{4}{H} \frac{\ddot{\xi} H + \dot{\xi} \dot{H}}{M_{\text{Pl}}^2 - 4\dot{\xi} H} , \quad (3.58)$$

$$\alpha_T \equiv -4 \frac{\ddot{\xi} - \dot{\xi} H}{M_{\text{Pl}}^2 - 4\dot{\xi} H} . \quad (3.59)$$

Eq. (3.57) can be derived in a different manner, and being acquainted with this procedure will prove to be of great use in models which lead to complicated field equations. We refer the reader to App. D. Also, Eq. (3.57) is valid when matter fields are included and no contribution in the form of an anisotropic stress tensor is considered, which shall be the case here.

The speed of GWs in the SGB model is

$$c_{\text{GW}}^2 \equiv 1 + \alpha_T , \quad (3.60)$$

and differs from the speed of light in vacuum generally (in the units where $c = 1$, which are the ones we adopt if the contrary is not stated; see discussion after Eq. (1.6)). Note also that the Hubble friction term is modified in the SGB model, same as in $F(\phi)R$ gravity (see Eq. (2.359)). If ξ is constant, then both, α_M and α_T , vanish and we recover the usual Hubble friction term and the speed of propagation of the tensor modes predicted by GR, as expected. However, we have an interesting possibility if ξ satisfies the following equation:

$$\ddot{\xi} = \dot{\xi} H . \quad (3.61)$$

In that case, $\alpha_T = 0$ and

$$\alpha_M = -\frac{\Omega_{\text{GB}}}{1 - \Omega_{\text{GB}}} (1 - \epsilon_1) , \quad (3.62)$$

where we used the first Hubble-flow parameter, defined in Eq. (1.70), and the density parameter assigned to the GB term (see Eq. (3.43)). Generally, we have

$$\alpha_M = \alpha_T - \frac{\Omega_{\text{GB}}}{1 - \Omega_{\text{GB}}} (1 - \epsilon_1) , \quad (3.63)$$

where (see Eqs. (3.44) and (1.17))

$$\frac{\Omega_{\text{GB}}}{1 - \Omega_{\text{GB}}} = \frac{\rho_{\text{GB}}}{\rho_{\phi} + \rho_{\text{M}}} . \quad (3.64)$$

Solving Eq. (3.61), one obtains

$$\xi(t) \propto \int^t d\tilde{t} a(\tilde{t}) . \quad (3.65)$$

GWs propagate at the speed of light for that choice of the coupling function, although the friction term is still altered. One can relate the α_T parameter with ϵ_1 using Eqs. (3.44) and (3.45), such that

$$\epsilon_1 = \frac{3}{2} \left(1 + \frac{P_{\phi} + P_{\text{M}}}{\rho_{\phi} + \rho_{\text{M}}} \right) + \frac{1}{2} \alpha_T . \quad (3.66)$$

Therefore, when GWs propagate at c (regardless of α_M), the GB term does not affect the acceleration of the spatial slices *directly* (this acceleration is parametrised by ϵ_1 , as we know already; see Eqs. (2.97) and (2.100)). It can have an effect on the scalar field dynamics though (see Eq. (3.34)), and hence on P_{ϕ} and ρ_{ϕ} in Eq. (3.66).

3.2 Gauss-Bonnet Dark Energy and the Speed of Gravitational Waves

A milestone in human history has been the detection of GWs [290, 291]. This event has opened a new window into the Universe and has caused an enormous impact on scalar-tensor theories and modified gravity models of DE at large. A case that concerns us³ is the multi-messenger observation of a binary neutron star system's merger in 2017 (GW170817) [292, 293]. The fact that it was possible to detect the GWs emitted by the merger of the two neutron stars as well as the electromagnetic counterpart has resulted in very tight constraints on the speed of the GWs, c_{GW} . The delay between arrival times of GWs and γ -rays led to the following bound on the α_T parameter (see Eq. (3.60)) [294]:

$$|\alpha_T| < 10^{-15} . \quad (3.67)$$

3. The work developed in Sec. 3.2 has resulted in a publication; see Ref. [3].

This constraint excluded a wide variety of well motivated DE models and narrowed down the space of available modifications to GR (see Refs. [295–301]). Among those models which were discarded, the *Gauss-Bonnet Dark Energy (GBDE)* one is claimed to be one of them [296].⁴ We however point out that the c_{GW} predicted by the GBDE model is not a constant over the expansion history of the Universe, while the constraint in Eq. (3.67) is an upper bound which is applicable only in the very latest stages of the evolution of the Universe. To be precise, it applies from redshift $z = 0.00980 \pm 0.00079$ [304], during DE domination already⁵ (see discussion after Eq. (1.45)), to $z = 0$ (today). Thus, to assess the implications of these constraints for the GBDE model, we need to consider such a bound more carefully.

In this section, we will use the *dynamical systems analysis* to search for viable GBDE models and determine the evolution of the α_T parameter. Before this however, we will review some aspects of the quintessential cosmology and introduce the dynamical systems analysis applied to GR.

3.2.1 Quintessential Cosmology

We present some known relevant aspects in the context of the study of the late Universe that will serve to set the stage for the eventual analysis of the GBDE model.

We assume a universe dominated by a fluid with constant EoS parameter w . Eq. (1.30) showed how the energy density behaved in this case and, assuming $w \neq -1$ too, we can obtain a power-law solution for the scale factor using the Friedmann equation (1.15) (with $K = 0$):

$$a(t) = a_0 t^{h_0}. \quad (3.68)$$

If $w = -1$, we would have an exponentially expanding universe, dominated by a CC, as was discussed already in preceding sections (see Eq. (1.40)). We can however express the EoS parameter in terms of the deviation from the value associated with the CC (see Eq. (1.70))

$$w = -1 + \frac{2}{3h_0}, \quad (3.69)$$

4. Modifications to the SGB model that include a coupling between the matter sector and the scalar field ϕ modelling DE, or a non-minimal coupling of this field to gravity, seem to satisfy the constraint (3.67) for very specific initial conditions set in radiation domination and a coupling function such that $\xi(\phi) \propto \phi^2$. These results are reported in Refs. [302, 303].

5. If one consults Ref. [305], the redshift value $z = 0.0099$ is the geocentric redshift of the host galaxy where the event occurred. The cosmological redshift by contrast, which is the one shown here and what we need in our study, is given in Ref. [304].

where the Hubble parameter is

$$H(t) = \frac{h_0}{t} . \quad (3.70)$$

It is clear that h_0 cannot be negative in an expanding universe but, if we were to assume that it is so, we could consider the following solution instead:

$$a(t) = a_0 (t_s - t)^{h_0} , \quad (3.71)$$

such that $t < t_s$. Therefore, the Universe expands when $h_0 < 0$ as the Hubble parameter is

$$H(t) = -\frac{h_0}{t_s - t} . \quad (3.72)$$

In either case, the EoS parameter is given by Eq. (3.69).

If $h_0 < 0$, we then see that $w < -1$ [306]. In this case, according to Eq. (1.30), $\rho(a)$ must increase over time, whereas the examples we had seen in the corresponding section (Eqs. (1.31) and (1.32)) showed that the energy densities were diluted by the expansion of the Universe. This fluid with gradually increasing energy density is referred to as ‘phantom’ in the literature [307] (see Ref. [308] and references therein as well), whilst the case $w > -1$ leads to what is called *quintessential cosmology* [309]. Quintessence⁶ is a form of DE, a field that has the EoS parameter given in Eq. (1.86) in the simplest models; i.e. a canonical scalar field with the aim of explaining the late time acceleration of the expansion of the Universe in a way similar to the inflaton in the early Universe. In some sense, the motivation for this late time scenario was grounded on the early developments in the inflationary theory [310–314], even when cosmic inflation was not generally accepted, or was not part of the conventional Cosmology (see Sec. 1.2). A clear difference between both frameworks however is the presence of matter fields in the late Universe, those playing no essential role during inflation, but that *cannot* be ignored in the study of the dynamics of the quintessence field.

Matter fields are then included. We first point out that the scale factor of Eq. (3.68) is the one of an expanding universe dominated by pressureless matter ($h_0 = 2/3$) or radiation ($h_0 = 1/2$), as we know. However, we can still have such a solution for the scale factor when the scalar field dominates. On one hand, when the potential vanishes, one has $w = 1$ (see Eq. (1.86)), which is a constant EoS parameter that leads to Eq. (3.68) with $h_0 = 1/3$.

6. The name ‘quintessence’ is reasonable because it would be the fifth element after baryonic matter, dark matter, neutrinos and radiation (photons).

This case does not trigger an accelerated expanding universe (see Eq. (1.71)). On the other hand, for a non-vanishing potential, when the scale factor takes on the power-law form, this corresponds to an exponential

$$V(\phi) = V_0 e^{-\lambda\phi/\phi_0}. \quad (3.73)$$

Indeed, using Eqs. (1.88) and (3.70), one obtains

$$\phi(t) = \frac{\phi_0}{\lambda} \ln \left(\frac{t}{t_0} \right)^2, \quad (3.74)$$

such that $\phi(t_0) = 0$, with $\lambda \equiv \pm(2\phi_0/M_{\text{Pl}})(2h_0)^{-1/2}$. Using the Friedmann equation (1.87) and solving for $V(\phi)$, we obtain the exponential potential shown above with $V_0 \equiv h_0(3h_0 - 1)M_{\text{Pl}}^2/t_0^2$. Notice that $h_0 = 1/3$ implies $V_0 = 0$ (i.e. $V(\phi) = 0$), which is the case considered earlier with $w = 1$. Also, $h_0 > 1/3$ (i.e. $w < 1$) in order for $V_0 > 0$. Setting $\phi_0 = M_{\text{Pl}}$ for simplicity, we see that when $\lambda^2 < 2$ (i.e. $h_0 > 1$) the Universe accelerates in its expansion (see Eq. (1.71)).

One important aspect of this exponential potential lies in the fact that the equations become *self-similar* and the dependence on the Hubble parameter drops out of them. It also gives rise to *scaling solutions* [315–319]. These cosmological solutions are given when the EoS parameter of the scalar field is equal to that of the dominant component among the matter fields and, therefore, the energy density of the scalar field scales in the same way as that of the matter component over the expansion of the Universe. The scaling solution is a *fixed point* of the autonomous system of equations which includes the matter fields and considers the exponential potential implicitly

$$3H^2 M_{\text{Pl}}^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_{\text{M}}, \quad (3.75)$$

$$2\dot{H} M_{\text{Pl}}^2 = -\dot{\phi}^2 - \rho_{\text{M}}(1 + w_{\text{M}}), \quad (3.76)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (3.77)$$

where w_{M} is constant (see below Eq. (1.29)), such that we assume $0 \leq w_{\text{M}} < 1$ which includes the cases of pressureless matter and radiation, as was commented already before Sec. 3.1.3. Although Eq. (3.77) was introduced in Sec. 1.2.3, we rewrite it here for convenience.

Introducing the dimensionless variables [309]

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}M_{\text{Pl}}H} , \quad (3.78)$$

$$y \equiv \frac{1}{M_{\text{Pl}}H} \sqrt{\frac{V}{3}} , \quad (3.79)$$

$$z \equiv \frac{1}{M_{\text{Pl}}H} \sqrt{\frac{\rho_{\text{M}}}{3}} , \quad (3.80)$$

we may perform the dynamical systems analysis: find the fixed points of interest and check their stability. $\Omega_{\phi} \equiv \rho_{\phi}/(3H^2M_{\text{Pl}}^2) = x^2 + y^2$ and $\Omega_{\text{M}} \equiv \rho_{\text{M}}/(3H^2M_{\text{Pl}}^2) = z^2$. The Friedmann equation (3.75) then reads

$$1 = x^2 + y^2 + z^2 . \quad (3.81)$$

Using this constraint equation, we can write $\epsilon_1 = -\dot{H}/H^2$ (see Eq. (1.70)) as

$$\epsilon_1 = 3x^2 + \frac{3}{2}z^2(1 + w_{\text{M}}) = \frac{3}{2} \left[x^2(1 - w_{\text{M}}) + (1 - y^2)(1 + w_{\text{M}}) \right] . \quad (3.82)$$

Taking the derivatives of x and y with respect to the number of e-folds $N = \ln a$ (we introduced N in Sec. 1.2.4), and using Eqs. (3.77) and (3.82) we obtain

$$x' = -3x + \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x \left[x^2(1 - w_{\text{M}}) + (1 - y^2)(1 + w_{\text{M}}) \right] , \quad (3.83)$$

$$y' = -\sqrt{\frac{3}{2}}\lambda xy + \frac{3}{2}y \left[x^2(1 - w_{\text{M}}) + (1 - y^2)(1 + w_{\text{M}}) \right] , \quad (3.84)$$

where the exponential potential is assumed already and primes are used to denote those derivatives. We assume that $\lambda > 0$.

Now, the fixed points x_c and y_c are obtained by setting $x' = y' = 0$, and these are (see Table I in Ref. [316]):

- $(x_c, y_c) = (0, 0)$, which gives $w|_c = w_{\text{M}}$; i.e. a universe dominated by the background fluid. This fixed point exists for all λ and w_{M} . We name it ‘**M**’.
- $(x_c, y_c) = (\pm 1, 0)$, which is a universe dominated by the kinetic energy of the scalar field (i.e. a regime of *kination*; see discussion before Eq. (1.143)). Hence $w|_c = 1$. These fixed points exist for all λ and w_{M} and we denote them by ‘**K** $\pm$ ’.

•

$$(x_c, y_c) = \left(\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}} \right), \quad (3.85)$$

which is when the scalar field dominates with $\lambda^2 = 2/h_0$ (see discussion below Eq. (3.74)). Therefore (see Eq. (3.69))

$$w|_c = -1 + \frac{\lambda^2}{3}. \quad (3.86)$$

This fixed point exists for $\lambda^2 < 6$ (so y_c is real) and we call it ‘**I**’ because, if $\lambda^2 < 2$, it corresponds to a regime of *power law inflation* [320] (given that $\epsilon_1|_c = \lambda^2/2$; see Eq. (1.70)).

•

$$(x_c, y_c) = \left(\sqrt{\frac{3}{2} \frac{(1 + w_M)^2}{\lambda^2}}, \sqrt{\frac{3}{2} \frac{1 - w_M^2}{\lambda^2}} \right), \quad (3.87)$$

which yields

$$\Omega_\phi|_c = 3 \frac{1 + w_M}{\lambda^2}. \quad (3.88)$$

And, using Eq. (3.82)

$$\epsilon_1|_c = \frac{3}{2} (1 + w_M). \quad (3.89)$$

Hence $w|_c = w_M$ (see Eq. (1.70)). This is the scaling solution with (see Eq. (1.86))

$$w_\phi|_c = \frac{x_c^2 - y_c^2}{x_c^2 + y_c^2} = w_M. \quad (3.90)$$

This fixed point exists for $\lambda^2 > 3(1 + w_M)$ (so $\Omega_\phi|_c < 1$; see Eq. (3.81)). We use ‘**Sc**’ to denote it.

The de Sitter solution, $(x_c, y_c) = (0, 1)$, is not considered as it corresponds to a constant potential ($\lambda = 0$) with $w = -1$, which does not lead to a power-law solution for the scale factor as was argued already.

It can be shown that the scaling solution is stable (and therefore an *attractor*) [316, 318].

If one perturbs the equations around the fixed point, these read

$$\delta x' = -3 \left[1 - \frac{3}{2}x_c^2(1 - w_M) - \frac{1}{2}(1 - y_c^2)(1 + w_M) \right] \delta x + y_c \left[\sqrt{6}\lambda - 3x_c(1 + w_M) \right] \delta y , \quad (3.91)$$

$$\delta y' = -y_c \left[\sqrt{\frac{3}{2}}\lambda - 3x_c(1 - w_M) \right] \delta x + \left[\frac{3}{2}x_c^2(1 - w_M) - \sqrt{\frac{3}{2}}\lambda x_c + \frac{3}{2}(1 - 3y_c^2)(1 + w_M) \right] \delta y . \quad (3.92)$$

We calculate the eigenvalues of the matrix of coefficients $\mathbb{M}$ from the equation $\delta \mathbf{x}' = \mathbb{M} \delta \mathbf{x}$, with $\delta \mathbf{x} = (\delta x, \delta y)$. The 2×2 matrix $\mathbb{M}$ has the following entries:

$$\mathbb{M}_{11} = -3 \left[1 - \frac{3}{2}x_c^2(1 - w_M) - \frac{1}{2}(1 - y_c^2)(1 + w_M) \right] , \quad (3.93)$$

$$\mathbb{M}_{12} = y_c \left[\sqrt{6}\lambda - 3x_c(1 + w_M) \right] , \quad (3.94)$$

$$\mathbb{M}_{21} = -y_c \left[\sqrt{\frac{3}{2}}\lambda - 3x_c(1 - w_M) \right] , \quad (3.95)$$

$$\mathbb{M}_{22} = \frac{3}{2}x_c^2(1 - w_M) - \sqrt{\frac{3}{2}}\lambda x_c + \frac{3}{2}(1 - 3y_c^2)(1 + w_M) . \quad (3.96)$$

The eigenvalues, which we denote by ‘ m ’, are

$$m_{1,2} = \frac{1}{2} \left[\mathbb{M}_{11} + \mathbb{M}_{22} \pm \sqrt{4\mathbb{M}_{12}\mathbb{M}_{21} + (\mathbb{M}_{11} - \mathbb{M}_{22})^2} \right] . \quad (3.97)$$

For $\mathbf{M}$, we have

$$m_{1,2} = \frac{3}{2}(w_M \mp 1) . \quad (3.98)$$

Since we assumed that $0 \leq w_M < 1$, this fixed point is a *saddle point* as $m_1 < 0$ and $m_2 > 0$ [318].

For $\mathbf{K}+$

$$m_1 = 3(1 - w_M) , \quad (3.99)$$

$$m_2 = 3 - \sqrt{\frac{3}{2}}\lambda . \quad (3.100)$$

This fixed point is always unstable given that $m_1 > 0$ (because $0 \leq w_M < 1$). If $\lambda > \sqrt{6}$, it

will be a saddle point. The other fixed point, $\mathbf{K}-$, gives

$$m_1 = 3(1 - w_M), \quad (3.101)$$

$$m_2 = 3 + \sqrt{\frac{3}{2}}\lambda, \quad (3.102)$$

and we readily see that it is unstable.

For the fixed point $\mathbf{I}$, we have

$$m_1 = \lambda^2 - 3(1 + w_M), \quad (3.103)$$

$$m_2 = \frac{1}{2}(\lambda^2 - 6). \quad (3.104)$$

Since $\lambda^2 < 6$ in order for the fixed point to exist (see Eq. (3.85)), then $m_2 < 0$ always. Since $0 \leq w_M < 1$, m_1 may be positive. In particular, power law inflation was possible if $\lambda^2 < 2$, and we can see that this implies that m_1 is definitely negative and the power law inflation regime is stable (the fixed point is a stable node [318]).

Finally, the scaling solution $\mathbf{Sc}$ yields

$$m_{1,2} = -\frac{3}{4}(1 - w_M) \left[1 \mp \sqrt{1 - 8 \frac{1 + w_M}{1 - w_M} \left(1 - \Omega_\phi|_{\mathbf{Sc}} \right)} \right], \quad (3.105)$$

where $\Omega_\phi|_{\mathbf{Sc}}$ is given in Eq. (3.88). Since $\Omega_\phi|_{\mathbf{Sc}} < 1$, and given that $0 \leq w_M < 1$, these eigenvalues are both complex and their real part is negative. The stability of the scaling solution is ensured then [318], same as in the power law inflation regime. However, this stability means that the scaling solution cannot account for the current acceleration of the Universe as $0 \leq w_M < 1$ implies that $\ddot{a} < 0$ (see Eqs. (1.70) and (1.71)).

A transition from this scaling solution to $\mathbf{I}$ with $\lambda^2 < 2$ (thus $w < -1/3$) may be viable though [318, 319]. This transition assumes that λ is no longer a constant. Defining λ as

$$\lambda \equiv -M_{\text{Pl}} \frac{V_{,\phi}}{V}, \quad (3.106)$$

we see that a constant λ gives the exponential potential of (3.73) (with $\phi_0 = M_{\text{Pl}}$). Deriving λ with respect to N , we obtain

$$\lambda' = -\lambda^2 \left(\frac{V_{,\phi\phi} V}{V_{,\phi}^2} - 1 \right) \frac{\phi'}{M_{\text{Pl}}} = \mp \lambda^2 \left(\frac{V_{,\phi\phi} V}{V_{,\phi}^2} - 1 \right) \sqrt{3\Omega_\phi (1 + w_\phi)}. \quad (3.107)$$

Given a convex potential function $V_{,\phi\phi} > 0$, for a field rolling down the potential, we have that $|\lambda|$ decreases in time if $V_{,\phi\phi}V/V_{,\phi}^2 > 1$, meaning that the potential becomes gradually constant and will eventually dominate over the background fluid under consideration. The Universe then abandons the scaling regime, approaches the fixed point **I** until the potential is flat enough to give way to the accelerated expansion in the late Universe [318, 319, 321]. Since the energy density of the scalar field gains upon that of the dominant background fluid, overtaking it, the condition above on $V_{,\phi\phi}V/V_{,\phi}^2$ is called *tracking condition* [318, 322, 323].

Another closed system of equations that can be obtained involves the density parameter of the scalar field and the EoS parameter w_ϕ , together with Eq. (3.107). Given that

$$-\frac{H'}{H} = \frac{3}{2} [\Omega_\phi (1 + w_\phi) + \Omega_M (1 + w_M)], \quad (3.108)$$

and the continuity equation $\rho'_\phi = -3\rho_\phi (1 + w_\phi)$, we derive Ω_ϕ with respect to N and obtain

$$\Omega'_\phi = -3\Omega_\phi (1 - \Omega_\phi) (w_\phi - w_M). \quad (3.109)$$

Regarding the EoS parameter, since $w_\phi = (x^2 - y^2)/(x^2 + y^2)$, we simply derive it with respect to N and use the equations x' and y' (Eqs. (3.83) and (3.84), respectively) such that

$$w'_\phi = (w_\phi - 1) \left[3(w_\phi + 1) - \lambda \sqrt{3\Omega_\phi (w_\phi + 1)} \right]. \quad (3.110)$$

The evolution of w_ϕ that can be inferred using these equations for different relevant quintessential models is outlined in Ref. [309]. The quintessential models in which the field rolls towards the minimum of its potential before the acceleration stage sets in and therefore $w \rightarrow -1$ are called *freezing models* [309]. There is also another set of models, *thawing models* [324], in which the field evolution is damped by the Hubble friction term of the scalar field equation (3.77), $3H\dot{\phi}$, so it remains at a value displaced from the minimum of the potential with $w_\phi \simeq -1$ until the current epoch, in which it begins to roll down to such a minimum.

3.2.2 Quintessence in the Scalar-Gauss-Bonnet Model and Constraints on

CGW

The presence of the field dependent coupling to the GB term brings in some new fixed points in the dynamical systems analysis, as well as altering the stability of the ones shown before.

The possibility of explaining DE with the GB term was first investigated in Refs. [270,

325–329]. As was pointed out before Sec. 3.2.1, quite generically GBDE models predict $\alpha_T \neq 0$. One choice yielding $\alpha_T = 0$, which was discussed already, is that ξ satisfies Eq. (3.61) (apart from the trivial one with constant ξ). This is discussed in Refs. [330–333] (other related references can be found therein) in the context of cosmic inflation (see also Refs. [281, 282, 287, 334, 335] where α_T was derived first). However, there is another, more generic option. We notice that α_T is suppressed in Eq. (3.59) by the Planck mass M_{Pl} . Therefore, as long as the conditions

$$\frac{|\ddot{\xi}|}{M_{\text{Pl}}^2}, \frac{H|\dot{\xi}|}{M_{\text{Pl}}^2} < 10^{-15}, \quad (3.111)$$

are satisfied, the action in Eq. (3.19) is compatible with the constraints on the speed of GWs (3.67). These conditions can be written in a more suggestive way

$$\frac{|\ddot{\xi}|}{H^2}, \frac{|\dot{\xi}|}{H} < 10^{-15} \left(\frac{M_{\text{Pl}}}{H} \right)^2. \quad (3.112)$$

This emphasises the change of the coupling function and the rate of change of this coupling over one Hubble time, $1/H$. When these conditions are imposed on large field models of inflation, $H_I \lesssim 10^{-5} M_{\text{Pl}}$ [64] (see Sec. 1.3.4), and hence we see that the bound can be tight. In that case then, it is reasonable to search for models that satisfy the condition we showed in Eq. (3.61). Nevertheless, the constraints do not apply to the period of inflation, so it is not very useful to use them in that context. Instead, they apply to the late Universe as was commented already (see discussion below Eq. (3.67)), within a very narrow range of e-folds (see Fig. 3.4). The Hubble parameter is then more than fifty orders of magnitude smaller than the inflation value cited above: $H_0 \sim 10^{-60} M_{\text{Pl}}$. Plugging this into Eq. (3.112), we obtain⁷

$$\frac{|\ddot{\xi}|}{H_0^2}, \frac{|\dot{\xi}|}{H_0} < 10^{105}. \quad (3.113)$$

This means that ξ and $\dot{\xi}$ need to vary by more than 100 orders of magnitude, $\Delta\xi, \Delta\dot{\xi} < 10^{105}$, over the age of the Universe to violate the bound. This seems to demonstrate that the constraint on c_{GW} is exceptionally weak and might give hope that the GBDE models remain viable after the 2017 detection. However, as we will show, at least for simple coupling functions $\xi(\phi)$, this is not the case.

7. This is only an order of magnitude estimate. Factors of order 1 do not change the conclusions reached here in any substantial way.

To analyse the dynamical system in the SGB model, we define x , y and z as in Eqs. (3.78), (3.79) and (3.80), respectively, but we add a new dimensionless variable that we call ‘ u ’:

$$u \equiv -2\sqrt{6}\frac{H^2\xi_{,\phi}}{M_{\text{Pl}}} . \quad (3.114)$$

The Friedmann equation, Eq. (3.44), can be written as (compare with Eq. (3.81))

$$1 = x^2 + y^2 + z^2 - 2ux . \quad (3.115)$$

This constraint equation fixes the dynamics onto the 3-dimensional hypersurface in the 4-dimensional phase space. Notice that if ξ is constant ($\xi_{,\phi} = 0$), we recover Eq. (3.81) of GR. Whereas y and z were constrained within the range $[0, 1]$ and $x \in [-1, 1]$ in General Relativistic quintessence models according to Eq. (3.81), in the SGB model however such a restriction does not apply. Due to an *a priori* undetermined sign of u , maximum values of $|x|$, y and z are not limited to 1.

The dimensionless variables defined in Eqs. (3.78)-(3.80) and (3.114) are related to the more familiar density parameters by $\Omega_\phi = x^2 + y^2$, $\Omega_{\text{M}} = z^2$ (see below Eq. (3.80)) and (see Eqs. (3.43), (3.78) and (3.114))

$$\Omega_{\text{GB}} = -2ux . \quad (3.116)$$

Following this we can write the constraint equation (3.115) as

$$1 = \Omega_\phi + \Omega_{\text{M}} + \Omega_{\text{GB}} . \quad (3.117)$$

In line with what was explained below Eq. (3.115), Ω_ϕ and Ω_{M} may exceed 1 due to the undetermined sign of Ω_{GB} (of u) *a priori*. This does not happen in GR where $|\Omega| \leq 1$.

Deriving x , y and z with respect to the number of e-folds, assuming the exponential potential (3.73) (with $\phi_0 = M_{\text{Pl}}$), and using Eqs. (3.34) and (3.47), we arrive at

$$x' = -(3 - \epsilon_1)x + \sqrt{\frac{3}{2}}\lambda y^2 + u(1 - \epsilon_1) , \quad (3.118)$$

$$y' = -\left(\sqrt{\frac{3}{2}}\lambda x - \epsilon_1\right)y , \quad (3.119)$$

$$z' = \left[\epsilon_1 - \frac{3}{2}(1 + w_{\text{M}})\right]z , \quad (3.120)$$

where (see Eqs. (1.70) and (3.45))

$$\epsilon_1 = \frac{1}{1+ux} \left[3x^2 + \frac{3}{2}z^2(1+w_M) - ux + (ux)' \right]. \quad (3.121)$$

It might appear that Eq. (3.121) is inconsistent because it diverges as $ux \rightarrow -1$. However, the constraint equation (3.115) forbids such values.

Besides the equations above, it is convenient to use the equation for u as well. If we take the derivative of u , we find

$$u' = -2\epsilon_1 u - 12H^2 \xi_{,\phi\phi} x. \quad (3.122)$$

A priori we do not impose the condition $u' = 0$ at fixed points, although it follows from the equations. We may see this if we take the derivative of the constraint equation (3.115) with respect to N . At a given fixed point, this equation reads

$$u'|_c x_c = 0. \quad (3.123)$$

This equation allows for $u'|_c \neq 0$ if $x_c = 0$. However, plugging $x' = x_c = 0$ into Eqs. (3.118) and (3.121), we find (and $y = y_c$ and $z = z_c$)

$$0 = \sqrt{\frac{3}{2}} \lambda y_c^2 + u(1 - \epsilon_1|_c). \quad (3.124)$$

$\epsilon_1|_c$ is the value of ϵ_1 at $x' = y' = z' = x_c = 0$

$$\epsilon_1|_c = \frac{3}{2} z_c^2 (1 + w_M). \quad (3.125)$$

As it is clear from the above equation, all quantities on the RHS are constant excepting u . Also, $\epsilon_1|_c \neq 1$. This can be seen from Eq. (3.120) where, in order for $z' = 0$, either $z_c = 0$ or $\epsilon_1|_c = \frac{3}{2}(1 + w_M) \neq 1$ (given that $0 \leq w_M < 1$). Hence, $u = u_c$ must be also a constant even for $x_c = 0$.

Knowing that u_c is a constant, we write the algebraic equations to find fixed point values

$$0 = -(3 - \epsilon_1|_c) x_c + \sqrt{\frac{3}{2}} \lambda y_c^2 + u_c (1 - \epsilon_1|_c), \quad (3.126)$$

$$0 = \left(\epsilon_1|_c - \sqrt{\frac{3}{2}} \lambda x_c \right) y_c, \quad (3.127)$$

$$0 = \left[\epsilon_1|_c - \frac{3}{2}(1 + w_M) \right] z_c. \quad (3.128)$$

$\epsilon_1|_c$ is given by

$$\epsilon_1|_c = \frac{1}{1 + u_c x_c} \left[3x_c^2 + \frac{3}{2} z_c^2 (1 + w_M) - u_c x_c \right]. \quad (3.129)$$

Four possible solutions of this system of equations (including the constraint equation (3.115)) are:

• **A:**

$$(x, y, u) = \left(x_c, 0, -\frac{3(1 - w_M)}{1 + 3w_M} x_c \right), \quad (3.130)$$

which is valid for any value of x_c . This gives $w|_c = w_M$ (see Eq. (1.70)) and hence we see that this is the scaling solution, where the scalar field adjusts to the EoS parameter of the background fluid.

• **B:**

$$(x, y, u) = \left(x_c, 0, -\frac{1 - x_c^2}{2x_c} \right), \quad (3.131)$$

with

$$w|_c = -1 + \frac{2}{3} \frac{1 + 5x_c^2}{1 + x_c^2}. \quad (3.132)$$

• **C:**

$$(x, y, u) = \left(x_c, \sqrt{1 - x_c^2 + 2x_c \frac{3x_c - \sqrt{\frac{3}{2}}\lambda}{1 + \sqrt{\frac{3}{2}}\lambda x_c}}, \frac{3x_c - \sqrt{\frac{3}{2}}\lambda}{1 + \sqrt{\frac{3}{2}}\lambda x_c} \right), \quad (3.133)$$

with

$$w|_c = -1 + \sqrt{\frac{2}{3}} \lambda x_c. \quad (3.134)$$

• **D:**

$$(x, y, u) = \left(\sqrt{\frac{3}{2}} \frac{1 + w_M}{\lambda}, \frac{\sqrt{\frac{3}{2}}(1 - w_M^2) + \frac{\lambda}{\sqrt{6}}(1 + 3w_M)u_c}{\lambda}, u_c \right), \quad (3.135)$$

with $w|_c = w_M$. This is a second scaling solution, valid for any value of u_c (allowed by the constraint equation (3.115)). Notice that if $u_c = 0$, we recover x and y from **Sc** of Sec. 3.2.1 (see Eq. (3.87)).

These fixed ‘points’ are actually *curves* in the 3-dimensional phase space. In the case of **A** to **C**, the curves are parametrised by the value of x_c , which can vary within the range where y , u and z remain real. In the case of **D**, the value of x is fixed to be $x_c = \sqrt{\frac{3}{2}} \frac{1 + w_M}{\lambda}$. $u = u_c$ however is free to vary within the similarly defined range.

3.2.3 Exponential Gauss-Bonnet Coupling: Fixed Points and Stability Analysis

The following coupling function:

$$\xi(\phi) = \xi_0 e^{\kappa\phi/\phi_0} , \quad (3.136)$$

is a good approximation to the one found in bosonic and heterotic string theories [279, 281–285, 328]. We study this case first, with $\phi_0 = M_{\text{Pl}}$ and $\kappa > 0$. Eq. (3.122) can be written as

$$u' = -2 \left(\epsilon_1 - \sqrt{\frac{3}{2}} \kappa x \right) u . \quad (3.137)$$

Notice that $u' = 0$ yields

$$\sqrt{\frac{3}{2}} \kappa x_c u_c = \epsilon_1|_c u_c . \quad (3.138)$$

$u_c = 0$ is an allowed solution. However, assuming that $\epsilon_1|_c u_c \neq 0$, we see that the corresponding fixed points exist only if the coupling function $\xi(\phi)$ takes on the form

$$\xi(\phi) = c_1 \frac{\phi}{M_{\text{Pl}}} + c_2 e^{\kappa\phi/M_{\text{Pl}}} , \quad (3.139)$$

where c_1 and $c_2 \neq 0$ are constants, and κ is determined to be

$$\kappa = \sqrt{\frac{2}{3}} \frac{\epsilon_1|_c}{x_c} . \quad (3.140)$$

To be precise, it is sufficient that the coupling function $\xi(\phi)$ approaches Eq. (3.139) asymptotically, as the trajectory in the phase space gets closer to those fixed points.

Plugging values consistent with Eq. (3.138) into the system **A-D** seen above, we obtain fixed points which are summarised in Table 3.1. In that table, one of the fixed points, **G**, has $x_c = \beta$. This is a solution of a cubic equation, and it is given by

$$\beta \equiv \frac{1}{3\sqrt{3}\kappa} \left(5\sqrt{2} + \frac{9\kappa^2 - 50}{\alpha^{1/3}} - \alpha^{1/3} \right) , \quad (3.141)$$

where α is a function of κ

$$\alpha = \sqrt{2} \left(27\kappa^2 - 250 \right) + \sqrt{2 \left(27\kappa^2 - 250 \right)^2 + \left(9\kappa^2 - 50 \right)^3} . \quad (3.142)$$

Looking at Table 3.1, we notice that all the fixed points that are present in the General

	x_c	y_c	u_c	z_c	$\epsilon_1 _c$	If κ is	$\epsilon_1 _c u_c$
M	0	0	0	1	$\frac{3}{2}(1+w_M)$	any κ	0
K$\pm$	± 1	0	0	0	3		
I	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\lambda^2$		
Sc	$\sqrt{\frac{3}{2}\frac{1+w_M}{\lambda}}$	$\frac{\sqrt{\frac{3}{2}(1-w_M^2)}}{\lambda}$	0	$\sqrt{1-\frac{3(1+w_M)}{\lambda^2}}$	$\frac{3}{2}(1+w_M)$		
dS	0	1	$-\sqrt{\frac{3}{2}}\lambda$	0	0		
S2	$\sqrt{\frac{3}{2}\frac{1+w_M}{\kappa}}$	0	$-\sqrt{\frac{3}{2}\frac{3(1-w_M^2)}{\kappa(1+3w_M)}}$	$\sqrt{1-\frac{3(7-3w_M)(1+w_M)^2}{2\kappa^2(1+3w_M)}}$	$\frac{3}{2}(1+w_M)$	$\kappa \neq \lambda$	$\neq 0$
G	β	0	$-\frac{1-\beta^2}{2\beta}$	0	$\frac{1+5\beta^2}{1+\beta^2}=\sqrt{\frac{3}{2}}\kappa\beta$		
S3	$\sqrt{\frac{3}{2}\frac{1+w_M}{\lambda}}$	$\frac{\sqrt{\frac{3}{2}(1-w_M^2)}+\frac{\lambda}{\sqrt{6}}(1+3w_M)u_c}{\lambda}$	u_c	$\sqrt{1-\frac{3(1+w_M)-\frac{\lambda}{\sqrt{6}}(5+3w_M)u_c}{\lambda^2}}$	$\frac{3}{2}(1+w_M)$	$\kappa = \lambda$	
IV	x_c	$\sqrt{1-x_c^2+2x_c\frac{3x_c-\sqrt{\frac{3}{2}}\lambda}{1+\sqrt{\frac{3}{2}}\lambda x_c}}$	$\frac{3x_c-\sqrt{\frac{3}{2}}\lambda}{1+\sqrt{\frac{3}{2}}\lambda x_c}$	0	$\sqrt{\frac{3}{2}}\lambda x_c$		

Table 3.1: Values of x , y , u and z variables at the fixed points and curves. The conditions for the existence of these fixed points and curves can be derived from the fact that all x_c , y_c , u_c and z_c values must be real. $\lambda > 0$, $\kappa > 0$ and β are defined in Eqs. (3.73), (3.136) (with $\phi_0 = M_{Pl}$) and (3.141) respectively.

Relativistic quintessence models (see Sec. 3.2.1), they are also present in the GBDE model under consideration. The fixed point **M** corresponds to the case of the background fluid domination. **K $\pm$** fixed points correspond to kination and, in the case of **I** and $\lambda < \sqrt{2}$, the corresponding fixed point represents the power law inflation regime [320]. In the scaling regime **S_c**, the evolution of the scalar field adjusts to mimic the behaviour of the background fluid.

As we commented already, the coupling function introduces new aspects regarding the fixed points and their stability. As to the fixed ‘points’, we have *two* more scaling solutions: **S₂** and **S₃**. The former exists for $\kappa \neq \lambda$ and **S₃** is a fixed curve which exists if $\xi(\phi)V(\phi) \equiv \text{constant}$. However, the most interesting fixed point is **dS** which corresponds to the de Sitter regime, where $\dot{H} = 0$. This fixed point is very robust, and exists for a wide variety of $\xi(\phi)$ coupling functions. Consequently, as opposed to what we saw in Sec. 3.2.1, $\lambda \neq 0$ in the SGB model allows for the existence of the de Sitter regime among the rest of fixed points (see comment in this regard after Eq. (3.90)).

Some discussion of DE models with $\kappa = \lambda$ and various solutions were provided in Ref. [325]. On the other hand, some dynamical analysis with $\kappa \neq \lambda$ was carried out in Ref. [328] (see also Ref. [336]). We shall modify and extend such an analysis to make it more generic. We take $\kappa \neq \lambda$.

We then write the linear equations to determine the stability of the fixed points of interest. In the case of the exponential coupling function, those read

$$\delta x' = -(3 - \epsilon_1|_c) \delta x + \sqrt{6} \lambda y_c \delta y + (1 - \epsilon_1|_c) \delta u + (x_c - u_c) \delta \epsilon_1 , \quad (3.143)$$

$$\delta y' = -\sqrt{\frac{3}{2}} \lambda y_c \delta x + \left(\epsilon_1|_c - \sqrt{\frac{3}{2}} \lambda x_c \right) \delta y + y_c \delta \epsilon_1 , \quad (3.144)$$

$$\delta u' = \sqrt{6} \kappa u_c \delta x + \left(\sqrt{6} \kappa x_c - 2 \epsilon_1|_c \right) \delta u - 2 u_c \delta \epsilon_1 , \quad (3.145)$$

$$\delta z' = \left[\epsilon_1|_c - \frac{3}{2} (1 + w_M) \right] \delta z + z_c \delta \epsilon_1 . \quad (3.146)$$

$\delta \epsilon_1$ is given by

$$\delta \epsilon_1 = \frac{1}{1 + u_c x_c} \left\{ u_c \delta x' + x_c \delta u' + [6 x_c - (1 + \epsilon_1|_c) u_c] \delta x - x_c (1 + \epsilon_1|_c) \delta u + 2 \epsilon_1|_c z_c \delta z \right\} . \quad (3.147)$$

We linearise the constraint equation (3.115) too

$$0 = (x_c - u_c) \delta x + y_c \delta y + z_c \delta z - x_c \delta u . \quad (3.148)$$

Next step is to compute the eigenvalues of the system of equations (3.143)-(3.146) above and determine their stability. In the case of $\mathbf{M}$, we have

$$m_3 = -3 (1 + w_M) , \quad (3.149)$$

in addition to Eq. (3.98). Hence, the fixed point is still unstable, as in GR. Regarding the kination fixed points $\mathbf{K}\pm$, we have (the upper sign corresponds to the point $\mathbf{K}+$)

$$m_1 = \frac{3}{2} (1 - w_M) , \quad (3.150)$$

$$m_2 = 3 \mp \sqrt{\frac{3}{2}} \lambda , \quad (3.151)$$

$$m_3 = -6 \pm \sqrt{6} \kappa . \quad (3.152)$$

Consequently, these fixed points remain unstable in the SGB model as well.

The results change when we consider the scaling solution $\mathbf{Sc}$ and the de Sitter one $\mathbf{dS}$ (especially because the latter did not exist in the quintessence model of GR with exponential potential, as we said already).

In the case of **Sc**, the eigenvalues (3.105) include

$$m_3 = 3(1 + w_M) \left(\frac{\kappa}{\lambda} - 1 \right). \quad (3.153)$$

We see that the stability of **Sc** is now determined by the sign of m_3 , which is non-negative if $\kappa \geq \lambda$ (again, remember that $\kappa, \lambda > 0$). For $\kappa > \lambda$, the scaling fixed point **Sc** is a saddle point and it is then *unstable*.

Furthermore, if we calculate the eigenvalues at the de Sitter fixed point **dS**, we obtain

$$m_1 = -\frac{3}{2}(1 + w_M), \quad (3.154)$$

$$m_{2,3} = -\frac{3}{2} \left[1 \pm \sqrt{1 - \frac{8\lambda^2}{3(2 + 3\lambda^2)} \left(\frac{\kappa}{\lambda} - 1 \right)} \right], \quad (3.155)$$

and we notice that the condition for the stability of this fixed point is exactly the opposite of that of the scaling regime. If $\kappa > \lambda$, the scaling fixed point is a saddle but the de Sitter one becomes an *attractor*. That is, only one fixed point is attractive, either **Sc** or **dS**, depending on the magnitude of the κ/λ ratio.

As was commented already, in the case of GR models and the exponential potential, the scaling fixed point **Sc** being an attractor was problematic because the Universe is not accelerating at **Sc** in contrast to what is required by observations (see discussion following Eq. (3.105)). Here however, the coupling function $\xi(\phi)$, coupled to the GB term, converts the scaling fixed point **Sc** into a saddle point. The trajectories are repelled and move to the de Sitter attractor **dS**, such that the scalar field potential energy comes to dominate and triggers the accelerated expansion of the Universe.

We integrate numerically a set of trajectories and show the phase portraits in Fig. 3.1. This will allow us to visualise the behaviour of the system under consideration. Since none of the solutions of cosmological interest pass through the fixed points **S2** and **G**, we do not analyse their stability in this work.

In all of the simulations, we have started with a negligible contribution from the GB term. To be precise, we set $u_0 = -10^{-25}$. Such small values are consistent with the scenario proposed in Ref. [329], where the GB contribution is negligible initially. All the trajectories in the plots in Fig. 3.1 start from $y_0 = 10^{-3}$ and approach the stable fixed point **dS** at $y = 1$. The trajectories of cosmological interest are those that start close to the **K $\pm$** or **M** fixed points, where the GB term contribution is negligible. The former couple of points corresponds to kination, as was pointed out already. This is the desirable starting point

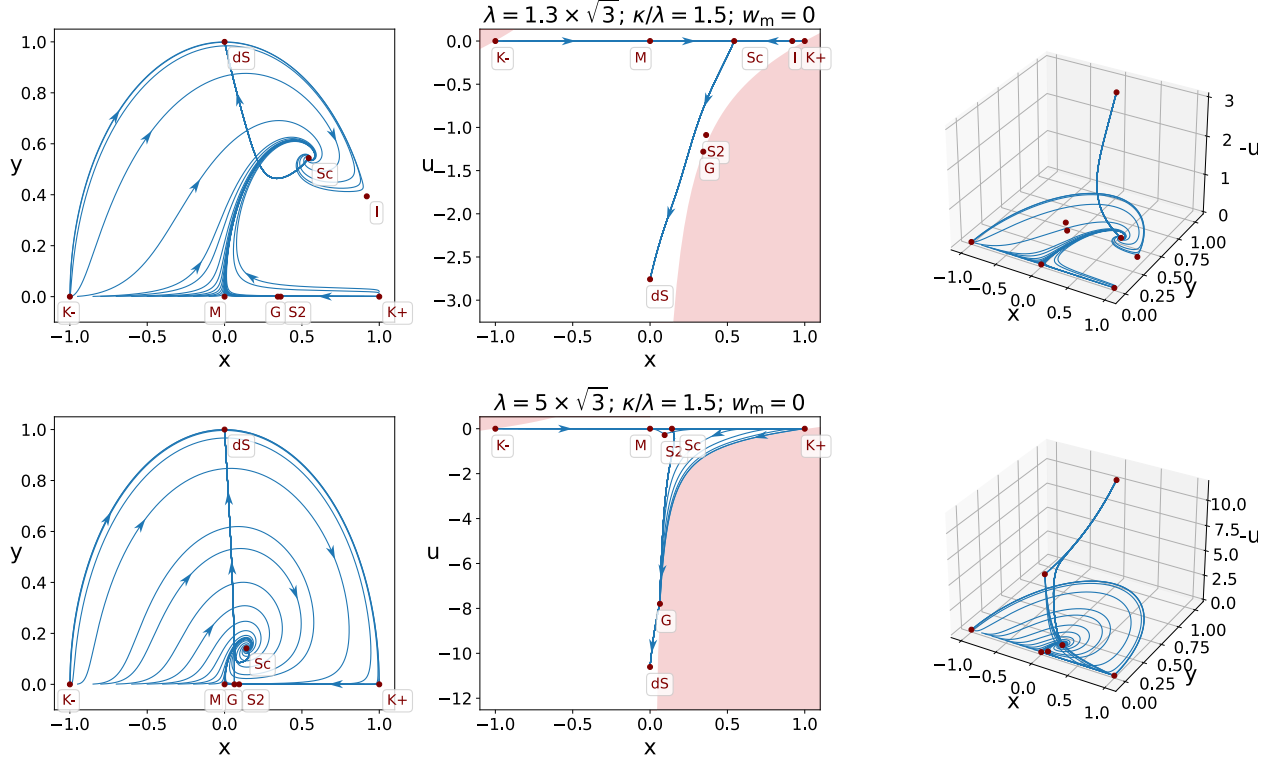


Figure 3.1: Phase portrait of the dynamical system of Eqs. (3.118), (3.119) and (3.137), with the background fluid component represented by pressureless matter, $w_M = w_m = 0$. All trajectories start with very small values of y and u . The function $\xi(\phi)$ coupled to the GB term converts the scaling fixed point **Sc** into a saddle one and provides an escape route in contrast to the case of GR models with the exponential potential. The scalar field may take over eventually and cause the Universe to expand in an accelerated fashion as the trajectories in the phase space converge onto the de Sitter attractor **dS** at $y = 1$. The first two columns show projections of the 3-dimensional phase space. The full 3D portrait is displayed in the last column. Red regions in the second column indicate forbidden regions of the phase space (for those values the constraint in Eq. (3.115) cannot be satisfied). The upper plot corresponds to $\lambda = 1.3\sqrt{3}$ and the lower one to $\lambda = 5\sqrt{3}$. In both cases, $\kappa/\lambda = 1.5$ so $\kappa > \lambda$.

in the case of *quintessential inflation* models [337–340], where the scalar field responsible for inflation takes the role of quintessence in the late Universe. Initially, the Universe is dominated by the kinetic energy of the scalar field. Then, the system moves into the scaling fixed point and, for a long time, the Universe evolves with an effective EoS parameter that corresponds to that of the background fluid (be it radiation or pressureless matter). In the case of **M**, the expected sequence of events is the same except that the system begins in a regime of background fluid domination, often found in the quintessence models.

Although in Fig. 3.1, all displayed trajectories start with initial value $u_0 = -10^{-25}$, this choice is also partly motivated by simplicity. As one can see in Fig. 3.2, the dynamics between the scaling fixed point **Sc** and the de Sitter one **dS** is not affected much, at least for some part of the trajectories. The sign of u_0 , on the other hand, is crucially important. We provide several plots in that figure with numerically integrated trajectories for several u_0 values. For clarity, we show the plots with negative and positive u_0 separately. Only those trajectories that satisfy $u_0 < 0$ (i.e. positive initial value of $\xi_{,\phi}$; see Eq. (3.114)) can eventually be attracted towards the **dS** fixed point. Also, negative values of u_0 such that $|u_0|$ is of order $\mathcal{O}(1)$ could provide promising candidate trajectories to explain DE. This is because a large portion of the phase space initially evolves towards smaller values of u , reaches the scaling fixed point **Sc** and follows the same evolution pattern as those trajectories with $|u_0| \ll 1$. However, for our ongoing analysis, we choose $|u_0| \ll 1$, as this increases the parameter space for viable candidate trajectories. Finally, we remark that the sign of u is found to be preserved by the evolution equations.

In Fig. 3.3, we display the time evolution of the density parameters Ω_ϕ , Ω_M and Ω_{GB} , as well as that of the effective EoS parameter of the *dark fluid* responsible for the current accelerated expansion. In order to apply the observational bounds on the EoS parameter of DE, we define ‘ w_f ’ which can be interpreted as the EoS parameter of the effective ‘dark fluid’. Using Eqs. (3.44), (3.59) and (3.66), we write

$$\epsilon_1 = \frac{3}{2} \frac{\rho_\phi (1 + w_f) + \rho_M (1 + w_M)}{\rho_\phi + \rho_M} , \quad (3.156)$$

where

$$w_f \equiv \frac{P_\phi - 4H^2 (\ddot{\xi} - \dot{\xi}H)}{\rho_\phi} . \quad (3.157)$$

ρ_ϕ and P_ϕ were defined in Eqs. (1.82) and (1.83) and are the energy density and isotropic pressure of the scalar field ϕ , respectively. In terms of the dimensionless variables defined in

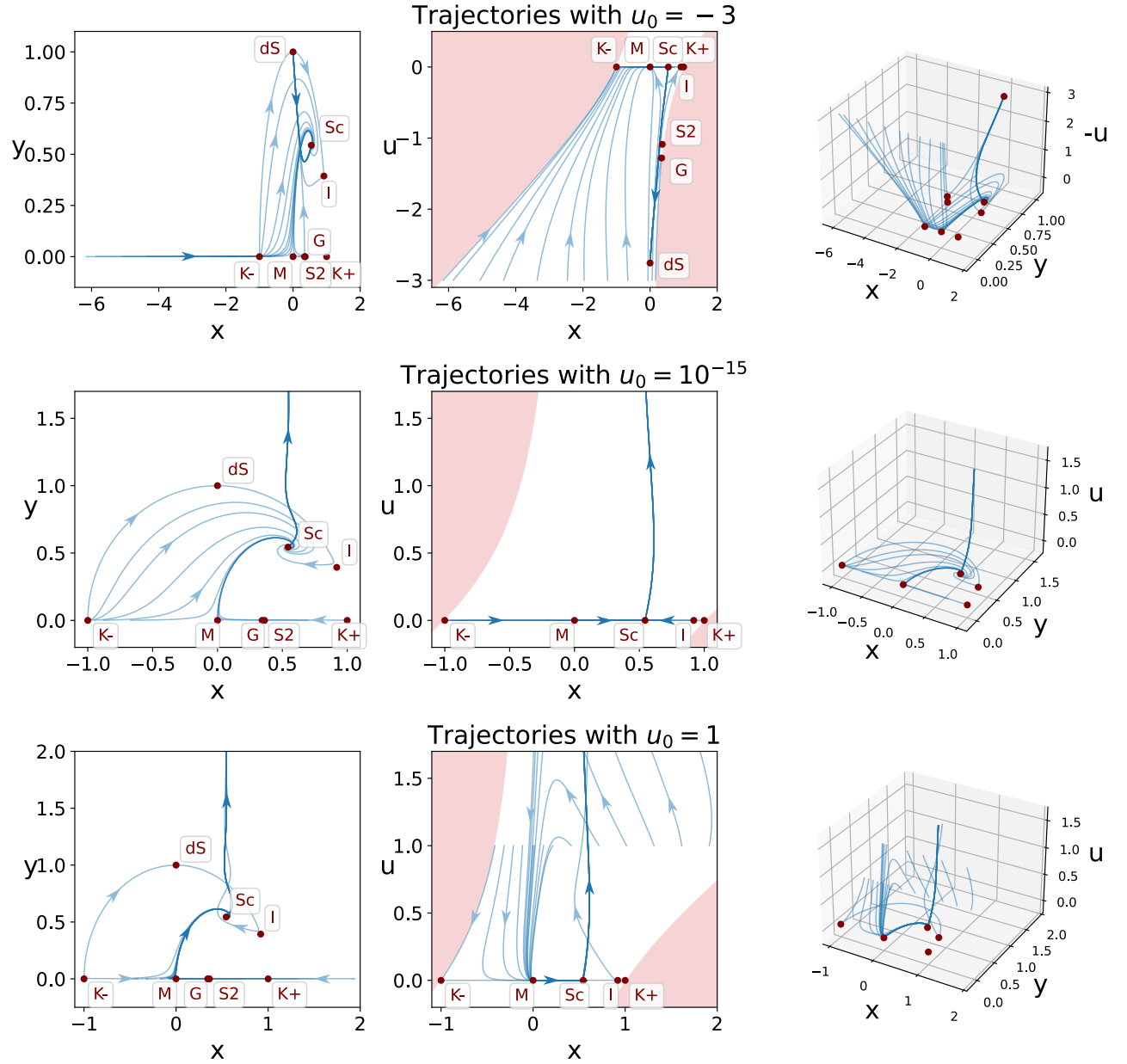


Figure 3.2: Some examples of phase portraits with different initial values u_0 . All models correspond to the same value $\lambda = 1.3\sqrt{3}$, $\kappa/\lambda = 1.5$ and $w_M = w_m = 0$. We choose the value of κ that is close to λ because, if the ratio κ/λ is too large, the attraction towards the **Sc** fixed point becomes weak, and most of the trajectories do not pass close to this point. This is valid for positive as well as negative u_0 . The values of λ , κ and w_M are also used in the upper row in Fig. 3.1. Notice that the vertical axis of the upper right column is $-u$.

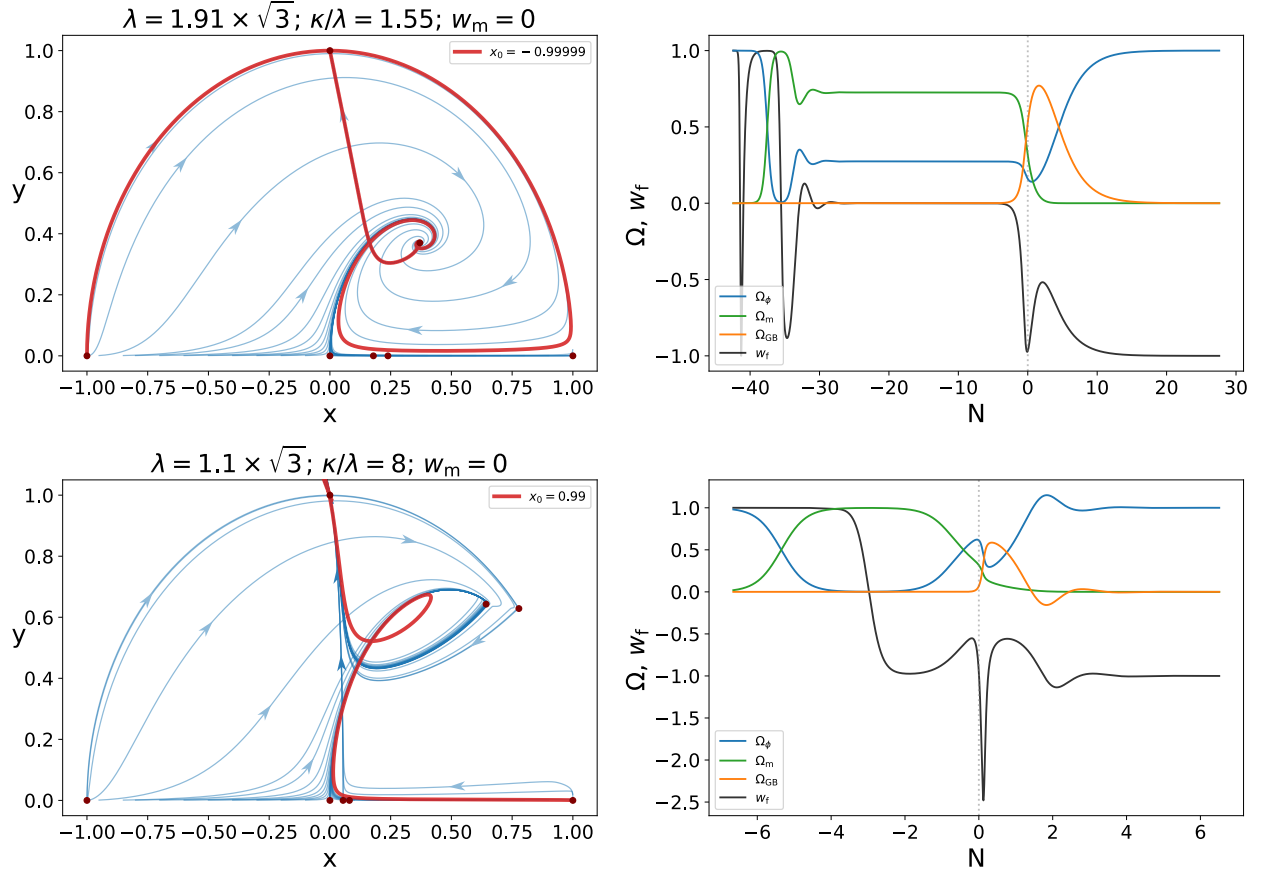


Figure 3.3: Time evolution of the density parameters and the EoS parameter of the ‘dark fluid’ for two selected trajectories (denoted by red curves on the left column). They are selected such that Ω_m and w_f are consistent with present day observations. We observe that both models have an initial period of kination that quickly gives way to pressureless matter domination. In the case of the upper plot, the model displays a long scaling period before the GB term contribution to the energy density comes into play. Another difference between the two models lies in the fact that, at the time Ω_m and w_f satisfy the constraints, either Ω_{GB} or Ω_ϕ are the dominant DE density contribution (upper and lower plots respectively). Eventually, all the models settle down at the de Sitter attractor fixed point, where $\Omega_\phi = 1$ and the rest of the density parameters vanish.

Eqs. (3.78), (3.79), (3.80) and (3.114), w_f can be written as

$$w_f = -1 + \frac{\frac{2}{3}\epsilon_1 (1 + 2ux) - z^2 (1 + w_M)}{x^2 + y^2} , \quad (3.158)$$

where ϵ_1 is meant to be substituted with Eq. (3.121), and we used the constraint equation (3.115) as well.

Keeping the initial condition for $u_0 = -10^{-25}$ and $w_M = w_m = 0$ but varying the λ and κ parameters and the initial conditions x_0 , we select models which have regions where $\Omega_m = 0.3147 \pm 0.0074$ and $w_f = -0.957 \pm 0.080$ [12] at present time, which will correspond to $N = 0$ (because the scale factor is normalised such that $a = 1$ at that moment and $dN = d \ln a$; see Eq. (1.107)). After many numerical simulations, we show in Fig. 3.3 two of such models. On the left column, we see the phase portraits from which these models are drawn, while on the right column we display the time evolution of the density parameters and w_f . As was discussed below Eq. (3.117), since Ω_{GB} can be positive as well as negative (see Eq. (3.116)), the value of each density parameter Ω is not necessarily bounded to lie at or below 1. In fact, we notice that, in the lower panel on the right column, $\Omega_\phi > 1$ for a brief period of time. Additionally, we observe in that very same plot a quite generic feature of GBDE models, namely, that the effective EoS parameter of the ‘dark fluid’ may drop below -1 [327, 328], so the fluid becomes phantom (see discussion after Eq. (3.72)).

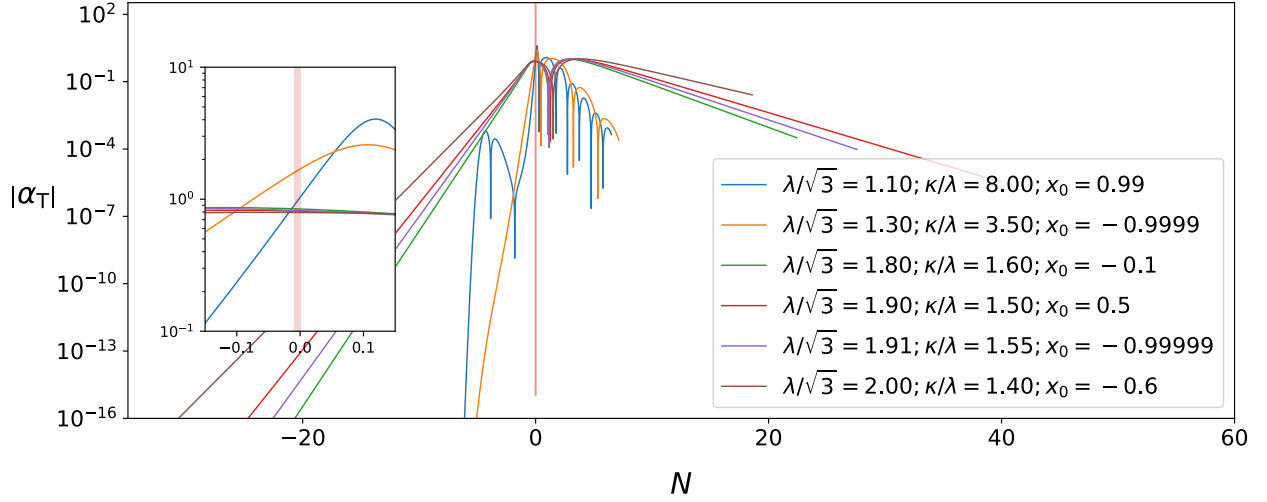
3.2.4 Exponential Gauss-Bonnet Coupling: The Speed of Gravitational Waves

As was pointed out earlier, SGB models generically predict $c_{GW} \neq 1$ [296]. Moreover, in those models c_{GW} is not a constant, but changes over time. The tight constraint in Eq. (3.67) applies only for a short period over the entire history of the Universe and, as we indicated already, one could naively conclude from Eq. (3.113) that the bound is not very constraining for the GBDE models.

To investigate this, we first write Eq. (3.59) in terms of the dimensionless variables (3.78)-(3.80) and (3.114)

$$\alpha_T = 2 \frac{(ux)' - (1 - \epsilon_1)ux}{1 + 2ux} = - \frac{\Omega'_{GB} - (1 - \epsilon_1)\Omega_{GB}}{1 - \Omega_{GB}} , \quad (3.159)$$

where Ω_{GB} was defined in Eq. (3.116). It might appear that α_T diverges when $ux = -1/2$ (or equivalently $\Omega_{GB} = 1$). But we can see from the constraint equation (3.115) that such a



value is not allowed.

If we evaluate it at the fixed points, α_T reads

$$\alpha_T|_c = -\frac{2u_c x_c}{1 + 2u_c x_c} (1 - \epsilon_1|_c), \quad (3.160)$$

and we see that it clearly vanishes at **Sc** and **dS**, which were the fixed points of interest. However, if the GBDE model under consideration is to be a good one of our Universe, the current stage of the evolution cannot be represented by any of these two fixed points, but by some point on a trajectory connecting those, as was demonstrated in Fig. 3.3. To determine the evolution of α_T , we selected a large number of numerical simulations satisfying the bounds on Ω_m and w_f discussed earlier. We ran those simulations with λ and κ values in the range $1/\sqrt{3} \leq \lambda \leq 20\sqrt{3}$ and $1.1 \leq \kappa/\lambda \leq 20$, and took $w_M = w_m = 0$ and $y_0 = 10^{-3}$ and $u_0 = -10^{-25}$, as well as a range of x_0 values. Obviously, the consistency with those constraints, though necessary, is not a sufficient condition for realistic models of DE. A negligibly small deviation of the speed of GWs from the speed of light is required. Drawing a few of those simulations for clarity, we plot the α_T parameter in Fig. 3.4. The curves

displayed in that plot are representative of the whole set.

We first observe that the maximum value of α_T is always very close to $N = 0$, where the observational bound (3.67) applies. As can be seen in the inset of Fig. 3.4, that value is always of order $\mathcal{O}(1)$, and clearly falls out of the allowed range. Consequently, with high confidence, we can conclude that the GBDE models with an exponential coupling function $\xi(\phi)$ are excluded by the constraint on the speed of GWs from observations. Moreover, we may try to figure out why α_T is so large precisely when the constraints apply. By writing Eq. (3.159) as

$$\alpha_T = -2(3 - \epsilon_1) + 3(1 - w_M) \frac{\Omega_M}{1 - \Omega_{GB}} + 6 \frac{y^2}{1 - \Omega_{GB}}, \quad (3.161)$$

we observe first that, at $N = 0$, since observations demand $\epsilon_1 \sim 0.5$, the absolute value of the first term is of order 1. Also, given the Friedmann equation in terms of the density parameters (see Eq. (3.117)), $\Omega_M \sim 0.3$ implies $\Omega_\phi + \Omega_{GB} \sim 0.7$. Since Ω_{GB} is positive between the scaling fixed point **Sc** and the de Sitter one **dS**, $0 < \Omega_{GB} < 0.7$ and the second term in Eq. (3.161) must be of order $\mathcal{O}(0.1)$ to $\mathcal{O}(1)$. Finally, $y^2 \leq \Omega_\phi < 0.7$ and the last term in that equation is of order $\mathcal{O}(1)$ at the maximum. Barring precise cancellations, this leads to the conclusion that $|\alpha_T|$ is of order $\mathcal{O}(0.1)$ - $\mathcal{O}(1)$ generically.⁸

The fact that Ω_M , of order $\mathcal{O}(0.1)$, quite generically leads to $|\alpha_T| \gtrsim 0.1$ raises another question regarding string theory corrections. As it follows from Eq. (3.113), such a large value of $|\alpha_T|$ implies a *very* large change in the coupling function ξ . However, the GB term is the lowest order term in the series of low-energy string theory corrections (see Refs. [274, 280, 288]). Given that ξ varies so much, one must wonder whether it is consistent to neglect the higher order corrections over the whole range of the evolution.

3.2.5 Linear Gauss-Bonnet Coupling

Although the exponential GB coupling might provide a reasonable history of the Universe and explain DE, such models predict the speed of GWs in the current Universe that violates the constraints from observations by many orders of magnitude. We then consider a *linear*

8. To obtain Eq. (3.161) we can use either the dynamical equations (3.118)-(3.122) and Eq. (3.115) to eliminate the time derivative from Eq. (3.159) or directly from Eq. (3.66). It is important to remark that Eq. (3.161) is very generic, valid for any potential $V(\phi)$ and any coupling function $\xi(\phi)$. When we make our estimations below Eq. (3.161) however, in particular the one regarding the sign of Ω_{GB} , it is based on the case being analysed.

coupling instead

$$\xi(\phi) \propto \phi. \quad (3.162)$$

Since $\xi_{,\phi\phi} = 0$, we may simply set $\kappa = 0$ in Eq. (3.138) given that $\kappa = M_{\text{Pl}}\xi_{,\phi\phi}/\xi_{,\phi}$ and $\xi_{,\phi} \neq 0$. Therefore, only the fixed points with $\epsilon_1|_c u_c = 0$ are present (see Eq. (3.122)). These are summarised in Table 3.1 too.

Given that the linear coupling function is equivalent to $\kappa = 0$ in the previous section, Sec. 3.2.3, we may use the results of that section. The eigenvalues at the scaling fixed point **Sc** are then

$$m_3 = -3(1 + w_{\text{M}}), \quad (3.163)$$

besides $m_{1,2}$ in Eq. (3.105). Since the real part of $m_{1,2}$ is always negative, as was discussed in Sec. 3.2.1, and m_3 is now negative as opposed to the exponential coupling case, we can see that **Sc** is always an attractor in the linear coupling case.

On the other hand, the eigenvalues at the de Sitter fixed point **dS** read (compare them with Eqs. (3.154) and (3.155))

$$m_1 = -\frac{3}{2}(1 + w_{\text{M}}), \quad (3.164)$$

$$m_{2,3} = -\frac{3}{2} \left[1 \pm \sqrt{1 + \frac{8\lambda^2}{3(2 + 3\lambda^2)}} \right]. \quad (3.165)$$

We clearly see that one of the eigenvalues is positive. Hence, this fixed point is a saddle.

The above result shows that a realistic cosmological scenario is impossible with the linear GB coupling. The scaling fixed point is an attractor as in the General Relativistic quintessence models and there are no solutions that display a long matter dominated period followed by an accelerated expansion. This is demonstrated in Fig. 3.5 as well.

We might consider higher-order polynomial or other functions $\xi(\phi)$ as, for example, the ones considered in Refs. [341, 342] in the context of astrophysical compact objects, and generalise the analysis we present here. However, other functional forms of $\xi(\phi)$ make the equations non-self-similar and demand a method different from the one applied here. Therefore, the relevance of other $\xi(\phi)$ functions in explaining DE is left for future investigations.

3.2.6 The Case of $\alpha_T = 0$

As was argued already, the α_T parameter (see Eq. (3.59)) can be made to vanish if the coupling function is chosen in such a way that it satisfies the condition in Eq. (3.61). Un-

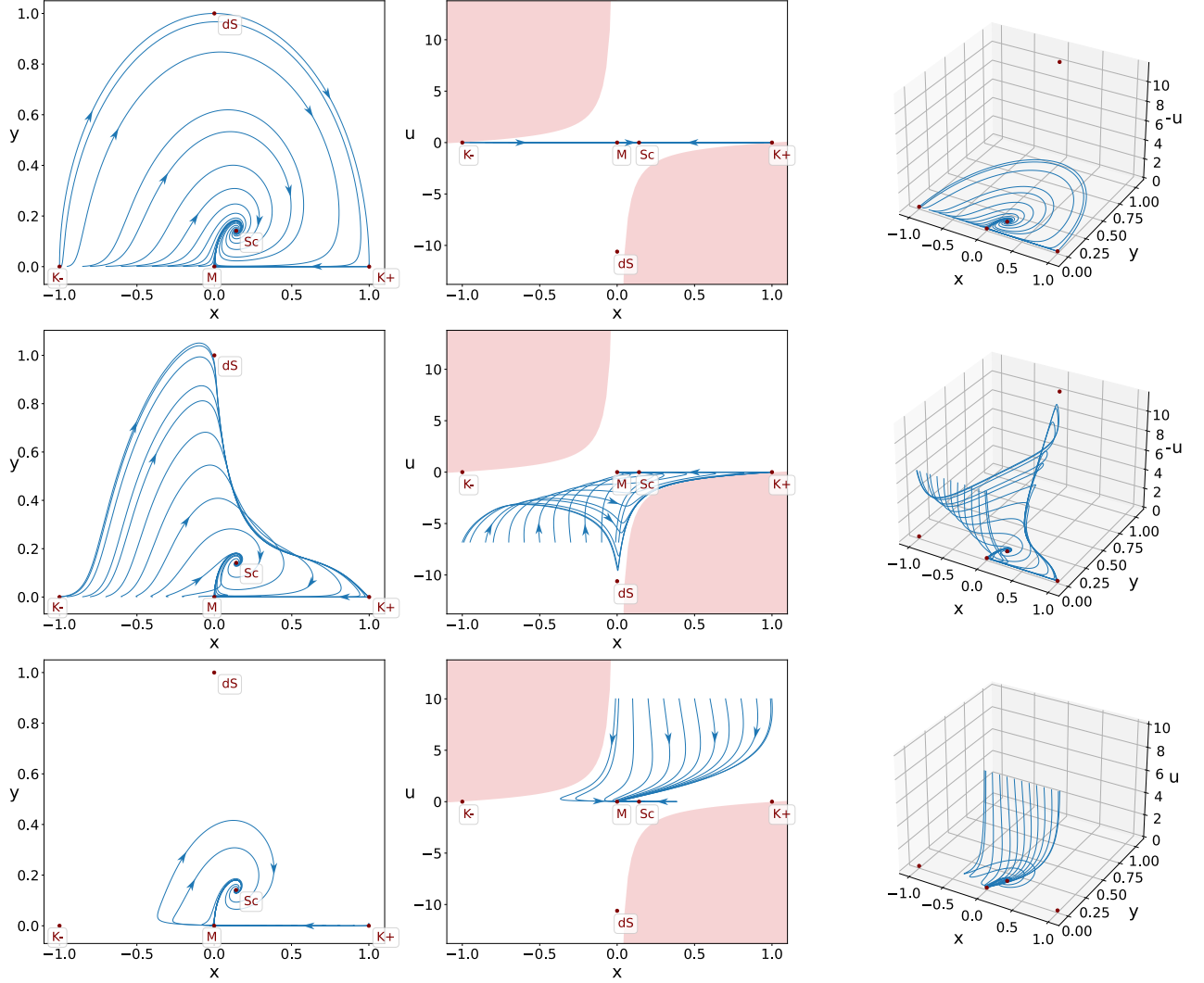


Figure 3.5: Phase space of the linear model, with $\xi(\phi)$ defined in Eq. (3.162). The notation is the same as in Figure 3.1. All three rows correspond to the same model with $\lambda = 5\sqrt{3}$ and $w_M = w_m = 0$, but different initial conditions of u . In the first row, $u_0 = -10^{-25}$ and, as we see, the evolution remains within the $u = 0$ plane. To demonstrate that the scaling fixed point **Sc** is an absolute attractor, we also run simulations with large initial values $|u_0|$, as can be seen in the other rows. In the second row in particular, we can see that some trajectories are initially attracted towards the de Sitter fixed point **dS**, but since for a linear $\xi(\phi)$ this point is a saddle, eventually all trajectories converge onto **Sc**. Notice that in the last column we plot $-u$ on the vertical axis of the first two plots and $+u$ of the third plot.

	x_c	y_c	u_c	z_c	$\epsilon_1 _c$
M	0	0	0	1	$\frac{3}{2}(1+w_M)$
K$\pm$	± 1	0	0	0	3
I	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1 - \frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\lambda^2$
Sc	$\sqrt{\frac{3}{2}} \frac{1+w_M}{\lambda}$	$\frac{\sqrt{\frac{3}{2}(1-w_M^2)}}{\lambda}$	0	$\sqrt{1 - \frac{3(1+w_M)}{\lambda^2}}$	$\frac{3}{2}(1+w_M)$
dS	0	1	$-\sqrt{\frac{3}{2}}\lambda$	0	0
$\overline{\text{IV}}$	$\sqrt{\frac{2}{3}} \frac{1}{\lambda}$	$\frac{2}{\sqrt{3}\lambda}$	$\sqrt{\frac{3}{2}} \frac{1}{\lambda} \left(1 - \frac{\lambda^2}{2}\right)$	0	1

Table 3.2: Fixed points of the dynamical system that satisfies $\alpha_T = 0$.

fortunately, this condition does not provide us with a functional form of $\xi(\phi)$. Nevertheless, it gives enough information to investigate relevant aspects of such a dynamical system.

First, we notice that Eq. (3.61), written in terms of the dimensionless variables (3.78)-(3.80) and (3.114), becomes

$$12H^2\xi_{,\phi\phi}x^2 = u \left[x' - (1 + \epsilon_1) x \right], \quad (3.166)$$

which can be used to eliminate the $\xi_{,\phi\phi}$ term in Eq. (3.122) as long as x does not vanish. At a fixed point, the equation above reads

$$12 \left(H^2\xi_{,\phi\phi} \right) \Big|_c x_c^2 = - (1 + \epsilon_1|_c) u_c x_c. \quad (3.167)$$

For those fixed points that satisfy $x_c = 0$, this equation vanishes identically. Looking at the system **A-D** (below Eq. (3.129)), only two of those points exist: $(x, y, u) = (0, 0, 0)$ and $(x, y, u) = \left(0, 1, -\sqrt{\frac{3}{2}}\lambda\right)$ (i.e. **M** and **dS** in Table 3.1, respectively). If $x_c \neq 0$, we equate Eq. (3.167) with

$$12 \left(H^2\xi_{,\phi\phi} \right) \Big|_c x_c = -2 \epsilon_1|_c u_c, \quad (3.168)$$

obtained from Eq. (3.122), such that

$$2 \epsilon_1|_c u_c = (1 + \epsilon_1|_c) u_c. \quad (3.169)$$

For $u_c = 0$ the equation vanishes identically and we obtain the same fixed points as those in Table 3.1 with $u_c = 0$. If $u_c \neq 0$, then $\epsilon_1|_c = 1$ (or $w|_c = -1/3$; see Eq. (1.70)). There is one fixed point that satisfies this (and $x_c \neq 0$), which is **C** with $x_c = \sqrt{2/3}/\lambda$ (remember that $0 \leq w_M < 1$); that is

$$(x, y, u) = \left(\sqrt{\frac{2}{3}} \frac{1}{\lambda}, \frac{2}{\sqrt{3}\lambda}, \sqrt{\frac{3}{2}} \frac{1}{\lambda} \left(1 - \frac{\lambda^2}{2} \right) \right). \quad (3.170)$$

We have summarised all the fixed point under $\alpha_T = 0$ in Table 3.2.

We now determine the stability of the fixed points **Sc** and **dS** as in the linear GB coupling case. In addition to Eqs. (3.118)-(3.122) and Eq. (3.115), we linearise Eq. (3.166). At **dS**, the linear version of Eq. (3.166) yields $\delta x' = \delta x$. Combined with Eqs. (3.143) and (3.148), we obtain $\delta u = (4 + 3\lambda^2)\delta x$. Consequently, the linear system reduces to

$$\delta x' = \delta x, \quad (3.171)$$

$$\delta u' = (4 + 3\lambda^2)\delta x, \quad (3.172)$$

$$\delta z' = -\frac{3}{2}(1 + w_M)\delta z. \quad (3.173)$$

We easily see that two eigenvalues have opposite signs

$$m_1 = 1, \quad (3.174)$$

$$m_2 = 0, \quad (3.175)$$

$$m_3 = -\frac{3}{2}(1 + w_M). \quad (3.176)$$

Therefore, this fixed point is a saddle, no longer an attractor as in the exponential GB coupling case.

In the vicinity of the scaling fixed point **Sc**, the linear equation for $\delta u'$ takes on the form

$$\delta u' = -\frac{1}{2}(1 + 3w_M)\delta u. \quad (3.177)$$

We may obtain this equation by linearising Eq. (3.166) around the scaling fixed point first:

$$12\delta(H^2\xi_{,\phi\phi}x)x_c = -(1 + \epsilon_1|_c)x_c\delta u, \quad (3.178)$$

where $12 (H^2 \xi_{,\phi\phi})|_{\mathbf{S_c}} x_{\mathbf{S_c}} = 0$ according to Eq. (3.168). Doing the same with Eq. (3.122)

$$\delta u' = -2 \epsilon_1|_c \delta u - 12 \delta(H^2 \xi_{,\phi\phi} x) , \quad (3.179)$$

and combining both (remember that $x_{\mathbf{S_c}} \neq 0$) we obtain Eq. (3.177). Given that $0 \leq w_M < 1$, the fixed point is stable, meaning that there are no trajectories that flow from the scaling fixed point to the de Sitter one.

The condition $c_{\text{GW}} = 1$ implies that the scaling fixed point is an attractor while de Sitter becomes a saddle point. Hence, there are no solutions that reproduce a long, matter-like domination period approaching asymptotically the de Sitter regime. Since this is the expected evolution of the Universe, the $\alpha_T = 0$ case for a coupling function $\xi(\phi)$ satisfying Eq. (3.61) is discarded. As was pointed out already, however, Eq. (3.67) does not require $\alpha_T = 0$ necessarily.

3.2.7 Concluding Remarks of Section 3.2

We have seen in Sec. 3.2 that, although the GBDE models predict the speed of GWs different from the speed of light, the deviation is time dependent and the bound is only applicable for the very late Universe. Moreover, if that bound is expressed in terms of the variation of the GB coupling function $\xi(\phi)$, then this appears to weaken the former considerably (see Eq. (3.113)). Therefore, before excluding the GBDE models, one needs to perform a more detailed dynamical systems analysis.

To that end, we assumed a scalar field with an exponential potential. We then found that the system quite generically has the scaling and the de Sitter fixed points, among others. In the former case, the EoS parameter of scalar field is the same as that of the background fluid, whereas the de Sitter corresponds to an exponentially expanding universe. While in the General Relativistic quintessence models, the scaling fixed point is stable, in the case of the exponential coupling function $\xi(\phi)$, depending on the magnitude of the exponent, it becomes a saddle point. Furthermore, de Sitter is a fixed point of the dynamical system and an attractor when trajectories are repelled from the scaling regime. Using numerical simulations, we showed that if the Universe starts with a very small GB contribution, near the kination or background fluid domination regimes, all solutions evolve towards the scaling fixed point and stay in its vicinity for a long period of time. Eventually they all leave the scaling regime and approach de Sitter, making this setup conducive to explaining the current accelerated expansion of the Universe. Besides, this system's behaviour serves to model

quintessential inflation [329], allowing to bridge the enormous energy density gap (more than a hundred orders of magnitude; see discussion following Eq. (3.112)) between early inflation and late DE. We found numerous models that follow this scenario and can predict observationally allowed values for the density parameter of matter and the EoS parameter of DE. However, it must be pointed out that if those models are aimed to represent the evolution of the actual Universe, we must find ourselves on some trajectory connecting the scaling fixed point and de Sitter, and not at these fixed points exactly.

Bearing this in mind, although the α_T parameter vanishes at the two fixed points mentioned above, unfortunately, as we showed in Fig. 3.4, $|\alpha_T|$ is predicted to be of order $\mathcal{O}(0.1)$ - $\mathcal{O}(1)$ at present time, when the density parameter of pressureless matter Ω_m is of order $\mathcal{O}(0.1)$. This leads us to conclude that the GBDE models with an exponential coupling function are not viable. When it comes to the linear GB coupling, the de Sitter fixed point becomes a saddle and the scaling regime is an attractor. This rules out any viable solution consistent with the known evolution of the Universe. Although more coupling functions may be studied (such as monomials or steeper than exponential examined, for example, in Refs. [341, 342]), equations lose their self-similar character and a different method is required.

One can also impose $\alpha_T = 0$, which we did above, but the stability of the scaling and de Sitter fixed points was shown to be very similar to the linear GB coupling case. Consequently, no solutions which provide a long period of matter-like domination and a subsequent asymptotic de Sitter regime are obtained.⁹

3.3 Horndeski's Theory and the α_T Parameter

As was commented at the beginning of this chapter, the GB term does not contribute to the metric field equations in four spacetime dimensions. Moreover, it is the second-order term in a generalisation of GR to higher spacetime dimensions called ‘Lovelock gravity’ (see Eq. (3.13)). When coupled to a scalar field, however, the SGB model is a subclass of a

9. After we completed this work, our attention was brought to Ref. [343]. In sum, they conclude that if the GB term comes from the breaking of the Weyl symmetry, a Weyl term should be added to the action. The theory however is different from the one considered in this part of the thesis. A relatively recent work [344] by D. Glavan and C. Lin noticed that one could rescale the constant coupled to the GB term (‘ ξ ’ in Eq. (3.2)) such that, when $D = 4$, the GB term *does* contribute to the field equations of the metric (see in Eq. (3.11) the factor that makes the contribution vanish in four dimensions), while the number of degrees of freedom remains the same and the theory is free from the Ostrogradsky instability (see **Appendix C** in Ref. [159] and Ref. [345]). As a result of that publication, there has been an intensive research along that line (see Ref. [262] for a review of the so-called ‘4D Einstein Gauss-Bonnet theory of gravity’), but no dynamical systems analysis has been performed including the seemingly unavoidable Weyl term to the best of our knowledge. Thus, this could be interesting for future investigation.

well-known scalar-tensor theory called ‘Horndeski’s theory’ [346]. This theory is the most general one in four dimensions leading to second-order field equations of the metric and a scalar field ϕ . The Lagrangian of the theory is given by the sum of

$$\mathcal{L}_2 = G_2(\phi, X) , \quad (3.180)$$

$$\mathcal{L}_3 = G_3(\phi, X)\square\phi , \quad (3.181)$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}(\phi, X) \left[(\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \right] , \quad (3.182)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}(\phi, X) \left[(\square\phi)^3 + 2\nabla_\mu\nabla^\nu\phi\nabla_\nu\nabla^\alpha\phi\nabla_\alpha\nabla^\mu\phi - \right. \\ & \left. - 3\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\square\phi \right] , \end{aligned} \quad (3.183)$$

where $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d’Alembert operator, introduced below Eq. (1.85), and X is used to denote the kinetic term $-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, for simplicity.

The α_T parameter in Horndeski’s theory is [347]

$$\alpha_T = X \frac{2(G_{4,X} - G_{5,\phi}) - (\ddot{\phi} - H\dot{\phi})G_{5,X}}{G_4 - X(2G_{4,X} + H\dot{\phi}G_{5,X} - G_{5,\phi})} . \quad (3.184)$$

With the following non-trivial choices of the G -functions:

$$G_2 = X - V(\phi) - 4\xi_{,\phi\phi\phi\phi}X^2 [3 - \ln(X)] , \quad (3.185)$$

$$G_3 = 2\xi_{,\phi\phi\phi}X [7 - 3\ln(X)] , \quad (3.186)$$

$$G_4 = \frac{M_{\text{Pl}}^2}{2} - 2\xi_{,\phi\phi}X [2 - \ln(X)] , \quad (3.187)$$

$$G_5 = 2\xi_{,\phi} \ln(X) , \quad (3.188)$$

one obtains the same field equations as those derived from the action (3.19) and recovers the α_T parameter in Eq. (3.59) [347]. We see that $\alpha_T = 0$ if $G_4 = G_4(\phi)$ and G_5 is independent of both, ϕ and X . Although, *a priori*, another possibility is

$$2(G_{4,X} - G_{5,\phi}) = (\ddot{\phi} - H\dot{\phi})G_{5,X} , \quad (3.189)$$

one can find e.g. in Ref. [297] arguments against this non-trivial cancellation based on its stability against radiative corrections and small changes in the energy density of the different components in the Universe. Then, one is left with conformally coupled theories (to gravity, with $G_4(\phi)R$ and constant G_5) as the only ones consistent with the observational constraints

if that precise cancellation is ruled out.¹⁰

However, it was found in Ref. [349] that, in contrast to the *metric formalism* of gravity, Horndeski’s theory in the *Palatini formalism* with $G_4 = G_4(\phi, X)$ does allow GWs propagating at the speed of light *exactly*. The reasoning is simple once one understands the basics of the Palatini formalism of gravity, and we will examine this in the next section. Given the connection between the SGB model and the Horndeski theory in the metric formalism (the one we have been studying until now), and the striking result of Ref. [349], one might wonder whether it is possible to amend the negative results obtained in Ref. [3] (see Sec. 3.2.4) such that the GBDE models which reproduce the known evolution of the Universe also evade the observational constraints on the speed of the GWs.¹¹ We will tackle this in the upcoming sections, once we have reviewed briefly the Palatini formalism of gravity.

3.4 Palatini Formalism of Gravity

3.4.1 Torsion and Non-Metricity Tensors

In the so-called *metric formalism of gravity* [351–355], the Riemann curvature tensor defined as [23]

$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta} , \quad (3.190)$$

is a function of the metric tensor and its derivatives. $\Gamma^\alpha_{\mu\nu}$ refers to the so-called ‘Christoffel symbols’ [23] of the Levi-Civita (LC) connection

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) , \quad (3.191)$$

which are not tensors under coordinate transformations, although the difference of two connections does transform as a mixed tensor of type (1, 2) [23]. The LC connection is said to

10. In addition to Horndeski’s theory, one may consider ‘beyond Horndeski’ theories [297] with third derivatives in the field equations and new constraints such that the number of propagating degrees of freedom is the same as in the usual Horndeski theory. Also, Degenerate Higher-Order Scalar-Tensor (DHOST) theories are a further generalisation of Horndeski’s theory, although one needs to include more new constraints to avoid instabilities [348].

11. It is worth noting that the bound on $|\alpha_T|$, Eq. (3.67), applies only to a limited range of GW frequencies (10-100 Hz). Horndeski’s theory applied to DE is a low energy effective field theory when compared to that range of frequencies. Then, it could be the case that high energy operators in the theory would drive the α_T parameter towards the constrained values. The consistency of such a bound across the whole spectrum of GW frequencies has been discussed within the context of Horndeski’s theory in Ref. [350] (where the previous range of frequencies is mentioned too).

be torsion-free and metric compatible [23, 354], meaning that

$$\Gamma_{\mu\nu}^{\alpha} = \Gamma_{\nu\mu}^{\alpha} , \quad (3.192)$$

$$\partial_{\alpha} g_{\mu\nu} = \Gamma_{\alpha\mu}^{\lambda} g_{\lambda\nu} + \Gamma_{\alpha\nu}^{\lambda} g_{\mu\lambda} , \quad (3.193)$$

respectively. If we define the torsion and non-metricity tensors as [356]

$$T_{\mu\nu}^{\alpha} \equiv 2\Gamma_{[\mu\nu]}^{\alpha} = \Gamma_{\mu\nu}^{\alpha} - \Gamma_{\nu\mu}^{\alpha} , \quad (3.194)$$

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu} , \quad (3.195)$$

respectively, this means that, for the LC connection, $T_{\mu\nu}^{\alpha} = 0$ and $Q_{\alpha\mu\nu} = 0$.¹² For a generic connection $\Gamma_{\mu\nu}^{\alpha}$ however, the torsion and non-metricity tensors do not have to vanish necessarily. One can derive a general relation between $\Gamma_{\mu\nu}^{\alpha}$ and the LC connection. We shall denote the LC connection by $\overset{\circ}{\Gamma}_{\mu\nu}^{\alpha}$ and use rings (overcircles) for any geometric quantity built upon it. Then $\overset{\circ}{\nabla}_{\alpha} g_{\mu\nu} = 0$, for instance.

Given that $\nabla_{\alpha} g_{\mu\nu} \neq 0$ for a generic connection, we have

$$Q_{\mu\alpha\nu} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu} = 2g_{\alpha\sigma} \left(\overset{\circ}{\Gamma}_{\mu\nu}^{\sigma} - \Gamma_{\mu\nu}^{\sigma} \right) + T_{\mu\nu\alpha} - T_{\mu\alpha\nu} - T_{\nu\alpha\mu} , \quad (3.196)$$

where $T_{\mu\nu\alpha} = T_{\mu\nu}^{\sigma} g_{\sigma\alpha}$ and so on. Then

$$\Gamma_{\mu\nu}^{\alpha} = \overset{\circ}{\Gamma}_{\mu\nu}^{\alpha} + \frac{1}{2} g^{\alpha\beta} (T_{\mu\nu\beta} - T_{\nu\beta\mu} + T_{\beta\mu\nu} - Q_{\mu\nu\beta} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}) . \quad (3.197)$$

Notice that $T_{\mu\nu}^{\alpha} = T_{[\mu\nu]}^{\alpha}$ and $Q_{\alpha\mu\nu} = Q_{\alpha(\mu\nu)}$. We define the *distortion tensor* $\kappa_{\mu\nu}^{\alpha}$ as [356]

$$\kappa_{\mu\nu}^{\alpha} \equiv \frac{1}{2} g^{\alpha\beta} (T_{\mu\nu\beta} - T_{\nu\beta\mu} + T_{\beta\mu\nu} - Q_{\mu\nu\beta} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}) , \quad (3.198)$$

such that the torsion and non-metricity tensors can be written as

$$T_{\mu\nu}^{\alpha} = 2\kappa_{[\mu\nu]}^{\alpha} , \quad (3.199)$$

$$Q_{\alpha\mu\nu} = -2\kappa_{\alpha(\mu\nu)} . \quad (3.200)$$

As was commented on some occasions and used explicitly in Eq. (1.151), the metric tensor

12. We remind the reader that $A_{(\mu\nu)} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ means symmetrisation and $A_{[\mu\nu]} \equiv \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ antisymmetrisation (see Eqs. (A.2) and (A.11), respectively).

transforms as a fully covariant tensor under coordinate transformations

$$g_{\tilde{\mu}\tilde{\nu}} = \frac{\partial \hat{x}^\alpha}{\partial \tilde{x}^\mu} \frac{\partial \hat{x}^\beta}{\partial \tilde{x}^\nu} g_{\hat{\alpha}\hat{\beta}} . \quad (3.201)$$

Taking the determinant on both sides, we obtain

$$\tilde{g} = J^{-2} \hat{g} , \quad (3.202)$$

J being the Jacobian of the transformation matrix $J \equiv \left| \frac{\partial \tilde{x}^\mu}{\partial \hat{x}^\mu} \right|$, and $\tilde{g} = g(\tilde{x}^\mu)$ and $\hat{g} = g(\hat{x}^\mu)$. This means that $\sqrt{-g}$ transforms as what is called a ‘scalar density of weight -1 ’ and not as a scalar under the coordinate transformations [23, 170]. Therefore, the covariant derivative of $\sqrt{-g}$ is not simply the partial derivative of it. To figure out the correct expression we calculate the partial derivative of $\tilde{g}$ with respect to $\tilde{x}$:

$$\partial_{\tilde{\mu}}(\sqrt{-\tilde{g}}) = J^{-1} \frac{\partial \hat{x}^\nu}{\partial \tilde{x}^\mu} \partial_{\hat{\nu}}(\sqrt{-\hat{g}}) - J^{-1} \sqrt{-\hat{g}} \frac{\partial \hat{x}^\nu}{\partial \tilde{x}^\mu} \frac{\partial \hat{x}^\sigma}{\partial \tilde{x}^\rho} \frac{\partial^2 \tilde{x}^\rho}{\partial \hat{x}^\nu \partial \hat{x}^\sigma} , \quad (3.203)$$

where, to obtain the second term, we used Jacobi’s formula on the Jacobian (see Eq. (1.75) where it applies to the determinant of the metric tensor). Notice that this term can be eliminated if we add $-\sqrt{-g} \Gamma_{\mu\rho}^\rho$ given that the coefficients of the connection transform as [23, 170]

$$\Gamma_{\tilde{\mu}\tilde{\nu}}^{\tilde{\alpha}} = \frac{\partial \hat{x}^\mu}{\partial \tilde{x}^\mu} \frac{\partial \hat{x}^\nu}{\partial \tilde{x}^\nu} \frac{\partial \hat{x}^\alpha}{\partial \tilde{x}^\alpha} \Gamma_{\hat{\mu}\hat{\nu}}^{\hat{\alpha}} - \frac{\partial \hat{x}^\mu}{\partial \tilde{x}^\mu} \frac{\partial \hat{x}^\nu}{\partial \tilde{x}^\nu} \frac{\partial^2 \tilde{x}^\alpha}{\partial \hat{x}^\mu \partial \hat{x}^\nu} . \quad (3.204)$$

We then have that

$$\nabla_\mu(\sqrt{-g}) \equiv \partial_\mu(\sqrt{-g}) - \sqrt{-g} \Gamma_{\mu\alpha}^\alpha , \quad (3.205)$$

transforms as a covariant vector of the same weight as $\sqrt{-g}$ under the coordinate transformations; i.e.

$$\nabla_{\tilde{\mu}}(\sqrt{-\tilde{g}}) = J^{-1} \frac{\partial \hat{x}^\mu}{\partial \tilde{x}^\mu} \nabla_{\hat{\mu}}(\sqrt{-\hat{g}}) . \quad (3.206)$$

Using again Jacobi’s formula, this time for the metric tensor, we write Eq. (3.205) as

$$\frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) = \mathring{\Gamma}_{\mu\alpha}^\alpha - \Gamma_{\mu\alpha}^\alpha = -\kappa_{\mu\alpha}{}^\alpha . \quad (3.207)$$

We see that $\mathring{\nabla}_\mu(\sqrt{-g}) = 0$. From Eq. (3.200) we obtain

$$\frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) = \frac{1}{2} Q_{\mu\alpha}{}^\alpha . \quad (3.208)$$

Hence

$$-\frac{1}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}g^{\mu\nu}) = Q_\alpha{}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}Q_{\alpha\sigma}{}^\sigma, \quad (3.209)$$

where $\nabla_\alpha g^{\mu\nu} = -Q_\alpha{}^{\mu\nu}$.

3.4.2 Metric-Affine Geometries

Similarly to what we did with the coefficients of the connection, we can now decompose the Riemann tensor into the LC part and the one depending on the distortion tensor and its derivatives

$$R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \kappa^\alpha{}_{\beta\mu\nu}, \quad (3.210)$$

where $\kappa^\alpha{}_{\beta\mu\nu}$ is given by

$$\kappa^\alpha{}_{\beta\mu\nu} = \mathring{\nabla}_\mu \kappa_\nu{}^\alpha{}_\beta - \mathring{\nabla}_\nu \kappa_\mu{}^\alpha{}_\beta + \kappa_{\mu\lambda}{}^\alpha \kappa_\nu{}^\lambda{}_\beta - \kappa_{\nu\lambda}{}^\alpha \kappa_\mu{}^\lambda{}_\beta. \quad (3.211)$$

It is clear that $R^\alpha{}_{\beta\mu\nu}$ bears only one symmetry as opposed to $\mathring{R}^\alpha{}_{\beta\mu\nu}$. We see that $R^\alpha{}_{\beta\mu\nu} = R^\alpha{}_{\beta[\mu\nu]}$, while $\mathring{R}^\alpha{}_{\beta\mu\nu}$ is known to satisfy [23]

$$\mathring{R}_{\alpha\beta\mu\nu} = -\mathring{R}_{\beta\alpha\mu\nu}, \quad (3.212)$$

$$\mathring{R}_{\alpha\beta\mu\nu} = \mathring{R}_{\mu\nu\alpha\beta}, \quad (3.213)$$

$$\mathring{R}_{\alpha\beta\mu\nu} + \mathring{R}_{\alpha\mu\nu\beta} + \mathring{R}_{\alpha\nu\beta\mu} = 0, \quad (3.214)$$

in addition to the antisymmetry under the interchange of the two last indices. The last equality can also be written as $\mathring{R}_{\alpha[\beta\mu\nu]} = 0$. The first equality above, in the case of a generic connection, becomes

$$R_{\alpha\beta\mu\nu} + R_{\beta\alpha\mu\nu} = -2\mathring{\nabla}_{[\mu}Q_{\nu]\alpha\beta} + 4\kappa_{[\mu|(\alpha}{}^\lambda Q_{|\nu]|\beta)\lambda}, \quad (3.215)$$

where Eq. (3.200) was employed. Notice that if the connection is metric compatible, then $R_{(\alpha\beta)\mu\nu} = 0$, even if it has non-zero torsion. An example of such a connection is the following one with distortion tensor:

$$\kappa_{\mu\nu}{}^\alpha = \delta_\mu^\alpha A_\nu - g_{\mu\nu} g^{\alpha\beta} A_\beta, \quad (3.216)$$

where A_μ is some arbitrary covariant vector. If we calculate the non-metricity tensor, it vanishes, while the torsion tensor reads (see Eqs. (3.200) and (3.199) respectively)

$$T_{\mu\nu}{}^\alpha = \delta_\mu^\alpha A_\nu - \delta_\nu^\alpha A_\mu . \quad (3.217)$$

Eq. (3.215) may be written as

$$R_{(\alpha\beta)\mu\nu} = -\nabla_{[\mu} Q_{\nu]\alpha\beta} - \frac{1}{2} T_{\mu\nu}{}^\lambda Q_{\lambda\alpha\beta} , \quad (3.218)$$

using Eq. (3.199).

The relation $\mathring{R}^\alpha_{[\beta\mu\nu]} = 0$, when the connection is not the LC connection, reads

$$R^\alpha_{[\beta\mu\nu]} = \mathring{\nabla}_{[\mu} T_{\nu\beta]}{}^\alpha + \kappa_{[\mu|\lambda}{}^\alpha T_{|\nu\beta]}{}^\lambda . \quad (3.219)$$

We see that $R^\alpha_{[\beta\mu\nu]} = 0$ (or $R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0$) if the connection under consideration is torsionless (although it may have non-zero non-metricity tensor). An example of this connection yields the following distortion tensor:

$$\kappa_{\mu\nu}{}^\alpha = \delta_\mu^\alpha A_\nu + \delta_\nu^\alpha A_\mu - g_{\mu\nu} g^{\alpha\beta} A_\beta . \quad (3.220)$$

The non-metricity tensor is (see Eq. (3.200))

$$Q_{\alpha\mu\nu} = -2A_\alpha g_{\mu\nu} . \quad (3.221)$$

Eq. (3.219) can be written as

$$R^\alpha_{[\beta\mu\nu]} = \nabla_{[\mu} T_{\nu\beta]}{}^\alpha + T_{[\mu\beta}{}^\lambda T_{\nu]\lambda}{}^\alpha , \quad (3.222)$$

and the covariant components read (remember that $\nabla_\alpha g_{\mu\nu} \neq 0$)

$$R_{\alpha[\beta\mu\nu]} = \nabla_{[\mu} T_{\nu\beta]\alpha} - T_{[\nu\beta}{}^\lambda Q_{\mu]\alpha\lambda} + T_{[\mu\beta}{}^\lambda T_{\nu]\lambda\alpha} . \quad (3.223)$$

The second equality (Eq. (3.213)), $\mathring{R}_{\alpha\beta\mu\nu} - \mathring{R}_{\mu\nu\alpha\beta} = 0$, can be written in terms of Eqs. (3.218) and (3.223). We obtain

$$R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = \frac{3}{2} \left(R_{\alpha[\beta\mu\nu]} + R_{\beta[\mu\alpha\nu]} + R_{\mu[\beta\alpha\nu]} + R_{\nu[\beta\mu\alpha]} \right) + R_{(\alpha\mu)\beta\nu} + R_{(\alpha\beta)\mu\nu} -$$

$$-R_{(\alpha\nu)\beta\mu} + R_{(\mu\beta)\nu\alpha} - R_{(\beta\nu)\mu\alpha} - R_{(\mu\nu)\alpha\beta} , \quad (3.224)$$

after some manipulations, when the connection is not the LC one.

In contrast to the metric formalism, when the connection is generic, there are several second-rank curvature tensors that can be obtained from the Riemann tensor. The Ricci tensor is given by

$$R_{\mu\nu} \equiv R^\sigma{}_{\mu\sigma\nu} = \mathring{R}_{\mu\nu} + \kappa_{\mu\nu} , \quad (3.225)$$

where $\kappa_{\mu\nu} \equiv \kappa^\sigma{}_{\mu\sigma\nu}$. The way it is defined is the same as in the metric formalism although $R_{\mu\nu}$ is not fully symmetric as $\mathring{R}_{\mu\nu}$. The antisymmetric part is

$$R_{[\mu\nu]} = R^\sigma{}_{[\mu|\sigma|\nu]} = \frac{3}{2} \left(\nabla_{[\sigma} T_{\nu\mu]}{}^\sigma + T_{[\sigma\mu}{}^\lambda T_{\nu]\lambda}{}^\sigma + \frac{1}{3} R^\sigma{}_{\sigma\mu\nu} \right) , \quad (3.226)$$

where we have considered Eq. (3.222). $R^\sigma{}_{\sigma\mu\nu}$ is not zero due to Eq. (3.218) and leads to a new second-rank tensor called ‘homothetic curvature’ tensor $\bar{R}_{\mu\nu}$ [357]:

$$\bar{R}_{\mu\nu} \equiv R^\sigma{}_{\sigma\mu\nu} = \bar{\kappa}_{\mu\nu} , \quad (3.227)$$

which is fully antisymmetric, where

$$\bar{\kappa}_{\mu\nu} \equiv \kappa^\sigma{}_{\sigma\mu\nu} = \mathring{\nabla}_\mu \kappa_{\nu\sigma}{}^\sigma - \mathring{\nabla}_\nu \kappa_{\mu\sigma}{}^\sigma . \quad (3.228)$$

Using Eq. (3.198) we have

$$\kappa_{\mu\sigma}{}^\sigma = -\frac{1}{2} Q_{\mu\sigma}{}^\sigma , \quad (3.229)$$

and hence

$$\bar{R}_{\mu\nu} = -\frac{1}{2} \left(\mathring{\nabla}_\mu Q_{\nu\sigma}{}^\sigma - \mathring{\nabla}_\nu Q_{\mu\sigma}{}^\sigma \right) . \quad (3.230)$$

Therefore, if we have a metric compatible connection, then $R^\sigma{}_{\sigma\mu\nu} = 0$. This is reasonable because $R_{(\alpha\beta)\mu\nu} = 0$ if $Q_{\mu\alpha\nu} = 0$, as we saw in Eq. (3.218). Notice also that, for a torsion-free connection, the Ricci tensor will still be asymmetric, as the antisymmetric part is now given by $\frac{1}{2} \bar{R}_{\mu\nu}$.

A third tensor derived from the Riemann tensor is the ‘co-Ricci’ tensor $\tilde{R}^\mu{}_\nu$ [357]:

$$\tilde{R}^\mu{}_\nu \equiv g^{\alpha\beta} R^\mu{}_{\alpha\nu\beta} , \quad (3.231)$$

where we used the inverse metric to obtain it, in contrast to the previous two tensors.

Lowering one of the indices, so it is covariant, the co-Ricci tensor reads

$$\tilde{R}_{\mu\nu} = \mathring{R}_{\mu\nu} + \tilde{\kappa}_{\mu\nu} , \quad (3.232)$$

where $\tilde{\kappa}_{\mu\nu} \equiv g^{\alpha\beta} \kappa_{\mu\alpha\nu\beta}$. In the metric formalism, the Ricci and co-Ricci tensors coincide again due to the antisymmetry in the interchange of the first two indices

$$\mathring{\tilde{R}}^\mu{}_\nu = g^{\alpha\beta} \mathring{R}^\mu{}_{\alpha\nu\beta} = g^{\alpha\beta} \mathring{R}_\alpha{}^\mu{}_{\beta\nu} = \mathring{R}^{\sigma\mu}{}_{\sigma\nu} = \mathring{R}^\mu{}_\nu . \quad (3.233)$$

It is not hard to check that $\tilde{R}_{\mu\nu} = R_{\mu\nu}$ if the connection is metric compatible (see Eq. (3.218)):

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + 2g^{\alpha\beta} R_{(\alpha\mu)\nu\beta} = R_{\mu\nu} - 2\nabla_{[\nu} Q_{\beta]\mu}{}^\beta - 2Q_{[\nu}{}^{\alpha\beta} Q_{\alpha]\beta\mu} - T_\nu{}^{\alpha\lambda} Q_{\lambda\alpha\mu} . \quad (3.234)$$

So we see that a metric compatible connection has only the Ricci curvature tensor because $\bar{R}_{\mu\nu} = 0$ and $\tilde{R}_{\mu\nu} = R_{\mu\nu}$. As an example, let us consider again Eq. (3.216), for which $Q_{\alpha\mu\nu} = 0$. We readily see that $\kappa_{\mu\sigma}{}^\sigma = 0$, as expected. Therefore $\bar{\kappa}_{\mu\nu} = 0$ and $\bar{R}_{\mu\nu} = 0$. $\kappa^\alpha{}_{\beta\mu\nu}$ is given by

$$\begin{aligned} \kappa^\alpha{}_{\beta\mu\nu} &= \delta_\nu^\alpha \mathring{\nabla}_\mu A_\beta - \delta_\mu^\alpha \mathring{\nabla}_\nu A_\beta + g_{\mu\beta} \mathring{\nabla}_\nu A^\alpha - g_{\nu\beta} \mathring{\nabla}_\mu A^\alpha + (\delta_\mu^\alpha A_\nu - \delta_\nu^\alpha A_\mu) A_\beta - \\ &- (\delta_\mu^\alpha g_{\nu\beta} - \delta_\nu^\alpha g_{\mu\beta}) A_\sigma A^\sigma + (A_\mu g_{\nu\beta} - A_\nu g_{\mu\beta}) A^\alpha . \end{aligned} \quad (3.235)$$

Taking the corresponding traces, we obtain

$$\kappa_{\mu\nu} = \tilde{\kappa}_{\mu\nu} = -2\mathring{\nabla}_\nu A_\mu + 2A_\mu A_\nu - g_{\mu\nu} \left(\mathring{\nabla}_\sigma A^\sigma + 2A_\sigma A^\sigma \right) . \quad (3.236)$$

Consequently, $R_{\mu\nu} = \tilde{R}_{\mu\nu}$.

Since $\bar{R}_{\mu\nu}$ is antisymmetric, the trace will be zero and the only scalar we may obtain from the curvature tensors is the Ricci scalar

$$R \equiv g^{\mu\nu} R_{\mu\nu} = \tilde{R}^\sigma{}_\sigma = \mathring{R} + \kappa , \quad (3.237)$$

where

$$\kappa \equiv g^{\mu\nu} \kappa_{\mu\nu} = \tilde{\kappa}^\sigma{}_\sigma = \mathring{\nabla}_\sigma (\kappa_\rho{}^{\rho\sigma} - \kappa_\rho{}^{\sigma\rho}) + \kappa_{\sigma\lambda}{}^\sigma \kappa_\rho{}^{\rho\lambda} - \kappa_{\rho\lambda}{}^\sigma \kappa_\sigma{}^{\rho\lambda} . \quad (3.238)$$

Eq. (3.198) again gives

$$\kappa_{\sigma\mu}{}^{\sigma} = -T_{\mu\sigma}{}^{\sigma} - \frac{1}{2}Q_{\mu\sigma}{}^{\sigma}, \quad (3.239)$$

$$\kappa_{\sigma}{}^{\sigma\mu} = -T_{\sigma}{}^{\mu\sigma} - Q_{\sigma}{}^{\mu\sigma} + \frac{1}{2}Q^{\mu}{}_{\sigma}{}^{\sigma}. \quad (3.240)$$

Then

$$\begin{aligned} R = \mathring{R} - \mathring{\nabla}_{\sigma} (2T_{\rho}{}^{\sigma\rho} + Q_{\rho}{}^{\sigma\rho} - Q^{\sigma}{}_{\rho}{}^{\rho}) - \left(T^{\lambda}{}_{\rho}{}^{\rho} + Q^{\lambda}{}_{\rho}{}^{\rho} \right) T_{\lambda\sigma}{}^{\sigma} + \left(T_{\lambda\sigma}{}^{\sigma} + \frac{1}{2}Q_{\lambda\sigma}{}^{\sigma} \right) Q_{\rho}{}^{\lambda\rho} - \\ - \frac{1}{4}Q_{\lambda\sigma}{}^{\sigma}Q^{\lambda}{}_{\rho}{}^{\rho} + \frac{1}{4}T_{\sigma\rho\lambda}T^{\sigma\rho\lambda} + \frac{1}{4}Q_{\sigma\rho\lambda}Q^{\sigma\rho\lambda} + \frac{1}{2}T_{\sigma\rho\lambda}T^{\sigma\lambda\rho} - \frac{1}{2}Q_{\sigma\rho\lambda}Q^{\rho\lambda\sigma} - T_{\sigma\rho\lambda}Q^{\rho\lambda\sigma}. \end{aligned} \quad (3.241)$$

The Riemann curvature tensor measures the change experienced by the components of a vector parallel transported around a closed curve [23, 356]. On the other hand, the torsion tensor refers to the non-closure of the infinitesimal parallelogram that is formed by two vectors parallel transported along each other [358]. Both quantities contribute to the commutator of the covariant derivatives (compare the following equation with Eq. (2.17); in that case, we were considering the LC connection, and the torsion was zero) [23]:

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma} - T_{\mu\nu}{}^{\lambda}\nabla_{\lambda}V^{\rho}. \quad (3.242)$$

The non-metricity however implies the variation of the length of vectors parallel transported [356, 358]. Let $x^{\mu}(\lambda)$ be a curve with parameter λ . Given the covariant derivative along the curve, defined as [23]

$$\frac{D}{d\lambda} \equiv \frac{dx^{\mu}}{d\lambda} \nabla_{\mu}, \quad (3.243)$$

a vector V^{μ} parallel transported along the curve satisfies [23]

$$\frac{D}{d\lambda} V^{\mu} = 0. \quad (3.244)$$

The same derivative applied to the norm of the vector $V_{\sigma}V^{\sigma}$ yields

$$\frac{D}{d\lambda} (V_{\sigma}V^{\sigma}) = Q_{\alpha\mu\nu} \frac{dx^{\alpha}}{d\lambda} V^{\mu}V^{\nu}. \quad (3.245)$$

The same is true for the inner product of two parallel transported vectors and therefore the angles are affected too [20]. See Fig. 3.6 for a schematic representation of the geometric

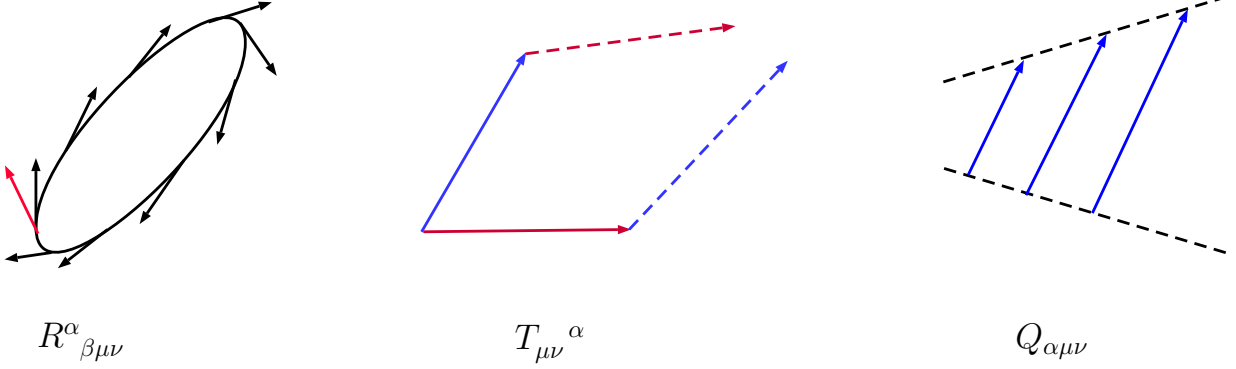


Figure 3.6: Schematic representation of the geometric meaning of the curvature, the torsion and the non-metricity from left to right, respectively.

meaning of each of the curvature, torsion and non-metricity tensors when the rest of them vanish.

By imposing vanishing curvature and a metric compatible connection, we have that $\mathring{R}$ is still not zero by virtue of Eq. (3.241) and it is given by

$$\mathring{R} = -\frac{1}{4}T + 2\mathring{\nabla}_\sigma T_\rho^{\sigma\rho}, \quad (3.246)$$

where T is

$$T \equiv T_{\mu\nu\alpha}T^{\mu\nu\alpha} + 4T_\sigma^{\mu\sigma}T_{\mu\rho}^{\rho} - 2T_{\mu\nu\alpha}T^{\alpha\mu\nu}, \quad (3.247)$$

which is a particular combination of all the scalars that can be built out of the torsion tensor, given its symmetries. The Einstein-Hilbert (EH) action of GR, with Lagrangian density $\sqrt{-g}\frac{M_{\text{Pl}}^2}{2}\mathring{R}$, is then equivalent to the action

$$S_{\text{TEGR}} = -\frac{M_{\text{Pl}}^2}{8} \int d^4x \sqrt{-g} T, \quad (3.248)$$

because $\mathring{R}$ and T only differ by a total derivative that can be integrated out. ‘TEGR’ stands for ‘Teleparallel Equivalent of GR’ [356]. ‘Teleparallel’ reflects the fact that the absence of curvature allows for distant or ‘absolute’ parallelism given that vectors no longer rotate when parallel transported.

Similarly, if we impose vanishing curvature and torsion, we have that

$$\mathring{R} = -\frac{1}{4}Q + \mathring{\nabla}_\sigma (Q_\rho^{\sigma\rho} - Q^\sigma{}_\rho{}^\rho), \quad (3.249)$$

where Q is

$$Q \equiv Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - 2Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} + 2Q_{\mu\sigma}{}^{\sigma}Q_{\rho}{}^{\mu\rho} - Q_{\mu\sigma}{}^{\sigma}Q^{\mu\rho}{}_{\rho}, \quad (3.250)$$

which, again, is a particular combination of all the scalars that can be built out of the non-metricity tensor, given its symmetries. Notice that there is one more term than in the case of the torsion because $T^{\sigma}{}_{\sigma}{}^{\mu} = 0$. The EH action of GR is equivalent to

$$S_{\text{STTEGR}} = -\frac{M_{\text{Pl}}^2}{8} \int d^4x \sqrt{-g} Q, \quad (3.251)$$

neglecting the total derivative. ‘STTEGR’ denotes ‘Symmetric Teleparallel Equivalent of GR’, meaning that the parallel transport is symmetric under the exchange of the transported vector and the direction of transport (ensuring the closure of the parallelogram that we mentioned earlier) [356], and ‘teleparallel’ refers to the same feature caused by the lack of curvature that we just commented.¹³

This equivalence among three theories formulated for different connections has been referred to as ‘geometric trinity of gravity’ in the literature [358]. It must be noted that the exclusion of boundary terms is not justified when spacetime boundaries are concerned, although we shall not discuss the non-equivalences this may give rise to. Also, the equivalence regarding the matter sector is tackled in Ref. [356] for torsional teleparallel geometries but we will not investigate them in this thesis either.

It is worth at this point summarising the so-called *metric-affine* geometries [152] which are relevant in the literature. Then, we will examine an EH-like action in the *first-order* or *Palatini formalism* as is known in the literature [152, 352, 354, 359–361], to eventually arrive at the notion of projective invariance which shall be of use later on. Those subclasses of the metric-affine geometries are [356]:

- General teleparallel geometry: $R^{\alpha}{}_{\beta\mu\nu} = 0$, while the torsion and non-metricity are non-zero.
- Riemann-Cartan geometry: $Q_{\alpha\mu\nu} = 0$, while the Riemann curvature tensor and the torsion do not vanish.
- Weyl geometry: $T_{\mu\nu}{}^{\alpha} = 0$, while the Riemann curvature tensor and the non-metricity tensor are present.

13. Note that the connections employed in those two cases are different (one is metric compatible and the other symmetric), despite the fact that we do not use carets or primes to differentiate them in order not to overdo the use of notation.

- Metric teleparallel geometry: $R^\alpha_{\beta\mu\nu} = 0$ and $Q_{\alpha\mu\nu} = 0$, while the torsion does not vanish.
- Symmetric teleparallel geometry: $R^\alpha_{\beta\mu\nu} = 0$ and $T_{\mu\nu}{}^\alpha = 0$, while the non-metricity tensor is non-vanishing.

We also have the well-known case of $T_{\mu\nu}{}^\alpha = 0$ and $Q_{\alpha\mu\nu} = 0$ that has been the arena on which the work developed in this thesis has been carried out until now. Also, the case of $R^\alpha_{\beta\mu\nu} = 0$, $T_{\mu\nu}{}^\alpha = 0$ and $Q_{\alpha\mu\nu} = 0$ corresponds to Minkowski's geometry. Among the geometries mentioned earlier, we will be interested in *Weyl geometry* and we shall comment on it once we examine the EH-like action in the Palatini formalism.

3.4.3 Einstein-Hilbert-Palatini Action

In the *second-order* formalism (or ‘metric formalism’ as named in the previous section), the EH action is a functional of the metric and its first and second derivatives. It is possible however to arrive at an equivalent action (in the sense that the field equations derived from the action are the same) which does not depend on the second derivative of the metric by simply integrating by parts the derivatives of the connection in the definition of the Ricci tensor (see Eq. (3.190) and remember that $R_{\mu\nu} \equiv R^\sigma_{\mu\sigma\nu}$), and using Jacobi's formula (1.75) and the Christoffel symbols (3.191). $\sqrt{-g}\mathring{R}$ becomes

$$\sqrt{-g}\mathring{R} = \sqrt{-g}g^{\mu\nu} \left(\mathring{\Gamma}^\sigma_{\mu\rho}\mathring{\Gamma}^\rho_{\sigma\nu} - \mathring{\Gamma}^\sigma_{\mu\nu}\mathring{\Gamma}^\rho_{\rho\sigma} \right) + \partial_\sigma \left[\sqrt{-g}g^{\mu\nu} \left(\mathring{\Gamma}^\sigma_{\mu\nu} - \delta^\sigma_\nu \mathring{\Gamma}^\rho_{\rho\mu} \right) \right]. \quad (3.252)$$

The resulting equivalent action is known as ‘Gamma squared action’ or ‘Einstein action’ [362, 363] and it is almost as old as the EH action of GR [364]. It reads

$$S_{\mathring{\Gamma}\mathring{\Gamma}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g}g^{\mu\nu} \left(\mathring{\Gamma}^\sigma_{\mu\rho}\mathring{\Gamma}^\rho_{\sigma\nu} - \mathring{\Gamma}^\sigma_{\mu\nu}\mathring{\Gamma}^\rho_{\rho\sigma} \right), \quad (3.253)$$

up to the boundary term. However, notice that the Lagrangian of this action is not a scalar under coordinate transformations (see Eq. (3.204)), as it needs to be combined with the total derivative term [363].

Another way for the action not to depend on second-order derivatives of the metric tensor is to consider the Palatini formalism that we mentioned in the previous section. In this formalism, the metric tensor and the connection are considered independent *a priori*. As we saw in the earlier section, the Riemann tensor is not a function of the metric if the connection is not the LC connection. Neither is the Ricci tensor because it is obtained by taking the

trace between the upper and lower indices of the Riemann tensor and does not require the use of the metric. This means that the dependence on the metric of the Lagrangian density only comes from the inverse metric and the determinant. The ‘Einstein-Hilbert-Palatini (EHP) action’ (the analogue of the EH action of GR) is then

$$S_{\text{EHP}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma) . \quad (3.254)$$

We include the matter action, S_{M} , and the CC, Λ , such that (compare with Eq. (2.1) and see Eq. (2.3) where S_{M} was introduced)

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} [g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda] + \int d^4x \sqrt{-g} \mathcal{L}_{\text{M}}(g_{\mu\nu}, \Psi) . \quad (3.255)$$

It is noteworthy that we have assumed that the matter fields are not coupled to the connection so the matter action is not a functional of the connection. In case we regard a dependence on the connection (which is not the case in this thesis), the variation of the matter action with respect to the connection will give rise to an extra tensor called ‘hypermomentum tensor’ [152]. The metric field equations are then

$$G_{(\mu\nu)} + \Lambda g_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} T_{\mu\nu}^{\text{M}} , \quad (3.256)$$

which are similar to Eq. (1.2) except for the Einstein tensor whose symmetric part is shown to enter into the equation explicitly given the fact that it is asymmetric *a priori*. $T_{\mu\nu}^{\text{M}}$ was defined in Eq. (2.2). Also, when performing the variation with respect to the inverse metric, we notice there is no boundary term given the independence of the connection from the metric and its derivatives. Indeed, the Ricci tensor $R_{\mu\nu}(\Gamma)$ is not varied with respect to the inverse metric because it depends only on the connection and its derivatives in the Palatini formalism. The full action S is not a functional of the second-order derivatives of the metric tensor and, consequently, no boundary terms arise when extremising the action with respect to the inverse metric (see Eq. (C.5) where the last two terms are responsible for the boundary terms in the metric formalism).

We now obtain the connection field equations. Since the CC term and the matter action do not depend on the connection, we extremise S_{EHP} in Eq. (3.254) only. To derive the equations, we use Eq. (3.237) to extract the part that depends on the distortion tensor $\kappa_{\mu\nu}{}^\alpha$ (see Eq. (3.198)) and its derivatives from the Ricci scalar R . The action in Eq. (3.254) is

then equivalent to

$$S_{\text{EHP}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left[\dot{R} - \kappa_{\mu\nu}{}^\alpha (\kappa_\alpha{}^{\mu\nu} - \delta_\alpha^\mu \kappa_\sigma{}^{\sigma\nu}) \right], \quad (3.257)$$

where the total derivative term has been ignored. We then obtain the Euler-Lagrange (EL) equations of the tensor $\kappa_{\mu\nu}{}^\alpha$. A simple variation of the Lagrangian yields

$$\frac{\partial \mathcal{L}_{\text{EHP}}}{\partial \kappa_{\mu\nu}{}^\alpha} = 0 \Rightarrow \kappa_\alpha{}^{\mu\nu} - \kappa_{\sigma\alpha}{}^\sigma g^{\mu\nu} + \kappa^\nu{}_\alpha{}^\mu - \delta_\alpha^\mu \kappa_\sigma{}^{\sigma\nu} = 0, \quad (3.258)$$

where $S_{\text{EHP}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{EHP}}$. Taking the traces, we obtain a trivial equation ($\alpha = \nu$) and two equations which combined lead to $\kappa^\sigma{}_{\mu\sigma} = \kappa^\sigma{}_{\sigma\mu} = \frac{1}{4} \kappa_{\mu\sigma}{}^\sigma$. Then, the equation above can be written as

$$\kappa_{\mu\nu}{}^\alpha + \kappa^\alpha{}_{\mu\nu} = \kappa^\sigma{}_{\beta\sigma} \left(\delta_\mu^\beta \delta_\nu^\alpha + g_{\mu\nu} g^{\alpha\beta} \right). \quad (3.259)$$

Writing the cyclic permutations, $\mu\nu\alpha$, $\nu\alpha\mu$ and $\alpha\mu\nu$, subtracting two of them and adding up the third one, we find that $\kappa_{\mu\nu}{}^\alpha = \delta_\nu^\alpha A_\mu$, where $A_\mu \equiv \kappa^\sigma{}_{\mu\sigma}$. We then show that the most general solution to the connection field equations obtained from the EHP action is [365]

$$\Gamma_{\mu\nu}^\alpha = \mathring{\Gamma}_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu. \quad (3.260)$$

The connection $\Gamma_{\mu\nu}^\alpha$ is not symmetric so it induces non-zero torsion, which reads

$$T_{\mu\nu}{}^\alpha = \delta_\nu^\alpha A_\mu - \delta_\mu^\alpha A_\nu. \quad (3.261)$$

This torsion tensor is the sign-reversed of Eq. (3.217). Also, it is not metric compatible, and the non-metricity tensor is

$$Q_{\alpha\mu\nu} = -2A_\alpha g_{\mu\nu}, \quad (3.262)$$

which is the same as Eq. (3.221).

We can calculate the corresponding Riemann tensor from the connection above. We obtain (see Eqs. (3.210) and (3.211))

$$R^\alpha{}_{\beta\mu\nu} = \mathring{R}^\alpha{}_{\beta\mu\nu} + \delta_\beta^\alpha A_{\mu\nu}, \quad (3.263)$$

where $A_{\mu\nu} \equiv \mathring{\nabla}_\mu A_\nu - \mathring{\nabla}_\nu A_\mu$. The Ricci tensor is $R_{\mu\nu} = \mathring{R}_{\mu\nu} + A_{\mu\nu}$, such that $R_{[\mu\nu]} = A_{\mu\nu}$. Hence it can be seen that the Ricci scalar is actually the same as in the metric formalism,

$R = \mathring{R}$. We can confirm this by plugging $\kappa_{\mu\nu}^\alpha = \delta_\nu^\alpha A_\mu$ into the action in Eq. (3.257), such that the Lagrangian density is given by $\sqrt{-g}\mathring{R}$ because the rest of terms vanish. This case reflects what is called a *projective*¹⁴ *invariance* of the EHP action [365, 367–370] in the sense that, given an arbitrary connection $\Gamma_{\mu\nu}^\alpha$, the action will be invariant if one considers another one $\hat{\Gamma}_{\mu\nu}^\alpha$ such that

$$\hat{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu , \quad (3.265)$$

A_μ being an arbitrary vector field (indeed, $\hat{R} = R$). Also, since $\hat{R}_{\mu\nu} = R_{\mu\nu} + A_{\mu\nu}$, the symmetric part of the Ricci tensor is invariant under the projective transformations and so are the metric field equations (Eq. (3.256)). The distortion field equations are invariant under the projective transformation as can be easily checked by replacing $\kappa_{\mu\nu}^\alpha \rightarrow \kappa_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu$ in Eq. (3.259).

That vector field A_μ can be chosen such that the tangent vector to a curve x^μ , in an *arbitrary* parametrisation, namely α , is parallel transported with respect to the connection $\hat{\Gamma}_{\mu\nu}^\alpha$ [371]. Indeed, by writing the equation for parallel transport along a curve $x^\mu(\lambda)$ with parameter λ of the tangent vector we have the geodesics equation (see Eq. (3.243); the parallel transport of the tangent vector corresponds to the autoparallelism mentioned in the footnote above [366]) [23]

$$\frac{D}{d\lambda} \left(\frac{dx^\mu}{d\lambda} \right) = 0 \Rightarrow \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = 0 . \quad (3.266)$$

This demands λ be an *affine* parameter. Any other arbitrary parameter α would lead to the following equation instead [23]:

$$\frac{d^2 x^\mu}{d\alpha^2} + \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\alpha} \frac{dx^\rho}{d\alpha} = f(\alpha) \frac{dx^\mu}{d\alpha} . \quad (3.267)$$

We recover however the equation in the affine parametrisation if $f(\alpha) = \lambda_{,\alpha\alpha}/\lambda_{,\alpha}$, where $\lambda_{,\alpha} = d\lambda/d\alpha$. Alternatively, we may also consider the projective transformation in Eq. (3.265)

14. The geometry of paths or *projective* geometry may be characterised by the process of *autoparallelism of directions*; that is, the infinitesimal displacement of the direction of a vector into its own direction. One then has that a transformation relating two connections is a projective transformation iff the vectors that result from the parallel transport under the two connections *do not differ in direction*. We see that this occurs when

$$\left(\hat{\Gamma}_{\mu\nu}^\alpha - \Gamma_{\mu\nu}^\alpha \right) V^\mu V^\nu \propto V^\alpha , \quad (3.264)$$

V^μ being an arbitrary vector. Eq. (3.265) would yield a vector $A_\sigma V^\sigma V^\mu$. See Ref. [366] for further details and the upcoming discussion about the geodesics equation.

and impose that the vector field A_μ satisfies [371]

$$A_\mu \frac{dx^\mu}{d\alpha} = -f(\alpha) . \quad (3.268)$$

Then, we easily see that the curves are geodesics in the presence of the connection $\hat{\Gamma}_{\mu\nu}^\alpha$ because

$$\frac{d^2 x^\mu}{d\alpha^2} + \hat{\Gamma}_{\sigma\rho}^\mu \frac{dx^\sigma}{d\alpha} \frac{dx^\rho}{d\alpha} = 0 \Rightarrow \frac{D}{d\alpha} \left(\frac{dx^\mu}{d\alpha} \right) = 0 , \quad (3.269)$$

or, in other words, α is an affine parameter of the connection $\hat{\Gamma}_{\mu\nu}^\alpha$ [365, 371].

To end this discussion, we go back to the Gamma squared action and attempt to write the Lagrangian density $\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)$ in a way similar to Eq. (3.252), such that

$$\begin{aligned} \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma) &= \sqrt{-g}g^{\mu\nu} \left[\Gamma_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda - \mathring{\Gamma}_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda - \Gamma_{\sigma\lambda}^\sigma \mathring{\Gamma}_{\nu\mu}^\lambda - \left(\Gamma_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda - \Gamma_{\nu\lambda}^\sigma \mathring{\Gamma}_{\sigma\mu}^\lambda - \mathring{\Gamma}_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda \right) \right] + \\ &+ \partial_\sigma \left[\sqrt{-g}g^{\mu\nu} \left(\Gamma_{\nu\mu}^\sigma - \delta_\nu^\sigma \Gamma_{\rho\mu}^\rho \right) \right] . \end{aligned} \quad (3.270)$$

Remember that $\Gamma_{\mu\nu}^\alpha$ is not symmetric *a priori*, so one has to be careful when writing the Lagrangian density above. If we set $\Gamma_{\mu\nu}^\alpha = \mathring{\Gamma}_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu$, we obtain Eq. (3.252) as expected. Moreover, if we introduce a new connection which is the result of the projective transformation of $\Gamma_{\mu\nu}^\alpha$ such that $\hat{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu$, then each of the two terms of Eq. (3.270) is *separately* invariant:

$$\begin{aligned} \sqrt{-g}g^{\mu\nu} \left[\hat{\Gamma}_{\sigma\lambda}^\sigma \hat{\Gamma}_{\nu\mu}^\lambda - \mathring{\Gamma}_{\sigma\lambda}^\sigma \hat{\Gamma}_{\nu\mu}^\lambda - \hat{\Gamma}_{\sigma\lambda}^\sigma \mathring{\Gamma}_{\nu\mu}^\lambda - \left(\hat{\Gamma}_{\nu\lambda}^\sigma \hat{\Gamma}_{\sigma\mu}^\lambda - \hat{\Gamma}_{\nu\lambda}^\sigma \mathring{\Gamma}_{\sigma\mu}^\lambda - \mathring{\Gamma}_{\nu\lambda}^\sigma \hat{\Gamma}_{\sigma\mu}^\lambda \right) \right] = \\ = \sqrt{-g}g^{\mu\nu} \left[\Gamma_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda - \mathring{\Gamma}_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda - \Gamma_{\sigma\lambda}^\sigma \mathring{\Gamma}_{\nu\mu}^\lambda - \left(\Gamma_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda - \Gamma_{\nu\lambda}^\sigma \mathring{\Gamma}_{\sigma\mu}^\lambda - \mathring{\Gamma}_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda \right) \right] , \end{aligned} \quad (3.271)$$

and

$$\partial_\sigma \left[\sqrt{-g}g^{\mu\nu} \left(\hat{\Gamma}_{\nu\mu}^\sigma - \delta_\nu^\sigma \hat{\Gamma}_{\rho\mu}^\rho \right) \right] = \partial_\sigma \left[\sqrt{-g}g^{\mu\nu} \left(\Gamma_{\nu\mu}^\sigma - \delta_\nu^\sigma \Gamma_{\rho\mu}^\rho \right) \right] . \quad (3.272)$$

This is in contrast to the invariance under coordinate transformations, which required the combination of the two as we said already.

3.4.4 Weyl Geometry

We now turn our attention to an example of scalar-tensor theory which was of great relevance in the most part of this thesis: the $F(\phi)R$ gravity theory. We review it in the Palatini formalism. The reason is that it motivates the discussion about the Weyl geometry, which will be of interest in the next part of the thesis, and provides the tools needed to understand

the argument given in Ref. [349] as to why Horndeski's theory with $G_4 = G_4(\phi, X)$ gives $c_{\text{GW}} = c$ (see paragraph before the beginning of Sec. 3.4).

The action of $F(\phi)R$ gravity in the Palatini formalism reads

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} F(\phi) g^{\mu\nu} R_{\mu\nu}(\Gamma) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_{\text{M}}[g_{\mu\nu}, \Psi], \quad (3.273)$$

where, as was pointed out below Eq. (2.1), ‘ Ψ ’ is used to denote collectively the matter fields of the matter sector. S_{M} is given in Eq. (2.3). The dependence on the connection comes from the Ricci tensor only, meaning that this action is projective invariant in view of what we discussed in the previous section. The metric field equations obtained by varying the action with respect to the inverse metric are

$$G_{(\mu\nu)} = \frac{1}{F} \left(T_{\mu\nu}^\phi + T_{\mu\nu}^{\text{M}} \right), \quad (3.274)$$

where $T_{\mu\nu}^\phi$ was given in Eq. (1.72) and $T_{\mu\nu}^{\text{M}}$ was defined in Eq. (2.2). These equations are projective invariant too for the reason explained in the previous section. However, we see that this equation does differ substantially from the one of the metric formalism due to the absence of an energy-momentum tensor induced by the gravity modifications (see Eqs. (2.8) and (2.10)). As was said in the previous section, below Eq. (3.256), the fact that the connection is *a priori* independent from the metric and its derivatives implies that no boundary terms will result when the variation with respect to the inverse metric is performed (the action S in Eq. (3.273) is not a functional of the derivatives of the metric and the Ricci tensor $R_{\mu\nu}$ in particular is not varied with respect to the inverse metric as in the metric formalism; see Eq. (C.5) where the variation of the Ricci scalar R of the metric formalism does yield boundary terms). Consequently, the time-varying Planck mass (or Newton's constant) will not yield an extra effective energy-momentum tensor in the equations. However, we will have these effects present in the connection field equations.

To derive the equations of the distortion tensor, we proceed as in the earlier section and use Eq. (3.237) to write the action as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} F(\phi) \mathring{R} - \frac{1}{2} [F \kappa_{\mu\nu}{}^\alpha (\kappa_\alpha{}^{\mu\nu} - \delta_\alpha^\mu \kappa_\sigma{}^{\sigma\nu}) + \partial_\sigma F (\kappa_\rho{}^{\rho\sigma} - \kappa_\rho{}^{\sigma\rho})] - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\} + S_{\text{M}}[g_{\mu\nu}, \Psi], \quad (3.275)$$

where we integrated by parts and remove total derivative terms. The equations of the

distortion tensor are

$$\kappa_{\mu\nu}{}^\alpha + \kappa_{\mu\nu}^\alpha - \kappa_{\mu\sigma}^\sigma \delta_\nu^\alpha - g_{\mu\nu} \kappa_\sigma^{\sigma\alpha} = \partial_\sigma \ln F (g_{\mu\nu} g^{\sigma\alpha} - \delta_\mu^\sigma \delta_\nu^\alpha). \quad (3.276)$$

These equations are invariant under projective transformations, same as those derived from the EHP action. Writing the cyclic permutations as we did below Eq. (3.259), and subtracting two of the equations and adding up the third one gives

$$\kappa_{\mu\nu}{}^\alpha = \delta_\nu^\alpha A_\mu + \delta_\mu^\alpha B_\nu - g_{\mu\nu} B^\alpha, \quad (3.277)$$

where

$$A_\mu \equiv \frac{1}{2} (\kappa_{\mu\sigma}^\sigma + \kappa_{\sigma\mu}^\sigma), \quad (3.278)$$

$$B_\mu \equiv \frac{1}{2} (\kappa_{\mu\sigma}^\sigma - \kappa_{\sigma\mu}^\sigma) - \partial_\mu \ln F. \quad (3.279)$$

Taking the traces yields $\kappa_{\sigma\mu}{}^\sigma = A_\mu + 3B_\mu$, $\kappa_{\mu\sigma}{}^\sigma = 4A_\mu$ and $\kappa_{\sigma\mu}^\sigma = A_\mu - 3B_\mu$. Hence $B_\mu = \frac{1}{2} \partial_\mu \ln F$ and we have the most general solution to the connection field equations:

$$\Gamma_{\mu\nu}^\alpha = \mathring{\Gamma}_{\mu\nu}^\alpha + \delta_\nu^\alpha A_\mu + \frac{1}{2} (\delta_\mu^\alpha \partial_\nu \ln F - g_{\mu\nu} \partial^\alpha \ln F). \quad (3.280)$$

If F is constant we recover Eq. (3.260) as expected. Given the invariance of the action and the field equations under projective transformations, we can write the equivalent connection

$$\Gamma_{\mu\nu}^\alpha = \mathring{\Gamma}_{\mu\nu}^\alpha + \frac{1}{2} (\delta_\nu^\alpha \partial_\mu \ln F + \delta_\mu^\alpha \partial_\nu \ln F - g_{\mu\nu} \partial^\alpha \ln F), \quad (3.281)$$

which is symmetric and still a solution to the field equations. This connection is an example of Eq. (3.220) with $A_\mu = \frac{1}{2} \partial_\mu \ln F$. For symmetric connections, we saw that $R_{[\mu\nu]} = \frac{1}{2} \bar{R}_{\mu\nu}$ (see Eq. (3.226)). For this connection, we have

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} - 3\mathring{\nabla}_\nu A_\mu + \mathring{\nabla}_\mu A_\nu + 2A_\mu A_\nu - g_{\mu\nu} (\mathring{\nabla}_\sigma A^\sigma + 2A_\sigma A^\sigma), \quad (3.282)$$

where

$$\bar{R}_{\mu\nu} = 4 (\mathring{\nabla}_\mu A_\nu - \mathring{\nabla}_\nu A_\mu). \quad (3.283)$$

Now, since $A_\mu = \frac{1}{2}\partial_\mu \ln F$, so the vector field is Abelian, then $\bar{R}_{\mu\nu} = 0$ and $R_{\mu\nu}$ reads

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} - \mathring{\nabla}_\mu \partial_\nu \ln F + \frac{1}{2}\partial_\mu \ln F \partial_\nu \ln F - \frac{1}{2}g_{\mu\nu} \left(\mathring{\square} \ln F + \partial_\sigma \ln F \partial^\sigma \ln F \right). \quad (3.284)$$

The Ricci scalar is

$$R = \mathring{R} - 3 \left(\mathring{\square} \ln F + \frac{1}{2}\partial_\sigma \ln F \partial^\sigma \ln F \right), \quad (3.285)$$

so the Einstein tensor reads

$$G_{\mu\nu} = \mathring{G}_{\mu\nu} - \mathring{\nabla}_\mu \partial_\nu \ln F + \frac{1}{2}\partial_\mu \ln F \partial_\nu \ln F + \frac{1}{2}g_{\mu\nu} \left(2\mathring{\square} \ln F + \frac{1}{2}\partial_\sigma \ln F \partial^\sigma \ln F \right). \quad (3.286)$$

This is nothing but the conformal transformation of the Einstein tensor when the conformal factor is $\Omega^2 = \alpha F(\phi)$ [172] (see Eq. (2.40)), α being a constant of dimension -2 (so Ω^2 is dimensionless; see Eq. (2.26) and discussion below Eq. (2.41)). Then, $G_{\mu\nu} = \hat{\mathring{G}}_{\mu\nu}$ given the conformal transformation of Eq. (2.26). Inserting this $G_{\mu\nu}$ in Eq. (3.274), we obtain

$$\mathring{G}_{\mu\nu} = \frac{1}{F} \left(T_{\mu\nu}^\phi + T_{\mu\nu}^M + \mathring{\nabla}_\mu \partial_\nu F - g_{\mu\nu} \mathring{\square} F - \frac{3}{2F} \partial_\mu F \partial_\nu F + \frac{3}{4F} g_{\mu\nu} \partial_\sigma F \partial^\sigma F \right). \quad (3.287)$$

Performing the following field redefinition:

$$\frac{d\phi}{d\varphi} = \sqrt{1 + \frac{3F_{,\varphi}^2}{2F}}, \quad (3.288)$$

one arrives at

$$\mathring{G}_{\mu\nu} = \frac{1}{F} \left(T_{\mu\nu}^\varphi + T_{\mu\nu}^M + \mathring{\nabla}_\mu \partial_\nu F - g_{\mu\nu} \mathring{\square} F \right), \quad (3.289)$$

which are the metric field equations of the metric formalism of the $F(\phi)R$ theory of gravity (see Eq. (2.7), with $f = FR$ and $\mathcal{K} = 1$), with the energy-momentum tensor $T_{\mu\nu}^\varphi$ being the same as Eq. (1.72), but with φ instead of ϕ and $V = V(\phi(\varphi))$.

There is one equation left to be checked: the scalar field equation. Varying the action in Eq. (3.273) with respect to the scalar field, we obtain

$$\int d^4x \sqrt{-g} \left[\left(\frac{1}{2} F_{,\phi} R - V_{,\phi} \right) \delta\phi - g^{\mu\nu} \partial_\mu \delta\phi \partial_\nu \phi \right] = 0. \quad (3.290)$$

Now, since the connection is not necessarily the LC connection, if we integrate by parts the

second term in the action we get

$$-\int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \delta\phi \partial_\nu \phi = \int d^4x \sqrt{-g} [(\Box\phi - Q_\sigma{}^{\sigma\mu} \partial_\mu \phi) \delta\phi - \nabla_\mu (g^{\mu\nu} \partial_\nu \phi \delta\phi)] , \quad (3.291)$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ (since $\nabla_\alpha g_{\mu\nu} \neq 0$ it is important to emphasise this definition), and the last term cannot be converted into a total derivative. We have instead

$$\sqrt{-g} \nabla_\mu V^\mu = \partial_\mu (\sqrt{-g} V^\mu) - \sqrt{-g} \left(T_{\mu\sigma}{}^\sigma + \frac{1}{2} Q_{\mu\sigma}{}^\sigma \right) V^\mu , \quad (3.292)$$

given a vector field V^μ . Therefore, the scalar field equation reads

$$\Box\phi - V_{,\phi} = -\frac{1}{2} F_{,\phi} R + \left(T_\sigma{}^{\mu\sigma} + Q_\sigma{}^{\mu\sigma} - \frac{1}{2} Q^\mu{}_\sigma{}^\sigma \right) \partial_\mu \phi . \quad (3.293)$$

However, the d'Alembert operator acting on ϕ gives

$$\Box\phi = \overset{\circ}{\Box}\phi - \kappa_\sigma{}^{\sigma\mu} \partial_\mu \phi , \quad (3.294)$$

and Eq. (3.240) yields

$$\overset{\circ}{\Box}\phi - V_{,\phi} = -\frac{1}{2} F_{,\phi} R . \quad (3.295)$$

If F is constant, we obtain the same equation as in the metric formalism. This is reasonable as the scalar field does not couple to the connection in the action. The only difference stems from the Ricci scalar. However, according to Eq. (3.285), we can write it as

$$\overset{\circ}{\Box}\phi - V_{,\phi} + \frac{1}{2} F_{,\phi} \overset{\circ}{R} = \frac{3F_{,\phi}}{2F} \left(\overset{\circ}{\Box}F - \frac{1}{2F} \partial_\sigma F \partial^\sigma F \right) . \quad (3.296)$$

Using the field redefinition in Eq. (3.288) and $\overset{\circ}{\Box}F = F_{,\varphi} \overset{\circ}{\Box}\varphi + F_{,\varphi\varphi} \partial_\sigma \varphi \partial^\sigma \varphi$, we arrive at

$$\overset{\circ}{\Box}\varphi - V_{,\varphi} + \frac{1}{2} F_{,\varphi} \overset{\circ}{R} = 0 , \quad (3.297)$$

which is the scalar field equation of a canonical scalar field φ with potential $V(\varphi)$ and non-minimally coupled to gravity. This is expected in view of the metric field equations (Eq. (3.289)), and the action (3.275) after we replace $\kappa_{\mu\nu}{}^\alpha$ by the distortion tensor in Eq. (3.281).

Eq. (3.281) is an example of what is called the *Weyl connection* [366], named after H. Weyl who proposed it in 1918 as an attempt to generalise Riemannian geometry and

conceive a ‘pure’ infinitesimal geometry which did not allow distance comparison of the length of vectors (we saw that this connection induced a non-zero non-metricity and hence a variation of the length of vectors parallel transported; see discussion below Eq. (3.242)). The resulting geometry is known as *Weyl geometry* [366]. In the concrete case seen here, given the fact that the vector field A_μ (Weyl vector) is the partial derivative of a scalar function, the connection $\Gamma_{\mu\nu}^\alpha$ is the conformal transformation of the LC connection [172], and this is in line with what we discussed in regard to the Einstein tensor below Eq. (3.286). The fact that this connection is obtained from the LC connection after a conformal transformation can be understood if we perform this conformal rescaling of the metric in the initial action in Eq. (3.273). Since $R_{\mu\nu} = R_{\mu\nu}(\Gamma)$, the calculation is considerably easier than in the metric formalism. Given the invariance of the Ricci tensor under conformal transformations, we have (with $\Omega^2(\phi) = F(\phi)/M_{\text{Pl}}^2$; see Eq. (2.26))

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{M_{\text{Pl}}^2}{2} \hat{g}^{\mu\nu} R_{\mu\nu}(\Gamma) - \frac{1}{2} \frac{M_{\text{Pl}}^2}{F} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{M_{\text{Pl}}^4}{F^2} V(\phi) \right] + S_{\text{M}} \left[\frac{M_{\text{Pl}}^2}{F} \hat{g}_{\mu\nu}, \Psi \right]. \quad (3.298)$$

The gravitational sector in this frame corresponds to the EHP action and we know that the solution to the connection field equations is the LC connection plus an arbitrary vector (see Eq. (3.260)), which is non-dynamical. Therefore $\Gamma_{\mu\nu}^\alpha = \hat{\Gamma}_{\mu\nu}^\alpha$. Reverting the conformal transformation [172]

$$\Gamma_{\mu\nu}^\alpha = \hat{\Gamma}_{\mu\nu}^\alpha + \frac{1}{\Omega} (\delta_\nu^\alpha \partial_\mu \Omega + \delta_\mu^\alpha \partial_\nu \Omega - g_{\mu\nu} \partial^\alpha \Omega). \quad (3.299)$$

Since $\Omega^2(\phi) = F(\phi)/M_{\text{Pl}}^2$, we obtain Eq. (3.281).

Notice that the Weyl connection

$$\Gamma_{\mu\nu}^\alpha = \hat{\Gamma}_{\mu\nu}^\alpha + \delta_\mu^\alpha A_\nu + \delta_\nu^\alpha A_\mu - g_{\mu\nu} A^\alpha, \quad (3.300)$$

is invariant under conformal transformations of the metric if the Weyl vector A_μ transforms as

$$\hat{A}_\mu(x) = A_\mu(x) - \partial_\mu \ln \Omega(x). \quad (3.301)$$

Therefore, there is a unique affine connection (the Weyl connection) in contrast to the LC connection, which is not unique when the conformal structure is taken into account [366].

We have then introduced the Weyl geometry in which the length of vectors (not only their direction) is path dependent and the Weyl connection is uniquely determined by the metric

structure of this geometry [366]. This Weyl connection is symmetric and hence torsionless. We shall employ this connection in the study of the SGB model in the Palatini formalism. Prior to this however, we shall very briefly comment on the theory of quadratic gravity in a space provided with a generic connection, before coupling the quadratic curvature scalars to the scalar field.

3.4.5 Quadratic Gravity

The Lagrangian of the theory of quadratic gravity is obtained by combining all the possible terms which are quadratic in curvature. In the metric formalism, this combination is [372]

$$\mathcal{L}_{\text{quad}} = \alpha \mathring{R}^2 + \beta \mathring{R}^{\mu\nu} \mathring{R}_{\mu\nu} + \gamma \mathring{R}^{\alpha\beta\mu\nu} \mathring{R}_{\alpha\beta\mu\nu} . \quad (3.302)$$

Given that in four dimensions, the GB term is a total derivative (see discussion below Eq. (3.11)), an alternative Lagrangian leading to the same field equations is

$$\mathcal{L}_{\text{quad}} = (\alpha - \gamma) \mathring{R}^2 + (\beta + 4\gamma) \mathring{R}^{\mu\nu} \mathring{R}_{\mu\nu} . \quad (3.303)$$

In the Palatini formalism, the combination includes further terms given the absence of symmetries as was discussed in the beginning of Sec. 3.4 already. Thus, the most general quadratic Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{quad}} = & \alpha R^2 + R^{\mu\nu} (\beta_1 R_{\mu\nu} + \beta_2 R_{\nu\mu}) + \tilde{R}^{\mu\nu} (\beta_3 R_{\mu\nu} + \beta_4 R_{\nu\mu} + \beta_5 \tilde{R}_{\mu\nu} + \beta_6 \tilde{R}_{\nu\mu}) + \\ & + \bar{R}^{\mu\nu} (\beta_7 R_{\mu\nu} + \beta_8 \tilde{R}_{\mu\nu} + \beta_9 \bar{R}_{\mu\nu}) + R^{\alpha\beta\mu\nu} (\gamma_1 R_{\alpha\beta\mu\nu} + \gamma_2 R_{\beta\alpha\mu\nu} + \gamma_3 R_{\mu\nu\alpha\beta} + \gamma_4 R_{\alpha\mu\nu\beta} + \\ & + \gamma_5 R_{\mu\alpha\nu\beta} + \gamma_6 R_{\nu\beta\mu\alpha} + \gamma_7 R_{\beta\nu\mu\alpha}) . \end{aligned} \quad (3.304)$$

Notice that the analogue of the second-order term of Lovelock's theory of gravity of the metric formalism (see Eq. (3.17)) corresponds to $\alpha = 1$, $\beta_2 = \beta_6 = -1$, $\beta_4 = -2$ and $\gamma_3 = 1$, the rest of the coefficients vanishing. This Lagrangian, which we write explicitly:

$$\mathcal{L}_{\text{Lovelock-Palatini}} = R^2 - (R^{\mu\nu} + \tilde{R}^{\mu\nu})(R_{\nu\mu} + \tilde{R}_{\nu\mu}) + R^{\alpha\beta\mu\nu} R_{\mu\nu\alpha\beta} , \quad (3.305)$$

turns out to be invariant under the projective transformations (see Eq. (3.265)). Indeed, as we saw, $\hat{R} = R$, $\hat{R}_{\mu\nu} = R_{\mu\nu} + A_{\mu\nu}$ (see below Eq. (3.265)), and $\hat{R}^{\alpha}_{\beta\mu\nu} = R^{\alpha}_{\beta\mu\nu} + \delta^{\alpha}_{\beta} A_{\mu\nu}$ and $\hat{\tilde{R}}_{\mu\nu} = \tilde{R}_{\mu\nu} - A_{\mu\nu}$. This 'Lovelock-Palatini' Lagrangian shares then this invariance with

the EHP Lagrangian, the analogue of the first-order term of Lovelock’s theory of gravity of the metric formalism (because $\mathcal{R}^1 = R$ in the Palatini formalism as well; see Eq. (3.16)).

Since we will turn our attention to the SGB model again, we can set the values of the coefficients in Eq. (3.304) such that when the distortion tensor is dislodged from the full connection, the terms depending on the LC connection give rise to the GB term of the metric formalism in the action. This requirement gives $(\alpha, \beta, \gamma) = (1, -4, 1)$, where

$$\beta \equiv \sum_{i=1}^6 \beta_i , \quad (3.306)$$

$$\gamma \equiv \gamma_1 - \gamma_2 + \gamma_3 , \quad (3.307)$$

and $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7$ (there is no need to fix β_7 , β_8 and β_9 because $\mathring{R}^\sigma_{\sigma\mu\nu} = 0$). Our interest in the next section lies in this quadratic Lagrangian (more generic than Lovelock-Palatini’s), coupled to the scalar field via a coupling function $\xi(\phi)$. We will perform this analysis assuming Weyl geometry.¹⁵ As was argued at the end of Sec. 3.3, and commented again in Sec. 3.4.4, we are chiefly motivated by the results obtained in Ref. [349], where the Horndeski models with $G_4 = G_4(\phi, X)$, that were known to be at odds with the observational constraints on the α_T parameter [293, 294], can be made viable in the Palatini formalism such that $c_{\text{GW}} = c$ (see more details in App. E).

3.5 Quintessence in the Weyl-Gauss-Bonnet Model

In Ref. [373], a model which consists of a Lagrangian with quadratic curvature tensors that reduces to the GB term in the metric formalism (referred to as ‘Extended Gauss-Bonnet’ gravity) is examined in Weyl geometry in arbitrary dimensions. We perform a similar analysis in four dimensions assuming the Weyl connection (3.300), but we couple a scalar field function $\xi(\phi)$ to the quadratic combination, in line with our discussion in previous sections.

The action takes on the form of Eq. (3.19), where $\mathcal{G}$ is now given by Eq. (3.304) with the corresponding values of the coefficients that give rise to $\mathring{\mathcal{G}}$ when the distortion tensor and the LC part are separated (and $\sqrt{-g}\frac{M_{\text{Pl}}^2}{2}R = \sqrt{-g}\frac{M_{\text{Pl}}^2}{2}\left(\mathring{R} + \kappa\right)$ in place of the EH term; see Eq. (3.237)).

15. A manuscript based on the work developed in Sec. 3.5 was submitted for publication and it is under review. The preprint can be found in Ref. [4].

3.5.1 Weyl-Gauss-Bonnet Model: Equations and Assumptions

The Weyl connection, which we impose in the action, leads to the following $\kappa^\alpha_{\beta\mu\nu}$ tensor (see Eq. (3.211)):

$$\begin{aligned} \kappa^\alpha_{\beta\mu\nu} = & \delta^\alpha_\nu \overset{\circ}{\nabla}_\mu A_\beta - \delta^\alpha_\mu \overset{\circ}{\nabla}_\nu A_\beta + \delta^\alpha_\beta \left(\overset{\circ}{\nabla}_\mu A_\nu - \overset{\circ}{\nabla}_\nu A_\mu \right) - g_{\nu\beta} \overset{\circ}{\nabla}_\mu A^\alpha + g_{\mu\beta} \overset{\circ}{\nabla}_\nu A^\alpha - \\ & - \left(\delta^\alpha_\nu A_\mu - \delta^\alpha_\mu A_\nu \right) A_\beta + \left(g_{\nu\beta} A_\mu - g_{\mu\beta} A_\nu \right) A^\alpha - \left(g_{\nu\beta} \delta^\alpha_\mu - g_{\mu\beta} \delta^\alpha_\nu \right) A_\sigma A^\sigma. \end{aligned} \quad (3.308)$$

$\kappa_{\mu\nu}$ and $\tilde{\kappa}_{\mu\nu}$ read (see below Eqs. (3.225) and (3.232), respectively)

$$\kappa_{\mu\nu} = \overset{\circ}{\nabla}_\mu A_\nu - 3\overset{\circ}{\nabla}_\nu A_\mu + 2A_\mu A_\nu - g_{\mu\nu} \left(\overset{\circ}{\nabla}_\sigma A^\sigma + 2A_\sigma A^\sigma \right), \quad (3.309)$$

$$\tilde{\kappa}_{\mu\nu} = -\overset{\circ}{\nabla}_\mu A_\nu - \overset{\circ}{\nabla}_\nu A_\mu + 2A_\mu A_\nu - g_{\mu\nu} \left(\overset{\circ}{\nabla}_\sigma A^\sigma + 2A_\sigma A^\sigma \right). \quad (3.310)$$

Firstly, we see that $\kappa_{\alpha\beta\mu\nu} \neq -\kappa_{\beta\alpha\mu\nu}$ as opposed to the tensor given in Eq. (3.235), which was calculated using a metric compatible connection. This is expected in view of Eq. (3.218) because $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ if $Q_{\alpha\mu\nu} = 0$. However, it can be noticed that the symmetric part $\kappa_{(\alpha\beta)\mu\nu}$ is simply

$$\kappa_{(\alpha\beta)\mu\nu} = g_{\alpha\beta} \left(\overset{\circ}{\nabla}_\mu A_\nu - \overset{\circ}{\nabla}_\nu A_\mu \right) \equiv \frac{1}{4} g_{\alpha\beta} \bar{\kappa}_{\mu\nu}, \quad (3.311)$$

which depends on the non-Riemannian contribution to the homothetic curvature tensor, defined in Eq. (3.228). This contribution is proportional to the field strength tensor $A_{\mu\nu}$, defined below Eq. (3.263), that reminds of the Faraday tensor of electromagnetism. We denote the antisymmetric part $\kappa_{[\alpha\beta]\mu\nu}$ as $\bar{\kappa}_{\alpha\beta\mu\nu}$. Also

$$\kappa_{\mu\nu} = \tilde{\kappa}_{\mu\nu} + \frac{1}{2} \bar{\kappa}_{\mu\nu}, \quad (3.312)$$

such that $\kappa_{(\mu\nu)} = \tilde{\kappa}_{\mu\nu}$ and $\kappa_{[\mu\nu]} = \frac{1}{2} \bar{\kappa}_{\mu\nu}$ ($\tilde{\kappa}_{\mu\nu}$ is fully symmetric as can be seen in Eq. (3.310)). Given the non-zero non-metricity, we have that $\kappa_{\alpha\beta\mu\nu} \neq \kappa_{\mu\nu\alpha\beta}$ (see Eq. (3.224)), but it can be easily found that

$$\kappa_{\alpha\beta\mu\nu} - \kappa_{\mu\nu\alpha\beta} = \frac{1}{4} \left(g_{\alpha\beta} \bar{\kappa}_{\mu\nu} - g_{\mu\nu} \bar{\kappa}_{\alpha\beta} - g_{\beta\nu} \bar{\kappa}_{\mu\alpha} + g_{\mu\alpha} \bar{\kappa}_{\beta\nu} - g_{\mu\beta} \bar{\kappa}_{\alpha\nu} + g_{\alpha\nu} \bar{\kappa}_{\mu\beta} \right). \quad (3.313)$$

Having all this in mind, we proceed to simplify the Lagrangian of the action. $\mathcal{G}$ becomes

$$\mathcal{G} = \overset{\circ}{\mathcal{G}} + 2\overset{\circ}{R}\kappa - 8\overset{\circ}{R}^{\mu\nu} \tilde{\kappa}_{\mu\nu} + 2\overset{\circ}{R}^{\alpha\beta\mu\nu} \bar{\kappa}_{\alpha\beta\mu\nu} + \kappa^2 - 4\tilde{\kappa}^{\mu\nu} \tilde{\kappa}_{\mu\nu} + \bar{\kappa}^{\alpha\beta\mu\nu} \bar{\kappa}_{\alpha\beta\mu\nu} - \frac{1}{4} \Upsilon \bar{\kappa}^{\mu\nu} \bar{\kappa}_{\mu\nu}, \quad (3.314)$$

where we defined

$$\Upsilon \equiv 1 - \beta_1 + \beta_2 - 2\beta_7 - 4\beta_9 - 2\gamma_1 + \gamma_4 . \quad (3.315)$$

$\mathring{\mathcal{G}}$ is the GB term of the metric formalism:

$$\mathring{\mathcal{G}} = \mathring{R}^2 - 4\mathring{R}^{\mu\nu}\mathring{R}_{\mu\nu} + \mathring{R}^{\alpha\beta\mu\nu}\mathring{R}_{\alpha\beta\mu\nu} . \quad (3.316)$$

The last term of $\mathcal{G}$ depends on the values of the arbitrary coefficients whereas the rest are oblivious to them. This was pointed out in Ref. [373].

We now write $\mathcal{G}$ in terms of the Weyl vector and its derivatives. The scalar κ is (see Eq. (3.238))

$$\kappa = -6 \left(\mathring{\nabla}_\sigma A^\sigma + A_\sigma A^\sigma \right) . \quad (3.317)$$

We then have

$$\begin{aligned} \mathcal{G} = \mathring{\mathcal{G}} + 8 \left[\mathring{G}^{\mu\nu} \mathring{\nabla}_\mu A_\nu - \mathring{R}^{\mu\nu} A_\mu A_\nu + \left(\mathring{\nabla}_\sigma A^\sigma + A_\sigma A^\sigma \right) \mathring{\nabla}_\rho A^\rho - \mathring{\nabla}^\mu A^\nu \left(\mathring{\nabla}_\nu A_\mu - 2A_\mu A_\nu \right) - \right. \\ \left. - \Upsilon \mathring{\nabla}^\mu A^\nu \left(\mathring{\nabla}_\mu A_\nu - \mathring{\nabla}_\nu A_\mu \right) \right] . \end{aligned} \quad (3.318)$$

Inserting this and $R = \mathring{R} + \kappa$ in Eq. (3.19):

$$\begin{aligned} S = \frac{1}{2} \int d^4x \sqrt{-g} \left[M_{\text{Pl}}^2 \mathring{R} - \xi(\phi) \mathring{\mathcal{G}} \right] - 4 \int d^4x \sqrt{-g} \left\{ \frac{3}{4} M_{\text{Pl}}^2 \left(\mathring{\nabla}_\sigma A^\sigma + A_\sigma A^\sigma \right) + \right. \\ \left. + \xi(\phi) \left[\mathring{G}^{\mu\nu} \mathring{\nabla}_\mu A_\nu - \mathring{R}^{\mu\nu} A_\mu A_\nu + \left(\mathring{\nabla}_\sigma A^\sigma + A_\sigma A^\sigma \right) \mathring{\nabla}_\rho A^\rho - \mathring{\nabla}^\mu A^\nu \left(\mathring{\nabla}_\nu A_\mu - 2A_\mu A_\nu \right) - \right. \right. \\ \left. \left. - \Upsilon \mathring{\nabla}^\mu A^\nu \left(\mathring{\nabla}_\mu A_\nu - \mathring{\nabla}_\nu A_\mu \right) \right] \right\} - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] . \end{aligned} \quad (3.319)$$

We drop the rings (overcircles) from now on for simplicity as it is clear that every geometric quantity involves only the LC connection after we have split the connection into the LC and non-Riemannian parts (the distortion tensor).

This action can be simplified further. We notice that the first $\sqrt{-g} \nabla_\sigma A^\sigma$ term can be integrated out as it becomes a total derivative. Besides, $G^{\mu\nu}$ is divergenceless by virtue of Bianchi's identity (see Eq. (2.13)), so by integrating by parts the covariant derivative acts on $\xi(\phi)$ and becomes a partial derivative, which does not involve the connection. Also, given Eq. (2.17), we may replace the Ricci tensor by the commutator of covariant derivatives such that

$$\nabla_\nu \nabla_\mu A^\nu - \nabla_\mu \nabla_\nu A^\nu = R_{\mu\nu} A^\nu . \quad (3.320)$$

Therefore

$$\int d^4x \sqrt{-g} \xi(\phi) (-R_{\mu\nu} A^\mu A^\nu + \nabla_\mu A^\mu \nabla_\nu A^\nu - \nabla^\mu A^\nu \nabla_\nu A_\mu) = \int d^4x \sqrt{-g} A^\nu (\partial_\mu \xi \nabla_\nu A^\mu - \partial_\nu \xi \nabla_\mu A^\mu). \quad (3.321)$$

We can also get rid of $\nabla_\mu A_\nu A^\mu A^\nu$ of the fourth term in the second integral. Notice that

$$\int d^4x \sqrt{-g} \xi(\phi) \nabla^\mu A^\nu A_\mu A_\nu = -\frac{1}{2} \int d^4x \sqrt{-g} (\partial_\mu \xi A^\mu + \xi \nabla_\mu A^\mu) A_\nu A^\nu, \quad (3.322)$$

hence

$$\int d^4x \sqrt{-g} \xi(\phi) (A_\mu A^\mu \nabla_\nu A^\nu + 2 \nabla^\mu A^\nu A_\mu A_\nu) = - \int d^4x \sqrt{-g} \partial^\mu \xi A_\mu A_\nu A^\nu. \quad (3.323)$$

Consequently, the action takes on the final form

$$\begin{aligned} S_{\text{WGB}} = & \frac{1}{2} \int d^4x \sqrt{-g} [M_{\text{Pl}}^2 R - \xi(\phi) \mathcal{G}] - 4 \int d^4x \sqrt{-g} \left\{ \frac{3}{4} M_{\text{Pl}}^2 A_\sigma A^\sigma - G^{\mu\nu} \partial_\mu \xi A_\nu - \right. \\ & - (\nabla_\sigma A^\sigma + A_\sigma A^\sigma) \partial_\rho \xi A^\rho + \nabla^\mu A^\nu [A_\mu \partial_\nu \xi - \xi(\phi) \Upsilon (\partial_\mu A_\nu - \partial_\nu A_\mu)] \left. \right\} - \\ & - \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V \right) + \int d^4x \sqrt{-g} \mathcal{L}_M(g_{\mu\nu}, \Psi), \end{aligned} \quad (3.324)$$

where the part that corresponds to the Weyl vector is denoted by ‘ S_A ’, such that

$$\begin{aligned} S_A \equiv & -4 \int d^4x \sqrt{-g} \left\{ \frac{3}{4} M_{\text{Pl}}^2 A_\sigma A^\sigma - (G^{\mu\nu} - \nabla^\mu A^\nu) A_\mu \partial_\nu \xi - (\nabla_\sigma A^\sigma + A_\sigma A^\sigma) \partial_\rho \xi A^\rho - \right. \\ & \left. - \xi(\phi) \Upsilon \nabla^\mu A^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \right\}, \end{aligned} \quad (3.325)$$

and a matter action with Lagrangian $\mathcal{L}_M$ was included given our interest in the late Universe (see Eq. (2.3)). Remember that ‘ Ψ ’ denotes collectively the matter fields (see Eq. (2.1)). Notice also that such an action is *not* a functional of the connection. As was commented already (see discussion following Eq. (3.256)), one could consider a dependence on the connection. In that case, an ‘hypermomentum tensor’ is obtained after varying the matter action with respect to the connection [152].

We refer to the model with the action S_{WGB} in Eq. (3.324) as the ‘Weyl-Gauss-Bonnet (WGB) model’.

To derive the equations of A_μ from S_A , the EL equations are used

$$\frac{\partial \mathcal{L}_A}{\partial A^\mu} = \nabla_\nu \left(\frac{\partial \mathcal{L}_A}{\partial (\nabla_\nu A^\mu)} \right), \quad (3.326)$$

where $S_A = \int d^4x \sqrt{-g} \mathcal{L}_A$ (see Eq. (3.325)). Taking the derivatives, we obtain

$$\frac{3}{2} \left(M_{\text{Pl}}^2 - \frac{4}{3} \partial_\nu \xi A^\nu \right) A_\mu = (G_\mu{}^\nu - 2 \nabla_\mu A^\nu) \partial_\nu \xi + (2 \nabla_\sigma A^\sigma + A_\sigma A^\sigma) \partial_\mu \xi + \frac{1}{2} \Upsilon \nabla_\nu (\xi \bar{\kappa}_\mu{}^\nu). \quad (3.327)$$

Assuming a spatially-flat FLRW metric and a homogeneous Weyl vector, such that $A_\mu(t) \equiv (-\Phi(t), A_i(t))$, one arrives at the equations

$$\Phi = 2 \frac{\dot{\xi}}{M_{\text{Pl}}^2} \left(W^2 - \frac{1}{3} a^{-2} A_i A^i \right), \quad (3.328)$$

$$\Upsilon \left[\xi \ddot{A}_i + (\dot{\xi} + \xi H) \dot{A}_i \right] = \left(\frac{3}{4} M_{\text{Pl}}^2 + \dot{\xi} W \right) A_i, \quad (3.329)$$

where we call $W \equiv H - \Phi$ the *Weyl parameter*. We see that the equation of Φ is algebraic while the equation of A_i is dynamical. This is reasonable as a Weyl vector solely with temporal components can be written as the partial derivative of a scalar function. This however corresponds to the conformal transformation of the LC connection, as we saw in Eq. (3.299), meaning that the non-metricity condition comes down to a conformal transformation. Thus Φ is not a propagating degree of freedom [373].

If ξ is constant, the action S_A becomes

$$S_A = - \int d^4x \sqrt{-g} \left(3 M_{\text{Pl}}^2 A_\sigma A^\sigma - 2 \xi \Upsilon A_{\mu\nu} A^{\mu\nu} \right), \quad (3.330)$$

where $\xi \Upsilon < 0$ and $A_{\mu\nu}$ was defined below Eq. (3.263) (remember that the rings or overcircles were dropped for simplicity because all geometric quantities are defined with respect to the LC connection at this stage). Then, by rescaling the Weyl vector, such that

$$A_\mu \rightarrow \frac{1}{2\sqrt{2|\xi\Upsilon|}} A_\mu, \quad (3.331)$$

the action takes on the form

$$S_A = \int d^4x \sqrt{-g} \left(-\frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{2} M^2 A_\sigma A^\sigma \right), \quad (3.332)$$

which is the action of a massive vector field of mass M given by

$$M^2 \equiv \frac{3M_{\text{Pl}}^2}{4|\xi\Upsilon|}. \quad (3.333)$$

This was discussed in Ref. [373] as explained at the beginning of this section. If $\dot{\xi} \neq 0$ however, we easily solve the equation of Φ , which can be written as

$$\Phi^2 - \Phi \left(2H + \frac{M_{\text{Pl}}^2}{2\dot{\xi}} \right) + H^2 - \frac{1}{3}a^{-2}A_iA^i = 0, \quad (3.334)$$

such that it is a quadratic equation. The solution in terms of the Weyl parameter is

$$W = -\frac{M_{\text{Pl}}^2}{4\dot{\xi}} \left[1 \pm \sqrt{1 + 8\frac{\dot{\xi}}{M_{\text{Pl}}^2} \left(H + \frac{2}{3}\frac{\dot{\xi}}{M_{\text{Pl}}^2}a^{-2}A_iA^i \right)} \right], \quad (3.335)$$

so we have W as a function of H , $\dot{\xi}$ and A_iA^i .

To derive the metric field equations, S_A is varied with respect to the inverse metric first. Integrating by parts and neglecting boundary terms, we arrive at

$$\begin{aligned} \delta S_A = -4 \int d^3\mathbf{x} \int dt a^3 \left\{ \left[\frac{3}{8}M_{\text{Pl}}^2\Phi^2 - \frac{1}{2}\dot{\xi}W a^{-2}A_iA^i - \frac{9}{2}\dot{\xi}H\Phi W - \frac{3}{2}\dot{\xi}\Phi^3 - \frac{1}{2}\xi\Upsilon a^{-2}\dot{A}_i\dot{A}^i + \right. \right. \\ \left. \left. + \frac{3}{8}M_{\text{Pl}}^2a^{-2}A_iA^i \right] \delta g^{00} - \frac{3}{2}M_{\text{Pl}}^2 \left[\Phi - 2\frac{\dot{\xi}}{M_{\text{Pl}}^2} \left(W^2 - \frac{1}{3}a^{-2}A_jA^j \right) \right] A_i \delta g^{0i} + \left[\frac{1}{2} \left(\frac{3}{2}M_{\text{Pl}}^2 + \ddot{\xi} + \right. \right. \\ \left. \left. + 3H\dot{\xi} - 2\dot{\xi}\Phi \right) A_iA_j + \dot{\xi}\dot{A}_iA_j + \xi\Upsilon\dot{A}_i\dot{A}_j + \left(\ddot{\xi}H\Phi - \frac{1}{2}\ddot{\xi}\Phi^2 + \frac{3}{8}M_{\text{Pl}}^2\Phi^2 - \frac{3}{8}M_{\text{Pl}}^2a^{-2}A_kA^k + \right. \right. \\ \left. \left. + \dot{\xi}H\dot{\Phi} + \dot{\xi}\dot{H}\Phi - \xi\Phi\dot{\Phi} - \frac{1}{2}\dot{\xi}\Phi^3 + \frac{3}{2}\dot{\xi}H^2\Phi - \frac{1}{2}\dot{\xi}W a^{-2}A_kA^k - \frac{1}{2}\xi\Upsilon a^{-2}\dot{A}_k\dot{A}^k \right) g_{ij} \right] \delta g^{ij} \right\}, \end{aligned} \quad (3.336)$$

at the background level. The off-diagonal part in the action vanishes because of Eq. (3.328).

The homogeneous equations (taking into account the rest of the full action S_{WGB} (3.324)) read

$$\begin{aligned} H^2 = \frac{1}{3M_{\text{Pl}}^2} (\rho_\phi + \rho_M) + \Phi^2 + a^{-2}A_iA^i - \frac{4}{3}\frac{\xi}{M_{\text{Pl}}^2}\Upsilon a^{-2}\dot{A}_i\dot{A}^i + \frac{4}{M_{\text{Pl}}^2}\dot{\xi}W \left(W^2 - \frac{1}{3}a^{-2}A_iA^i \right), \\ - M_{\text{Pl}}^2 \left(2\dot{H} + 3H^2 \right) g_{ij} = - \left(4\ddot{\xi}H^2 + 8\dot{\xi}H^3 + 8\dot{\xi}H\dot{H} - P_\phi - P_M \right) g_{ij} + 4 \left(\frac{3}{2}M_{\text{Pl}}^2 + \ddot{\xi} + 3H\dot{\xi} - \right. \end{aligned} \quad (3.337)$$

$$-2\Phi\dot{\xi}) A_i A_j + 8\xi\dot{A}_{(i} A_{j)} + 8\xi\Upsilon\dot{A}_i\dot{A}_j + 8\left(\ddot{\xi}H\Phi - \frac{1}{2}\ddot{\xi}\Phi^2 + \frac{3}{8}M_{\text{Pl}}^2\Phi^2 - \frac{3}{8}M_{\text{Pl}}^2a^{-2}A_kA^k + \right. \\ \left. + \xi H\dot{\Phi} + \xi\dot{H}\Phi - \xi\Phi\dot{\Phi} - \frac{1}{2}\dot{\xi}\Phi^3 + \frac{3}{2}\dot{\xi}H^2\Phi - \frac{1}{2}\dot{\xi}W a^{-2}A_kA^k - \frac{1}{2}\xi\Upsilon a^{-2}\dot{A}_k\dot{A}^k\right) g_{ij} , \quad (3.338)$$

where ρ_ϕ and P_ϕ were defined in Eqs. (1.82) and (1.83), and are the energy density and isotropic pressure of the scalar field ϕ , respectively. ρ_{M} and P_{M} are the same respective quantities of the background fluid under consideration, with EoS parameter $w_{\text{M}} = P_{\text{M}}/\rho_{\text{M}}$. This w_{M} will be assumed to lie in the range $0 \leq w_{\text{M}} < 1$, same as in Sec. 3.2.

The last equation (3.338) can be written as $M_{\text{Pl}}^2 G_{ij} = T_{ij}^{\text{eff}}$ (see Eq. (A.35)), where the effective energy-momentum tensor does not have the same diagonal components. If we assume, for example, that the homogeneous spatial vector A_i lies along the z -direction, such that $\mathbf{A}(t) = (0, 0, A_z(t))$, then we have $T_{xx}^{\text{eff}} = T_{yy}^{\text{eff}} \neq T_{zz}^{\text{eff}}$. In order to preserve isotropy at background level, and to simplify the equations, we assume that the background Weyl vector has zero spatial components; i.e. $A_i(t) = 0$.

Setting $A_i(t) = 0$ in the homogeneous equations above and taking the trace of Eq. (3.338), we arrive at

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} (\rho_\phi + \rho_{\text{M}} + 12\xi\dot{W}^3) + \Phi^2, \quad (3.339)$$

$$2\dot{H} + 3H^2 = -\frac{1}{M_{\text{Pl}}^2} (P_\phi + P_{\text{M}}) - 3\Phi^2 + \frac{4}{M_{\text{Pl}}^2} \left[(\xi W^2)^\bullet + 2\xi H W^2 + \xi \Phi W^2 \right]. \quad (3.340)$$

Plugging the first equation into the latter one

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2} \left[\dot{\phi}^2 + (1 + w_{\text{M}}) \rho_{\text{M}} + 4\xi\dot{W}^3 - 12\xi\Phi W^2 - 4(\xi W^2)^\bullet \right] - 3\Phi^2. \quad (3.341)$$

Now, the equation of Φ which is given by (see Eq. (3.328))

$$\Phi = 2\frac{\dot{\xi}}{M_{\text{Pl}}^2} W^2, \quad (3.342)$$

can be used to rewrite Eqs. (3.339) and (3.341) in a simpler manner. These become

$$3W^2 = \frac{1}{M_{\text{Pl}}^2} \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_{\text{M}} \right], \quad (3.343)$$

$$\dot{W} + \Phi W = -\frac{1}{2M_{\text{Pl}}^2} \left[\dot{\phi}^2 + (1 + w_{\text{M}}) \rho_{\text{M}} \right]. \quad (3.344)$$

Finally, the scalar field equation has the same form as that of the metric formalism

$$\Box\phi - V_{,\phi} = \frac{1}{2}\xi_{,\phi}\mathcal{G} , \quad (3.345)$$

as was argued below Eq. (3.290). The analogue of the GB term reads (compare with Eq. (3.35))

$$\mathcal{G} = 24W^2 \left(\dot{W} + HW \right) . \quad (3.346)$$

The homogeneous scalar field equation is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + 12\xi_{,\phi}W^2 \left(\dot{W} + HW \right) = 0 . \quad (3.347)$$

The Hubble friction term can be replaced by $3W\dot{\phi}$, and using Eq. (3.342), we obtain

$$\ddot{\phi} + 3W\dot{\phi} + V_{,\phi} + 12\xi_{,\phi}W^2 \left(\dot{W} + W^2 + \Phi W + \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2} \right) = 0 . \quad (3.348)$$

Using Eq. (3.344)

$$\ddot{\phi} + 3W\dot{\phi} + V_{,\phi} + 12\xi_{,\phi}W^2 \left(W^2 - \frac{1+w_{\text{M}}}{2M_{\text{Pl}}^2}\rho_{\text{M}} \right) = 0 . \quad (3.349)$$

We see that there are no H and Φ terms in this equation. The same applies to Eqs. (3.343) and (3.344) if we use Eq. (3.342) to write the latter as

$$\dot{W} = -\frac{1}{2M_{\text{Pl}}^2} \left[\dot{\phi}^2 + (1+w_{\text{M}})\rho_{\text{M}} + 4\xi W^3 \right] . \quad (3.350)$$

According to Eq. (3.335), the ratio between the Weyl and Hubble parameters, W/H , can be written in terms of $\mu \equiv \dot{\xi}H/M_{\text{Pl}}^2$ solely (remember that $A_i(t) = 0$):

$$\frac{W}{H} = -\frac{1}{4\mu} \left(1 - \sqrt{1+8\mu} \right) . \quad (3.351)$$

Notice that the positive root has been discarded. If we take $\dot{\xi} = 0$ in Eq. (3.342), then $\Phi = 0$ (that is, $W = H$). Only the negative root in Eq. (3.351) yields $W \simeq H$ when $|\mu| \ll 1$.

Also, if $|\mu| \gg 1$, then

$$\frac{W}{H} \simeq \frac{1}{\sqrt{2\mu}} , \quad (3.352)$$

meaning that $W \ll H$. We observe that μ cannot be smaller than $-1/8$, otherwise W/H becomes complex. Given that we shall consider expanding universes ($H > 0$), this bound must be taken into account if $\dot{\xi}$ turns negative.

We will make use of the inverse relation to Eq. (3.351) later. Using Eq. (3.342), we can write the $H(W, \dot{\xi})$ dependence as

$$H = W + \Phi = W \left(1 + 2 \frac{\dot{\xi} W}{M_{\text{Pl}}^2} \right). \quad (3.353)$$

Notice that, if $\Phi = 0$ (i.e. $W = H$) in Eqs. (3.339), (3.341) and (3.347), we recover the homogeneous equations of the SGB model (see Eqs. (3.44), (3.45) and (3.34) respectively). However, if $\Phi = 0$ in Eqs. (3.343), (3.344) and (3.349), the equations of GR are obtained instead. This is due to Eq. (3.342), as $\Phi = 0$ implies $\dot{\xi} = 0$ (constant ξ), which allows us to recover GR. Another possibility for $\Phi = 0$ is $H = \Phi$ in Eq. (3.342), but this leads to $H = 0$, that we do not consider because $a = a(t)$.

3.5.2 Equations for Tensor Perturbations

We will now extend the analysis to tensor perturbations and derive the equation of GWs. Quadratic terms are retained in the action as we do in App. D (see Eqs. (D.12), (D.26) and (D.29)). Bearing in mind that the FLRW metric is conformally equivalent to the Minkowski one (see App. D), we readily find that the action S_A can be written as [172]

$$\begin{aligned} S_A = & -4 \int d^4x \sqrt{-g} \left[\left(\frac{3}{4} M_{\text{Pl}}^2 \Omega^2 - \partial_\rho \xi \partial^\rho \ln \Omega \right) A_\sigma A^\sigma - A^\mu \partial^\nu \xi (G_{\mu\nu} - \nabla_\mu A_\nu) + \right. \\ & + 2A^\mu \partial^\nu \xi (\nabla_\mu \partial_\nu \ln \Omega - \partial_\mu \ln \Omega \partial_\nu \ln \Omega) - (\nabla_\sigma A^\sigma + 2A^\sigma \partial_\sigma \ln \Omega + A_\sigma A^\sigma + \partial_\sigma \ln \Omega \partial^\sigma \ln \Omega + \\ & \left. + 2\nabla_\sigma \partial^\sigma \ln \Omega) \partial_\rho \xi A^\rho - \frac{1}{2} \xi \Upsilon A^{\mu\nu} A_{\mu\nu} \right], \end{aligned} \quad (3.354)$$

where the conformal factor Ω is meant to be the scale factor $a(\eta)$ according to Eq. (D.1) and the metric corresponds to $\eta_{\mu\nu} + h_{\mu\nu}$. $A_{\mu\nu}$ is defined below Eq. (3.263). The background action is

$$\bar{S}_A = -4 \int d\mathbf{x} \int d\eta \left[-\frac{3}{4} a^2 M_{\text{Pl}}^2 (\Phi^2 - A_i A^i) + 3\mathcal{H} \xi' \Phi \mathcal{W} + \xi' \mathcal{W} A_i A^i + \xi' \Phi^3 + \xi \Upsilon \delta^{ij} A'_i A'_j \right], \quad (3.355)$$

where $\mathcal{W} \equiv \mathcal{H} - \Phi$. Primes denote derivatives with respect to the conformal time η . It must be pointed out that Φ in this action is not the same as in earlier calculations. Since we have conformally transformed the metric, the Φ in the background Minkowski is related to the Φ in the background FLRW such that

$$\Phi_{\text{Minkowski}} = a\Phi_{\text{FLRW}} . \quad (3.356)$$

Keeping this in mind, $\mathcal{W} = aW$. Then, we rewrite $\bar{S}_A$ in terms of the coordinate time and use the EL equations to obtain Eqs. (3.328) and (3.329), as expected.

Let us now proceed with perturbations. There is no need to perturb the scalar field ϕ and the vector field A_μ . This is because, for the vector field with background temporal components only, those perturbations do not affect the equation of GWs. The first term in the integral is simple. It yields

$$\left(\frac{3}{4} M_{\text{Pl}}^2 \Omega^2 - \partial_\rho \xi \partial^\rho \ln \Omega \right) A_\sigma A^\sigma = - \left(\frac{3}{4} M_{\text{Pl}}^2 a^2 + \mathcal{H} \xi' \right) \Phi^2, \quad (3.357)$$

contributing only at background level. The next term with the Einstein tensor is slightly more complicated

$$\begin{aligned} -A^\mu \partial^\nu \xi (G_{\mu\nu} - \nabla_\mu A_\nu) = & \Phi \xi' \left(\Phi' + \frac{1}{2} h^{ij} \partial_k \partial^k h_{ij} - \frac{1}{8} h^{ij'} h'_{ij} + \frac{3}{8} \partial^k h^{ij} \partial_k h_{ij} - \right. \\ & \left. - \frac{1}{4} \partial^i h^{jk} \partial_j h_{ik} \right), \end{aligned} \quad (3.358)$$

and includes quadratic terms. Next term is

$$2A^\mu \partial^\nu \xi (\nabla_\mu \partial_\nu \ln \Omega - \partial_\mu \ln \Omega \partial_\nu \ln \Omega) = 2 (\mathcal{H}^2 - \mathcal{H}') \xi' \Phi, \quad (3.359)$$

and it does not yield perturbation terms. Finally, we have

$$\begin{aligned} -(\nabla_\sigma A^\sigma + 2A^\sigma \partial_\sigma \ln \Omega + A_\sigma A^\sigma + \partial_\sigma \ln \Omega \partial^\sigma \ln \Omega + 2\nabla_\sigma \partial^\sigma \ln \Omega) \partial_\rho \xi A^\rho = & - \left(\Phi' - \right. \\ & \left. - 2\mathcal{H}' - \mathcal{W}^2 - \frac{1}{2} \Phi h^{ij} h'_{ij} + \mathcal{H} h^{ij} h'_{ij} \right) \xi' \Phi. \end{aligned} \quad (3.360)$$

Notice that $-\frac{1}{2}\xi(\phi)\Upsilon A^{\mu\nu} A_{\mu\nu}$ does not contribute when $A_i(t) = 0$.

$\sqrt{-g}\mathcal{L}_A$ containing the tensor perturbation terms is then

$$\begin{aligned} \sqrt{-g}\mathcal{L}_A = & -\frac{1}{2}\Phi\xi' \left(h^{ij}\partial_k\partial^k h_{ij} - h^{ij'}h'_{ij} \right) + \left[-\frac{3}{4}a^2 M_{\text{Pl}}^2 \Phi^2 + 3\mathcal{H}\xi'\Phi\mathcal{W} + \xi'\Phi^3 + \xi''\Phi^2 - \right. \\ & \left. -2\xi'\Phi\mathcal{W}' - 2\xi''\Phi\mathcal{H} - 2\xi'\Phi'\mathcal{H} \right] h^{ij}h_{ij} , \end{aligned} \quad (3.361)$$

where some quadratic terms have been removed integrating by parts and the total derivatives are not shown. If we consider the $h^{ij}h_{ij}$ term of the full action S_{WGB} (3.324) (see App. D for further details), then

$$\begin{aligned} & -\frac{1}{4}a^2 \left[M_{\text{Pl}}^2 \left(2\mathcal{H}' + \mathcal{H}^2 \right) + 4\mathcal{H}a^{-2} \left(\xi'\mathcal{H}^2 - \xi''\mathcal{H} - 2\xi'\mathcal{H}' \right) + a^{-2} \left(a^2\frac{1}{2}\phi'^2 - U + a^2 P_{\text{M}} \right) + \right. \\ & + 3M_{\text{Pl}}^2 \Phi^2 - 12\mathcal{H}a^{-2}\xi'\Phi\mathcal{W} - 4a^{-2}\xi'\Phi^3 - 4a^{-2}\xi''\Phi^2 + 8a^{-2}\xi'\Phi\mathcal{W}' + 8a^{-2}\xi''\mathcal{H}\Phi + \\ & \left. + 8a^{-2}\xi'\mathcal{H}\Phi' \right] h^{ij}h_{ij} , \end{aligned} \quad (3.362)$$

i.e.

$$\begin{aligned} & -\frac{1}{4}a^2 \left[M_{\text{Pl}}^2 \left(2\mathcal{H}' + \mathcal{H}^2 + 3\Phi^2 \right) + a^{-2} \left(a^2\frac{1}{2}\phi'^2 - U + a^2 P_{\text{M}} \right) - 4a^{-2} \left(\xi'\mathcal{W}^2 \right)' + \right. \\ & \left. + 4a^{-2}\xi'\mathcal{W}^3 \right] h^{ij}h_{ij} . \end{aligned} \quad (3.363)$$

We have assumed there is no contribution from the matter action in the form of an anisotropic stress tensor (see comment below Eq. (3.59)). Eq. (3.340) in conformal time reads (remember that $\Phi \rightarrow \Phi/a$)

$$M_{\text{Pl}}^2 \left(2\mathcal{H}' + \mathcal{H}^2 \right) = -a^{-2} \left(a^2\frac{1}{2}\phi'^2 - U + a^2 P_{\text{M}} \right) - 3M_{\text{Pl}}^2 \Phi^2 + 4a^{-2} \left(\xi'\mathcal{W}^2 \right)' - 4a^{-2}\mathcal{W}^3\xi' , \quad (3.364)$$

which makes the $h^{ij}h_{ij}$ term vanish.

Then, the full action for TT tensor perturbations takes on the form (compare with Eq. (D.32))

$$S_h = \frac{1}{8} \int d\eta d\mathbf{x} a^2 \left\{ \left(M_{\text{Pl}}^2 - 4a^{-2}\xi'\mathcal{W} \right) h^{ij'}h'_{ij} + \left[M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' - \xi'\mathcal{W} \right) \right] h^{ij}\partial_k\partial^k h_{ij} \right\} . \quad (3.365)$$

We then derive the GWs equation using the EL equations as in previous examples (see

Eq. (D.33))

$$\begin{aligned} & \left(M_{\text{Pl}}^2 - 4a^{-2}\xi' \mathcal{W} \right) h_{ij}'' + \left[2\mathcal{H} M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' \mathcal{W} + \xi' \mathcal{W}' \right) \right] h_{ij}' - \left[M_{\text{Pl}}^2 - \right. \\ & \left. - 4a^{-2} \left(\xi'' - \xi' \mathcal{W} \right) \right] \partial_k \partial^k h_{ij} = 0 . \end{aligned} \quad (3.366)$$

In coordinate time:

$$\begin{aligned} & \left(M_{\text{Pl}}^2 - 4\dot{\xi} W \right) h_{ij}'' + \left[3H \left(M_{\text{Pl}}^2 - 4\dot{\xi} W \right) - 4 \left(\ddot{\xi} W + \dot{\xi} \dot{W} \right) \right] h_{ij}' - \left[M_{\text{Pl}}^2 - \right. \\ & \left. - 4 \left(\ddot{\xi} + \dot{\xi} \Phi \right) \right] a^{-2} \partial_k \partial^k h_{ij} = 0 . \end{aligned} \quad (3.367)$$

This equation resembles Eq. (3.56) almost if we replace H by W . There is however the prefactor in the gradient term, where $\ddot{\xi}$ is replaced by $\ddot{\xi} + \dot{\xi} \Phi$.

Eq. (3.367) may be rewritten in the following way (*assuming that* $4\dot{\xi} W \neq M_{\text{Pl}}^2$):

$$h_{ij}'' + (3 + \alpha_M) H h_{ij}' - (1 + \alpha_T) a^{-2} \partial_k \partial^k h_{ij} = 0 , \quad (3.368)$$

as we did with Eq. (3.57), where

$$\alpha_M \equiv -\frac{4}{H} \frac{\ddot{\xi} W + \dot{\xi} \dot{W}}{M_{\text{Pl}}^2 - 4\dot{\xi} W} , \quad (3.369)$$

$$\alpha_T \equiv -4 \frac{\ddot{\xi} - (H - 2\Phi) \dot{\xi}}{M_{\text{Pl}}^2 - 4\dot{\xi} W} . \quad (3.370)$$

Using Eq. (3.342), the α_T parameter predicted by the WGB model can be written entirely in terms of W , ξ and $\ddot{\xi}$:

$$\alpha_T = -\frac{4}{M_{\text{Pl}}^2} \frac{M_{\text{Pl}}^2 \left(\ddot{\xi} - \dot{\xi} W \right) + 2\dot{\xi}^2 W^2}{M_{\text{Pl}}^2 - 4\dot{\xi} W} . \quad (3.371)$$

The speed of GWs in the WGB model takes on the same form as that of the SGB model (see Eq. (3.60)). It can be seen that, in opposition to the results in Ref. [349], the Palatini version of this scalar-tensor theory does not show any seeming improvement so the propagation speed matches that of light exactly. We remember that the SGB model is a subclass of Horndeski theory in the metric formalism which includes further G -functions than those considered in Ref. [349] (see Eqs. (3.185)-(3.188)), so it could be the case that this fact prevents $c_{\text{GW}} = c$ in the Palatini formalism. Generally, when Horndeski models more

complicated than in Ref. [349] are considered [289, 374], the connection field equations turn out to be *differential* instead of algebraic. Hence the connection $\Gamma_{\mu\nu}^\alpha$ becomes a propagating degree of freedom of the theory. This involves relevant new aspects in the physical content of the theory [375, 376].¹⁶

Another possible explanation for $c_{\text{GW}} \neq c$ is the connection we have chosen. As it is shown in App. F however, the homogeneous equations in the previous section, as well as the tensor perturbations equation, are valid for a metric compatible connection but with non-zero torsion. This connection was considered already in Eq. (3.216), and the fact that it is related to the Weyl connection via a projective transformation (see Eq. (3.265)) is crucial for the validity of the equations and, therefore, of the results we shall derive in the next sections (see App. F for further details).

Although $c_{\text{GW}} \neq c$, such a constraint can be imposed, same as in the SGB model. Also, one can perform a dynamical systems analysis and check if the presence of the Weyl vector A_μ in the new action (3.324) prevents $|\alpha_T|$ from violating the observational bounds at present time. This will be our first goal, before imposing $c_{\text{GW}} = c$.

3.5.3 Dynamical Systems Analysis: Dynamical System

Given the homogeneous equations (3.343), (3.344) and (3.349), we proceed to analyse the behaviour of the dynamical system. By defining the dimensionless variables

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}WM_{\text{Pl}}} , \quad y \equiv \frac{1}{WM_{\text{Pl}}} \sqrt{\frac{V}{3}} , \quad z \equiv \frac{1}{WM_{\text{Pl}}} \sqrt{\frac{\rho_{\text{M}}}{3}} , \quad (3.372)$$

the constraint equation (3.343), analogous to the Friedmann equation in GR (3.75), reads

$$1 = x^2 + y^2 + z^2. \quad (3.373)$$

It can be noticed that, in contrast to the definitions of x , y and z in Eqs. (3.78)-(3.80), here the dimensionless variables are given in terms of the Weyl parameter W , not the Hubble one. W and H do not generally coincide and are related by Eq. (3.353). Using the dimensionless variables (3.372), this relation can be written as

$$H = W(1 + ux), \quad (3.374)$$

16. Even the fact of whether the SGB model remains a subclass of Horndeski's theory in the Palatini formalism is an unexplored issue in the literature to the best of our knowledge.

where another dimensionless variable,

$$u \equiv 2\sqrt{6} \frac{W^2 \xi_{,\phi}}{M_{\text{Pl}}} , \quad (3.375)$$

was defined. If $|ux| \ll 1$, then $H \simeq W$.

Another difference between those definitions and the ones in Eqs. (3.78)-(3.80) and (3.114) is that u is not constrained by Eq. (3.373). This independence is related to the fact that the modifications of gravity are included in the definition of x , y and z in Eq. (3.372) via the Weyl parameter. On the contrary, in the SGB model, the effects of modified gravity are considered solely by the u dimensionless variable (see Eq. (3.114)).

With the definitions above, and Eq. (3.374), we can write Eq. (3.373) as

$$\Omega_\phi + \Omega_{\text{M}} = \frac{1}{(1+ux)^2} . \quad (3.376)$$

Ω_ϕ and Ω_{M} , the density parameters of the scalar field and of the background matter fluid, were defined above Eq. (3.81). In terms of x , y and z in the WGB model, these read

$$\Omega_\phi \equiv \frac{x^2 + y^2}{(1+ux)^2} , \quad (3.377)$$

$$\Omega_{\text{M}} \equiv \frac{z^2}{(1+ux)^2} = \frac{1 - x^2 - y^2}{(1+ux)^2} . \quad (3.378)$$

We may define the ‘effective’ density parameter associated with ξ as

$$\Omega_\xi \equiv \frac{(2+ux)ux}{(1+ux)^2} , \quad (3.379)$$

such that $1 = \Omega_\phi + \Omega_{\text{M}} + \Omega_\xi$ (compare with Eq. (3.117)).

If $|ux| \ll 1$, notice that

$$\Omega_\xi \simeq 2ux , \quad (3.380)$$

which is Ω_{GB} introduced in Eq. (3.43) and written in terms of the dimensionless variables of the SGB model in Eq. (3.116) (see Eq. (3.117) also).¹⁷ Also, $\Omega_\phi \simeq x^2 + y^2$ and $\Omega_{\text{M}} \simeq z^2$. This is sensible because $H \simeq W$ in that case, as we said earlier. However, notice that when $|ux|$ becomes very large, then $\Omega_\xi \rightarrow 1$, in contrast to the SGB model where it diverges (see

17. Notice the difference of sign between Eq. (3.380) and (3.116) due to the distinct definition of u in Eqs. (3.114) and (3.375).

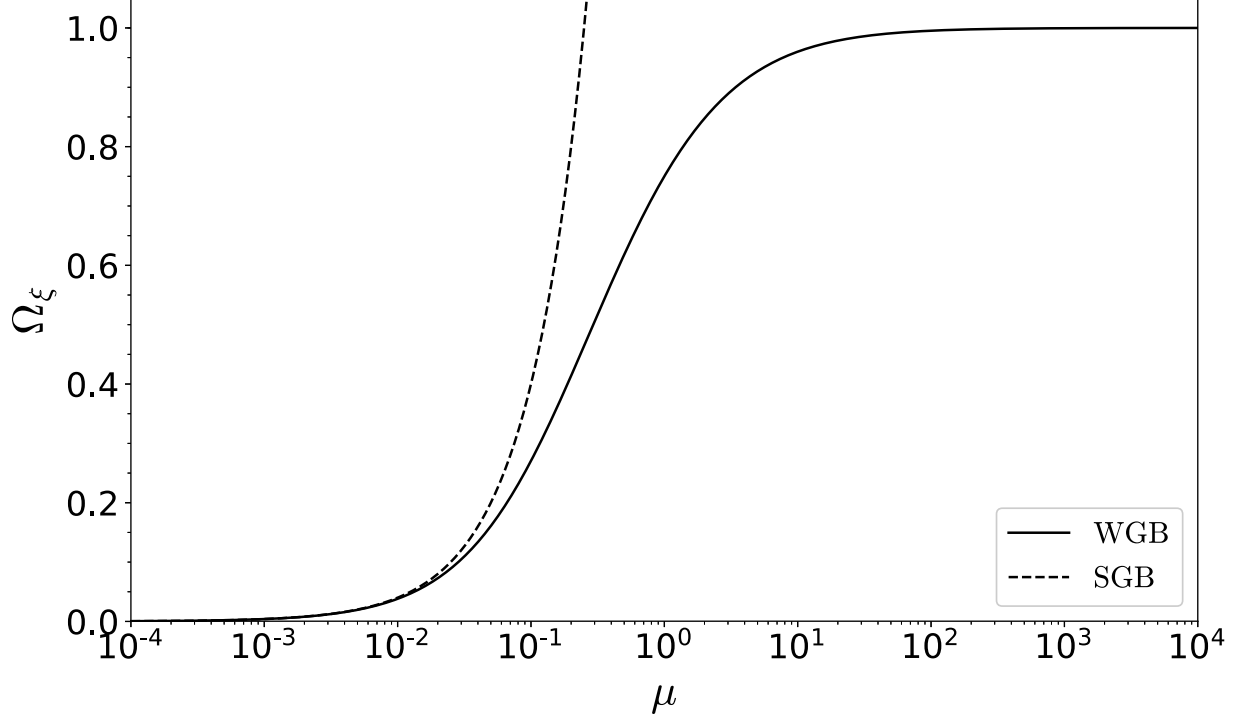


Figure 3.7: ‘Effective’ density parameter of ξ as a function of $\mu = \frac{1}{2}ux(1+ux)$ in the WGB model (see definition above Eq. (3.351)). For small values of $|\mu|$ (i.e. of $|ux|$), Ω_ξ approaches $4\mu \simeq 2ux$, which is the density parameter Ω_{GB} in the SGB model (see Eq. (3.43)). For large values of $|\mu|$ (i.e. of $|ux|$), Ω_ξ tends to 1 in the WGB model while the corresponding density parameter in the SGB model is known to diverge.

Eq. (3.116) and Fig. 3.7).

We assume an exponential potential (see Eq. (3.73), with $\phi_0 = M_{Pl}$), such that the dynamical equations become self-similar. The constant factor λ in the exponential is assumed to be positive, same as in our previous study of the SGB model (see Sec. 3.2.2). Taking the derivatives of x , y , z and u , and using Eq. (3.347), the continuity equation (3.47), and the relation between H and W , Eq. (3.374), we arrive at

$$x' = -\frac{1}{1+ux} \left[(3x+u)(1+ux) - \sqrt{\frac{3}{2}}\lambda y^2 + (x+u)\frac{\dot{W}}{W^2} \right], \quad (3.381)$$

$$y' = -\frac{y}{1+ux} \left(\frac{\dot{W}}{W^2} + \sqrt{\frac{3}{2}}\lambda x \right), \quad (3.382)$$

$$z' = -z \left[\frac{\dot{W}}{W^2} \frac{1}{1+ux} + \frac{3}{2}(1+w_M) \right], \quad (3.383)$$

$$u' = \frac{2}{1+ux} \left(u \frac{\dot{W}}{W^2} + 6\xi_{,\phi\phi} W^2 x \right). \quad (3.384)$$

Primes denote derivatives with respect to the number of e-folds, same as in Eqs. (3.118)-(3.120) and (3.122) (where $dN = Hdt$).

$\dot{W}/W^2$ can be expressed in terms of the dimensionless variables using Eq. (3.344). Plugging Eq. (3.342) into Eq. (3.344) we find

$$-\frac{\dot{W}}{W^2} = x(3x+u) + \frac{3}{2}z^2(1+w_M). \quad (3.385)$$

$\dot{W}/W^2$ is related to the Hubble-flow parameter ϵ_1 (defined in Eq. (1.71)) by

$$\epsilon_1 = -\frac{1}{1+ux} \left[\frac{\dot{W}}{W^2} + \frac{(ux)^\bullet}{H} \right]. \quad (3.386)$$

Although Eqs. (3.381)-(3.383) are self-similar, Eq. (3.384) is not unless we assume an exponential GB coupling function, which we will do from now on. Another possibility is the linear coupling, which was briefly considered in Sec. 3.2.5. However, it was found that the linear coupling case was discarded on the basis of our knowledge of the past history of the Universe. We will see later on that the same applies in the WGB model. Consequently, we take the exponential GB coupling function of Eq. (3.136) (with $\phi_0 = M_{\text{Pl}}$),¹⁸ and the constant factor κ in the exponential is taken positive again. Eq. (3.384) then becomes

$$u' = \frac{2u}{1+ux} \left(\frac{\dot{W}}{W^2} + \sqrt{\frac{3}{2}}\kappa x \right), \quad (3.387)$$

where $\dot{W}/W^2$ is meant to be replaced by Eq. (3.385), same as in Eqs. (3.381)-(3.383).

3.5.4 Dynamical Systems Analysis: Fixed Points for the Exponential Coupling

Fixed points of the dynamical system of equations (3.381), (3.382) and (3.387) are denoted by x_c , y_c and u_c , respectively, and correspond to regions of the phase space where $x' = y' = u' = 0$, as was first explained in Sec. 3.2.1. Among those, we can list the following ones to

18. Do not confuse the strength of the exponential κ in Eq. (3.136) with the scalar κ defined in Eq. (3.238), which accounts for the non-Riemannian contribution to the Ricci scalar.

begin with:

- **M**: $(x_c, y_c, u_c) = (0, 0, 0)$ and $w|_c = w_M$;
- **K $\pm$** : $(x_c, y_c, u_c) = (\pm 1, 0, 0)$ and $w|_c = 1$;
- **I**: $(x_c, y_c, u_c) = \left(\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}, 0\right)$ and $w|_c = -1 + \frac{\lambda^2}{3}$;
- **ScI**: $(x_c, y_c, u_c) = \left(\sqrt{\frac{3}{2} \frac{(1+w_M)^2}{\lambda^2}}, \sqrt{\frac{3}{2} \frac{1-w_M^2}{\lambda^2}}, 0\right)$ and $w|_c = w_M$;

which are fixed points in GR too because $W_c = H_c$ at those fixed points (see Eq. (3.374)). w is the effective EoS parameter first related to ϵ_1 in Eq. (1.71). z_c is determined by the constraint equation (3.373), so we do not really need to impose $z' = 0$; it follows from the constraint equation.

Notice also from Eq. (3.383) that, in order for $z' = 0$, either $z_c = 0$ or $\epsilon_H|_c = \frac{3}{2}(1 + w_M)$ (see Eq. (3.386)). Both are satisfied (although not simultaneously) by the fixed points above and by the ones we shall discuss later. Besides this, we remark that, unlike z , $u' = 0$ is imposed as $x' = y' = 0$ are satisfied e.g. for $u = u(N)$ and $x_c = \pm 1$ and $y_c = 0$. As was commented already (see discussion below Eq. (3.375)), this is due to the fact that the modifications of gravity are included in all the definitions of the dimensionless variables, Eqs. (3.372) and (3.375), whilst in Sec. 3.2.2, these were only accounted for by u (see Eq. (3.114)).

Similarly to Sec. 3.2.1, **M** here represents the regime of the background fluid domination. **K $\pm$** stands for the period of kination (dominance of the kinetic energy of the scalar field $\dot{\phi}^2/2$) and **I** corresponds to a regime of power law inflation (when $\lambda < \sqrt{2}$, so that $\epsilon_H|_I < 1$) where matter fields do not contribute ($z_I = 0$; see Eq. (3.373)). Notice that this latter fixed point exists if $\lambda < \sqrt{6}$, so y_I is real (remember that $\lambda > 0$). Same existence condition is found in Sec. 3.2.1 and in the SGB model. Lastly, we have the scaling fixed point **ScI**. In this case, the EoS parameter of the scalar field (see Eq. (1.86)) equates that of the background fluid, w_M , as explained in Sec. 3.2.1 too (see below Eq. (3.87)). This fixed point yields $\Omega_\phi|_{ScI}$ in Eq. (3.88) and, by virtue of Eq. (3.373), we obtain following the existence condition:

$$\lambda > \sqrt{3(1 + w_M)} \geq \sqrt{3} . \quad (3.388)$$

Therefore, we observe that **I** and **ScI** coexist whenever $\sqrt{6} > \lambda > \sqrt{3}$. If **I** corresponds to power law inflation, then both fixed points cannot coexist because $\lambda < \sqrt{2}$ is required. The

same conclusions can be drawn in GR and in the SGB model because, as was said already, $W_c = H_c$ given that $u_c = 0$ for those fixed points.

In addition to the fixed points listed above, we find five other points for which u_c no longer vanishes:

- **dS**: $(x_c, y_c, u_c) = \left(0, 1, \sqrt{\frac{3}{2}}\lambda\right)$ and $w|_c = -1$;
- **ScII $\pm$** : $(x_c, y_c, u_c) = \left(\frac{\kappa}{\sqrt{6}} \frac{1+3w_M}{3(1-w_M^2)} \Delta_{\pm}(\kappa, w_M), 0, \frac{\kappa}{\sqrt{6}} \frac{1}{1+w_M} \Delta_{\pm}(\kappa, w_M)\right)$, where

$$\Delta_{\pm}(\kappa, w_M) \equiv 1 \pm \sqrt{1 - \frac{18(1+w_M)(1-w_M^2)}{\kappa^2(1+3w_M)}}, \quad (3.389)$$

and $w|_c = w_M$;

- **$\hat{\mathbf{K}}\pm$** : $(x_c, y_c, u_c) = \left(\pm 1, 0, \sqrt{\frac{3}{2}}\kappa \mp 3\right)$ and $w|_c = -\frac{1 \mp \frac{\kappa}{2\sqrt{6}}}{1 \mp \frac{3\kappa}{2\sqrt{6}}}$.

The first fixed point corresponds to the de Sitter expansion, where the total energy density consists only of the CC. Although it is not found in GR when $\lambda \neq 0$, we saw that it did exist in the SGB model for an exponential potential (see Table 3.1). Notice that, since $x_{\mathbf{dS}} = 0$, then $W_{\mathbf{dS}} = H_{\mathbf{dS}}$ (see Eq. (3.374)). Consequently, this fixed point is the same as in the SGB model.

On top of the de Sitter fixed point, there is a second scaling regime which consists of two fixed points (**ScII $\pm$**), depending on the root sign in Eq. (3.389). These fixed points exist if

$$\kappa^2 \geq 18 \frac{(1+w_M)(1-w_M^2)}{1+3w_M}, \quad (3.390)$$

so that $x_{\mathbf{ScII}\pm}$ and $u_{\mathbf{ScII}\pm}$ are real. Then $1 \leq \Delta_+ < 2$ and $0 < \Delta_- \leq 1$.

Eq. (3.373) demands

$$x_{\mathbf{ScII}}^2 = \frac{\kappa^2}{54} \frac{(1+3w_M)^2}{(1-w_M^2)^2} \Delta_{\pm}^2 < 1. \quad (3.391)$$

Combining it with Eq. (3.390) we obtain

$$\Delta_{\pm}^2 < \frac{3(1-w_M)}{1+3w_M}. \quad (3.392)$$

Given that $1 \leq \Delta_+^2 < 4$ and $0 < \Delta_-^2 \leq 1$, we have $0 \leq w_M < \frac{1}{3}$ for Δ_+ and $0 \leq w_M < 1$ for Δ_- (remember that the whole range of values of w_M was assumed to be $0 \leq w_M < 1$; see

below Eq. (3.338)). Consequently, the scaling fixed point **ScII** $+$ in particular does not exist if the background fluid under consideration corresponds to radiation, where $w_M = w_r = 1/3$ (just to mention a cosmologically relevant background fluid). In that case however, the scaling regime is still represented by the fixed point **ScII** $-$.

The fourth and fifth fixed points listed above yield vanishing potential energy and energy density of the background fluid. We denote both by ' $\hat{\mathbf{K}}_{\pm}$ ' and call the corresponding regime 'pseudo-kination' since $3W_{\hat{\mathbf{K}}_{\pm}}^2 M_{\text{Pl}}^2 = \dot{\phi}_{\hat{\mathbf{K}}_{\pm}}^2/2$, although $W_{\hat{\mathbf{K}}_{\pm}} \neq H_{\hat{\mathbf{K}}_{\pm}}$. Irrespective of the sign, if $\kappa \gg 1$, then $\epsilon_1|_{\hat{\mathbf{K}}_{\pm}} \rightarrow 1$. This value of the Hubble-flow parameter, for large κ , leads into a universe that does not accelerate (see Eq. (1.71)).

For $x_c = +1$ in particular (i.e. for $\hat{\mathbf{K}}_+$) and $\kappa < 2\sqrt{2/3}$, $\epsilon_1|_{\hat{\mathbf{K}}_+} < 0$ (we remind the reader that $\kappa > 0$; see below Eq. (3.136)). This means that the Universe accelerates faster than during de Sitter, where it accelerates exponentially [377, 378]. This phase of acceleration faster than exponential can be seen by solving $\epsilon_1 = -\alpha$ (see Eq. (1.70)), $\alpha > 0$ being a constant (indeed, ϵ_1 is constant at the fixed points $\hat{\mathbf{K}}_{\pm}$; see $w|_{\hat{\mathbf{K}}_{\pm}}$ below Eq. (3.389))

$$H(t) = \frac{H_0}{1 - t/t_s}, \quad (3.393)$$

where $H_0 = H(t=0)$ and $t_s \equiv 1/(\alpha H_0)$. Solving for $a(t)$ (see Eq. (1.11)), we obtain

$$a(t) = a_0 \left(1 - \frac{t}{t_s}\right)^{-1/\alpha}, \quad (3.394)$$

such that $a_0 = a(t=0)$. Assuming an expanding universe, $H_0 > 0$ and $t < t_s$. Then, $a(t)$ diverges (approaches $a(t_s)$) faster than if $a(t) = a_0 e^{H_0 t}$.

In Fig. 3.8, the effective EoS parameter w at the pseudo-kination fixed points is plotted as a function of κ (remember that κ is taken positive). While the pseudo-kination fixed point $\hat{\mathbf{K}}_-$ never features a *super-accelerating* [377, 378] universe, $w|_{\hat{\mathbf{K}}_+} < -1$ when $\kappa < 2\sqrt{2/3}$. Notice that, for that super-accelerated phase (see Eq. (3.374))

$$\left.\frac{H}{W}\right|_{\hat{\mathbf{K}}_+} = 1 + u_{\hat{\mathbf{K}}_+} x_{\hat{\mathbf{K}}_+} = 1 + \left(\sqrt{\frac{3}{2}}\kappa - 3\right), \quad (3.395)$$

is negative. For the rest of the fixed points treated above, the ratio H/W is 1 (positive in the case of **ScII** $\pm$ as one can check from the values of x and u at those fixed points; see

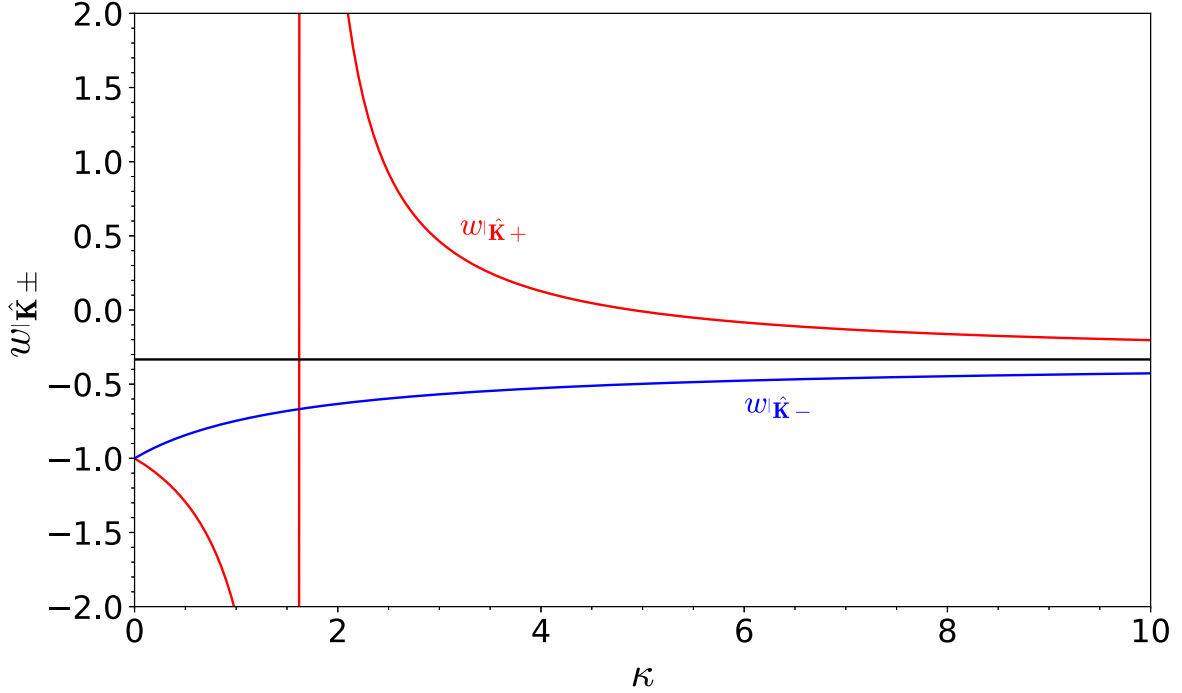


Figure 3.8: Effective EoS parameter at the fixed points $\hat{\mathbf{K}}_{\pm}$ for different values of $\kappa > 0$ (see Eq. (1.71) where the relation between w and ϵ_1 was introduced and find $w|_{\hat{\mathbf{K}}_{\pm}}$ as a function of $\kappa > 0$ below Eq. (3.389)). We observe that, when $x_c = +1$, then $w|_{\hat{\mathbf{K}}_+} < -1$ for $\kappa < 2\sqrt{\frac{2}{3}}$ (meaning that $\epsilon_1|_{\hat{\mathbf{K}}_+} < 0$), while for $x_c = -1$, $w|_{\hat{\mathbf{K}}_-} > -1$ for all κ . If the constant w is smaller than -1 , that means a constant $\epsilon_1 < 0$ and a phase where the Universe accelerates faster than during de Sitter, where the acceleration is exponential (see Eq. (3.394)). Regardless of the sign of the field velocity at the fixed point, w tends to $-1/3$ for large values of κ (i.e. ϵ_1 tends to 1). This asymptotic value, marked by the horizontal black line, leads to a universe where $\ddot{a} = 0$, which does not accelerate (see Eq. (1.71)).

before Eq. (3.389)) because $u_c = 0$. The same holds for $\hat{\mathbf{K}}_-$ given that

$$\left. \frac{H}{W} \right|_{\hat{\mathbf{K}}_-} = 1 + u_{\hat{\mathbf{K}}_-} x_{\hat{\mathbf{K}}_-} = 1 - \left(\sqrt{\frac{3}{2}} \kappa + 3 \right), \quad (3.396)$$

is always negative for $\kappa > 0$.

Choosing $\lambda = \kappa$ allows for all x_c , y_c and u_c being non-zero at once as can be seen from Eqs. (3.383) and (3.387). The same occurs in the SGB model (see Table 3.1). The corresponding fixed ‘curves’ are:

• **ScIII:**

$$(x, y, u) = \left(\frac{\frac{3}{2}(1+w_M)}{\sqrt{\frac{3}{2}\lambda - \frac{3}{2}u_c(1+w_M)}}, \sqrt{\frac{\sqrt{\frac{3}{2}\lambda u_c + \frac{3}{2}(1+w_M)} \left(\frac{3}{2}(1-w_M) - \sqrt{\frac{3}{2}\lambda u_c + \frac{1}{2}(1+3w_M)u_c^2} \right)}{\left(\sqrt{\frac{3}{2}\lambda - \frac{3}{2}u_c(1+w_M)} \right)^2}}, u_c \right)$$

and $w|_c = w_M$;

• $\widehat{\mathbf{IV}}$: $(x, y, u) = \left(x_c, \sqrt{1-x_c^2}, -3x_c + \sqrt{\frac{3}{2}\lambda} \right)$ and $w|_c = -\frac{1-3x_c^2 + \frac{1}{3}\sqrt{\frac{3}{2}\lambda}x_c}{1-3x_c^2 + \sqrt{\frac{3}{2}\lambda}x_c}$.

We then have a third scaling solution, **ScIII**, and a regime of zero energy density of the background fluid, corresponding to the fixed ‘curve’ $\widehat{\mathbf{IV}}$. This curve is denoted that way following the convention in Table 3.1, although neither of those two fixed ‘curves’ are found in the SGB model strictly because $W_c \neq H_c$. This is the reason why carets are used on the latter to differentiate it from the fixed curve in the SGB model (similarly to what was done with the pseudo-kination fixed points we discussed above).

Although those two regimes might be of cosmological interest (especially the scaling one), we shall consider $\kappa \neq \lambda$ and leave the stability analysis of these two fixed curves and possible solutions for future projects.

3.5.5 Stability of the Fixed Points for the Exponential Coupling

To perform the stability analysis of the fixed points shown above, we perturb Eqs. (3.381), (3.382) and (3.387) up to linear order

$$(1 + u_c x_c) \delta x' = - \left[3 - 9x_c^2 - 2u_c x_c - \frac{3}{2}(1 + w_M) \left(1 - 3x_c^2 - y_c^2 - 2u_c x_c \right) \right] \delta x + \\ + y_c \left[\sqrt{6}\lambda - 3(1 + w_M)(x_c + u_c) \right] \delta y - \left[1 - x_c^2 - \frac{3}{2}(1 + w_M) \left(1 - x_c^2 - y_c^2 \right) \right] \delta u, \quad (3.397)$$

$$(1 + u_c x_c) \delta y' = -y_c \left[\sqrt{\frac{3}{2}} \lambda - u_c - 3x_c(1 - w_M) \right] \delta x - \left[\left(\sqrt{\frac{3}{2}} \lambda - 3x_c - u_c \right) x_c - \frac{3}{2}(1 + w_M) (1 - x_c^2 - 3y_c^2) \right] \delta y + y_c x_c \delta u, \quad (3.398)$$

$$(1 + u_c x_c) \delta u' = 2u_c \left[\sqrt{\frac{3}{2}} \kappa - u_c - 3x_c(1 - w_M) \right] \delta x + 6u_c y_c (1 + w_M) \delta y + 2 \left[\left(\sqrt{\frac{3}{2}} \kappa - 3x_c - 2u_c \right) x_c - \frac{3}{2}(1 + w_M) (1 - x_c^2 - y_c^2) \right] \delta u, \quad (3.399)$$

where the constraint equation (3.373) was used to replace z^2 in Eq. (3.385).

Our stability analysis begins with $\mathbf{M}$ and $\mathbf{K}\pm$ first, given the relevance of the latter fixed points in particular in early stages in models of quintessential inflation [329]. In the case of $\mathbf{M}$, which corresponds to the regime of the background fluid domination as was commented already, the eigenvalues, which we denote by ‘ m ’ as in previous studies [3] (see Secs. 3.2.1, 3.2.3 and 3.2.5 too), are

$$m_1 = -\frac{3}{2}(1 - w_M), \quad (3.400)$$

$$m_2 = \frac{3}{2}(1 + w_M), \quad (3.401)$$

$$m_3 = -3(1 + w_M). \quad (3.402)$$

One of the eigenvalues, m_2 , which corresponds to the eigenvector $\mathbf{v} = (0, 1, 0)$,¹⁹ is non-negative, whereas $m_1, m_3 < 0$. Therefore $\mathbf{M}$ is a saddle point and is unstable in the y direction.

Regarding the kination fixed points $\mathbf{K}\pm$, the eigenvalues are given by

$$m_1 = 3(1 - w_M), \quad (3.403)$$

$$m_2 = 3 \mp \sqrt{\frac{3}{2}} \lambda, \quad (3.404)$$

$$m_3 = -6 \pm \sqrt{6} \kappa, \quad (3.405)$$

where the upper sign corresponds to the fixed point $\mathbf{K}+$. m_1 , corresponding to the eigenvector $\mathbf{v} = (1, 0, 0)$, is always positive, irrespective of λ and κ . Thus, the kination period is

19. We order eigenvector components as $\mathbf{v} = (v_x, v_y, v_u)$.

unstable in the x direction. The stability of these fixed points, $\mathbf{M}$ and $\mathbf{K}\pm$, that is shown here, resembles the one in the SGB model [3, 328] (see Sec. 3.2.3).

The eigenvalues for the scaling fixed point $\mathbf{ScI}$ can be easily calculated to be

$$m_1 = 3(1 + w_M) \left(\frac{\kappa}{\lambda} - 1 \right), \quad (3.406)$$

$$m_{2,3} = -\frac{3}{4}(1 - w_M) \left\{ 1 \mp \sqrt{1 - 8 \frac{1 + w_M}{1 - w_M} \left[1 - \frac{3}{\lambda^2}(1 + w_M) \right]} \right\}. \quad (3.407)$$

Given Eq. (3.388), $m_{2,3}$ are complex eigenvalues whose real part is always negative. On the other hand, if $\kappa > \lambda$, m_1 will be positive and hence the scaling fixed point will be a saddle (unstable). We find that same feature in the SGB model [3, 328] (see Sec. 3.2.3).

Regarding the de Sitter fixed point, the eigenvalues are

$$m_1 = -3(1 + w_M), \quad (3.408)$$

$$m_{2,3} = -\frac{3}{2} \left[1 \mp \sqrt{1 - \frac{4}{3}\lambda^2 \left(\frac{\kappa}{\lambda} - 1 \right)} \right]. \quad (3.409)$$

As long as $\kappa > \lambda$, all the eigenvalues will have negative real parts, and $\mathbf{dS}$ will be a stable fixed point (an attractor). The condition $\kappa > \lambda$ was sufficient in order for $\mathbf{ScI}$ to be unstable (see Eq. (3.406)). Therefore, the Universe may transition from $\mathbf{ScI}$ to $\mathbf{dS}$. This reproduces the evolution of the Universe as we know it: from radiation/matter domination to an eventual exponentially expanding universe. The fact that $\mathbf{dS}$ may be a stable fixed point is a well-known result in the SGB model as well [3, 328] (see Sec. 3.2.3).

For completeness, before we consider the fixed points at which $W_c \neq H_c$, we include the stability analysis of $\mathbf{I}$. In this case, the eigenvalues read

$$m_1 = \lambda^2 \left(\frac{\kappa}{\lambda} - 1 \right), \quad (3.410)$$

$$m_2 = \frac{1}{2} (\lambda^2 - 6), \quad (3.411)$$

$$m_3 = \lambda^2 - 3(1 + w_M). \quad (3.412)$$

Since $\lambda^2 < 6$ in order for $\mathbf{I}$ to exist (so that $y_{\mathbf{I}}$ is real, as was commented earlier), we have that $m_2 < 0$. Given the range of values of w_M , m_3 may be non-negative (λ^2 would have to be smaller than $3(1 + w_M)$ otherwise). In the case of power law inflation, where $\lambda^2 < 2$, we see that m_3 is definitely negative. Regardless of w_M and λ , m_1 will be positive if $\kappa > \lambda$.

Then, $\mathbf{I}$ will be unstable if $\kappa > \lambda$.

As shown above, the fixed point $\hat{\mathbf{K}}_+$ could lead to a phase of faster than exponential acceleration, $\epsilon_1|_{\hat{\mathbf{K}}_+} < 0$, if $\kappa < 2\sqrt{\frac{2}{3}}$. The eigenvalues for both fixed points are

$$m_1 = -\frac{2}{2 \mp \sqrt{\frac{3}{2}}\kappa} \left[\pm \sqrt{\frac{3}{2}}\kappa + \frac{3}{2}(1 + w_M) \left(2 \mp \sqrt{\frac{3}{2}}\kappa \right) \right], \quad (3.413)$$

$$m_2 = \frac{\mp \sqrt{\frac{3}{2}}\lambda}{2 \mp \sqrt{\frac{3}{2}}\kappa} \left(\frac{\kappa}{\lambda} - 1 \right), \quad (3.414)$$

$$m_3 = -\frac{2}{2 \mp \sqrt{\frac{3}{2}}\kappa} \left(3 \mp \sqrt{\frac{3}{2}}\kappa \right), \quad (3.415)$$

the upper sign corresponding to $\hat{\mathbf{K}}_+$. In the case of $\hat{\mathbf{K}}_-$, the fixed point will be unstable whenever $\kappa > \lambda$ (because $m_2 > 0$), regardless of the magnitude of κ (given that m_1 and m_3 are always negative for $\hat{\mathbf{K}}_-$). For $\hat{\mathbf{K}}_+$ and $\kappa < 2\sqrt{\frac{2}{3}}$, m_1 and m_3 are both negative too. On top of that, if we assume that $\kappa > \lambda$, then m_2 , corresponding to the eigenvector $\mathbf{v} = (0, 1, 0)$, will be negative and the phase of super-acceleration will be *stable*. If not, it will be unstable in the y direction.

Irrespective of the sign of x_c , if $\kappa \gg 1$, but $\lambda > \kappa$, the asymptotic phase of zero acceleration, corresponding to $\epsilon_1|_{\hat{\mathbf{K}}_{\pm}} = 1$, will be stable as well.

Finally, for the second scaling solution, $\mathbf{ScII}_{\pm}$, the eigenvalues are

$$m_1 = \frac{3}{2}(1 + w_M) \left(1 - \frac{\lambda}{\kappa} \right), \quad (3.416)$$

$$m_{2,3} = -\frac{3}{4}(1 - w_M) \left\{ 1 \mp \sqrt{1 + 8 \frac{1 + w_M}{1 - w_M} \left[1 - \frac{1 - 3w_M}{3(1 - w_M)} \Delta_{\pm} - \frac{12(1 + w_M)^2}{\kappa^2(1 + 3w_M)} \right]} \right\}. \quad (3.417)$$

$\Delta_{\pm}$ was defined in Eq. (3.389). We see that m_1 , which corresponds to the eigenvector $\mathbf{v} = (0, 1, 0)$, is positive (therefore, neither of the scaling fixed points $\mathbf{ScII}_{\pm}$ is stable) whenever $\kappa > \lambda$. Notice that this is the same condition as that of the former scaling solution (see Eq. (3.406)). However, the $m_{2,3}$ eigenvalues do not depend on the magnitude of λ now.

From the existence condition of the fixed points (see Eq. (3.390)) we arrive at

$$\frac{1 - 3w_M}{3(1 - w_M)}\Delta_{\pm} + \frac{12}{\kappa^2} \frac{(1 + w_M)^2}{1 + 3w_M} \leq \frac{(1 - 3w_M)\Delta_{\pm} + 2}{3(1 - w_M)}. \quad (3.418)$$

Given that $1 \leq \Delta_+ < 2$ and $0 \leq w_M < \frac{1}{3}$ for the fixed point **ScII+** (see Eq. (3.392)), then

$$\frac{1 - 3w_M}{3(1 - w_M)}\Delta_+ + \frac{12}{\kappa^2} \frac{(1 + w_M)^2}{1 + 3w_M} < \frac{2}{3} \frac{2 - 3w_M}{1 - w_M} \leq \frac{4}{3}. \quad (3.419)$$

On the other hand, in the case of **ScII-**, we assume that $0 \leq w_M < \frac{1}{3}$ as well, although w_M was constrained to be smaller than 1 (see Eq. (3.392)). Since $0 < \Delta_- \leq 1$, we have

$$\frac{1 - 3w_M}{3(1 - w_M)}\Delta_- + \frac{12}{\kappa^2} \frac{(1 + w_M)^2}{1 + 3w_M} \leq 1. \quad (3.420)$$

Therefore only in the case of **ScII-** will the eigenvalues $m_{2,3}$ always be real (assuming $0 \leq w_M < \frac{1}{3}$). If these eigenvalues are complex, the real part of both eigenvalues is negative and **ScII+** is stable if $\lambda > \kappa$. When it comes to the other cases, the stability shall depend on the value of the EoS parameter of the background fluid, w_M , under consideration.

A summary of all the fixed points we studied in this section, as well as their stability, can be found in Table 3.3.

3.5.6 The Case of the Linear Coupling

For a linear coupling function $\xi(\phi) \propto \phi$, we have that $\kappa = 0$ because $\xi_{,\phi\phi} = 0$ (see Eq. (3.136)). This implies that $m_1 < 0$ in Eq. (3.406), irrespective of λ . Then, the scaling fixed point is *stable*. Moreover, we see that **dS** is *unstable* (see Eq. (3.409), where $m_2 > 0$). As was explained in Sec. 3.2.5, the stability of the scaling fixed point **ScI** contradicts observations indicating that the Universe undergoes a phase of current accelerating expansion, which cannot correspond to the scaling regime with $\epsilon_1 = \frac{3}{2}(1 + w_M)$. Therefore, the linear coupling can be ruled out. Also, $\kappa = 0$ implies that either $u_c = 0$ or $\epsilon_1|_c = 0$ (see Eqs. (3.387) and (3.386)). This implies that no other scaling regime with $u_c \neq 0$ exists. The same conclusions can be drawn in the SGB model [3].

		Fixed point: (x_c, y_c, u_c)	Stability
M		$(0, 0, 0)$	Unstable
K$\pm$		$(\pm 1, 0, 0)$	Unstable
I		$\left(\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \frac{\lambda^2}{6}}, 0\right)$	Unstable whenever $\kappa > \lambda$
ScI		$\left(\sqrt{\frac{3}{2} \frac{(1+w_M)^2}{\lambda^2}}, \sqrt{\frac{3}{2} \frac{1-w_M^2}{\lambda^2}}, 0\right)$	Unstable whenever $\kappa > \lambda$
dS		$\left(0, 1, \sqrt{\frac{3}{2}\lambda}\right)$	Stable unless $\kappa < \lambda$
ScII$\pm$		$\left(\frac{\kappa}{\sqrt{6}} \frac{1+3w_M}{3(1-w_M^2)} \Delta_{\pm}, 0, \frac{\kappa}{\sqrt{6}} \frac{1}{1+w_M} \Delta_{\pm}\right)$	Unstable whenever $\kappa > \lambda$
$\hat{\mathbf{K}}_{\pm}$	$\hat{\mathbf{K}}+$	$\left(+1, 0, \sqrt{\frac{3}{2}\kappa - 3}\right)$	Unstable whenever $\kappa < \lambda$ and $\kappa < 2\sqrt{\frac{2}{3}}$
	$\hat{\mathbf{K}}-$	$\left(-1, 0, \sqrt{\frac{3}{2}\kappa + 3}\right)$	Unstable whenever $\kappa > \lambda$

Table 3.3: Summary of the fixed points considered in this section and their stability according to κ and λ for an exponential coupling function (see Eq. (3.136)). We determine the stability of each fixed point assuming an EoS parameter of the background fluid in the range $0 \leq w_M < 1$ and $\kappa \neq \lambda$. $\Delta_{\pm}(\kappa, w_M)$ is defined in Eq. (3.389).

3.5.7 Numerical Simulations: Two Scaling Solutions

We now solve Eqs. (3.381), (3.382) and (3.387) (we use the constraint equation (3.373) to replace z^2) numerically and consider trajectories which enter the two scaling solutions (one of them consisted of two fixed points). In line with our analysis of the SGB model [3], we are interested in the trajectories that begin close to kination (where the kinetic energy of the quintessence field dominates after inflation in quintessential inflation models [329]), go through the scaling regime **ScI** (so $\epsilon_1 = \frac{3}{2}(1 + w_M)$) and approach the de Sitter fixed point (where the CC becomes the only contribution to the energy density of the Universe), as suggested by the observations. However, we will examine numerically those that pass near the two scaling fixed points **ScII** $\pm$ too because the effective EoS parameter w equates w_M as well.

In our numerical simulations, the background fluid is chosen to be a pressureless one, such that $w_M = w_m = 0$ and $\Omega_M = \Omega_m$. We select those trajectories in the phase space that satisfy $\Omega_m = 0.3147 \pm 0.0074$ and $w_f = -0.957 \pm 0.080$ [12] at $N = 0$, which corresponds to the present time (see below Eq. (3.158)). w_f was defined in Eq. (3.156) in terms of ϵ_1 . We remind the reader that w_f is interpreted as the EoS parameter of the DE fluid triggering the accelerated expansion of the Universe. In the WGB model, it reads

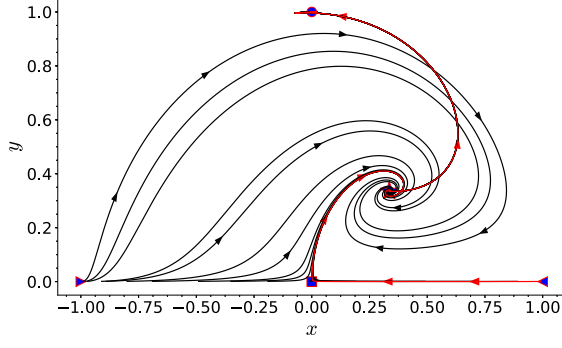
$$w_f = -1 + \frac{1}{x^2 + y^2} \left[\frac{2}{3}\epsilon_1 - \left(1 - x^2 - y^2\right)(1 + w_M) \right], \quad (3.421)$$

where ϵ_1 is meant to be given in Eqs. (3.386) and (3.385).

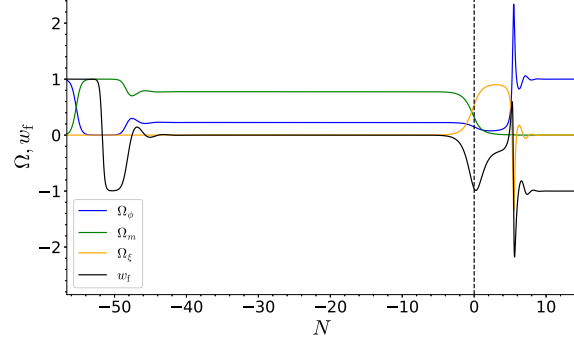
The initial values of x are chosen differently within the range $(-1, 1)$, while we set $u_0 = 10^{-23}$ and $y_0 = 10^{-3}$. The small value of u_0 is imposed to ensure a negligible initial contribution from the GB coupling (similar argument to the given in Sec. 3.2.3). In that case, the trajectory starts near the kination fixed points **K** $\pm$, and the background fluid domination one **M** (depending on the value of x_0).

We see in Figs. 3.9a and 3.9b the results of the simulations for $\lambda = 4.47\sqrt{\frac{2}{3}}$ and $\kappa/\lambda = 1.4$. For these values of λ and κ parameters, a trajectory reproduces the evolution of interest which enters the scaling regime **ScI** and reaches de Sitter asymptotically. All the trajectories in Fig. 3.9a (which correspond to the different initial values of x) end up at the de Sitter fixed point $y_{\text{ds}} = 1$. Since $\delta u \propto \exp \left[3 \left(\frac{\kappa}{\lambda} - 1 \right) N \right]$ during the scaling regime (see Eq. (3.406) and remember that $w_M = w_m = 0$) and κ/λ is of order $\mathcal{O}(1)$, it will take long for the Universe to abandon the scaling regime.

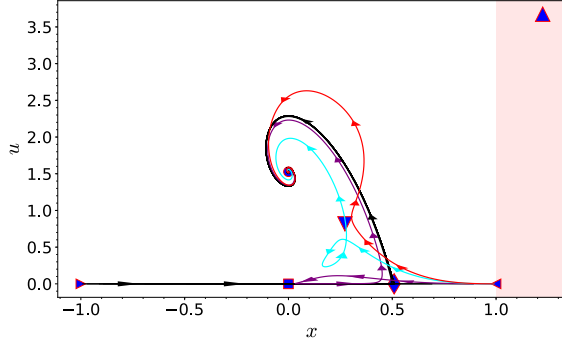
The 3D phase portrait is shown in Fig. 3.9e. To clarify the convention, we remark that



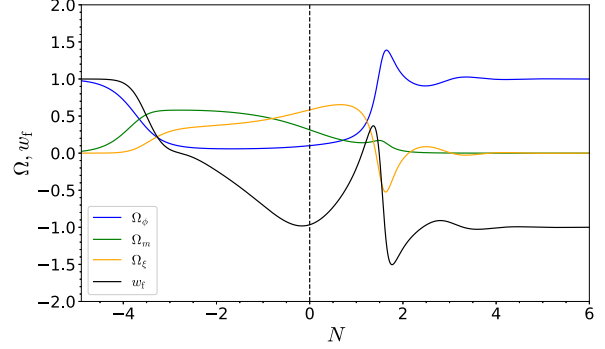
(a) Phase portrait for $\lambda = 4.47\sqrt{\frac{2}{3}}$ and $\kappa/\lambda = 1.4$.



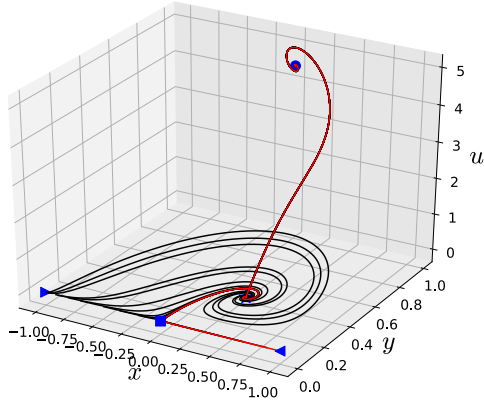
(b) Ω 's and w_f drawn from the red trajectory.



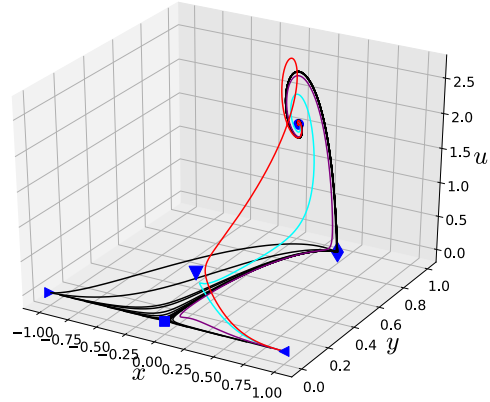
(c) Phase portrait for $\lambda = 0.72\sqrt{3}$ and $\kappa/\lambda = 4.41$.



(d) Ω 's and w_f drawn from the red trajectory.



(e) 3D phase portrait for $\lambda = 4.47\sqrt{\frac{2}{3}}$ and $\kappa/\lambda = 1.4$.



(f) 3D phase portrait for $\lambda = 0.72\sqrt{3}$ and $\kappa/\lambda = 4.41$.

Figure 3.9: Phase portraits and density parameters and w_f for some values of the parameters λ and κ . In Fig. 3.9a, all the trajectories begin either close to $\mathbf{K}\pm$ or near $\mathbf{M}$. They all converge at $\mathbf{ScI}$ and transition to $\mathbf{dS}$. In Fig. 3.9c, the red and cyan trajectories in particular pass near the scaling regime $\mathbf{ScII-}$ (triangle pointing downwards) before reaching $\mathbf{dS}$. The rest converge at $\mathbf{I}$ instead (the diamond in between $\mathbf{K+}$ and $\mathbf{M}$ in the plot). The stripe indicates the forbidden region explicitly due to the constraint equation (3.373) and, the triangle pointing upwards, the scaling fixed point $\mathbf{ScII+}$. Same phase portraits in 3D are shown in Figs. 3.9e and 3.9f.

the **M** fixed point has been represented by a square in Figs. 3.9a and 3.9e. The kination fixed points **K** $\pm$ in those figures correspond to the left and right leaning triangles, depending on the sign. The de Sitter fixed point is given by the circle instead, and the scaling regime corresponds to a star onto which all the trajectories converge. We first notice that, before the de Sitter phase is reached, the density parameter of the scalar field Ω_ϕ (see Eq. (3.377)) becomes larger than 1 given that Ω_ξ (see Eq. (3.379)) turns negative. This is obtained in many other simulations with different values of λ and κ . Likewise, w_f grows and surpasses the value of the EoS parameter of the background fluid for a brief moment of time. There is an equally short period where it takes on phantom values (i.e. $w_f > -1$) before settling down at $w_{\text{ds}} = -1$, which is the expected value during de Sitter. Note that slightly phantom DE is preferred by the observational data [12].

We observe another feature shared with other simulations: the density parameter Ω_ξ is always the dominant contribution to DE at the present time. This occurs as long as the scalar field energy density *does not* take over during the scaling regime, which is sensible because the quintessence field ϕ is supposed to be light and, in case it dominates, it may alter the successful predictions of BBN (see Sec. 1.1.3 and Fig. 1.4).

The Universe may enter the scaling regime **ScII** $\pm$, where $y_{\text{ScII}\pm} = 0$, instead of **ScI**, starting from kination or from the background fluid domination regime. If the density parameter of the background fluid, Ω_M (see Eq. (3.378)), is larger than the sum of the rest of the density parameters, $\Omega_\phi + \Omega_\xi$, during the scaling regime, then it must satisfy $\Omega_M|_{\text{ScII}\pm} > 0.5$.²⁰ For pressureless matter, $\Omega_M|_{\text{ScII}\pm} = \Omega_m|_{\text{ScII}\pm}$ reads

$$\Omega_m|_{\text{ScII}\pm} = \frac{54(54 - \kappa^2\Delta_\pm^2)}{(54 + 3\kappa^2\Delta_\pm^2)^2}. \quad (3.423)$$

Then $\Omega_m|_{\text{ScII}\pm} > 0.5$ implies $\kappa\Delta_\pm < \sqrt{6}$. The fixed point **ScII** $+$ can be discarded because $\kappa > 3\sqrt{2}$ in order for the fixed point to exist, and $1 \leq \Delta_+$ (see Eq. (3.390) and discussion below), which means that $\kappa \leq \kappa\Delta_+ < \sqrt{6}$. Regarding the fixed point **ScII** $-$, we have (see Eq. (3.389))

$$\kappa\Delta_- = \kappa - \sqrt{\kappa^2 - 18} < \sqrt{6}. \quad (3.424)$$

20. One might argue that this is not necessarily true if Ω_ξ is negative. However, this is discarded in this situation given that $\Omega_\xi < 0$ implies $-2 < ux < 0$ (see Eq. (3.379)). At **ScII** $\pm$, we obtain (see above Eq. (3.389))

$$u_{\text{ScII}\pm}x_{\text{ScII}\pm} = \frac{1 + 3w_M}{3(1 - w_M)}u_{\text{ScII}\pm}^2, \quad (3.422)$$

and because $0 \leq w_M < 1$, $u_{\text{ScII}\pm}x_{\text{ScII}\pm}$ is never negative and, in consequence, neither is $\Omega_\xi|_{\text{ScII}\pm}$.

Hence $\kappa > 2\sqrt{6}$ for Ω_m to be larger than $\Omega_\phi + \Omega_\xi$ during the scaling regime, in agreement with the BBN predictions.

On top of that, it can be noticed that if the fixed points **ScII** $\pm$ are unstable only in the y direction (see discussion below Eq. (3.417)), then we should find complex $m_{2,3}$, given that the real part in that case is always negative (see Eq. (3.417)). Unfortunately, this cannot happen unless we consider the fixed point **ScII** $+$ (see Eq. (3.420) and discussion below). But only at **ScII** $-$ can Ω_m dominate over $\Omega_\phi + \Omega_\xi$.

An example of trajectories going through the second scaling solution (corresponding to the fixed point **ScII** $-$, which is being represented by a triangle pointing downwards) are the red and cyan ones in Fig. 3.9c, where $\lambda = 0.72\sqrt{3}$, $\kappa/\lambda = 4.41$ and $u_0 = 10^{-5}$. The purple line no longer passes near the scaling regime, but approaches the fixed point **I**, represented by the diamond in the plot. The 3D phase portrait is shown in Fig. 3.9f. In Fig. 3.9c we show the density parameters and w_f corresponding to the red trajectory, which are compatible with the constraints on Ω_m and w_f (see values above Eq. (3.421)). Unfortunately, the scaling regime does not last long enough for this particular case to reproduce the matter domination period in the history of the Universe, although we do see that Ω_m takes on the expected value in view of Eq. (3.423) and of the value of κ . The fixed point **ScII** $+$, represented by a triangle pointing upwards, falls within the forbidden region because $\kappa > 3\sqrt{2}$, as explained above. The forbidden region corresponds to the pale red coloured stripe, which excludes values of $x > 1$ that violate the constraint equation (3.373).

3.5.8 The Speed of Gravitational Waves

There are works in which the constraint on $|\alpha_T|$, Eq. (3.67), has been examined in the context of the SGB model of the metric formalism [379, 380].²¹ However, when it comes to the Palatini formalism, no analysis has been performed in that sense to the best of our knowledge. We then proceed to examine the evolution of the α_T parameter in this WGB model.

Written in terms of the dimensionless variables defined in Eqs. (3.372) and (3.375), and of the Hubble-flow parameter ϵ_1 (see Eqs. (3.385) and (3.386)), α_T is given by

$$\alpha_T = -\frac{2}{1-2ux} \left\{ (1+2ux)(ux)' - ux [1-ux-\epsilon_1(1+ux)] \right\}. \quad (3.425)$$

21. Ref. [380] in particular includes a case of kinetic coupling to curvature besides the coupling to the GB term.

Notice that for $|ux| \ll 1$, we recover Eq. (3.159), as expected (see discussion following Eq. (3.379) and remember the difference in sign between the definition of u in this model and that of the SGB model that we chose in Ref. [3]). It can be readily seen that

$$\alpha_T|_{\mathbf{M}} = \alpha_T|_{\mathbf{K}\pm} = \alpha_T|_{\mathbf{I}} = \alpha_T|_{\mathbf{ScI}} = \alpha_T|_{\mathbf{dS}} = 0 . \quad (3.426)$$

For the rest of the fixed points

$$\alpha_T|_{\mathbf{ScII}\pm} = -\frac{18(1+w_{\mathbf{M}})(1-w_{\mathbf{M}}^2) - 2\kappa^2(1+3w_{\mathbf{M}})\Delta_{\pm}}{27(1+w_{\mathbf{M}})(1-w_{\mathbf{M}}^2) - 2\kappa^2(1+3w_{\mathbf{M}})\Delta_{\pm}} \left[2 - \frac{\kappa^2}{18} \frac{(1+3w_{\mathbf{M}})(5+3w_{\mathbf{M}})}{(1+w_{\mathbf{M}})(1-w_{\mathbf{M}}^2)} \Delta_{\pm} \right] , \quad (3.427)$$

$$\alpha_T|_{\hat{\mathbf{K}}\pm} = -\frac{(6 \mp \sqrt{6}\kappa)(4 \mp \sqrt{6}\kappa)}{7 \mp \sqrt{6}\kappa} . \quad (3.428)$$

$\alpha_T|_{\hat{\mathbf{K}}+}$ in particular is zero if $\kappa = 3\sqrt{\frac{2}{3}}$, which does not satisfy the bound $\kappa < 2\sqrt{\frac{2}{3}}$ and falls out of the super-accelerated phase region. Another possibility is $\kappa = 2\sqrt{\frac{2}{3}}$, but in that case $u_c x_c = -1$ and $H_c = 0$ (see Eq. (3.374)). Since $\kappa > 0$, $\alpha_T|_{\hat{\mathbf{K}}-}$ never vanishes exactly.

Regarding $\alpha_T|_{\mathbf{ScII}\pm}$, in the case of the fixed point $\mathbf{ScII}+$, the α_T parameter cannot be zero due to the existence conditions of the fixed point derived below Eq. (3.389) and the range of values of $w_{\mathbf{M}}$ under consideration. Indeed, if

$$\frac{\kappa^2}{18} \frac{(1+3w_{\mathbf{M}})(5+3w_{\mathbf{M}})}{(1+w_{\mathbf{M}})(1-w_{\mathbf{M}}^2)} \Delta_+ = 2 , \quad (3.429)$$

$\alpha_T|_{\mathbf{ScII}+} = 0$. This implies that (see Eq. (3.390) and remember that $0 \leq w_{\mathbf{M}} < 1$)

$$5\Delta_+ \leq (5+3w_{\mathbf{M}})\Delta_+ \leq 2 . \quad (3.430)$$

Thus $\Delta_+ \leq 0.4$. But this is not possible because $1 \leq \Delta_+$ (see Eq. (3.389)). Another possibility for $\alpha_T|_{\mathbf{ScII}+} = 0$ is

$$\frac{\kappa^2}{18} \frac{1+3w_{\mathbf{M}}}{(1+w_{\mathbf{M}})(1-w_{\mathbf{M}}^2)} \Delta_+ = \frac{1}{2} . \quad (3.431)$$

However, using Eq. (3.390), we see that this is discarded because one obtains $\Delta_+ \leq 0.5$.

In the case of $\mathbf{ScII}-$, one might have $\alpha_T|_{\mathbf{ScII}-} = 0$. However, given $w_{\mathbf{M}} = w_m = 0$, $\alpha_T|_{\mathbf{ScII}-} = 0$ if $\kappa^2 \Delta_-(\kappa, 0) = 36/5$ or $\kappa^2 \Delta_-(\kappa, 0) = 9$ (see Eq. (3.427)). By looking at

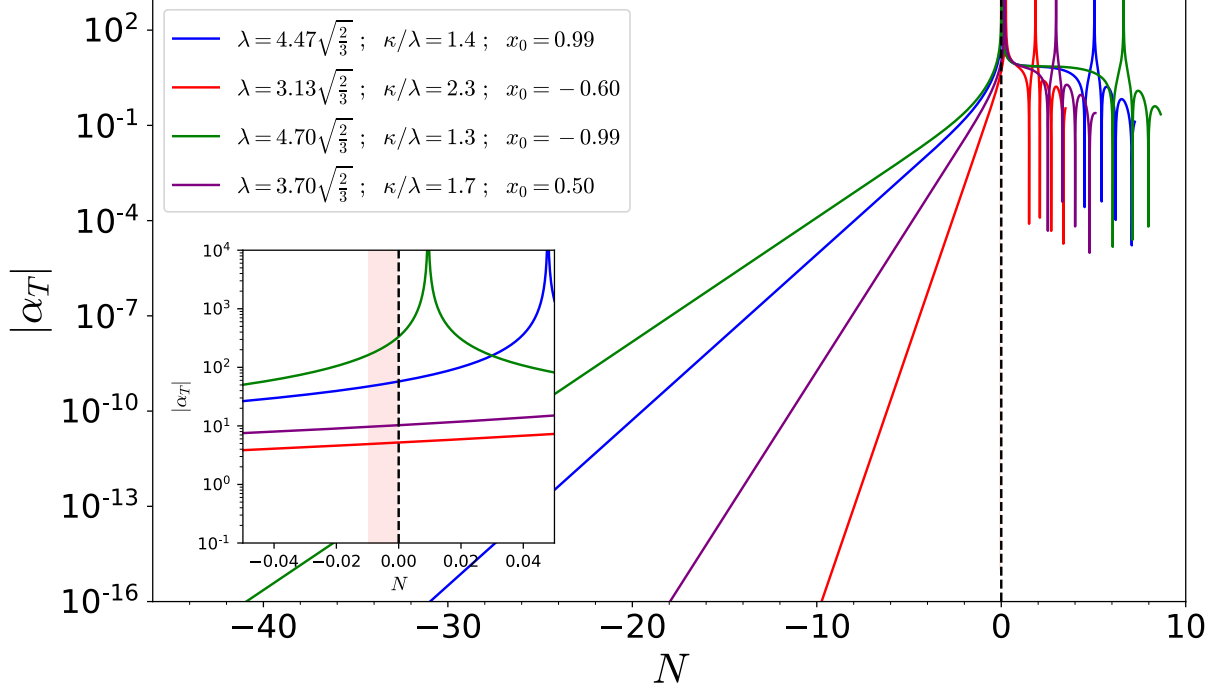


Figure 3.10: $|\alpha_T|$ predicted by the WGB model for some values of the parameters λ and κ and different initial conditions of x . The stripe in the smallest graph on the left indicates the excluded region starting at redshift $z = 0.00980 \pm 0.00079$ [304]. We see that the bound $|\alpha_T| < 10^{-15}$ is grossly violated for the cases under consideration. These cases have been chosen given that they did satisfy the constraints on Ω_m and w_f .

Eq. (3.389), one can see that no solutions for κ are obtained in those cases. Thus, α_T never vanishes at **ScII**— when $w_M = w_m = 0$.

Although the value of $|\alpha_T|$ at the fixed points is useful to have an indication of its evolution, the current state of the Universe cannot be described by any of those fixed points (as we pointed out in Sec. 3.2.4). Moreover, $|\alpha_T|$ does not have to be exactly zero but satisfy the bound in Eq. (3.67).

We then show the evolution of α_T connecting the different fixed points and make sure it does not violate the observational constraint (3.67). The upper bound is applicable from redshift $z = 0.00980 \pm 0.00079$ [304] (much after the beginning of DE domination in the Universe) to $z = 0$ (today), as was argued in Sec. 3.2.4 already. Choosing models that reproduce the trajectories going through the scaling solution **ScI**, and then entering the de Sitter regime, with different values of the parameters κ and λ , and different initial values for x , we plot the evolution of the absolute value of α_T over the number of e-folds N (see Fig. 3.10). We place an inset plot of the region around $N = 0$. The excluded region, marked by a vertical, pale red coloured stripe shows that the bound is violated by many orders of

magnitude, similarly to the SGB model [3] (see Sec. 3.2.4 and Fig. 3.4). The cases shown in the plot were selected because they did satisfy the constraints on Ω_m and w_f at the present time (see values before Eq. (3.421)).

By examining Eq. (3.425), we understand why the bound is so strongly violated. For ux of order $\mathcal{O}(0.1)$ - $\mathcal{O}(1)$, Ω_ξ is of order $\mathcal{O}(0.1)$ according to Eq. (3.379). Also, we observed numerically that Ω_ξ is always the density parameter that dominates over the rest by the time Ω_m and w_f satisfy the constraints (present time). On top of this, we now remark that Ω_ξ grows at that time. Then $ux \rightarrow 1/2$ (i.e. $\Omega_\xi \rightarrow 0.6$) while the numerator in Eq. (3.425) remains of order $\mathcal{O}(0.1)$ - $\mathcal{O}(1)$. This causes $|\alpha_T|$ to increase near the present time, although it does not diverge because $(ux)'$ increases eventually as well, even though Ω_ξ remains constant. The constancy of Ω_ξ can be seen by simply taking the derivative of Eq. (3.379)

$$\Omega'_\xi = \frac{2(1 - \Omega_\xi)}{1 + ux} (ux)'. \quad (3.432)$$

Hence when $\Omega_\xi \rightarrow 1$, $\Omega'_\xi \rightarrow 0$ (see Fig. 3.7 also).

The fact that $|\alpha_T|$ does not remain below the observational bound 10^{-15} while we approach the present time indicates that the exponential GB coupling function in the WGB model must be discarded, in the same way as in the SGB model [3] (see Sec. 3.2.4).

3.5.9 The Case of $\alpha_T = 0$ in the Weyl-Gauss-Bonnet Model

We now impose the condition $\alpha_T = 0$ without making any assumption on the functional form of $\xi(\phi)$. The exponential GB coupling function has been shown not to satisfy all the observational constraints, but we can examine the case of $\alpha_T = 0$ which would automatically fulfill Eq. (3.67) at least.

In the SGB model, we saw that such a condition translated into Eq. (3.61). In the WGB model however, $\alpha_T = 0$ yields

$$\ddot{\xi} = \dot{\xi} W \left(1 - 2 \frac{\dot{\xi} W}{M_{\text{Pl}}^2} \right). \quad (3.433)$$

Using Eq. (3.351) we may write

$$\ddot{\xi} = -\frac{\dot{\xi} H}{2\mu} \left(1 + 2\mu - \sqrt{1 + 8\mu} \right). \quad (3.434)$$

If $|\mu| \ll 1$ (or $|ux| \ll 1$; see caption in Fig. (3.7)), we recover Eq. (3.61).

The equation of $\dot{H}$ in the SGB model when $\ddot{\xi} = H\dot{\xi}$ reads (see Eq. (3.45))

$$-2\dot{H} \left(M_{\text{Pl}}^2 - 4\dot{\xi}H \right) = \dot{\phi}^2 + \rho_{\text{M}}(1 + w_{\text{M}}). \quad (3.435)$$

This implies that, in de Sitter, $\dot{\phi} = 0$ (so $\dot{H} = 0$). However, this is problematic when we integrate Eq. (3.61). The integral on the LHS of the equation

$$\int_0^{\dot{\xi}_1} d \ln(|\dot{\xi}|) = \int_{a_{\text{dS}}}^{a_1} d \ln a, \quad (3.436)$$

which includes the de Sitter value, does not converge. In the WGB model however, since we have Φ in addition to H , there is a chance that $\dot{H} = 0$ but still $\dot{\phi} \neq 0$. To verify this, an equation analogous to Eq. (3.435) is derived now. Bearing Eq. (3.434) in mind, we write Eq. (3.341) as

$$-\frac{2M_{\text{Pl}}^2 \dot{H}}{\sqrt{1+8\mu}} = \dot{\phi}^2 + \rho_{\text{M}}(1 + w_{\text{M}}) - 4\ddot{\xi}H^2 \frac{(1 - \sqrt{1+8\mu})^2}{16\mu^2 \sqrt{1+8\mu}} - 4\dot{\xi}H^3 \frac{(1 - \sqrt{1+8\mu})^3}{64\mu^3}, \quad (3.437)$$

where Eq. (3.351) was used. Plugging Eq. (3.434) into Eq. (3.437), we obtain

$$-\frac{2M_{\text{Pl}}^2 \dot{H}}{\sqrt{1+8\mu}} = \dot{\phi}^2 + \rho_{\text{M}}(1 + w_{\text{M}}) + 3H^2 M_{\text{Pl}}^2 \frac{(1 - \sqrt{1+8\mu})^2}{16\mu^2 \sqrt{1+8\mu}} \left(1 + 4\mu - \sqrt{1+8\mu} \right). \quad (3.438)$$

It can be seen that the last term vanishes when $|\mu| \ll 1$, meaning that one recovers Eq. (3.435) for $|\dot{\xi}H| \ll M_{\text{Pl}}^2$, as expected from previous examples (see e.g. Eq. (3.434)). Unfortunately, each term on the RHS is non-negative meaning that $\dot{H} = 0$ (de Sitter) implies $\dot{\phi} = 0$ and hence $\dot{\xi} = 0$, same as in the SGB model.

Knowing this, we analyse the stability of the de Sitter **dS** and scaling **ScI** fixed points when the condition $\alpha_T = 0$ is imposed. An analogous analysis was performed in the SGB model in Ref. [3] (see Sec. 3.2.6). We then write Eq. (3.433) as

$$(ux)' = \frac{ux}{1+ux} \left[1 - (3x+2u)x - \frac{3}{2}(1+w_{\text{M}}) \left(1 - x^2 - y^2 \right) \right], \quad (3.439)$$

where we used Eq. (3.385), the constraint equation (3.373) and Eqs. (3.375) and (3.384), because we make no assumption on the functional form of the GB coupling. Eq. (3.439) is equivalent to $\alpha_T = 0$ in Eq. (3.425). However, we wish to make it clear that we use Eq. (3.384) to write it that way.

We perturb Eq. (3.439) around the de Sitter fixed point (which is still a fixed point of the system despite Eq. (3.439)), that can only correspond to $x = 0$ and $z = 0$ as we just argued. The resulting equation reads

$$\delta x' = \delta x . \quad (3.440)$$

This equation implies that $\delta x \propto e^N$. We see that it is the same equation as in the SGB model (see Sec. 3.2.6 and Eq. (3.171) in particular). The rest of the equations are²²

$$\delta x' = -3\delta x - \sqrt{\frac{3}{2}}\lambda(1 + 3w_M)\delta y - \delta u , \quad (3.441)$$

$$\delta y' = -3(1 + w_M)\delta y . \quad (3.442)$$

The second equation gives $\delta y \propto e^{-3(1+w_M)N}$. Using Eq. (3.440), Eq. (3.441) gives δu . Since δy dies out but δx grows, δu will increase too. We see then that the de Sitter fixed point **dS** is unstable, as in the SGB model [3] (see Sec. 3.2.6).

Regarding the scaling regime **ScI**, since $u_{\text{ScI}} = 0$, this is still a fixed point despite Eq. (3.439). Linearising Eq. (3.439), we arrive at

$$\delta u' = -\frac{1}{2}(1 + 3w_M)\delta u . \quad (3.443)$$

Hence $\delta u \propto e^{-\frac{1}{2}(1+3w_M)N}$, and given the eigenvalues in Eq. (3.407) (which remain the same despite $\alpha_T = 0$), we have that **ScI** becomes stable, a result similar to the one obtained in the SGB model in Ref. [3] (see Sec. 3.2.6) and in quintessence models in GR (see Sec. 3.2.1).

When it comes to the second scaling regime, where $y_{\text{ScII}\pm} = 0$, we can see that they are *not* fixed points of the new system of equations that includes Eq. (3.439).

Consequently, the WGB model with $\alpha_T = 0$ cannot reproduce a long scaling regime with small u followed by a period of accelerated expansion, and hence can be ruled out.

Despite the negative results of this work, similar to the ones we obtained in Ref. [3] (see also Secs. 3.2.4, 3.2.5 and 3.2.6), there were several assumptions made which could be reconsidered in future publications. To begin with, we considered no coupling functions other than the exponential and linear cases. The reason was that equations become self-similar (along with the exponential potential) and can be studied via dynamical systems analysis.

22. Although we have not assumed a specific coupling function $\xi(\phi)$ in this section, $u_{\text{dS}} = \sqrt{\frac{3}{2}}\lambda$, as in the case of the exponential function. This can be seen from Eq. (3.381) with $x_{\text{dS}} = 0$ and $y_{\text{dS}} = 1$.

In this section in particular, we also restricted ourselves to the case of $\kappa \neq \lambda$. If $\kappa = \lambda$, some fixed ‘curves’ were found at the end of Sec. 3.5.4 but no further analysis was done. In the metric formalism, the case of $\kappa = \lambda$ was considered already in Ref. [325] analytically, as was commented already (see above Eq. (3.143)). Since one of the fixed ‘curves’ is a scaling fixed point, **ScIII**, it would be interesting to explore this further in the vein of Ref. [325].

Also, we considered a Weyl vector with vanishing homogeneous spatial components under the assumption of spatial isotropy; i.e. $A_\mu(t) = (-\Phi(t), \mathbf{0})$. However, one might regard the presence of such components so that they are relevant in the dynamical analysis but subdominant enough not to generate a large scale anisotropy. During inflation for example, this and even the fully dominant presence of the vector field were shown to be consistent with observations in Ref. [101]. We leave this for future work.

3.5.10 Concluding Remarks of Section 3.5

The goal of Sec. 3.5 was to investigate whether the SGB gravity, as a DE model, when formulated in a different geometry, fulfilled the constraint (3.67) and yet displayed a period of matter domination followed by an accelerated expanding phase of the Universe in accordance with other constraints on the density and EoS parameters. The reason was that some Horndeski models were found to be consistent with Eq. (3.67) in the Palatini formalism (see App. E), where the connection is independent of the metric and the underlying geometry no longer corresponds to the pseudo-Riemannian one. Since the SGB model is a subclass of Horndeski’s theory (see Sec. 3.3), there could be the case that one avoids or alleviates the constraint on $|\alpha_T|$ if the geometry is different from the one of the metric formalism.

We assumed a spatially-flat FLRW metric and a background Weyl vector with temporal component $\Phi(t)$ and no spatial components breaking spatial isotropy. The action was written with the LC and non-Riemannian parts shown explicitly (see Eq. (3.324)). The α_T parameter was calculated in Sec. 3.5.2. We then performed a dynamical systems analysis assuming an exponential potential (see Eq. (3.73), with $\phi_0 = M_{\text{Pl}}$). The dimensionless variables in Eqs. (3.372) and (3.375) were formulated in terms of the *Weyl parameter* W , defined below Eq. (3.329). Thus, all these variables include the modifications to Einstein’s gravity in contrast to the SGB model of the metric formalism, where only u accounts for them. The fixed points were calculated in Sec. 3.5.4 under the assumption that $\xi(\phi)$ is an exponential function (see Eq. (3.136), with $\phi_0 = M_{\text{Pl}}$), which makes the equations amenable to the method that was used in Ref. [3] already (see Sec. 3.2.3). Another possibility is the linear GB coupling (where $\xi_{,\phi\phi} = 0$) but, in that case, in line with the conclusions reached in

Ref. [3] (see Sec. 3.2.5), one does not reproduce the expected evolution of the Universe. A stability analysis of the fixed points was done in Sec. 3.5.5.

Amongst the fixed points calculated in Sec. 3.5.4, **ScI** and **dS** had been found in the SGB model of the metric formalism too [3], and were shown to satisfy the same stability conditions in both the WGB and SGB models. These fixed points correspond to the scaling and de Sitter regimes, respectively. While the scaling fixed point is unstable if $\kappa > \lambda$, **dS** becomes an attractor and the Universe is predicted to abandon a period of matter domination and enter a regime of accelerated expansion. κ and λ are the strength of the exponentials involved in the GB coupling and the scalar potential, as shown in Eqs. (3.136) and (3.73), respectively. If $\kappa < \lambda$, **dS** becomes a saddle point whereas **ScI** is a stable fixed point. In addition to these points, we found a second scaling regime that consists of two fixed point **ScII $\pm$** . Whereas $u_{\text{ScI}} = 0$, $y_{\text{ScII}\pm} = 0$. In the case of **ScII $+$** , it was shown not to exist if the EoS parameter of the background fluid fell within the range $\frac{1}{3} \leq w_M < 1$. The scaling regime **ScII $\pm$** was shown to be unstable whenever $\kappa > \lambda$, same as in the case of **ScI**. The instability in the case of **ScII $\pm$** was found to be in the y direction of the phase space. We found also that **ScII $+$** is always stable only if $\lambda > \kappa$ because two of the three eigenvalues, which do not depend on λ , are complex with negative real parts (see Eq. (3.417)).

Finally, we calculated another couple of fixed points, **K $\pm$** , corresponding to a regime of ‘pseudo-kination’. We called it that way because $3W_{\text{K}\pm}^2 M_{\text{Pl}}^2 = \dot{\phi}_{\text{K}\pm}^2/2$ but $W_{\text{K}\pm} \neq H_{\text{K}\pm}$. It was found that **K $+$** led to a universe that accelerated faster than during de Sitter, where it accelerates exponentially, if $\kappa < 2\sqrt{\frac{2}{3}}$. Regardless of the sign, if $\kappa \gg 1$, then the Universe was predicted not to accelerate. Both scenarios, the super-accelerated and uniform expansion phases, were determined to be stable if $\kappa > \lambda$ and $\kappa < \lambda$, respectively. A summary of the fixed points and their stability was provided in Table 3.3.

In Sec. 3.5.7 we performed some numerical simulations assuming pressureless background matter such that $w_M = w_m = 0$ and examined trajectories crossing both scaling regimes, **ScI** and **ScII $\pm$** . When it comes to those approaching **ScI**, we plotted the 2D phase portrait in Fig. 3.9a and the 3D one in Fig. 3.9e for some values of λ and κ . These were chosen among other simulations given that they satisfied the constraints on Ω_m and w_f [12] at present time ($N = 0$). w_f is the EoS parameter of the DE fluid responsible for the current accelerated expansion of the Universe (see Eq. (3.421)). The evolution of the density parameters and w_f for one of the trajectories was displayed in Fig. 3.9b. The system’s behaviour is reproduced as expected: starting from a period of kination, the Universe enters a long period of background fluid domination which gives way to the present epoch of DE domination. An eventual

period of de Sitter is predicted after a transient epoch where the effective density parameter associated with the GB coupling $\xi(\phi)$, Ω_ξ , dominates and w_f crosses the phantom barrier (i.e. $w_f < -1$). A common feature of our simulations was that, during DE domination, Ω_ξ is always dominant over the density parameter of the scalar field Ω_ϕ .

A similar but much shorter period of matter domination was reproduced in the case of the second scaling solution **ScII**– (see Fig. 3.9d). The corresponding 2D phase diagram was displayed in Fig. 3.9c and the 3D phase space in Fig. 3.9f. In order for Ω_m to dominate over the sum $\Omega_\phi + \Omega_\xi$ during the scaling regime (so BBN predictions are not affected; see Fig. 1.4 and Sec. 1.1.3 for more details), it was shown that $\kappa > 2\sqrt{6}$. Regarding **ScII**+, the bound implied that $\kappa < \sqrt{6}$, this being at odds with the existence condition of that fixed point (see Eq. (3.390) and discussion below). We indeed saw that **ScII**+ was located in the excluded region of the phase diagram 3.9c (this region corresponds to the coloured stripe).

The α_T parameter was written in terms of the dimensionless variables (which we remind the reader were defined in Eqs. (3.372) and (3.375)), their derivatives, and the Hubble flow parameter ϵ_1 (see Eq. (1.70)). α_T was shown to be non-vanishing for $\hat{\mathbf{K}}\pm$ (in the case of $\hat{\mathbf{K}}+$ only if one considers the super-accelerated phase). Also, α_T never vanishes at **ScII**– if $w_M = 0$. However, it was argued that the value of $|\alpha_T|$ at the fixed points was not enough because the current Universe had to be somewhere in between the scaling and de Sitter fixed points in the phase space. The evolution of $|\alpha_T|$ for different initial conditions x_0 and values of the constants λ and κ was displayed in Fig. 3.10. It was found that the constraint (3.67) is violated by many orders of magnitude at present time, when Ω_m and w_f satisfy the aforementioned constraints as well. This was clearly indicated in an inset plot, where the constrained region corresponds to the coloured narrow stripe.

Without any assumption on the functional form of $\xi(\phi)$, we regarded the case of $\alpha_T = 0$ in Sec. 3.5.9, and analysed the stability of the scaling and de Sitter fixed points in particular given their relevance in late time cosmology. Despite the presence of a new time dependent variable $\Phi(t)$ in the equations, we found that de Sitter is only possible if $\dot{\phi} = 0$, same as in the SGB model. The subsequent analysis of the stability indicated that, whereas the de Sitter regime became unstable, the scaling one was represented by a stable fixed point, contradicting the past evolution of the Universe as we know it. Consequently, when $\alpha_T = 0$ for a non-constant coupling function $\xi(\phi)$, we encountered the same stability of **ScI** and **dS** as that in Ref. [3] (see Sec. 3.2.6).

Although one of the assumptions in this work relies on spatial isotropy, such that $A_i(t) = 0$, one might regard the presence of the spatial components of the Weyl vector so that they

are relevant in the dynamical analysis but subdominant enough not to induce a large scale anisotropy. When it comes to cosmic inflation, the subdominant $A_i(t)$ and even the fully dominant presence of those components were shown to be consistent with observations in Ref. [101]. It would be interesting to see if something similar holds in the late Universe in the WGB model. Also, we discarded the case of $\kappa = \lambda$. However, this was considered in Ref. [325] in the metric formalism and some solutions were found analytically (we mentioned this in Sec. 3.2.2). We leave a similar analytical treatment in the WGB model for future publications, although two fixed ‘curves’ were calculated at the end of Sec. 3.5.4 already. One of such ‘curves’ corresponds to a third scaling solution **ScIII** that might be of cosmological interest.

CHAPTER 4

CONCLUSIONS

Cosmological inflation stands out as the most likely candidate for a reliable description of the early Universe. As realised by Guth [50], the horizon and flatness problems of the Hot Big Bang (HBB) model are resolved within the framework of inflation. Moreover, inflation provides an explanation for the origin of the Large Scale Structures in the Universe by virtue of the theory of cosmological perturbations, the quantum theory of fields and the equivalence principle of General Relativity (GR). This and the fact that inflation can be realised by multitude of field models bridge the study of gravity and particle physics.

When confronted with observations, slow-roll (SR) models are the most favoured ones. In particular, those predicting an almost scale invariant spectrum of scalar perturbations, nearly Gaussian and adiabatic, fit *Planck* 2018 and lensing data well (see Fig. 1.9). The amplitude of tensor perturbations in particular is bounded to be at most of order $\mathcal{O}(0.01)$ times smaller than that of the scalar ones, and this allows to constrain the energy scale of inflation. The reheating process [69] following inflation is necessary given that we know with high accuracy that the Universe was dominated by radiation at the time of the early formation of the light chemical elements (the so-called Big Bang Nucleosynthesis (BBN)). BBN theory provides successful predictions regarding the abundances of those elements (see Fig. 1.4) and the Universe after inflation differs greatly from radiation domination. This period of reheating is still under investigation [71].

Gravity modifications play an important role in models of inflation at present. Starobinsky's R^2 model [163] is the oldest and yet most successful realisation of inflation when compared with observations. Another (related) example of modified gravity inflation model is the Higgs inflation [181] which exploits the presence of a field dependent, non-minimal coupling to gravity, $F(\phi)R$, expected from quantum field theories on curved spacetime (classical) backgrounds. These modified gravities belong to a class of theories called 'scalar-tensor' which were developed earlier than the moment at which the idea of inflation was conceived. The motivations were different back then [193], but these theories are now virtually ubiquitous in the study of early and late cosmology.

The most general scalar-tensor theory that leads to second order equations of the metric and the scalar field is Horndeski's gravity [346] and one relevant subclass is the scalar-Gauss-

Bonnet (SGB) model [347]. This gravity theory displays a coupling to the Gauss-Bonnet (GB) term $\mathcal{G} \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$. This combination of quadratic curvature scalars does not contribute to the field equations of the metric in four spacetime dimensions, but a field dependent coupling of the form $\xi(\phi)\mathcal{G}$ introduces non-trivial contributions in those dimensions.

The notion of conformal frames gains special relevance in scalar-tensor theories of the $f(\phi, R)$ type. These modified gravity models are essentially formulated in the so-called Jordan frame, where the scalar field does not couple to the matter sector but pertains to the gravity one, modifying the Einstein-Hilbert (EH) action of GR. However, a conformal rescaling of the metric (that is, one which does not alter the angles) transforms the gravity sector so it resembles that of GR at the expense of bringing in a coupling between the scalar field and the fields of the matter sector. The issue regarding the physical equivalence between this frame (known as ‘Einstein frame’) and Jordan’s can be addressed by noticing that a conformal transformation of the metric amounts to a transformation of the units of measure. This was illustrated in Deruelle and Sasaki’s paper [214] in the context of the ‘veiled’ version of GR and discussed in the paradigm of cosmic inflation in Ref. [215]. One then concludes that inflation, as a physical phenomenon, and hence independent of the units of the measure, occurs in all frames but the way it is interpreted changes depending on the frame. On this basis, by keeping the slicing fixed, we mapped SR inflation conditions, clearly defined in the Einstein frame, to the Jordan frame. We introduced three sets of flow parameters but found that only two are needed to describe the system with the most compact equations. We derived the SR equations in the Jordan frame and determined the SR conditions that must be satisfied in order for those to hold. The first order equation of $\dot{\phi}$ that was found to be the SR attractor in Ref. [223] was explicitly obtained from the homogeneous second order scalar field equation by applying the SR conditions consistently.

An advantage of the SR approximation is to express inflation observables, which allow to make predictions, in terms of homogeneous quantities alone. Therefore, we mapped the homogeneous relations in order to derive observable models predictions as well. There is also an issue regarding the number of e-folds N , an essential time coordinate during inflation, because of the difference between the predicted N in Jordan and Einstein frames. The difference is proportional to the logarithm of the varying Newton’s gravitational constant, which was illustrated in Fig. 2.2. However, this apparent issue is not problematic because the Jordan and Einstein frame numerical results were compared on the *same time slice*. These numerical simulations were performed assuming a generalisation of the induced gravity

inflation model [226, 227], where $F(\phi) = \xi\phi^2$ and the potential is the symmetry breaking one $V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$, ξ being a coupling constant and λ and v the self-coupling constant and the vacuum expectation value, respectively. These generalised induced gravity inflation models were classified into *hilltop-like* and *chaotic-like* regimes. The analytical results were summarised in Table 2.1. We found that, among the many regions where SR inflation was possible to be realised, only certain cases in the chaotic type setup fell within the 2σ region of the BICEP/Keck results [243] (see Fig. 2.4b). These cases under consideration were used to learn how to compute the main spectral properties in terms of Jordan frame variables by means of the formalism developed previously. The results after comparing Jordan and Einstein frame predictions (again, on the same time slice) can be found in Fig. 2.7. All this work summarised in this and the previous paragraph resulted in a paper which was published in Physical Review D. See Ref. [1].

Cosmological perturbations are subject to gauge transformations and to the conformal transformation too. One gauge invariant perturbation in particular is the linear comoving curvature perturbation $\mathcal{R}$, which corresponds to the linear curvature perturbation on comoving slicings, and it is of extreme importance in inflation [146]. In the case of $F(\phi)R$ gravity, this perturbation was shown to transform under the conformal rescaling of the metric when defined on hypersurfaces that comove with the effective fluid with conserved energy and momentum in the Jordan frame. We derived the conformal relation between $\mathcal{R}$ in Jordan and Einstein frames. This relation holds irrespective of the dynamics of the background but we applied it to SR and made use of the formalism developed in Ref. [1]. It was found that, while the conformal equivalence between the two frames does not hold on the subhorizon regime during SR inflation, $\mathcal{R}$ is conformally invariant on superhorizon scales up to a term which is quadratic in gradients. This conformal relation includes a ratio between the variation of the two time dependent background scales of the theory: the Hubble parameter H and the effective Planck's mass $\sqrt{F}$. This ratio is of order $\mathcal{O}(1)$ in the generalised induced gravity inflation models that agree with observations (the chaotic type ones). Thus, the equivalence of $\mathcal{R}$ shortly after horizon exit in those models is ensured. As a demonstration of this equivalence, we derived the expressions for the inflation observables of Ref. [1] using the comoving curvature perturbation of the Jordan frame and the analogues of Mukhanov-Sasaki (for scalar and tensor perturbations). We demonstrated that it is possible to formulate the inflation observables using $\mathcal{R}$ in Jordan and Einstein frames and hence the conformal equivalence on the superhorizon regime in SR inflation. As a result of this work, a paper was published in Physical Review D. See Ref. [2].

Some aspects of modified gravity involve the late Universe as well. An example is the SGB model which was applied to the study of the dark energy (DE) in Refs. [325, 327]. Late DE models have become quite popular after the observational discovery of the current accelerated expansion of the Universe from Type Ia supernovae [51, 58]. Although the cosmological constant (CC) is arguably the simplest candidate for DE, no convincing scenario has been found in which the observed energy scale of DE can be naturally accounted for by the vacuum energy associated with a particle physics model [8, 10]. Alternatively, quintessence [314, 322] models provide a sensible mechanism to explain DE while allowing for a unified description of the late accelerated expansion era and early inflation (called ‘quintessential inflation’ [337]). These lead to stable scaling solutions in GR, where the scalar field equation of state (EoS) parameter adjusts to that of the dominant component in the matter sector. The scalar field potential is exponential so equations become self-similar and the dependence on the Hubble parameter H disappears. The scaling solution is of cosmological relevance because it provides prolonged radiation and pressureless matter dominated eras. However, the stability of the scaling fixed point is troubling if one intends to describe the current accelerated expansion of the Universe. The SGB model provides an escape route when an exponential GB coupling function $\xi(\phi)$ is considered [328], leading to a stable de Sitter period following the scaling regime when the strength of the exponentials, κ for $\xi(\phi)$ and λ for $V(\phi)$, are such that $\kappa > \lambda$.

Nevertheless, these Gauss-Bonnet Dark Energy (GBDE) models are strongly constrained by the multi-messenger observation of a binary neutron star system’s merger reported in 2017 [293]. This merger occurred during DE domination [304] and is a low-redshift event, having consequences on DE models that modify the speed of gravitational waves (GWs). The delay between arrival times of the GWs and γ -rays emitted by the system’s merger [294] yielded the bound $|\alpha_T| < 10^{-15}$ on the α_T parameter defined as $\alpha_T \equiv c_{\text{GW}}^2 - 1$, in units where $c = 1$. The SGB model predicts $\alpha_T \neq 1$ generally, and this fact led some authors to exclude such DE models. However, we came to the realisation that the bound does not seem to be very constraining for those models. To be precise, we found that $\xi(t)$ and $\dot{\xi}(t)$ need to vary by more than 100 orders of magnitude over the age of the Universe to violate the observational bound on $|\alpha_T|$. Motivated by this, we performed a dynamical systems analysis (see the fixed points and ‘curves’ in Table 3.1) and examined the evolution of $|\alpha_T|$ connecting the scaling and de Sitter regimes, these represented by fixed points of the system that we labelled as **Sc** and **dS**, respectively. Although $\alpha_T = 0$ at **Sc** and **dS**, the current state of our Universe cannot be represented by neither of those points exactly, but we must find ourselves somewhere in between. If the density parameter of matter fields Ω_m is of

order $\mathcal{O}(0.1)$, corresponding to the current value, it was shown that $|\alpha_T| \sim \mathcal{O}(0.1)\text{--}\mathcal{O}(1)$ for numerous models with exponentials $\xi(\phi)$ and $V(\phi)$ (see Fig. 3.4). This is clearly ruled out by the constraint $|\alpha_T| < 10^{-15}$ and makes GBDE of this type unviable. We investigated the linear GB coupling case, which also makes the equations self-similar, but found that the de Sitter fixed point is a saddle and the scaling one an attractor. Consequently, no viable solution under this choice of the GB coupling function would be consistent with the past history of the Universe as we know it. Other coupling functions were left for future publications because the dynamical equations lose their self-similar character and are not amenable to the methods employed in this project.

Another possibility, which was considered too, is to impose $\alpha_T = 0$ and have no explicit functional form of $\xi(\phi)$ instead. We then obtained an additional constraint equation and studied the stability of the scaling and de Sitter fixed points (which still are fixed points of the system despite the new constraint; see Table 3.2). It was shown that the stability is similar to the one determined under the linear GB coupling function. Hence no solutions providing a long period of matter domination followed by an accelerated expansion comply with the imposed constraint $c_{\text{GW}} = c$. Since the constraint $|\alpha_T| < 10^{-15}$ is so strong, and in view of those results, we concluded that it might be very difficult to find GBDE models consistent with other cosmological observations if they are constrained to satisfy the bound on $|\alpha_T|$. Also, given that the SGB model action contains the lowest order term in the series of low energy effective string theory corrections [274, 280, 288], the fact that ξ varies so much (because $|\alpha_T|$ is predicted to be of order $\mathcal{O}(0.1)$ or larger) makes one wonder whether it is consistent to dismiss the higher order terms over the entire evolution of the Universe. We conclude that higher order operators might be relevant in line with the discussion in Ref. [350], where C. de Rham and S. Melville pointed out that the bound on $|\alpha_T|$ applies to a limited range of GW frequencies, higher than the range of energies in which Horndeski's DE models at large are applied actually. The work summarised in this and the two paragraphs above gave rise to a paper, published in the Journal of Cosmology and Astroparticle Physics. See Ref. [3].

While Horndeski's theory (and hence the SGB model) seems to be at odds with the observational constraint on the speed of GWs, this was recently found to be resolved in particular cases when formulated in the Palatini formalism [349]. To be precise, those Horndeski's models that consist of $G(\phi, X)R$ plus a so-called 'k-essence' term $K(\phi, X)$, $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ being the kinetic term, predict $c_{\text{GW}} = c$ in that formalism in sharp contrast to the prediction in the metric formalism, where c_{GW} does not coincide exactly with c because of the

dependence on X of the non-minimal coupling function G . Some quintessence models have been developed in the Palatini formalism already,¹ but the GBDE models in particular still lack a formulation in that formalism to the best of our knowledge.

In the Palatini formalism, the metric $g_{\mu\nu}(x)$ and the connection $\Gamma_{\mu\nu}^\alpha(x)$ are independent variables and the geometry differs from the pseudo-Riemannian one. The formulation of quadratic gravity in the Weyl geometry, where the connection is symmetric (so it does not generate torsion) but not metric compatible and includes a vector field A_μ , was considered in Ref. [373]. The Lagrangian of the theory is more complicated than that of the metric formalism because more quadratic curvature scalars appear due to the lack of symmetries which are granted by the metric formalism. We considered a similar Lagrangian and the same Weyl geometry but with a field dependent coupling function to the collection of quadratic curvature scalars, and focus on four dimensions (in the aforementioned paper, the dimensions were arbitrary). We named this model ‘Weyl-Gauss-Bonnet’ (WGB). Our goal was then to determine whether the SGB gravity, as a model of DE, when formulated in a geometry different from that prescribed by the metric formalism, fitted the bound on $|\alpha_T|$ and still was in accordance with other constraints (on the density and EoS parameters in particular). In order not to break spatial isotropy, we assumed a background Weyl vector A_μ which did not have non-vanishing spatial components, such that $A_\mu(t) \equiv (-\Phi(t), \mathbf{0})$. The corresponding α_T parameter of the theory was derived and we performed a dynamical systems analysis and found the fixed points of the autonomous system. We assumed the same exponential coupling and potential functions as in our previous project [3]. A stability analysis of the fixed points was carried out too. Among those points, the scaling **ScI** (where $\xi(\phi)$ is constant) and de Sitter **dS** ones had been found in the SGB model of the metric formalism already [3], and the results from the corresponding stability analysis were essentially the same in both models (the WGB and SGB ones). This ensured an evolution of the Universe consistent with cosmological observations; that is, from matter domination to accelerated expansion. On top of those and other known fixed points such as that representing a phase of the background fluid domination (**M**) or a phase of kinetic energy domination (**K $\pm$**), we found a second scaling solution consisting of two fixed points **ScII $\pm$** (where $V(\phi)$ vanishes). Remarkably, the fixed point **ScII+** was shown not to exist if the background fluid corresponds to radiation, among other values of the EoS parameter which do not include pressureless matter. This fixed point was found to be stable only if $\lambda > \kappa$ but the same cannot be concluded of **ScII-**.

1. See Refs. [381, 382], where $f(R)$ and $f(\phi, R)$ theories are considered, respectively. In the case of $F(\phi)R$ gravity, an example of quintessence scalar field couplings with metric and torsion was put forward in Ref. [383], where the growth of linear matter perturbations is examined.

Whenever $\kappa > \lambda$, we showed that **ScII** $\pm$ was negative definitely, and the unstable direction was determined to be that of the potential energy. Along with **ScII** $\pm$, we found a second couple of fixed points, $\hat{\mathbf{K}}\pm$, that we associated with a regime of ‘pseudo-kination’. We called it that way because an equation similar to the one during kination was obtained but with a new parameter, the Weyl parameter $W \equiv H - \Phi$, replacing the Hubble parameter. Regarding this regime, we found that $\hat{\mathbf{K}}+$ led to a universe that accelerated faster than during de Sitter, where it accelerates exponentially, if $\kappa < 2\sqrt{\frac{2}{3}}$. If $\kappa \gg 1$, both fixed points $\hat{\mathbf{K}}\pm$ led into a uniformly expanding universe. Both regimes, the super-accelerated and uniform expansion ones, were shown to be stable if $\kappa > \lambda$ and $\kappa < \lambda$, respectively. The summary of all the fixed points found in this project and their stability was given in Table 3.3.

With the aim of illustrating the dynamical behaviours determined previously, we did some numerical simulations assuming pressureless background matter, similarly to Ref. [3]. We looked into the trajectories crossing or approaching the fixed points **ScI** and **ScII** $\pm$ mostly, given the theoretical interest in the cosmological expansion history. When it comes to **ScI**, the system’s behaviour was reproduced as expected: from the initial period of kination, the Universe enters a long period of background fluid domination where the EoS parameter of the quintessence field adjusts so it mimics that of matter, and then a period of accelerated expansion with $\Omega_m \sim 0.1$ follows. We observed a common feature which was that the effective density parameter associated with the GB coupling, Ω_ξ , dominated and the quintessence field took on phantom values (i.e. smaller than -1). Regarding **ScII** $\pm$, we argued that **ScII** $+$ placed itself out of the region allowed by the constraints if Ω_m dominated over the sum of the rest of the density parameters (so BBN predictions were not altered). Some trajectories then approached and crossed the scaling fixed point **ScII** $-$ only but, unfortunately, the period of matter domination was found to be too short to agree with the past history of the Universe as we know it, making this second scaling regime unfeasible (for the range of parameters under analysis).

The evolution of the α_T parameter was determined via numerical simulations, and models that satisfied the constraints on Ω_m and the EoS parameter of the effective fluid causing the present accelerated expansion were chosen and plotted in Fig. 3.10 in the vein of Ref. [3]. Unfortunately, the constraint $|\alpha_T| < 10^{-15}$ was found to be grossly violated by many orders of magnitude, in a way similar to that reflected in Ref. [3]. The reason was that, while the numerator in the expression of α_T in the WGB model remained of order $\mathcal{O}(0.1)$ - $\mathcal{O}(1)$ when Ω_ξ comes to dominate, the denominator decreased and the ratio blew up. We then considered $\alpha_T = 0$, same as in Ref. [3]. Despite the presence of the new variable $\Phi(t)$ in the equations,

we found that the de Sitter regime is possible only if $\dot{\phi} = 0$, same as in the SGB model. We argued this by deriving the corresponding equation. Having this in mind, we studied the stability of the fixed points **ScI** and **dS** with the new constraint equation from $\alpha_T = 0$ and found essentially the same stability as that prescribed by the SGB model which, we remembered, was at odds with the known evolution of the Universe. Despite these negative results, we explained that many assumptions made in this work could be reconsidered, these including the fact that the components $A_i(t)$ vanish or are negligible. We concluded however that the results obtained in this work for the Weyl connection applied to a metric compatible connection but with non-zero torsion. As a result of this investigation, summarised in this and the three previous paragraphs, a paper was written and submitted for publication. This manuscript is under review at present, and the latest preprint can be found on the arxiv; see Ref. [4].

APPENDIX A

LINEAR PERTURBATIONS ABOUT THE FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER METRIC

In this section, we calculate the linear perturbations around a spatially-flat FLRW metric of all the important geometric quantities. We consider scalar, vector and tensor perturbations, the scalar ones being relevant in Sec. 1.3. The line element reads [24, 41]

$$ds^2 = -(1 + 2A) dt^2 - 2(\partial_i B + S_i) dx^i dt + a^2(t) \left[(1 - 2\psi) \delta_{ij} + 2(\partial_i \partial_j E + \partial_{(i} F_{j)}) + h_{ij} \right] dx^i dx^j, \quad (\text{A.1})$$

where $\partial^i S_i = \partial^i F_i = 0$ and $\partial_i h^i_j = h^i_i = 0$ (see below Eq. (1.283)). $\partial_{(i} F_{j)}$ means symmetrisation

$$\partial_{(i} F_{j)} \equiv \frac{1}{2} (\partial_i F_j + \partial_j F_i). \quad (\text{A.2})$$

The background Christoffel symbols which are non-vanishing read

$$\Gamma_{ij}^0 = a^2 H \delta_{ij}, \quad (\text{A.3})$$

$$\Gamma_{j0}^i = \Gamma_{0j}^i = H \delta_j^i. \quad (\text{A.4})$$

The perturbations at linear order are (see Eq. (C.1))

$$\delta\Gamma_{00}^0 = \dot{A}, \quad (\text{A.5})$$

$$\delta\Gamma_{i0}^0 = \delta\Gamma_{0i}^0 = \partial_i (A - HB) - HS_i, \quad (\text{A.6})$$

$$\delta\Gamma_{00}^i = a^{-2} \left[\partial^i (A - \dot{B}) - \dot{S}^i \right], \quad (\text{A.7})$$

$$\begin{aligned} \delta\Gamma_{ij}^0 = & -a^2 \left\{ \left[\dot{\psi} + 2H(\psi + A) \right] \delta_{ij} - 2H \partial_i \partial_j E \right\} + \partial_i \partial_j \chi + \partial_{(i} \chi_{j)} + 2H a^2 \partial_{(i} F_{j)} + \\ & + \frac{1}{2} a^2 (\dot{h}_{ij} + 2H h_{ij}), \end{aligned} \quad (\text{A.8})$$

$$\delta\Gamma_{0j}^i = \delta\Gamma_{j0}^i = -\dot{\psi} \delta_j^i + \partial^i \partial_j \dot{E} + a^{-2} \delta^{ik} \left(\partial_{[k} S_{j]} + a^2 \partial_{(k} \dot{F}_{j)} \right) + \frac{1}{2} \dot{h}^i_j, \quad (\text{A.9})$$

$$\begin{aligned}\delta\Gamma_{ij}^k &= \delta_{ij}\partial^k(\psi + HB) - \left(\delta_j^k\partial_i\psi + \delta_i^k\partial_j\psi\right) + \partial_i\partial_j\partial^k E + \partial_i\partial_j F^k + H\delta_{ij}S^k + \\ &+ \frac{1}{2}\left(\partial_i h^k{}_j + \partial_j h^k{}_i - \partial^k h_{ij}\right),\end{aligned}\tag{A.10}$$

where $\partial_{[k}S_{j]}$ means antisymmetrisation

$$\partial_{[k}S_{j]} \equiv \frac{1}{2}(\partial_k S_j - \partial_j S_k).\tag{A.11}$$

χ is defined in Eq. (1.161) and χ_i such that

$$\chi_i \equiv S_i + a^2 \dot{F}_i,\tag{A.12}$$

is the vector shear perturbation [41]. In contrast to χ , χ_i is gauge-invariant. To see this, we parametrise the gauge transformation vector ξ^μ as

$$\xi^\mu \equiv (0, \beta^i),\tag{A.13}$$

such that $\partial_i \beta^i = 0$. Then, from Eq. (1.154), we obtain

$$S_i \rightarrow S_i + a^2 \dot{\beta}_i,\tag{A.14}$$

$$F_i \rightarrow F_i - \beta_i.\tag{A.15}$$

Finally, h_{ij} can be shown to be gauge-invariant (see Eq. (1.284)).

The Riemann tensor background components which do not vanish are

$$R^0{}_{i0j} = -R^0{}_{ij0} = a^2 (\dot{H} + H^2) \delta_{ij},\tag{A.16}$$

$$R^i{}_{00j} = -R^i{}_{0j0} = (\dot{H} + H^2) \delta_j^i,\tag{A.17}$$

$$R^i{}_{jkl} = a^2 H^2 (\delta_k^i \delta_{jl} - \delta_l^i \delta_{jk}).\tag{A.18}$$

The linear perturbations that are non-vanishing can be easily calculated once we have the perturbed Christoffel symbols (see Eq. (C.2)). These are

$$\delta R^0{}_{00i} = -\delta R^0{}_{0i0} = -(\dot{H} + H^2)(\partial_i B + S_i),\tag{A.19}$$

$$\begin{aligned}\delta R^0{}_{i0j} &= -\delta R^0{}_{ij0} = -a^2 \left[\ddot{\psi} + H(2\dot{\psi} + \dot{A}) \right] \delta_{ij} + \partial_i \partial_j (\dot{\chi} - A) - \\ &- 2a^2 (\dot{H} + H^2) \left[(\psi + A) \delta_{ij} - \partial_i \partial_j E - \partial_{(i} F_{j)} \right] + \partial_{(i} \dot{\chi}_{j)} +\end{aligned}$$

$$+ \frac{1}{2} a^2 \left[\ddot{h}_{ij} + 2H\dot{h}_{ij} + 2 \left(\dot{H} + H^2 \right) h_{ij} \right], \quad (\text{A.20})$$

$$\delta R^i_{00j} = -\delta R^i_{0j0} = - \left[\ddot{\psi} + H \left(2\dot{\psi} + \dot{A} \right) \right] \delta^i_j + a^{-2} \left[\partial_j \partial^i (\dot{\chi} - A) + \delta^{ik} \partial_{(k} \dot{\chi}_{j)} \right] + \frac{1}{2} \left(\ddot{h}^i_j + 2H\dot{h}^i_j \right), \quad (\text{A.21})$$

$$\delta R^i_{jk0} = -\delta R^i_{j0k} = \delta^i_k \left[\partial_j \left(\dot{\psi} + HA \right) - H^2 \left(\partial_j B + S_j \right) \right] - \delta_{jk} \left[\partial^i \left(\dot{\psi} + HA \right) + \dot{H} \left(\partial^i B + S^i \right) \right] - a^{-2} \delta^{is} \partial_k \partial_{[j} \chi_{s]} - \frac{1}{2} \left(\partial_j \dot{h}^i_k - \partial^i \dot{h}_{jk} \right), \quad (\text{A.22})$$

$$\delta R^i_{0jk} = -\delta^i_k \left[\partial_j \left(\dot{\psi} + HA \right) - H^2 \left(\partial_j B + S_j \right) \right] + \delta^i_j \left[\partial_k \left(\dot{\psi} + HA \right) - H^2 \left(\partial_k B + S_k \right) \right] + a^{-2} \partial^i \partial_{[j} \chi_{k]} + \frac{1}{2} \left(\partial_j \dot{h}^i_k - \partial_k \dot{h}^i_j \right), \quad (\text{A.23})$$

$$\delta R^0_{ijk} = a^2 \left[\delta_{ij} \partial_k \left(\dot{\psi} + HA \right) - \delta_{ik} \partial_j \left(\dot{\psi} + HA \right) \right] + \partial_i \partial_{[j} \chi_{k]} - \frac{1}{2} a^2 \left(\partial_k \dot{h}_{ij} - \partial_j \dot{h}_{ik} \right), \quad (\text{A.24})$$

$$\begin{aligned} \delta R^i_{jkl} &= \delta^i_k \partial_l \partial_j \psi - \delta^i_l \partial_k \partial_j \psi + \delta_{jl} \partial_k \partial^i \psi - \delta_{jk} \partial_l \partial^i \psi - 2a^2 H \left(\delta^i_k \delta_{jl} - \delta^i_l \delta_{jk} \right) \times \\ &\times \left[\dot{\psi} + H \left(\dot{\psi} + A \right) \right] + H \left(\delta^i_k \partial_l \partial_j \chi - \delta^i_l \partial_k \partial_j \chi + \delta_{jl} \partial_k \partial^i \chi - \delta_{jk} \partial_l \partial^i \chi \right) + \\ &+ 2a^2 H^2 \left(\delta^i_k \partial_l \partial_j E - \delta^i_l \partial_k \partial_j E \right) + H \left(\delta^i_k \partial_{(l} \chi_{j)} - \delta^i_l \partial_{(k} \chi_{j)} + \delta_{jl} \delta^{is} \partial_{(s} \chi_{k)} - \delta_{jk} \delta^{is} \partial_{(s} \chi_{l)} \right) + \\ &+ 2a^2 H^2 \left(\delta^i_k \partial_{(l} F_{j)} - \delta^i_l \partial_{(k} F_{j)} \right) + \frac{1}{2} \left(\partial_j \partial_k \dot{h}^i_l - \partial_k \partial^i h_{lj} - \partial_j \partial_l \dot{h}^i_k + \partial_l \partial^i h_{kj} \right) + \\ &+ \frac{1}{2} H a^2 \left[\delta^i_k \left(\dot{h}_{lj} + 2H h_{lj} \right) - \delta^i_l \left(\dot{h}_{kj} + 2H h_{kj} \right) + \delta_{jl} \dot{h}^i_k - \delta_{jk} \dot{h}^i_l \right]. \end{aligned} \quad (\text{A.25})$$

In regard to the Ricci tensor, the background values which are non-vanishing read

$$R_{00} = -3 \left(\dot{H} + H^2 \right), \quad (\text{A.26})$$

$$R_{ij} = a^2 \left(\dot{H} + 3H^2 \right) \delta_{ij}. \quad (\text{A.27})$$

The linear perturbations are ($\delta R_{\mu\nu} \equiv \delta R^\sigma_{\mu\sigma\nu}$)

$$\delta R_{00} = 3 \left[\ddot{\psi} + H \left(2\dot{\psi} + \dot{A} \right) \right] - a^{-2} \partial_i \partial^i (\dot{\chi} - A), \quad (\text{A.28})$$

$$\delta R_{i0} = \delta R_{0i} = 2\partial_i \left(\dot{\psi} + HA \right) - \left(\dot{H} + 3H^2 \right) \left(\partial_i B + S_i \right) + \frac{1}{2} a^{-2} \partial_j \partial^j \chi_i, \quad (\text{A.29})$$

$$\begin{aligned} \delta R_{ij} &= -a^2 \left[\ddot{\psi} + H \left(6\dot{\psi} + \dot{A} \right) + 2 \left(\dot{H} + 3H^2 \right) \left(\dot{\psi} + A \right) \right] \delta_{ij} + \partial_i \partial_j \psi + \delta_{ij} \partial_k \partial^k \psi + \\ &+ \partial_i \partial_j (\dot{\chi} - A) + H \left(\partial_i \partial_j \chi + \delta_{ij} \partial_k \partial^k \chi \right) + 2a^2 \left(\dot{H} + 3H^2 \right) \left(\partial_i \partial_j E + \partial_{(i} F_{j)} \right) + \partial_{(i} \dot{\chi}_{j)} + \\ &+ H \partial_{(i} \chi_{j)} + \frac{1}{2} a^2 \left[\ddot{h}_{ij} + 3H\dot{h}_{ij} - a^{-2} \partial_k \partial^k h_{ij} + 2 \left(\dot{H} + 3H^2 \right) h_{ij} \right]. \end{aligned} \quad (\text{A.30})$$

The Ricci scalar has the following homogeneous value:

$$R = 12H^2 + 6\dot{H}, \quad (\text{A.31})$$

and the linear perturbation is ($\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}$)

$$\delta R = 2 \left[\dot{\kappa} + 4H\kappa - \left(a^{-2} \partial_i \partial^i A + 3\dot{H}A \right) + 2a^{-2} \partial_i \partial^i \psi \right], \quad (\text{A.32})$$

κ being the scalar expansion rate perturbation at linear order in the comoving frame, where $V = 0$ (see Eq. (1.175)):

$$\kappa \equiv -3 \left(HA + \dot{\psi} \right) + a^{-2} \partial_i \partial^i \chi. \quad (\text{A.33})$$

The non-vanishing components of the Einstein tensor, defined in Eq. (1.9), are

$$G_{00} = 3H^2, \quad (\text{A.34})$$

$$G_{ij} = -a^2 \left(2\dot{H} + 3H^2 \right) \delta_{ij}, \quad (\text{A.35})$$

and the linear perturbations are

$$\delta G_{00} = \dot{\kappa} + 4H\kappa + 3 \left[\ddot{\psi} + H \left(2\dot{\psi} + \dot{A} \right) + \left(\dot{H} + 4H^2 \right) A \right] - a^{-2} \partial_i \partial^i (\dot{\chi} - 2\psi), \quad (\text{A.36})$$

$$\delta G_{i0} = \delta G_{0i} = 2\partial_i \left(\dot{\psi} + HA \right) + \left(2\dot{H} + 3H^2 \right) (\partial_i B + S_i) + \frac{1}{2} a^{-2} \partial_j \partial^j \chi_i, \quad (\text{A.37})$$

$$\begin{aligned} \delta G_{ij} = & -a^2 \left[\dot{\kappa} + 3H\kappa + \ddot{\psi} + H \left(3\dot{\psi} + \dot{A} \right) - \dot{H} (4\psi + A) - 3H^2 (2\psi - A) \right] \delta_{ij} + \\ & + \partial_i \partial_j \psi - \delta_{ij} \partial_k \partial^k \psi - \partial_i \partial_j A + \delta_{ij} \partial_k \partial^k A + \partial_i \partial_j \dot{\chi} + H \partial_i \partial_j \chi - 2a^2 \left(2\dot{H} + 3H^2 \right) \times \\ & \times \left(\partial_i \partial_j E + \partial_{(i} F_{j)} \right) + \partial_{(i} \dot{\chi}_{j)} + H \partial_{(i} \chi_{j)} + \frac{1}{2} a^2 \left[\ddot{h}_{ij} + 3H \dot{h}_{ij} - a^{-2} \partial_k \partial^k h_{ij} - \right. \\ & \left. - 2 \left(2\dot{H} + 3H^2 \right) h_{ij} \right]. \end{aligned} \quad (\text{A.38})$$

APPENDIX B

FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER METRIC WITH LAPSE FUNCTION $\mathcal{N}$

In this brief section of the **Appendix**, we derive the homogeneous geometric variables when the spatially-flat FLRW metric includes an arbitrary lapse function $\mathcal{N}$. The line element is then

$$ds^2 = -\mathcal{N}^2 dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \quad (\text{B.1})$$

the determinant being $\sqrt{-g} = \mathcal{N}a^3$. The Christoffel symbols that do not vanish are

$$\Gamma_{00}^0 = \frac{\dot{\mathcal{N}}}{\mathcal{N}}, \quad (\text{B.2})$$

$$\Gamma_{j0}^i = \Gamma_{0j}^i = \mathcal{N}H\delta_j^i, \quad (\text{B.3})$$

$$\Gamma_{ij}^0 = \frac{H}{\mathcal{N}} g_{ij}, \quad (\text{B.4})$$

where $g_{ij} = a^2 \delta_{ij}$ and $H = \dot{a}/(a\mathcal{N})$.

The non-vanishing Riemann tensor components are

$$R_{i00j} = R_{0ij0} = -R_{i0j0} = -R_{0i0j} = \mathcal{N}^2 \left(\frac{\dot{H}}{\mathcal{N}} + H^2 \right) g_{ij}, \quad (\text{B.5})$$

$$R_{ijkl} = H^2 (g_{ik}g_{jl} - g_{il}g_{jk}). \quad (\text{B.6})$$

The non-zero Ricci tensor components read

$$R_{00} = -3\mathcal{N}^2 \left(\frac{\dot{H}}{\mathcal{N}} + H^2 \right), \quad (\text{B.7})$$

$$R_{ij} = \left(\frac{\dot{H}}{\mathcal{N}} + 3H^2 \right) g_{ij}. \quad (\text{B.8})$$

Finally, the Ricci scalar is

$$R = 6 \left(\frac{\dot{H}}{\mathcal{N}} + 2H^2 \right), \quad (\text{B.9})$$

and we can calculate the Einstein tensor components

$$G_{00} = 3\mathcal{N}^2 H^2, \quad (\text{B.10})$$

$$G_{ij} = - \left(2\frac{\dot{H}}{\mathcal{N}} + 3H^2 \right) g_{ij} . \quad (\text{B.11})$$

APPENDIX C

VARIATION OF THE GAUSS-BONNET TERM IN THE METRIC FORMALISM

We consider the gravitational action in Eq. (3.2) where $\mathcal{G}$ is the GB term defined in Eq. (3.1) and ξ a dimensionless constant. To derive the field equations, we first vary the coefficients of the LC connection $\Gamma_{\mu\nu}^\alpha$

$$\delta\Gamma_{\mu\nu}^\alpha = \frac{1}{2}g^{\alpha\beta} (\nabla_\mu\delta g_{\beta\nu} + \nabla_\nu\delta g_{\beta\mu} - \nabla_\beta\delta g_{\mu\nu}), \quad (\text{C.1})$$

which is a tensor as it corresponds to the difference between two connections. The variation of the Riemann tensor $R^\alpha_{\beta\mu\nu}$ reads

$$\delta R^\alpha_{\beta\mu\nu} = \nabla_\mu\delta\Gamma_{\nu\beta}^\alpha - \nabla_\nu\delta\Gamma_{\mu\beta}^\alpha, \quad (\text{C.2})$$

or, in terms of the metric tensor

$$\delta R^\alpha_{\beta\mu\nu} = g^{\alpha\sigma} \left(\nabla_{[\mu} \nabla_{\nu]} \delta g_{\sigma\beta} + \nabla_\mu \nabla_{[\beta} \delta g_{\sigma]\nu} - \nabla_\nu \nabla_{[\beta} \delta g_{\sigma]\mu} \right). \quad (\text{C.3})$$

Therefore, the variation of the Ricci tensor $R_{\mu\nu} \equiv R^\sigma_{\mu\sigma\nu}$ is

$$\delta R_{\mu\nu} = g^{\alpha\beta} \left(\nabla_\alpha \nabla_{(\mu} \delta g_{\nu)\beta} - \frac{1}{2} \nabla_\nu \nabla_\mu \delta g_{\alpha\beta} \right) - \frac{1}{2} \square \delta g_{\mu\nu}, \quad (\text{C.4})$$

where $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the d'Alembert operator introduced in Sec. 1.2.3. The variation of the Ricci scalar $R \equiv g^{\mu\nu} R_{\mu\nu}$ is given by

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + \nabla_\alpha \left(2g^{\alpha[\mu} g^{\beta]\nu} \nabla_\beta \delta g_{\mu\nu} \right) = R_{\mu\nu} \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu} \square \delta g^{\mu\nu}. \quad (\text{C.5})$$

We define the vector γ^μ as

$$\gamma^\mu \equiv g^{\alpha\beta} \delta\Gamma_{\beta\alpha}^\mu - g^{\mu\alpha} \delta\Gamma_{\sigma\alpha}^\sigma = 2g^{\mu[\alpha} g^{\sigma]\beta} \nabla_\sigma \delta g_{\alpha\beta}, \quad (\text{C.6})$$

and we notice that the covariant divergence of a vector multiplied by the square root of the

determinant of the metric becomes a total derivative

$$\sqrt{-g}\nabla_\mu\gamma^\mu = \partial_\mu(\sqrt{-g}\gamma^\mu) - \gamma^\mu\partial_\mu(\sqrt{-g}) + \sqrt{-g}\Gamma_{\mu\nu}^\mu\gamma^\nu = \partial_\mu(\sqrt{-g}\gamma^\mu), \quad (\text{C.7})$$

where the Jacobi's formula was used (see Eq. (1.75))

$$\Gamma_{\sigma\mu}^\sigma = -\frac{1}{2}g_{\alpha\beta}\partial_\mu g^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}). \quad (\text{C.8})$$

Consequently, the variation of the $\sqrt{-g}R$ term reads

$$\delta(\sqrt{-g}R) = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} = \sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}, \quad (\text{C.9})$$

up to the total derivative which is not included.

Regarding the GB term, we may analyse each term separately:

- R^2 term:

$$\delta(\sqrt{-g}R^2) = 2\sqrt{-g}R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right)\delta g^{\mu\nu} + 2\sqrt{-g}R\nabla_\mu\gamma^\mu. \quad (\text{C.10})$$

The second term cannot be converted into a total derivative due to R . We then have to use the chain rule

$$2\sqrt{-g}R\nabla_\mu\gamma^\mu = -2\sqrt{-g}(\nabla_\mu\partial_\nu R - g_{\mu\nu}\square R)\delta g^{\mu\nu}, \quad (\text{C.11})$$

up to total derivatives that have been ignored. Thus

$$\delta(\sqrt{-g}R^2) = 2\sqrt{-g}(RS_{\mu\nu} - \nabla_\mu\partial_\nu R + g_{\mu\nu}\square R)\delta g^{\mu\nu}, \quad (\text{C.12})$$

where we introduced the traceless tensor $S_{\mu\nu}$ in four spacetime dimensions

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R. \quad (\text{C.13})$$

- $R^{\mu\nu}R_{\mu\nu}$ term: we simply notice that $R^{\mu\nu}R_{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta}R_{\mu\nu}$, and then

$$\delta(\sqrt{-g}R^{\mu\nu}R_{\mu\nu}) = 2\sqrt{-g}\left(R_{\mu\alpha}R^\alpha{}_\nu - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}\right)\delta g^{\mu\nu} + 2\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu}. \quad (\text{C.14})$$

The second term yields

$$2\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu} = 2\sqrt{-g}R^{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\mu\delta g_{\beta\nu} - \sqrt{-g}R^{\mu\nu}\left(g^{\alpha\beta}\nabla_\mu\nabla_\nu\delta g_{\alpha\beta} + \square\delta g_{\mu\nu}\right). \quad (\text{C.15})$$

We apply the chain rule and obtain

$$2\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\left(\square R_{\mu\nu} + g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta} - 2\nabla_\alpha\nabla_\nu R^\alpha{}_\mu\right)\delta g^{\mu\nu}, \quad (\text{C.16})$$

where we have ignored the total derivatives. Therefore

$$\begin{aligned} \delta(\sqrt{-g}R^{\mu\nu}R_{\mu\nu}) &= \sqrt{-g}\left[2\left(R_{\mu\alpha}R^\alpha{}_\nu - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}\right) - 2\nabla_\alpha\nabla_\nu R^\alpha{}_\mu + \right. \\ &\quad \left. + g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta} + \square R_{\mu\nu}\right]\delta g^{\mu\nu}. \end{aligned} \quad (\text{C.17})$$

- $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ term: this term can be written as

$$R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = g^{\beta\lambda}g^{\mu\sigma}g^{\nu\rho}g_{\alpha\delta}R^\alpha{}_{\lambda\sigma\rho}R^\delta{}_{\beta\mu\nu}. \quad (\text{C.18})$$

Then

$$\begin{aligned} \delta(\sqrt{-g}R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}) &= 2\sqrt{-g}\left(R_{\alpha\beta\sigma\mu}R^{\alpha\beta\sigma}{}_\nu - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta\sigma\rho}R_{\alpha\beta\sigma\rho}\right)\delta g^{\mu\nu} + \\ &\quad + 2\sqrt{-g}R_\alpha{}^{\beta\mu\nu}\delta R^\alpha{}_{\beta\mu\nu}, \end{aligned} \quad (\text{C.19})$$

where we exploited the symmetries of the Riemann tensor to simplify the result. The second term becomes

$$2\sqrt{-g}R_\alpha{}^{\beta\mu\nu}\delta R^\alpha{}_{\beta\mu\nu} = 4\sqrt{-g}R^{\alpha\beta\mu\nu}\nabla_\mu\nabla_\beta\delta g_{\alpha\nu}, \quad (\text{C.20})$$

where

$$R^{\alpha\beta\mu\nu}\nabla_\mu\nabla_\nu\delta g_{\alpha\beta} = 0. \quad (\text{C.21})$$

Applying the chain rule

$$2\sqrt{-g}\xi R_\alpha{}^{\beta\mu\nu}\delta R^\alpha{}_{\beta\mu\nu} = 4\sqrt{-g}\nabla_\alpha\nabla_\beta R_\mu{}^\alpha{}_\nu{}^\beta\delta g^{\mu\nu}, \quad (\text{C.22})$$

where the total derivatives are omitted. Hence

$$\begin{aligned} \delta(\sqrt{-g}R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}) &= 2\sqrt{-g}\left(R_{\alpha\beta\sigma\mu}R^{\alpha\beta\sigma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta\sigma\rho}R_{\alpha\beta\sigma\rho} + \right. \\ &\quad \left. + 2\nabla_{\alpha}\nabla_{\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\right)\delta g^{\mu\nu}. \end{aligned} \quad (\text{C.23})$$

Then, the variation of the GB term yields (up to the total derivatives)

$$\begin{aligned} \delta(\sqrt{-g}\mathcal{G}) &= 2\sqrt{-g}\left[RS_{\mu\nu} - 4\left(R_{\mu\alpha}R^{\alpha}{}_{\nu} - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}\right) + R_{\alpha\beta\sigma\mu}R^{\alpha\beta\sigma}{}_{\nu} - \right. \\ &\quad \left. - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta\sigma\rho}R_{\alpha\beta\sigma\rho} - \nabla_{\mu}\partial_{\nu}R + 4\nabla_{\alpha}\nabla_{\nu}R^{\alpha}{}_{\mu} - 2g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}R^{\alpha\beta} - 2\Box G_{\mu\nu} + \right. \\ &\quad \left. + 2\nabla_{\alpha}\nabla_{\beta}R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\right]\delta g^{\mu\nu}, \end{aligned} \quad (\text{C.24})$$

or

$$\begin{aligned} \delta(\sqrt{-g}\mathcal{G}) &= 2\sqrt{-g}\left[RR_{\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} + R_{\alpha\beta\sigma\mu}R^{\alpha\beta\sigma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{G} + 4\nabla_{\alpha}\nabla_{\nu}S^{\alpha}{}_{\mu} - \right. \\ &\quad \left. - 2\Box G_{\mu\nu} - 2\nabla_{\alpha}\nabla_{\beta}\left(g_{\mu\nu}R^{\alpha\beta} - R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\right)\right]\delta g^{\mu\nu}. \end{aligned} \quad (\text{C.25})$$

If we interchange the covariant derivatives of $\nabla_{\alpha}\nabla_{\nu}S^{\alpha}{}_{\mu}$

$$\nabla_{\alpha}\nabla_{\nu}S^{\alpha}{}_{\mu} = \nabla_{\nu}\nabla_{\alpha}S^{\alpha}{}_{\mu} + R_{\sigma\nu}S^{\sigma}{}_{\mu} - R^{\sigma}{}_{\mu\alpha\nu}S^{\alpha}{}_{\sigma} = \nabla_{\nu}\nabla_{\alpha}S^{\alpha}{}_{\mu} + R_{\sigma\nu}R^{\sigma}{}_{\mu} - R^{\sigma}{}_{\mu\alpha\nu}R^{\alpha}{}_{\sigma}, \quad (\text{C.26})$$

where we used Eq. (2.17) applied to tensors [23]

$$\begin{aligned} [\nabla_{\mu}, \nabla_{\nu}]T^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m} &= R^{\alpha_1}{}_{\sigma\mu\nu}T^{\sigma\alpha_2\ldots\alpha_n}{}_{\beta_1\ldots\beta_m} + \ldots + R^{\alpha_n}{}_{\sigma\mu\nu}T^{\alpha_1\ldots\alpha_{n-1}\sigma}{}_{\beta_1\ldots\beta_m} - \\ &\quad - R^{\sigma}{}_{\beta_1\mu\nu}T^{\alpha_1\ldots\alpha_n}{}_{\sigma\beta_2\ldots\beta_m} - \ldots - R^{\sigma}{}_{\beta_m\mu\nu}T^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_{m-1}\sigma}. \end{aligned} \quad (\text{C.27})$$

Bianchi's identity (see Eq. (2.13)) gives

$$\nabla_{\nu}\nabla_{\alpha}S^{\alpha}{}_{\mu} = \frac{1}{4}\nabla_{\nu}\partial_{\mu}R, \quad (\text{C.28})$$

and hence

$$\nabla_{\alpha}\nabla_{\nu}S^{\alpha}{}_{\mu} = \frac{1}{4}\nabla_{\mu}\partial_{\nu}R + R_{\mu\alpha}R^{\alpha}{}_{\nu} - R^{\alpha\beta}R_{\alpha\mu\beta\nu}. \quad (\text{C.29})$$

Bianchi's identity and the symmetries of the Riemann tensor lead to

$$\nabla_\alpha \nabla_\beta R_\mu{}^\alpha{}_\nu{}^\beta = \nabla_\alpha \nabla_\beta R_\nu{}^\beta{}_\mu{}^\alpha = -\nabla_\alpha \nabla_\mu R^\alpha{}_\nu + \square R_{\mu\nu} . \quad (\text{C.30})$$

We interchange the covariant derivatives of $\nabla_\alpha \nabla_\mu R^\alpha{}_\nu$

$$\nabla_\alpha \nabla_\mu R^\alpha{}_\nu = \frac{1}{2} \nabla_\mu \partial_\nu R + R_{\mu\alpha} R^\alpha{}_\nu - R^{\alpha\beta} R_{\alpha\mu\beta\nu} , \quad (\text{C.31})$$

and then

$$\nabla_\alpha \nabla_\beta R_\mu{}^\alpha{}_\nu{}^\beta = -\frac{1}{2} \nabla_\mu \partial_\nu R - R_{\mu\alpha} R^\alpha{}_\nu + R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \square R_{\mu\nu} . \quad (\text{C.32})$$

Bianchi's identity again gives

$$g_{\mu\nu} \nabla_\alpha \nabla_\beta R^{\alpha\beta} = \frac{1}{2} g_{\mu\nu} \square R . \quad (\text{C.33})$$

Hence Eq. (C.25) becomes

$$\delta(\sqrt{-g}\mathcal{G}) = 2\sqrt{-g} \left(R R_{\mu\nu} - 2R_{\mu\alpha} R^\alpha{}_\nu - 2R^{\alpha\beta} R_{\alpha\mu\beta\nu} + R_{\alpha\beta\sigma\mu} R^{\alpha\beta\sigma}{}_\nu - \frac{1}{4} g_{\mu\nu} \mathcal{G} \right) \delta g^{\mu\nu} . \quad (\text{C.34})$$

APPENDIX D

GRAVITATIONAL WAVES EQUATION IN THE SCALAR-GAUSS-BONNET MODEL OBTAINED FROM THE ACTION

In this section of the **Appendix**, we derive the GWs equation in the SGB model again but, rather than perturbing the equations, we now perturb the action and retain quadratic terms.

Since we perturb around the FLRW metric, and this metric is conformally equivalent to the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, we shall perform a conformal rescaling of the metric and perturb around Minkowski, which is easier. Given the metric $g_{\mu\nu}$, we have

$$g_{\mu\nu} = a^2(\eta) (\eta_{\mu\nu} + h_{\mu\nu}), \quad (\text{D.1})$$

where $|h_{\mu\nu}| \ll 1$ and η is the conformal time, introduced in Eq. (1.12). The conformal factor is the homogeneous scale factor a . The inverse metric reads (see the appendix of Ref. [289])²

$$g^{\mu\nu} = a^{-2} \left(\eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda} h_{\lambda}^{\nu} \right), \quad (\text{D.2})$$

where the quadratic terms are included. The square root of the determinant of the metric is given by (up to quadratic terms)

$$\sqrt{-g} = a^4 \left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} \right), \quad (\text{D.3})$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ is the trace of $h_{\mu\nu}$. Given these expressions, it is not hard to find the Christoffel symbols perturbed around Minkowski up to quadratic terms. These are

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} \left(\eta^{\alpha\beta} - h^{\alpha\beta} \right) \left(\partial_{\mu}h_{\beta\nu} + \partial_{\nu}h_{\beta\mu} - \partial_{\beta}h_{\mu\nu} \right). \quad (\text{D.4})$$

2. Notice that, following the same convention as in the appendix of Ref. [289], we are considering that the spacetime dependent inverse metric reads $\bar{g}^{\mu\nu} - \delta g^{\mu\nu}$, $\bar{g}^{\mu\nu}$ being the background inverse metric and $\delta g^{\mu\nu} = \bar{g}^{\mu\sigma} \bar{g}^{\nu\rho} \delta g_{\sigma\rho}$. The minus sign is shown explicitly already, in contrast to what has been our convention in earlier calculations, where $\delta g^{\mu\nu} = -\bar{g}^{\mu\sigma} \bar{g}^{\nu\rho} \delta g_{\sigma\rho}$, and therefore the full inverse metric read $\bar{g}^{\mu\nu} + \delta g^{\mu\nu}$. Neither of these choices is wrong as long as the convention is consistently applied throughout the calculation.

The Riemann tensor is

$$\begin{aligned}
R^\alpha_{\beta\mu\nu} = & \frac{1}{2} (\eta^{\alpha\sigma} - h^{\alpha\sigma}) (\partial_\mu \partial_\beta h_{\sigma\nu} - \partial_\nu \partial_\beta h_{\sigma\mu} - \partial_\mu \partial_\sigma h_{\beta\nu} + \partial_\nu \partial_\sigma h_{\beta\mu}) + \\
& + \frac{1}{4} [\partial_\nu h^{\alpha\sigma} (\partial_\mu h_{\sigma\beta} + \partial_\beta h_{\sigma\mu} - \partial_\sigma h_{\beta\mu}) - \partial_\mu h^{\alpha\sigma} (\partial_\nu h_{\sigma\beta} + \partial_\beta h_{\sigma\nu} - \partial_\sigma h_{\beta\nu}) - \\
& - \partial_\sigma h^\alpha_{\nu} (\partial_\mu h^\sigma_{\beta} + \partial_\beta h^\sigma_{\mu} - \partial^\sigma h_{\beta\mu}) + \partial_\sigma h^\alpha_{\mu} (\partial_\nu h^\sigma_{\beta} + \partial_\beta h^\sigma_{\nu} - \partial^\sigma h_{\beta\nu}) + \\
& + \partial^\alpha h_{\nu\sigma} (\partial_\mu h^\sigma_{\beta} + \partial_\beta h^\sigma_{\mu} - \partial^\sigma h_{\beta\mu}) - \partial^\alpha h_{\mu\sigma} (\partial_\nu h^\sigma_{\beta} + \partial_\beta h^\sigma_{\nu} - \partial^\sigma h_{\beta\nu})], \quad (D.5)
\end{aligned}$$

while the Ricci tensor reads

$$\begin{aligned}
R_{\mu\nu} = & \frac{1}{2} (\eta^{\alpha\beta} - h^{\alpha\beta}) (\partial_\alpha \partial_\mu h_{\beta\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\alpha \partial_\beta h_{\mu\nu}) + \\
& + \frac{1}{2} [\partial^\alpha h^\beta_{\nu} (\partial_\alpha h_{\beta\mu} - \partial_\beta h_{\alpha\mu}) - \partial_\alpha h^{\alpha\beta} (\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu})] + \\
& + \frac{1}{4} [\partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} + \partial_\alpha h (\partial_\mu h^\alpha_{\nu} + \partial_\nu h^\alpha_{\mu} - \partial^\alpha h_{\mu\nu})], \quad (D.6)
\end{aligned}$$

and the Ricci scalar is given by

$$\begin{aligned}
R = & \partial_\mu \partial_\nu h^{\mu\nu} - \square h + h^{\mu\nu} \square h_{\mu\nu} - 2h^{\mu\nu} \partial_\mu \partial_\sigma h^\sigma_{\nu} + h^{\mu\nu} \partial_\mu \partial_\nu h - \frac{1}{4} \partial_\sigma h \partial^\sigma h + \partial_\mu h^{\mu\nu} \partial_\nu h + \\
& + \frac{3}{4} \partial^\sigma h^{\mu\nu} \partial_\sigma h_{\mu\nu} - \frac{1}{2} \partial^\sigma h^{\mu\nu} \partial_\mu h_{\sigma\nu} - \partial_\mu h^{\mu\nu} \partial_\sigma h^\sigma_{\nu}, \quad (D.7)
\end{aligned}$$

where $\square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$.

Assuming only TT tensor perturbations, the expressions of the Ricci tensor and Ricci scalar above become greatly simplified, and correspond to the following ones:

$$\begin{aligned}
R^{\text{TT}}_{\mu\nu} = & -\frac{1}{2} \square h_{\mu\nu} - \frac{1}{2} h^{\alpha\beta} (\partial_\alpha \partial_\mu h_{\beta\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\mu \partial_\nu h_{\alpha\beta} - \partial_\alpha \partial_\beta h_{\mu\nu}) + \\
& + \frac{1}{2} \partial^\alpha h^\beta_{\nu} (\partial_\alpha h_{\beta\mu} - \partial_\beta h_{\alpha\mu}) + \frac{1}{4} \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta}, \quad (D.8)
\end{aligned}$$

$$R^{\text{TT}} = h^{\mu\nu} \square h_{\mu\nu} + \frac{3}{4} \partial^\sigma h^{\mu\nu} \partial_\sigma h_{\mu\nu} - \frac{1}{2} \partial^\sigma h^{\mu\nu} \partial_\mu h_{\sigma\nu}, \quad (D.9)$$

respectively. Bearing this in mind, we conformally transformed the EH action part of Eq. (3.19) to begin with (see Ref. [172], where the conformal relations are shown)

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} a^2 (R + 6\partial_\sigma \ln a \partial^\sigma \ln a). \quad (D.10)$$

For TT tensor perturbations, $\sqrt{-g}$ is simply

$$\sqrt{-g}^{\text{TT}} = 1 - \frac{1}{4} h^{ij} h_{ij} . \quad (\text{D.11})$$

Then, the EH action for TT tensor perturbations reads

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d\eta d\mathbf{x} a^2 \left(-h^{ij} h_{ij}'' + h^{ij} \partial_k \partial^k h_{ij} - \frac{3}{4} h^{ij'} h_{ij}' + \frac{3}{4} \partial^k h^{ij} \partial_k h_{ij} - \frac{1}{2} \partial^i h^{jk} \partial_j h_{ik} + \frac{3}{2} \mathcal{H}^2 h^{ij} h_{ij} \right), \quad (\text{D.12})$$

where primes denote derivatives with respect to the conformal time and $\mathcal{H}$ was defined in Eq. (1.13). Integrating by parts and integrating out total derivatives, we arrive at

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{8} \int d\eta d\mathbf{x} a^2 \left[h^{ij'} h_{ij}' + h^{ij} \partial_k \partial^k h_{ij} - 2 \left(2\mathcal{H}' + \mathcal{H}^2 \right) h^{ij} h_{ij} \right]. \quad (\text{D.13})$$

The GB term transforms as

$$\hat{\mathcal{G}} = \Omega^{-4} \left\{ \mathcal{G} + 8\Omega^{-1} G_{\mu\nu} \nabla^\mu \nabla^\nu \Omega + 2\Omega^{-2} \left[4(\Box\Omega)^2 - 4\nabla_\mu \nabla_\nu \Omega \nabla^\mu \nabla^\nu \Omega - 8S_{\mu\nu} \nabla^\mu \Omega \nabla^\nu \Omega \right] + 8\Omega^{-3} (4\nabla_\mu \nabla_\nu \Omega \nabla^\mu \Omega \nabla^\nu \Omega - \Box\Omega \nabla_\sigma \Omega \nabla^\sigma \Omega) - 24\Omega^{-4} \nabla_\sigma \Omega \nabla^\sigma \Omega \nabla_\rho \Omega \nabla^\rho \Omega \right\}, \quad (\text{D.14})$$

under the conformal rescaling of the metric [172], where $\nabla^\mu = g^{\mu\nu} \nabla_\nu$. Ω^2 is the conformal factor (i.e. $\Omega^2 = a^2$ in our case). Assuming that the GB term without the caret, $\mathcal{G}$, is obtained from the perturbed metric around Minkowski, we have

$$\begin{aligned} \mathcal{G}^{\text{TT}} = & -\Box h^{\mu\nu} \Box h_{\mu\nu} + \partial^\mu \partial^\nu h^{\alpha\beta} (\partial_\mu \partial_\nu h_{\alpha\beta} + \partial_\alpha \partial_\beta h_{\mu\nu} - 2\partial_\mu \partial_\alpha h_{\beta\nu}) = 2h^{ij''} \partial_k \partial^k h_{ij} - \\ & - \partial_k \partial^k h^{ij} \partial_l \partial^l h_{ij} - 2\partial^k h^{ij'} \partial_k h_{ij}' + 2\partial^i h^{jk'} \partial_j h_{ik}' + \partial^l \partial^k h^{ij} \partial_l \partial_k h_{ij} + \partial^i \partial^j h^{kl} \partial_k \partial_l h_{ij} - \\ & - 2\partial^i \partial^l h^{jk} \partial_j \partial_l h_{ik}, \end{aligned} \quad (\text{D.15})$$

for TT tensor perturbations. We then calculate each term of Eq. (D.14) (assuming the TT tensor perturbations). The first term involving the Einstein tensor yields:

$$8\Omega^{-1} G^{\mu\nu} \nabla_\mu \partial_\nu \Omega = 2\mathcal{H} h^{ij'} \Box h_{ij} + \left(\mathcal{H}' + \mathcal{H}^2 \right) \left(4h^{ij} \partial_k \partial^k h_{ij} - h^{ij'} h_{ij}' + 3\partial^k h^{ij} \partial_k h_{ij} - 2\partial^i h^{jk} \partial_j h_{ik} \right), \quad (\text{D.16})$$

where the following relations have been considered:

$$\delta^{(1)}\Gamma_{\mu\nu}^0 = \frac{1}{2}h'_{\mu\nu} , \quad (\text{D.17})$$

$$\delta^{(2)}\Gamma_{\mu\nu}^0 = 0 , \quad (\text{D.18})$$

$$\eta^{\mu\nu}\Gamma_{\mu\nu}^\alpha = 0 . \quad (\text{D.19})$$

$\delta^{(1)}$ and $\delta^{(2)}$ indicate that we are considering the linear and quadratic perturbation terms, respectively. Besides these, in regard to the Ricci tensor, we have

$$\delta^{(1)}R_{00} = 0 , \quad (\text{D.20})$$

$$\delta^{(2)}R_{00} = \frac{1}{2}h^{ij}h''_{ij} + \frac{1}{4}h^{ij'}h'_{ij} . \quad (\text{D.21})$$

Next term is

$$\begin{aligned} 2\Omega^{-2} \left[4(\Box\Omega)^2 - 4\partial_\mu\partial_\nu\Omega\nabla^\mu\partial^\nu\Omega - 8S_{\mu\nu}\partial^\mu\Omega\partial^\nu\Omega \right] = & -8\mathcal{H}(\mathcal{H}' + \mathcal{H}^2)h^{ij}h'_{ij} - \\ & -\mathcal{H}^2 \left(3h^{ij'}h'_{ij} + 4h^{ij}h''_{ij} + 4h^{ij}\partial_k\partial^kh_{ij} + 3\partial^kh^{ij}\partial_k h_{ij} - 2\partial^ih^{jk}\partial_j h_{ik} \right) . \end{aligned} \quad (\text{D.22})$$

Finally

$$8\Omega^{-3} (4\nabla_\mu\partial_\nu\Omega\partial^\mu\Omega\partial^\nu\Omega - \Box\Omega\partial_\sigma\Omega\partial^\sigma\Omega) - 24\Omega^{-4}\partial_\sigma\Omega\partial^\sigma\Omega\partial_\rho\Omega\partial^\rho\Omega = 4\mathcal{H}^2 (6\mathcal{H}' + \mathcal{H}h^{ij}h'_{ij}) . \quad (\text{D.23})$$

The corresponding GB term with caret is then

$$\begin{aligned} \hat{\mathcal{G}} = a^{-4} \Big\{ & 24\mathcal{H}^2\mathcal{H}' - 4\mathcal{H}^3h^{ij}h'_{ij} - 8\mathcal{H}\mathcal{H}'h^{ij}h'_{ij} - 4\mathcal{H}^2h^{ij'}h'_{ij} - \mathcal{H}'h^{ij'}h'_{ij} - 4\mathcal{H}^2h^{ij}h''_{ij} + \\ & + 4\mathcal{H}'h^{ij}\partial_k\partial^kh_{ij} + 3\mathcal{H}'\partial^kh^{ij}\partial_k h_{ij} - 2\mathcal{H}h^{ij'}h''_{ij} + 2\mathcal{H}h^{ij'}\partial_k\partial^kh_{ij} + 2h^{ij''}\partial_k\partial^kh_{ij} - \\ & - 2\mathcal{H}'\partial^ih^{jk}\partial_j h_{ik} - \partial_k\partial^kh^{ij}\partial_l\partial^lh_{ij} - 2\partial^kh^{ij'}\partial_k h'_{ij} + 2\partial^ih^{jk'}\partial_j h'_{ik} + \partial^l\partial^kh^{ij}\partial_l\partial_k h_{ij} + \\ & + \partial^i\partial^jh^{kl}\partial_k\partial_l h_{ij} - 2\partial^i\partial^lh^{jk}\partial_j\partial_l h_{ik} \Big\} . \end{aligned} \quad (\text{D.24})$$

We see that the background value of $\mathcal{G}$ (we remove the caret) is $\bar{\mathcal{G}} = 24a^{-4}\mathcal{H}^2\mathcal{H}'$, which is Eq. (3.35) written in terms of the conformal time derivatives and the conformal Hubble parameter, as we would expect. There are no linear terms however, so $\mathcal{G} = \bar{\mathcal{G}} + \delta^{(2)}\mathcal{G}$.

Therefore

$$-\frac{1}{2} \int d^4x \sqrt{-g} \xi(\phi) \mathcal{G} \rightarrow -\frac{1}{2} \int d\eta d\mathbf{x} a^4 \xi(\phi) \left(\bar{\mathcal{G}} + \delta^{(2)} \mathcal{G} - \frac{1}{4} \bar{\mathcal{G}} h^{ij} h_{ij} \right), \quad (\text{D.25})$$

where $\xi(\phi)$ denotes the homogeneous coupling function. Knowing that there are no terms from the scalar field Lagrangian that act as source terms in the GWs equation, we do not perturb the scalar field and keep only the corresponding homogeneous quantities for simplicity. The second term in the integral, after integrating by parts and dismissing total derivatives, becomes

$$\begin{aligned} -\frac{1}{2} \int d\eta d\mathbf{x} a^4 \xi \delta^{(2)} \mathcal{G} &= -\frac{1}{2} \int d\eta d\mathbf{x} \left\{ \xi' \mathcal{H} h^{ij'} h'_{ij} + \left(\xi'' - \xi' \mathcal{H} \right) h^{ij} \partial_k \partial^k h_{ij} + \right. \\ &\quad \left. + 2\mathcal{H} \left(\xi' \mathcal{H}^2 + 3\xi \mathcal{H} \mathcal{H}' - \xi'' \mathcal{H} - 2\xi' \mathcal{H}' \right) h^{ij} h_{ij} \right\}. \end{aligned} \quad (\text{D.26})$$

The scalar field Lagrangian

$$\sqrt{-g} \mathcal{L}_\phi = -\frac{1}{2} \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2V(\phi) \right], \quad (\text{D.27})$$

once we revert the conformal rescaling, can be written as

$$S_\phi = -\frac{1}{2} \int d^4x \sqrt{-g} \left[a^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2U(\phi) \right], \quad (\text{D.28})$$

where $U(\phi) \equiv a^4 V(\phi)$. Perturbing the action

$$S_\phi = -\frac{1}{2} \int d\eta d\mathbf{x} \left[-a^2 \phi'^2 + 2U + \frac{1}{2} \left(a^2 \frac{1}{2} \phi'^2 - U \right) h^{ij} h_{ij} \right], \quad (\text{D.29})$$

where ϕ denotes the background scalar field and $U(\phi)$ is the background potential.

Putting all together, we write the GB action part (including the action of the scalar field) for TT tensor perturbations

$$\begin{aligned} \frac{1}{8} \int d\eta d\mathbf{x} a^2 \left\{ -4a^{-2} \xi' \mathcal{H} h^{ij'} h'_{ij} - 4a^{-2} \left(\xi'' - \xi' \mathcal{H} \right) h^{ij} \partial_k \partial^k h_{ij} - \right. \\ \left. - 2 \left[4\mathcal{H} a^{-2} \left(\xi' \mathcal{H}^2 - \xi'' \mathcal{H} - 2\xi' \mathcal{H}' \right) + a^{-2} \left(a^2 \frac{1}{2} \phi'^2 - U \right) \right] h^{ij} h_{ij} \right\}. \end{aligned} \quad (\text{D.30})$$

The full action of the SGB model, S_{SGB} , for TT tensor perturbations reads

$$S_{\text{SGB}} = \frac{1}{8} \int d\eta d\mathbf{x} a^2 \left\{ \left(M_{\text{Pl}}^2 - 4a^{-2} \xi' \mathcal{H} \right) h^{ij'} h'_{ij} + \left[M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' - \xi' \mathcal{H} \right) \right] h^{ij} \partial_k \partial^k h_{ij} - \right. \\ \left. - 2 \left[M_{\text{Pl}}^2 \left(2\mathcal{H}' + \mathcal{H}^2 \right) + 4\mathcal{H} a^{-2} \left(\xi' \mathcal{H}^2 - \xi'' \mathcal{H} - 2\xi' \mathcal{H}' \right) + a^{-2} \left(a^2 \frac{1}{2} \phi'^2 - U \right) \right] h^{ij} h_{ij} \right\}. \quad (\text{D.31})$$

Using the homogenous equation (3.37) in conformal time, we see that the mass term in the action (the term proportional to $h^{ij} h_{ij}$) vanishes. The action takes on the simple form

$$S_h = \frac{1}{8} \int d\eta d\mathbf{x} a^2 \left\{ \left(M_{\text{Pl}}^2 - 4a^{-2} \xi' \mathcal{H} \right) h^{ij'} h'_{ij} + \left[M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' - \xi' \mathcal{H} \right) \right] h^{ij} \partial_k \partial^k h_{ij} \right\}. \quad (\text{D.32})$$

We derive the GWs equation using the EL equation

$$\frac{\partial(\sqrt{-g}\mathcal{L}_h)}{\partial h^{ij}} - \frac{d}{d\eta} \left[\frac{\partial(\sqrt{-g}\mathcal{L}_h)}{\partial h^{ij'}} \right] = 0, \quad (\text{D.33})$$

such that $S_h = \int d^4x \sqrt{-g} \mathcal{L}_h$. This leads to

$$\left(M_{\text{Pl}}^2 - 4a^{-2} \xi' \mathcal{H} \right) h''_{ij} + \left[2\mathcal{H} M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' \mathcal{H} + \xi' \mathcal{H}' \right) \right] h'_{ij} - \\ - \left[M_{\text{Pl}}^2 - 4a^{-2} \left(\xi'' - \mathcal{H} \xi' \right) \right] \partial_k \partial^k h_{ij} = 0. \quad (\text{D.34})$$

In coordinate time, this equation becomes Eq. (3.56). Consequently, we obtain Eq. (3.57) when $4H\dot{\xi} \neq M_{\text{Pl}}^2$. As we indicated already (see below Eq. (3.59)), this way of deriving the equation of GWs is sometimes preferred over perturbing the full field equations, mostly when the latter are tedious.

APPENDIX E

PALATINI-HORNDESKI GRAVITY MODEL WITH $G_4(\phi, X)$ AND THE SPEED OF GRAVITATIONAL WAVES

In this section of the **Appendix**, we explain briefly how those Horndeski models evade the observational constraints as shown in Ref. [349]. The key is related to the discussion following Eq. (3.298).

The action of this ‘Palatini-Horndeski’ model of gravity reads [349]

$$S = \int d^4x \sqrt{-g} [G_4(\phi, X) g^{\mu\nu} R_{\mu\nu}(\Gamma) + G_2(\phi, X)] , \quad (\text{E.1})$$

where $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic energy. In the metric formalism, we should include the term multiplying $G_{4,X}$ in Eq. (3.182). However, in the Palatini formalism, the Ricci tensor does not depend on the metric and its derivatives as explained in previous sections already. Then, there is no need for such a ‘counterterm’ to get rid of higher-order derivatives [349].

This subclass of Horndeski theory in the metric formalism is known to lead to $c_{\text{GW}} \neq c$ (see Eq. (3.184)). In the Palatini formalism however, one can perform a conformal rescaling of the metric such that (see Eq. (2.26))

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{M_{\text{Pl}}^2}{2} \hat{g}^{\mu\nu} R_{\mu\nu}(\Gamma) + \frac{M_{\text{Pl}}^4}{4} \frac{G_2(\phi, \Omega^2 \hat{X})}{G_4^2(\phi, \Omega^2 \hat{X})} \right] , \quad (\text{E.2})$$

where

$$\Omega^2(\phi, X) = \frac{2}{M_{\text{Pl}}^2} G_4(\phi, X) , \quad (\text{E.3})$$

is the chosen conformal factor. Action (E.2) is the EHP one (see Eq. (3.254)) in addition to the second term proportional to G_2/G_4^2 which can be interpreted as a k-essence term [349]

$$K(\phi, X) \equiv \frac{M_{\text{Pl}}^4}{4} \frac{G_2(\phi, X)}{G_4^2(\phi, X)} . \quad (\text{E.4})$$

If we obtain the connection field equations, then the solution is Eq. (3.260), which is the LC

connection essentially (given the invariance of Eq. (E.2) under projective transformations; see discussion below Eq. (3.263)). Therefore, the EHP term corresponds to the EH term of the metric formalism because $R = \mathring{R}$. Since the k-essence term does not contribute to the TT tensor perturbation equations, then these take on the same form as those of GR. On top of this, we know that tensor perturbations are invariant under conformal transformations to fully non-linear order (see below Eq. (2.366)). By rewriting the equations for TT tensor perturbation in terms of unhatted variables, it can be readily seen that the speed of propagation of the GWs (TT tensor perturbations) does not differ from that of the hatted frame. Since in that frame the equation of GWs was the same as that in GR, the speed of propagation of GWs prescribed by this Palatini-Horndeski gravity model is that of light in vacuum; i.e. $c_{\text{GW}} = c$. Further details involving calculations can be found in Ref. [349].

In that same reference, a calculation of the GWs equation is performed in the unhatted frame (which they call ‘Jordan frame’ as in scalar-tensor theories of the metric formalism; see Sec. 2.1.5) in order to determine the speed of GWs following a different procedure. They confirm that it coincides with that of light in vacuum. Additionally, they discuss the presence of the analogue of the $\mathcal{L}_3$ Lagrangian, Eq. (3.181), in the Palatini formalism. However, since $G_3(\phi, X)$ does not play any role in the speed of GWs in the metric formalism (see Eq. (3.184)), we do not address it further.

Consequently, as shown in Ref. [349], the Horndeski gravity model with $G_4 = G_4(\phi, X)$, which is discarded in the metric formalism due to the observational constraint on $|\alpha_T|$ (see Eq. (3.67)), can be rescued in the Palatini formalism. This serves as motivation to study the popular SGB model in this latter formalism, although we commented in the paragraph below Eq. (3.371) the difficulty it implies given that the analogue of $\mathcal{L}_5$ in the Palatini formalism leads to differential equations for the connection $\Gamma_{\mu\nu}^\alpha$ and a whole (exciting) new physics.

APPENDIX F

PROJECTIVE TRANSFORMATIONS AND TORSION

The homogeneous equations that we have derived for the Weyl connection in the WGB model, Eqs. (3.343), (3.344) and (3.349), can be shown to be exactly the same for a metric compatible connection but with non-zero torsion (Eq. (3.216), to be precise). We can check this by resorting to the projective transformations (that we introduced in Sec. 3.4.3, below Eq. (3.263))

$$\Gamma_{\mu\nu}^{\alpha} \rightarrow \Gamma_{\mu\nu}^{\alpha} + \mathcal{A}_{\mu} \delta_{\nu}^{\alpha} , \quad (\text{F.1})$$

where $\mathcal{A}_{\mu}$ is an arbitrary vector. This amounts to (see Eqs. (3.197) and (3.198))

$$\kappa_{\mu\nu}^{\alpha} \rightarrow \kappa_{\mu\nu}^{\alpha} + \mathcal{A}_{\mu} \delta_{\nu}^{\alpha} . \quad (\text{F.2})$$

The κ tensors, defined in Sec. 3.4.2, below Eq. (3.210), transform as

$$\kappa^{\alpha}_{\beta\mu\nu} \rightarrow \kappa^{\alpha}_{\beta\mu\nu} + \delta_{\beta}^{\alpha} \mathcal{A}_{\mu\nu} , \quad (\text{F.3})$$

$$\kappa_{\mu\nu} \rightarrow \kappa_{\mu\nu} + \mathcal{A}_{\mu\nu} , \quad (\text{F.4})$$

$$\tilde{\kappa}_{\mu\nu} \rightarrow \tilde{\kappa}_{\mu\nu} - \mathcal{A}_{\mu\nu} , \quad (\text{F.5})$$

$$\bar{\kappa}_{\mu\nu} \rightarrow \bar{\kappa}_{\mu\nu} + 4\mathcal{A}_{\mu\nu} , \quad (\text{F.6})$$

where $\mathcal{A}_{\mu\nu} \equiv \mathring{\nabla}_{\mu} \mathcal{A}_{\nu} - \mathring{\nabla}_{\nu} \mathcal{A}_{\mu}$. The scalar κ is invariant under these transformations as the tensor $\mathcal{A}_{\mu\nu}$ is antisymmetric. Notice also that the tensor $\bar{\kappa}_{\alpha\beta\mu\nu}$, which was defined as the antisymmetrisation of the first two indices of $\kappa_{\alpha\beta\mu\nu}$ below Eq. (3.311), is invariant as well.

Now, bearing this in mind, let us apply this transformation to the Weyl connection. The $\mathcal{G}$ in Eq. (3.314) becomes

$$\mathcal{G} \rightarrow \mathcal{G} - 4\mathcal{A}^{\mu\nu} \mathcal{A}_{\mu\nu} - 4\Upsilon \left(\frac{1}{2} \bar{\kappa}^{\mu\nu} + \mathcal{A}^{\mu\nu} \right) \mathcal{A}_{\mu\nu} . \quad (\text{F.7})$$

Under this transformation, the distortion tensor of the Weyl connection transforms as (see

Eq. (3.220))

$$\kappa_{\mu\nu}{}^\alpha \rightarrow \delta_\mu^\alpha A_\nu + \delta_\nu^\alpha (A_\mu + \mathcal{A}_\mu) - g_{\mu\nu} A^\alpha, \quad (\text{F.8})$$

where A^μ is the Weyl vector (see paragraph below Eq. (3.297)). If we set $\mathcal{A}_\mu = -A_\mu$, we obtain the connection compatible with the metric but with non-zero torsion which was put as an example in Eq. (3.216), and (see Eq. (3.228))

$$\mathcal{A}_{\mu\nu} = -\frac{1}{4}\bar{\kappa}_{\mu\nu}. \quad (\text{F.9})$$

Therefore, the $\mathcal{G}$ with torsion and vanishing non-metricity is

$$\mathcal{G} = \mathcal{G}^{\text{Weyl}} - \frac{1-\Upsilon}{4}\bar{\kappa}^{\mu\nu}\bar{\kappa}_{\mu\nu}. \quad (\text{F.10})$$

It can be seen that $\mathcal{G}$ and $\mathcal{G}^{\text{Weyl}}$ only differ by the term $\bar{\kappa}^{\mu\nu}\bar{\kappa}_{\mu\nu}$. In fact, the new $\mathcal{G}$ does not depend on Υ

$$\mathcal{G} = \mathring{\mathcal{G}} + 2\mathring{R}\kappa - 8\mathring{R}^{\mu\nu}\tilde{\kappa}_{\mu\nu} + 2\mathring{R}^{\alpha\beta\mu\nu}\bar{\kappa}_{\alpha\beta\mu\nu} + \kappa^2 - 4\tilde{\kappa}^{\mu\nu}\tilde{\kappa}_{\mu\nu} + \bar{\kappa}^{\alpha\beta\mu\nu}\bar{\kappa}_{\alpha\beta\mu\nu} - \frac{1}{4}\bar{\kappa}^{\mu\nu}\bar{\kappa}_{\mu\nu}. \quad (\text{F.11})$$

So the calculations are the same as in the case of the Weyl connection if one simply sets $\Upsilon = 1$ (see Eq. (3.314)). Moreover, we notice that if the homogeneous spatial components of the Weyl vector are set equal to zero, $A_i(t) = 0$, the Υ term does not contribute to the homogeneous equations as it multiplies $\bar{\kappa}_{\mu\nu}$ (which vanishes in that case; see definition of $\bar{\kappa}_{\mu\nu}$ in Eq. (3.228)). The same occurs in the equations for tensor perturbations (3.367) (see Sec. 3.5.2). This means that, for a background Weyl vector that has solely a non-zero temporal component, the results derived in Sec. 3.5 are the same as those obtained assuming Eq. (3.216) with non-zero torsion and metric compatible.

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