CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE FÍSICA

"Violación de sabor leptónico dentro del modelo más sencillo de little Higgs"

Tesis que presenta

Enrique Ramírez Flores

para obtener el Grado de

Doctor en Ciencias

en la Especialidad de

Física

Director de tesis: Dr. Pablo Roig Garcés

Ciudad de México Diciembre, 2022

CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

PHYSICS DEPARTMENT

"Lepton flavor violation within the simplest little Higgs model"

by

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In order to obtain the

Doctor of Science

degree, speciality in

Physics

Advisor: Ph. D. Pablo Roig Garcés

Mexico City December, 2022

Enrique Ramirez Flores: *Lepton Flavor violation within the simplest little Higgs Model.* *Mexico City, december, 2022*.

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ACKNOWLEDGEMENTS

I am so clever that sometimes
I don't understand a single word
of what I am saying.

— Oscar Wilde

I would like to thank my family for their support, to my PhD advisor Pablo Roig Who taught and guided me throughout these years of work. To my partner Lucy, without her I would not have come this far. This work was supported by the CONACyT graduate grants program.

Mexico City, december, 2022

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[1]The material in the rest of this chapter and the next ones is based on our paper [157].

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ABSTRACT

We carry out an exhaustive analysis of lepton flavor violating processes within the Simplest Little Higgs model. Its discovery could be expected from either $\mu \to e$ conversion in nuclei, $\mu \to e\gamma$ or $\mu \to 3e$ decays. Then, the tau sector could help discriminating this model not only via $\tau \to \ell\gamma$ ($\ell = \mu, e$) and $\tau \to 3\ell$ decays, but also by means of $\ell \to \tau$ conversion in nuclei, which is promising in this respect. Although the model violates slightly custodial symmetry, acommodating the recent CDF M_W measurement is in tension with electroweak precision data. In addition, we show how SM neutrino masses could be introduced in this model.

RESUMEN

Llevamos a cabo un análisis exhaustivo de los procesos que violan el sabor de los leptones dentro del modelo más sencillo de Little Higgs. Su descubrimiento podría esperarse de la conversión de $\mu \to e$ en núcleos, $\mu \to e\gamma$ o desintegraciones de $\mu \to 3e$. Entonces, el sector tau podría ayudar a discriminar este modelo no solo a través de los decaimientos $\tau \to \ell\gamma$ ($\ell = \mu, e$) y $\tau \to 3\ell$, sino también por medio de conversión de $\ell \to \tau$ en núcleos, lo cual es prometedor a este aspecto. Aunque el modelo viola ligeramente la simetría de custodia, acomodar la medición CDF M_W reciente está en tensión con los datos de precisión electrodébil. Además, mostramos cómo las masas de neutrinos del modelo estándar podrían introducirse en este modelo.

INTRODUCTION

With the discovery of the Higgs boson [1, 2] the Standard Model of particle physics (SM) [3, 4, 5] was completed. Powerful and praiseworthy as it is, a Higgs mass value in the electroweak scale, v, calls for a deeper understanding of the hierarchy concept.

Assuming the SM is a low-energy effective theory of a more general high-energy theory, generalizing it at a high-energy scale $\Lambda >> v$, raises the question: Is there new physics between these two energy scales? *Hierarchy problem* or *the fine tuning problem* [6] is the reason motivating the existence of new physics, that lies between v and Λ. In the context of the SM, it means that the Higgs boson mass receives quadratically divergent loop contributions which are much larger than its measured value, and require a correspondingly large bare mass value so that the fine-tuned cancellation between both yields the observed $m_H \sim 125$ GeV. These leading quantum corrections are only cancelled when the parameters are fine-tuned. Nevertheless after the LEP experiment and their Electroweak Precision Data (EWPD) [7] a Little hierarchy problem emerged [8], given that LEP measurements prevented new physics near v. There must be then a little hierarchy between v and the lightest new physics scale, which should lie above the TeV.

Many beyond SM theories have been used to alleviate the hierarchy problem, like supersymmetry, technicolor, extra dimensions and Little Higgs. Our approach belongs to the last one. Little Higgs models [9, 10, 11, 12, 13, 14] postpone the hierarchy problem in the SM, introducing adequate new particles under an enlarged symmetry at an energy scale of some TeVs. All these models are based on the idea that the Higgs is a Pseudo Nambu-Goldstone boson (pNGB), which arises from some approximate spontaneously broken symmetry at a scale $f \gtrsim$ TeV. This new symmetry is introduced to protect the Higgs mass from large quantum corrections and the Higgs fields are taken to be NGBs corresponding to a spontaneously broken global symmetry of a new strongly interacting sector. As a result of this novel non-perturbative dynamics, additional new physics is expected at a scale $\sim 4\pi f$.

Through the years many Little Higgs models have already been constructed, in which the new particles depend on the particular symmetry of the model. We can divide Little Higgs models into two categories [15]: *product group models*, where the SM gauge group arises from the diagonal breaking of two or more gauge groups, i.e, $(SU(2) \times U(1))^N$ and *Simple group models* where the SM gauge group stems from the breaking of a single larger group, i.e, $SU(N) \times U(1)$. One the most important product group model realizations of the Little Higgs model is the T-parity extension proposed by Cheng and Low [16]. LFV has been extensively studied within this model [17, 18, 19, 20, 21, 22, 23]. In these models, there is no need to enlarge the SM matter sector and the collective breaking can be realized with just one sigma model, although there is more freedom related to the extra gauge couplings, and a discrete symmetry needs to be imposed to comply with EWPD. On the other hand, a simple group model that is popular by its minimality ($N = 3$ above) is the Simplest Little Higgs (SLH) that was proposed by Kaplan and Schmaltz [24, 25], which we will use. In this case an additional fermion field is needed for the SM doublets to become triplets. In the lepton sector this is a heavy quasi-Dirac neutrino, which drives LFV. The situation is more involved in the quark counterpart, where there are two possible embeddings, as we will explain. LFV has already been studied within the SLH in slightly different approaches [26, 27, 28, 29, 30] [2]. However, leptonic tau decays and $\ell - \tau$ conversion in nuclei did not receive much attention because they are less restrictive than the analogous muon processes, according to experimental limits, several orders of magnitude weaker for taus. We will include them here for the first time, mainly to increase the model-discriminating power adding these observables to our toolkit. We will not discuss Z [29] or Higgs [28] LFV decays as their branching fractions turn out to be $\leqslant 10^{-11}$ and $\leqslant 10^{-12}$, respectively[3]., far away from current or near-future bounds. Similarly, we will not address semileptonic LFV τ decays as purely lepton LFV τ decays have always a few orders of magnitude larger branching fractions [26, 27] (see, however ref. [31]).

LFV in the charged lepton sector is long sought as it will surely be due to new physics, given the GIM-like [32] suppression of SM contributions in presence of massive neutrinos [33, 34, 35]. There are very stringent bounds [36] from MEG [37], SINDRUM [38], SINDRUM-II [39], BaBar [40] and Belle [41]. There also is and will be a plethora of experiments contributing to this quest: MEG-II [42], PRISM/PRIME [43], Mu2e [44], Mu3e [45], COMET [46], DeeMe [47], Belle II [48], ... enhancing the case for studying the related phenomenology. In the case of $\ell - \tau$ nuclei conversion there are still no experimental limits for this phenomenon (recently been studied [49, 50, 51]). Ref. [52] pointed out that the NA64 experiment could be able to search for it, as well as the proposed muon collider [53] or the electron-ion collider [54, 55], among others. Indeed, as $\mu \to e$ conversion in nuclei is synergic with the LFV μ decays in $\mu \to e$ transitions; we will find that in the $\tau \leftrightarrow \ell$ ($\ell = e, \mu$) processes, conversion in nuclei will put significant constraints together with the purely leptonic τ LFV decays.

This work is divided into the following parts:

THE FIRST CHAPTER offers a general vision of quantum field theory and particle physics, including some problems of the Standard Model.

THE SECOND CHAPTER gives the introduction to the hierarchy problem and the solution proposed by Little Higgs models.

[2] Only ref. [28] considered three heavy neutrinos with general mixing, as we do here.

[3] These upper bounds correspond to the range of the SLH model parameters that we study, see chapter 5

THE THIRD CHAPTER develops the full structure of the LFV processes.

THE FOURTH CHAPTER contains the formalism to introduce mass to the neutrinos in the SLH model.

THE FIFTH CHAPTER shows the numerical results of this work.

THE SIXTH CHAPTER shows the conclusions and future work perspectives.

THE APPENDIX A shows some useful mathematical results for our work.

THE APPENDIX B shows some extra plots that were not considered in the main discussion.

1 FOUNDATIONS

Men of learning suspect a little about
this world, and ignore it mostly.
Wise men have interpreted
dreams, and the gods
have laughed.

— H.P. Lovecraft

The Standard Model (SM) is the most rigorous theory of particle physics, incredibly precise and accurate in its predictions. It mathematically lays out the 17 building blocks of nature: six quarks, six leptons, four force-carrier particles, and the Higgs boson. These are ruled by the electromagnetic, weak and strong forces. The known elementary interactions (except for gravitational interactions) are described by a Lagrangian density defined in terms of a set of local symmetries and an observed particle spectrum.

In 2012 CMS and ATLAS collaboration data showed the existence of a spin zero resonance of mass around 125 GeV [1], [2]. This particle has been identified as the boson of the Higgs mechanism, responsible for the mass of the SM particles. Nevertheless, there remain observations without explanation within the SM. Only around 5% of the Universe energy content is explained by ordinary matter. Then, around 27% of the energy seems to require an explanation in terms of the so-called "dark matter", while the rest is known as the "dark energy", which explains the accelerated expansion of the Universe. Besides, the SM does not explain massive neutrinos and it contains no mechanism to describe the matter/antimatter asymmetry present in the Universe either. So although the Standard Model accurately describes the phenomena within its domain, it is still incomplete. Perhaps it is only a part of a bigger picture that includes new physics hidden deep in the subatomic world or in the dark recesses of the universe.

This chapter is mainly based on the references [56, 57, 58, 59, 60].

1.1 STANDARD MODEL OF PARTICLE PHYSICS

All the matter that is contained in the universe is made of small "blocks" of matter, the elementary particles. These particles can be divided into two groups: *leptons* and *quarks*, each group contains six particles (and their antiparticles), which are grouped in pairs or *generations*. From galaxies and stars to molecules and atoms, all the stable matter that exists in our universe is made from particles that belong to the first generation. Leptons and quarks belong to a broader group called *fermions*. By the other hand all the interactions in the universe are represented by the four fundamental forces: he strong force, the weak force, the electromagnetic force, and the gravitational force. Three of the fundamental forces result from the exchange of force-carrier particles which belong to another group called *bosons*. Elementary particles interact between them exchanging discrete amounts of energy by exchanging bosons with each other. Each fundamental force has its own corresponding boson: the strong force is carried by the *gluon*, the electromagnetic force is carried by the *photon*, and the W and Z bosons are responsible for the weak force, although not yet found, the "graviton" should be the corresponding force-carrying particle of gravity, see figure 1.

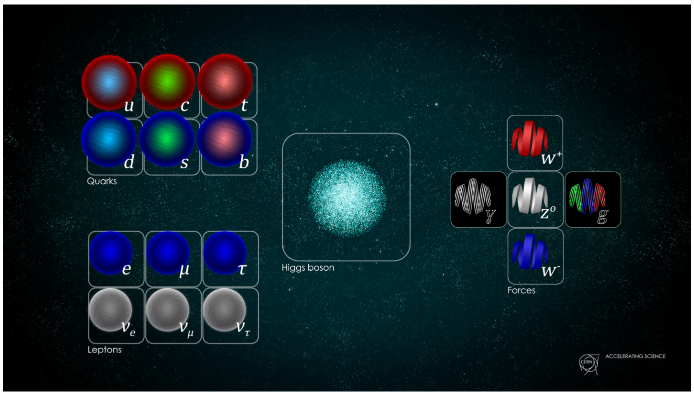


Figure 1: Elementary particles [61].

1.1.1 Lagrangian density and Particle content

The Standard Model explains the strong and electroweak forces, which are incorporated via the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. We can construct a universal gauge-covariant derivative in any representation of the local gauge group:

$$\mathcal{D}_\mu = \partial_\mu + \frac{ig}{2} W_{a\mu}\tau_a + \frac{ig'}{2} B_\mu Y + \frac{ig_s}{2}\lambda_a G_{a\mu} \tag{1}$$

where: τ_a are the three generator of $SU(2)$, λ_a are the eight generators of $SU(3)$, g, g', g_s are the weak, hypercharge, strong coupling constants; finally W_a, Y and G_q are the gauge bosons fields. This description of the gauge-covariant derivative will work for any particle in any representation of the Standard Model, for example, in the fundamental representation the generators τ_a are the Pauli matrices and λ_a are the Gell-Mann matrices.

The full Lagrangian density can be broken down into three pieces:

$$\mathcal{L}_{SM} = \mathcal{L}_{fermions} + \mathcal{L}_{scalar} + \mathcal{L}_{gauge\ bosons}, \tag{2}$$

where $\mathcal{L}_{fermions}$ contains Dirac Lagrangian terms with gauge-covariant derivatives, Yukawa mass terms, (and possible neutrino mass terms of any type), $\mathcal{L}_{scalar}$ contains the Higgs kinetic term with gauge-covariant derivative and the Higgs self-potential terms and $\mathcal{L}_{gauge-bosons}$ contains the gauge boson kinetic terms, including the self-interactions necessary for local gauge invariance. We will use the notation for the multiplets:

$$(\text{color representation}, \text{weak multiplet dimension})_{\text{Hypercharge}}\,. \tag{3}$$

- **Fermions**

There are the three generations of the left-handed leptons:

$$\begin{aligned} \Psi_{eL} &= (1,2)_{-\frac{1}{3}}, \\ \Psi_{\mu L} &= (1,2)_{-\frac{1}{3}}, \\ \Psi_{\tau L} &= (1,2)_{-\frac{1}{3}}, \end{aligned} \tag{4}$$

each doublet contains one charged lepton and its corresponding neutrino. There are the right-handed leptons:

$$\begin{aligned} \Psi_{eR} &= (1,1)_{-1}, \\ \Psi_{\mu R} &= (1,1)_{-1}, \\ \Psi_{\tau R} &= (1,1)_{-1}. \end{aligned} \tag{5}$$

Then there are the quarks, which are color triplets:

$$\begin{aligned} \Psi_{uL} &= (3,2)_{\frac{1}{3}}, & \Psi_{uR} &= (3,1)_{\frac{2}{3}}, & \Psi_{sR} &= (3,1)_{-\frac{1}{3}}, \\ \Psi_{cL} &= (3,2)_{\frac{1}{3}}, & \Psi_{dR} &= (3,1)_{-\frac{1}{3}}, & \Psi_{tR} &= (3,1)_{\frac{2}{3}}, \\ \Psi_{tL} &= (3,2)_{\frac{1}{3}}, & \Psi_{cR} &= (3,1)_{\frac{2}{3}}, & \Psi_{bR} &= (3,1)_{-\frac{1}{3}}. \end{aligned} \tag{6}$$

All the multiplets so far are composed of spinors. The Standard Model requires one scalar Higgs doublet:

$$\Phi = (1,2)_1. \tag{7}$$

that's the complete matter content of the SM but we need to take into account the gauge bosons which are necessary for the local gauge invariance. Using the matter and covariant derivative, the first piece of the Lagrangian density can be written:

$$\mathcal{L}_{fermions} = i\bar{\Psi}_j\gamma^\mu\mathcal{D}_\mu\Psi'_j - \frac{\sqrt{2}m_f}{v}\left(\bar{\Psi}_{fL}\Phi\Psi_{fR} + \bar{\Psi}_{fL}\bar{\Phi}\Psi_{fR} + \text{h.c.}\right), \tag{8}$$

where the index j runs over each fermion multiplet, f runs over each fermion type, and the prime indicates the weak interaction eigenstates, which are related to the mass eigenstates by unitary mixing matrices[1] and v is the electroweak symmetry-breaking vacuum expectation value (vev).

[1]For quarks, the mixing matrix is the CKM matrix, and for leptons (if we consider right-handed neutrinos) it is the analogous PMNS matrix.

- **Scalar**

The Lagrangian Density terms pertaining to the scalar doublet are:

$$\mathcal{L}_{scalar} = (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - V\left(\Phi^\dagger \Phi\right), \tag{9}$$

scalar-vector interaction terms arise here because of the covariant derivatives. The scalar potential must be of a form which leads to spontaneous symmetry breaking of the $SU(2)_L \times U(1)_Y$ symmetry and renormalizability, and is taken to be:

$$V\left(\Phi^\dagger \Phi\right) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi\Phi)^2, \tag{10}$$

where $\mu^2 < 0$ and $\lambda > 0$.

- **Gauge Bosons**

Finally, there are the gauge boson terms:

$$\mathcal{L}_{gauge\ bosons} = -\frac{1}{4} W_{a\mu\nu} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu}, \tag{11}$$

where the weak, strong and hypercharge field strength tensors are, respectively,

$$\begin{aligned} W_{a\mu\nu} &= \partial_\mu W_{a\nu} - \partial_\nu W_{a\mu} - g\epsilon_{abc} W_{b\mu} W_{c\nu}, \\ G_{a\mu\nu} &= \partial_\mu G_{a\nu} - \partial_\nu G_{a\mu} - g_s f_{abc} G_{b\mu} G_{c\nu}, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \end{aligned} \tag{12}$$

were ϵ_{abc} and f_{abc} are the structure constants of $SU(2)$ and $SU(3)$ respectively. The Lagrangian density interpreted as a quantum field theory, describes the fundamental theory of particle physics.

1.1.2 Renormalizability and Unitarity

Adding the Higgs sector, the SM contains 19 free parameters: the 9 fermion masses, 3 gauge couplings, the Higgs mass and vev, 3 mixing angles and 1 CPV phase from the CKM matrix, and θ_{QCD}, which characterizes the QCD vacuum. All free parameters in the theory have to be fixed by experimental observables. The renormalization procedure is used to absorb the divergences (infinities) arising in computations of physical quantities in perturbative quantum field theory. This implies that the original parameters defining the theory in the Lagrangian will have a dependence on the energy scale after renormalization is performed, so they are said to "run" with energy; i.e. have different values in the infrared (IR) with respect to the ultraviolet (UV). This running is given by the *Renormalization Group Equations* (RGEs).

There is the possibility of having massive gauge bosons in a gauge-invariant way without a Higgs particle in the spectrum, this constitutes a non renormalizable theory. For instance, it turns out that the scattering of longitudinally polarized W bosons grows with the square of the energy, which eventually violates the perturbative unitarity of the theory:

$$\mathcal{A}\left(W_L^+W_L^- \to W_L^+W_L^-\right) \xrightarrow{E \gg M_W} \frac{g^2}{4M_W^2}(s+t) \propto E^2 \tag{13}$$

where M_W is the W mass and s and t are the so-called Mandelstam variables, related to the kinematics of the process. In the case of the SM this loss of perturbativity occurs at around $2-3$ TeV. There are then two possibilities: either the theory just becomes strongly coupled at high energies, thus appearing to lose unitarity in the perturbative expansion, or new degrees of freedom appear to restore unitarity, which is what indeed happens in the SM Higgs mechanism.

1.2 SPONTANEOUS SYMMETRY BREAKING OF A GLOBAL SYMMETRY

A *symmetry* is said to be present when there is a transformation on the variables of a system leaving the essential physics unchanged and one of the most important developments in quantum field theory has been the realization that there is more than one way in which symmetries and broken symmetries can manifest themselves in physical system. Sometimes it is useful to think in terms of symmetries in situations when the physics is invariant under a transformation just *approximately*. Usually in these cases the Lagrangian contains parameters such that when set to zero the symmetry is recovered.

Depending on the dynamics of the theory, a symmetry in the Lagrangian may either remain exact or be broken:

- The symmetry can be *explicitly* broken if there are terms in the Lagrangian that are not invariant under the associated transformation. These terms might be small, as in the case of an approximate symmetry.
- It may happen that the ground state of the theory does not display the full symmetry of the Lagrangian and, as a consequence, the symmetry is lost in the spectrum of physical states; so the symmetry can be said to be *hidden*.
- Finally, it is possible that a symmetry in the Lagrangian is conserved at the classical level; but it is broken by quantum effects through an anomaly.

About 1960 Nambu and Goldstone realised the significance of this notion in condensed matter physics [62, 63, 64, 65, 66], and Nambu in particular speculated on its applications to particle physics. In 1964 Higgs pointed out that the consequences of spontaneous symmetry breaking in gauge theories are very different from those in non-gauge theories [67, 68].

1.2.1 Classical symmetry

Eugene Wigner defined what symmetry meant and explained how a symmetry transformation could be represented by either a unitary or anti-unitary operator in the Hilbert space in elementary quantum mechanics [69]. Its characteristic signature is degenerate multiplet structure for the spectrum, and the violation of this kind of symmetry involves explicit symmetry breaking terms in the Hamiltonian that lift the multiplet degeneracies. A good example is provided by a spherical quantum-mechanical system such as an atom.

In the present discussion there are other two symmetry modes of more interest: the realization of a symmetry in the Goldstone or Higgs mode is commonly termed *spontaneous symmetry breaking*. The difference between the Goldstone and Higgs modes is simply that the spontaneous symmetry breaking occurs in the presence of a local gauge symmetry for the Higgs mode. The crucial distinction between symmetry implementation in the Wigner, Goldstone, and Higgs modes lies in the structure of the vacuum. The Lagrangian of a system may be invariant under transformation by some unitary representation U of a symmetry group, however, for a perturbative quantum field theory we build states from the vacuum and the theory requires specification of the symmetry for this state, as well as that of the Lagrangian. If the Lagrangian is invariant under a set of transformations the symmetry is implemented in the

- *Wigner mode* if the vacuum $|0\rangle$ is also invariant:

$$U|0\rangle = |0\rangle\,, \tag{14}$$

- The *Goldstone mode* if the vacuum is not invariant and the Lagrangian symmetry is global:

$$U|0\rangle \neq |0\rangle \qquad \text{(global symmetry)}, \tag{15}$$

- The *Higgs mode* if the vacuum is not invariant and the Lagrangian symmetry is local gauge symmetry:

$$U|0\rangle \neq |0\rangle \qquad \text{(local symmetry)}. \tag{16}$$

We begin with an analysis of spontaneous symmetry breaking in classical field theory. Consider a real scalar ϕ theory Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \tag{17}$$

this Lagrangian has a discrete symmetry: it is invariant under the operation $\phi \to -\phi$.

Two qualitatively different cases may be distinguished, depending on the sign of the coefficient μ^2, the potential for $\mu^2 > 0$ and $\mu^2 < 0$ is shown in fig.2

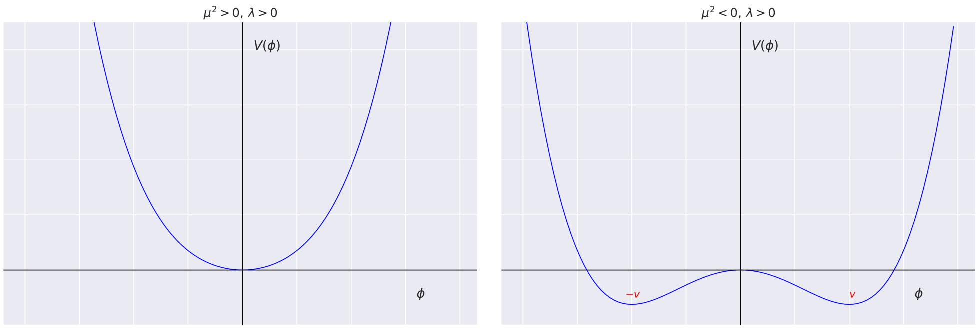


Figure 2: Potential for $\mu^2 > 0$ and $\mu^2 < 0$.

The case with $\mu^2 > 0$ corresponds to the Wigner mode, so we have for the vacuum expectation value of the field:

$$\langle \phi \rangle_0 \equiv \langle 0|\phi|0\rangle = 0. \tag{18}$$

Now consider the case $\mu^2 < 0$, the potential has minima at:

$$\langle \phi \rangle_0 = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v. \tag{19}$$

Now there are two degenerate vacuum states, the minima at $\langle \phi \rangle_0 = \pm v$ are equivalent and either may be chosen as the classical ground (vacuum) state of the system. The original Lagrangian is invariant under the discrete symmetry; however once the vacuum is chosen it is no longer invariant under the transformation $\phi \to -\phi$. This is a typical case of spontaneous symmetry breaking. To interpret this theory, suppose that the system is near one of the minima (say the positive one) and an infinitesimal fluctuation is sufficient to drive the system into this minimum. Then it is convenient to define

$$\xi(x) = \phi(x) - v. \tag{20}$$

In terms of this new variable the vacuum state is $\langle \xi \rangle_0 = 0$, and the Lagrangian density is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \xi \partial^\mu \xi - \lambda v^2 \xi^2 - \lambda v \xi^3 - \frac{1}{4}\lambda \xi^4, \tag{21}$$

which has no apparent reflection symmetry. In fact, the symmetry is there because the original Lagrangian possessed such a symmetry, but it has been hidden. For small oscillations about the classical vacuum:

$$\mathcal{L} \simeq \frac{1}{2}\partial_\mu \xi \partial^\mu \xi - \lambda v^2 \xi^2, \tag{22}$$

which is the Lagrangian density of a free scalar field of mass $m_\xi = \sqrt{-2\mu^2}$. This example contains most of the features that characterize spontaneous symmetry breaking:

- There is a nonzero expectation value of some field in the vacuum state.
- The resulting classical theory has a degenerate vacuum, with the choice among the equivalent vacua being completely arbitrary.
- The chosen vacuum state does not possess the same symmetry as the Lagrangian.
- Expanded around the chosen vacuum, the original symmetry of the Lagrangian is no longer apparent. The degenerate vacua are related to each other by symmetry operations, which tells us that the symmetry is still there, but it is not manifest; it is hidden.
- Once the theory develops degenerate vacua the origin becomes an unstable point. Thus the symmetry may be "broken spontaneously" in the absence of external intervention.
- If the spontaneously broken symmetry is a continuous global symmetry, one massless scalar field (Nambu-Goldstone boson) must appear in the theory for each group generator that has been broken.
- If a continuous local gauge symmetry is spontaneously broken no Nambu-Goldstone bosons are produced, and the gauge bosons may acquire a mass without spoiling gauge invariance (Higgs mechanism).

1.2.2 The linear Sigma Model

A more interesting theory arises when the broken symmetry is continuous, rather than discrete. The most important example is a generalization of the preceding theory called the *linear sigma model*. This model was introduced to study the spontaneous breaking of QCD chiral symmetry [70]. The Lagrangian of the linear sigma model involves a set of N real scalar fields $\phi^i(x)$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2, \tag{23}$$

the Lagrangian (23) is invariant under the symmetry:

$$\phi^i = R^{ij}\phi^j, \tag{24}$$

for any $N \times N$ orthogonal matrix R. The group of transformations (24) is just the rotation group in N dimensions, also known as the N dimensional *orthogonal group* or simply $O(N)$, which has $\frac{1}{2}N(N-1)$ generators.

Again the lowest energy classical configuration is a constant field $\langle\phi^i\rangle_0$, whose value minimizes the potential (see figure 3):

$$V\left(\phi^i\right) = -\frac{\mu^2}{2}\left(\phi^i\right)^2 + \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2. \tag{25}$$

This potential is minimized for any $\langle\phi^i\rangle_0$ that satisfies:

$$\langle\phi^i\rangle_0 = \pm\sqrt{\frac{\mu^2}{\lambda}} = \pm\nu. \tag{26}$$

This condition determines only the length of the vector $\langle\phi^i\rangle_0$; its direction is arbitrary. It is conventional to choose coordinates so that $\langle\phi^i\rangle_0$ points in the Nth direction:

$$\langle\phi^i\rangle_0 = (0, 0, \ldots, 0, \nu)\,, \qquad \text{where } \nu = \frac{\mu}{\sqrt{\lambda}}. \tag{27}$$

In contrast to our earlier example, this vacuum state is *invariant under a subgroup of the original group*: the group $O(N-1)$, which does not mix the last field with the others. The generators of $O(N-1)$ are $\frac{1}{2}(N-1)(N-2)$ and the difference between the number of generators for the original group $O(N)$ and the residual group $O(N-1)$ is $N-1$; thus, there are $N-1$ broken generators.

We can now define a set of shifted fields by writing:

$$\phi^i = \left(\pi^k, \nu + \sigma(x)\right), \qquad k = 1, \ldots, N-1. \tag{28}$$

It is straightforward to rewrite the Lagrangian (23) in terms of the π and σ fields:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\left(\partial_\mu\pi^k\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma\right)^2 - \frac{1}{2}\left(2\mu^2\right)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 \\ &- \sqrt{\lambda}\mu\left(\pi^k\right)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \frac{\lambda}{4}\left[\left(\pi^k\right)^2\right]^2.\end{aligned} \tag{29}$$

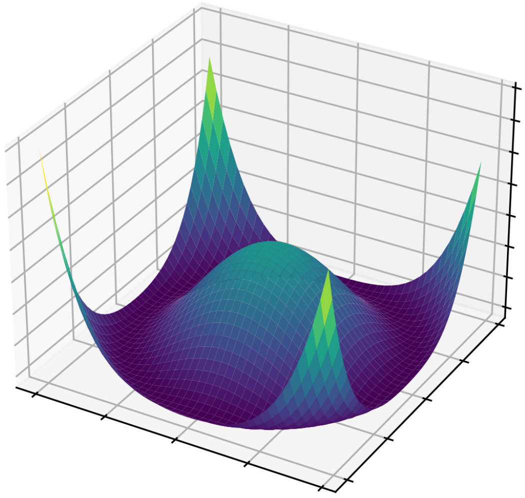


Figure 3: Potential for spontaneous breaking of a continuous $O(N)$ symmetry, drawn for the case $N = 2$. Oscillations along the trough of the potential correspond to the massless ϕ fields.

We obtain a massive σ field just as in the previous section and also a set of $N-1$ massless π fields, these are the *Nambu-Goldstone bosons*. The original $O(N)$ symmetry is hidden, leaving only the subgroup $O(N-1)$, which rotates the π fields among themselves. The ϕ fields describe oscillations of ϕ^i in the radial direction, the massless ϕ fields describe oscillations of ϕ^i in the tangential directions, along the trough of the potential. The trough is an $(N-1)$ dimensional surface, and all $N-1$ directions are equivalent, reflecting the unbroken $O(N-1)$ symmetry.

The appearance of massless scalar fields is a specific example of a general phenomenon in the spontaneous breaking of global symmetries that is important enough to have achieved the status of a theorem [62, 63, 64, 65, 66] (see appendix A for the proof):

- **Goldstone Theorem**: *If a continuous global symmetry is broken spontaneously, for each broken group generator there must appear a massless particle in the theory*.

This theorem can also be understood in terms of a coset space: The Goldstone bosons associated with the spontaneous symmetry breaking from $G \to H$ are described by the coset space G/H, and their number is $\dim G/H = \dim G - \dim H$.

In any field theory that obeys the usual axioms, including locality, Lorentz invariance, and positive-definite norm on the Hilbert space, if an exact continuous symmetry of the Lagrangian is not a symmetry of the physical vacuum, then the theory must contain a massless spin zero particle whose quantum numbers are those of the broken group generators.

1.2.3 Non Linear Sigma Model

The previous discussion is an example of a spontaneous breaking of a symmetry that is realized in a linear way. It is however possible to implement the spontaneous breaking in a non-linear fashion by integrating out the radial field. In this limit, the remaining Lagrangian consists of the kinetic term on the ϕ^i, subject to the constraint that the potential on these fields equals its minimum value:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i, \quad \text{and} \quad V\left(\phi^i\right) = V\left(\langle\phi^i\rangle_0\right). \tag{30}$$

The Lagrangian is still invariant under G, and so is the condition for the minimum. But G is spontaneously broken to H, and the model only describes the Goldstone bosons. The true dynamical fields are no longer the fields ϕ^i, since they are related to one another by the constraint $V\left(\phi^i\right) = V\left(\langle\phi^i\rangle_0\right)$. Introducing true dynamical fields may be done by solving the constraint.

Consider the previous example, a vector of N real scalar fields ϕ^i with a quartic potential:

$$V\left(\phi^i\right) = \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2, \quad \nu = \sqrt{\frac{\mu^2}{\lambda}}, \tag{31}$$

now we know that the vacuum expectation value can be written as $\langle\phi^i\rangle_0 = \nu\delta_{i,N}$. The constraint over the potential is given by $\phi^i\phi^i = \nu^2$. Eliminating the only non-Goldstone field ϕ^N (in the previous section this field is represented by the σ field) using:

$$\sigma \equiv \phi^N = \sqrt{\nu^2 - \phi^i\phi^i}, \quad \text{with } i = 1, \ldots, N-1. \tag{32}$$

In the previous section the fields ϕ^i with $i = 1, \ldots, N-1$ is represented by the π fields. The resulting Lagrangian is now a *non linear sigma model*, with Lagrangian:

$$\mathcal{L}_{GB} = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{1}{2}\frac{\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2}{\nu^2 - \vec{\pi}^2}. \tag{33}$$

Before eliminating ϕ_N, the Lagrangian $\mathcal{L}$ was invariant under SO(N). What happened with this symmetry? Has it completely disappeared or is its presence just more difficult to detect?

Since SO(N − 1) leaves ϕ_N invariant, the only concern is with the transformations in SO(N) modulo those in SO(N − 1). Using SO(N − 1) symmetry, these are equivalent to rotations g between ϕ_1 and ϕ_N, which leave ϕ^i with $i = 1, \ldots, N-1$ invariant:

$$g\begin{pmatrix}\phi_1\\ \phi_N\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_N\end{pmatrix}, \tag{34}$$

Since ϕ_N has been eliminated from the Lagrangian $\mathcal{L}_{GB}$, only the action of g on ϕ_1 is of interest in examining the symmetries of $\mathcal{L}_{GB}$,

$$g(\phi_1) = \phi_1\cos\alpha + \sqrt{\nu^2 - \vec{\phi}^2}\sin\alpha, \tag{35}$$

Notice that from this relation the one for ϕ_N follows:

$$g(\phi_N) = \sqrt{\nu^2 - \vec{\phi}^2 + \phi_1^2 - (\phi_1\cos\alpha + \phi_N\sin\alpha)^2} = \phi_N\cos\alpha - \phi_1\sin\alpha, \tag{36}$$

as a result, the Lagrangian is still invariant $g(\mathcal{L}_{GB}) = \mathcal{L}_{GB}$. But, the novelty is that the symmetry is realized in a *nonlinear* way.

1.3 SPONTANEOUS SYMMETRY BREAKING OF A GAUGE SYMMETRY

The Goldstone theorem applies to any field theory obeying the "normal postulates" such as locality, Lorentz invariance, and positive definite norm on the Hilbert space but the paucity of massless scalar or pseudoscalar particles in nature would seem to preclude the use of spontaneous symmetry breaking in realistic quantum field theories. However gauge field theories do not fit into that category: there is no single gauge in which such theories simultaneously fulfill each of these conditions.

Thus we are led to investigate whether the Goldstone theorem works in a theory possessing a local gauge invariance. We will find that there is an unexpected relation between the massless gauge fields and the Goldstone bosons produced by the spontaneous symmetry breaking that can be arranged so as to eliminate the massless Goldstone bosons and give a mass to the gauge quanta without spoiling the gauge invariance or renormalizability of the theory. This formalism is called the *Higgs mechanism* [67, 68].

1.3.1 Spontaneous breaking of a gauge symmetry

The simplest example of the Higgs mechanism is provided by the *Abelian Higgs model*, which is the gauge invariant $U(1)$ theory. In the absence of spontaneous symmetry breaking it would describe the ordinary electrodynamics of charged scalars. The Lagrangian density is:

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{37}$$

where λ is positive and

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \tag{38}$$

$$D^{\mu} = \partial^{\mu} + iqA^{\mu}, \tag{39}$$

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{40}$$

The Lagrangian is invariant under global $U(1)$ rotations and under the local gauge transformations:

$$\begin{aligned} \phi(x) &\to e^{iq\alpha(x)}\phi(x), \\ A_{\mu}(x) &\to A_{\mu}(x) - \partial_{\mu}\alpha(x). \end{aligned} \tag{41}$$

As in the first example, two possibilities may be distinguished:

- $\mu^2 > 0$

  The potential has a minimum at $\phi = \phi^{\dagger} = 0$ that is unique. The symmetry of the Lagrangian is also the symmetry of the ground state, and the spectrum is simply that of ordinary QED of charged scalars with a massless photon A^{μ} and a pair of scalar fields ϕ and $\phi^{\dagger}$ with a common mass μ.

- $\mu^2 < 0$

  This case corresponds to a spontaneously broken local symmetry. We must analyse this situation carefully.

The potential has a continuum of absolute minima, corresponding to a continuum of degenerate vacua, at

$$\langle \phi^{i} \rangle_0 = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{\nu}{\sqrt{2}} \tag{42}$$

with ν real and positive, and expanding in polar coordinates:

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}\left[\nu + \eta(x)\right] e^{i\xi(x)/\nu} \\ &\frac{1}{\sqrt{2}}\left[\nu + \eta(x) + i\xi(x) + \ldots\right]. \end{aligned} \tag{43}$$

Then the Lagrangian appropriate for the study of small oscillations is:

$$\begin{aligned} \mathcal{L} \simeq\, & \frac{1}{2}\left(\partial_{\mu}\eta\right)\left(\partial^{\mu}\eta\right) + \mu^2\eta^2 + \frac{1}{2}\left(\partial_{\mu}\xi\right)\left(\partial^{\mu}\xi\right) + q\nu A_{\mu}\partial^{\mu}\xi \\ & + \frac{1}{2}q^2\nu^2 A_{\mu}A^{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{44}$$

now this looks like the Lagrangian density of a quantum field theory with three fields: η, ξ and A^{μ}. From our study of the Goldstone phenomenon, the η field, which corresponds to radial oscillations, has a mass:

$$m_{\eta} = \sqrt{-2\mu^2} \tag{45}$$

as implied by the term $\mu^2\eta^2$. The gauge field A^{μ} appears to have acquired a mass as implied by the term $\frac{1}{2}q^2\nu^2 A_{\mu}A^{\mu}$, but it is mixed up in the penultimate term with the ξ field which seems to be massless.

An astute choice of gauge will make it easier to sort out the spectrum of the spontaneously broken theory. To this end, it is convenient to rewrite the terms involving A^μ and ξ as:

$$\frac{q^2 v^2}{2}\left(A_\mu + \frac{1}{qv}\partial_\mu \xi\right)\left(A^\mu + \frac{1}{qv}\partial^\mu \xi\right), \tag{46}$$

a form that pleads for the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{qv}\partial^\mu \xi, \tag{47}$$

which corresponds to the phase rotation on the scalar field

$$\phi \to \phi' e^{-i\xi(x)/v}\phi(x) = \frac{v+\eta}{\sqrt{2}}. \tag{48}$$

Knowing that the Lagrangian is locally gauge invariant, we may return to the original expression (37) to compute:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \eta)(\partial^\mu \eta) + 2\mu^2\eta^2\right] + \frac{q^2 v^2}{2}A'_\mu A'^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{49}$$

In this gauge the particle spectrum is manifest:

- An η field with mass $m_\eta^2 = -2\mu^2$,
- A massive vector field A'_μ with mass $m^2 = q^2 v^2$,
- No ξ field.

By our choice of gauge, the ξ field has disappeared from the Lagrangian (we say that it has been gauged away). The gauge transformation (47) shows that what was formerly the ξ field has become the longitudinal component of the massive vector field A'_μ. Before spontaneous symmetry breaking, the theory had four particle degrees of freedom: two scalars ϕ and $\phi^\dagger$ plus two helicity states of the massless gauge field A_μ. After spontaneous symmetry breaking, we are left with one scalar particle η plus three helicity states of the massive gauge field A'_μ, for a total of four particle degrees of freedom.

Thus no massless fields appear, and *the Goldstone theorem does not apply to a local gauge theory*. The massless field ξ that was gauged away reappears as a *longitudinal polarization degree of freedom for the vector field*, giving it a mass. The massive scalar field η is called the physical *Higgs field*, and the special gauge in which the particle spectrum is manifest for spontaneous breaking of a local gauge symmetry is called the Unitary Gauge. This type of symmetry realization is called the *Higgs mode* and the distinctive of this signature is the acquisition of mass by gauge bosons at the expense of would-be Goldstone bosons, which vanish from the theory.

1.3.2 Higgs Mechanism in the Electroweak Gauge Group

The SM Lagrangian density as presented above is not realistic because bare mass terms are not allowed for the electroweak gauge bosons or for the fermions. We want to generate masses for three gauge fields: W^+, W^- and Z; while keeping the photon, A, massless (since the symmetry breaking only affects the EW gauge sector, the gluons will remain massless all along). That makes three degrees of freedom will be needed to become the longitudinal modes of the massive bosons. As explained throughout the previous sections, one needs to introduce a new field with a potential that keeps the Lagrangian invariant under $SU(2)_L \times U(1)_Y$; while making the vacuum not invariant under the EW gauge symmetry. The simplest solution to this that also preserves the renormalizability of the Lagrangian is to introduce a complex $SU(2)_L$ doublet Φ.

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \Phi^\dagger = \begin{pmatrix} \phi^- \\ \phi^{0\dagger} \end{pmatrix}. \tag{50}$$

The most general gauge invariant Lagrangian for the Higgs field that can be constructed with renormalizable interactions is:

$$\mathcal{L}_\phi = D_\mu \Phi^\dagger D^\mu \Phi - V(\Phi) - \left[\bar{Q}_L \tilde{\Phi} Y_U U_R + \bar{Q}_L \Phi Y_D D_R + \bar{L}_L \Phi Y_E E_R + \text{h.c.}\right], \tag{51}$$

the gauge covariant derivative is:

$$\mathcal{D}_\mu \Phi = \left(\partial_\mu + \frac{ig}{2}\tau_a W_{a\mu} + \frac{ig'}{2} B_\mu\right)\Phi, \tag{52}$$

and Y_U, Y_D, Y_E are the so-called Yukawa matrices, which contain the couplings of the Higgs boson with the three families of fermions. The Yukawa term for the up-type quark has to be constructed with the charge conjugate of the Higgs field in order to preserve the EW gauge group:

$$\tilde{\Phi} = i\tau^2 \Phi^\dagger = \begin{pmatrix} \phi^{0\dagger} \\ -\phi^- \end{pmatrix}, \tag{53}$$

where the complex conjugation makes the term invariant under hypercharge and the antisymmetric tensor, $\epsilon_{ij} = \left(i\tau^2\right)_{ij}$, contracts the left-handed doublets in a way $SU(2)_L$ is preserved.

The square of the covariant derivative leads to three and four-points interactions between the gauge and Higgs fields. The equation (10) is the Higgs potential, for $\mu^2 < 0$ there will be spontaneous symmetry breaking, and the vev of $\langle 0|\phi^0|0\rangle$ will generate the W and Z masses. The λ term describes a quartic self-interaction $\lambda\left(\phi^-\phi^+ + \phi^{0\dagger}\phi^0\right)^2$ between the Higgs fields. Vacuum stability requieres $\lambda > 0$.

It is convenient to rewrite Φ in a hermitian basis as:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{pmatrix}, \tag{54}$$

where $\phi_i = \phi_i^\dagger$ represents four hermitian fields. In this new basis the Higgs potential becomes[2]:

$$V\left(\Phi^\dagger\Phi\right) = \frac{\mu^2}{2}\left(\sum_{i=1}^{4}\phi_i^2\right) + \frac{\lambda}{4}\left(\sum_{i=1}^{4}\phi_i^2\right)^2, \tag{55}$$

now we can choose the axis in this four dimensional space so that $\langle 0|\phi_i|0\rangle = 0$, $i = 1,2,4$ and $\langle 0|\phi_3|0\rangle = \nu$, thus:

$$\begin{aligned}\Phi &\to \langle 0|\Phi|0\rangle = \nu = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \nu\end{pmatrix},\\ V\left(\Phi^\dagger\Phi\right) &\to V(\nu) = \frac{\mu^2}{2}\nu^2 + \frac{\lambda}{4}\nu^4,\end{aligned} \tag{56}$$

which must be minimized with respect to ν. Two important cases are illustrated in figure 2. For $\mu^2 > 0$ the minimum occurs at $\nu = 0$ and $SU(2)_L \times U(1)_Y$ is unbroken at the minimum. On the other hand, for $\mu^2 < 0$ the $\nu = 0$ symmetric point is unstable, and the minimum occurs for $\nu \neq 0$, breaking the $SU(2)_L \times U(1)_Y$ symmetry.

The point is found by requiring

$$\left.\frac{\partial V}{\partial \Phi}\right|_\nu = \nu\left(\mu^2 + \lambda\nu^2\right) = 0 \longrightarrow \nu = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{57}$$

The dividing point $\mu^2 = 0$ cannot be treated classically, it is necessary to consider the one loop corrections to the effective potential, in which case it is found that the symmetry is again spontaneously broken [71].

Using the case $\mu^2 < 0$, the generators corresponding to L^1, L^2 and $L^3 - Y$ are spontaneously broken,

$$L^i\nu = \frac{\tau^i}{2\sqrt{2}}\begin{pmatrix}0\\ \nu\end{pmatrix} \neq 0, \quad i = 1,2,3 \qquad Y\nu = \frac{1}{2\sqrt{2}}\begin{pmatrix}0\\ \nu\end{pmatrix} \neq 0. \tag{58}$$

However the vacuum carries no electric charge $\left(Q\nu \equiv (L^3 + Y)\nu = 0\right)$, so the $U(1)_Q$ of electromagnetism is not broken.

To find the masses of the physical particles it is necessary to quantize around the classical vacuum, i.e., write $\phi = \nu + \phi'$, where ϕ' are quantum fields with zero vacuum expectation value. To display the physical particles it is useful to rewrite the four components of Φ in terms of a new set of variables using the Kibble transformation [72]:

$$\Phi = \frac{1}{\sqrt{2}}e^{i\left(\sum_{j=1}^{3}\xi^j L'^j\right)}\begin{pmatrix}0\\ \nu + H\end{pmatrix}, \tag{59}$$

where the L'^j are the three broken generators and H is the physical Higgs boson. The three hermitian fields ξ^j would be the massless pseudoscalar Goldstones bosons, these would have no potential and would only appear

[2] The potential V is $O(4) \sim SU(2) \times SU(2)$ invariant, this is an accidental symmetry.

as derivatives. In this parametrization is made explicit that Φ contains four real degrees of freedom: three phases (the would-be-NGBs), and a radial excitation, the physical Higgs boson field that remains in the particle spectrum.

A theory can be quantized in the unitary gauge and the 't Hooft gauge. As we will see in the next chapter, each gauge has its advantages and disadvantages, for the purpose of this section we will use the unitary gauge for displaying the particle content of the theory, this is possible doing a gauge transformation, the NGBs are "eaten" by the gauge boson fields:

$$\Phi \rightarrow \Phi' = e^{i\left(\sum_{j=1}^{3} \xi^j L'^j\right)} \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}. \tag{60}$$

The Higgs covariant kinetic energy term takes the following simple form in this gauge:

$$(\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) = \frac{1}{2} (0 \;\; v) \left[\frac{g}{2} \tau_a W_{a\mu} + \frac{g'}{2} B_\mu \right]^2 \begin{pmatrix} 0 \\ v \end{pmatrix} + H \text{ terms}. \tag{61}$$

For now we concentrate on the part depending only on v. The equation (61) can be rewritten using:

$$\tau^a W^a = \tau^3 W^3 + \sqrt{2}\tau^+ W^+ + \sqrt{2}\tau^- W^-, \tag{62}$$

where

$$W^\pm = \frac{W^1 \mp i W^2}{\sqrt{2}}, \qquad \tau^\pm = \frac{\tau^1 \pm i\tau^2}{\sqrt{2}}, \tag{63}$$

to obtain

$$\begin{aligned} &\frac{g^2 v^2}{4} W^{+\mu} W^-_\mu + \frac{1}{2}\left(g^2 + g'^2\right) \frac{v^2}{4} \left[\frac{-g' B_\mu + g W^3_\mu}{\sqrt{g^2 + g'^2}} \right]^2 \\ &\equiv M_W^2 W^{+\mu} W^-_\mu + \frac{M_Z^2}{2} Z^\mu Z_\mu, \end{aligned} \tag{64}$$

where $W^\pm$ are the complex charged gauge bosons which will mediate the charged current interactions, and:

$$Z \equiv \frac{-g' B_\mu + g W^3_\mu}{\sqrt{g^2 + g'^2}} = -\sin\theta_w B + \cos\theta_w W^3, \tag{65}$$

is a massive hermitian vector boson which will mediate the new neutral current interaction predicted by $SU(2)_L \times U(1)_Y$. In the second form, θ_w is the weak mixing angle, that rotates the mass matrix to provide the neutral mass eigenstates and it is related to the gauge coupling constants as:

$$\tan\theta_w \equiv \frac{g'}{g} \Rightarrow \quad \sin\theta_w = \frac{g'}{g_Z}, \quad \cos\theta_w = \frac{g}{g_Z}, \tag{66}$$

where

$$g_Z = \sqrt{g^2 + g'^2}. \tag{67}$$

The weak angle is not a prediction of the SM; but a free parameter, with $\sin^2\theta_w \simeq 0.23$ from experimental data [36]. The combination of B and W^3 orthogonal to Z is the photon, with field

$$A = \cos\theta_w B + \sin\theta_w W^3, \tag{68}$$

which remains massless, and the electromagnetic coupling is:

$$e = g\sin\theta_w = g'\cos\theta_w. \tag{69}$$

The (tree-level) masses are predicted to be:

$$M_W = \frac{g\nu}{2}, \quad = \frac{g_Z\nu}{2} = \frac{M_W}{\cos\theta_w}, \quad M_A = 0, \tag{70}$$

implying the relation

$$\sin^2\theta_w = 1 - \frac{M_W^2}{M_Z^2}. \tag{71}$$

Going back to the scalar sector, the full Higgs part of the Lagrangian is:

$$\begin{aligned}\mathcal{L} &= D_\mu\Phi^\dagger D^\mu\Phi - V(\Phi)\\ &= M_W^2 W^{\mu+}W_\mu^-\left(1+\frac{H}{\nu}\right)^2 + \frac{1}{2}M_Z^2 Z^\mu Z_\mu\left(1+\frac{H}{\nu}\right)^2 + \frac{1}{2}\left(\partial_\mu H\right)^2 - V\left(\Phi\right),\end{aligned} \tag{72}$$

where in unitary gauge the Higgs potential (55) becomes

$$V\left(\Phi\right) \to -\frac{\mu^4}{4\lambda} - \mu^2H^2 + \lambda\nu H^3 + \frac{\lambda}{4}H^4. \tag{73}$$

The equation (72) describes the quartic interactions ZZH^2, W^+W^- and the trilinear ones ZZH and W^+W^-H. Also this equation includes the canonical kinetic energy term and potential for the Higgs. In the equation (73) the first term is constant, which reflects the fact that V is defined so that $V(0) = 0$, and therefore $V < 0$ at minimum. A constant term is irrelevant to physics in the absence of gravity[3]. The second term in V represents a (tree level) mass

$$M_H^2 = -2\mu^2 = 2\lambda\nu^2, \tag{74}$$

for the Higgs boson, notice that both M_H and λ are unknown parameters in the SM. However, it turns out that the vacuum expectation value of the Higgs field, ν is precisely the EW energy scale, ν_{EW}, which can be related through the Fermi constant, G_F:

[3] But is one of the most serious problems of the SM when gravity is incorporated because it acts like a cosmological constant much larger (and of opposite sign) than allowed by observations.

$$\frac{G_F}{\sqrt{2}} = \frac{1}{2v_{EW}^2} = \frac{g^2}{8M_W^2} \Rightarrow v = v_{EW} = 246\text{GeV}, \tag{75}$$

thus the EW scale v can be determined form the EW gauge bosons masses and be extracted for instance from the muon decay rate [36]. The last two terms in V are, respectively, the induced cubic and quartic self-interactions of the Higgs.

The fermions acquire a mass through the Yukawa interactions with the Higgs field. In the unitary gauge the Yukawa Lagrangian reads:

$$\mathcal{L}_Y = -\frac{v+H}{\sqrt{2}}\left[\bar{U}_L Y_U U_R + \bar{D}_L Y_D D_R + \bar{L}_L Y_E E_R + \text{h.c.}\right], \tag{76}$$

therefore, once the Higgs boson acquires a vev, and after rotation to the fermion mass eigenstate basis all fermions are given a mass that can be schematically written as:

$$m_f = \frac{y_f v}{\sqrt{2}}, \tag{77}$$

where y_f is the Yukawa coupling for each fermion. Note that the Higgs mechanism does not provide any explanation for the disparity of fermion masses, often referred to as the *flavor puzzle*.

1.4 WHY BEYOND STANDARD MODEL?

The success of this theory as a description of the observed behaviour of fundamental particles has been extraordinary. Yet there is universal consensus that the theory as it stands cannot be complete, regardless of its remarkable success within its domain of applicability. Instead, the Standard Model must be the low energy effective manifestation of a more comprehensive theory; some arguments are "empirical", in the sense that the theory does not explain all the observed properties of matter in the universe, as we would expect of a theory of fundamental particles.

Although the original formulation did not provide for massive neutrinos,they are easily incorporated by the addition of right-handed states ν_R (in the case that the neutrinos are of Dirac, but then the are *also* Majorana) or as higher-dimensional operators (in the case that the neutrinos are of Majorana), however this model has too many free parameters to be a final theory, for massless neutrinos the SM has 20 free parameters, but taking into account massive neutrinos there are another 7 (9) in the case of Dirac (Majorana). The complications of the Standard Model can also be described in terms of a number of problems.

1.4.1 Charge Quantization

One of the most important empirical observations about nature is that the electric charges of elementary particles appear to be quantized. All SM particles have charges which are integer multiples of $e/3$, which thus seems to be the fundamental unit of charge. A deeper theoretical understanding of this electric charge quantization is necessary because it allows the electrical neutrality of atoms ($|q_p| = |q_e|$). Historically, one of the first attempts to solve this puzzle was proposed by Klein [73] in the context of theories of gravitation in higher dimensions, also Dirac found that using his theory of magnetic monopoles [74] the existence of a fundamental unit of charge was necessary.

Electric charge quantization is considered to be a problem in the standard model because the complete gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is not a compact simple group. This problem is specifically related to the group $U(1)_Y$. It was point out in Ref. [75] that group theory may be used to show that observables from the generators of simple groups can always be chosen to have rational eigenvalues, but $U(1)$ is not a simple group and therefore there is no group-theoretic reason for the eigenvalues of the generator to be rational.

1.4.2 Matter-Antimatter Asymmetry

Physicists believe that the Big Bang should have created equal amounts of matter and antimatter in the early universe, however, everything in our universe, from the smallest cell in the Earth to the biggest galaxy is made entirely of matter which means that there is not much antimatter to be found in our universe. If n_b is the number density of the baryons (matter), $n_{\bar{b}}$ is the number density of the antibaryons (antimatter) and n_γ is the number density of the photons then the baryon asymmetry parameter η is defined by:

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} = \frac{n_B}{n_\gamma}, \tag{78}$$

where $n_B = n_b - n_{\bar{b}}$ is the net baryon number density. The WMAP experiment [76] found that the baryon-photon ratio for our universe is:

$$\eta \sim 6.1 \times 10^{-10}. \tag{79}$$

This experimental evidence of more matter than antimatter in our universe is known as the matter-antimatter asymmetry (or the baryon asymmetry) of the universe. Neither the standard model of particles or the general relativity can explain this experimental evidence and for this reason one of the greatest challenges in physics is to figure out what happened to the antimatter, in other words, why we see an matter-antimatter asymmetry.

Andrei Sakharov in 1967, proposed [77] three necessary conditions that a baryon-generating interaction must satisfy to produce a matter-antimatter asymmetry in the universe.

- Baryon number B violation.
- C symmetry and CP symmetry violation.
- Interactions out of thermal equilibrium.

These conditions were inspired by the discoveries of the cosmic background radiation [78] and CP-violation in the neutral kaon system [79]. The Standard Model imply that there are just as many antimatter in the universe as matter because provides processes which create (or destroy) matter and antimatter in equal amounts. But if the Standard Model is a true description of reality, where is all this antimatter? Processes which do not treat matter and antimatter symmetrically are possible at the high energies which characterized the early universe but are not observed at the lower energies scales of the present universe, such events (including proton decay) can occur in Grand Unification Theories (GUTs) and supersymmetric (SUSY) models via hypothetical massive bosons such as the X boson.

1.4.3 Dark Matter and Dark Energy

Vera Rubin and W. Kent Ford confirmed its existence by the observation that the stars are orbiting the galaxy's centre with such speed that the gravity generated by their observable matter could not possibly hold them together. Some non luminous matter that we have yet to detect directly is giving these galaxies extra mass, generating the extra gravity they need to stay intact; this unknown matter was called *dark matter* since it is not visible (it does not appear to interact with the electromagnetic field and is, therefore, difficult to detect) [80]. The Standard Model has no explanation for the observed dark matter, which represent approximately the 27% of all the matter in the universe. An attractive possibility for dark matter is some new kind of elementary particle that has not yet been discovered and it must barely interact with ordinary matter and radiation, except (of course) through gravity. This possibility is the *weakly interacting massive particles*, one early candidate for a WIMP was the *axion*, a very light pseudoscalar arising from a new global $U(1)$ symmetry introduced to solve the strong CP problem of QCD [81]. Another approach is to introduce a discrete, conserved *R parity* into supersymmetric models, which leads to a stable dark matter candidate, such as a *neutralino* or a *gravitino*. However, there are still a few dark matter possibilities that are viable, this may consist of small primordial dark holes, relic extinguished stars of various types, and "Jupiter-like objects", collectively these possiblities are referred to as *Massive Compact Halo Objects* (MACHOs).

The universe is full of matter and the attractive force of gravity pulls all matter together, one might think that the universe has enough energy density to stop its expansion and collapse. The universe is full of matter and the force of gravity pulls all matter together, for this reason that physicists assumed that the attractive force of gravity would slow down the expansion of the universe over time. Observations of supernovae in 1998 from the Hubble Space Telescope (HST) telescope showed that the expansion of the universe has not been slowing due to gravity, it has been accelerating instead [82, 83]. The name given to the mysterious force that is causing the rate of expansion of our universe to accelerate over time is *Dark energy*. Dark energy is one of the most relevant mysteries for theoretical physics, the only proof of its existence it is the effect on the universe's expansion. It turns out that roughly 68% of the universe is dark energy. Dark matter makes up about 27%. The rest (mainly baryons) adds up to less than 5% of the universe. Two proposals for explication of the dark energy are the cosmological constant, representing a constant energy density related whit the space itself, and it would explain why it is not diluted with the expansion of the universe. Another explanation is that it is a new kind of dynamical energy, like a fifth fundamental force which remains unknown, called quintessence", after the fifth element of the Greek philosophers.

1.4.4 The Gravity Problem

All the fundamental particles described by the Standard Model represent ordinary matter, so according with gravity force, hey must interact gravitationally. However general relativity is not a quantum theory, and there is no simple way to introduce one within the SM context, in general grounds we need to adopt as a local gauge symmetry the proper orthochronous Lorentz group. Some possible solutions include Kaluza-Klein and supergravity theories but do not yield renormalizable theories of quantum gravity. Another set of solutions are superstring theories, which unify gravity with the SM and may be finite theories of quantum gravity. This problem is the most obvious deficiency of the Standard Model.

In addition to the fact that gravity is not unified and quantized there is another difficult mismatch between the vacuum energy it predicts with that estimated from cosmological measurements [83, 84]. The first source to consider is the ground state energy density of the Higgs field potential [85, 86]. After the spontaneous symmetry breaking we have the minimum of the Higgs field at the electroweak scale ν, such that $\langle 0|V(\nu)|0\rangle = -\mu^4/4\lambda$. This number is of great importance when the SM is coupled to gravity, because it contributes to the cosmological constant:

$$\Lambda_{cosm} = \Lambda_{bare} + \Lambda_{SSB}, \tag{80}$$

where $\Lambda_{bare} = 8\pi G_N V(0)$ is the primordial constant, i.e, is the value of the energy of the vacuum in the absence of spontaneous symmetry breaking. Λ_{SSB} is the part generated by the Higgs mechanism:

$$|\Lambda_{SSB}| = 8\pi G_N |\langle 0|V|0\rangle| \sim 10^{56}\Lambda_{obs}. \tag{81}$$

It is some 10^{56} times larger in magnitude than the observed value $\Lambda_{obs} \sim (0.0042eV)^4/8\pi G_N$and it is of the wrong sign. The problem becomes even worse in superstring theories,where one expects a vacuum energy of $\mathcal{O}(M_P^4)$, leading to $|\Lambda_{SSB}| \geqslant 10^{123}\Lambda_{obs}$. This is unacceptable and some physicists call it "the largest discrepancy between theory and experiment in all of science".

1.4.5 Hierarchy Problem

The last problem that I will discuss is the so called *hierarchy* or *fine tuning* problem associated with the mass of the Higgs boson. In the standard Model one introduces an elementary Higgs to generate masses for the W, Z and fermions, but scalar fields in quantum field theory, and so the Higgs boson, are subject to quadratically divergent contributions to their mass squared from loop processes:

$$M_H^2 = \left(M_H^2\right)^2_{bare} + \mathcal{O}\left(\lambda, g^2, h^2\right)\Lambda^2, \tag{82}$$

where Λ is the next higher scale in the theory. If the next scale is gravity, Λ is the Planck scale $M_P \sim 10^{19}$ GeV or in a GUTs, Λ is to be order the unification scale $\Lambda \sim 10^{14}$GeV. Hence, the natural scale for M_H is $\mathcal{O}(\Lambda)$, which is much larger that the measured value. These large mass contributions must then be offset by precise values of the Lagrangian coefficients μ and λ in order to achieve the observed electroweak scale, in other words there must be a fine-tuned and apparently highly contrived cancellation between the bare value and the correction to more than 30 decimals places in the case of gravity.

One proposed solution is that one may solve the hierarchy problem via supersymmetry, the superpartners of the SM particles, having different statistics, contribute to the radiative corrections to the Higgs mass with the opposite sign than the SM particles. In the limit of exact supersymmetry, all corrections to M_H^2 cancel. Another way to avoid the problem is the Higgs boson being is a composite state, and the compositeness scale is around the TeV, then the corrections to its mass are cutoff at the TeV scale, one example of these models are the *Little Higgs models*, in which the Higgs is a pseudo-Nambu-Godstone boson and its mass is protected by some global symmetry. This thesis focuses in the study of lepton flavor violating within one of such models.

1.5 HUNTING FOR NEW PHYSICS

Now that we have discussed the SM of particle physics elements and its problems, we are going to explore the quest for new physics beyond the SM (BSM). Search for new physics effects could be done in two ways, direct and indirect tests, the former is carried out testing specific models and the latter is undertaken through precision measurements of SM processes. Over time, high-precision measurements of electroweak observables have found remarkable agreement with the SM, leading to stringent constraints on BSM effects [36, 87, 88].

In the following we are going to discuss (briefly) the precision electroweak physics, which is related with leptons and electroweak gauge bosons (quark-based observables are also important but are not relevant for this work). Some quantities that are sensitive to electroweak physics and have been very well measured are [36]:

- **The electromagnetic coupling**

The fine structure constant, α, quantifies the strength of the electromagnetic interaction between elementary charged particles and can be estimated from the electron anomalous magnetic moment [89], $a_e = (1159652180.91 \pm 0.26) \times 10^{-12}$, but it is also possible to obtain α from the combined measurements of the Rydberg constant and atomic masses with interferometry of atomic recoil kinematics with ^{87}Rb [90] and ^{133}Cs [91]. Combining these methods leads to the world average of $\alpha^{-1} = 137.035999084(21)$. This value of the fine-structure constant corresponds to very low energy limit. In electroweak renormalization schemes, it is convenient to define a running α dependent on the energy scale of the process. In the modified minimal subtraction ($\overline{\text{MS}}$) scheme we have $\hat{\alpha}^{(5)}(M_Z)^{-1} = 127.952 \pm 0.009$.

- **Fermi Constant**

The Fermi constant can be obtained from the μ lifetime formula:

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)\left[1 + H_1\left(\frac{m_e^2}{m_\mu^2}\right)\frac{\hat{\alpha}(m_\mu)}{\pi} + H_2\left(\frac{m_e^2}{m_\mu^2}\right)\frac{\hat{\alpha}^2(m_\mu)}{\pi^2}\right], \quad (83)$$

where

$$\begin{aligned} F\left(\frac{m_e^2}{m_\mu^2}\right) &= 0.99981295, \quad H_1\left(\frac{m_e^2}{m_\mu^2}\right) = -1.80793, \\ H_2\left(\frac{m_e^2}{m_\mu^2}\right) &= 6.64, \qquad \hat{\alpha}^{-1}(m_\mu) = 135.901. \end{aligned} \quad (84)$$

The first term in eq. (83) represents the tree level contribution. H_1 and H_2 contain QED corrections (H_2 also captures hadronic vacuum polarization effects) [92, 93]. Altogether, this yields the Fermi constant value $G_F = 1.1663787(6) \times 10^{-5}\ \text{GeV}^{-2}$.

- **The weak mixing angle**

The weak mixing angle is a parameter of the electroweak interaction, and is usually denoted as θ_W. This weak mixing angle can be measured using the on-shell or the $\overline{\text{MS}}$ scheme, here we are going to follow the last one. In this prescription we introduce the quantity:

$$\sin^2 \hat{\theta}_W(\mu) = \frac{\hat{g}'^2(\mu)}{\hat{g}^2(\mu) + \hat{g}'^2(\mu)}, \tag{85}$$

where the couplings $\hat{g}$ and $\hat{g}'$ are defined by modified minimal subtraction and for most of the electroweak processes we define the scale $\mu \equiv M_Z$. The on-shell and $\overline{\text{MS}}$ definitions are related by:

$$\sin^2 \hat{\theta}_W(M_Z) \equiv \hat{s}_Z^2 = c\,(m_t, M_H)\, s_W^2 = (1.0351 \pm 0.0003)\, s_W^2. \tag{86}$$

For low-energy physics we use the weak mixing angle as $\hat{s}_0^2 \equiv \sin^2 \hat{\theta}_W(M_0)$. In the table 1 we show the numerical value of the weak mixing angle in the different schemes [36].

Scheme	Notation	Value	Uncertainty
On-shell	s_W^2	0.22337	± 0.00010
$\overline{\text{MS}}$	$\hat{s}_Z^2$	0.23121	± 0.00004
$\overline{\text{MS}}$	$\hat{s}_0^2$	0.23857	± 0.00005

Table 1: Numerical values for $\sin^2 \theta_W$ in different schemes.

Of course, there are a lot more electroweak precision data like the muon anomalous magnetic moment, the polarization asymmetry in Z boson production, Z and W pole physics among others, see [36] for more details.

We note that there are many loops that can contribute to radiative corrections of the electroweak observables (discussed above), since they are given at tree-level by gauge boson exchange, the major contributions come from vacuum polarization amplitudes, which involves the third family of quarks and Higgs boson. These are called *oblique corrections*.

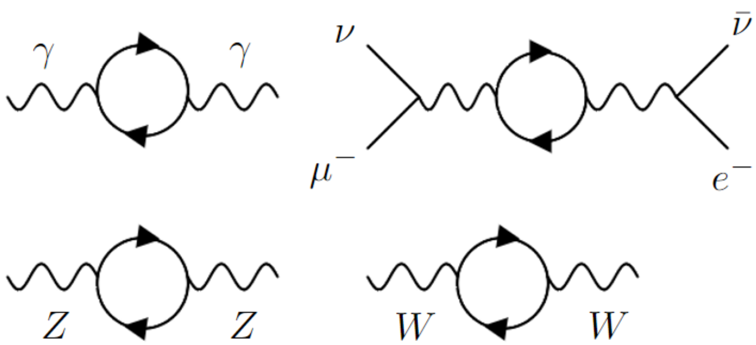


Figure 4: One-loop corrections to weak-interactions observables: electromagnetic coupling, G_F, Z and W pole mass.

All these corrections have been taken into account in the numerical results shown above

1.5.1 Custodial symmetry, T, S and U

From the SM we know that the mass ratio of W and Z gauge boson at tree level, is the quotient between weak and electromagnetic gauge couplings

$$\frac{M_W^2}{M_Z^2} = \frac{g^2}{g^2 + g'^2} = \cos^2\theta_W. \tag{87}$$

The relation (87) is a consequence of the spontaneous symmetry breaking of the gauge symmetry through the Higgs mechanism with a single $SU(2)$ doublet. It is useful to define the ρ-parameter as:

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_w}, \tag{88}$$

in the SM the tree-level value is $\rho_0 = 1$, but what guarantees that $\rho_0 = 1$ holds?, the answer lies in an accidental symmetry. To see this symmetry we need explore the Higgs field and potential

$$\begin{aligned} H &= \frac{1}{\sqrt{2}}\begin{pmatrix} h_3 + ih_4 \\ h_1 + ih_2 \end{pmatrix}, \\ V(H) &= \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2 = \frac{\lambda}{2}\left(h_1^2 + h_2^2 + h_3^2 + h_4^2 - v^2\right)^2, \end{aligned} \tag{89}$$

the potential in eq (89) is invariant under a $SO(4)$ symmetry, under which the quadruplet (h_1, h_2, h_3, h_4) transforms in the fundamental representation. Note that $SO(4)$ has twice generators than $SU(2)$. When H gets its vev $(h_1 = v, h_2 = h_3 = h_4 = 0)$ the symmetry is broken as $SO(4) \to SO(3)$. Thus there are actually three unbroken (global) symmetry directions in the Higgs sector of the SM. In other words, there is a residual global $SU(2)$ symmetry after electroweak symmetry breaking. This is known as custodial $SU(2)$.

The custodial $SU(2)$ symmetry relates $SU(2)_{weak}$ partners, such as the up and down quarks. The Yukawa couplings generally do not respect custodial $SU(2)$, however, the breaking is a small effect since most of the Yukawa couplings are small (this is not true for the top quark). To see how the top quark affects the ρ parameter, we first need a better definition, in terms of the $\overline{MS}$ scheme [36],

$$\rho_0 = \frac{M_W^2}{M_Z^2 \hat{c}_Z^2 \hat{\rho}}, \quad \text{with} \quad \hat{\rho} = 1.01019 \pm 0.00009, \tag{90}$$

which describes new sources of $SU(2)$ breaking that cannot be accounted for by the SM Higgs doublet or by m_t effects. The current experimental value is [36]

$$\rho_0 = 1.00038 \pm 0.00020. \tag{91}$$

Looking for deviations of $\rho_0 = 1$ can tell us about custodial-SU(2) violating interactions, although it is necessary to have some additional ways to constrain new physics. In a more general analysis thus we classify those results in different classes of BSM theories. A classic example is the S, T, U parameters, also known as oblique parameters, proposed by Peskin and Takeuchi [94] (and generalized by others [95, 96]):

$$\begin{aligned}\frac{\hat{\alpha}(M_Z)}{4\hat{s}_Z^2}(S+U) \equiv \frac{\Pi_{WW}^{new}(M_W^2) - \Pi_{WW}^{new}(0)}{M_W^2} - \frac{\hat{c}_Z}{\hat{s}_Z}\frac{\Pi_{Z\gamma}^{new}(M_Z^2)}{M_Z^2} \\ - \frac{\Pi_{\gamma\gamma}^{new}(M_Z^2)}{M_Z^2},\end{aligned} \tag{92}$$

$$\begin{aligned}\frac{\hat{\alpha}(M_Z)}{4\hat{s}_Z^2\hat{c}_Z^2}S \equiv \frac{\Pi_{ZZ}^{new}(M_Z^2) - \Pi_{ZZ}^{new}(0)}{M_Z^2} - \frac{\hat{c}_Z^2 - \hat{s}_Z^2}{\hat{c}_Z\hat{s}_Z}\frac{\Pi_{Z\gamma}^{new}(M_Z^2)}{M_Z^2} \\ - \frac{\Pi_{\gamma\gamma}^{new}(M_Z^2)}{M_Z^2},\end{aligned} \tag{93}$$

$$\hat{\alpha}(M_Z)T \equiv \frac{\Pi_{WW}^{new}(0)}{M_W^2} - \frac{\Pi_{ZZ}^{new}(0)}{M_Z^2} = \frac{\rho - 1}{\hat{\alpha}(M_Z)}. \tag{94}$$

Here *new* means that S, T and U are normalized by subtracting off the Standard Model prediction. The standard model value is $T = S = U = 0$. T parameter measures custodial-SU(2) violation, since it is equivalent to ρ. S parameter can be used to constrain the number of fermion families (even if custodial-SU(2) were preserved), under the assumption that there are no new contributions to T or U. S and T tend to give stronger constraints on BSM physics than U. The current experimental value for these parameters are [36]:

$$\begin{aligned}S &= -0.01 \pm 0.10, \\ T &= 0.03 \pm 0.12, \\ U &= 0.02 \pm 0.11,\end{aligned} \tag{95}$$

if we set $U = 0$ (U is suppressed by and additional factor M_{new}^2/M_Z^2) we get [36]

$$\begin{aligned}S &= 0.00 \pm 0.07, \\ T &= 0.05 \pm 0.06.\end{aligned} \tag{96}$$

2 BACKGROUND

There are more things in Heaven
and Earth, Horatio, than are
dreamt of in your
philosophy.

— William Shakespeare

2.1 NATURALNESS AND THE HIERARCHY PROBLEM

Hierarchy is an important concept in theoretical physics, comes from the mathematical structure of physics and enables us to understand the domain of validity of certain theories using their energy scales

Physical Theory	Energy Scale
Atomic Physics	~ 1 eV - 10 KeV
Nuclear Physics	~ 1 MeV - 1 GeV
Electro-Weak theory	~ 10^2 GeV - (?) GeV
Grand Unified Theories	~ 10^{14} GeV
String Theory	~ 10^{19} GeV

Table 2: Physical theories and their relevant domain.

As already mentioned in the previous chapter, most physicist regard the SM as an effective theory, that at some higher energy scale Λ must become inadequate. This scale must be in the multi-TeV range and, therefore, higher than the Electroweak scale v. Thus, Λ acts as a cut-off which separates low-energy interaction sector and the strongly-interacting sector at higher energies. One might ask: what kind of new particles and interactions lie beyond the electroweak scale? Decades of effort and experimental searching suggest that the answer is the simplest one, nothing. Except for the well-known standard model particles there is nothing all the way (in principle) up to a scale of GUTs. One reason that there must be new physics between the Electroweak scale v and the high energy scale Λ, is the hierarchy problem. This problem is also called *the fine tuning problem* and it was described by Georgi, Quinn and Weinberg in 1974 [6], which says that scalar particles in an effective field theory would have fine tuning due the hierarchy between different energy scales. In the context of the SM, the problem arises from the fact that there are quadratically divergent loop contributions to the Higgs mass, which are much greater that the expected (and measured) value of the Higgs mass itself. These large quantum corrections are only cancelled when the parameters in the theory (such as the tree-level mass parameter) are finely tuned [1]. This is considered unnatural.

What is the meaning of *technical-natural* in physics?, this notion in particle physics is often formulated in terms of the definition given by t'Hooft [97]:

- At any energy scale μ, a physical parameter or set of physical parameters $\alpha_i(\mu)$ is allowed to be very small only if the replacement $\alpha_i(\mu) = 0$ would increase the symmetry of the system.

For example m_ℓ is, technically natural. Setting $m_\ell \to 0$ introduces Chiral symmetry. The ratio M_Z/M_{Planck} is an unnatural example. There has been occasions where naturalness has been a guide for new physics, an example of this is the prediction of the charm quark mass [98]. A quite accurate estimation of the charm quark mass was proposed in order to explain the suppression of flavor changing neutral currents in kaon physics through the GIM mechanism [32].

Here the Higgs mass is very small compared to the high energy scale Λ and the replacement by 0 does not increase the symmetry. Hence it is also (technically) unnatural. Suppose the standard model has a cutoff Λ. Unknown physics resides in the gauge couplings, Yukawa couplings and the Higgs potential, these parameters depend on the structure of a more complete theory.

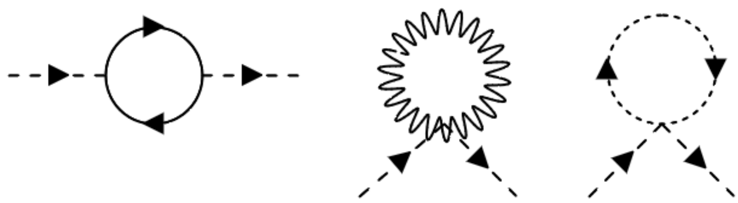


Figure 5: The one loop corrections to the Higgs mass parameter in the SM.

For a Higgs potential of the form (73):

$$V(H) = -\mu^2|H|^2 + \lambda|H|^4, \tag{97}$$

the loops of the figure 5 will contribute to $-\mu^2 \to \left(-\mu^2\right)_{bare} + \delta\mu^2$ (see [99] for more details) where

$$\delta\mu^2 = \frac{\Lambda^2}{16\pi^2}\left[-6y_t^2 + \frac{9g^2}{4} + \lambda^2\right], \tag{98}$$

where Λ is cut off of the theory. The energy scale of the standard model or the mass of every ordinary particle in the standard model is proportional

[1] It should be emphasized that (in principle) it is not a real physical problem however, for there is no physical law or principle that would forbid such a fine tuning. It can be seen as an aesthetic problem.

to the vacuum expectation value of the Higgs field. This is obtained by minimizing the Higgs potential. From electro-weak measurements on gauge boson masses we know that $v \approx 246$ GeV. Similarly the physical Higgs mass is:

$$M_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}v = 125\text{GeV}, \tag{99}$$

which implies $\lambda \sim 0.13 \sim 1/8$. The physical value of v and λ embrace all the quantum corrections in the potential, which in principle, include the quadratically sensitive shift to the mass parameter μ. If $\Lambda \gg$ TeV we would find $\delta\mu^2 \gg \left(-\mu^2\right)_{bare}$, leading to the so-called hierarchy problem. Therefore, the bare mass parameter $(-\mu)_{bare}$ must be fine-tuned to cancel large quantum corrections to get the correct (physical) μ parameter, the bigger the cutoff, the bigger the problem. Suppose we had considered some grand unified theory to replace the standard model at a scale of $\Lambda = 10^{14}$ GeV.

$$-\frac{\mu^2}{\Lambda^2} = \frac{\left(-\mu^2\right)_{bare}}{\Lambda^2} + \frac{1}{16\pi^2}\left[-6y_t^2 + \frac{9g^2}{4} + \lambda^2\right] \sim 10^{-26}. \tag{100}$$

From equation (100) we find that the bare Higgs mass parameter would have to be finely tuned 26 orders of magnitude to cancel all the quantum corrections and reproduce a low energy scale, a small physical Higgs mass and therefore, the masses of the standard model particles. In other words, If the Higgs boson is coupled to the new physics (NP) sector, its mass gets loop corrections from the new heavy particles which are quadratic in their mass $M \sim \Lambda$, this is considered as a problem for a UV completion of the SM.

Note that the hierarchy problem is related to elementary scalars. Why not with fermions or gauge bosons? the reason is that in the limit when their masses go to zero, a new symmetry will appear in the Lagrangian. A new chiral symmetry will appear for fermion masses in the $m \to 0$ limit, protecting their masses from large unsuppressed corrections. This is achieved because the quantum correctios are proportional to the mass itself: $\delta m \propto m \log\frac{\Lambda}{m}$. For gauge bosons there is an unbroken gauge symmetry in the $M_W \to 0$ limit, and similarly, it ensure that $\delta M_W^2 \propto M_W^2 \log\frac{\Lambda}{M_w}$.

In conclusion, the hierarchy problem appears whenever there is NP with particles of mass Λ that couple to the Higgs proportionally to its own mass. This means that the Higgs is sensitive to any beyond standard model (BSM) scale coupled to it.

2.1.1 Little Hierarchy Problem

If the SM is an effective theory, with new degrees of freedom at higher energies, these heavy degrees of freedom are integrated out to generate a theory that cuts off divergent momentum integrals at the scale of these heavy fields. However, the validity of this theory, diminishes as the scale of new physics is approached. After the LEP experiments [7] a *little hierarchy problem* emerged [8], LEP results forbade new physics close to the Electroweak scale, and so there must be a little hierarchy between the Electroweak scale and the scale of new physics. In the years following the Higgs boson discovery [1, 2] the absence of new physics near the Electroweak scale leading us to believe that the cut-off of the SM is not near the weak scale and that new physics will appears above the TeV scale.

Let us assume that the SM is valid up to the energy scale $\Lambda = 10$ TeV. The three most significant quadratically divergent contributions involve the top quark[2], the gauge bosons and the Higgs boson itself. The contributions of those diagrams in figure 5 are [13]:

$$\text{top loop} \qquad -\frac{3}{8\pi^2} y_t^2 \Lambda^2 \sim -(2 \text{ TeV})^2 \,, \tag{101}$$

$$\text{gauge loop} \qquad \frac{1}{16\pi^2} g^2 \Lambda^2 \sim (700 \text{ GeV})^2 \,, \tag{102}$$

$$\text{Higgs loop} \qquad \frac{1}{16\pi^2} \lambda^2 \Lambda^2 \sim (500 \text{ GeV})^2 \,, \tag{103}$$

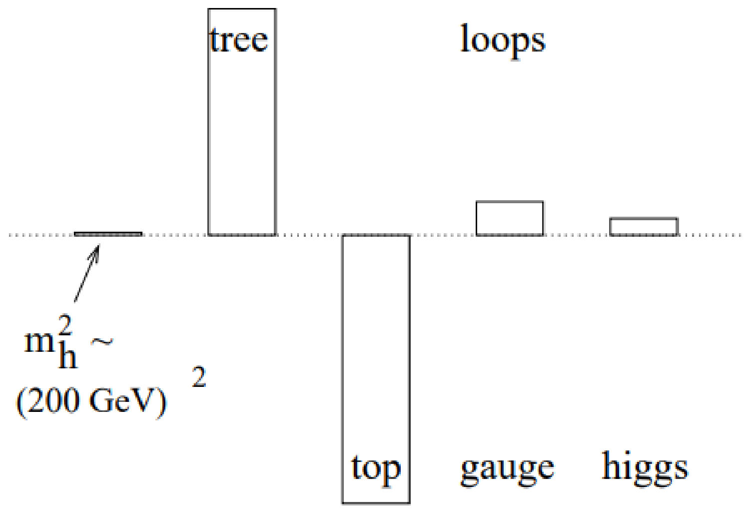


Figure 6: A graphical illustration of the fine tuning of the Higgs boson mass in a SM with a cut off of $\Lambda = 10$ TeV [13].

To obtain the value of the Higgs mass fine tuning at the 1% level is required. If the SM has been proved experimentally near to 1 TeV, why physics beyond the SM has not been found? If we take $\Lambda = 1$ TeV, the contribution from the top loop (which is the largest) is about $(200 \text{ GeV})^2$, so there is no need for fine tuning and therefore, we would expect find new physics

[2] In principle we have to take the other quark loops into account, but we see that these contributions depend on the mass. So for the bottom quark we have: $\frac{-3m_b^2\Lambda^2}{16\pi^2 v^2} \sim -(45 \text{ GeV})^2$, which does not really require fine tuning. The other quarks also contribute, but even less.

around 1 TeV. To obtain a weak-scale expectation value for the Higgs with no less than 10% fine tuning, the naturalness arguments are used to predict aspects of the new physics, also predict bounds for the energy scale at which they arise

$$\lambda_{top} \lesssim 2 \text{ TeV}, \quad \lambda_{gauge} \lesssim 5 \text{ TeV}, \quad \lambda_{Higgs} \lesssim 10 \text{ TeV}. \tag{104}$$

Equation (104) means that the top, gauge, and Higgs loops are bounded by their own scales. This implies that there are new particles with masses not greater than the corresponding cut offs, that couple to the Higgs and that cancel the quadratically divergent contributions from the SM. EW precision measurements put significant constrains on higher dimensional operators, that need to be suppressed by energy scales of $5 - 10$ TeV. This seems to indicate that there is no new physics up to ~ 10 TeV, which creates a tension with the above naturalness requirement that new physics should appear at ~ 1 TeV. This tension is known as the little hierarchy problem.

A resolution to the little hierarchy problem requires a model that proposes new physics at a scale high enough to be allowed by current observational limits while requiring as little fine tuning as possible. Popular theories included supersymmetric theories, technicolor theories and large extra dimensions. Another type of theory that solves this problem stabilizing the Higgs mass in a natural way and exploring the possibility that the Higgs is a pseudo Nambu Goldstone boson, are called *Little Higgs models*.

2.2 LITTLE HIGGS MODELS

A Little Higgs model is a beyond Standard Model (SM) theory that postpones the *hierarchy Problem* that comes up in the SM introducing new particles at an energy of 1 TeV up to a higher scale of energy. These models are based in an old idea which is inspired by the Pions of QCD, that the (light) Higgs is a Pseudo Nambu-Goldstone boson (PNGB), that arises from some spontaneously broken (approximate) symmetry, the first attempts to construct this model were not successful, however a new kind of mathematical structure like *dimensional deconstruction* and *collective symmetry breaking* allowed Nima Arkani-Hamed, Andy Cohen and Howard Georgi create the fist prototype of this theory [10], but it was a "toy" model. The first realistic model (Little Higgs) was develop by Nima Arkani-Hamed, Andy Cohen and Ann Nelson [12]. The key ideas of Little Higgs theories can be summarized by the following points [100]:

- The Higgs fields are Goldstone bosons, associated with some global symmetry breaking at a higher scale Λ.
- The Higgs fields acquire a mass through (collective) symmetry breaking at the electroweak scale and then become pseudo-Goldstone bosons.

- The Higgs fields are protected by the approximate global symmetry, and it is free from one-loop quadratic sensitivity up to the scale Λ, and therefore remain light.

To protect the Higgs mass from large quantum corrections, a global symmetry of a new strongly interacting sector is necessary and then, the Higgs fields are taken to be NGBs corresponding to this spontaneously broken global symmetry. As seen in the previous chapter, an exact NGB only has derivative interactions, therefore a tree-level mass for the Higgs boson cannot be generated. However, due to the Higgs boson couplings with gauge bosons, fermions, and itself, the symmetry is only approximate because these terms explicitly break the global symmetry. A potential and a mass term for the Higgs are generated via quantum effects involving the symmetry breaking interactions.

In consequence, these models are characterized by the existence of separate energy scales in the theory. On the one hand, there is the characteristic scale of the strongly interacting sector, Λ. On the other hand, below, there is the characteristic scale of the vacuum expectation value which breaks the global symmetry, denoted as f. By naive dimensional analysis [101] one can establish a relation between the two scales:

$$\Lambda \sim 4\pi f, \tag{105}$$

these two scales are separated by a gap, alike the gap between the light mesons and the other heavier resonances that exist in QCD. The low energy degrees of freedom in Little Higgs models are described by non linear sigma models. In the energy gap between f and Λ, Little Higgs models are weakly interacting but above the compositeness scale $\Lambda = 4\pi f$ the theory becomes strongly interacting. These models contain the SM particles and extra vector bosons, scalars and fermions. These new particles can have order f masses.

Many Little Higgs models have already been constructed in the last few years, the new particles depend on the particular symmetry of the model. As shown in Ref. [15], without loss of generality, we can divide Little Higgs models into two categories:

- **Product group Models:** In these models the SM $SU(2)$ gauge group arises from the diagonal breaking of two or more gauge groups. These models contain a product of multiple $SU(2)$ gauge bosons at the TeV scale which are obtained from the diagonal breaking of two or even more gauge groups down to $SU(2)_L$. When we expand the fields around the vacuum expectation value, all the generators that are not affected by the diagonal breaking of the $SU(2)$'s to $SU(2)_L$ are still Nambu-Golsdtone bosons and we can choose one as the SM Higgs to achieve the spontaneous electroweak symmetry breaking. All this can be embedded in one non-linear sigma model field.

- **Simple group Models:** In these models the SM $SU(2)$ gauge group arises from the breaking of a single larger group down to an $SU(2)$

subgroup. These kinds of models have two major differences that distinguish them from the product group models. First these models contain an $SU(N) \times U(1)$ gauge symmetry that is broken down to $SU(2)_L \times U(1)_Y$. There are not free parameters because the gauge couplings of $SU(N) \times U(1)$ are fixed in terms of the $SU(2)_L \times U(1)_Y$ gauge couplings. In these models the SM gauge bosons and fermions representations must be extended to transform as a representation of $SU(N)$. This gives rise to additional heavy fermions per generation and forbids mixing between the SM W gauge bosons and their TeV scale counterparts, in contrast to the product group model. Second, to implement the collective symmetry breaking, we require at least two sigma-model multiplets.

Here we will give some examples of Little Higgs models in addition to the Littlest Higgs model introduced by Arkani-Hamed [12] and the T-parity extension to that model proposed by Cheng and Low [16]. In these models the heavy particles are odd under T-parity and the light SM particles are even, but we will not discuss T-parity here.

- **Simplest Little Higgs**: This model is the smallest simple group model. It is represented by two non linear sigma models and the high energy breaking scales f_i of the NGBs fields ϕ_i differ from each other [24, 25]. This model has difficulties in adding a Higgs quartic potential without introducing additional fine tuning.

- $SU(4) \times U(1) \rightarrow SU(2) \times U(1)$: Another simple group model also described by [25]. This model is very closely related to the Simplest Little Higgs. It has 4 instead of 2 ϕ fields which make easier to implement the Higgs quartic term.

- **Minimal moose model**: The minimal moose model is a product group model which transforms under two different gauge groups. Generally a moose model[3] consists of a global G^N symmetry which transform as bifundamental representation under its composite fields ϕ_i. At some high energy breaking scale a subset of each symmetry is then gauged. This forces the bifundamental condensates to break down to the SM at the TeV scale. The minimal moose is just the smallest mooose model [10, 102].

- **Antisymmetric Condensate model**: This model breaks a global $SU(6)$ to a global $Sp(6)$ subgroup at a high energy breaking scale f. Simultaneously a $[SU(2) \times U(1)]^2$ contained in $SU(6)$ is broken down diagonally to the Electroweak SM group [103].

[3]It is called a moose model after the *moose diagrams* that can be formed from its symmetry groups.

2.2.1 CCWZ formalism for Goldstone bosons

The general formalism for effective Lagrangians for spontaneously broken symmetries was worked out by Callan, Coleman, Wess, and Zumino [104, 105]. This formalism allows to write general low-energy effective Lagrangians for strongly (or weakly) coupled theories whit a generic $G \to H$ spontaneous symmetry breaking pattern, describing the Goldstone bosons associated with the symmetry breaking and the heavy resonances. This formalism can be extended to incorporate explicit symmetry breaking.

Consider a classical field theory with n scalar fields ϕ^A, $A = 1, \ldots, n$ with a Lagrangian:

$$\mathcal{L} = \mathcal{L}_{kin} - V\left(\phi^A\right), \tag{106}$$

which is invariant under the Lie group G acting on the scalar fields. The minimum-energy state of the potential is given by $\langle\phi^A\rangle$, and we assume there is a subgroup $H \epsilon G$ under which the vacuum configuration is invariant, i.e. $h\langle\phi^A\rangle = \langle\phi^A\rangle\ \forall\, h \,\epsilon\, H$. There is a spontaneous symmetry breaking of the global symmetry group G down to a subgroup H, in the vacuum configuration. From the Goldstone's theorem we know that there is a zero eigenvalue of the scalar mass matrix for each generator of the coset G/H. It is straightforward to introduce a parametrisation of the Goldstone bosons for a generic spontaneous symmetry breaking pattern G/H. This prescription is given by the CCWZ formalism, which we will review in the following (See [106] for a more pedagogical and explicit discussion).

Let $\phi(x)$ be a set of scalar fields transforming linearly under the continuous global symmetry group G:

$$g : \phi \to g\phi. \tag{107}$$

We denote the generators of the subgroup H and the coset G/H as T^a and X^a respectively, the CCWZ prescription is to parametrise $\phi(x)$ as:

$$\phi(x) = \xi(x)\langle\phi\rangle = e^{i\pi^a(x)\cdot X^a/f}\langle\phi\rangle, \tag{108}$$

Were $\pi^a(x)$ are the Nambu-Goldstone bosons fields, $\langle\phi\rangle$ is the vacuum expectation value which realizes the symmetry breaking $G \to H$, and f sets the scale of the symmetry breaking, is a parameter with mass-dimension. The CCWZ formalism is independent of the representation of ϕ under G.

One might think that also $\xi(x)$ transforms linearly as $\phi(x)$ under the action of $g \,\epsilon\, G$, but this is not true (not in generally): under a global symmetry transformation g, the matrix $\xi(x)$ is transformed to a new matrix $g\xi(x)$, which is in general cannot be parameterize as $\phi(x)$:

$$g : \xi(x) \to g\xi(x) \neq e^{i\pi'(x)\cdot X^a/f}. \tag{109}$$

We can use the fact that the vacuum $\langle\phi\rangle$ is invariant under transformations of the subgroup H, to have a well-defined linear transformation for $\phi(x)$

$$h\langle\phi\rangle = \langle\phi\rangle\,, \quad \forall\, h \in H\,. \tag{110}$$

To find a matrix $U_H \in H$ such that $g\xi(x)U_H^\dagger(g,\pi)$ is in standard form:

$$\begin{aligned} g\phi(x) &= g\left[\xi(x)\langle\phi\rangle\right] = g\xi(x)U_H^\dagger(g,\pi)U_H(g,\pi)\langle\phi\rangle \\ &= g\xi(x)U_H^\dagger(g,\pi)\langle\phi\rangle = \xi'(x)\langle\phi\rangle\,. \end{aligned} \tag{111}$$

The matrix $U_H^\dagger(g,\pi)$ depends on g and ξ: so, under a transformation $g \in G$ the Nambu-Goldstone boson fields transform non-linearly as:

$$g : \xi(x) \to g\xi(x)U_H^\dagger(g,\pi). \tag{112}$$

At the same time, ξ transforms linearly under transformations of the unbroken subgroup H

$$h : \xi(x) \to h\xi(x)h^{-1}. \tag{113}$$

We are going to use an example to show how works the CCWZ parametrisation. Consider a theory whit a symmetry breaking pattern $SU(N) \to SU(N-1)$ and a single scalar field ϕ. Each symmetry can be represented with the generators number, so in order to determining the number of broken generators we need to look the difference in number of generators between the original group $SU(N)$ and the new one $SU(N-1)$:

$$\left(N^2-1\right) - \left[(N-1)^2 - 1\right] = 2N-1, \tag{114}$$

we should thus obtain $2N-1$ Nambu-Goldstone bosons. We are going to choose the following parametrization of our vacuum expectation value f in order to break the group $SU(N)$ to $SU(N-1)$:

$$\phi = \xi\langle\phi\rangle = e^{i\pi^a X^a/f}\langle\phi\rangle = \exp\left[\frac{i}{f}\begin{pmatrix} \pi^0 & & & \pi_1 \\ & \ddots & & \vdots \\ & & \pi^0 & \pi_{N-1} \\ \pi_1^* & \cdots & \pi_{N-1}^* & -(N-1)\,\pi^0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{115}$$

where $\bar{\pi} = (\pi_1, \ldots, \pi_{N-1})$ are complex fields, while π^0 is real, representing the $2N-1$ Nambu-Goldstone bosons of the theory. This way of parametrizing symmetry breaking in a non linear sigma model is called the *CCWZ parametrization*.

We are interesting on the transformation properties of the complex Nambu-Goldstone boson fields. Let's focus on how they transform under the broken and unbroken symmetry groups. Under the unbroken $SU(N-1)$ group, ϕ transforms as:

$$\phi \xrightarrow{T^a} U_{N-1}\phi = \left(U_{N-1}e^{i\pi^a X^a/f}U_{N-1}^\dagger\right)U_{N-1}\langle\phi\rangle = e^{i/f\left(U_{N-1}\pi^a X^a U_{N-1}^\dagger\right)}\langle\phi\rangle\,, \tag{116}$$

We have used that the vacuum $\langle\phi\rangle$ is invariant under unbroken U_{N-1} transformations. Therefore the Nambu-Goldstone bosons transform linearly under the unbroken $SU(N-1)$ group as:

$$\pi^a X^a \xrightarrow{T^a} U_{N-1}\pi^a X^a U_{N-1}^{\dagger}. \tag{117}$$

Where, a $SU(N-1) \to SU(N)$ transformation can be written as:

$$U_{N-1} = \begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{118}$$

from equation (118) we can see that the $N-1$ complex Goldstone bosons transform in the fundamental representation of $SU(N-1)$:

$$\begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^{\dagger} & 0 \end{pmatrix} \to U_{N-1} \begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^{\dagger} & 0 \end{pmatrix} U_{N-1}^{\dagger} = \begin{pmatrix} 0 & \hat{U}_{N-1}\bar{\pi} \\ \bar{\pi}^{\dagger}\hat{U}_{N-1}^{\dagger} & 0 \end{pmatrix}. \tag{119}$$

Under a symmetry transformation of the broken coset group[4] G/H we have, on the other hand:

$$\begin{aligned} U_{G/H} e^{i\pi^a X^a/f} \langle\phi\rangle &= \exp\left[i \begin{pmatrix} 0 & \bar{\alpha} \\ \bar{\alpha}^{\dagger} & 0 \end{pmatrix}\right] \exp\left[\frac{i}{f} \begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^{\dagger} & 0 \end{pmatrix}\right] \langle\phi\rangle \\ &= \exp\left[i \begin{pmatrix} 0 & \bar{\alpha} \\ \bar{\alpha}^{\dagger} & 0 \end{pmatrix}\right] \exp\left[\frac{i}{f} \begin{pmatrix} 0 & \bar{\pi} \\ \bar{\pi}^{\dagger} & 0 \end{pmatrix}\right] U_H^{\dagger}(\alpha, \pi) \langle\phi\rangle \\ &= \exp\left[\frac{i}{f} \begin{pmatrix} 0 & \bar{\pi}' \\ \bar{\pi}'^{\dagger} & 0 \end{pmatrix}\right] \langle\phi\rangle , \end{aligned} \tag{120}$$

to get the equation (120) we use the Baker-Campbell-Hausdorff formula until the second order. This equation defining a non–linear transformation law for the Goldstone bosons. Note that, to linear order in α, the transformation (120) reduces to a shift transformation (Nambu-Goldstone bosons shift under the group $SU(N)$, because at leading order they only have derivative couplings):

$$\bar{\pi} \xrightarrow{T^a} \bar{\pi}' = \bar{\pi} + \bar{\alpha} \cdot f + \mathcal{O}\left(\alpha^2\right). \tag{121}$$

The most general effective field theory for Nambu-Goldstone boson (all the heavy fields are integrated out) can be constructed write down all Lorentz and G invariant terms that has only derivative couplings, however, for general G and H, this is not trivial. For example, if we consider two-derivatives term, one could use the field $\xi(x)$ in the parameterization (108)

$$f^2 \mathrm{Tr}|\partial_\mu \xi|^2, \tag{122}$$

but in general this term is not G invariant

$$f^2 \mathrm{Tr}|\partial_\mu \xi|^2 \to f^2 \mathrm{Tr}\left|\partial_\mu \left(g\xi(x)U^{\dagger}(x)\right)\right|^2 \neq f^2 \mathrm{Tr}|\partial_\mu \xi|^2, \tag{123}$$

[4] Remember that $G \equiv SU(N)$ and $H \equiv SU(N-1)$.

because of the x dependence in $U(g,\pi)\epsilon H$. It can be shown that:

$$\mathrm{Tr}|\partial_\mu \xi|^2 = \mathrm{Tr}\left[\left(\partial_\mu \xi^\dagger\right) \xi\xi^\dagger \left(\partial^\mu \xi\right)\right] = \mathrm{Tr}\left[\left(\xi^\dagger \partial_\mu \xi\right)^\dagger \left(\xi^\dagger \partial^\mu \xi\right)\right], \qquad (124)$$

the object $\xi^\dagger \partial_\mu \xi$ decomposes as:

$$\xi^\dagger \partial_\mu \xi = v_\mu^a T^a + p_\mu^a X^a, \qquad (125)$$

it can be shown that $v_\mu \equiv v_\mu^a T^a$ and $p_\mu \equiv p_\mu^a X^a$ are transformed as

$$\begin{aligned} v_\mu &\to U\left(v_\mu + \partial_\mu\right) U^\dagger \\ p_\mu &\to U p_\mu U^\dagger. \end{aligned} \qquad (126)$$

The field v_μ transforms like a connection, while with p_μ we are able to construct a G invariant two-derivatives term, but it has a non-trivial term is given by:

$$\mathcal{L} = f^2 \mathrm{Tr}\left[p_\mu p_\mu^\dagger\right]. \qquad (127)$$

However the form of p_μ and v_μ depends on the specific groups G and H.

If the Lie algebra of the coset space G/H is a *symmetric space*[5] everything can be simplified, roughly speaking, a symmetric space is a coset space whit an involutive automorphism on the generators:

$$T^a \to T^a, \qquad X^a \to -X^a, \qquad (128)$$

applying the automorphism to eq. (125), we find that p_μ is given by:

$$p_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger\right). \qquad (129)$$

We can thus rewrite the two-derivative term of eq. (127) as:

$$\mathcal{L} = \frac{f^2}{4} \mathrm{Tr}|\partial_\mu \Sigma|^2, \qquad (130)$$

the equation (130) contains the NGB kinetic term canonically normalized, where we have defined

$$\Sigma = \xi\tilde{\xi}^\dagger = \xi^2 = e^{2i\pi^a X^a/f}, \qquad (131)$$

with $\tilde{\xi}$ the image of ξ under the automorphism. From eq. (112), we see that Σ transforms as

$$\Sigma \to g\Sigma\tilde{g}^\dagger, \qquad (132)$$

[5] In mathematics, a symmetric space is a pseudo-Riemannian manifold whose group of symmetries contains an inversion symmetry about every point. From the point of view of Lie theory, a symmetric space is the quotient G/H of a connected Lie group G by a Lie subgroup H which is (a connected component of) the invariant group of an involution of G.

where $\tilde{g}$ is the image of g under the automorphism. Therefore, in symmetric spaces we can construct a Goldstone matrix Σ that is an element of G/H but transforms linearly under G.

We can then summarize the CCWZ formalism with a prescription for constructing the most general effective field theory of only NGBs degrees of freedom:

- Identify the groups G and H describing the spontaneous symmetry breaking pattern.
- Construct the NGB matrix $\xi(x)$ and consequently the quantities p_μ, ν_μ or $\Sigma(x)$ depending on whether the coset G/H is a symmetric group or not.
- Write all Lorentz and G invariant terms with p_μ, ν_μ (or $\partial_\mu\Sigma$) as building blocks, with increasing number of derivatives.
- Identify the finite cut off up to which the theory is valid.

2.2.2 Collective Symmetry Breaking

We are going to use as an example the symmetry breaking of a global group $SU(3)$ down to $SU(2)$ to give a notion about the collective symmetry breaking. According with the previous section, there are five broken generators, so five massless NGB's will appear, four of them form a $SU(2)$ (complex) doublet, which only has derivative terms. However, when we add couplings to the gauge bosons and fermions, we are adding non-derivative terms which explicitly do not respect the $SU(3)$ symmetry [107]. This results in the Nambu-Goldstone bosons becoming pseudo- Nambu-Goldstone bosons. If we assume that these terms are added with a very small coupling ϵ, all the non-derivative interactions will be proportional to ϵ.

Collective symmetry breaking works with two (or more) couplings in such a way that both couplings on their own preserve the original symmetry, but together they break the symmetry. Roughly speaking, we can consider the $SU(3)$ invariant Lagrangian $\mathcal{L}_0$, and then, we add two sets of interactions $\mathcal{L}_{1,2}$:

$$\mathcal{L} = \mathcal{L}_0 + \epsilon_1\mathcal{L}_1 + \epsilon_2\mathcal{L}_2. \tag{133}$$

We see that each term with the ϵ_i with $i = 1, 2$ separately conserve an $SU(3)$ symmetry, in consequence there are two $SU(3)$ symmetries. Only when both terms are taken together the symmetries are broken to the diagonal subgroup. It is said that the symmetry is collectively broken by the action of the two terms: if one of the two ϵ_i is set to zero, restores the full two-group symmetry, leaving us with exactly massless Nambu-Goldstone bosons in the coset space. Only when both terms are non-zero the remaining symmetry is the diagonal subgroup, leaving us with massive pseudo Nambu-Goldstone bosons. In the following section we will present illustrations of the Little Higgs idea and collective symmetry breaking.

2.2.3 The road to the Simplest Little Higgs model

In this section we will consider a toy model to illustrate the idea of collective symmetry breaking and then, we will see how this model is related with the Simplest Little Higgs model [25, 24]. Consider a theory with a global $SU(3)$ symmetry, spontaneously broken to an $SU(2)$ subgroup at the scale f, by a vacuum condensate transforming in the fundamental representation. This condensate can be written as:

$$\langle\phi\rangle = \phi_0 = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \tag{134}$$

The first thing we can do is count the number of broken generators, which by Goldstone's theorem equals the number of NGBs:

$$N_{NGBs} = \left(3^2 - 1\right) - \left(2^2 - 1\right) = 5. \tag{135}$$

Following the CCWZ prescription we can describe this theory of NGBs by the $SU(3)/SU(2)$ non-linear sigma model

$$\phi = e^{\frac{i}{f}\Pi}\phi_0 = e^{\frac{i}{f}\pi^a T_a} \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}, \tag{136}$$

with $\Pi(x) = \sum_{a=4}^{8} \pi^a(x) T_a$ and $\pi^a(x)$ the 5 NBGs. The generators $T_4, \ldots, T_8$ correspond to the broken generators of $SU(3)$ and the ones T_1, T_2, T_3 leave the ground state invariant. Writing Π as a matrix gives:

$$\Pi = \begin{pmatrix} \frac{\pi_5}{2\sqrt{3}} & 0 & \frac{1}{2}\left(\pi_1 - i\pi_2\right) \\ 0 & \frac{\pi_5}{2\sqrt{3}} & \frac{1}{2}\left(\pi_3 - i\pi_4\right) \\ \frac{1}{2}\left(\pi_1 + i\pi_2\right) & \frac{1}{2}\left(\pi_3 + i\pi_4\right) & -\frac{\pi_5}{\sqrt{3}} \end{pmatrix}. \tag{137}$$

We redefine the NGBs, $\pi_5 \to \frac{\sqrt{3}}{2}\eta$, $\frac{1}{2}\left(\pi_1 - i\pi_2\right) \to h_1$, $\frac{1}{2}\left(\pi_3 - i\pi_4\right) \to h_2$ to obtain:

$$\Pi = \frac{\eta}{4}\begin{pmatrix} \mathbb{I}_{2\times 2} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & -2 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{h}_{2\times 1} \\ \mathbf{h}^\dagger_{1\times 2} & 0 \end{pmatrix}, \tag{138}$$

note that h is a doublet under the unbroken $SU(2)$ symmetry, i.e. h is a complex scalar like the Standard Model Higgs, but it is also a NGB that, shifts under "broken" $SU(3)$ transformations. Moreover, we have the η state which transforms as a singlet under $SU(2)$ and for now we may ignore it. To see what kind of self-interactions we get for h, we expand the eq. (136) with Π as in eq.(138):

$$\phi = \exp\left[\frac{i}{f}\begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{h}_{2\times 1} \\ \mathbf{h}^\dagger_{1\times 2} & 0 \end{pmatrix}\right]\begin{pmatrix} \mathbf{0}_{2\times 1} \\ f \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2\times 1} \\ f \end{pmatrix} + i\begin{pmatrix} \mathbf{h}_{2\times 1} \\ 0 \end{pmatrix} - \frac{1}{2f}\begin{pmatrix} \mathbf{0}_{2\times 1} \\ h^\dagger h \end{pmatrix} + \ldots \tag{139}$$

Therefore, we would only expect derivative terms for π_i to appear. This NLSM involves non-renormalizable interactions, because we are working with an effective theory that is valid below a cut off, Λ.

To obtain a low-energy effective interactions we write all $SU(3)$ invariant terms constructed with ϕ and with increasing powers of ϕ derivatives. The first two terms with zero derivatives are $\phi^\dagger\phi = f^2$ and $\epsilon^{abc}\phi_a\phi_b\phi_c \equiv 0$ (because of symmetry and antisymmetry). The next relevant term is one with two derivatives, and turns be the most important one: $\partial_\mu\phi^\dagger\partial^\mu\phi$:

$$\partial_\mu\phi^\dagger\partial^\mu\phi = \partial_\mu h^\dagger\partial^\mu h + \frac{1}{4f^2}\partial_\mu\left(h^\dagger h\right)\partial^\mu\left(h^\dagger h\right), \tag{140}$$

There is a kinetic term for the Higgs doublet and interactions which are suppressed by the symmetry breaking scale f. The second term in equation (140) produces a loop correction to the kinetic term, and as an effective theory we expect that the interactions to become comparable to the tree-level kinetic term at the cut off scale Λ. We can make a rough estimate of the cut off in this theory by computing the loop diagram in the figure 7:

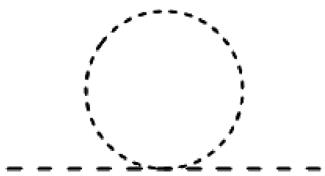


Figure 7: One loop corrections to the kinetic term for h.

This correction becomes $\mathcal{O}(1)$ if:

$$\begin{aligned}\frac{1}{f^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2} &= \frac{1}{f^2\,(2\pi)^4}\int_{S^3}d\Omega\int_0^\Lambda d|k|\frac{|k|^3}{|k|^2} = \frac{2\pi^2\Lambda^2}{f^2\,(2\pi)^4\,2} \\ &= \frac{\Lambda^2}{16\pi^2 f^2}\sim 1,\end{aligned} \tag{141}$$

so as a relation between the cut off and f we find $\Lambda \sim 4\pi f$ (as found by naive dimensional analysis). For energies above Λ, interactions in this model violate unitarity and the theory must be UV completed.

Until now, we have a theory which produces a Higgs-like doublet which is a NGB that transforms under an unbroken $SU(2)$. As we know the doublet only have derivative interactions (due to the shift symmetry), therefore no mass term can arise. In addition, there can be no gauge or Yukawa couplings or a quartic potential for this reason. To obtain the standard model

Higgs boson we need to incorporate these interactions in the theory. We know that in order to get a Higgs potential the $SU(3)$ group must be explicitly broken, this can be done by gauging an $SU(2)$ subgroup[6]. The explicit breaking generates a potential for the (p)NGBs through $SU(2)$ gauge bosons loops. Mathematically we do this by replacing partial derivatives with covariant ones, the gauge fields live in the unbroken $SU(2)$ subspace (ignoring hypercharge for the moment):

$$\partial_\mu \to \partial_\mu - igW_\mu^a(x)T^a, \tag{142}$$

where g is the gauge coupling constant, the W_μ^a are the $SU(2)$ gauge fields, and the $T^a (a = 1, 2, 3)$ are gauged generators given by:

$$T^a = \begin{pmatrix} \sigma^a/2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{143}$$

the σ^a being the Pauli matrices. Expanding the Higgs kinetic term, we now find:

$$\begin{aligned} \mathcal{L}_H = |D_\mu\phi|^2 &= \left|\partial_\mu\phi - ig\begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix}\phi\right|^2 \\ &= |\partial_\mu\phi|^2 + ig\phi^\dagger\begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix}\partial^\mu\phi - ig\partial_\mu\phi^\dagger\begin{pmatrix} W^\mu & 0 \\ 0 & 0 \end{pmatrix}\phi + g^2\left|\begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix}\phi\right|^2 \\ &= \partial_\mu h^\dagger\partial^\mu h + \frac{1}{4f^2}\partial_\mu\left(h^\dagger h\right)\partial^\mu\left(h^\dagger h\right) + igh^\dagger W_\mu\partial^\mu h - ig\partial_\mu h^\dagger W^\mu h \\ &+ g^2 W_\mu^\dagger W^\mu h^\dagger h, \end{aligned} \tag{144}$$

The interactions in equation (144), lead to quadratically divergent Feynman diagrams (see figure 8) which are present in the Standard Models. The global $SU(3)$ invariance is explicitly broken via the interaction $|gW_\mu h|^2$ with the $SU(2)$ gauge bosons W.

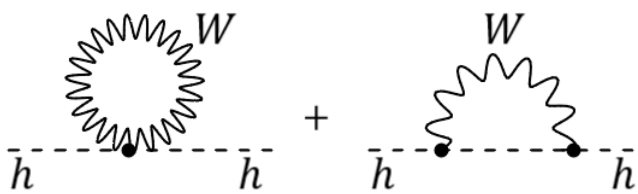


Figure 8: Quadratically divergent one loop contributions to the Higgs boson mass and quartic coupling from the gauge sector.

Both diagrams are quadratically divergent, and the loop momentum integrals need to be cut off at a certain ultraviolet energy scale to obtain a finite result:

[6] Note that the $SU(2)$ subgroup does not generically coincide with the subgroup $H = SU(2)$ that survives the spontaneous breaking at f.

$$V_{1-\text{loop}}(\phi) \propto \frac{g^2}{16\pi^2}\Lambda^2 h^\dagger h, \tag{145}$$

The result above is the kind of quadratic dependence on the cut off scale, that we were trying to solve. Now, let us calculate the Higgs potential in a more efficient way following the Reference [99], the first step is to extract the term quadratic in W from the eq. (144)

$$\mathcal{L}_H \supset g^2 \left| \begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix} \phi \right|^2 = M^2(h)\, W_\mu^\dagger W^\mu, \tag{146}$$

where

$$M(h) = gP\phi \equiv g \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix} \phi, \tag{147}$$

is a Higgs dependent mass matrix for W_μ, and we use an $SU(3)$ breaking spurion[7] P from this term we can compute the Coleman-Weinberg potential [71] for the Higgs, which arise to radiative corrections from the gauge bosons. This potential is given by:

$$V_{CW}(h) = \frac{\Lambda^2}{16\pi^2}\text{Tr}\left[M^\dagger M\right] + \frac{3}{64\pi^2}\text{Tr}\left[M^\dagger M\right]^2 \log\left[\frac{M^\dagger M}{\Lambda^2}\right], \tag{148}$$

The Higgs potential is quadratically divergent, because $M^\dagger M$ is not $\propto 1$. Which means,

$$V_{CW}(h) = \frac{\Lambda^2}{16\pi^2}\text{Tr}\left[M^\dagger M\right] = \frac{g^2\Lambda^2}{16\pi^2}\text{Tr}\left[\phi^\dagger P^\dagger P\phi\right] = \frac{g^2\Lambda^2}{16\pi^2} h^\dagger h. \tag{149}$$

If not for the spurion P, the quadratic divergence would be proportional to $\phi^\dagger\phi$ which is independent of h. We see that gauging only the $SU(2)$ causes the appearance of quadratically divergent 1-loop contributions from the gauge bosons, despite the fact that we have the right gauge couplings. Now if we gauge the entire $SU(3)$ group, we have the following covariant derivative:

$$\partial_\mu\phi \to \left(\partial_\mu - igG_\mu^a T^a\right)\phi \equiv \left(\partial_\mu - igG_\mu\right)\phi, \tag{150}$$

that containing the 8 $SU(3)$ gauge bosons, so the Lagrangian for the kinetic Higgs term is:

$$\begin{aligned} \mathcal{L}_H = |D_\mu\phi|^2 &= \left(\partial_\mu\phi^\dagger + ig\phi^\dagger G_\mu\right)\left(\partial^\mu\phi - igG_\mu\phi\right) \\ &= \phi_0^T e^{-\frac{i}{f}\Pi}\left(g\sum_{a=1}^{8} G_\mu^a T^a - \frac{1}{f}\sum_{a=4}^{8}\left(\partial_\mu\pi_a\right)T^a\right)^2 e^{\frac{i}{f}\Pi}\phi_0 \\ &= g^2 f^2\left(\sum_{a=4}^{8}\left(G_\mu^a - \frac{1}{gf}\partial_\mu\pi^a\right)^2\right) + \text{h.o.t.} \end{aligned} \tag{151}$$

[7] A spurion is a fictitious, auxiliary field that can be used to parameterize any symmetry breaking and to determine all operators invariant under the symmetry

We can redefine our gauge bosons as:

$$G'^{a}_{\mu} = G^{a}_{\mu} - \frac{1}{gf}\partial_{\mu}\pi^{a}, \tag{152}$$

so we have eaten our Goldstone bosons and generated mass terms for 5 of our gauge bosons (those lying in the coset space $SU(3)/SU(2)$). However, our Higgs field has also vanished. In other words, we applied a gauge transformation, by redefining our gauge fields, and as a result our Higgs field was lost. As our expression is gauge invariant, this means our Higgs field is unphysical. In terms of the Coleman-Weinberg potential, we get rid of the spurious $P^{\dagger}P$ in eq.(149) and the quadratic divergence becomes:

$$V_{CW}(h) = \frac{\Lambda^2}{16\pi^2}\mathrm{Tr}\left[\phi^{\dagger}\phi\right] \sim \frac{g^2 f^2 \Lambda^2}{16\pi^2} + \dots \tag{153}$$

which is independent on $h^{\dagger}h$ and the Higgs boson becomes into a Goldstone boson which gives mass of order f to the heavy gauge bosons. Finally, we introduce the gauge couplings in a collective manner to preserves the Higgs boson uneaten and avoid quadratically divergent contributions to the Higgs mass. We introduce two copies of the non linear sigma field $\phi_{1,2}$. The two fields form condensates at the scales f_1 and f_2, this can be written as:

$$\langle\phi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}, \quad \langle\phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix}. \tag{154}$$

We are assuming for simplicity identical and aligned vacuum expectation values[8] so that $f_1 = f_2 = f$. The global symmetry is therefore extended to $SU(3)_1 \times SU(3)_2$ and is spontaneously broken by $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ to $SU(2)_1 \times SU(2)_2$, generating 10 NGBs. We then gauge the diagonal subgroup $SU(3)_D \subset SU(3)_1 \times SU(3)_2$, as a result the global symmetry is also explicitly broken, leaving a singlet and complex doublet as massless NGBs. We can parametrize this breaking by two non linear sigma fields ϕ_1 and ϕ_2, in the first and second coset space $SU(3)/SU(2)$, respectively:

$$\phi_1 = e^{\frac{i}{f}\Pi_1}\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}, \quad \phi_2 = e^{\frac{i}{f}\Pi_2}\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}. \tag{155}$$

Now we must add two covariant derivative terms to the Lagrangian. Note that the covariant derivative is the same for both fields

$$\mathcal{L}_H = |D_{\mu}\phi_1|^2 + |D_{\mu}\phi_2|^2, \quad \text{with} \quad D_{\mu} = \partial_{\mu} - igG^{a}_{\mu}T^{a}. \tag{156}$$

If we assume that the gauge coupling of ϕ_1 is set to zero, the $SU(3)_1$ global symmetry will be restore, and we would have a complex doublet as an exact NGB associated with the coset $SU(3)_1/SU(2)_1$, by the other hand, the other complex doublet would be eaten by the heavy $SU(3)$ gauge bosons. The same argument is valid if we turn off the gauge coupling to ϕ_2. Only when

[8] Later we will work with an misalignment between vevs.

the gauge couplings for both ϕ_1 and ϕ_2 are present the global symmetry is broken, massive $SU(3)$ gauge bosons can be generated and the Higgs (now a pseudo–Nambu-Goldstone boson) can develop a potential. This implies that any term contributing to the mass of the Higgs must come from diagrams that contain both g_1 and g_2. The two Π_i filds can be parametrized as:

$$\phi_1 = e^{\frac{i}{f}\Pi_\perp} e^{\frac{i}{f}\Pi_\parallel} \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}, \qquad \phi_2 = e^{-\frac{i}{f}\Pi_\perp} e^{\frac{i}{f}\Pi_\parallel} \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}, \tag{157}$$

in eq. (156) $\Pi_\parallel$ represent the part that is coaligned with the $SU(3)$ gauge bosons and can be eaten by them (like in eq. (151)), i.e, $\Pi_\parallel$ is an unphysical matrix which means that 5 $SU(3)$ gauge bosons in $SU(3)/SU(2)$ acquire mass of order gf. This matrix can be removed by a gauge transformation. The perpendicular $\Pi_\perp$ is the part where ϕ_1 and ϕ_2 are perpendicular to the gauge bosons, leading as a result that this part remains massless and it cannot be removed by a gauge transformation and thus represents a physical matrix. The action of both fields can generate a potential for these NGBS, which break the symmetry explicitly leading a set of pNGBs that can then cause spontaneous electroweak symmetry breaking, as the Higgs in the Standard Model.

We are going to work in the unitary gauge because has the advantage of working with the physical spectrum of particles. So, we rotate $\Pi_\parallel$ away, this leaves us with the two perpendicular fields (also, we leave out the singlet state):

$$\phi_1 = \exp\left[\frac{i}{f}\begin{pmatrix} \mathbf{0}_{2\times2} & \mathbf{h}_{2\times1} \\ \mathbf{h}^\dagger_{1\times2} & 0 \end{pmatrix}\right]\begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix}, \qquad \phi_2 = \exp\left[-\frac{i}{f}\begin{pmatrix} \mathbf{0}_{2\times2} & \mathbf{h}_{2\times1} \\ \mathbf{h}^\dagger_{1\times2} & 0 \end{pmatrix}\right]\begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix}, \tag{158}$$

the different possible products of ϕ_1 and ϕ_2 yields:

$$\begin{aligned}
|\phi_1^\dagger \phi_1| &= (\mathbf{0}_{1\times2} \quad f)\, e^{-\frac{i}{f}\Pi_\perp} e^{\frac{i}{f}\Pi_\perp} \begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix} = f^2, \\
|\phi_2^\dagger \phi_2| &= (\mathbf{0}_{1\times2} \quad f)\, e^{-\frac{i}{f}\Pi_\perp} e^{\frac{i}{f}\Pi_\perp} \begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix} = f^2, \\
|\phi_1^\dagger \phi_2| &= (\mathbf{0}_{1\times2} \quad f)\, e^{-\frac{i}{f}\Pi_\perp} e^{-\frac{i}{f}\Pi_\perp} \begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix}, \\
&= (\mathbf{0}_{1\times2} \quad f)\left[\mathbb{I} - \frac{2i}{f}\begin{pmatrix} \mathbf{0}_{2\times2} & \mathbf{h}_{2\times1} \\ \mathbf{h}^\dagger_{1\times2} & 0 \end{pmatrix} - \frac{2}{f^2}\begin{pmatrix} \mathbf{h}\mathbf{h}^\dagger{}_{2\times2} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{1\times2} & h^\dagger h \end{pmatrix} + \ldots\right]\begin{pmatrix} \mathbf{0}_{2\times1} \\ f \end{pmatrix} \\
&= f^2 - 2h^\dagger h + \ldots
\end{aligned} \tag{159}$$

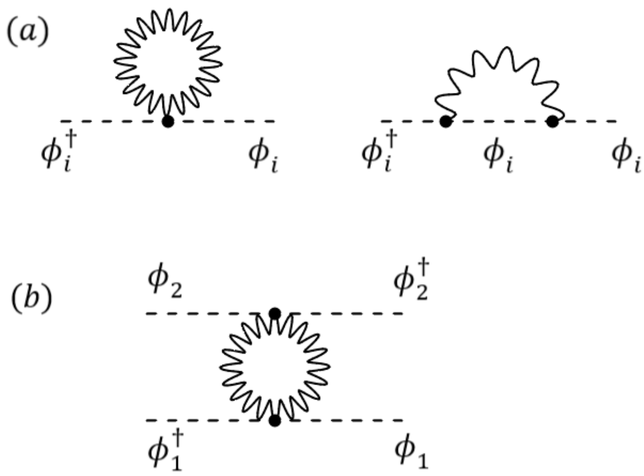


Figure 9: (a) Quadratically divergent one loop contributions which do not contribute to the Higgs potential and (b) logarithmic divergent contributions to the Higgs mass.

Diagrams (a) in fig. 9 (come from terms like $|D_\mu \phi_i|^2$) do not generate a quadratic divergent contribution to the Higgs mass at one loop level as in eq. (153) because they have no h dependence, and these diagrams contribute

$$V_{CW}(\phi_i) = \frac{\Lambda^2}{16\pi^2} \sum_{i=1}^{2} \mathrm{Tr}\left[\phi_i^\dagger \phi_i\right] \sim \frac{\Lambda^2 f^2}{16\pi^2}\left(g_1^2 + g_2^2\right) + \ldots . \quad (160)$$

However, we have a third diagram that contribute to the radiative generation of a Higgs potential, which is the diagram (b) of the figure 9, where both ϕ_1 and ϕ_2 are directly coupled through a gauge boson loop. By counting powers of momentum, we see that diagrams with both ϕ fields are logarithmically divergent and give contribution to the Higgs mass-squared at one-loop level as:

$$V_{CW}\left(\phi_{1,2}\right) = -\frac{3 g_1^2 g_2^2}{64\pi^4} \mathrm{Tr}\left[|\phi_1^\dagger \phi_2|^2\right] \log\frac{\Lambda^2}{\mu^2} \sim \frac{g_1^2 g_2^2 f^2}{4\pi^2} \log\frac{\Lambda^2}{\mu^2} h^\dagger h + \ldots . \quad (161)$$

Finally, we avoid one loop quadratic divergences to the Higgs mass with the collective symmetry breaking mechanism. For example if we take $g_1 = g_2 = g$ equal to the $SU(2)$ gauge coupling, $\mu \sim \nu$ and $f \sim 1$ TeV this gives a contribution to the Higgs mass-squared of the order $M_H^2 \sim \mathcal{O}\left(100\mathrm{GeV}\right)^2$, which looks pretty well. Summarizing, if two ϕ fields are involved the $\Pi_\perp$ can contribute to the Higgs mass, but not quadratically, only logarithmically which ensure that the little hierarchy problem is solved up to the cut off energy scale Λ. It is important to highlight that in this model the cancellation of the gauge bosons quadratic divergences happens between same spin partners (spin 1 heavy particles) due to the collective symmetry breaking, unlike other models with supersymmetry where the cancellation happens between opposite spin partners. In the Following we are going to introduce fermions to this model.

We have removed the quadratic correction to the Higgs potential in the gauge sector, nevertheless the largest contribution to quadratic corrections to the Higgs mass comes from the fermionic sector, more specifically from the top quark due to its large Yukawa coupling, for this reason we are going to focus only on this family. We introduce $SU(3)$ symmetries in the Yukawa couplings which are broken collectively. To do this we need enlarge the standard model $SU(2)$ doublet into $SU(3)$ triplets of the gauge group. We introduce heavy degrees of freedom, to cancel large Higgs mass contributions from SM degrees of freedom, therefore we add a new left-handed fermion T with charge 2/3 which will mix with the SM top quark:

$$\Psi = \begin{pmatrix} t_L \\ b_L \\ T_L \end{pmatrix} \equiv \begin{pmatrix} Q \\ T \end{pmatrix} . \tag{162}$$

To get the right Yukawa couplings, we need to include the corresponding right-handed partners t_1^c, b^c, t_2^c, where t_2^c is the Dirac partner of T, so that we can write

$$\mathcal{L}_Y = \lambda_1 \phi_1^\dagger \Psi t_1^c + \lambda_2 \phi_2^\dagger \Psi t_2^c + \text{h.c.} = (t_1^c, t_2^c) \begin{pmatrix} \lambda_1 \phi_1^\dagger \\ \lambda_2 \phi_2^\dagger \end{pmatrix} \Psi + \text{h.c.} \tag{163}$$

The Lagrangian above exhibits collective breaking in a very similar manner to eq. (156), if one of the couplings is set to zero, the symmetry of the above equation (163) is enhanced to $SU(3) \times SU(3)$,only when both couplings are present the diagonal symmetry $SU(3)_D$ is preserved. In a similar way to the gauge bosons only those terms that include both λ_1 and λ_2 will contribute to the Higgs potential at one-loop level. For simplicity, we consider the simple case of symmetric couplings $\lambda_1 = \lambda_2 \equiv \lambda/\sqrt{2}$. We expand the ϕ_i fields in terms of the Higgs:

$$\begin{aligned}
\mathcal{L}_Y = \frac{\lambda}{\sqrt{2}} \Bigg[& (0\ \ 0\ \ f) \left\{ \mathbb{I} - \frac{i}{f} \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{h}_{2\times 1} \\ \mathbf{h}^\dagger_{1\times 2} & 0 \end{pmatrix} - \frac{1}{2f^2} \begin{pmatrix} \mathbf{h}\mathbf{h}^\dagger{}_{2\times 2} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & h^\dagger h \end{pmatrix} + \dots \right\} \begin{pmatrix} Q \\ T \end{pmatrix} t_1^c \\
& + (0\ \ 0\ \ f) \left\{ \mathbb{I} + \frac{i}{f} \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{h}_{2\times 1} \\ \mathbf{h}^\dagger_{1\times 2} & 0 \end{pmatrix} - \frac{1}{2f^2} \begin{pmatrix} \mathbf{h}\mathbf{h}^\dagger{}_{2\times 2} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & h^\dagger h \end{pmatrix} + \dots \right\} \begin{pmatrix} Q \\ T \end{pmatrix} t_2^c \Bigg] + \text{h.c.} \\
= & \frac{\lambda}{\sqrt{2}} \left[fT\,(t_2^c + t_1^c) + i h^\dagger Q\,(t_2^c - t_1^c) - \frac{h^\dagger h}{2f^2} T\,(t_2^c + t_1^c) + \dots \right] \text{h.c.} \\
= & \lambda f \left(1 - \frac{h^\dagger h}{2f^2} \right) T T^c + \lambda h^\dagger Q t^c + \dots + \text{h.c.},
\end{aligned} \tag{164}$$

where we defined the singlet partners of heavy top and top quark respectively:

$$\begin{aligned} T^c &= \frac{t_2^c + t_1^c}{\sqrt{2}}, \\ t^c &= \frac{t_2^c - t_1^c}{\sqrt{2}}, \end{aligned} \tag{165}$$

Since the last term of the second line in eq. (164) is the standard model top Yukawa coupling, we identify $\lambda = \lambda_t$. Now, we found that the tree level mass of the heavy top quark T is $\lambda_t f$ and the T fermion couples to the Higgs doublet with a coupling constant of $\lambda_t/2f$. We can now draw the Feynman diagrams given in the figure 10, and we will see that the quadratic divergent contributions to the Higgs mass-squared cancel.

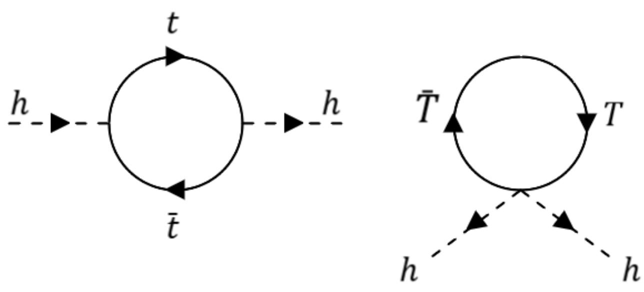


Figure 10: Quadratically divergent one loop contributions to the Coleman Weinberg scalar potential from the top sector.

At one-loop level these two contributions cancel, because the quadratic divergent part of the Coleman Weinberg effective potential contains only terms which are independent of the Higgs mass:

$$\begin{aligned} V_{CW}(\phi) &\sim -\frac{2}{64\pi^2}\Lambda^2 \mathrm{Tr}\left[M_t^\dagger(\phi)\, M_t(\phi)\right] \\ &\sim \lambda^2\Lambda^2\left(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2\right) \sim 2\lambda^2\Lambda^2 f^2 + \ldots \end{aligned} \tag{166}$$

where the top mass matrix is

$$M_t(\phi) = \begin{pmatrix} \lambda_1\phi_1^\dagger \\ \lambda_2\phi_2^\dagger \end{pmatrix}. \tag{167}$$

In other words, the heavy top partner loop quadratic divergence has compensated the contribution coming from the SM loop. As we point out before this cancellation occurs between same spin particles, unlike for example in Supersymmetry. According to the collective symmetry breaking structure, the only terms that can contribute to the Higgs boson potential are those proportional to both λ_1 and λ_2. The lowest order relevant loop diagram is proportional to $(\lambda_1\lambda_2)^2$ and generates a Higgs mass and quartic self-interaction. This one-loop diagram (see figure 11) contains four fermionic propagators and is, therefore, only logarithmically divergent.

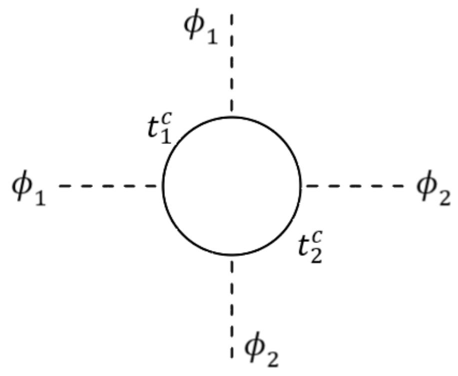


Figure 11: Logarithmically divergent one loop contribution to the Coleman Weinberg scalar potential from the top sector.

Similar to the gauge sector, the mass contribution is

$$V_{CW}(\phi_{1,2}) \sim \frac{\lambda^4 f^2}{4\pi^2} \log\left[\frac{\Lambda^2}{\mu^2}\right] h^\dagger h + \ldots \tag{168}$$

Again, if we take the value for $f \sim 1$ TeV as in the case of gauge boson the Higgs mass-squared is order $\mathcal{O}\left(M^2_{weak}\right)$. We have shown an explicit example of a Little Higgs theory with a toy model to demonstrate the way in which collective symmetry breaking is used to reduce the fine tuning in models that realize the Higgs as a pseudo Nambu-Goldstone boson of a spontaneously broken approximate global symmetry. In a realistic physical model we need to take the hypercharge into account, so we must enlarge $SU(3) \to SU(2)$ to $SU(3) \times U(1) \to SU(2)_L \times U(1)_Y$. This will be done in the next section with the full Simplest Little Higgs model.

2.3 SIMPLEST LITTLE HIGGS

We are going to develop the main characteristics of this model first introduced in refs. [24, 25]. The Higgs fields are Goldstone bosons which are associated with a new global symmetry breaking at a high scale $f \sim \mathcal{O}(10$ TeV). The Higgs fields will acquire a mass and become pseudo-Goldstone bosons via collective symmetry breaking at the electroweak scale, v. This mass will be light compared to f, since it is protected by the approximate global symmetry and is free from quadratic divergences at one-loop. Through this section we develop the fields expansion of the theory. We remind this was done in ref. [15] using the unitary gauge, though we follow the notation of ref. [26] and use the 't Hooft-Feynman gauge.

The SLH model is constructed by expanding the SM $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group to $SU(3)_c \times SU(3)_L \times U(1)_x$. In this case the $SU(2)$ doublets of the SM have to be enlarged to $SU(3)$ triplets and additional $SU(3)_L$ gauge bosons appear. The subscript x indicates a new *x-hypercharge*. Following the usual convention, the quantum numbers of the fundamental fermions in the model will be indicated using the notation:

$$(\text{color representation, weak multiplet representation})_{\text{x-Hypercharge}}\,. \tag{169}$$

The $SU(3)_L \times U(1)_x$ gauge symmetry is broken down to the SM electroweak gauge group by two scalar fields Φ_1 and Φ_2, which transform as complex triplets. The model contains a global $[SU(3) \times U(1)]^2$ symmetry. The diagonal subgroup [9] is gauged so the gauge symmetry is $[SU(3) \times U(1)]$. The symmetry is spontaneously broken by the vacuum in that the scalar fields assume vacuum expectation values. These vacuum expectation values are not assumed to be equal and designated f_1 and f_2. So schematically

$$[SU(3) \times U(1)]^2 \longrightarrow [SU(2) \times U(1)]^2\,. \tag{170}$$

In the above, the gauge symmetry is also broken: $[SU(3) \times U(1)] \longrightarrow [SU(2) \times U(1)]$, where the latter is the gauge symmetry of the SM. From this symmetry breaking we get five Nambu-Goldstone bosons from each scalar field (so 10 group generators are broken). Five linear combination of NGB's (corresponding to the diagonal symmetry) are eaten by the Higgs mechanism, they are in a sense, non-physical and form the longitudinal degrees of freedom of new massive gauge fields with masses of order f. The five orthogonal linear combinations are PNGB's, corresponding to the symmetry which is explicitly broken by the gauge interactions. So, after the symmetry breaking we are left with 5 physical NGB's, 4 of these form the complex Higgs doublet (h) and the last is a real scalar field called η. The $SU(3)$ symmetries of the Φ fields are spontaneously broken by non-zero vacuum expectation values, which are chosen to be aligned but not necessarily equal in magnitude:

$$\langle \Phi_1 \rangle = \begin{pmatrix} 0 \\ 0 \\ f c_\beta \end{pmatrix}_{(\mathbf{3},\mathbf{1})} \qquad \langle \Phi_2 \rangle = \begin{pmatrix} 0 \\ 0 \\ f s_\beta \end{pmatrix}_{(\mathbf{1},\mathbf{3})} \tag{171}$$

Originally, $f \sim 1$ TeV was considered but larger values are assumed nowadays, according to LHC searches [36]. The subscripts indicate the $[SU(3) \times U(1)]_1 \times [SU(3) \times U(1)]_2$ transformation properties of each condensate. However, under the full gauge group $SU(3)_c \times SU(3)_L \times U(1)_x$, Φ_1 and Φ_2 have quantum numbers $(\mathbf{1}, \mathbf{3})_{-\frac{1}{3}}$.

2.3.1 Particle Content

The SM fermions are embedded into $SU(3)_L$ triplets. For the lepton sector case the enlarging is straightforward, but for the quark sector this is not

[9] In group theory, given a group G, the diagonal subgroup of the n-fold direct product is the subgroup $(g, g, \dots, g)$, with g an element of G.

obvious. There are two choices of representations for the quarks. In the *Universal Embedding*, the representation is the same for each generation, but not all gauge anomalies are cancelled within the model [10]. The other representation is *the Anomaly-free embedding* where all gauge anomalies are cancelled. The cost, however, is placing the first and second generation quarks in a different representation than the third generation quarks. In both embeddings the lepton sector remains equal, but right-handed neutrinos are omitted, so neutrinos are treated as massless [11].

Leptons

In all the cases, the first generation of leptons is

$$\begin{aligned} \Psi_L &= (\mathbf{1}, \mathbf{3})_{-\frac{1}{3}} , \\ e_R &= (\mathbf{1}, \mathbf{1})_{-1} , \\ N_R &= (\mathbf{1}, \mathbf{1})_{0} , \end{aligned} \tag{172}$$

Explicitly:

$$\Psi_L = \begin{pmatrix} \nu_e \\ e \\ iN_e \end{pmatrix}_L \tag{173}$$

where a phase i, is needed to get real masses and lepton mixing angles. Here we get a new massive lepton N_e with the usual standard model electron and electron neutrino. This structure will be replicated for the second and third generations.

Quarks

In the *Universal Embedding* the first generation of quarks have the following quantum numbers:

$$\begin{aligned} \Psi_{uL} &= (\mathbf{3}, \mathbf{3})_{\frac{1}{3}} , \\ u_R &= (\mathbf{3}, \mathbf{1})_{\frac{2}{3}} , \\ d_R &= (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}} , \\ U_R &= (\mathbf{3}, \mathbf{1})_{\frac{2}{3}} \end{aligned} \tag{174}$$

where

$$\Psi_{uL} = \begin{pmatrix} u \\ d \\ iU \end{pmatrix}_L \tag{175}$$

later, we can note (with the charge operator) that the quarks u and d belong to the SM and that the new quark U has charge $+\frac{2}{3}$. For the other

[10] In this case there must be new physics, beyond the SLH, obviously. In any case, the sensitivity of the Higgs mass to the cutoff at two loops, within the SLH, requires this additional new physics at scales not much larger than f (typically $\Lambda \sim 4\pi f$).

[11] Ref. [108] extended the SLH accounting for the measured neutrino masses.

two generations this structure reproduces the quarks of the SM c,s,b,t, and new massive quarks C and T.

In the *anomaly-free embedding*, the third generation of quarks is the same as in the universal representation, but the first two generations are in the representation:

$$\begin{aligned} \Psi_{uL} &= (\mathbf{3}, \bar{\mathbf{3}})_0 \,, \\ u_R &= (\mathbf{3}, \mathbf{1})_{\frac{2}{3}} \,, \\ d_R &= (\mathbf{3}, \mathbf{1})_{\frac{-1}{3}} \,, \\ D_R &= (\mathbf{3}, \mathbf{1})_{\frac{-1}{3}} \end{aligned} \tag{176}$$

That is, the triplets are in the anti-fundamental representation of $SU(3)$, and the $x-$hypercharges have been modified. The new massive quark is now labeled D instead of U (and S replaces C) because it has a charge of $-\frac{1}{3}$. Explicitly the triplet is :

$$\Psi_{uL} = \begin{pmatrix} d \\ -u \\ iD \end{pmatrix}_L . \tag{177}$$

again, in both embeddings, the phase i is needed to produce real masses and mixing angles.

2.4 FEYNMAN RULES FOR THE SLH MODEL

The nonlinear fields Φ_1 and Φ_2 have six real components each, but are also subject to a normalization constraint, so for the two fields we have 10 real degrees of freedom[12], which will emerge as the longitudinal modes of the three massive gauge bosons of the SM: (Z^0, W^+, W^-), also the longitudinal modes of five new massive gauge bosons: $(Z', Y, Y^\dagger, X^-, X^+)$, the Standard Model Higgs scalar (H), and one additional massive pseudoscalar (η).

The two scalar triplets are introduced as nonlinear sigma fields and they can be parameterized in the following manner, to realize the spontaneous global symmetry breaking pattern:

$$\Phi_1 = \exp\left(\frac{i\Theta'}{f}\right) \exp\left(\frac{i t_\beta \Theta}{f}\right) \begin{pmatrix} 0 \\ 0 \\ f c_\beta \end{pmatrix}, \tag{179}$$

$$\Phi_2 = \exp\left(\frac{i\Theta'}{f}\right) \exp\left(-\frac{i\Theta}{f t_\beta}\right) \begin{pmatrix} 0 \\ 0 \\ f s_\beta \end{pmatrix}, \tag{180}$$

12

$$\begin{aligned} [SU(3) \times U(1)]^2 &\longrightarrow 2[(3^2-1)+1] = 18 \\ [SU(2) \times U(1)]^2 &\longrightarrow 2[(2^2-1)+1] = 8 \end{aligned} \tag{178}$$

so we have $18 - 8 = 10$ NGB's that are not physical particles.

where we have introduced the short notation: $s_\beta = \sin\beta$, $c_\beta = \cos\beta$, $t_\beta = \tan\beta$. This parametrization has the form of an $SU(3)$ (broken) transformation. Θ and Θ' are 3×3 matrix fields, parametrized as:

$$\begin{aligned} \Theta &= \frac{\eta}{\sqrt{2}}\mathbf{1}_{3\times3} + \begin{pmatrix} \mathbf{0}_{2\times2} & h \\ h^\dagger & 0 \end{pmatrix}, \\ \Theta' &= \frac{\xi}{\sqrt{2}}\mathbf{1}_{3\times3} + \begin{pmatrix} \mathbf{0}_{2\times2} & k \\ k^\dagger & 0 \end{pmatrix}, \end{aligned} \tag{181}$$

where

$$\begin{aligned} h &= \begin{pmatrix} h^0 \\ h^- \end{pmatrix}, \quad h^0 = \frac{1}{\sqrt{2}}(v + H - i\chi), \quad h^\pm = -\phi^\pm, \\ k &= \begin{pmatrix} k^0 \\ x^- \end{pmatrix}, \quad k^0 = \frac{\sigma - i\omega}{\sqrt{2}}. \end{aligned} \tag{182}$$

Here h is an $SU(2)$ doublet, becoming the SM Higgs doublet, and η is a real $SU(2)$ singlet, that will play no role in the next development (see [15, 109, 110, 111, 112] for details). We will assume that only the real part of h^0 may acquire a non-zero vacuum expectation value. At this point you can follow two paths:

- **Unitary Gauge**

The nonphysical eaten fields (Θ') must be rotated away through a $SU(3)_L \times U(1)_x$ transformation and gives only the physical particle spectrum. The physical fields can not be simultaneously rotated away because of the sign difference. What remains still is an $SU(2)$ symmetry:

$$\Phi_1 = \exp\left(\frac{it_\beta\Theta}{f}\right)\begin{pmatrix} 0 \\ 0 \\ fc_\beta \end{pmatrix}, \tag{183}$$

$$\Phi_2 = \exp\left(-\frac{i\Theta}{ft_\beta}\right)\begin{pmatrix} 0 \\ 0 \\ fs_\beta \end{pmatrix}, \tag{184}$$

with

$$\Theta = \begin{pmatrix} 0 & 0 & \frac{(v+H)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{(v+H)}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \tag{185}$$

The Higgs scalar H is then expressed as an excitation around the vacuum expectation value:

$$h = \begin{pmatrix} \frac{v+H}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{186}$$

with the electroweak scale given by $v = \sqrt{2}\langle h^0\rangle \simeq 246$ GeV[13].

[13] Remember that $v = \left(\sqrt{2}G_F\right)^{-1/2}$.

- **'t Hooft-Feynman Gauge**

Nonphysical fields are preserved in this gauge and the fields expansion are the same that in the equations (179), (180). In this gauge, loop calculations are easier than in the unitary gauge but there exist more Feynman diagrams. In the following work we use this gauge.

The fields Φ_i can be expanded in powers of $\frac{v}{f}$, in this work only the order $\mathcal{O}\left(\frac{v^2}{f^2}\right)$ is necessary, then the fields will be expanded to the fourth order:

$$\Phi_1 = \exp\left(\frac{i\Theta'}{f}\right)\begin{pmatrix} \mathbb{1}_{2\times 2} - \frac{t_\beta^2}{2f^2}\mathbf{hh}^\dagger{}_{2\times 2} + \frac{t_\beta^4}{24f^4}(\mathbf{hh}^\dagger)^2_{2\times 2} & \frac{it_\beta}{f}\mathbf{h}_{2\times 1} - \frac{it_\beta^3}{6f^3}\mathbf{hh}^\dagger\mathbf{h}_{2\times 1} \\ \frac{it_\beta}{f}\mathbf{h}^\dagger_{1\times 2} - \frac{it_\beta^3}{6f^3}\mathbf{h}^\dagger\mathbf{hh}^\dagger{}_{1\times 2} & 1 - \frac{t_\beta^2}{2f^2}h^\dagger h + \frac{t_\beta^4}{24f^4}(h^\dagger h)^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ fc_\beta \end{pmatrix},$$

$$\Phi_2 = \exp\left(\frac{i\Theta'}{f}\right)\begin{pmatrix} \mathbb{1}_{2\times 2} - \frac{1}{2f^2t_\beta^2}\mathbf{hh}^\dagger{}_{2\times 2} + \frac{1}{24f^4t_\beta^4}(\mathbf{hh}^\dagger)^2_{2\times 2} & -\frac{i}{ft_\beta}\mathbf{h}_{2\times 1} + \frac{i}{6f^3t_\beta^3}\mathbf{hh}^\dagger\mathbf{h}_{2\times 1} \\ -\frac{i}{ft_\beta}\mathbf{h}^\dagger_{1\times 2} + \frac{i}{6f^3t_\beta^3}\mathbf{h}^\dagger\mathbf{hh}^\dagger{}_{1\times 2} & 1 - \frac{1}{2f^2t_\beta^2}h^\dagger h + \frac{1}{24f^4t_\beta^4}(h^\dagger h)^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ fs_\beta \end{pmatrix}. \quad (187)$$

2.4.1 Gauge Sector

The $SU(3)_L \times U(1)_x$ is made a local symmetry by the introduction of the gauge-covariant derivative

$$D_\mu = \partial_\mu - igA_\mu^a T_a + ig_x Q_x B_\mu^x, \quad g_x = \frac{gt_w}{\sqrt{1-t_w^2/3}}, \quad (188)$$

where g is the Standard Model weak couplings constant and g_x is a new $U(1)_x$ coupling constant. A_μ^a and B_μ^x denote $SU(3)_L$ and $U(1)_x$ gauge fields.

The kinetic terms for the ϕ field can be written as:

$$\mathcal{L}_\Phi = (D^\mu\Phi_1)^\dagger(D_\mu\Phi_1) + (D^\mu\Phi_2)^\dagger(D_\mu\Phi_2). \quad (189)$$

The $SU(3)_L$ gauge fields can be written, in the fundamental representation, as:

$$\begin{aligned} A_\mu^a T_a &= \frac{A_\mu^3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{A_\mu^8}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & A_\mu^1 - iA_\mu^2 & A_\mu^4 - iA_\mu^5 \\ A_\mu^1 + iA_\mu^2 & 0 & A_\mu^6 - iA_\mu^7 \\ A_\mu^4 + iA_\mu^5 & A_\mu^6 + iA_\mu^7 & 0 \end{pmatrix} \\ &= \frac{A_\mu^3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{A_\mu^8}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & W_\mu^+ & Y_\mu^0 \\ W_\mu^- & 0 & X_\mu^- \\ Y_\mu^{0\dagger} & X_\mu^+ & 0 \end{pmatrix}. \end{aligned} \quad (190)$$

The first diagonal terms will join with the generator of $U(1)_x$ to form the neutral gauge bosons A_μ, Z_μ^0, Z_μ'. In the third term three pairs of conjugate particles can be recognized, since the upper left 2×2 sub-matrix contains the unbroken $SU(2)$, we can identify $W^\pm$ as the Standard Model charged gauge bosons.

From the covariant derivative on the Φ_i fields one can find the particular spectrum of the gauge fields and their masses:

$$\sum_{j=1}^{2}\left(D^{\mu}\Phi_j\right)^{\dagger}\left(D_{\mu}\Phi_j\right)=\sum_{j=1}^{2}\Bigg[\Phi_j^{\dagger}\overleftarrow{\partial}_{\mu}\partial^{\mu}\Phi_j-i\Phi_j^{\dagger}\overleftarrow{\partial}_{\mu}\left(gA_{\mu}^{a}T_a-\frac{1}{3}g_xB_{\mu}^{x}\right)\Phi_j$$
$$+i\Phi_j^{\dagger}\left(gA_{\mu}^{a}T_a-\frac{1}{3}g_xB_{\mu}^{x}\right)\partial^{\mu}\Phi_j$$
$$+\Phi_j^{\dagger}\left(gA_{\mu}^{a}T_a-\frac{1}{3}g_xB_{\mu}^{x}\right)^2\Phi_j\Bigg],\tag{191}$$

the first part of the equation (191) gives the kinetic terms of the Higgs boson, the second and third terms give the interaction between the gauge bosons and the NGBs and the last term gives us the masses of the physical particles to the desired order. The following arrangement can be made in the fourth term:

$$\sum_{i=1}^{2}\Phi_j^{\dagger}\left(gA_{\mu}^{a}T_a-\frac{1}{3}g_xB_{\mu}^{x}\right)^2\Phi_j\longrightarrow\text{Trace}\left[\left(gA_{\mu}^{a}T_a-\frac{1}{3}g_xB_{\mu}^{x}\right)^2\sum_{i=1}^{2}\Phi_i\Phi_i^{\dagger}\right]\tag{192}$$

The trace is a linear operator and the important thing that is necessary to calculate is:

$$\sum_{i=1}^{2}\Phi_i\Phi_i^{\dagger}=\begin{pmatrix} h^{\dagger}h-\frac{(h^{\dagger}h)^2}{3f^2}\left(\frac{s_\beta^4}{c_\beta^2}+\frac{c_\beta^4}{s_\beta^2}\right) & 0 & -i\frac{2hh^{\dagger}h}{3f}\left(\frac{s_\beta^3}{c_\beta}-\frac{c_\beta^3}{s_\beta}\right)\\ 0 & 0 & 0\\ i\frac{2hh^{\dagger}h}{3f}\left(\frac{s_\beta^3}{c_\beta}-\frac{c_\beta^3}{s_\beta}\right) & 0 & f^2-h^{\dagger}h+\frac{(h^{\dagger}h)^2}{3f^2}\left(\frac{s_\beta^4}{c_\beta^2}+\frac{c_\beta^4}{s_\beta^2}\right)\end{pmatrix}$$
$$=\begin{pmatrix} \frac{(v+H)^2}{2}-\frac{(v+H)^4}{12f^2}\left(\frac{s_\beta^4}{c_\beta^2}+\frac{c_\beta^4}{s_\beta^2}\right) & 0 & -i\frac{(v+H)^3}{3\sqrt{2}f}\left(\frac{s_\beta^3}{c_\beta}-\frac{c_\beta^3}{s_\beta}\right)\\ 0 & 0 & 0\\ i\frac{(v+H)^3}{3\sqrt{2}f}\left(\frac{s_\beta^3}{c_\beta}-\frac{c_\beta^3}{s_\beta}\right) & 0 & f^2-\frac{(v+H)^2}{2}+\frac{(v+H)^4}{12f^2}\left(\frac{s_\beta^4}{c_\beta^2}+\frac{c_\beta^4}{s_\beta^2}\right)\end{pmatrix}\tag{193}$$

here it is important note that the quantity $hh^{\dagger}h$ has a different meaning from $\mathbf{hh^{\dagger}h}_{2\times1}$, the latter is a vector and the former is the scalar: $\frac{(v+H)^3}{2\sqrt{2}}$.

Inserting the equation (193) into (192), but in order to relate the X-hypercharge and $SU(3)_L$ gauge couplings to the Standard Model counterparts, we do not yet assume a vacuum expectation value for the Higgs, the mass eigenstates (and masses) that are present before spontaneous electroweak symmetry breaking can be found, in other words, the spontaneous breaking of the $SU(3)_L\times U(1)_x\longrightarrow SU(2)_L\times U(1)_Y$ symmetry is made:

$$
\begin{aligned}
&\text{Trace}\left[\left(gA^a_\mu T_a - \frac{1}{3}g_x B^x_\mu\right)^2 \sum_{i=1}^{2} \Phi_i \Phi_i^\dagger\right] \\
&= \left[\frac{g^2}{2}X^+_\mu X^{-\mu} + \left(-\frac{A^8_\mu g}{\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2\right]\left[f^2 - h^\dagger h + \frac{(h^\dagger h)^2}{3f^2}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right] \\
&+ \left[\frac{g^2}{2}W^+_\mu W^{-\mu} + \left(\frac{A^3_\mu g}{2} + \frac{A^8_\mu g}{2\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2\right]\left[h^\dagger h - \frac{(h^\dagger h)^2}{3f^2}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right] \\
&+ i\left[\frac{g^2}{2}W^-_\mu X^{+\mu} + \frac{g}{\sqrt{2}}Y^{0\dagger}_\mu\left(\frac{A^{3\mu}g}{2} - \frac{A^{8\mu}g}{2\sqrt{3}} + \frac{2g_x B^{x\mu}}{3}\right)\right]\frac{2hh^\dagger h}{3f}\left(\frac{c^3_\beta}{s_\beta} - \frac{s^3_\beta}{c_\beta}\right) + h.c \\
&+ \frac{g^2 f^2}{2}Y^{0\dagger}_\mu Y^{0\mu}.
\end{aligned}
\tag{194}
$$

At this point, we can see that only five of the nine gauge bosons acquire masses of the order $f \sim$ TeV: two combinations of A^4_μ and A^5_μ which form $Y^{0(\dagger)}_\mu$ and two combination of A^6_μ and A^7_μ which form $X^\pm_\mu$. Together, these particles form an SU(2) doublet of new heavy gauge bosons with the same masses; also it is possible to find neutral gauge bosons, the gauge fields which correspond to the diagonal generators T^8 and Q_x mix to form mass eigenstates, which are:

$$
\begin{aligned}
\mathcal{Z}_\mu &= \frac{1}{\sqrt{3g^2 + g^2_x}}\left[\sqrt{3}gA^8_\mu - g_x B^x_\mu\right], \\
B_\mu &= \frac{1}{\sqrt{3g^2 + g^2_x}}\left[g_x A^8_\mu + \sqrt{3}gB^x_\mu\right].
\end{aligned}
\tag{195}
$$

We can proof that the boson $\mathcal{Z}_\mu$ acquires a mass term [14] meanwhile the orthogonal combination of $\mathcal{Z}_\mu$ remains massless and it represents the hypercharge gauge boson B_μ.

So, we obtained five combinations of gauge fields that acquire mass, which correspond to the broken generators:

$$
\begin{aligned}
Y^{0(\dagger)}_\mu &= \frac{1}{\sqrt{2}}\left(A^4_\mu \mp iA^5_\mu\right), \qquad M^2_{Y^0} = \frac{g^2 f^2}{2} \\
X^\pm_\mu &= \frac{1}{\sqrt{2}}\left(A^6_\mu \pm iA^7_\mu\right), \qquad M^2_X = \frac{g^2 f^2}{2} \\
\mathcal{Z}_\preceq &= \frac{\sqrt{3}gA^8_\mu - g_x B^x_\mu}{\sqrt{3g^2 + g^2_x}}, \qquad M^2_{\mathcal{Z}} = \frac{f^2}{9}\left(3g^2 + g^2_x\right)
\end{aligned}
\tag{196}
$$

and three combinations are massless, among them the bosons $W^\pm$ of the standard model :

[14] This gauge boson does not represent the Z boson of the standard model or his heavy partner Z' that will be found after consider a vacuum expectation value for the Higgs field.

$$
\begin{aligned}
W^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}\left(A^{1}_{\mu} \mp iA^{2}_{2}\right) , \\
B_{\mu} &= \frac{g_x A^{8}_{\mu} + \sqrt{3} g B^{x}_{\mu}}{\sqrt{3g^2 + g_x^2}} , \\
A^{3}_{\mu} & .
\end{aligned} \tag{197}
$$

The covariant derivative can be rewritten in terms of the mass eigenstates:

$$
\begin{aligned}
D_{\mu} = \partial_{\mu} &- \frac{ig}{\sqrt{2}}\left[W^{+}_{\mu}\left(T_1 + iT_2\right) + W^{-}_{\mu}\left(T_1 - iT_2\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[Y^{0}_{\mu}\left(T_4 + iT_5\right) + Y^{0\dagger}_{\mu}\left(T_4 - iT_5\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[X^{-}_{\mu}\left(T_6 + iT_7\right) + X^{+}_{\mu}\left(T_6 - iT_7\right)\right] \\
&+ \frac{igg_x\sqrt{3}}{\sqrt{3g^2 + g_x^2}} B_{\mu}\left[-\frac{T_8}{\sqrt{3}} + Q_x\right] \\
&- \frac{i\mathcal{Z}_{\mu}}{\sqrt{3g^2 + g_x^2}}\left[g^2\sqrt{3}T_8 + g_x^2 Q_x\right] \\
&- \frac{ig}{\sqrt{2}} A^{3}_{\mu}
\end{aligned} \tag{198}
$$

from the last equation we can find the hypercharge coupling, that is the coupling to B, and is equal to

$$
g^{'} = \frac{gg_x\sqrt{3}}{\sqrt{3g^2 + g_x^2}} = \frac{g}{\sqrt{1 + \frac{g_x^2}{3g^2}}} \tag{199}
$$

and also we find the hypercharge generator:

$$
Y = -\frac{T_8}{\sqrt{3}} + Q_x . \tag{200}
$$

We note that the masses of the heavy bosons are uniquely determined once the symmetry breaking scale f is fixed. Now we break the electroweak symmetry if we assume a vacuum expectation value for the Higgs in the equation (194), in other words, we consider the spontaneous breaking of the symmetry $SU(2)_L \times U(1)_Y \longrightarrow U(1)_{EM}$. We will find that the vector bosons $W^{\pm}$ get mass and the mass of the heavy doublet is split:

$$
\begin{aligned}
&\text{Trace}\left[\left(gA^a_\mu T_a - \frac{1}{3}g_x B^x_\mu\right)^2 \sum_{i=1}^{2}\Phi_i\Phi_i^\dagger\right] \\
&= \left[\frac{g^2}{2}X^+_\mu X^{-\mu} + \left(-\frac{A^8_\mu g}{\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2\right]\left[f^2 - \frac{(v+H)^2}{2} + \frac{(v+H)^4}{12f^2}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right] \\
&+ \left[\frac{g^2}{2}W^+_\mu W^{-\mu} + \left(\frac{A^3_\mu g}{2} + \frac{A^8_\mu g}{2\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2\right]\left[\frac{(v+H)^2}{2} - \frac{(v+H)^4}{12f^2}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right] \\
&+ i\left[\frac{g^2}{2}W^-_\mu X^{+\mu} + \frac{g}{\sqrt{2}}Y^{0\dagger}_\mu\left(\frac{A^{3\mu}g}{2} - \frac{A^{8\mu}g}{2\sqrt{3}} + \frac{2g_x B^{x\mu}}{3}\right)\right]\frac{(v+H)^3}{3\sqrt{2}f}\left(\frac{c^3_\beta}{s_\beta} - \frac{s^3_\beta}{c_\beta}\right) + h.c \\
&+ \frac{g^2f^2}{2}Y^{0\dagger}_\mu Y^{0\mu}.
\end{aligned}
\tag{201}
$$

The masses of the unmixed gauge bosons up to order $\mathcal{O}(v^4/f^4)$ can now be read directly from this Lagrangian term [26]:

$$
\begin{aligned}
\mathcal{L}_{mass} \supset & \frac{g^2v^2}{4}\left[1 - \frac{v^2}{6f^2}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right]W^+_\mu W^{-\mu} + \frac{g^2f^2}{2}\left[1 - \frac{v^2}{2f^2} + \frac{v^4}{12f^4}\left(\frac{s^4_\beta}{c^2_\beta} + \frac{c^4_\beta}{s^2_\beta}\right)\right]X^+_\mu X^{-\mu} \\
&+ \left[\frac{if^2g^2v^3}{6\sqrt{2}f^3}\left(\frac{c^3_\beta}{s_\beta} - \frac{s^3_\beta}{c_\beta}\right)W^-_\mu X^{+\mu} + h.c\right],
\end{aligned}
\tag{202}
$$

We need to rotate the original fields to eliminate the mixed terms as follows:

$$
\begin{aligned}
W^\pm &\longrightarrow W^\pm \pm \frac{iv^3}{3\sqrt{2}f^3}\left(\frac{c^3_\beta}{s_\beta} - \frac{s^3_\beta}{c_\beta}\right)X^\pm, \\
X^\pm &\longrightarrow X^\pm \pm \frac{iv^3}{3\sqrt{2}f^3}\left(\frac{c^3_\beta}{s_\beta} - \frac{s^3_\beta}{c_\beta}\right)W^\pm.
\end{aligned}
\tag{203}
$$

The physical states W and X differ from the interaction states only by a term of order v^3/f^3 [26]. This does not matter for the following calculations, but is important in determining the Goldstone bosons states. The masses of the physical fields are [26]:

$$
\begin{aligned}
M_W &= \frac{gv}{2}\left[1 - \frac{v^2}{12f^2}\left(\frac{c^4_\beta}{s^2_\beta} + \frac{s^4_\beta}{c^2_\beta}\right)\right], \\
M_X &= \frac{gf}{\sqrt{2}}\left[1 - \frac{v^2}{4f^2} + \frac{v^4}{24f^4}\left(\frac{c^4_\beta}{s^2_\beta} + \frac{s^4_\beta}{c^2_\beta}\right)\right] \sim \frac{gf}{\sqrt{2}}\left[1 - \frac{v^2}{4f^2}\right].
\end{aligned}
\tag{204}
$$

The neutral gauge bosons sector is more complicated because is non-diagonal at order $\mathcal{O}(v^4/f^4)$:

$$
\mathcal{L}_{mass} \supset M^2_Y Y^{0\mu}Y^{0\dagger}_\mu + (A_3\ \ , A_8,\ \ B_x)\,\mathcal{M}\begin{pmatrix} A_3 \\ A_8 \\ B_x \end{pmatrix}
\tag{205}
$$

with

$$\mathcal{M} = \begin{pmatrix} \frac{g^2\Delta}{4} & \frac{g^2\Delta}{4\sqrt{3}} & \frac{gg_x\Delta}{6} \\ \frac{g^2\Delta}{4\sqrt{3}} & \frac{g^2f^2}{3} - \frac{g^2\Delta}{4} & \frac{gg_x\Delta}{2\sqrt{3}} - \frac{gg_xf^2}{3\sqrt{3}} \\ \frac{gg_x\Delta}{6} & \frac{gg_x\Delta}{2\sqrt{3}} - \frac{gg_xf^2}{3\sqrt{3}} & \frac{g_x^2f^2}{9} \end{pmatrix}, \tag{206}$$

where

$$\Delta = \frac{v^2}{2} - \frac{v^4}{12f^2}\left(\frac{c_\beta^4}{s_\beta^2} + \frac{s_\beta^4}{c_\beta^2}\right). \tag{207}$$

The matrix $\mathcal{M}$ needs to be diagonalized (see apenddix A) to get the physical fields. Masses at order $\mathcal{O}(v^2/f^2)$ are [26]:

$$\mathcal{L}_{mass} \supset M_Y^2 Y^{0\mu} Y^{0\dagger}_\mu + \frac{1}{2}M_Z^2 Z^\mu Z_\mu + \frac{1}{2}M_{Z'}^2 Z'^\mu Z'_\mu + \frac{1}{2}M_A^2 A^\mu A_\mu, \tag{208}$$

$$\begin{aligned} M_A &= 0, \\ M_Y &= \frac{gf}{\sqrt{2}}, \\ M_{Z'} &= \frac{\sqrt{2}fg}{\sqrt{3-t_W^2}}\left(1 - \frac{(3-t_W^2)\,v^2}{16c_W^2f^2}\right), \\ M_Z &= \frac{gv}{2c_W}\left(1 - \frac{v^2}{16f^2}\left(1-t_W^2\right)^2 - \frac{v^2}{12f^2}\left(\frac{s_\beta^4}{c_\beta^2} + \frac{c_\beta^4}{s_\beta^2}\right)\right), \end{aligned} \tag{209}$$

where the first order mixing matrix for gauge bosons is:

$$\begin{pmatrix} A_3 \\ A_8 \\ B_x \end{pmatrix} = \begin{pmatrix} 0 & c_W & -s_W \\ -\sqrt{\frac{3-t_W^2}{3}} & \frac{s_W^2}{c_W\sqrt{3}} & \frac{s_W}{\sqrt{3}} \\ \frac{t_W}{\sqrt{3}} & s_W\sqrt{\frac{3-t_W^2}{3}} & c_W\sqrt{\frac{3-t_W^2}{3}} \end{pmatrix} \begin{pmatrix} Z' \\ Z \\ A \end{pmatrix}. \tag{210}$$

It is important to recall that the SLH model has no custodial symmetry [113, 114], i.e. there cannot be a $SU(2)_L \times SU(2)_R$ embedded into the $SU(2)_L \times U(1)_Y$ to which the $SU(3)_L \times U(1)_x$ breaks spontaneously. However the parameter $\rho \equiv \frac{M_W^2}{c_W^2 M_Z^2} \simeq 1$ only gets corrections at $\mathcal{O}(v^2/f^2)$ and the breaking of this symmetry is very small (a model with custodial symmetry has been proposed in ref. [115]), see section 5.1.

Using the equation (210) it is possible to rewrite the covariant derivative in terms of the mass eigenstates after the spontaneous electroweak symmetry breaking:

$$
\begin{aligned}
D_\mu = \partial_\mu &- \frac{ig}{\sqrt{2}}\left[W^+_\mu\left(T_1 + iT_2\right) + W^-_\mu\left(T_1 - iT_2\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[Y^0_\mu\left(T_4 + iT_5\right) + Y^{0\dagger}_\mu\left(T_4 - iT_5\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[X^-_\mu\left(T_6 + iT_7\right) + X^+_\mu\left(T_6 - iT_7\right)\right] \\
&+ \frac{ig}{c_w} Z_\mu\left(-c_w^2 Q + Y\right) + \frac{ig}{\sqrt{3 - t_3^2}} Z'_\mu T^z \\
&+ igs_w A_\mu Q,
\end{aligned}
\tag{211}
$$

it was found that the coupling of the photon is equal to $e = gs_w$ and the generators have the form:

$$
\begin{aligned}
Q &= T^3 + Y, \\
T^Z &= \sqrt{3}T^8 + t_w^2 Y.
\end{aligned}
\tag{212}
$$

The quadratic couplings of the Higgs with one heavy and one SM gauge boson induce, after the electroweak symmetry breaking, a mixing between them. But in our model this only affects the definition of the Z_μ and Z'_μ bosons, so, the physical states require the replacements:

$$
Z'_\mu \longrightarrow Z'_\mu + \delta_Z Z_\mu, \qquad Z_\mu \longrightarrow Z_\mu - \delta_Z Z'_\mu, \tag{213}
$$

where

$$
\delta_Z = \frac{\left(1 - t_w^2\right)\sqrt{3 - t_w^2}\, v^2}{8 c_w f^2}. \tag{214}
$$

Finally, the appropriate form of the covariant derivative that will be used to find the interactions between the heavy and SM gauge bosons is:

$$
\begin{aligned}
D_\mu = \partial_\mu &- \frac{ig}{\sqrt{2}}\left[W^+_\mu\left(T_1 + iT_2\right) + W^-_\mu\left(T_1 - iT_2\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[Y^0_\mu\left(T_4 + iT_5\right) + Y^{0\dagger}_\mu\left(T_4 - iT_5\right)\right] \\
&- \frac{ig}{\sqrt{2}}\left[X^-_\mu\left(T_6 + iT_7\right) + X^+_\mu\left(T_6 - iT_7\right)\right] \\
&+ \frac{ig}{c_w} Z_\mu\left(-c_w^2 Q + Y + \frac{\delta_Z c_w}{\sqrt{3 - t_w^2}} T^Z\right) \\
&+ \frac{ig}{c_w} Z'_\mu\left(\delta_Z\left(c_w^2 Q - Y\right) + \frac{c_w}{\sqrt{3 - t^2}} T^Z\right) \\
&+ igs_w A_\mu Q,
\end{aligned}
\tag{215}
$$

Taking the gauge bosons rotations (210) and (213) into account, the relevant Feynman rules can be obtained. Now we only need the charged Goldstone eigenstates since neutral pNGBs do not contribute to LFV processes. The mixing of pNGBs and gauge bosons are of the form $V^\mu \partial_\mu \phi$. The kinetic terms for the pNGBs and the Goldstones-gauge mixing terms read [26]:

$$\mathcal{L}_\Phi \supset \left[1-\frac{v^2}{6f^2}\left(\frac{c_\beta^4}{s_\beta^2}+\frac{s_\beta^4}{c_\beta^2}\right)\right]\partial_\mu\phi^+\partial^\mu\phi^- + \left[1-\frac{v^2}{2f^2}\right]\partial_\mu x^+\partial^\mu x^- \\ -\frac{v^2}{3f^2}\left(\frac{c_\beta^3}{s_\beta}-\frac{s_\beta^3}{c_\beta}\right)\left(\partial_\mu x^+\partial^\mu\phi^- + \partial_\mu x^-\partial^\mu\phi^+\right), \tag{216}$$

$$\mathcal{L}_\Phi \supset iW_\mu^-\frac{gv}{2}\left[\left(1-\frac{v^2}{6f^2}\left(\frac{c_\beta^4}{s_\beta^2}+\frac{s_\beta^4}{c_\beta^2}\right)\right)\partial^\mu\phi^+ - \frac{v^2}{3f^2}\left(\frac{c_\beta^3}{s_\beta}-\frac{s_\beta^3}{c_\beta}\right)\partial^\mu x^+\right] \\ + X_\mu^-\frac{gf}{\sqrt{2}}\left[\frac{v^2}{3f^2}\left(\frac{c_\beta^3}{s_\beta}-\frac{s_\beta^3}{c_\beta}\right)\partial^\mu\phi^+ - \left(1-\frac{v^2}{2f^2}\right)\partial^\mu x^+\right] + \text{h.c.} \tag{217}$$

As in the case of W and X gauge bosons, it is necessary to rotate their would-be longitudinal degrees of freedom, to express the interaction eigenstates in terms of the final pNGB eigenstates up to order $\mathcal{O}(v^2/f^2)$:

$$x^\pm \to -\left(1+\frac{v^2}{4f^2}\right)x^\pm \mp i\frac{v^2}{3f^2}\left(\frac{c_\beta^3}{s_\beta}-\frac{s_\beta^3}{c_\beta}\right)\phi^\pm, \\ \phi^\pm \to \mp i\left[1+\frac{v^2}{12f^2}\left(\frac{c_\beta^4}{s_\beta^2}+\frac{s_\beta^4}{c_\beta^2}\right)\right]\phi^\pm. \tag{218}$$

For the calculation of these states we use the relations (203) to obtain the v^2/f^2 corrections. Taking eqs. (210), (213) and (218) into account, the relevant Feynman rules can now be obtained.

2.4.2 Vector-Boson Lagrangian

The kinetic Lagrangian of the gauge bosons gives rise to the trilinear[15] gauge bosons couplings necessary for our calculation. It can be written as[16]:

$$\mathcal{L}_V = -\frac{1}{2}\text{Tr}\left[\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}\right] - \frac{1}{4}B_x^{\mu\nu}B_{x\mu\nu}, \qquad \tilde{G}_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right], \tag{219}$$

the results[17] to order $\mathcal{O}(v^2/f^2)$ are given in tables 3 and 4 [26]:

[15] Quartic gauge bosons couplings also arise, but are irrelevant for our work.
[16] Another way to write the lagrangian is: $\mathcal{L}_G = -\frac{1}{4}B_x^{\mu\nu}B_{x\mu\nu} - \frac{1}{4}A^{a\mu\nu}A^a_{\mu\nu}$ with $A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu$.
[17] Through this work we have found a few typos in the Feynman rules and form factors given previously.

$V_iV_jV_j$	$g^{V_iV_jV_j}$	$V_iV_jV_j$	$g^{V_iV_jV_j}$
AX^+X^-	-1	AW^+W^-	-1
ZX^+X^-	$\frac{1}{2c_W s_W}\left[c_W^2 - s_W^2 + c_W\delta_Z\sqrt{3-t_W^2}\right]$	ZW^+W^-	$\frac{c_W}{s_W}$
$Z'X^+X^-$	$\frac{1}{2c_W s_W}\left[-\delta_Z\left(c_W^2 - s_W^2\right) + c_W\sqrt{3-t_W^2}\right]$	$Z'W^+W^-$	$-\frac{\delta_Z c_W}{s_W}$

Table 3: Feynman rules for the trilinear gauge boson couplings $V_\mu(p_1)\, V_\nu^+(p_2)\, V_\rho^-(p_3)$. All these couplings have the generic form: $ieg^{V_iV_jV_j}\left[g_{\mu\nu}(p_2-p_1)_\rho + g_{\nu\rho}(p_3-p_2)_\mu + g_{\mu\rho}(p_1-p_3)_\nu\right]$ (j labels the particle-antiparticle gauge boson pair in the vertex). All four-momenta are taken incoming.

SVV	K	VSS	G
$x^\pm X^\mp \gamma$	$\pm iM_X$	$\gamma x^\pm x^\mp$	± 1
$\phi^\pm W^\mp \gamma$	$\pm iM_W$	$\gamma \phi^\pm \phi^\mp$	± 1
$x^\pm X^\mp Z$	$\mp iM_X \frac{c_W^2 - s_W^2}{2c_W s_W} \pm i\delta_Z \frac{M_X(1+t_W^2)}{2s_W\sqrt{3-t_W^2}}$	$Zx^\pm x^\mp$	$\mp \frac{c_W^2 - s_W^2}{2s_W c_W} \mp \delta_Z \frac{1-t_W^2}{2s_W\sqrt{3-t_W^2}}$
$\phi^\pm W^\mp Z$	$\pm iM_W t_W \mp i\delta_Z \frac{M_W(1-t_W^2)}{s_W\sqrt{3-t_W^2}}$	$Z\phi^\pm\phi^\mp$	$\mp \frac{c_W^2 - s_W^2}{2s_W c_W} \mp \delta_Z \frac{1-t_W^2}{2s_W\sqrt{3-t_W^2}}$
$x^\pm X^\mp Z'$	$\pm i\frac{M_X(1+t_W^2)}{2s_W\sqrt{3-t_W^2}} \pm i\delta_Z M_X \frac{c_W^2 - s_W^2}{2s_W c_W}$	$Z'x^\pm x^\mp$	$\mp \frac{1-t_W^2}{2s_W\sqrt{3-t_W^2}} \pm \delta_Z \frac{c_W^2 - s_W^2}{2s_W c_W}$
$\phi^\pm W^\mp Z'$	$\mp iM_W \frac{1-t_W^2}{s_W\sqrt{3-t_W^2}} \mp i\delta_Z M_W t_W$	$Z'\phi^\pm\phi^\mp$	$\mp \frac{1-t_W^2}{2s_W\sqrt{3-t_W^2}} \pm \delta_Z \frac{c_W^2 - s_W^2}{2s_W c_W}$

Table 4: Vertices $[SV^\mu V^\nu] = ieKg^{\mu\nu}$ and $[V^\mu S(p_1)S(p_2)] = ieG\,(p_1-p_2)^\mu$.

Since the W and Z bosons embody the $SU(2)$ subalgebra in $SU(3)$, it is not surprising that its trilinear couplings is unchanged to the leading order. Further, a check of the available structure constants [18] of $SU(3)$ will show that none of the new off-diagonal gauge bosons $X^\pm$, Y^0, $Y^{0\dagger}$ can decay to any combination of only Standard Model gauge bosons A, Z, $W^\pm$, and it can also be noted that, since there is no $SU(3)$ structure constant with indices 3 and 8, there can not be trilinear coupling with any two of the diagonal gauge bosons A, Z, Z'.

2.4.3 Fermion Sector

As anticipated, the SM fermion $SU(2)$ doublets must be enlarged to $SU(3)$ triplets. In addition, in order to give mass to the new third components of the $SU(3)$-triplet fermions, new $SU(3)$-singlet fermions must be introduced.

[18] The independent antisymmetric structure constants of $SU(3)$ are: $f_{123} = 1$, $f_{147} = f_{426} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}$, $f_{458} = f_{678} = \frac{\sqrt{3}}{2}$.

Then each lepton family consists of an $SU(3)$ left-handed triplet **3** and two right-handed singlets **1**. A right-handed neutrino is not included, leaving them massless as in the SM:

$$L_m^T = (\nu_L \quad \ell_L \quad iN_L)_m, \qquad \ell_{Rm}, \qquad N_{Rm}, \tag{220}$$

where m is the generation index. There are three new heavy neutral states N_m, defined with a phase i, necessary to get real masses and lepton mixing angles. In the case that we want to give mass to the SM neutrinos, one would need extra singlets to define Dirac neutrinos or new terms that break lepton number to introduce Majorana masses for the SM neutrinos, as shown in refs. [108, 116, 117]. The structure of the quark fields depends on the embedding we select:

- **Universal embedding**

All generations carry identical gauge quantum numbers and the $SU(3)_L \times U(1)_X$ gauge group is anomalous (the SM $SU(2) \times U(1)_Y$ gauge group remains anomaly-free). Each quark family consists of an $SU(3)$ left-handed triplet **3** and three right-handed singlets **1**:

$$Q_m^T = (u_L \quad d_L \quad iU_L)_m, \qquad u_{Rm}, \qquad d_{Rm}, \qquad U_{Rm}. \tag{221}$$

The new massive quarks U, C and T have charge $+\frac{2}{3}$.

- **Anomaly free embedding**

In this configuration we take different charge assignments for the different generations of quarks triplets, the third generation of quarks is the same as in the Universal representation, but the first two generations are in the anti-fundamental representation of $SU(3)$ [118]:

$$\begin{aligned} Q_1^T &= (d_L \quad -u_L \quad iD_L), \quad d_R, \quad u_R, \quad D_R, \\ Q_2^T &= (s_L \quad -c_L \quad iS_L), \quad s_R, \quad c_R, \quad S_R, \\ Q_3^T &= (t_L \quad b_L \quad iT_L), \quad t_R, \quad b_R, \quad T_R, \end{aligned} \tag{222}$$

such that with this new charge assignment all anomalies cancel [119, 120]. The new massive quarks are now labeled D and S because of their charge of $-\frac{1}{3}$, and we have again a massive quark T. In both embeddings, the phase i is needed to produce real masses and mixing angles. Table 5 collects the gauge representations and hypercharges for the fermion sector in both embeddings.

Universal Embedding							
Fermion	$Q_{1,2}$	Q_3	u_{Rm}, U_{Rm}	d_{Rm}	L_m	N_{Rm}	e_{Rm}
Q_x charge	1/3	1/3	2/3	−1/3	−1/3	0	−1
$SU(3)$ rep.	**3**	**3**	**1**	**1**	**3**	**1**	**1**
Anomaly free Embedding							
Fermion	$Q_{1,2}$	Q_3	u_{Rm}, T_{Rm}	d_{Rm}, D_{Rm}, S_{Rm}	L_m	N_{Rm}	e_{Rm}
Q_x charge	0	1/3	2/3	−1/3	−1/3	0	−1
$SU(3)$ rep.	$\bar{\mathbf{3}}$	**3**	**1**	**1**	**3**	**1**	**1**

Table 5: Quark quantum numbers in different embeddings.

Leptons

The Yukawa sector of the SLH model collects the flavour structure of the theory. Lepton masses follow from the Yukawa Lagrangian, and are generated by two types of terms: linear and bilinear in the Φ fields. This Lagrangian can be written as:

$$\mathcal{L}_Y \supset i\lambda_N^m \bar{N}_{Rm}\Phi_2^\dagger L_m + \frac{\lambda_\ell^{mn}}{\Lambda}\bar{\ell}_{Rm}\epsilon_{ijk}\Phi_1^i\Phi_2^j L_n^k + \text{h.c.}, \tag{223}$$

where Λ is the ultraviolet cut-off of the theory. Here m and n are generation indices, whereas i, j, k are $SU(3)$ indices. Notice that λ_N has been taken diagonal. However λ_ℓ does not need to be aligned in flavor space. After spontaneous electroweak symmetry breaking, this Lagrangian yields the lepton masses and the heavy masses up to $\mathcal{O}(v^2/f^2)$ [26]:

$$\begin{aligned}\mathcal{L}_Y \supset & -fs_\beta\lambda_N^m\left[\left(1-\frac{\delta_\nu^2}{2}\right)\bar{N}_{Rm}N_{Lm} - \delta_\nu\bar{N}_{Rm}\nu_{Lm}\right] \\ & + \xi_\beta\frac{fv}{\sqrt{2}\Lambda}\lambda_\ell^{mn}\bar{\ell}_{Rm}\ell_{Ln} + \text{h.c.},\end{aligned} \tag{224}$$

where

$$\delta_\nu = -\frac{v}{\sqrt{2}ft_\beta}, \quad \xi_\beta = \left[1 - \frac{v^2}{4f^2} - \frac{v^2}{12f^2}\left(\frac{s_\beta^4}{c_\beta^2} + \frac{c_\beta^4}{s_\beta^2}\right)\right], \tag{225}$$

here δ_ν represents the mixing angle between a heavy neutrino and a SM neutrino of the same generation. Notice that the rotation that diagonalizes λ_N does not necessarily diagonalize λ_ℓ, meaning that there is a mixing between the charged leptons and heavy neutrinos mediated by the charged gauge bosons. Charged leptons mass eigenstates and flavour eigenstates are related by the rotation:

$$\ell_{Lm} \longrightarrow (V_\ell\ell_L)_m = V_\ell^{mi}\ell_{Li}, \tag{226}$$

where V^{mi} is a CKM-like matrix. Furthermore, according to (224) each heavy neutrino is mixed just with the light neutrino of the same family. To separate them, we rotate only the left-handed sector. To order $\mathcal{O}(v^2/f^2)$, the physical states for the neutrinos are given by:

$$\begin{pmatrix} \nu_L \\ N_L \end{pmatrix}_m = \left[\begin{pmatrix} 1-\frac{\delta_\nu^2}{2} & -\delta_\nu \\ \delta_\nu & 1-\frac{\delta_\nu^2}{2} \end{pmatrix} \begin{pmatrix} V_\ell \nu_L \\ N_\ell \end{pmatrix} \right]_m . \tag{227}$$

After the Spontaneous Symmetry Breaking, in the mass eigenstates basis, the matrix λ_ℓ^{mn} is diagonal. The lepton masses up to $\mathcal{O}(v^2/f^2)$ are [26]:

$$m_{\ell_i} = -\xi_\beta \frac{fv}{\sqrt{2}\Lambda} y_{\ell_i}, \tag{228}$$

where y_ℓ is the eigenvalue of the λ_ℓ matrix, and we rotate in the same way the SM charged leptons and neutrinos, because in this work we consider massless SM neutrinos. We note that, in the physical basis, Higgs LFV interactions arise at one loop [28], which makes Higgs-mediated contributions negligible in the processes under study. Finally, the heavy neutrino masses are:

$$m_{N_i} = f s_\beta \lambda_N^i . \tag{229}$$

For a complete description of the lepton sector it is necessary to calculate the vertices of a Goldstone boson with a lepton pair. These vertices are obtained from the lepton kinetic Lagrangian, which can be written as:

$$\mathcal{L}_F = \bar{\psi}_m i \not{D} \psi_m, \qquad \psi = (L_m, \ell_{Rm}, N_{Rm}) . \tag{230}$$

The covariant derivative was given in eq. (188) with the Q_x charges in Table 5. The vertices of Goldstone bosons and leptons are collected in the following [26]:

We highlight that the non-chirally suppressed couplings of the heavy neutrinos showcase their non-decoupling behaviour, which was stressed before (see e.g. [17, 18]). To get those couplings it is necessary to use eqs. (218), (226) and (227). Some couplings vanish because they would be proportional to SM neutrino masses, that we neglect.

SFF	g_L	g_R
$x^+\bar{N}_m\ell_i$	$-\frac{1}{\sqrt{2}s_W}\frac{M_{N_m}}{M_X}\left(1-\delta_\nu^2/2\right)V_\ell^{mi}$	$\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_X}\left(1-\delta_\nu^2/2\right)V_\ell^{mi}$
$x^-\bar{\ell}_i N_m$	$\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_X}\left(1-\delta_\nu^2/2\right)V_\ell^{im*}$	$-\frac{1}{\sqrt{2}s_W}\frac{M_{N_m}}{M_X}\left(1-\delta_\nu^2/2\right)V_\ell^{im*}$
$\phi^+\bar{N}_m\ell_i$	$\delta_\nu\frac{i}{\sqrt{2}s_W}\frac{M_{N_m}}{M_W}V_\ell^{mi}$	$\delta_\nu\frac{i}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_W}V_\ell^{mi}$
$\phi^-\bar{\ell}_i N_m$	$-\delta_\nu\frac{i}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_W}V_\ell^{im*}$	$-\delta_\nu\frac{i}{\sqrt{2}s_W}\frac{M_{N_m}}{M_W}V_\ell^{im*}$
$x^+\bar{\nu}_i\ell_i$	0	$\delta_\nu\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_X}$
$x^-\bar{\ell}_i\nu_i$	$\delta_\nu\frac{1}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_X}$	0
$\phi^+\bar{\nu}_i\ell_i$	0	$\frac{i}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_W}\left(1-\delta_\nu^2/2\right)$
$\phi^-\bar{\ell}_i\nu_i$	$-\frac{i}{\sqrt{2}s_W}\frac{m_{\ell_i}}{M_W}\left(1-\delta_\nu^2/2\right)$	0

Table 6: Vertices $[\text{SFF}] = ie\left(g_L P_L + g_R P_R\right)$ for the lepton sector.

$V_\mu\bar{f}_i f_m$ Vertex	$g_L^{V\bar{f}_i f_m}$	$g_R^{V\bar{f}_i f_m}$
$A\bar{\ell}_i\ell_i$	1	1
$W^+\bar{\nu}_i\ell_i$	$\frac{1}{\sqrt{2}s_W}\left(1-\frac{\delta_\nu^2}{2}\right)$	0
$W^+\bar{N}_m\ell_i$	$-\delta_\nu\frac{1}{\sqrt{2}s_W}V_\ell^{mi}$	0
$Z\bar{\ell}_i\ell_i$	$\frac{2s_W^2-1}{2c_W s_W}+\frac{\delta_Z\left(2s_W^2-1\right)}{2s_W c_W^2\sqrt{3-t_W^2}}$	$t_W+\frac{\delta_Z s_W}{c_W^2\sqrt{3-t_W^2}}$
$Z\bar{\nu}_i\nu_i$	$\frac{1-\delta_\nu^2}{2c_W s_W}-\frac{\delta_Z\left(1-2s_W^2\right)}{2s_W c_W^2\sqrt{3-t_W^2}}$	0
$Z\bar{N}_i N_i$	$\frac{\delta_Z}{s_W\sqrt{3-t_W^2}}+\frac{\delta_\nu^2}{2c_W s_W}$	0
$Z\bar{N}_m\nu_i$	$-\delta_\nu\frac{1}{2c_W s_W}V_\ell^{mi}$	0
$X^+\bar{\nu}_i\ell_i$	$-\delta_\nu\frac{i}{\sqrt{2}s_W}$	0
$X^+\bar{N}_m\ell_i$	$-\frac{i}{\sqrt{2}s_W}\left(1-\frac{\delta_\nu^2}{2}\right)V_\ell^{mi}$	0
$Y^0\bar{\nu}_i\nu_i$	$\delta_\nu\frac{i}{\sqrt{2}s_W}$	0
$Y^0\bar{N}_i N_i$	$-\delta_\nu\frac{i}{\sqrt{2}s_W}$	0
$Y^0\bar{\nu}_i N_m$	$\frac{i}{\sqrt{2}s_W}\left(1-\delta_\nu^2\right)V_\ell^{mi\dagger}$	0
$Y^0\bar{N}_m\nu_i$	$-\delta_\nu^2\frac{i}{\sqrt{2}s_W}V_\ell^{mi}$	0
$Z'\bar{\ell}_i\ell_i$	$\frac{2s_W^2-1}{2s_W c_W^2\sqrt{3-t_W^2}}+\frac{\delta_Z\left(1-2s_W^2\right)}{2s_W c_W}$	$\frac{s_W}{c_W^2\sqrt{3-t_W^2}}-\delta_Z t_W$
$Z'\bar{\nu}_i\nu_i$	$\frac{2s_W^2-1}{2s_W c_W^2\sqrt{3-t_3^2}}\left(1-\frac{\left(3-t_W^2\right)\delta_\nu^2 c_W^2}{1-2s_W^2}\right)-\frac{\delta_Z}{2c_W s_W}$	0
$Z'\bar{N}_i N_i$	$\frac{1}{2s_W\sqrt{3-t_W^2}}\left[2-\delta_\nu^2\left(3-t_W^2\right)\right]$	0
$Z'\bar{N}_m\nu_i$	$\frac{\delta_\nu\sqrt{3-t_W^2}}{2s_W}V_\ell^{mi}$	0

Table 7: Vertices $[V^\mu ff] = ie\gamma^\mu\left(g_L P_L + g_R P_R\right)$ for the lepton sector [26].

It is possible to find the SM couplings with v^2/f^2 corrections (in δ_ν and δ_Z), new couplings of the heavy gauge bosons to leptons, couplings of the SM gauge bosons to the new heavy neutral leptons and couplings of the new gauge bosons with the new heavy neutral leptons. Entries of the table 7 were obtained using equations (226) and (227).

Quarks in the Anomaly free Embedding

The Yukawa couplings are found by contracting the fermion fields with the scalar sets into singlets in all possible ways. For the *anomaly free embedding*, the basic Yukawa Lagrangian reads [26]:

$$\begin{aligned}\mathcal{L}_Y \supset\ & i\lambda_1^t \bar{u}_{R3}^1 \Phi_1^\dagger Q_3 + i\lambda_2^t \bar{u}_{R3}^2 \Phi_2^\dagger Q_3 + i\frac{\lambda_b^m}{\Lambda}\bar{d}_{Rm}\epsilon_{ijk}\Phi_1^i\Phi_2^j Q_3^k \\ & + i\lambda_1^{dn}\bar{d}_{Rn}^1 Q_n^T \Phi_1 + i\lambda_2^{dn}\bar{d}_{Rn}^2 Q_n^T\Phi_2 + i\frac{\lambda_u^{mn}}{\Lambda}\bar{u}_{Rm}\epsilon_{ijk}\Phi_1^{*i}\Phi_2^{*j}Q_n^k,\end{aligned} \tag{231}$$

where $n = 1, 2$; $i, j, k = 1, 2, 3$ are $SU(3)$ indices; $d_{Rm} = \{d_R, s_R, b_R, D_R, S_R\}$; $u_{Rm} = \{u_R, c_R, t_R, T_R\}$; u_{R3}^1 and u_{R3}^2 are linear combinations of t_R and T_R; d_{R1}^n and d_{R2}^n are linear combinations of d_R and D_R for $n = 1$ and of s_R and S_R for $n = 2$:

$$\begin{aligned}
T_R &= \frac{\lambda_1^t c_\beta u_{R3}^1 + \lambda_2^t s_\beta u_{R3}^2}{\sqrt{(\lambda_1^t)^2 c_\beta^2 + (\lambda_2^t)^2 s_\beta^2}}, & t_R &= \frac{-\lambda_2^t s_\beta u_{R3}^1 + \lambda_1^t c_\beta u_{R3}^2}{\sqrt{(\lambda_1^t)^2 c_\beta^2 + (\lambda_2^t)^2 s_\beta^2}}, \\
D_R &= \frac{\lambda_1^{d1} c_\beta d_{R1}^1 + \lambda_2^{d1} s_\beta d_{R1}^2}{\sqrt{(\lambda_1^{d1})^2 c_\beta^2 + (\lambda_2^{d1})^2 s_\beta^2}}, & d_R &= \frac{-\lambda_2^{d1} s_\beta d_{R1}^1 + \lambda_1^{d1} c_\beta d_{R1}^2}{\sqrt{(\lambda_1^{d1})^2 c_\beta^2 + (\lambda_2^{d1})^2 s_\beta^2}}, \\
S_R &= \frac{\lambda_1^{d2} c_\beta d_{R2}^1 + \lambda_2^{d2} s_\beta d_{R2}^2}{\sqrt{(\lambda_1^{d2})^2 c_\beta^2 + (\lambda_2^{d2})^2 s_\beta^2}}, & s_R &= \frac{-\lambda_2^{d2} s_\beta d_{R2}^1 + \lambda_1^{d2} c_\beta d_{R2}^2}{\sqrt{(\lambda_1^{d2})^2 c_\beta^2 + (\lambda_2^{d2})^2 s_\beta^2}}.
\end{aligned} \tag{232}$$

We have obtained heavy states with corresponding large mass and light orthogonal states which remain massless at this point. In general, λ_1^d can be taken diagonal and, to avoid large quark flavour changing effects, we also assume λ_2^d to be diagonal [15]. Corrections of the order v^2/f^2 to vertices are only needed for particles involved in triangle diagrams and, since quarks only appear in box diagrams, $\mathcal{O}(v/f)$ precision is sufficient. Then, before the SEWSB we obtain the following masses for the heavy quarks:

$$\begin{aligned}
m_T &= f\sqrt{(\lambda_1^t)^2 c_\beta^2 + (\lambda_2^t)^2 s_\beta^2}, \\
m_D &= f\sqrt{(\lambda_1^{d1})^2 c_\beta^2 + (\lambda_2^{d1})^2 s_\beta^2}, \\
m_S &= f\sqrt{(\lambda_1^{d2})^2 c_\beta^2 + (\lambda_2^{d2})^2 s_\beta^2}.
\end{aligned} \tag{233}$$

After the SEWSB, the quark mass terms work out as follows to leading order [26]:

$$\mathcal{L}_Y^{mass} \supset -m_T\bar{T}_R T_L + \frac{v}{\sqrt{2}}\frac{s_\beta c_\beta\left[\left(\lambda_1^t\right)^2-\left(\lambda_2^t\right)^2\right]}{\sqrt{\left(\lambda_1^t\right)^2 c_\beta^2+\left(\lambda_2^t\right)^2 s_\beta^2}}\bar{T}_R t_L - \frac{v}{\sqrt{2}}\frac{\lambda_1^t\lambda_2^t}{\sqrt{\left(\lambda_1^t\right)^2 c_\beta^2+\left(\lambda_2^t\right)^2 s_\beta^2}}\bar{t}_R t_L$$
$$- m_D\bar{D}_R D_L - \frac{v}{\sqrt{2}}\frac{s_\beta c_\beta\left[\left(\lambda_1^{d1}\right)^2-\left(\lambda_2^{d1}\right)^2\right]}{\sqrt{\left(\lambda_1^{d1}\right)^2 c_\beta^2+\left(\lambda_2^{d1}\right)^2 s_\beta^2}}\bar{D}_R d_L + \frac{v}{\sqrt{2}}\frac{\lambda_1^{d1}\lambda_2^{d1}}{\sqrt{\left(\lambda_1^{d1}\right)^2 c_\beta^2+\left(\lambda_2^{d1}\right)^2 s_\beta^2}}\bar{d}_R d_L$$
$$- m_S\bar{S}_R S_L - \frac{v}{\sqrt{2}}\frac{s_\beta c_\beta\left[\left(\lambda_1^{d2}\right)^2-\left(\lambda_2^{d2}\right)^2\right]}{\sqrt{\left(\lambda_1^{d2}\right)^2 c_\beta^2+\left(\lambda_2^{d2}\right)^2 s_\beta^2}}\bar{S}_R s_L + \frac{v}{\sqrt{2}}\frac{\lambda_1^{d2}\lambda_2^{d2}}{\sqrt{\left(\lambda_1^{d2}\right)^2 c_\beta^2+\left(\lambda_2^{d2}\right)^2 s_\beta^2}}\bar{s}_R s_L$$
$$+ \frac{vf}{\sqrt{2}\Lambda}\lambda_u^{mn}\bar{u}_{Rm}u_{Ln} + \frac{vf}{\sqrt{2}\Lambda}\lambda_b^m\bar{d}_{Rm}b_L + \text{h.c.} \tag{234}$$

In general, the couplings λ_b^m and λ_u^{mn} generate a misalignment between the up and down sectors in the mass basis, causing the CKM matrix to appear, these couplings also provoke a misalignement between heavy and SM quarks, but since in this work we are interested in LFV, we will assume no flavour mixing in the quark sector for simplicity, so we demand $\lambda_b^m = \lambda_u^{mn} \equiv 0$ for all the couplings that mix different families or heavy and light quarks. SEWSB also induces mixing between heavy left-handed quarks and the SM quarks, mixing that we keep. We rotate the left-handed fields to obtain the physical quarks states [26]:

$$\begin{aligned}
T_L &\to T_L + \delta_t t_L,\\
t_L &\to t_L - \delta_t T_L,\\
D_L &\to D_L + \delta_d d_L,\\
d_L &\to d_L - \delta_d D_L,\\
S_L &\to S_L + \delta_s s_L,\\
s_L &\to s_L - \delta_s S_L,
\end{aligned} \tag{235}$$

where

$$\begin{aligned}
\delta_t &= \frac{v}{\sqrt{2}f}\frac{s_\beta c_\beta\left[\left(\lambda_1^t\right)^2-\left(\lambda_2^t\right)^2\right]}{\left(\lambda_1^t\right)^2 c_\beta^2+\left(\lambda_2^t\right)^2 s_\beta^2},\\
\delta_d &= -\frac{v}{\sqrt{2}f}\frac{s_\beta c_\beta\left[\left(\lambda_1^{d1}\right)^2-\left(\lambda_2^{d1}\right)^2\right]}{\left(\lambda_1^{d1}\right)^2 c_\beta^2+\left(\lambda_2^{d1}\right)^2 s_\beta^2},\\
\delta_s &= -\frac{v}{\sqrt{2}f}\frac{s_\beta c_\beta\left[\left(\lambda_1^{d2}\right)^2-\left(\lambda_2^{d2}\right)^2\right]}{\left(\lambda_1^{d2}\right)^2 c_\beta^2+\left(\lambda_2^{d2}\right)^2 s_\beta^2},
\end{aligned} \tag{236}$$

are complex in general. Taking all this into account we get the SM quark masses:

$$
\begin{aligned}
m_u &= -\frac{vf}{\sqrt{2}\Lambda}\lambda_u^{11}, \\
m_c &= -\frac{vf}{\sqrt{2}\Lambda}\lambda_u^{22}, \\
m_b &= -\frac{vf}{\sqrt{2}\Lambda}\lambda_b^{3}, \\
m_t &= \frac{v}{\sqrt{2}}\frac{\lambda_1^t\lambda_2^t}{\sqrt{\left(\lambda_1^t\right)^2 c_\beta^2 + \left(\lambda_2^t\right)^2 s_\beta^2}}, \\
m_d &= -\frac{v}{\sqrt{2}}\frac{\lambda_1^{d1}\lambda_2^{d1}}{\sqrt{\left(\lambda_1^{d1}\right)^2 c_\beta^2 + \left(\lambda_2^{d1}\right)^2 s_\beta^2}}, \\
m_s &= -\frac{v}{\sqrt{2}}\frac{\lambda_1^{d2}\lambda_2^{d2}}{\sqrt{\left(\lambda_1^{d2}\right)^2 c_\beta^2 + \left(\lambda_2^{d2}\right)^2 s_\beta^2}}.
\end{aligned}
\tag{237}
$$

Like for the lepton sector we need the quark-gauge Lagrangian to complete the review of the quark couplings. In the anomaly free embedding we have:

$$
\begin{aligned}
\mathcal{L}_F = &\bar{Q}_m i\not{D}_m^L Q_m + \bar{u}_{Rm} i\not{D}^u u_{Rm} + \bar{d}_{Rm} i\not{D}^d d_{Rm} \\
&+ \bar{T}_R i\not{D}^u T_R + \bar{D}_R i\not{D}^d D_R + \bar{S}_R i\not{D}^d S_R.
\end{aligned}
\tag{238}
$$

Remembering that the first two families are in the anti-fundamental representation:

$$
\begin{aligned}
D_{(1,2)\mu}^L &= \partial_\mu + igA_\mu^a T_a^*, \\
D_{3\mu}^L &= \partial_\mu - igA_\mu^a T_a + \frac{ig_x}{3}B_\mu^x, \\
D_\mu^u &= \partial_\mu + \frac{2ig_x}{3}B_\mu^x, \\
D_\mu^d &= \partial_\mu - \frac{ig_x}{3}B_\mu^x.
\end{aligned}
\tag{239}
$$

With this information and redefining the Goldstone fields as given in (218), we can obtain the relevant quark-Goldstone boson couplings[19] for our processes, which are given in table 8 [26]:

[19] We found a few typos in these Feynman rules given in the work [26].

SFF	g_L	g_R
$x^-\bar{D}_m u_m$	$-\frac{M_{D_m}}{M_X}\frac{1}{\sqrt{2}s_W}$	$\frac{m_{u_m}}{M_X}\frac{1}{\sqrt{2}s_W}$
$x^-\bar{d}_m u_m$	0	$\delta^*_{d_m}\frac{m_{u_m}}{M_X}\frac{1}{\sqrt{2}s_W}$
$\phi^-\bar{D}_m u_m$	$\delta_{d_m}\frac{i}{\sqrt{2}s_W}\frac{M_{D_m}}{M_W}$	$-\delta^*_{d_m}\frac{i m_{u_m}}{M_W}\frac{1}{\sqrt{2}s_W}$
$\phi^-\bar{d}_m u_m$	$-\frac{i m_{d_m}}{M_W}\frac{1}{\sqrt{2}s_W}$	$\frac{i m_{u_m}}{M_W}\frac{1}{\sqrt{2}s_W}$

(a) First and second family, where $u_m = u, c$ and $d_m(D_m) = d, s(D, S)$.

SFF	g_L	g_R
$x^+\bar{T}b$	$-\frac{M_T}{M_X}\frac{1}{\sqrt{2}s_W}$	$\frac{m_b}{M_X}\frac{1}{\sqrt{2}s_W}$
$x^+\bar{t}b$	0	$\delta^*_t\frac{m_b}{M_X}\frac{1}{\sqrt{2}s_W}$
$\phi^+\bar{T}b$	$\delta_t\frac{i}{\sqrt{2}s_W}\frac{M_T}{M_W}$	$\delta^*_t\frac{i m_b}{M_W}\frac{1}{\sqrt{2}s_W}$
$\phi^+\bar{t}b$	$-\frac{i m_t}{M_W}\frac{1}{\sqrt{2}s_W}$	$-\frac{i m_b}{M_W}\frac{1}{\sqrt{2}s_W}$

(b) Third family.

Table 8: Vertices $[SFF] = ie\,(g_L P_L + g_R P_R)$ for the quark sector in the anomaly-free embedding entering in our calculation.

We remind that all quark flavor changing vertices were removed, so there is no CKM-like matrix. For the anomaly-free embedding the vector-quark interactions are given in table 9 [26]:

VFF	g_L	g_R
$\gamma\bar{u}_m u_m$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\gamma\bar{d}_m d_m$	$\frac{1}{3}$	$\frac{1}{3}$
$W^-\bar{D}_m u_m$	$-\delta^*_{d_m}\frac{1}{\sqrt{2}s_W}$	0
$W^-\bar{d}_m u_m$	$\frac{1}{\sqrt{2}s_W}$	0
$Z\bar{u}_m u_m$	$\frac{4c_W^2-1}{6c_W s_W}$	$-\frac{2s_W}{3c_W}$
$Z\bar{d}_m d_m$	$\frac{-1-2c_W^2}{6c_W s_W}$	$\frac{s_W}{3c_W}$
$X^-\bar{D}_m u_m$	$-\frac{i}{\sqrt{2}s_W}$	0
$X^-\bar{d}_m u_m$	$-\delta^*_{d_m}\frac{i}{\sqrt{2}s_W}$	0
$Z'\bar{u}_m u_m$	$\frac{\sqrt{3-t_W^2}}{6s_W}$	$\frac{-2t_W^2}{3s_W\sqrt{3-t_W^2}}$
$Z'\bar{d}_m d_m$	$\frac{\sqrt{3-t_W^2}}{6s_W}$	$\frac{t_W^2}{3s_W\sqrt{3-t_W^2}}$

(a) First and second family, where $u_m = u, c$ and $d_m(D_m) = d, s(D, S)$.

VFF	g_L	g_R
$\gamma\bar{t}t$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\gamma\bar{b}b$	$\frac{1}{3}$	$\frac{1}{3}$
$W^+\bar{T}b$	$-\delta^*_t\frac{1}{\sqrt{2}s_W}$	0
$W^+\bar{t}b$	$\frac{1}{\sqrt{2}s_W}$	0
$Z\bar{t}t$	$\frac{4c_W^2-1}{6c_W s_W}$	$-\frac{2s_W}{3c_W}$
$Z\bar{b}b$	$\frac{-1-2c_W^2}{6c_W s_W}$	$\frac{s_W}{3c_W}$
$X^+\bar{T}b$	$-\frac{i}{\sqrt{2}s_W}$	0
$X^+\bar{t}b$	$-\delta^*_t\frac{i}{\sqrt{2}s_W}$	0
$Z'\bar{t}t$	$\frac{-3-t_W^2}{6s_W\sqrt{3-t_W^2}}$	$\frac{-2t_W^2}{3s_W\sqrt{3-t_W^2}}$
$Z'\bar{b}b$	$\frac{-3-t_W^2}{6s_W\sqrt{3-t_W^2}}$	$\frac{t_W^2}{3s_W\sqrt{3-t_W^2}}$

(b) Third family.

Table 9: Vertices $[V^\mu FF] = ie\gamma^\mu\,(g_L P_L + g_R P_R)$ for the quark sector in the Anomaly-free embedding.

Quarks in the Universal Embedding

The situation is similar in the *universal embedding* although the Yukawa Lagrangian is different:

$$\mathcal{L}_Y^{mass} \supset i\lambda_1^{un}\bar{u}_{Rn}^1\Phi_1^\dagger Q_n + i\lambda_2^{un}\bar{u}_{Rn}^2\Phi_2^\dagger Q_n + i\frac{\lambda_d^{mn}}{\Lambda}\bar{d}_{Rm}\epsilon_{ijk}\Phi_1^i\Phi_2^j Q_n^k + \text{h.c.} \tag{240}$$

Here $m, n = 1, 2, 3$ are generation indices and $i, j, k = 1, 2, 3$ are $SU(3)$ indices; d_m runs over the down quarks (d, s, b) and $u_n^{1,2}$ are linear combinations of the orthogonal light and heavy up quarks states:

$$\begin{aligned} U_{Rn} &= \frac{\lambda_1^{un}c_\beta u_{Rn}^1 + \lambda_2^{un}s_\beta u_{Rn}^2}{\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}}, \\ u_{Rn} &= \frac{-\lambda_2^{un}s_\beta u_{Rn}^1 + \lambda_1^{un}c_\beta u_{Rn}^2}{\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}}. \end{aligned} \tag{241}$$

Analogously to the anomaly free case, λ_1^u can be made diagonal by a field redefinition and λ_2^u is also taken diagonal to avoid large quark flavor effects. The mass terms are [26]:

$$\begin{aligned} \mathcal{L}_Y^{mass} \supset & -f\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}\,\bar{U}_{Rn}U_{Ln} + \frac{v}{\sqrt{2}}\frac{s_\beta c_\beta\left[\left(\lambda_1^{un}\right)^2 - \left(\lambda_2^{un}\right)^2\right]}{\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}}\bar{U}_{Rn}u_{Ln} \\ & - \frac{v}{\sqrt{2}}\frac{\lambda_1^{un}\lambda_2^{un}}{\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}}\bar{u}_{Rn}u_{Ln} + \frac{vf}{\sqrt{2}\Lambda}\lambda_d^{ij}\bar{d}_{Ri}d_{Lj} + \text{h.c.} \end{aligned} \tag{242}$$

We have neglected terms proportional to v^2/f^2. We will again ignore all generation mixing terms. This means setting $\lambda_d^{ij} = \lambda_d^i\delta_{ij}$. The only mixing effect in which we are interested corresponds to terms involving the light and heavy up quarks of each generation. The following rotation of the left-handed fields is required to obtain diagonal mass terms:

$$\begin{aligned} U_{Ln} &\to U_{Ln} + \delta_{u_n}u_{Ln}, \\ u_{Ln} &\to U_{Ln} - \delta_{u_n}U_{Ln}, \end{aligned} \tag{243}$$

where

$$\delta_{u_n} = \frac{v}{\sqrt{2}f}\frac{s_\beta c_\beta\left[\left(\lambda_1^{un}\right)^2 - \left(\lambda_2^{un}\right)^2\right]}{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}. \tag{244}$$

The quark masses to order v/f are:

$$\begin{aligned} M_{U_n} &= f\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}, \\ m_{u_n} &= \frac{v}{\sqrt{2}}\frac{\lambda_1^{un}\lambda_2^{un}}{\sqrt{\left(\lambda_1^{un}\right)^2 c_\beta^2 + \left(\lambda_2^{un}\right)^2 s_\beta^2}}, \\ m_{d_n} &= \frac{vf}{\sqrt{2}\Lambda}\lambda_d^n. \end{aligned} \tag{245}$$

The quark-gauge Lagrangian is more symmetric for the universal embedding:

$$\mathcal{L} = \bar{Q}_m i\not{D}^L Q_m + \bar{u}_{Rm} i\not{D}^u u_{Rm} + \bar{d}_{Rm} i\not{D}^d d_{Rm} + \bar{U}_{Rm} i\not{D}^u U_{Rm} \qquad (246)$$

where

$$\begin{aligned} D^L_\mu &= \partial_\mu - igA^a_\mu T^a + \frac{ig_x}{3} B^x_\mu, \\ D^u_\mu &= \partial_\mu + \frac{2ig_x}{3} B^x_\mu, \\ D^d_\mu &= \partial_\mu - \frac{ig_x}{3} B^x_\mu. \end{aligned} \qquad (247)$$

The Feynman rules[20] for quark-Goldstone couplings to order $\mathcal{O}(v/f)$ are given in table 10 [26]:

SFF	g_L	g_R
$x^+\overline{U}_m d_m$	$-\frac{M_{U_m}}{M_X}\frac{1}{\sqrt{2}s_W}$	$\frac{m_{d_m}}{M_X}\frac{1}{\sqrt{2}s_W}$
$x^+\bar{u}_m d_m$	0	$\delta^*_{u_m}\frac{m_{d_m}}{M_X}\frac{1}{\sqrt{2}s_W}$
$\phi^+\overline{U}_m d_m$	$\delta_{u_m}\frac{i}{\sqrt{2}s_W}\frac{M_{U_m}}{M_W}$	$\delta^*_{u_m}\frac{im_{d_m}}{M_W}\frac{1}{\sqrt{2}s_W}$
$\phi^+\bar{u}_m d_m$	$-\frac{im_{u_m}}{M_W}\frac{1}{\sqrt{2}s_W}$	$-\frac{im_{d_m}}{M_W}\frac{1}{\sqrt{2}s_W}$

Table 10: Vertices $[SFF] = ie(g_L P_L + g_R P_R)$ for the quark sector in the universal embedding entering our calculations, where $u_m(U_m) = u, c, t(U, C, T)$ and $d_m = d, s, b$.

For the universal embedding the vector-quark interactions are given in table 11 [26]:

[20] We found differences in these Feynman rules and those given in [26].

VFF	g_L	g_R
$\gamma\bar{u}_m u_m$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\gamma\bar{d}_m d_m$	$\frac{1}{3}$	$\frac{1}{3}$
$W^+\overline{U}_m d_m$	$-\delta^*_{u_m}\frac{1}{\sqrt{2}s_W}$	0
$W^+\bar{u}_m d_m$	$\frac{1}{\sqrt{2}s_W}$	0
$Z\bar{u}_m u_m$	$\frac{4c_W^2-1}{6c_W s_W}$	$-\frac{2s_W}{3c_W}$
$Z\bar{d}_m d_m$	$\frac{-1-2c_W^2}{6c_W s_W}$	$\frac{s_W}{3c_W}$
$X^+\overline{U}_m d_m$	$-\frac{i}{\sqrt{2}s_W}$	0
$X^+\bar{u}_m d_m$	$-\delta^*_{u_m}\frac{i}{\sqrt{2}s_W}$	0
$Z'\bar{u}_m u_m$	$-\frac{3+t_W^2}{6s_W\sqrt{3-t_W^2}}$	$\frac{-2t_W^2}{3s_W\sqrt{3-t_W^2}}$
$Z'\bar{d}_m d_m$	$-\frac{3+t_W^2}{6s_W\sqrt{3-t_W^2}}$	$\frac{t_W^2}{3s_W\sqrt{3-t_W^2}}$

Table 11: Vertices $[V^\mu FF] = ie\gamma^\mu\,(g_L P_L + g_R P_R)$ for the quark sector in the universal embedding, where $u_m(U_m) = u, c, t(U, C, T)$ and $d_m = d, s, b$.

3 LEPTON FLAVOUR VIOLATION PROCESSES

We must know. We will know.

— David Hilbert

Although the particle physics is well understood and described by the Standard Model, there remain many unanswered questions. One example lies in the lepton sector; in the SM three generations of leptons exist, which appear in doublets as shown below:

First generation	e	ν_e
Second generation	μ	ν_μ
Third generation	τ	ν_τ

Table 12: Generations for the lepton sector.

These three families of leptons seem to be an exact replica of one another (except for the mass), because the weak coupling strengths to the gauge bosons are the same, within the experimental precision [36]. This property, rooted in gauge symmetry, is known as *Lepton universality*. A quantum number is associated to each lepton called *Lepton Flavour*, which is different for each generation: this number is 1 for leptons, -1 for antileptons and 0 otherwise.

Experimentally lepton flavour is a conserved quantity, however within the SM nothing tells us that lepton flavour needs to be a conserved quantity, so one can speculate that some global or accidental symmetry is responsible for the apparent conservation of lepton flavour. In the past, it was assumed that lepton flavour is a conserved quantity because of the degeneracy of the neutrino masses, i.e, they are massless. This assumption implies that there is not a mass matrix that needs to be diagonalized (this constrasts with the CKM matrix in the quark sector), then by definition, neutrinos are flavour eigenstates. Currently processes have been found in which the eigenstates of a certain flavour of the initial neutrinos are different from the final eigenstates, so, as with the quarks and the CKM matrix, the flavour eigenstates are not identical to the mass eigenstates because the mass of the neutrinos is non-zero and there are mixing angles and therefore, the violation of leptonic flavour is allowed in the sector of neutral leptons.

No processes have been detected in which there is a violation of flavour in the charged sector, however, due to the aforementioned, it is hoped to see processes in which the violation of the lepton flavour in the charged sector is manifested. The principal motivation to study processes with lepton flavour violation in the charged sector (*cLFV*) is that it will be an immediate proof of physics beyond the SM (and not simply in the form of neutrino masses, as we will discuss). There are many models of new physics that can lead to "observable" *cLFV* introducing new sources of lepton flavour violation, as well as new operators at the origin of the flavour violating transitions and decays. A first generic approach (independent of the model) consists in describing the interactions with *cLFV* using higher-dimensional operators ($d > 4$), so the new low-energy effective Lagrangian can be written as:

$$\mathcal{L}^{eff} = \mathcal{L}^{SM} + \sum_{n \geqslant 1} \frac{\mathcal{C}_{ij}^{4+n}}{\Lambda^n} \mathcal{O}_{ij}^{4+n}, \tag{248}$$

in which Λ denotes the scale of new physics and $\mathcal{C}$, $\mathcal{O}$ the effective couplings and operators, with the former corresponding to complex matrices in flavour space, but there exist a large number of dimension six operators, whose low-energy effects include *cLFV*. Regarding the *cLFV* dimension-six operators, these can be loosely classified as dipole, four-fermion and scalar/vector operators. A second phenomenological approach consists in considering specific new physics models or theories, some examples are: supersymmetric (SUSY) extensions of the SM, little(st) Higgs models, extra-dimensional models, among others.

In minimal extensions of the Standard Model (that includes right handed neutrinos allowing for small neutrino Dirac masses), a loop is induced at electroweak energy scales that allows decays with flavour violation, the branching ratio of these processes are not possible to measure, for example for a decay of the type $\ell_\alpha \to \ell_\beta \gamma$ we have [34, 35, 121]:

$$\mathcal{B}(\ell_\alpha \to \ell_\beta \gamma) = \frac{3\alpha}{32\pi} \left| \sum_{2,3} U_{\alpha i}^* U_{\beta i} \frac{\Delta m_{i1}^2}{M_W^2} \right|^2 \leqslant 10^{-54}, \tag{249}$$

where Δm_{i1}^2 are the differences among the neutrino squared masses, M_W is the mass of the W boson and $U_{\ell i}$ are the elements of the neutrino mixing matrix, this branching ratio is outside the experimental limit by many orders of magnitude (more than forty). In the following decade, better precision is expected in branching ratios where flavour violation is allowed which could eventually unveil physics beyond the SM.

3.1 FLAVOR VIOLATION IN CHARGED LEPTONIC PROCESSES

As it was mentioned already *cLFV* is a clear signal of new physics and its search has continued from the early 1940's. The most important searches have used the muon state because of the high statistics available in muon beams.

The experimental observation of *cLFV* is the goal of a bunch of excellent dedicated experiments like, for instance, MEG [37], SINDRUM [38] and Mu3e [45] in the search of muon decays, and those looking for muon conversion in the presence of nuclei, SINDRUM II [39], Mu2e [44] or COMET [46], PRISM/PRIME [43]. The first generation of B-factories, (that stand for τ factories too) like BaBar [40] or Belle [41], have joined in the pursuit of *cLFV* decays coming from the τ lepton. The study of *LFV* in decays of the tau lepton are also one of the main goals of the future SuperKEKB/Belle II project at KEK in Japan [48].

3.1.1 Experimental situation of the processes $\mu \to e\gamma$, $\mu \to 3e$ and $\mu - e$ conversion

The first search for the process $\mu \to e\gamma$, was performed by Hincks and Pontecorvo [122], now MEG II at Paul Scherrer Institute searches for the $\mu^+ \to e^+\gamma$ decay with a design sensitivity of 6×10^{-14} [42]. There are two major backgrounds: one is a prompt background from radiative muon decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu\gamma$. The other background is an accidental coincidence of a e^+ in a normal μ-decay. The current limits for LFV processes in muons are:

Process	Current Limit
BR $(\mu \to e\gamma)$	4.2×10^{-13} [123]
BR $(\mu \to eee)$	1×10^{-12} [38]

Table 13: Expected 90% CL upper limits on LFV decays with muons.

The decay $\mu \to 3e$ is of great interest; it is sensitive to supersymmetry, Little Higgs scenarios, leptoquarks, and other new physics models. This mode has been examined in Littlest Higgs scenarios [124] and in the Simplest Little Higgs model(SLH) [26]. The Mu3e experiment [45] in under construction at Paul Scherrer Institut and aims at reaching a 10^{-16} sensitivity in two successive phases and improving the former result by 3 orders of magnitude.

Another process that has close relation with $\mu \to e\gamma$ is the coherent $\mu - e$ conversion[1], in these processes negative muons are captured in a target of atomic nuclei to form muonic atoms. The muon then converts into an electron in the nuclear field without creating a neutrino. The present sensitivities of the $\mu - e$ conversion rates in different nuclei are collected here [39]:

[1]Coherent conversion is the process $\mu^- N \to e^- N$ in which the nucleus N remains in its initial state.

Process	current sensitivity
BR $(\mu \to e : \mathrm{Au})$	$< 7 \times 10^{-13}$
BR $(\mu \to e : \mathrm{Ti})$	$< 4.3 \times 10^{-12}$
BR $(\mu \to e : \mathrm{Pb})$	$< 4.6 \times 10^{-11}$

Table 14: Current bounds at 90% on the branching ratios for various targets in $\mu - e$ conversion.

We see that the strongest limit is set by the gold target. In the future PRISM/PRIME collaboration is expected to improve the bounds on $\mu N \to eN$ by several orders of magnitude.

In the minimal SM with massless neutrinos, lepton flavour is conserved separately for each generation. This is not necessarily true if new particles or new interactions beyond the SM are introduced. In SUSY models, the origin of *LFV* could be interactions at a very high energy scale, or the mass scale of a heavy right handed Majorana neutrino that appears in the see-saw mechanism. The effective Lagrangians for muon LFV process can be grouped into two types: photonic and four-fermion interactions [125]. The effective Lagrangian for $\mu^+ \to e^+\gamma$ is given by:

$$\mathcal{L}_{\mu\to e\gamma} = -\frac{4G_F}{\sqrt{2}}\left(m_\mu A_R \bar{\mu}_R \sigma_{\mu\nu} e_L F^{\mu\nu} + m_\mu A_L \bar{\mu}_L \sigma^{\mu\nu} e_R F_{\mu\nu} + \mathrm{h.c}\right), \quad (250)$$

where A_R and A_L are coupling constants that correspond to $\mu^+ \to e_R^+\gamma$ and $\mu^+ \to e_L^+\gamma$ processes, respectively. For the $\mu^+ \to e^+e^+e^-$ decay and $\mu^- - e^-$ conversion, off-shell photon emission also contributes, the general photonic amplitude is:

$$\begin{aligned}\mathcal{M}_{\mathrm{photonic}} = &-eA_\mu^*(q)\bar{u}_e(p_e)\Big[\left(f_{E0}(q^2) + \gamma_5 f_{M0}(q^2)\right)\gamma_\nu \left(g^{\mu\nu} - \frac{q^\nu q^\mu}{q^2}\right) \\ &+ \left(f_{M1}(q^2) + \gamma_5 f_{E1}(q^2)\right)\frac{i\sigma_{\mu\nu}q^\nu}{m_\mu}\Big]u_\mu(p_\mu),\end{aligned} \quad (251)$$

The electromagnetic form factors $f_{E0}, f_{E1}, f_{M0}, f_{M1}$ are functions of momentum transfer. The general four fermion couplings could introduce $\mu^+ \to e^+e^+e^-$ decay and $\mu^- - e^-$ conversion, and are given by the interactions:

$$\begin{aligned}\mathcal{L}_{\mu\to 3e}^{\mathrm{non-photo}} = &-\frac{G_F}{\sqrt{2}}\Big(g_1 \bar{\mu}_R e_L \bar{e}_R e_L + g_2 \bar{\mu}_L e_R \bar{e}_L e_R + g_3 \bar{\mu}_R \gamma^\mu e_R \bar{e}_R \gamma_\mu e_R \\ &+ g_4 \bar{\mu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu e_L + g_5 \bar{\mu}_R \gamma^\mu e_R \bar{e}_L \gamma_\mu e_L + g_6 \bar{\mu}_L \gamma^\mu e_L \bar{e}_R \gamma_\mu e_R \\ &+ \mathrm{h.c.}\Big).\end{aligned} \quad (252)$$

For the $\mu^- - e^-$ conversion process, the relevant interactions are written as:

$$\begin{aligned}
\mathcal{L}_{\mu-e}^{\text{non-photo}} = &-\frac{G_F}{\sqrt{2}}\Sigma_{q=u,d,s}\Big[\left(g_{LS(q)}\bar{e}_L\mu_R + g_{RS(q)}\bar{e}_R\mu_L\right)\bar{q}q \\
&+\left(g_{LP(q)}\bar{e}_L\mu_R + g_{RP(q)}\bar{e}_R\mu_L\right)\bar{q}\gamma_5 q \\
&+\left(g_{LV(q)}\bar{e}_L\gamma^{\mu}\mu_L + g_{RV(q)}\bar{e}_R\gamma^{\mu}\mu_R\right)\bar{q}\gamma_{\mu} q \\
&+\left(g_{LA(q)}\bar{e}_L\gamma^{\mu}\mu_L + g_{RA(q)}\bar{e}_R\gamma^{\mu}\mu_R\right)\bar{q}\gamma_{\mu}\gamma_5 q \\
&+\frac{1}{2}\left(g_{LT(q)}\bar{e}_L\sigma^{\mu\nu}\mu_R + g_{RT(q)}\bar{e}_R\sigma^{\mu\nu}\mu_L\right)\bar{q}\sigma_{\mu\nu}q + \text{h.c.}\Big].
\end{aligned} \tag{253}$$

Here, the flavor-changing quark currents are not included and the four fermion coupling constants g_n depend on the model beyond the SM [126, 127].

3.1.2 Flavor Violation in the Tau physics

The EW interactions of the SM has been successfully tested with a very high accuracy, however the physics of the tau lepton is very important for studying strong interactions effects at low energies [128], because the τ is the only lepton that can decay into hadrons and with its semileptonic decays we can study the hadronic weak currents.

Actually the B factories have explored *LFV* τ decays, the reason is that its decays provide a great variety of opportunities to probe new interactions and new sources of *CP* violation and these processes are theoretically clean compared to charm and bottom decay processes. There are three important physical observables potentially sensitive to new physics effects in charged lepton processes, namely the anomalous magnetic moment $(g-2)$, the *EDM*, and *cLFV*, the main goal of this work is studying the *cLFV* processes. An important feature of τ *cLFV* searches is the range of processes that can be studied. If we compare to the μ *cLFV* case where $\mu^+ \to e^+\gamma$, $\mu^+ \to e^+e^+e^-$ and $\mu - e$ conversion in a nuclei are the three major processes, there are many possible τ *cLFV* decay modes in which searches can be carried out at *B* factories. For example, there are six different flavor combinations in τ to three-charged-lepton decay processes, to be compared in the muon case to only $\mu^+ \to e^+e^+e^-$, and there are many cLFV τ decay modes with hadrons in the final state that have been searched for experimentally [36]:

Process	Present Bound
BR $(\tau \to e\gamma)$	3.3×10^{-8}
BR $(\tau \to \mu\gamma)$	4.4×10^{-8}
BR $(\tau \to eee)$	2.7×10^{-8}
BR $(\tau \to \mu\mu\mu)$	2.1×10^{-8}
BR $(\tau^- \to e^-\mu^+\mu^-)$	2.7×10^{-8}
BR $(\tau^- \to \mu^-e^+e^-)$	1.8×10^{-8}
BR $(\tau^- \to e^+\mu^-\mu^-)$	1.7×10^{-8}
BR $(\tau^- \to \mu^+e^-e^-)$	1.5×10^{-8}

Table 15: Current experimental bounds for several LFV observables of interest at 90% CL.

With 50 ab^{-1} of data, the Belle II experiment is expected to be able to probe *LFV* in τ decays at the level of 10^{-9}.The expected sensitivities for some processes in the Belle II experiment [48] are shown in Table 16:

Process	5 ab^{-1}	50 ab^{-1}
BR $(\tau \to \mu\gamma)$	10×10^{-9}	3×10^{-9}
BR $(\tau \to \mu\mu\mu)$	3×10^{-9}	1×10^{-9}
BR $(\tau \to \mu\eta)$	5×10^{-9}	2×10^{-9}

Table 16: Expected 90% CL upper limits on $\tau \to \mu\gamma$, $\tau \to \mu\mu\mu$, and $\tau \to \mu\eta$ with $5ab^{-1}$ and $50ab^{-1}$ data sets from Belle II and Super KEKB.

In the case of the $\mu - e$ nuclei conversion the experiments are performed at low energy and the muon becomes bounded before decaying in orbit or being captured by the nucleus. The experiments with τ leptons are different, for the $\mu - \tau$ conversion is expected to occur by deep inelastic scattering of the lepton off the nucleus, thus these experiments are based on a fixed-target nucleus hit by an incoming lepton beam of a given flavour ℓ. There are still no experimental limits for this phenomenon, but its feasibility at NA64 has been pointed out in ref. [52]:

$$\mathcal{R}\left(\ell - \tau\right) \sim 10^{-13} - 10^{-12}. \tag{254}$$

It is expected that future fixed target experiments like the muon collider [53] or the electron-ion collider (EIC) [54, 55] could consider to look for this conversion.

3.2 LEPTON FLAVOUR VIOLATING DECAYS IN THE SIMPLEST LITTLE HIGGS MODEL[2]

The SLH model extends the SM group, $SU(2)_L \times U(1)_Y$, to a gauge group $SU(3)_L \times U(1)_X$, that requires to enlarge the $SU(2)$ doublets of the SM into $SU(3)$ triplets, adding also other $SU(3)$ gauge bosons. Then the $SU(3)_L \times U(1)_X$ gauge symmetry breaks down spontaneously to the SM EW group by two complex scalar fields $\Phi_{1,2}$, triplets under SU(3).

LFV decays in the SLH model arise at one loop level and they are driven by the presence of the heavy neutrinos N_i in connivance with the rotation of light lepton fields V_ℓ^{ij}. There are two generic topologies participating in this amplitude:

- **Penguin diagrams**, namely $\ell \to \ell_k\{\gamma, Z, Z'\}$, followed by $\{\gamma, Z, Z'\} \to \ell_a \bar{\ell_b}$,
- **Box diagrams**.

In principle there should be also a penguin contribution with a Higgs boson $\ell \to \ell_k H$, however the coupling of the Higgs to the light fermions $H \to \ell_a \bar{\ell_a}$, has an intrinsic suppression due to the small mass of the fermions and, therefore, we do not take this into account. As there is no contribution from the SLH model to our processes at tree level, the calculation, at one loop, has to be finite. In the following:

- $\ell(p) \to \ell_k(p_1) + \dots$
- We calculate our amplitudes as an expansion, until the second order of the parameter: ν/f,
- $p^2 = m_\ell^2$,
- $p_1^2 = m_{\ell_k}^2 = 0$.

In the Unitary Gauge only physical states appear in the calculation. The price to pay is that we have to use the unitary propagator for the gauge vector bosons and the calculation looks more divergent. However the best strategy is to use the 't Hooft-Feynman gauge for easier divergence cancellation.

3.3 GENERAL STRUCTURE OF THE LFV PROCESSES

The contributions of the SM to the *LFV* processes $\ell \to \ell_a \gamma$ and $\ell \to \ell_k \ell_a \bar{\ell}_b$ are negligible for they are proportional to the observed neutrino masses [33, 34, 35, 129], nevertheless the new Little Higgs contributions can be a priori

[2] the material in the rest of this chapter and the next ones is based on our paper [**ramirez:2022zpk**].

large. The effective LFV $V_\mu \ell \ell_a$ vertex with $V_\mu = \gamma, Z, Z'$ is sketched in figure 12.

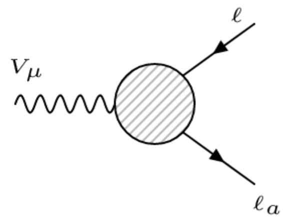


Figure 12: Effective LFV vertex, where $V_\mu = \gamma, Z, Z'$.

The most general structure for on-shell fermions can be written as :

$$i\Gamma^\mu(p, p_1) = ie\Big[\gamma^\mu \left(F_L^V P_L + F_R^V P_R\right) - \left(iF_M^V + F_E^V \gamma_5\right)\sigma^{\mu\nu} Q_\nu + \left(iF_S^V + F_P^V \gamma_5\right) Q^\mu\Big], \tag{255}$$

where $Q = p - p_1$ is the vector boson momentum. Three body lepton decays $\ell \to \ell_k \ell_a \bar{\ell}_b$ receive contributions from penguin and box diagrams as we show in figure 13.

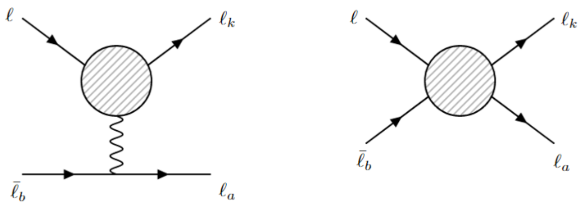


Figure 13: Generic penguin and box diagrams for $\ell \to \ell_k \ell_a \bar{\ell}_b$.

There are seven possible decays:

(a) $\tau^- \to \mu^- \mu^- \mu^+$,
(b) $\tau^- \to \mu^- \mu^- e^+$,
(c) $\tau^- \to e^- \mu^- \mu^+$,
(d) $\tau^- \to e^- e^- \mu^+$,
(e) $\tau^- \to \mu^- e^- e^+$,
(f) $\tau^- \to e^- e^- e^+$,
(g) $\mu^- \to e^- e^- e^+$.

We divide them into three categories according to the leptonic flavours in the final state: Category (*i*) comprises all the decays where $\ell_k = \ell_a = \ell_b$ (i.e. the decays (a), (f) and (g)), this kind of decays receive the name of *same-flavors* decays. Category (*ii*) contains all the decays where either $\ell_k \neq \ell_b$ and $\ell_a = \ell_b$, or $\ell_k = \ell_b$ and $\ell_a \neq \ell_b$ (i.e. the decays (c) and (e)), this category is known as *same-sign* decays. And lastly, all the decays with final leptons having $\ell_k \neq \ell_b$, $\ell_a \neq \ell_b$ belong to the category (*iii*) (i.e. the decays (b) and (d)), the so-called *wrong-sign* decays. Finally, we studied the $\ell N - \ell_a N$ conversion processes ($\ell = \mu, e$, $\ell_a = \tau, e$) whose form factors look very similar to the same flavors category decays.

3.3.1 $\ell \to \gamma \ell_a$ decays

The amplitude $\ell \to \ell_a \gamma$ is proportional to the vertex in figure 12, however as shown in refs. [17, 130] the form factors $F^V_{L,R} = 0$ when V is an on-shell photon. The scalar and pseudoscalar form factors $F^V_{S,P}$ do not contribute for real V and are negligible for virtual V in the processes under study. Neglecting $m_{\ell_a} \ll m_\ell$ the total width for $\ell \to \ell_a \gamma$ is given by:

$$\Gamma\left(\ell \to \ell_a \gamma\right) = \frac{\alpha m_\ell^3}{2}\left(\left|F_M^\gamma\right|^2 + \left|F_E^\gamma\right|^2\right). \tag{256}$$

The branching ratio is obtained dividing by the SM decay width which, at leading order, is:

$$\begin{aligned} &\Gamma\left(\ell_j \to \ell_i \nu_j \bar{\nu}_i\right) = \frac{G_F^2 m_{\ell_j}^5}{192\pi^3}, \quad G_F = \frac{\pi \alpha_W}{\sqrt{2} M_W^2}, \\ &\alpha_W = \frac{\alpha}{s_W^2}. \end{aligned} \tag{257}$$

In the case of τ decays the SM branching ratio must be multiplyied by ~ 0.17 to take into account both lepton Michel and hadron decay channels. For these calculations we have approximated all the integrations until $\mathcal{O}\left(v^2/f^2\right)$ and then neglected the ratios:

$$\frac{m_\ell^2}{M_{N_i}^2} = \frac{m_\ell^2}{M_W^2} = \frac{m_\ell^2}{M_X^2} = 0\,. \tag{258}$$

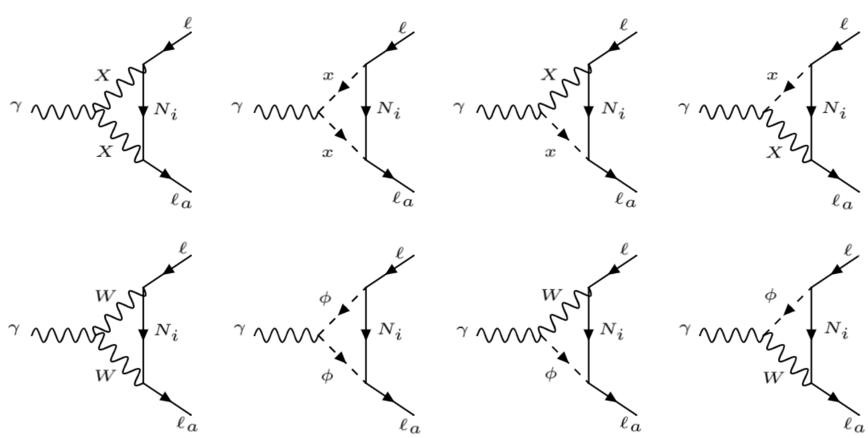


Figure 14: Feynman diagrams for $\ell \to \gamma \ell_a$

We can classify the contributions to $\ell \to \ell_a \gamma$ into two types of topologies (see figure 14): loop diagrams with heavy X gauge bosons and with W bosons (with corresponding Goldstone bosons x and ϕ, equivalent to their longitudinal polarizations). Since only dipole form factors contribute to this process, we have:

$$\begin{aligned} F_M^\gamma &= F_M^\gamma|_X + F_M^\gamma|_W, \\ F_E^\gamma &= F_E^\gamma|_X + F_E^\gamma|_W. \end{aligned} \tag{259}$$

Defining the mass ratios:

$$x_i = \frac{M_{N_i}^2}{M_X^2} \simeq \mathcal{O}(1), \quad \omega = \frac{M_W^2}{M_X^2} \simeq \mathcal{O}\left(v^2/f^2\right), \tag{260}$$

we find the following contribution to the dipole form factors for the X-mediated diagrams [26]:

$$F_M^\gamma|_X = -iF_E^\gamma|_X = \frac{\alpha_W}{16\pi}\frac{m_\ell}{M_X^2}\left(1-\delta_\nu^2\right)\sum_i V^{\ell_a i*}V^{i\ell}F_X\left(x_i\right), \tag{261}$$

where

$$\begin{aligned} F_X(x) &= M_X^2\left[2\bar{C}_1 - 3\bar{C}_{11} - x\left(\bar{C}_0 + 3\bar{C}_1 + \frac{3}{2}\bar{C}_{11}\right)\right] \\ &= \frac{5}{6} - \frac{3x - 15x^2 - 6x^3}{12(1-x)^3} + \frac{3x^2}{2(1-x)^4}\log[x]. \end{aligned} \tag{262}$$

These contributions are equal to those of the SM with massive neutrinos, replacing $X \to W$, $N_i \to \nu_i$ and $V^{im} \to V_{PMNS}$. For tiny neutrino masses, $x_i = m_{\nu_i}^2/M_W \ll 1$,

$$F_W(x) \to \frac{5}{6} - \frac{x}{4} + \mathcal{O}(x^2) \tag{263}$$

and we recover a well known result [34, 35, 121] bounded by neutrino oscillation experiments:

$$\mathcal{B}\left(\mu \to e\gamma\right)_{SM} = \frac{3\alpha}{32\pi}\left|\sum_{i=2,3} V_{PMNS}^{ei*}V_{PMNS}^{i\mu}\frac{\Delta m_{i1}^2}{M_W^2}\right|^2 \leqslant 10^{-54}. \tag{264}$$

Expressions for the loop functions are collected in Appendix A and take the value with $Q^2 = 0$ for an on-shell photon. For the W-based diagrams, we obtain [26]:

$$F_M^\gamma|_W = -iF_E^\gamma|_W = \frac{\alpha_W}{16\pi}\frac{m_\ell}{M_W^2}\delta_\nu^2\sum_i V^{\ell_a i*}V^{i\ell}F_W\left(x_i/\omega\right), \tag{265}$$

with

$$F_W(x) = M_W^2\left(-2\bar{C}_1 + 3\bar{C}_{11}\right) + M_{N_i}^2\left(\bar{C}_0 + \bar{C}_1 - \frac{3}{2}\bar{C}_{11}\right), \tag{266}$$

that we rewrite as:

$$F_W(x) = \frac{M_W^2}{M_X^2} M_X^2 \left(-2\bar{C}_1 + 3\bar{C}_{11}\right) + \frac{M_{N_i}^2}{M_X^2} M_X^2 \left(\bar{C}_0 + \bar{C}_1 - \frac{3}{2}\bar{C}_{11}\right), \quad (267)$$

to keep leading order terms at $\mathcal{O}(v^2/f^2)$. The first term of the form factor in (267) is already of order $\mathcal{O}(v^2/f^2)$, so when multiplied by δ_ν^2 the result becomes order $\mathcal{O}(v^4/f^4)$, which we neglect. Then, this contribution to the form factor is:

$$\begin{aligned} F_W(x) &= x M_X^2 \left(\bar{C}_0 + \bar{C}_1 - \frac{3}{2}\bar{C}_{11}\right) \\ &= \frac{x\left(8x^2 + 5x - 7\right)}{12(1-x)^3} + \frac{x^2\left(3x-2\right)}{2(1-x)^4} \log[x], \end{aligned} \quad (268)$$

in agreement with ref. [26]. The whole dipole form factors are thus:

$$F_M^\gamma = -iF_E^\gamma = \frac{\alpha_W}{4\pi} \frac{m_\ell}{M_W^2} \sum_i V_\ell^{\ell_a i *} V_\ell^{i\ell} \left[\frac{v^2}{2f^2} F_X\left(x_i\right) + \delta_\nu^2 F_W\left(x_i/\omega\right)\right]. \quad (269)$$

3.3.2 $\ell \to \ell_a \ell_a \bar{\ell}_a$ decays

The contributions to the transition amplitude of the *LFV* three body decays can be summarized as [26]:

$$\mathcal{M} = \mathcal{M}_{\gamma\,\text{penguin}} + \mathcal{M}_{Z\,\text{penguin}} + \mathcal{M}_{Z'\,\text{penguin}} + \mathcal{M}_{\text{boxes}}\,. \quad (270)$$

We define the amplitudes and form factors as:

$$
\begin{aligned}
\mathcal{M}_{\gamma\text{penguin}} =& \frac{e^2}{Q^2}\bar{u}(p_1)\left[Q^2\gamma^\mu\left(A_1^L P_L + A_1^R P_R\right) + m_\ell i\sigma^{\mu\nu}Q_\nu\left(A_2^L P_L + A_2^R P_R\right)\right]u(p) \\
& \times \bar{u}(p_2)\gamma_\mu v(p_3) - (p_1 \leftrightarrow p_2)\,, \\
\mathcal{M}_{Z\text{penguin}} =& \frac{e^2}{M_Z^2}\bar{u}(p_1)\left[\gamma^\mu\left(F_L P_L + F_R P_R\right)\right]u(p)\bar{u}(p_2)\left[\gamma_\mu\left(Z_L^a P_L + Z_R^a P_R\right)\right]v(p_3) \\
& - (p_1 \leftrightarrow p_2)\,, \\
\mathcal{M}_{Z'\text{penguin}} =& \frac{e^2}{M_{Z'}^2}\bar{u}(p_1)\left[\gamma^\mu\left(F_L' P_L + F_R' P_R\right)\right]u(p)\bar{u}(p_2)\left[\gamma_\mu\left(Z_L'^a P_L + Z_R'^a P_R\right)\right]v(p_3) \\
& - (p_1 \leftrightarrow p_2)\,, \\
\mathcal{M}_{\text{boxes}} =& e^2 B_1^L\left[\bar{u}(p_1)\gamma^\mu P_L u(p)\right]\left[\bar{u}(p_2)\gamma_\mu P_L v(p_3)\right] \\
&+ e^2 B_1^R\left[\bar{u}(p_1)\gamma^\mu P_R u(p)\right]\left[\bar{u}(p_2)\gamma_\mu P_R v(p_3)\right] \\
&+ e^2 B_2^L\{\left[\bar{u}(p_1)\gamma^\mu P_L u(p)\right]\left[\bar{u}(p_2)\gamma_\mu P_R v(p_3)\right] - (p_1 \leftrightarrow p_2)\} \\
&+ e^2 B_2^R\{\left[\bar{u}(p_1)\gamma^\mu P_R u(p)\right]\left[\bar{u}(p_2)\gamma_\mu P_L v(p_3)\right] - (p_1 \leftrightarrow p_2)\} \\
&+ e^2 B_3^L\{\left[\bar{u}(p_1)P_L u(p)\right]\left[\bar{u}(p_2)P_L v(p_3)\right] - (p_1 \leftrightarrow p_2)\} \\
&+ e^2 B_3^R\{\left[\bar{u}(p_1)P_R u(p)\right]\left[\bar{u}(p_2)P_R v(p_3)\right] - (p_1 \leftrightarrow p_2)\} \\
&+ e^2 B_4^L\{\left[\bar{u}(p_1)\sigma^{\mu\nu}P_L u(p)\right]\left[\bar{u}(p_2)\sigma_{\mu\nu}P_L v(p_3)\right] - (p_1 \leftrightarrow p_2)\} \\
&+ e^2 B_4^R\{\left[\bar{u}(p_1)\sigma^{\mu\nu}P_R u(p)\right]\left[\bar{u}(p_2)\sigma_{\mu\nu}P_R v(p_3)\right] - (p_1 \leftrightarrow p_2)\},
\end{aligned}
\tag{271}
$$

where

$$
\begin{aligned}
A_2^L &= -\left(F_M^\gamma + iF_E^\gamma\right)/m_\ell & A_2^R &= -\left(F_M^\gamma - iF_E^\gamma\right)/m_\ell \\
A_1^L &= F_L^\gamma/Q^2 & A_1^R &= F_R^\gamma/Q^2 \\
F_L &= -F_L^Z & F_R &= -F_R^Z \\
F_L' &= -F_L^{Z'} & F_R' &= -F_R^{Z'}.
\end{aligned}
\tag{272}
$$

We can use eqs. (271) to obtain the partial decay width for the *same flavors* decays [26]:

$$
\begin{aligned}
\Gamma\left(\ell \to \ell_a \ell_a \bar{\ell}_a\right) = & \frac{\alpha^2 m_\ell^5}{32\pi}\Bigg[\left|A_1^L\right|^2 + \left|A_1^R\right|^2 - 2\left(A_1^L A_2^{*R} + A_2^L A_1^{*R} + \text{h.c.}\right) \\
& + \left(\left|A_2^L\right|^2 + \left|A_2^R\right|^2\right)\left(\frac{16}{3}\log\left[\frac{m_\ell}{m_{\ell_a}}\right] - \frac{22}{3}\right) + \frac{1}{6}\left(\left|\hat{B}_1^L\right|^2 + \left|\hat{B}_1^R\right|^2\right) + \frac{1}{3}\left(\left|\hat{B}_2^L\right|^2 + \left|\hat{B}_2^R\right|^2\right) \\
& + \frac{1}{24}\left(\left|B_3^L\right|^2 + \left|B_3^R\right|^2\right) + 6\left(\left|B_4^L\right|^2 + \left|B_4^R\right|^2\right) - \frac{1}{2}\left(B_3^L B_4^{L*} + B_3^R B_4^{R*} + \text{h.c.}\right) \\
& + \frac{1}{3}\left(A_1^L \hat{B}_1^{L*} + A_1^R \hat{B}_1^{R*} + A_1^L \hat{B}_2^{L*} + A_1^R \hat{B}_2^{R*} + \text{h.c.}\right) \\
& - \frac{2}{3}\left(A_2^R \hat{B}_1^{L*} + A_2^L \hat{B}_1^{R*} + A_2^L \hat{B}_2^{R*} + A_2^R \hat{B}_2^{L*} + \text{h.c.}\right) \\
& + \frac{1}{3}\Big\{2\left(\left|F_{LL}\right|^2 + \left|F_{RR}\right|^2\right) + \left|F_{LR}\right|^2 + \left|F_{RL}\right|^2 \\
& + \left(\hat{B}_1^L F_{LL}^* + \hat{B}_1^R F_{RR}^* + \hat{B}_2^L F_{LR}^* + \hat{B}_2^R F_{RL}^* + \text{h.c.}\right) + 2\left(A_1^L F_{LL}^* + A_1^R F_{RR}^* + \text{h.c.}\right) \\
& + \left(A_1^L F_{LR}^* + A_1^R F_{RL}^* + \text{h.c.}\right) - 4\left(A_2^R F_{LL}^* + A_2^L F_{RR}^* + \text{h.c.}\right) \\
& - 2\left(A_2^L F_{RL}^* + A_2^R F_{LR}^* + \text{h.c.}\right)\Big\}\Bigg],
\end{aligned}
\tag{273}
$$

where

$$
\begin{aligned}
F_{LL} &= \frac{F_L Z_L^a}{M_Z^2}, \quad F_{RR} = \frac{F_R Z_R^a}{M_Z^2}, \\
F_{LR} &= \frac{F_L Z_R^a}{M_Z^2}, \quad F_{RL} = \frac{F_R Z_L^a}{M_Z^2}.
\end{aligned}
\tag{274}
$$

Some box form factors have been redefined to include the contributions from the Z' penguins:

$$
\begin{aligned}
B_1^L &\to \hat{B}_1^L = B_1^L + 2F'_{LL}, \\
B_1^R &\to \hat{B}_1^R = B_1^R + 2F'_{RR}, \\
B_2^L &\to \hat{B}_2^L = B_2^L + F'_{LR}, \\
B_2^R &\to \hat{B}_2^R = B_2^R + F'_{RL},
\end{aligned}
\tag{275}
$$

with

$$
\begin{aligned}
F'_{LL} &= \frac{F'_L Z'^a_L}{M_{Z'}^2}, \quad F'_{RR} = \frac{F'_R Z'^a_R}{M_{Z'}^2}, \\
F'_{LR} &= \frac{F'_L Z'^a_R}{M_{Z'}^2}, \quad F'_{RL} = \frac{F'_R Z'^a_L}{M_{Z'}^2}.
\end{aligned}
\tag{276}
$$

In our case many of the form factors vanish. The relevant penguin diagrams are listed in figure 15.

PHOTON PENGUINS

The dipole form factors are the same as in the $\ell \to \ell_a \gamma$ case and for those terms we can set $Q^2 = 0$, since Q^2 is small in these processes. The form factors $F_{L,R}$ are linear in Q^2 and we neglect terms of order m_ℓ^2/M^2, which means that $F_R \simeq 0$. The contribution of diagrams with X bosons is:

$$F_L^\gamma|_X = \frac{\alpha_W}{4\pi}(1-\delta_\nu^2)\sum_i V^{\ell_a i*}V^{i\ell}G_X(x_i), \tag{277}$$

where

$$\begin{aligned} G_X(x) &= -\frac{1}{2} + \overline{B}_1 + 6\overline{C}_{00} + x\left(\frac{1}{2}\overline{B}_1 + \overline{C}_{00} - M_X^2\overline{C}_0\right) \\ &\quad - Q^2\left(2\overline{C}_1 + \frac{1}{2}\overline{C}_{11}\right) \\ &= \Delta^X + \frac{Q^2}{M_X^2}G_X^{(1)}(x) + \mathcal{O}\left(\frac{Q^2}{M_X^4}\right), \end{aligned} \tag{278}$$

where (Δ^W is defined analogously below)

$$\begin{aligned} \Delta^X &= \frac{2}{\epsilon} - \gamma_E + \log(4\pi) - \log\left(\frac{M_X^2}{\mu^2}\right) \\ &= \Delta_\epsilon - \log\left(\frac{M_X^2}{\mu^2}\right) \end{aligned} \tag{279}$$

diverges in four dimensions.

The terms that are not proportional to Q^2 are cancelled by GIM mechanism, therefore the contribution to the form factor is [26]:

$$G_X(x) = \frac{Q^2}{M_X^2}G_X^{(1)}(x) + \mathcal{O}\left(\frac{Q^4}{M_X^4}\right), \tag{280}$$

$$G_X^{(1)}(x) = -\frac{5}{18} + \frac{x(12+x-7x^2)}{24(1-x)^3} + \frac{x^2(12-10x+x^2)}{12(1-x)^4}\log[x]\,. \tag{281}$$

Then, the contributions of the X-diagrams is:

$$F_L^\gamma\big|_X = \frac{\alpha_W}{4\pi}\frac{Q^2}{M_X^2}\left(1-\delta_\nu^2\right)\sum_i V_\ell^{\ell_a i*}V_\ell^{i\ell}G_X^{(1)}(x_i)\,. \tag{282}$$

The W-based diagrams contribute with [26][4]:

$$F_L^\gamma|_W = \frac{\alpha_W}{4\pi}\delta_\nu^2\sum_i V^{\ell_a i*}V^{i\ell}G_W(x_i/\omega), \tag{283}$$

where

$$\begin{aligned} G_W(x) &= \frac{1}{2} - \overline{B}_1 - 6\overline{C}_{00} + \frac{M_N^2}{M_W^2}\left(\overline{C}_{00} + \frac{1}{2}\overline{B} + M_W^2\overline{C}_0\right) \\ &= -\Delta^W + \frac{Q^2}{M_W^2}G_W^{(1)}(x) + \mathcal{O}\left(\frac{Q^2}{M_W^4}\right), \end{aligned} \tag{284}$$

[4] It is important to note that in this work we found a discrepancy in the contribution of the W boson based diagrams and the paper [26].

$$G_W^{(1)}(x) = -\frac{1}{6} - \frac{x(-2+7x-11x^2)}{72(1-x)^3} + \frac{x^4}{12(1-x)^4}\log[x], \tag{285}$$

that is:

$$F_L^{\gamma}|_W = \frac{\alpha_W}{4\pi}\frac{Q^2}{M_W^2}\delta_\nu^2 \sum_i V_\ell^{\ell_a i*} V_\ell^{i\ell} G_W^{(1)}(x_i/\omega). \tag{286}$$

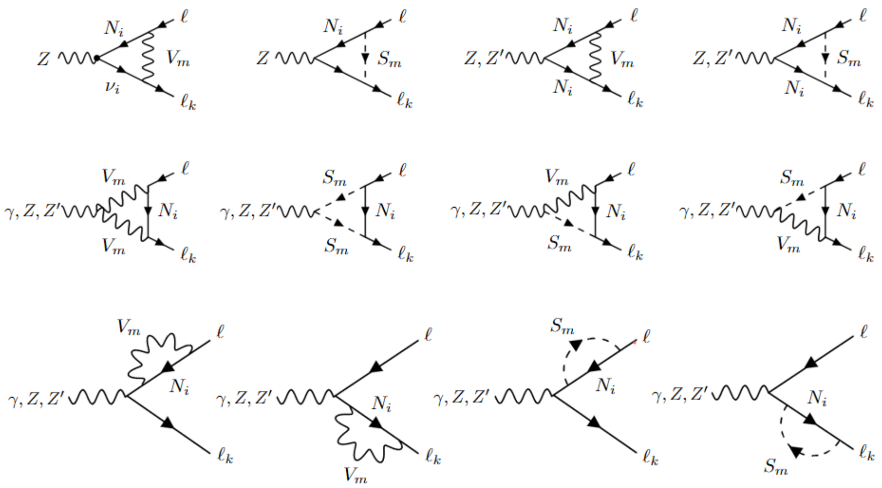


Figure 15: Relevant triangle and self-energy diagrams for $\ell \to \ell_k \ell_a \bar{\ell}_b$ decays, where $V_m = X, W$ and $S_m = x, \phi$.

Z PENGUINS

In this case there are three pieces: two of them involve only heavy neutrinos in the loop ($F_L^Z|_X$, $F_L^Z|_W$), and the third contains one heavy and one light neutrino ($F_L^Z|_{hl}$), with either gauge boson,

$$F_L^Z = F_L^Z|_X + F_L^Z|_W + F_L^Z|_{hl}. \tag{287}$$

Again we neglect m_ℓ^2/M^2 terms. The Z dipole form factors $F_{M,E}^Z$ (which are chirality flipping) can be neglected as compared to F_L^Z. The X-based diagrams result in [26][5]:

$$F_L^Z|_X = \frac{\alpha_W}{4\pi}\frac{1}{c_W s_W}\sum_i V^{\ell_a i*} V^{i\ell}\left[\frac{c_W \delta_Z}{\sqrt{3-t_W^2}} I_X(x_i) + \delta_\nu^2 H_X(x_i)\right], \tag{288}$$

where

$$I_X(x) = \frac{6x-x^2}{2(1-x)} + \frac{8x}{2(1-x)^2}\log[x],$$
$$H_X(x) = \frac{x}{4} + \frac{x}{4(1-x)}\log[x]. \tag{289}$$

[5] in this work we found differences in the terms $I_X(x)$ and $H_X(x)$ with respect to the published work [26].

The W boson diagrams give the following contribution [26][6]:

$$F_L^Z|_W = \frac{\alpha_W}{4\pi}\frac{\delta_\nu^2}{c_W s_W}\sum_i V^{\ell_a i*}V^{i\ell}\left[-c_W^2 H_W(x_i/\omega) + \delta_\nu^2\frac{(2+(1-t_W^2)t_\beta^2)}{8} I_W(x_i/\omega)\right.$$
$$\left. + \left(-\frac{s_W^2}{2} + \frac{(1-2s_W^2)^2 t_\beta^2 \delta_\nu^2}{8c_W^4}\right) R_W(x_i/\omega)\right], \tag{290}$$

where (the R_W contribution turns out to be negligible)

$$\begin{aligned} H_W(x) &= \frac{1}{8} + \frac{5x}{4(1-x)} + \frac{5x^2}{4(1-x)^2}\log[x], \\ I_W(x) &= -\frac{2x^2+3x}{2(1-x)} - \frac{x^2}{(1-x)^2}\log[x], \\ R_W(x) &= \frac{x}{4(1-x)} + \frac{x^2}{2(1-x)^2}\log[x]. \end{aligned} \tag{291}$$

Diagrams where the Z couples one heavy to one light neutrino contribute with [26]:

$$\begin{aligned} F_L^Z\big|_{hl} &= \frac{\alpha_W}{4\pi}\frac{\delta_\nu^2}{s_W c_W}\Sigma_i V^{\ell_a i*}V^{i\ell}\left[\hat{C}_{00}(M_W^2, 0; x_i/\omega) - \hat{C}_{00}(M_X^2, 0; x_i)\right], \\ &= \frac{\alpha_W}{4\pi}\frac{\delta_\nu^2}{s_W c_W}\Sigma_i V^{\ell_a i*}V^{i\ell}\left[H_Z(x_i/\omega) - H_Z(x_i)\right], \end{aligned} \tag{292}$$

where[7]:

$$H_z(x) = \frac{x\log[x]}{4(1-x)}. \tag{293}$$

Z' penguins

Here we have two contributions:

$$F_L^{Z'} = F_L^{Z'}|_X + F_L^{Z'}|_W, \tag{294}$$

there is no piece analogous to $F_L^Z\big|_{hl}$ since the Z' has an additional v^2/f^2 suppression from its propagator that makes those terms subleading. The form factors read [26][8]:

$$F_L^{Z'}|_X = \frac{\alpha_W}{4\pi}\frac{1}{s_W\sqrt{3-t_W^2}}\sum_i V^{\ell_a i*}V^{i\ell} I_X(x_i), \tag{295}$$

---

[6] In this part, we found that the factor $-c_W^2$ that multiplies to H_W was omitted in [26].
[7] Here we found a difference by a factor of 2 with respect to [26].
[8] Here we found again the same term R_w that was omitted in [26].

$$F_L^{Z'}|_W = \frac{\alpha_W}{8\pi} \frac{\delta_\nu^2}{s_W\sqrt{3-t_W^2}} \sum_i V^{\ell_a i *} V^{i\ell} \left[I_W(x_i/\omega) + \frac{(1-2s_W^2)}{c_W^2} R_W(x_i/\omega) \right], \tag{296}$$

where I_X, I_W and R_W are defined in equation (289) and (291). We note that the pieces with a $(1-2s_W^2)$ prefactor are numerically suppressed and can be neglected.

BOX DIAGRAMS

Only W and X particles can be involved in the loop (see figure 16). Crossed diagrams, not shown in the figure, contribute a factor 2 due to Fierz identities [131]. In the limit of zero external momenta (all internal masses are much larger than the muon or tau mass) all of them have the same form (being proportional to a scalar integral over the internal momentum).

Neglecting m_ℓ/M, we have contributions only to the B_1^L form factor, divided in three terms:

$$B_1^L = B^L|_X + B^L|_W + B^L|_{WX}, \tag{297}$$

where (only the numerically relevant terms below were quoted in ref. [26], as it happens in other box contributions)[9]: :

$$B_1^L|_W = \frac{\alpha_W}{8\pi}\frac{\delta_\nu^4}{s_W^2}\frac{1}{M_W^2}\sum_{ij}\chi_{ij}\left[\left(1+\frac{x_ix_j}{4\omega^2}\right)\tilde{d}_0(x_i/\omega, x_j/\omega) + \frac{2x_ix_j}{\omega^2} d_0(x_i/\omega, x_j/\omega)\right], \tag{298}$$

$$B_1^L|_X = \frac{\alpha_W}{8\pi}\frac{(1-2\delta_\nu^2)}{s_W^2}\frac{1}{M_X^2}\sum_{ij}\chi_{ij}\left[\left(1+\frac{x_ix_j}{4}\right)\tilde{d}_0(x_i, x_j) - 2x_ix_j d_0(x_i, x_j)\right], \tag{299}$$

$$B_1^L|_{WX} = \frac{\alpha_W}{8\pi}\frac{\delta_\nu^2}{s_W^2}\frac{1}{M_W^2}\sum_{ij}\chi_{ij}x_ix_j\left[\frac{1}{2}\tilde{d}_0(\omega, x_i, x_j) - 2d_0(\omega, x_i, x_j)\right], \tag{300}$$

and

$$\chi_{ij} = V^{\ell_a i *} V^{i\ell} |V^{j\ell_a}|^2 \tag{301}$$

encodes all flavor mixing.

[9] In this case, the first and third contribution to $B_1^L|_W$ are numerically irrelevant in the form factor.

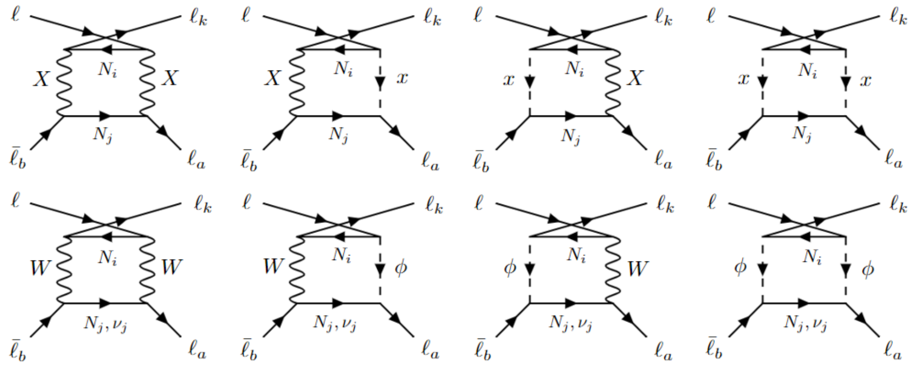


Figure 16: Relevant box diagrams for $\ell \to \ell_k \ell_a \bar{\ell}_b$ decays.

3.3.3 $\ell \to \ell_k \ell_a \bar{\ell}_a$ decays

Now, we analyse the *same-sign* category, i.e. the decays of the form $\ell(p) \to \ell_k(p_1)\ell_a(p_2)\bar{\ell}_a(p_3)$. We note that in this case the amplitude has no crossed penguin diagrams contributions due to swapping ℓ_k and ℓ_a because it will be a two-loop process. Therefore, the amplitude for the *same-sign* decays has no $p_1 \leftrightarrow p_2$ term in eqs. (271). However, for the box amplitudes there are additional diagrams for swapping ℓ_k and ℓ_a. Furthermore, now there is no symmetry factor of 1/2 in the phase space integration because all three final leptons are distinguishable. The final decay width can be written as [20]:

$$\begin{aligned}\Gamma\left(\ell \to \ell_k \ell_a \bar{\ell}_a\right) = \frac{\alpha^2 m_\ell^5}{96\pi}\Big[2\left(|A_1^L|^2 + |A_1^R|^2\right) - 4\left(A_1^L A_2^{*R} + A_2^L A_1^{*R} + \text{h.c.}\right) \\ + 4\left(|A_2^L|^2 + |A_2^R|^2\right)\left(4\log\left[\frac{m_\ell}{m_{\ell_a}}\right] - 7\right) + |F_{LL}|^2 + |F_{RR}|^2 + |F_{LR}|^2 + |F_{RL}|^2 \\ + |\hat{B}_{L1}|^2 + |\hat{B}_{R1}|^2 + |\hat{B}_{L2}|^2 + |\hat{B}_{R2}|^2 + \frac{1}{4}\left(|B_{L3}|^2 + |B_{R3}|^2\right) + 12\left(|B_{L4}|^2 + |B_{R4}|^2\right) \\ + \left(A_1^L F_{LL}^* + A_1^R F_{RR}^* + A_1^L F_{LR}^* + A_1^R F_{RL}^* + \text{h.c.}\right) - 2\left(A_2^R F_{LL}^* + A_2^L F_{RR}^* + A_2^R F_{LR}^* + A_2^L F_{RL}^* + \text{h.c.}\right) \\ + \left(A_1^L \hat{B}_1^{L*} + A_1^R \hat{B}_1^{R*} + A_1^L \hat{B}_2^{L*} + A_1^R \hat{B}_2^{R*} + \text{h.c.}\right) - 2\left(A_2^R \hat{B}_1^{L*} + A_2^L \hat{B}_1^{R*} + A_2^L \hat{B}_2^{R*} + A_2^R \hat{B}_2^{L*} + \text{h.c.}\right) \\ + \left(F_{LL}\hat{B}_1^{L*} + F_{RR}\hat{B}_1^{R*} + F_{LR}\hat{B}_2^{L*} + F_{RL}\hat{B}_2^{R*} + \text{h.c.}\right)\Big],\end{aligned} \tag{302}$$

with the same simplifying definitions as in eqs. (274) and (276), however the redefinitions of the box form factors are almost the same that in the eqs. (275) but, in this case we use:

$$\begin{aligned} B_1^L \to \hat{B}_1^L &= B_1^L + F'_{LL}, \\ B_1^R \to \hat{B}_1^R &= B_1^R + F'_{RR}, \\ B_2^L \to \hat{B}_2^L &= B_2^L + F'_{LR}, \\ B_2^R \to \hat{B}_2^R &= B_2^R + F'_{RL}. \end{aligned} \tag{303}$$

As in the example above many of the form factors are zero so the decay width will be simplified. These decays receive contributions from the penguin diagrams in figure 15. We take the following approximations:

$$\frac{m_\ell^2}{M_X^2} = \frac{m_\ell^2}{M_W^2} = \frac{m_{\ell_k}^2}{M_X^2} = \frac{m_{\ell_k}^2}{M_W^2} = 0, \tag{304}$$

which means that the form factor in equations (277)-(296) are the same in this category, also the dipole form factors are the same. On the other hand the contributions from four-point form factors can be written in the generic form [20]:

$$\mathcal{F}_4 = \sum_{ij} \chi_{ij}^{\ell\ell_k\ell_a\ell_a} F_4\left(M_{N_i}, M_{N_j}, \dots\right), \tag{305}$$

where we have defined the flavor mixing coefficients:

$$\chi_{ij}^{\ell\ell_k\ell_a\ell_a} = V^{\ell_k i*} V^{i\ell} |V^{j\ell_a}|^2 + V^{\ell_a i*} V^{i\ell} V^{\ell_k j*} V^{j\ell_a}, \tag{306}$$

and the second term exchanges ℓ_k and ℓ_a. Thus, the box form factors[10] contributing in this category are represented in figure 16:

$$B_1^L|_X = \frac{\alpha_W}{16\pi} \frac{(1-2\delta_\nu^2)}{s_W^2 M_X^2} \sum_{ij} \chi_{ij}^{\ell\ell_k\ell_a\ell_a} \left[\left(1 + \frac{x_i x_j}{4}\right) \tilde{d}_0(x_i, x_j) - 2x_i x_j d_0(x_i, x_j) \right], \tag{307}$$

$$B_1^L|_W = \frac{\alpha_W}{16\pi} \frac{\delta_\nu^4}{s_W^2} \frac{1}{M_W^2} \sum_{ij} \chi_{ij}^{\ell\ell_k\ell_a\ell_a} \left[\left(1 + \frac{x_i x_j}{4\omega^2}\right) \tilde{d}_0(x_i/\omega, x_j/\omega) + \frac{2x_i x_j}{\omega^2} d_0(x_i/\omega, x_j/\omega) \right], \tag{308}$$

$$B_1^L|_{WX} = \frac{\alpha_W}{16\pi} \frac{\delta_\nu^2}{s_W^2 M_W^2} \sum_{ij} \chi_{ij}^{\ell\ell_k\ell_a\ell_a} x_i x_j \left[\frac{1}{2} \tilde{d}_0(\omega, x_i, x_j) - 2d_0(\omega, x_i, x_j) \right]. \tag{309}$$

Another difference between these box form factors and those given in the equations (298)-(300) is that here we do not have crossed diagrams, so no factor of 2 comes from the Fierz identities.

3.3.4 $\ell \to \ell_a \ell_a \bar{\ell}_b$ decays

In the category of *wrong-sign* decays we have the *double flavor* violating three-body process: $\ell(p) \to \ell_a(p_1)\ell_a(p_2)\bar{\ell}_b(p_3)$, in this case the amplitude does not receive contributions from the three-point form factors[11], the box

[10] It is important to note that all the footnotes given in the decays of category *same-flavors* are the same in the case of the decays in the category *same-sign*.

[11] Contributions of Z, Z′ and γ penguin diagrams start at 2 loops.

contributions on the other hand are the same that in *same flavors* category. The total decay width is [20]:

$$\Gamma\left(\ell \to \ell_a \ell_a \bar{\ell}_b\right) = \frac{\alpha^2 m_\ell^5}{192\pi}\left[\left|B_1^L\right|^2 + \left|B_1^R\right|^2 + 2\left(\left|B_2^L\right|^2 + \left|B_2^R\right|^2\right) + \frac{1}{4}\left(\left|B_3^L\right|^2 + \left|B_3^R\right|^2\right) + \frac{1}{36}\left(\left|B_4^L\right|^2 + \left|B_4^R\right|^2\right) - 3\left(B_3^L B_4^{*L} + B_3^R B_4^{*R} + \text{h.c.}\right)\right], \tag{310}$$

as in the previous cases, many of the form factors are zero. These kind of processes can receive contributions from box diagrams that conserve lepton number (LNC) like in figure 16 and diagrams with explicit lepton number violation (LNV), but in our setting we cannot construct box diagrams with LNV vertices because we lack Majorana particles for these contributions. The relevant box form factors are almost the same that in the equations (298)-(300) but the flavor mixing coefficient now differs:

$$\chi^{\ell\ell_a\ell_a\ell_b} = V^{\ell_a i*} V^{i\ell} V^{\ell_a j*} V^{i\ell_b}. \tag{311}$$

Then, the box form factors are:

$$B_1^L|_W = \frac{\alpha_W}{8\pi}\frac{\delta_\nu^4}{s_W^2}\frac{1}{M_W^2}\sum_{ij}\chi^{\ell\ell_a\ell_a\ell_b}\left[\left(1+\frac{x_i x_j}{4\omega^2}\right)\tilde{d}_0(x_i/\omega, x_j/\omega) + \frac{2x_i x_j}{\omega^2} d_0(x_i/\omega, x_j/\omega)\right], \tag{312}$$

$$B_1^L|_X = \frac{\alpha_W}{8\pi}\frac{(1-2\delta_\nu^2)}{s_W^2 M_X^2}\sum_{ij}\chi^{\ell\ell_a\ell_a\ell_b}\left[\left(1+\frac{x_i x_j}{4}\right)\tilde{d}_0(x_i, x_j) - 2x_i x_j d_0(x_i, x_j)\right], \tag{313}$$

$$B_1^L|_{WX} = \frac{\alpha_W}{8\pi}\frac{\delta_\nu^2}{s_W^2}\frac{1}{M_W^2}\sum_{ij}\chi^{\ell\ell_a\ell_a\ell_b} x_i x_j\left[\frac{1}{2}\tilde{d}_0(\omega, x_i, x_j) - 2d_0(\omega, x_i, x_j)\right]. \tag{314}$$

In the *wrong sign* decays there is a mixing of the three families in the flavor coefficients, unlike in the *same flavors* and *same sign* decays where only two families of leptons are mixing.

3.3.5 $\mu - e$ conversion in nuclei

As we have already said $\mu - e$ nuclei conversion is similar to $\mu \to ee\bar{e}$ and differs only in that the lower part of the diagrams is coupled to quarks. It does not have crossed penguin and box diagrams because we have a coherent sum of quarks composing the probed nucleus. Also, we do not have identical particles in the final state. We will write the amplitudes as follows [26]:

$$
\begin{aligned}
\mathcal{M}_{\gamma peng} =& -\frac{e^2}{Q^2}\bar{u}_{\ell_a}(p_1)\left[Q^2\gamma^\mu\left(A_1^L P_L + A_1^R P_R\right) + i m_\ell \sigma^{\mu\nu} Q_\nu \left(A_2^L P_L + A_2^R P_R\right)\right] u_\ell(p) \\
& \times \bar{u}_q(p_2) Q_q \gamma_\mu v_q(p_3), \\
\mathcal{M}_{Z peng} =& \frac{e^2}{M_Z^2}\bar{u}_{\ell_a}(p_1)\left[\gamma^\mu\left(F_L P_L + F_R P_R\right)\right] u_\ell(p)\bar{u}_q(p_2)\gamma_\mu\left(Z_L^q P_L + Z_R^q P_R\right) v_q(p_3), \\
\mathcal{M}_{Z' peng} =& \frac{e^2}{M_{Z'}^2}\bar{u}_{\ell_a}(p_1)\left[\gamma^\mu\left(F_L' P_L + F_R' P_R\right)\right] u_\ell(p)\bar{u}_q(p_2)\gamma_\mu\left(Z_L'^q P_L + Z_R'^q P_R\right) v_q(p_3), \\
\mathcal{M}^q_{box} =& e^2 B^L_{1q}\bar{u}_{\ell_a}(p_1)\gamma^\mu P_L u_\ell(p)\bar{u}_q(p_2)\gamma^\mu P_L v_q(p_3).
\end{aligned}
\tag{315}
$$

We have already taken into account that the only non-zero form factor is B_1^L due to the fact that the SLH couplings are primarily left-handed. This gives for the corresponding conversion width in a nucleus with Z protons and N neutrons [26]:

$$
\begin{aligned}
\Gamma(\mu N \to e N) = \alpha^5 \frac{Z_{eff}^4}{Z}|F(q)|^2 m_\mu^5 \Big| & 2Z\left(A_1^L - A_2^R\right) \\
& - (2Z+N)\,\bar{B}^L_{1u} - (Z+2N)\,\bar{B}^L_{1d}\Big|^2,
\end{aligned}
\tag{316}
$$

where Z_{eff} is the nucleus effective charge for the lepton ℓ and $F(q)$ the associated form factor. The vertex form factors are as for $\ell \to \ell_a \ell_a \bar{\ell}_a$ and were given in (272). We have also defined:

$$
\bar{B}^L_{1q} = B^L_{1q} + \frac{\left(Z_L^q + Z_R^q\right) F_L}{M_Z^2} + \frac{\left(Z_L'^q + Z_R'^q\right) F_L'}{M_{Z'}^2},
\tag{317}
$$

to include the contributions from the Z' penguins. In the case of muons the conversion rate is obtained by dividing by the muon capture rate:

$$
\mathcal{R} = \frac{\Gamma(\mu \to e)}{\Gamma_{capt}}.
\tag{318}
$$

The nuclei we will consider are $^{22}_{48}$Ti and $^{79}_{197}$Au, whose relevant parameters are listed in table 17.

Nucleus	Z	N	Z_{eff}	F(q)	Γ_{capt} [GeV]
$^{22}_{48}$Ti	22	26	17.6	0.54	1.7×10^{-18}
$^{79}_{197}$Au	79	118	33.5	0.16	8.6×10^{-18}

Table 17: Relevant input parameters for the nuclei under study (see refs. [132, 133]).

Only the box form factors need to be recalculated and these are of course embedding-dependent. We stress that we neglect any quark mixing effect for simplicity.

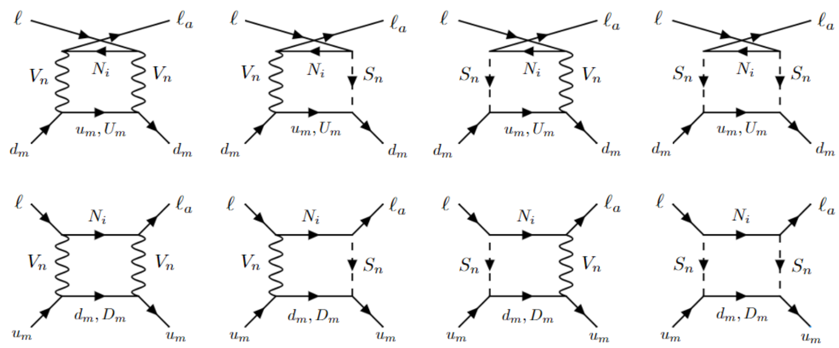


Figure 17: Relevant box diagrams for $\ell N \to \ell_a N$ conversion, where $V_n(S_n) = X, W(x, \phi)$, $u_m(d_m) = u, c(d, s)$ and $U_m(D_m) = U, C(D, S)$.

In this approximation, only diagrams with a D quark appear in the anomaly-free embedding while only diagrams with a U quark are included for the universal embedding. Diagrams proportional to ω and with light quarks appear in both embeddings but will be found to be a subleading contribution.

In the anomaly-free embedding we obtain [26][12]:

$$
\begin{aligned}
B^L_{1u_m}|_X &= \frac{\alpha_W}{16\pi}\frac{(1-\delta_\nu^2)}{s_W^2 M_X^2}\sum_i V^{\ell_a i *}V^{i\ell}\Big[-\Big(4+\frac{x_{D_m}x_i}{4}\Big)\,\tilde{d}_0\left(x_i, x_{D_m}\right)+2x_i x_{D_m} d_0\left(x_i, x_{D_m}\right) \\
&\quad -4|\delta_{d_m}|^2\tilde{d}_0\left(x_i, x_{d_m}\right)\Big], \\
B^L_{1u_m}|_W &= \frac{\alpha_W}{16\pi}\frac{\delta_\nu^2}{s_W^2 M_W^2}\sum_i V^{\ell_a i *}V^{i\ell}\Big[-|\delta_{d_m}|^2\Big(4+\frac{x_{D_m}x_i}{4\omega^2}\Big)\,\tilde{d}_0\left(x_i/\omega, x_{D_m}/\omega\right)-4\tilde{d}_0\left(x_i/\omega, x_{d_m}/\omega\right) \\
&\quad +\frac{x_i x_{D_m}}{\omega^2}(\delta_{d_m}^2+\delta_{d_m}^{*2})d_0\left(x_i/\omega, x_{D_m}/\omega\right)\Big], \\
B^L_{1u_m}|_{XW} &= \frac{\alpha_W}{16\pi}\frac{(1-\delta_\nu^2)}{s_W^2}\frac{1}{M_W^2}\sum_i V^{\ell_a i *}V^{i\ell}\Big[-\delta_\nu\left(\delta_{d_m}+\delta_{d_m}^*\right)x_{D_m}x_i\Big(d_0\left(\omega, x_i, x_{D_m}\right) \\
&\quad -\frac{\tilde{d}_0\left(\omega, x_i, x_{D_m}\right)}{4}\Big) \\
&\quad +\omega\delta_\nu\left(\delta_{d_m}+\delta_{d_m}^*\right)\left[4\left(\tilde{d}_0\left(\omega, x_i, x_{D_m}\right)-\tilde{d}_0\left(\omega, x_i, x_{d_m}\right)\right)-x_i x_{D_m} d_0\left(\omega, x_i, x_{D_m}\right)\right]\Big],
\end{aligned}
\tag{319}
$$

$$B^L_{1d_m} = 0. \tag{320}$$

In the universal embedding we find [26]:

$$B^L_{1u_m} = 0, \tag{321}$$

[12] It is important to remark that, in the form factors, terms related with the light quark are numerically negligible compared with the others terms, as we have already pointed out. In the form factor $B^L_{1u}|_W$, like in the equation (298), the first and third contribution do not appear, also in $B^L_{1u}|_{XW}$ contributions proportional to ω do not appear in ref. [26], but all those contributions are numerically irrelevant. This also occurs in the Universal embedding.

$$
\begin{aligned}
B^L_{1d_m}|_X &= \frac{\alpha_W}{16\pi}\frac{(1-\delta_\nu^2)}{s_W^2 M_X^2}\sum_i V^{\ell_a i*}V^{i\ell}\Big[\left(1+\frac{x_{U_m}x_i}{4}\right)\tilde{d}_0\left(x_i,x_{U_m}\right)-2x_i x_{U_m} d_0\left(x_i,x_{U_m}\right)\\
&+|\delta_{u_m}|^2\tilde{d}_0\left(x_i,x_{u_m}\right)\Big],\\
B^L_{1d_m}|_W &= \frac{\alpha_W}{16\pi}\frac{\delta_\nu^2}{s_W^2 M_W^2}\sum_i V^{\ell_a i*}V^{i\ell}\Big[|\delta_{u_m}|^2\left(1+\frac{x_{U_m}x_i}{4\omega^2}\right)\tilde{d}_0\left(x_i/\omega,x_{U_m}/\omega\right)\\
&+\tilde{d}_0\left(x_i/\omega,x_{u_m}/\omega\right)+\frac{x_i x_{U_m}}{\omega^2}(\delta_{u_m}^2+\delta_{u_m}^{*2})d_0\left(x_i/\omega,x_{U_m}/\omega\right)\Big],\\
B^L_{1d_m}|_{XW} &= \frac{\alpha_W}{16\pi}\frac{(1-\delta_\nu^2)}{s_W^2 M_W^2}\sum_i V^{\ell_a i*}V^{i\ell}\Big[-\delta_\nu\left(\delta_{u_m}+\delta_{u_m}^*\right)x_{U_m}x_i\Big(d_0\left(\omega,x_i,x_{U_m}\right)\\
&-\frac{\tilde{d}_0\left(\omega,x_i,x_{U_m}\right)}{4}\Big)\\
&+\omega\delta_\nu\left(\delta_{u_m}+\delta_{u_m}^*\right)\left(\tilde{d}_0\left(\omega,x_i,x_{u_m}\right)-\tilde{d}_0\left(\omega,x_i,x_{U_m}\right)+x_i x_{U_m} d_0\left(\omega,x_i,x_{U_m}\right)\right)\Big],
\end{aligned}
\tag{322}
$$

with

$$
\begin{aligned}
x_{D_m} &= M_{D_m}^2/M_X^2, \quad x_{U_m} = M_{U_m}^2/M_X^2,\\
x_{d_m} &= m_{d_m}^2/M_X^2, \quad x_{u_m} = m_{u_m}^2/M_X^2.
\end{aligned}
\tag{323}
$$

3.3.6 $\ell-\tau$ conversion in nuclei

The study of $\ell-\tau$ conversion differs from the well-known $\mu-e$ case. The latter is a low-energy process, while the former could be probed via a deep inelastic scattering (DIS) of the initial lepton beam. In a DIS the leptons break the nucleons inside the atomic nuclei and interact with the partons (quarks and gluons) leading to any hadronic final state; thus we are only interested in the $\ell+\mathcal{N}(A,Z)\to\tau+X$ case, where X could be any hadrons in which we do not focus. One important piece in this analysis corresponds to the parton distribution functions (PDFs) encoding the low-energy non-perturbative QCD effects. Thus, perturbative effects ($\hat{\sigma}$) and the long-distance contributions (H) are splitted, via QCD factorization theorems, in the following way:

$$
\sigma_{\ell-\tau}=\hat{\sigma}\otimes H\,. \tag{324}
$$

Such oversimplified form of the convolution cannot hold because $\sigma_{\ell-\tau}$ depends on all partons inside the nucleons, so this calculation is correct up to some scale, which is usually taken as $Q^2=-q^2$ where q^2 is the momentum transfer of the process. In addition, PDFs are characterized by the Lorentz invariant quantity ξ, which represents the fraction of the momentum carried by the interacting parton. Considering both quantities, one should write:

$$
\sigma_{\ell-\tau}=\hat{\sigma}\left(\xi,Q^2\right)\otimes H\left(\xi,Q^2\right)\,. \tag{325}
$$

In this work we are dealing with heavy nuclei (Fe and Pb) and as pointed out in ref. [134], binding effects alter the long-distance behavior at different

ξ scales. To account for this, we use the fit of the nuclear parton distribution functions (nPDFs) provided by the NCTQ15 project [135] which is incorporated in the ManeParse Mathematica package [136]. Also, to consider the running of the quark masses with the scale Q^2 we incorporate the RunDec Mathematica package [137]. The perturbative cross sections are (no contribution from gluons arises at lowest order in the SLH and there are no quark FCNCs in our setting) [50]:

$$\begin{aligned} \frac{d^2\hat{\sigma}\left(\ell q_i(\xi P) \to \tau q_i\right)}{d\xi dQ^2} &= \frac{1}{16\pi\lambda\left(s(\xi), m_\ell^2, m_i^2\right)}\overline{|\mathcal{M}_{qq}(\xi, Q^2)|^2}, \\ \frac{d^2\hat{\sigma}\left(\ell \bar{q}_i \to \tau \bar{q}_i(\xi P)\right)}{d\xi dQ^2} &= \frac{1}{16\pi\lambda\left(s(\xi), m_\ell^2, m_i^2\right)}\overline{|\mathcal{M}_{\bar{q}\bar{q}}(\xi, Q^2)|^2}, \end{aligned} \tag{326}$$

where $p_i = \xi P$ is the fraction of the nucleus total momentum P carried by the parton, thus we consider $m_i^2 = \xi^2 M^2$. It is necessary to add the same processes but with anti-quarks because the cross section and the non-perturbative behavior is not the same in both cases. The total cross section can be expressed as the sum of the cross section over the nucleons of the nuclei [52]:

$$\sigma\left(\ell + (A, Z) \to \tau + X\right) = Z\sigma\left(\ell + p \to \tau + X\right) + (A - Z)\sigma\left(\ell + n \to \tau + X\right), \tag{327}$$

here the nucleon cross section is [50], [52]:

$$\begin{aligned} \sigma\left(\ell + N \to \tau + X\right) = \sum_i \int_{\xi_{min}}^{1} \int_{Q_-^2(\xi)}^{Q_+^2(\xi)} d\xi dQ^2 \Big[&\frac{d^2\hat{\sigma}\left(\ell q_i(\xi P) \to \tau q_i\right)}{d\xi dQ^2} H_{q_i}\left(\xi, Q^2\right) \\ &+ \frac{d^2\hat{\sigma}\left(\ell \bar{q}_i \to \tau \bar{q}_i(\xi P)\right)}{d\xi dQ^2} H_{\bar{q}_i}\left(\xi, Q^2\right) \Big], \end{aligned} \tag{328}$$

where $H_{q_i}\left(\xi, Q^2\right)$ and $H_{\bar{q}_i}\left(\xi, Q^2\right)$ are the quark and anti-quark PDFs, respectively, and nuclear targets under consideration are Fe with $A = 56$, $Z = 26$ and Pb with $A = 207$, $Z = 82$. The integration limits can be found in ref.[50]. Penguin form factors (quarks and anti-quarks) are the same than for the $\mu - e$ conversion and quark box form factors are computed from the Feynman diagrams in figure 17 (for anti-quarks we need to invert the lower fermion line). Quark box form factors are the same than in eqs. (319)-(323), and the anti-quarks box form factors can be obtained from those equations as well. When we invert the lower fermion line we exchange the diagrams for the different embeddings: quark diagrams in the Anomaly free (Universal) embedding are then related to antiquark diagrams in the Universal (Anomaly free) embedding, therefore we need to change $\{u_m, U_m\} \leftrightarrow \{d_m, D_m\}$ in those equations (and the overall mixing coefficient, so that it corresponds to $\ell \to \tau$ transitions), to get the anti-quark box form factors. Again, diagrams with light quarks and those which are proportional to ω give subleading corrections. The squared amplitude can be computed from eqs.(315), leading to the result (we use the Mandelstam variables $s=(p_\ell+p_i)^2$, $t=(p_\ell-p_\tau)^2=-Q^2$, $u=(p_i-p_\tau)^2$):

$$
\begin{aligned}
\overline{|\mathcal{M}_{qq}(\xi,Q^2)|^2} = & -\frac{4e^4Q_q^2}{(Q^2)^2}\Bigg[(Q^2)^2|A_{L1}|^2\Big[2m_q\xi M\left(m_\ell^2+m_\tau^2+Q^2\right)\\
&+\left(m_\tau^2+\xi^2M^2+Q^2-s\right)\left(m_\tau^2+\xi^2M^2-u\right)+\left(m_\ell^2+\xi^2M^2-s\right)\left(m_\ell^2+\xi^2M^2+Q^2-u\right)\Big]\\
&+\frac{|A_{R2}|^2m_\ell^2}{2}\Big[\left(m_\ell^2+m_\tau^2+Q^2\right)^2\left(\xi^2M^2+m_q^2+Q^2-2\xi Mm_q\right)\\
&+4m_\ell^2\left(\xi^2M^2+m_\tau^2-u\right)\left(\xi^2M^2+m_\ell^2+Q^2-u\right)\\
&+\left(m_\ell^2+m_\tau^2+Q^2\right)\Big[\left(m_\ell^2+m_\tau^2\right)\left(6\xi Mm_q-\xi^2M^2-m_q^2-Q^2\right)\\
&-\left(s-m_\ell^2+m_\tau^2-u\right)\left(s-m_\tau^2+m_\ell^2-u\right)\Big]\\
&+4m_\tau^2\Big[\left(s-m_\ell^2-\xi^2M^2\right)\left(s-m_\tau^2-\xi^2M^2-Q^2\right)-4\xi Mm_qm_\ell^2\Big]\Big]\\
&+\frac{m_\ell^2Q^2}{2}\left(A_{L1}A_{R2}^*+A_{R2}A_{L1}^*\right)\Big[\left(6\xi Mm_q-\xi^2M^2-m_q^2-Q^2\right)\left(m_\ell^2+Q^2-m_\tau^2\right)\\
&+\left(u-\xi^2M^2-m_\ell^2-Q^2\right)\left(s-m_\ell^2-3\xi^2M^2-2m_\tau^2+2u\right)\\
&+\left(s-\xi^2M^2-m_\tau^2-Q^2\right)\left(u-\xi^2M^2-m_\tau^2\right)\Big]\Bigg]\\
&-4e^4\Big[|\hat{B}^q_{L1}|^2\left(\xi^2M^2+m_\ell^2-s\right)\left(\xi^2M^2+m_\ell^2+Q^2-u\right)\\
&+|\hat{B}^q_{L2}|^2\left(\xi^2M^2+m_\tau^2-u\right)\left(\xi^2M^2+m_\tau^2+Q^2-s\right)\\
&+\xi Mm_q\left(m_\ell^2+m_\tau^2+Q^2\right)\left(\hat{B}^q_{L1}\hat{B}^{q*}_{L2}+\hat{B}^q_{L2}\hat{B}^{q*}_{L1}\right)\Big]\\
&+4e^4Q_q\left(A_{L1}\hat{B}^{q*}_{L2}+\hat{B}^q_{L2}A^*_{L1}\right)\Big[\left(\xi^2M^2+m_\tau^2+Q^2-s\right)\left(\xi^2M^2+m_\tau^2-u\right)\\
&+\xi Mm_q\left(m_\ell^2+m_q^2+Q^2\right)\Big]\\
&+4e^4Q_q\left(A_{L1}\hat{B}^{q*}_{L1}+\hat{B}^q_{L1}A^*_{L1}\right)\Big[\left(\xi^2M^2+m_\ell^2+Q^2-u\right)\left(\xi^2M^2+m_\ell^2-s\right)\\
&+\xi Mm_q\left(m_\ell^2+m_q^2+Q^2\right)\Big]\\
&-\frac{e^4Q_qm_\ell^2}{Q^2}\left(A_{R2}\hat{B}^{q*}_{L2}+\hat{B}^q_{L2}A^*_{R2}\right)\Big[\left(m_\ell^2+Q^2-m_\tau^2\right)\left(\xi^2M^2+m_q^2+Q^2-6\xi Mm_q\right)-4Q^2\left(s+m_\tau^2\right)\\
&+\left(u-m_\tau^2-\xi^2M^2\right)\left(\xi^2M^2+2m_\tau^2+Q^2+s+2u\right)+3\left(s+u-\xi^2M^2\right)\left(\xi^2M^2+m_\tau^2+Q^2-u\right)\\
&+u\left(\xi^2M^2+m_\tau^2+Q^2-u\right)+m_\tau^2\left(s+u-m_\tau^2-m_\ell^2-\xi^2M^2\right)+m_\ell^2\left(m_\ell^2+Q^2+u-m_\tau^2-s\right)\Big]\\
&+\frac{e^4Q_qm_\ell^2}{Q^2}\left(A_{R2}\hat{B}^{q*}_{L1}+\hat{B}^q_{L1}A^*_{R2}\right)\Big[-\left(m_\ell^2+Q^2+m_\tau^2\right)\left(\xi^2M^2+m_q^2+Q^2-6\xi Mm_q\right)\\
&+\left(\xi^2M^2+m_\tau^2-u\right)\left(\xi^2M^2+2m_\ell^2-m_\tau^2+Q^2+s-2u\right)+3\left(\xi^2M^2+m_\ell^2-s\right)\left(\xi^2M^2+m_\ell^2+Q^2-u\right)\\
&+2m_\tau^2\left(\xi^2M^2+m_q^2+Q^2-6\xi Mm_q\right)\Big],
\end{aligned}
\tag{329}
$$

where

$$\hat{B}^q_{L1} = B^q_{L1} + F^q_{LL} + F'^q_{LL}, \quad \hat{B}^q_{L2} = F^q_{LR} + F'^q_{LR}. \tag{330}$$

To find the squared amplitude for anti-quarks we only replace $s \leftrightarrow u$ in eq. (329). We note that the dominant contributions come from the $|\hat{B}^q_{L1,2}|^2$ terms.

4 NEUTRINOS IN THE SIMPLEST LITTLE HIGGS

What we observe is not nature itself,
but nature exposed to our method
of questioning.

— Werner Heisenberg

In the original formulation of the SM model the neutrinos are massless, because in nature there aren not ν_R singlet states which form a Dirac pair with the right-handed electrons, i.e., neutrinos are left-handed. However, the SM must be extended to account for the evidence of neutrino oscillations [138, 139, 140] which implies non-zero masses and mixings for the active neutrinos. The most straightforward extension is the so-called νSM [141], which adds right-handed gauge singlets for the three neutrinos, thus (just like for the other fermions) we can generate a Dirac mass for the neutrinos via Yukawa couplings. Nevertheless, if we take into account the tiny observed mass for neutrinos, we would require extremely suppressed Yukawa couplings: $\lambda_\nu \sim 10^{-12}$, this value suggests that other mechanism could be responsible for neutrinos masses, as it is unnaturally small.

Another way to introduce neutrino masses is given by the see-saw mechanism [142, 143, 144] which predicts the existence of two kind of neutrinos: the left-handed (which have been observed) and the heavy right-handed neutrinos (which are yet undiscovered) and the later could be Majorana particles (they are their own antiparticles if lepton number is not conserved). The spectrum for these models are three Majorana fields with mass $\approx M$ and the three light neutrinos with mass $m_\nu \approx \lambda_\nu^2/M$, which means the higher the mass of the right-handed neutrino, the lower the mass of the left-handed neutrinos. In order to obtain the observed light neutrino mass around eV, M is required to be of order $10^{14} - 10^{15}$GeV. Nevertheless, Ref. [108] pointed out that this realization of neutrino masses is inconsistent with Little Higgs models, because the M scale gives a large contribution to the Higgs mass, proportional to M^2. Recently, it was shown in [21] that the inverse see-saw of type I is able to reproduce current data in the Littlest Higgs model with T-parity [1]. In this work we are going to follow the proposal given in Ref. [108], which involves a quasi-Dirac field at the TeV scale and a small Majorana mass terms that breaks lepton number. In this way, novel sources of LFV will be generated, still LNV effects will remain suppressed.

[1] We have verified, however, that it is not possible to apply it to the Simplest Little Higgs model, because of the different multiplet structure.

4.1 INTRODUCING MAJORANA NEUTRINOS

As we can see from equations (223) and (224) neutrinos in the SLH model arise from the linear term in the Yukawa Lagrangian. Those neutrinos could be accommodated in the following way to first order in ν/f:

$$-\mathcal{L}_y = \begin{pmatrix}\bar{\nu}_L & \bar{N}_L & \bar{N}_R\end{pmatrix}\begin{pmatrix}0 & 0 & \frac{c_\beta \nu \lambda}{\sqrt{2}} \\ 0 & 0 & f s_\beta \lambda \\ \frac{c_\beta \nu\lambda}{\sqrt{2}} & f s_\beta \lambda & 0\end{pmatrix}\begin{pmatrix}\nu_L \\ N_L \\ N_R\end{pmatrix}, \tag{331}$$

each entry standing for a 3×3 matrix accounting for the 3 lepton families. If we diagonalize this matrix, we find the following eigenvalues:

$$x_1 = 0, \;\; x_{2,3} = \mp f s_\beta \lambda \sqrt{1+\delta_\nu^2}, \tag{332}$$

at this stage we have 3 strictly massless neutrinos and three exact heavy Dirac fields with mass $M_D \approx \lambda f s_\beta$ (the sign in the second and third eigenvalue correspond to particle and antiparticle). The relation between flavor and mass eigenstates is given by:

$$\begin{pmatrix}\nu_L \\ N_L \\ N_R\end{pmatrix} = \begin{pmatrix}-(1-\delta_\nu^2/2) & \frac{\delta_\nu}{\sqrt{2}} & -\frac{\delta_\nu}{\sqrt{2}} \\ -\delta_\nu & -\frac{(1-\delta_\nu^2/2)}{\sqrt{2}} & \frac{(1-\delta_\nu^2/2)}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\nu_L \\ N_L \\ N_R\end{pmatrix}, \tag{333}$$

equation (333) is equivalent to eq. (227). To introduce a Majorana mass for the SM neutrinos we allow the breaking of lepton number symmetry, this is done by introducing a Majorana mass term for the N_R singlet, which is allowed because it transforms trivially under the gauge group. The Yukawa Lagrangian now reads

$$\mathcal{L}_y \supset i\lambda_N^m \bar{N}_{Rm}\Phi_2^\dagger L_m + \frac{\lambda_\ell^{mn}}{\Lambda}\bar{\ell}_{Rm}\epsilon_{ijk}\Phi_1^i\Phi_2^j L_n^k - \frac{1}{2}\mu \bar{N}_R N_R^c + \text{h.c.}, \tag{334}$$

the mass matrix now has an additional element:

$$-\mathcal{L}_y = \begin{pmatrix}\bar{\nu}_L^c & \bar{N}_L^c & \bar{N}_R\end{pmatrix}\begin{pmatrix}0 & 0 & \frac{c_\beta \nu \lambda}{\sqrt{2}} \\ 0 & 0 & M_D \\ \frac{c_\beta \nu\lambda}{\sqrt{2}} & M_D & \mu\end{pmatrix}\begin{pmatrix}\nu_L \\ N_L \\ N_R^c\end{pmatrix}, \tag{335}$$

the eigenvalues for eq. (335) now are:

$$m_\nu = 0, \;\; M_{L,R} = \frac{1}{2}\left(\mu \mp 2M_D\sqrt{1+\delta_\nu^2+\frac{\mu^2}{4M_D^2}}\right). \tag{336}$$

Here still we have SM massless neutrinos, but now we have a quasi-Dirac field because the mass of the left-handed N_L and the right-handed singlet N_R are split due to the introduction of the Majorana mass for the singlet. The masses are split at order $\mathcal{O}(\mu/f)$. The SM neutrinos get mass at the loop level (we do not dwell into this here as it is immaterial for the discussion of LFV in the charged lepton sector). The diagram in figure 18 generates the effective dimension five operator which gives rise to a radiatively induced Majorana mass for the light neutrinos.

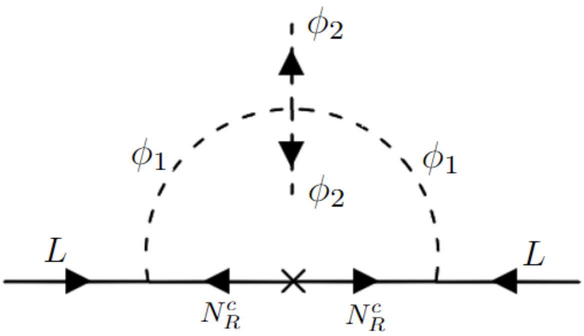


Figure 18: Diagram for Majorana mass.

The effective dimension five operator is [108]:

$$\begin{aligned}\mathcal{L}_5 &= \frac{\lambda^2}{\Lambda_\nu}\left(\bar{L}^c\Phi_1\right)\left(\Phi_1^\dagger L\right) + \text{h.c.}\\ &= \frac{\lambda^2}{\Lambda_\nu}\left(\frac{s_\beta^2 v^2}{2}\bar{\nu}_L^c\nu_L - \frac{vfs_\beta c_\beta}{\sqrt{2}}\bar{\nu}_L^c N_L - \frac{vfs_\beta c_\beta}{\sqrt{2}}\bar{N}_L^c\nu_L + f^2c_\beta^2\bar{N}_L^c N_L\right) + \text{h.c.}\end{aligned} \tag{337}$$

With the equations (337) and (335) we can construct the full mass matrix:

$$-\mathcal{L}_y = \left(\bar{\nu}_L^c \quad \bar{N}_L^c \quad \bar{N}_R\right)\mathcal{M}_{DM}\begin{pmatrix}\nu_L\\ N_L\\ N_R^c\end{pmatrix}, \tag{338}$$

with

$$\mathcal{M}_{DM} = \begin{pmatrix}\frac{\lambda^2 s_\beta^2 v^2}{2\Lambda_\nu} & -\frac{\lambda^2}{\Lambda_\nu}\frac{vfs_\beta c_\beta}{\sqrt{2}} & \frac{c_\beta v\lambda}{\sqrt{2}}\\ -\frac{\lambda^2}{\Lambda_\nu}\frac{vfs_\beta c_\beta}{\sqrt{2}} & \frac{\lambda^2 c_\beta^2 f^2}{\Lambda_\nu} & \lambda f s_\beta\\ \frac{c_\beta v\lambda}{\sqrt{2}} & \lambda f s_\beta & \mu\end{pmatrix}. \tag{339}$$

To obtain the eigenvalues of this matrix, it is necessary to make some approximations, which are justified by the following hierarchy between the matrix elements:

$$\lambda f > \lambda v \gg \mu \gg \left[\frac{\lambda^2 s_\beta^2 v^2}{2\Lambda_\nu}, -\frac{\lambda^2}{\Lambda_\nu}\frac{vfs_\beta c_\beta}{\sqrt{2}}, \frac{\lambda^2 c_\beta^2 f^2}{\Lambda_\nu}\right], \tag{340}$$

now the eigenvalues have the approximate values of:

$$m_\nu \approx \frac{\lambda^2 s_\beta^2 v^2}{2\Lambda_\nu}, \quad M_{L,R} \approx \frac{1}{2}\left(\mu \mp 2M_D\sqrt{1+\delta_\nu^2+\frac{\mu^2}{4M_D^2}}\right), \tag{341}$$

we have finally achieved massive SM neutrinos and we found that the two previously non-zero eigenvalues barely change, this is because the 2×2 upper block in the matrix is small enough to appreciably alter the result given in eq. (336). Now, we solve the loop diagram in figure 18 to estimate the Λ_ν cut-off, which value is [108]:

$$\frac{1}{\Lambda_\nu} = \frac{\mu\kappa}{4\pi^2 f^2 s_\beta^2}\left[\frac{x-1-\log[x]}{(1-x)^2}\right], \quad \text{with} \quad x = \frac{M_{\text{Higgs}}^2}{M_D^2}, \tag{342}$$

where κ is the Higgs quartic coupling. With the eq. (342), the heavy and light neutrino masses per family are:

$$\begin{aligned} m_\nu &\approx \frac{v^2}{f^2}\frac{\lambda^2\kappa\mu}{8\pi^2}\left[\frac{x-1-\log[x]}{(1-x)^2}\right], \\ M_{L,R} &\approx \frac{1}{2}\left(\mu \mp 2M_D\sqrt{1+\delta_\nu^2+\frac{\mu^2}{4M_D^2}}\right), \end{aligned} \tag{343}$$

from eq. (343) we see that if we take $\mu \to 0$ light neutrino mass is exactly zero, lepton number symmetry is restored, and the heavy neutrinos are an exact Dirac field (as it must be). As we explained before, the 2×2 upper block in the mass matrix barely changes the eigenvalues and the same is true for the eigenvectors therefore the flavor and mass eigenstates are related (approximately) by the following transformation:

$$\begin{pmatrix} \nu_L \\ N_L \\ N_R^c \end{pmatrix} = \begin{pmatrix} -(1-\delta_\nu^2/2) & \frac{\delta_\nu}{\sqrt{2}} & -\frac{\delta_\nu}{\sqrt{2}} \\ -\delta_\nu & -\frac{(1-\delta_\nu^2/2)}{\sqrt{2}} & \frac{(1-\delta_\nu^2/2)}{\sqrt{2}} \\ 0 & \frac{(1-\delta_\nu^2/2)}{\sqrt{2}} & \frac{(1-\delta_\nu^2/2)}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} V\nu_L \\ N_L \\ N_R^c \end{pmatrix}, \tag{344}$$

the transformation given in (344) is almost the same that the equation (333), the only difference is in the state associated with the component N_R^c. As in the chapter 2, we again have the same CKM-like matrix which encodes the flavor-mixing coefficients. It is also straightforward to see that the previous results are unchanged, because we have the same transformation in the fields ν_L and N_L, but now we have one more contribution from the component N_R^c. In the following we present the relevant Feynman rules involving the new state N_R^c.

$V_\mu \bar{f}_i f_m$ Vertex	$g_L^{V\bar{f}_i f_m}$	$g_R^{V\bar{f}_i f_m}$
$W^+ \bar{\mathcal{N}}_m \ell_i$	$-\delta_\nu \frac{1}{\sqrt{2} s_W} V_\ell^{mi}$	0
$Z \bar{\mathcal{N}}_i \mathcal{N}_i$	$\frac{\delta_Z}{2 s_W \sqrt{3 - t_W^2}} + \frac{\delta_\nu^2}{4 c_W s_W}$	0
$Z \bar{\mathcal{N}}_m \nu_i$	$\delta_\nu \frac{1}{\sqrt{2} 2 c_W s_W} V_\ell^{mi}$	0
$Z \bar{\mathcal{N}}_i N_i$	$-\frac{\delta_\nu^2}{4 c_w s_w} - \frac{\delta_Z}{2 s_w \sqrt{3 - t_w^2}}$	0
$X^+ \bar{\mathcal{N}}_m \ell_i$	$-\frac{i}{2 s_W} \left(1 - \frac{\delta_\nu^2}{2}\right) V_\ell^{mi}$	0
$Z' \bar{\mathcal{N}}_i \mathcal{N}_i$	$\frac{1}{4 s_W \sqrt{3 - t_W^2}} \left[2 - \delta_\nu^2 \left(2 - t_W^2\right)\right]$	0
$Z' \bar{\mathcal{N}}_i N_i$	$\frac{1}{4 s_W \sqrt{3 - t_W^2}} \left[-2 + \delta_\nu^2 \left(2 - t_W^2\right)\right]$	0
$Z' \bar{\mathcal{N}}_m \nu_i$	$-\frac{\delta_\nu \sqrt{3 - t_W^2}}{2\sqrt{2} s_W} V_\ell^{mi}$	0

Table 18: Vertices $[V^\mu f f] = i e \gamma^\mu (g_L P_L + g_R P_R)$ for the new state N_R^c denoted as $\mathcal{N}$ (not to be confused with N, which is already in the original version of the SLH).

SFF	g_L	g_R
$x^+ \bar{N}_m \ell_i$	0	$\frac{1}{2 s_W} \frac{m_{\ell_i}}{M_X} \left(1 - \delta_\nu^2/2\right) V_\ell^{mi}$
$x^- \bar{\ell}_i \mathcal{N}_m$	$\frac{1}{2 s_W} \frac{m_{\ell_i}}{M_X} \left(1 - \delta_\nu^2/2\right) V_\ell^{im*}$	0
$\phi^+ \bar{\mathcal{N}}_m \ell_i$	0	$-\delta_\nu \frac{i}{2 s_W} \frac{m_{\ell_i}}{M_W} V_\ell^{mi}$
$\phi^- \bar{\ell}_i \mathcal{N}_m$	$\delta_\nu \frac{i}{2 s_W} \frac{m_{\ell_i}}{M_W} V_\ell^{im*}$	0

Table 19: Vertices $[SFF] = i e (g_L P_L + g_R P_R)$ for the new state N_R^c denoted as $\mathcal{N}$.

With the Feynman rules given in the tables 18 and 19 we can calculate the new contributions to the LFV processes studied before. We expect an increase in the results presented (which are displayed in the next chapter), however that analysis will be presented elsewhere.

5 NUMERICAL RESULTS

It doesn't matter how beautiful your theory is,
it doesn't matter how smart you are.
If it doesn't agree with experiment,
it's wrong.

— R. P. Feynman

In this section we show and discuss the numerical results of our 16 LFV processes exposed above. The first step is setting the range for the free parameters of SLH model: f, t_β, M_{N_i}, δ_ν, $V^{\ell i}$ and δ_q, as we motivate in the following. Dependence of the $\mu \to e$ observables on these parameters is studied in detail in ref. [17] for the case of two heavy neutrinos. Approximate cancellations between the $\gamma + Z$ penguins and box contributions were already noted in this reference (for $\mu \to e$ transitions). These effects strongly depend on the specific region in the parameter space of the SLH model and, because of that, we will not dwell into them here.

The scale of compositeness, f, can be estimated through the direct search of Z' bosons at LHC [145], where the lower limit is set as [146] $f \gtrsim 7.5$ TeV at 95% C.L. Following the analysis given in ref. [147] we fix the upper limit $f \lesssim 85$ TeV. Above these energies, SLH loses consistency.

The ratio of the two vevs $t_\beta = f_1/f_2$ is another important free parameter of this model. A perturbative unitarity analysis [147] binds $1 \leqslant t_\beta \leqslant 9$. For small f ($10 \leqslant f(\text{TeV}) \leqslant 20$), t_β can vary freely in this interval, while for $20 \leqslant f(\text{TeV}) \leqslant 80$, the approximate relation $t_\beta = \frac{2}{15} f(\text{TeV}) - \frac{25}{15}$ holds.

Heavy neutrinos are responsible for the LFV lepton decays, however this "little" neutrino masses are unknown. We will, nevertheless, follow ref. [28] and take the ratios involving them as: $0.1 \leqslant x_1 \leqslant 0.25$, $1.1x_1 \leqslant x_2 \leqslant 10x_1$, $1.1x_2 \leqslant x_3 \leqslant 10x_2$ (we remind that x_i depends quadratically on the N_i mass), where we include the cases of a small splitting $x_2 = 1.1x_1$ and a large one $x_2 = 10x_1$ (analogously for x_3).

The mixing of the "little" and light neutrino of the same family is encapsulated in δ_ν and, according to data, $\delta_\nu \lesssim 0.05$ [26, 148, 149], that we will take.

We do not have any information of the mixing matrix $V^{\ell i}$ between charged leptons and "little" neutrinos, which can be parameterized in the standard form [36]. According to ref. [28], we have scanned over $-1 \leqslant s_{ij} \leqslant 1$ ensuring the low-energy restrictions and, in addition, we assumed for simplicity CP conservation (phase in $V^{\ell i}$ set to zero).

Finally, the mixing between the heavy quarks and the corresponding SM quarks is parameterized by the δ_q parameters. We follow the arguments of ref.[15] and assume that the mixing effects are suppressed in the down-quark (up-quark) sector for the Anomaly free (Universal) embedding in the $t_\beta > 1$ regime, so it implies: $\delta_q = \mp\delta_\nu$, where the upper sign is for the Anomaly free $(q = D, S)$ and the lower sign is for the Universal embedding $(q = U, C)$. We also follow the proposal of ref.[26] and take the reference values for the ratios of heavy quark masses as $x_U = x_D = x_C = x_S \equiv 1$.

As we have already mentioned, there are still no experimental limits for $\ell - \tau$ conversion, but if we consider the expected sensitivity of NA64 experiment we can express the conversion probability as the ratio [52]:

$$\mathcal{R} = \frac{\sigma\left(\ell + N \to \tau + X\right)}{\sigma\left(\ell + N \to \ell + X\right)} \sim 10^{-13} - 10^{-12}, \tag{345}$$

where the denominator is the dominant contribution to the inclusive $\ell + N$ processes due to the bremsstrahlung of leptons off nuclei [52]:

$$\begin{aligned} &\sigma\left(e + \mathrm{Fe} \to e + X\right) = 0.129 \times 10^5 \ \ \mathrm{GeV}^{-2}, \\ &\sigma\left(\mu + \mathrm{Fe} \to \mu + X\right) = 0.692 \ \ \mathrm{GeV}^{-2}, \\ &\sigma\left(e + \mathrm{Pb} \to e + X\right) = 1.165 \times 10^5 \ \ \mathrm{GeV}^{-2}, \\ &\sigma\left(\mu + \mathrm{Pb} \to \mu + X\right) = 6.607 \ \ \mathrm{GeV}^{-2}. \end{aligned} \tag{346}$$

As representative energy for the initial electron or muon beam we take $E_e = 100$ GeV and $E_\mu = 150$ GeV. We do the analysis within a Monte Carlo simulation with all channels sharing the free parameters enumerated above. In the following table we summarize our mean values, the present experimental bounds [36] and the future expected sensitivities (whose values were taken from ref. [150] and references therein).

LFV decays	Experimental Limits	Our mean values	Future sensitivity
$\mu \to e\gamma$	4.2×10^{-13}	2.1×10^{-14}	6×10^{-14}
$\mu \to ee\bar{e}$	1.0×10^{-12}	5.7×10^{-15}	10^{-16}
$\tau \to e\gamma$	3.3×10^{-8}	5.6×10^{-12}	3×10^{-9}
$\tau \to \mu\gamma$	4.4×10^{-8}	2.3×10^{-12}	10^{-9}
$\tau \to ee\bar{e}$	2.7×10^{-8}	3.2×10^{-12}	$(2-5) \times 10^{-10}$
$\tau \to \mu\mu\bar{\mu}$	2.1×10^{-8}	1.6×10^{-12}	$(2-5) \times 10^{-10}$
$\tau \to e\mu\bar{\mu}$	2.7×10^{-8}	2.1×10^{-12}	$(2-5) \times 10^{-10}$
$\tau \to \mu e\bar{e}$	1.8×10^{-8}	1.0×10^{-12}	$(2-5) \times 10^{-10}$
$\tau \to \mu\mu\bar{e}$	1.7×10^{-8}	3.8×10^{-18}	$(2-5) \times 10^{-10}$
$\tau \to ee\bar{\mu}$	1.5×10^{-8}	5.6×10^{-18}	$(2-5) \times 10^{-10}$
$\mu Ti \to eTi$	4.3×10^{-12}	6.8×10^{-14} (AF), 8.6×10^{-14} (U)	10^{-18}
$\mu Au \to eAu$	7.0×10^{-13}	8.2×10^{-14} (AF), 1.1×10^{-13} (U)	-
$eFe \to \tau Fe$	-	9.2×10^{-20} (AF), 9.3×10^{-20} (U)	-
$ePb \to \tau Pb$	-	1.6×10^{-19} (AF), 1.6×10^{-19} (U)	-
$\mu Fe \to \tau Fe$	-	6.2×10^{-16} (AF), 6.2×10^{-16} (U)	-
$\mu Pb \to \tau Pb$	-	9.6×10^{-16} (AF), 9.8×10^{-16} (U)	-

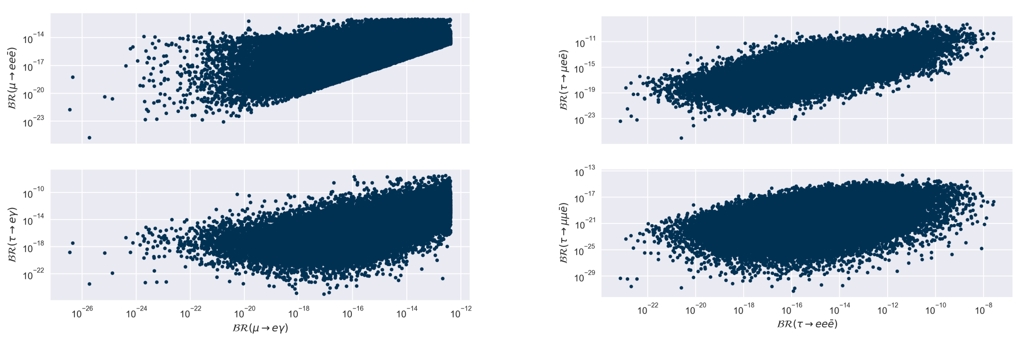
Table 20: Mean values of branching ratios and conversion rates (where AF stands for Anomaly Free embedding and U stands for Universal embedding) against current upper limits at 90 % confidence level and future sensitivities.

In the case of muon decays, our results are below the experimental limit by one ($\mu \to e\gamma$) and three ($\mu \to 3e$) orders of magnitude, the mean values of nuclei conversion in both embeddings are below the upper bounds by one or two orders of magnitude. For the case of Au nuclei conversion in the Universal embedding our mean value is only a factor ~ 7 below the experimental limit.

We turn now to LFV transitions involving the tau flavour. For the cases of $\tau \to \ell\gamma$ ($\ell = e, \mu$), same flavor and same sign decays, our results are below the experimental limits by four orders of magnitude (not for wrong sign decays, which are six orders of magnitude further suppressed). For the case of $\ell - \tau$ conversion, we find that the mean values with electrons are too small for the expected sensitivity of the NA64 experiment. However, the analogous processes with muons are only slightly below their forecasted sensitivity, and in principle could be tested with future experiments.

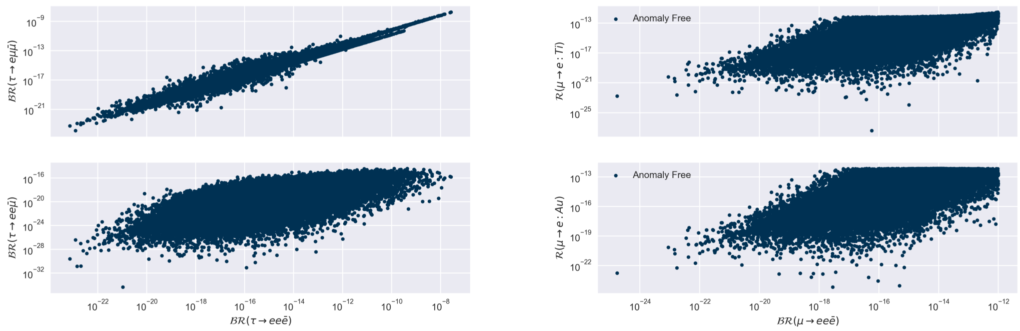
From these results we verify that muon physics is the best candidate to test LFV; our mean values are of the same order ($\mu \to e\gamma$) or higher ($\mu \to 3e$, $\mu\text{Ti} \to e\text{Ti}$) than the future sensitivity and will set the strongest limits. For tau decays, our mean values are still below the future sensitivity and only next generations of B factories could be able to search for them, according to the SLH model. Still, $\mu \to \tau$ conversion in nuclei appears promising as a discovery tool to first measure LFV involving the tau flavor. Consequently, it can play a significant role in characterizing the underlying new physics causing charged LFV, as can be checked from the correlations amid processes, that we discuss next.

We show in figs. 19-23 a selected set of scatter plots comparing the different processes. Fig.19a plots $\mathcal{BR}(\mu \to e\gamma)$ vs. $\mathcal{BR}(\mu \to ee\bar{e})$, which are moderately correlated. A similar, though softened, trend is observed in the accompanying figure $\text{BR}(\mu \to e\gamma)$ vs. $\text{BR}(\tau \to e\gamma)$ (despite they differ in the flavor coefficients). Figure 19b and 20a should be understood together. In these plots, we show the correlation between same flavor decay against the same sign and wrong sign decays, respectively. We see that $\mathcal{BR}(\tau \to ee\bar{e})$ keeps a big correlation with $\mathcal{BR}(\tau \to e\mu\bar{\mu})$ and the opposite happens with $\mathcal{BR}(\tau \to \mu e\bar{e})$. This is also caused by the corresponding flavor coefficients, as expected. The comparison between same flavor and wrong sign decays is quite different, because wrong sign are decays with only box contributions, so no correlation is expected in those plots (however the lower panel of fig. 20a exhibits a small one, albeit this will be very challenging to probe, given the big suppression of the wrong-sign decays in our setting). We do not show other analogous plots including, for instance, the $\tau \to 3\mu$ decay (see appendix B).



(a) $\mathcal{BR}(\mu \to e\gamma)$ vs. $\mathcal{BR}(\mu \to ee\bar{e})$, $\mathcal{BR}(\tau \to e\gamma)$ **(b)** $\mathcal{BR}(\tau \to ee\bar{e})$ vs. $\mathcal{BR}(\tau \to \mu\mu\bar{e})$, $\mathcal{BR}(\tau \to \mu e\bar{e})$

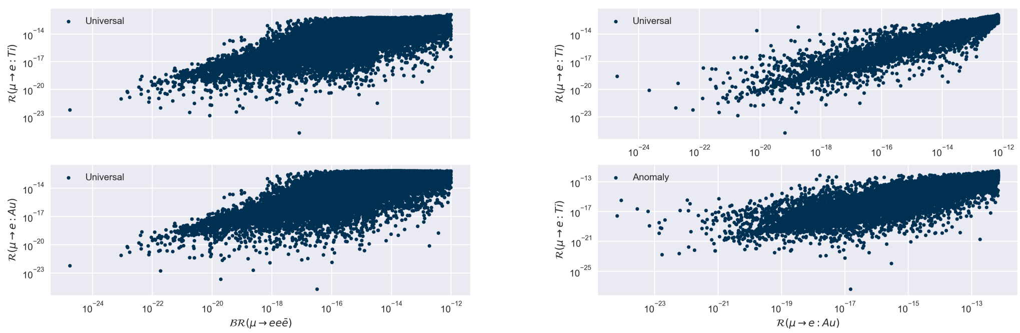
Figure 19: Scatter plots for $\ell \to \ell'\gamma$ and some $\ell \to 3\ell'$ decays.



(a) $\mathcal{BR}(\tau \to ee\bar{e})$ vs. $\mathcal{BR}(\tau \to ee\bar{\mu})$, $\mathcal{BR}(\tau \to e\mu\bar{\mu})$ **(b)** $\mathcal{BR}(\mu \to ee\bar{e})$ vs. $\mathcal{R}(\mu \to e : \mathrm{Ti})$, $\mathcal{R}(\mu \to e : \mathrm{Au})$

Figure 20: Scatter plots for $\ell \to 3\ell'$ decays and $\ell \to \ell'$ nuclei conversion.

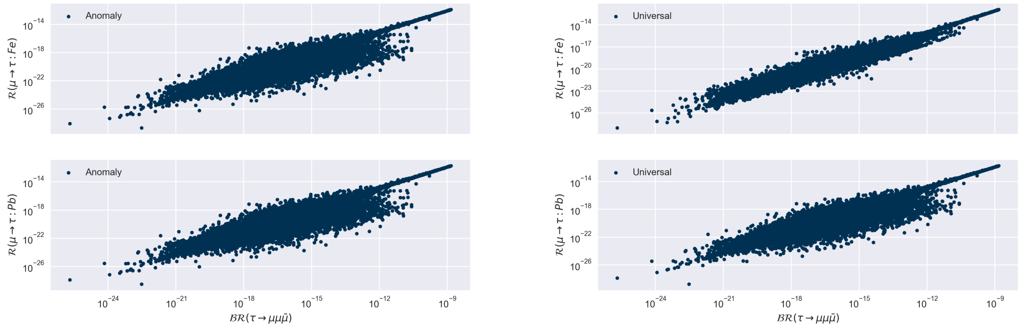
In plots 20b and 21a we show the scatter plot for the $\mathcal{BR}(\mu \to 3e)$ against $\mathcal{R}(\mu \to e)$ nuclei conversion. The result for Ti and Au nuclei are almost the same, and in general the outcome for both embeddings is very similar. However, results for nuclei conversion in Ti show a bigger correlation than the results for Au, being the dependence stronger in the Anomaly free embedding. Plot 21b shows the comparison of nuclei conversion in both embeddings (which is the reason why we draw the x-axis in both plots). Our results are alike in both, with stronger correlation in the Universal embedding.



(a) $\mathcal{BR}(\mu \to ee\bar{e})$ vs. $\mathcal{R}(\mu \to e : \mathrm{Ti})$, $\mathcal{R}(\mu \to e : \mathrm{Au})$ **(b)** $\mathcal{R}(\mu \to e : \mathrm{Ti})$ vs. $\mathcal{R}(\mu \to e : \mathrm{Au})$

Figure 21: Scatter plots for $\mu \to 3e$ and $\mu \to e$ nuclei conversion.

The scatter plots in fig. 22 show the comparison of $\mathcal{BR}(\tau \to 3\mu)$ with $\mu \to \tau$ nuclei conversion, the general behavior in both embeddings is the same, but small differences lie in the non-perturbative behavior of quarks and anti-quarks inside the heavy nuclei under consideration. However, as we can see in table 20 these differences are negligible in the expected probabilities. Again, the strongest correlations are found in the Universal embedding. Finally, the scatter plot in figure 23 compares the $\mu \to \tau$ conversion in both embeddings. We see a perfect correlation in the Anomaly free embedding that is a bit degraded for the universal case. We do not show analogous correlations (even if with three orders of magnitude smaller probabilities) for $e \to \tau$ nuclei conversion.



(a) $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$ vs. $\mathcal{R}(\mu \to \tau : \mathrm{Fe})$, $\mathcal{R}(\mu \to \tau : \mathrm{Pb})$ **(b)** $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$ vs. $\mathcal{R}(\mu \to \tau : \mathrm{Fe})$, $\mathcal{R}(\mu \to \tau : \mathrm{Pb})$

Figure 22: Scatter plots for $\tau \to 3\mu$ and $\mu \to \tau$ nuclei conversion.

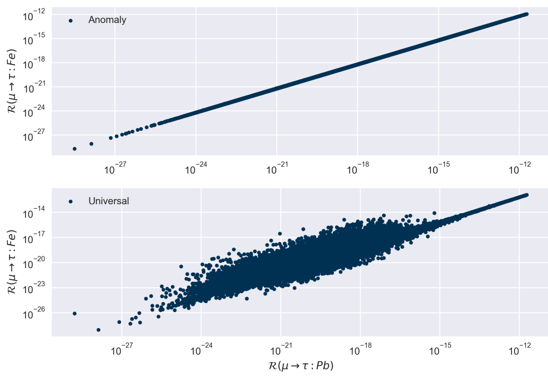


Figure 23: $\mathcal{R}(\mu \to \tau : \mathrm{Pb})$ vs. $\mathcal{R}(\mu \to \tau : \mathrm{Fe})$

5.1 THE CDF M_W MEASUREMENT WITHIN THE SLH MODEL

As a timely topic, we will finally discuss the implications, within the SLH, of the new measurement of the W boson mass given by the CDF collaboration with a result $M_W = (80.4335 \pm 0.0094)$ GeV [151], that shows a discrepancy of 7σ with the SM prediction $M_W = (80.357 \pm 0.006)$ GeV and is also in tension with respect to the world average $M_W = (80.379 \pm 0.012)$ GeV [36]. Including the CDF measurement, the new world average would be $M_W = (80.4242 \pm 0.0087)$ GeV. As already mentioned before, SLH does not have $SU(2)$ custodial symmetry, and the tree-level SM relation $\rho = 1$ is no longer valid:

$$\rho = 1 + \frac{v^2}{8f^2}\left(1 - t_W^2\right)^2 . \tag{347}$$

In the SM the ρ parameter is $\rho \equiv 1$ at tree level, and the EWPD, upon the inclusion of radiative corrections, yields $\rho \equiv 1.00038 \pm 0.00020$ [36], such deviation from the SM value can be encoded as:

$$\delta\rho = \frac{v^2}{8f^2}\left(1 - t_W^2\right)^2 \equiv \alpha T, \tag{348}$$

where T together with S and U are the oblique parameters [152, 94] which can parametrize potential new physics affecting electroweak two-point Green functions. We are going to use the formalism given in references [153, 154] to show how heavy Z' bosons can modify the oblique parameters and, since SLH is not an Universal theory[1], these corrections cannot be fully represented with four universal parameters: $\hat{S}$, $\hat{T}$, Y, W. Nevertheless, the corresponding effects can be well approximated with this formalism. SM corresponds to $\hat{T} = \hat{S} = W = Y = 0$ and these new parameters are related to the usual oblique ones as $S = 4s_W^2\hat{S}/\alpha$ and $T = \hat{T}/\alpha$, the U parameter corresponds to dimension-eight operators, and because $f \gg v$, we can neglect it.

A generic model with a Z' boson is determined by few quantities: gauge coupling $g_{Z'}$, Mass $M_{Z'}$, the couplings to the Higgs boson Z'_H, and to the left and right fermion multiplets Z'_L, Z'_e, Z'_Q, Z'_u, Z'_d (but we can omit quark data because they are less precise that the leptons ones). From ref. [153] we get that a generic model with Z' boson contributes to the universal parameters as [153]:

$$\begin{aligned}
\hat{S} &\approx \frac{M_W^2}{M_{Z'}^2}\left(b_W - b_y/t_W\right)\left(b_W - b_y t_W - \frac{2g_{Z'}Z'_H}{g}\right) \\
\hat{T} &\approx \frac{M_W^2}{M_{Z'}^2}\left[\left(c_y t_W + \frac{2g_{Z'}Z'_H}{g}\right)^2 - b_W^2\right] \\
W &\approx \frac{M_W^2}{M_{Z'}^2} b_W^2, \\
Y &\approx \frac{M_W^2}{M_{Z'}^2} b_y^2, \\
b_W &= \frac{2g_{Z'}}{Y_e g}\left(Z'_e Y_L - Z'_L Y_e\right) \\
b_y &= \frac{g_{Z'}Z'_e}{g' Y_e}.
\end{aligned} \tag{349}$$

In chapter 2 we can find all the necessary coefficients:

$$\begin{aligned}
g_{Z'} &= \frac{g}{\sqrt{1 - \frac{t_W^2}{3}}}, \quad Z'_L = Z'_H = \frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2}s_{Z'}^2, \quad \frac{g'}{g} = t_W \\
s_{Z'}^2 &= \frac{t_W^2}{3}, \qquad\qquad Z'_e = \sqrt{3}s_{Z'}^2.
\end{aligned} \tag{350}$$

With eqs. (350), eqs. (349) reduce to:

[1] An universal theory is such that Z′ couples to the fermions universally (which does not occur in SLH) or proportionally to the SM vectors.

$$\hat{T} \approx 0, \quad \hat{S} \approx \frac{4M_W^2}{M_{Z'}^2(3-t_W^2)} = 4W = \frac{4Y}{t_W^2}. \qquad (351)$$

From the above equations, we see that corrections to the T parameter due to the Z' boson are negligible, however we could estimate them using the fact that $\hat{T} = \alpha T = \delta\rho$. We present our results taking into account the PDG average and the update including the new measurement of the *W* boson mass.

	SM	EWPD	$M_W = 80.357$ GeV	$M_W = 80.4242$ GeV
ρ	1	1.00038 ± 0.00020	1.0004758	1.0016013
$\hat{T}$	0	-	5×10^{-4}	1.6×10^{-3}
$\hat{S}$	0	-	7×10^{-5}	7×10^{-5}
T	0	0.03 ± 0.12 0.05 ± 0.06	0.07	0.22
S	0	-0.01 ± 0.10 0.00 ± 0.07	0.008	0.008

Table 21: Values of oblique parameters according to EWPD and using instead M_W as in the PDG [36], or from the CDF measurement [151]. Two values are given for T and S. The upper one is obtained fitting also U (for which 0.02 ± 0.11 is obtained) and the second one setting $U = 0$ [36].

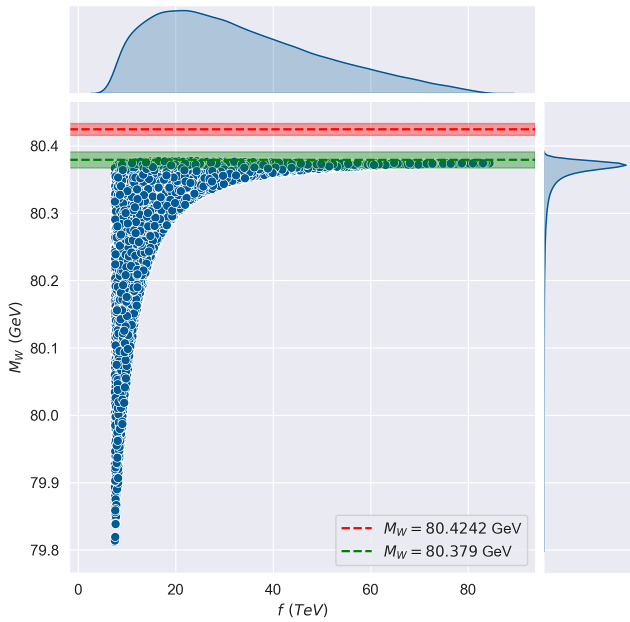


Figure 24: Corrections to the *W* boson mass provided by the SLH compared to its measurement, using $M_W = 80.379$ GeV.

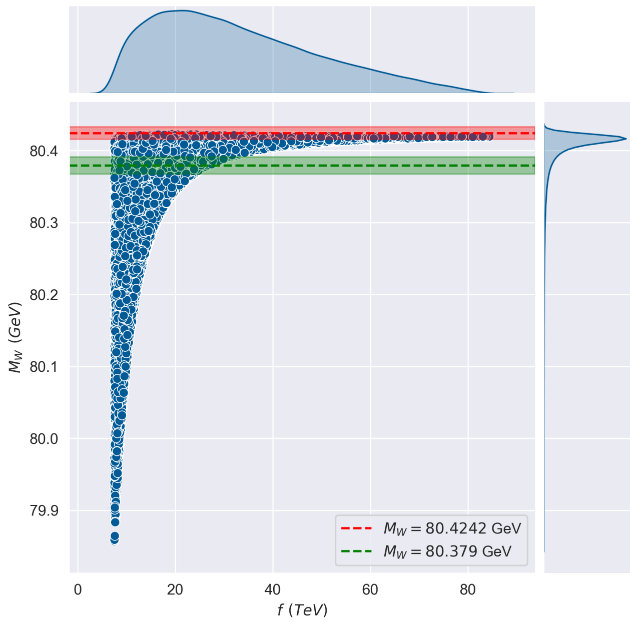


Figure 25: Corrections to the W boson mass provided by the SLH compared to its measurement, using $M_W = 80.4242$ GeV.

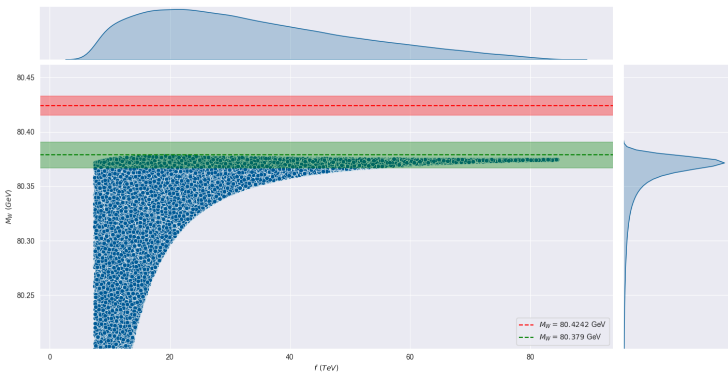


Figure 26: Zoom in on scatter plot using $M_W = 80.379$ GeV.

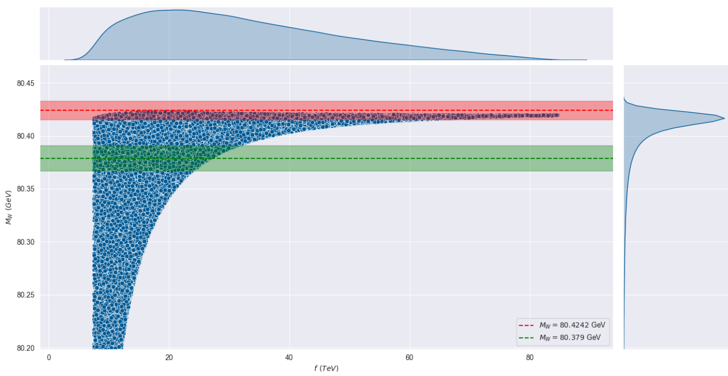


Figure 27: Zoom in on scatter plot using $M_W = 80.4242$ GeV.

Plots 24-27 show the corrections that SLH provided to the W boson mass for different values of f [2]. When we use as input the PDG average, the corrections to M_W agree within the uncertainties. In the supplementary axes of plot 24, we show the distribution for the values of M_W and f. For M_W most of the values are around the PDG average, and for f most of them are within $[8.5, 40]$ TeV. Then, in plot 26 we zoom in to show that SLH reproduces the world average for the W mass in the range $f \in [16, 22]$ TeV. Also for larger f, values of M_W are inside the uncertainties. For plots 25 and 27 we use as input the new world average including the CDF II result. For the range $f \in [8, 27]$ TeV, the PDG world average and the new one can be reproduced in the SLH model but the marginal distribution shows that getting the PDG average is unlikely. In the range $f \in [23, 84]$ TeV, the W mass is always below the central value, but still within its uncertainties.

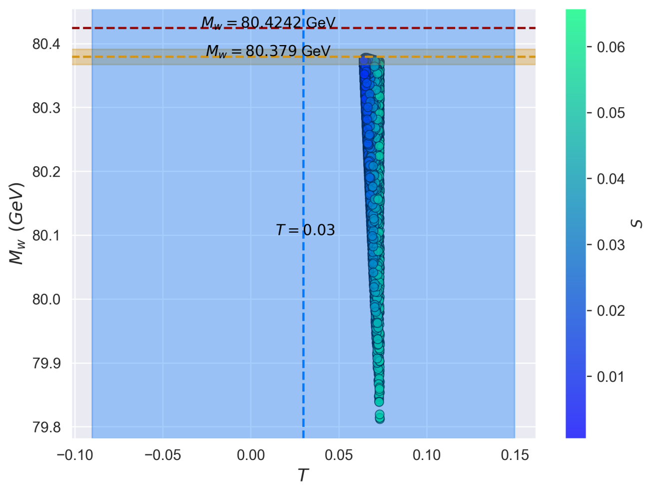


Figure 28: Correction to the oblique parameters S and T in the SLH, using $M_W = 80.379$ GeV.

[2] $\tan\beta$ is also varied, although not shown. All pairs $(f, \tan\beta)$ considered satisfy experimental limits on the LFV processes studied before. Collider limits and unitarity bounds are also respected.

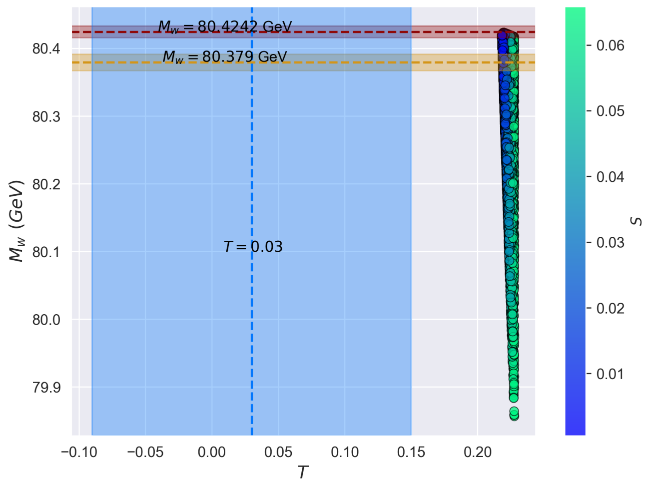


Figure 29: Correction to the oblique parameters S and T in the SLH, using $M_W = 80.4242$ GeV.

Finally, figures 28 and 29 shows the corrections to the oblique parameters S and T in the SLH, and for the different values of the M_W. In figure 28 we use the PDG value, and show that the oblique parameters agree with the EWPD within uncertainties. This means that although SLH modifies the ρ and T parameters, it is only slightly, without conflicting with the SM. For the S parameter all values agree with the SM as well. Now in plot 29 we show the corrections to the oblique parameters with the average including the CDF II measurement, and show that T is outside the EWPD (corrections to the S parameter are negligible, as shown in table 21) confidence interval.

6 CONCLUSIONS AND PERSPECTIVES

One of the virtues of the SLH model is its minimality, extending the SM gauge group to $SU(3)_c \times SU(3)_L \times U(1)_x$ attempting to understand the hierarchy problem related to the Higgs boson mass value in presence of generic new physics coupling to it. As a simple group model, it has a small number of unknown parameters and new heavy particles. These allow the appearance of lepton flavor violation processes driven by three heavy neutrinos, with signals that could in principle be probed in current and forthcoming experiments. In this work we have examined the most relevant LFV processes: $\ell \to \ell_a \gamma$, $\ell \to \ell_k \ell_a \bar{\ell}_b$ and $\ell N \to \ell_a N$, which arise at one loop within the SLH. We have computed the relevant observables as an expansion in v/f, keeping the results at $\mathcal{O}(v^2/f^2)$. To carry out our numerical calculations, we have floated the free parameters within the allowed region (by other measurements and perturbativity), ensuring that all experimental upper limits were satisfied.

As it is well-known, processes with muons would most likely be the discovery channels for LFV. Within the SLH, this can be expected either from conversions in nuclei, $\mu \to e\gamma$ or $\mu \to 3e$. However, those with taus (not considered exhaustively in previous SLH analyses) will then be needed for characterizing the underlying new physics. We have found that -in analogy with the role of $\mu \to e$ conversion in nuclei amid $\mu \to e$ transitions-, $\ell \to \tau$ conversion in nuclei are synergic with the $\tau \to \ell\gamma$ and $\tau \to 3\ell$ decays in probing LFV transitions involving taus. We hope that our work and other recent related studies motivate the experimental collaborations (Belle-II, NA64, EIC, muon collider, etc.) to pursue the corresponding dedicated searches. Finally, we verified that -although the SLH modifies the ρ parameter and can in principle accommodate the recent CDF M_W measurement-, this is in tension with electroweak precision data.

The most straightforward perspective of this work (for which preliminary results are included in this thesis) is its extension including three quasi-Dirac TeV scale neutrinos, to account for the observed neutrino masses without spoiling the naturally small radiative corrections to the Higgs mass that arise within the traditional SLH. Based on the experience with the Littlest Higgs model with T parity, we envisage that enlarging the SLH this way, will result in predictions for the different LFV processes closer to the current experimental limits. In addition to this virtue, it would provide a common solution to the naturalness of the Higgs mass and to the tiny neutrino masses, which makes it theoretically appealing, and it may also explain the baryon asymmetry of the universe via Leptogenesis. This interesting analysis will be completed and presented elsewhere.

A MATHEMATICAL RESULTS

Here we show some useful mathematical results for this work.

A.1 GOLDSTONE THEOREM

In this section we will see a general proof of Goldstone's theorem for classical scalar field theories. Consider a Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - V(\phi), \tag{352}$$

and is invariant under a global symmetry group whose algebra is spanned by N generators, iT^a_{ij} with $a \epsilon [1, \dots, N]$. Under an infinitesimal transformation, parametrized by some $\varepsilon^a \ll 1$, the fields transform according to:

$$\phi^i(x) \to \phi_i - \varepsilon^a T^a_{ij} \phi^j(x). \tag{353}$$

The invariance of the Lagrangian under the transformation imposes a condition on the potential:

$$\begin{aligned} &\mathcal{L}\left(\phi^i - \varepsilon^a T^a_{ij}\phi^j\right) = \mathcal{L}\left(\phi^i\right) \\ &\Rightarrow V\left(\phi^i - \varepsilon^a T^a_{ij}\phi^j\right) - V\left(\phi^i\right) = -\frac{\delta V}{\delta \phi^i}\varepsilon^a T^a_{ij}\phi^j = 0, \quad \text{for any } \varepsilon^a. \end{aligned} \tag{354}$$

On the other hand, the vacuum state corresponds to a field configuration that minimizes the potential:

$$\left.\frac{\delta V(\phi)}{\delta \phi^i}\right|_{\langle \phi^i \rangle_0} = 0. \tag{355}$$

If the symmetry is spontaneously broken so the vacuum is invariant only under a subset of transformations, and there are $(N-k)$ broken generators, then the conditions are:

$$\begin{aligned} &T^a_{ij}\langle \phi^j \rangle = 0, \quad (\text{for any } i), \qquad a \epsilon [1, \dots, k], \\ &\exists i \mid T^{\hat{a}}_{ij}\langle \phi^j \rangle \neq 0, \qquad \hat{a} \epsilon [k+1, \dots, N]. \end{aligned} \tag{356}$$

Differentiating the equation (354), the potential invariance condition, with respect to ϕ^k and evaluating at the vacuum, gives a new condition:

$$\frac{\delta V}{\delta \phi^i} T^a_{ij}\phi^j = 0 \Rightarrow \left.\frac{\delta^2 V}{\delta \phi^i \delta \phi^k}\right|_{\langle \phi^i \rangle_0} T^a_{ij}\langle \phi^j \rangle + \left.\frac{\delta V}{\delta \phi^i}\right|_{\langle \phi^i \rangle_0} T^a_{ik} = 0, \tag{357}$$

whose last term vanishes due to equation (355):

$$\left.\frac{\delta^2 V}{\delta\phi^i \delta\phi^k}\right|_{\langle\phi^i\rangle_0} T^{a}_{ij}\langle\phi^j\rangle_0 = 0. \tag{358}$$

According to $T^{\hat{a}}_{ij}\langle\phi^j\rangle \neq 0$ for $(N-K)$ generators, so the equation (358) implies that the second derivative of the potential at the vacuum has precisely $(N-K)$ eigenstates with vanishing eigenvalue. From the variation of the action, it can be seen that the excitations around the vacuum, $\pi^i(x) \equiv \phi^i(x) - \langle\phi^i(x)\rangle$, must satisfy the Klein Gordon equation:

$$\left(\delta_{ij}\partial_\mu\partial^\mu + \left.\frac{\delta^2 V}{\delta\phi^i \delta\phi^j}\right|_{\langle\phi^i\rangle_0}\right)\pi^j(x) = 0. \tag{359}$$

Therefore, the $(N-k)$ zero eigenvalues of the second derivative of the potential should correspond to $(N-k)$ massless particles in the spectrum: the Goldstone bosons. Finally, the fact that the Lagrangian is nevertheless invariant under the $(N-k)$ transformations, means that it has to be invariant under shifts of the corresponding Goldstone fields. Therefore, when the Lagrangian is written in terms of the physical fields, it can only contain derivative couplings of the GBs. For this reason, it can be said that the shift symmetry forbids a mass term for them.

A.2 DIAGONALIZING MATRICES

Consider an $n \times n$ matrix A that has n linearly independent eigenvectors v_i corresponding to eigenvalues λ_i. Let P the matrix which has the v_i's as its columns. P is invertible ($\det P \neq 0$). Notice that $P^{-1}v_i$ is the i-th column of $P^{-1}P = 1$. This means that $P^{-1}v_i$ is the i-th column of the identity matrix e_i.

$$P^{-1}v_i = e_i, \tag{360}$$

This implies:

$$\begin{aligned}
P^{-1}AP &= P^{-1}A\,[v_1, v_2, \ldots, v_n] \\
&= P^{-1}\,[Av_1, Av_2, \ldots, Av_n] \\
&= P^{-1}\,[\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n] \\
&= \left[\lambda_1 P^{-1}v_1, \lambda_2 P^{-1}v_2, \ldots, \lambda_n P^{-1}v_n\right] \\
&= [\lambda_1 e_1, \lambda_2 e_2, \ldots, \lambda_n e_n] \\
&= \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.
\end{aligned} \tag{361}$$

So A can be diagonalized by a matrix P as in the foregoing such that:

$$P^{-1}AP = D, \tag{362}$$

where D is a diagonal matrix with the eigenvalues of A on the diagonal. The eigenvectors are automatically linearly independent if the eigenvalues are distinct, meaning that they are all different. If A is a symmetric matrix $\left(A = A^T\right)$ then the eigenvectors of different eigenvalues are orthogonal to each other. Namely, suppose that we have eigenvalues λ_1 and λ_2 corresponding to eigenvalues v and w:

$$\lambda_1 v \cdot w = Av \cdot w = v \cdot A^T w = v \cdot Aw = v \cdot \lambda_2 w = \lambda_2 v \cdot w. \tag{363}$$

This can only true if $v \cdot w = 0$, thus if v and w are orthogonal. There are always n linearly independent eigenvectors if A is symmetric, so A will be diagonalizable. If the entries of A are all real, then there exist n mutually orthogonal eigenvectors. The matrix P then contains n orthogonal columns and also its inverse is equal to cP^T, where c is a normalization factor. Matrices for which $P^{-1} = P^T$ are called orthogonal matrices. Thus real symmetric matrices can be diagonalized by the orthogonal matrices P.

$$P^T AP = D, \quad \text{if } A = A^T. \tag{364}$$

A.2.1 Diagonalizing Matrix before the SEWSB

In this part we obtain the mass eigenstates and the masses of the neutral gauge bosons before the spontaneous electroweak symmetry breaking. From the equation (194) we recognize the mixing between weak eigenstates of the bosons A^8_μ and B^x_μ:

$$f^2\left(-\frac{A^8_\mu g}{\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2 = f^2\left(\frac{g^2}{3}A^8_\mu A^{8\mu} + \frac{g_x^2}{9}B^x_\mu B^{x\mu} - \frac{2gg_x}{3\sqrt{3}}A^8_\mu B^{x\mu}\right) \tag{365}$$

we can reordering the last equation in the following form

$$(A_8 \;\; B_x)\,\mathcal{M}\begin{pmatrix} A_8 \\ B_x \end{pmatrix}, \tag{366}$$

with

$$\mathcal{M} = \begin{pmatrix} \frac{g^2 f^2}{3} & -\frac{g g_x f^2}{3\sqrt{3}} \\ -\frac{g g_x f^2}{3\sqrt{3}} & \frac{g_x^2 f^2}{9} \end{pmatrix}. \tag{367}$$

The eigenvalues of the matrix $\mathcal{M}$ are:

$$\lambda_1 = 0, \quad \lambda_2 = \frac{f^2}{9}\left(3g^2 + g_x^2\right), \tag{368}$$

and the normalized corresponding eigenvectors are:

$$v_1 = \frac{1}{\sqrt{3g^2+g_x^2}}\begin{pmatrix} g_x \\ \sqrt{3}g \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{3g^2+g_x^2}}\begin{pmatrix} \sqrt{3}g \\ -g_x \end{pmatrix}, \tag{369}$$

the basis change matrix is:

$$P = \frac{1}{\sqrt{3g^2+g_x^2}}\begin{pmatrix} g_x & \sqrt{3}g \\ \sqrt{3}g & -g_x \end{pmatrix} \tag{370}$$

so, the mass matrix is:

$$P^T\mathcal{M}P = D = \begin{pmatrix} 0 & 0 \\ 0 & \frac{f^2}{9}\left(3g^2+g_x^2\right) \end{pmatrix} \tag{371}$$

then the gauge bosons A^8_μ and B^x_μ in terms of the mass eigenstates are:

$$\begin{aligned} \begin{pmatrix} A^8_\mu \\ B^x_\mu \end{pmatrix} &= P\begin{pmatrix} B_\mu \\ \mathcal{Z}_\mu \end{pmatrix}, \\ A^8_\mu &= \frac{1}{\sqrt{3g^2+g_x^2}}\left[g_x B_\mu + \sqrt{3}g\mathcal{Z}_\mu\right], \\ B^x_\mu &= \frac{1}{\sqrt{3g^2+g_x^2}}\left[\sqrt{3}gB_\mu - g_x\mathcal{Z}_\mu\right], \end{aligned} \tag{372}$$

or

$$\begin{aligned} B_\mu &= \frac{1}{\sqrt{3g^2+g_x^2}}\left[g_x A^8_\mu + \sqrt{3}gB^x_\mu\right], \\ \mathcal{Z}_\mu &= \frac{1}{\sqrt{3g^2+g_x^2}}\left[\sqrt{3}gA^8_\mu - g_x B^x_\mu\right]. \end{aligned} \tag{373}$$

A.2.2 Diagonalizing Matrix after the SEWSB

Now we obtain the mass eigenstates and the masses of the neutral gauge bosons after the spontaneous electroweak symmetry breaking. From the equation (201) we recognize the mixing between weak eigenstates of the bosons A^3_μ, A^8_μ and B^x_μ:

$$\begin{aligned} &\left(-\frac{A^8_\mu g}{\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2 (f^2-\Delta) + \left(\frac{A^3_\mu g}{2} + \frac{A^8_\mu g}{2\sqrt{3}} + \frac{B^x_\mu g_x}{3}\right)^2 \Delta, \\ &= \frac{g^2\Delta}{4}A^{3\mu}A^3_\mu + \frac{2g^2\Delta}{4\sqrt{3}}A^{3\mu}A^8_\mu + \frac{2gg_x\Delta}{6}A^{3\mu}B^x_\mu + \left(\frac{g^2f^2}{3} - \frac{g^2\Delta}{4}\right)A^{8\mu}A^8_\mu \\ &+ 2gg_x\left(\frac{\Delta}{2\sqrt{3}} - \frac{f^2}{3\sqrt{3}}\right)A^{8\mu}B^x_\mu + \frac{g_x^2f^2}{9}B^{x\mu}B^x_\mu \end{aligned} \tag{374}$$

reordering the last equation we obtain:

$$(A_3 \;\; A_8 \;\; B_x)\,\mathcal{M}\begin{pmatrix} A_3 \\ A_8 \\ B_x \end{pmatrix} \tag{375}$$

with

$$\mathcal{M} = \begin{pmatrix} \frac{g^2\Delta}{4} & \frac{g^2\Delta}{4\sqrt{3}} & \frac{gg_x\Delta}{6} \\ \frac{g^2\Delta}{4\sqrt{3}} & \frac{g^2f^2}{3} - \frac{g^2\Delta}{4} & \frac{gg_x\Delta}{2\sqrt{3}} - \frac{gg_xf^2}{3\sqrt{3}} \\ \frac{gg_x\Delta}{6} & \frac{gg_x\Delta}{2\sqrt{3}} - \frac{gg_xf^2}{3\sqrt{3}} & \frac{g_x^2f^2}{9} \end{pmatrix}, \quad \text{where } \Delta = \frac{v^2}{2} - \frac{v^4}{12f^2}\left(\frac{c_\beta^4}{s_\beta^2} + \frac{s_\beta^4}{c_\beta^2}\right). \tag{376}$$

The eigenvalues of the matrix $\mathcal{M}$ are:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{1}{2}\left[\frac{f^2}{3}\left(g^2 + \frac{g_x^2}{3}\right) \mp \sqrt{\frac{f^4}{9}\left(g^2 + \frac{g_x^2}{3}\right)^2 + \frac{4g^2}{3}\left(\frac{g_x^2}{3} + \frac{g^2}{4}\right)(\Delta^2 - \Delta f^2)}\,\right]. \tag{377}$$

for the main purpose of this work, we need to take the mass of the gauge bosons up to order $\mathcal{O}(v^2/f^2)$, so we expand the eigenvalues $\lambda_{2,3}$ in a Taylor's series:

$$\begin{aligned} \lambda_2 &= \frac{1}{2}\frac{g^2v^2}{4c_w^2}\left[1 - \frac{v^2}{6f^2}\left(\frac{s_\beta^4}{c_\beta^2} + \frac{c_\beta^4}{s_\beta^2}\right) - \frac{v^2}{8f^2}\left(1 - t_w^2\right)^2\right], \\ \lambda_3 &= \frac{1}{2}\frac{2f^2g^2}{3 - t_w^2}\left[1 - \frac{\left(3 - t_w^2\right)v^2}{8f^2c_w^2}\right]. \end{aligned} \tag{378}$$

and the corresponding first order normalized eigenvectors are:

$$v_1 = \begin{pmatrix} -s_w \\ \frac{s_w}{\sqrt{3}} \\ \frac{c_w}{\sqrt{3}}\sqrt{3 - t_w^2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} c_w \\ \frac{s_w^2}{c_w\sqrt{3}} \\ \frac{s_w}{\sqrt{3}}\sqrt{3 - t_w^2} \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -\frac{\sqrt{3 - t_w^2}}{\sqrt{3}} \\ \frac{t_w}{\sqrt{3}} \end{pmatrix}, \tag{379}$$

the basis change matrix is:

$$P = \begin{pmatrix} 0 & c_w & -s_w \\ -\sqrt{\frac{3 - t_w^2}{3}} & \frac{s_w^2}{c_w\sqrt{3}} & -\frac{s_w}{\sqrt{3}} \\ \frac{t_w}{\sqrt{3}} & s_w\sqrt{\frac{3 - t_w^2}{3}} & c_w\sqrt{\frac{3 - t_w^2}{3}} \end{pmatrix}, \tag{380}$$

so, the mass matrix is:

$$P^T \mathcal{M} P = D = \begin{pmatrix} \frac{1}{2}\frac{2f^2g^2}{3-t_w^2}\left[1-\frac{(3-t_w^2)v^2}{8f^2c_w^2}\right] & 0 & 0 \\ 0 & \frac{1}{2}\frac{g^2v^2}{4c_w^2}\left[1-\frac{v^2}{6f^2}\left(\frac{s_\beta^4}{c_\beta^2}+\frac{c_\beta^4}{s_\beta^2}\right)-\frac{v^2}{8f^2}\left(1-t_w^2\right)^2\right] & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{381}$$

then the gauge bosons A^3_μ, A^8_μ and B^x_μ in terms of the mass eigenstates are:

$$\begin{aligned} A^3_\mu &= c_w Z_\mu - s_w A_\mu, \\ A^8_\mu &= -\sqrt{1-\frac{t_w^2}{3}}Z'_\mu + \frac{s_w^2}{c_w\sqrt{3}}Z_\mu + \frac{s_w}{\sqrt{3}}A_\mu, \\ B^x_\mu &= \frac{t_w}{\sqrt{3}}Z'_\mu + s_w\sqrt{1-\frac{t_w^2}{3}}Z_\mu + c_w\sqrt{1-\frac{t_w^2}{3}}A_\mu, \end{aligned} \tag{382}$$

or

$$\begin{aligned} Z'_\mu &= -\sqrt{1-\frac{t_w^2}{3}}A^8_\mu + \frac{t_w}{\sqrt{3}}B^x_\mu, \\ Z_\mu &= c_w A^3_\mu + \frac{s_w^2}{c_w\sqrt{3}}A^8_\mu + s_w\sqrt{1-\frac{t_w^2}{3}}B^x_\mu, \\ A_\mu &= -s_w A^3_\mu - \frac{s_w}{\sqrt{3}}A^8_\mu + c_w\sqrt{1-\frac{t_w^2}{3}}B^x_\mu. \end{aligned} \tag{383}$$

A.3 LOOP FUNCTIONS

A general one-loop tensor integral in D dimensions with N legs and P integration momenta in the numerator is represented as [155]:

$$\frac{i}{16\pi^2}T^N_{\mu_1,\ldots,\mu_P} \equiv \mu^{4-D}\int \frac{d^Dq}{(2\pi)^D}\frac{q_{\mu_1}\ldots q_{\mu_P}}{D_0D_1\ldots D_{N-1}}, \tag{384}$$

where

$$D_0 = q^2 - m_0^2 + i\epsilon, \quad D_i = (q+k_i)^2 - m_i^2 + i\epsilon, \quad i = 1,\ldots,N-1. \tag{385}$$

The vectors K_i are the sum of external momenta p_i:

$$k_1 = p_1, \quad k_2 = p_1 + p_2, \quad \ldots, \quad k_{N-1} = \sum_{i=1}^{N-1} p_i. \tag{386}$$

These tensor integrals are invariant under permutations of propagators D_i and symmetric in the Lorentz indices. Generally we define $T^1 = A$, $T^2 = B$, etc. The scalar integrals are A_0, B_0, etc. Lorentz covariance allows decomposing equation (384) into a linear combination of tensors constructed with the metric and the external momenta, however this basis is not unique, we could use any set of linearly independent momenta and $g_{\mu\nu}$ [156]. For this work we use the decompositions:

$$\begin{aligned}
B_\mu &= k_{1\mu} B_1, \\
C_\mu &= k_{1\mu} C_1 + k_{2\mu} C_2, \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + \sum_{i,j=1}^{2} k_{i\mu} k_{j\nu} C_{ij}, \\
D_\mu &= \sum_{i=1}^{3} k_{i\mu} D_i, \\
D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^{3} k_{i\mu} k_{j\nu} D_{ij}.
\end{aligned} \tag{387}$$

These functions have been calculated for the argument configuration required by the processes under study, obtaining the results presented in the following. All of them agree with those collected in the appendix B of ref. [17].

Two-point functions

There appear:

$$\frac{i}{16\pi^2}\{B_0, B^\mu\}(\text{args}) = \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\{1, q^\mu\}}{\left(q^2 - m_0^2\right)\left[(q+p)^2 - m_1^2\right]}. \tag{388}$$

Their tensor coefficients depend on the invariant quantities (args)= $\left(p^2, m_0^2, m_1^2\right)$. Functions $B \equiv B\left(0, M_1^2, M_2^2\right)$ and $\overline{B} \equiv B\left(0, M_2^2, M_1^2\right)$ read

$$B_0 = \overline{B}_0 = \Delta_\epsilon + 1 - \frac{M_1^2 \log\left(\frac{M_1^2}{\mu^2}\right) - M_2^2 \log\left(\frac{M_2^2}{\mu^2}\right)}{M_1^2 - M_2^2}, \tag{389}$$

$$\begin{aligned}
B_1 &= -\frac{\Delta_\epsilon}{2} + \frac{4M_1^2 M_2^2 - 3M_1^4 - M_2^4 + 2M_1^4 \log\left(\frac{M_1^2}{\mu^2}\right) + 2M_2^2\left(M_2^2 - 2M_1^2\right)\log\left(\frac{M_2^2}{\mu^2}\right)}{4\left(M_1^2 - M_2^2\right)^2} \\
&= -\overline{B}_0 - \overline{B}_1,
\end{aligned} \tag{390}$$

with $\Delta_\epsilon = \frac{1}{\epsilon} - \gamma + \log(4\pi)$ encoding the ultraviolet divergences in $D = 4$ dimensions.

Three-point functions

In this case there arise:

$$\frac{i}{16\pi^2}\{C_0, C^\mu, C^{\mu\nu}\}(args) = \mu^{4-D}\int \frac{d^D q}{(2\pi)^D}\frac{\{1, q^\mu, q^\mu q^\nu\}}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]}, \tag{391}$$

with $p^2 = p_1^2 = 0$ and $Q^2 = (p-p_1)^2$, so that only the following general types are necessary for us: $C = C\left(0, Q^2, 0, M_1^2, M_2^2, M_2^2\right)$ (we define the mass ratio $x = M_2^2/M_1^2$):

$$\begin{aligned}
C_0 &= \frac{1}{M_1^2}\left[\frac{1-x+\log(x)}{(1-x)^2} + \frac{Q^2}{M_1^2}\left(\frac{-2-3x+6x^2-x^3-6x\log(x)}{12(1-x)^4}\right)\right] + \mathcal{O}(Q^4),\\
C_1 = C_2 &= \frac{1}{M_1^2}\frac{4x-3-x^2-2\log(x)}{4(1-x)^3} + \mathcal{O}(Q^2),\\
C_{11} = C_{22} = 2C_{12} &= \frac{1}{M_1^2}\frac{11-18x+9x^2-2x^3+6\log(x)}{18(1-x)^4} + \mathcal{O}(Q^2),\\
C_{00} &= -\frac{1}{2}B_1 - \frac{Q^2}{M_1^2}\left(\frac{11-18x+9x^2-2x^3+6\log(x)}{72(1-x)^4}\right) + \mathcal{O}(Q^4).
\end{aligned} \tag{392}$$

Now, defining $\overline{C} = C\left(0, Q^2, 0, M_2^2, M_1^2, M_1^2\right)$:

$$\begin{aligned}
\overline{C}_0 &= \frac{1}{M_1^2}\left[\frac{-1+x-x\log(x)}{(1-x)^2} + \frac{Q^2}{M_1^2}\left(\frac{-1+6x-3x^2-2x^3+6x^2\log(x)}{12(1-x)^4}\right)\right] + \mathcal{O}(Q^4),\\
\overline{C}_1 = \overline{C}_2 &= \frac{1}{M_1^2}\frac{1-4x+3x^2-2x^2\log(x)}{4(1-3)^3} + \mathcal{O}(Q^2),\\
\overline{C}_{11} = \overline{C}_{22} = 2\overline{C}_{12} &= \frac{1}{M_1^2}\frac{-2+9x-18x^2+11x^3-6x^3\log(x)}{18(1-x)^4} + \mathcal{O}(Q^2),\\
\overline{C}_{00} &= -\frac{1}{2}\overline{B}_1 - \frac{Q^2}{M_1^2}\left(\frac{-2+9x-18x^2+11x^3-6x^3\log(x)}{72(1-x)^4}\right) + \mathcal{O}(Q^4).
\end{aligned} \tag{393}$$

Alternatively, defining $\hat{C} = \left(0, Q^2, 0, M_1^2, M_2^2, 0\right)$, which is symmetric under the exchange $M_1 \leftrightarrow M_2$,

$$\hat{C}_{00} = \frac{1}{8}\left(3 + 2\Delta_\epsilon - 2\log\left(\frac{M_1^2}{\mu^2}\right)\right) + \frac{x\log(x)}{4(1-x)} + \mathcal{O}(Q^2). \tag{394}$$

Four-point functions

Those needed are all ultraviolet convergent:

$$\frac{i}{16\pi^2}\{D_0, D^\mu, D^{\mu\nu}\}(args) = \mu^{4-D}\int \frac{d^D q}{(2\pi)^D}\frac{\{1, q^\mu, q^\mu q^\nu\}}{D_0 D_1 D_2 D_3}, \tag{395}$$

where:

$$D_0 = q^2 - m_0^2, \quad D_i = (q+p_i)^2 - m_i^2, \quad i = 1, 2, 3. \tag{396}$$

with $(\text{args}) = \left(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2; m_0^2, m_1^2, m_2^2, m_3^2\right)$. Zero external momenta will be set, so we only need:

$$\begin{aligned}
\frac{i}{16\pi^2} D_0 &= \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{\left(q^2 - m_0^2\right)\left(q^2 - m_1^2\right)\left(q^2 - m_2^2\right)\left(q^2 - m_3^2\right)}, \\
\frac{i}{16\pi^2} D_{00} &= \frac{\mu^{4-D}}{4} \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{\left(q^2 - m_0^2\right)\left(q^2 - m_1^2\right)\left(q^2 - m_2^2\right)\left(q^2 - m_3^2\right)}.
\end{aligned} \tag{397}$$

In terms of the mass ratios $x = m_1^2/m_0^2$, $y = m_2^2/m_0^2$, $z = m_3^2/m_0^2$ the previous integrals read:

$$\begin{aligned}
d_0(x, y, z) \equiv m_0^4 D_0 &= \frac{x \log[x]}{(1-x)(x-y)(x-z)} - \frac{y \log[y]}{(1-y)(x-y)(y-z)} \\
&+ \frac{z \log[z]}{(1-z)(x-z)(y-z)} \\
\tilde{d}_0(x, y, z) \equiv 4 m_0^2 D_{00} &= \frac{x^2 \log[x]}{(1-x)(x-y)(x-z)} - \frac{y^2 \log[y]}{(1-y)(x-y)(y-z)} \\
&+ \frac{z^2 \log[z]}{(1-z)(x-z)(y-z)}.
\end{aligned} \tag{398}$$

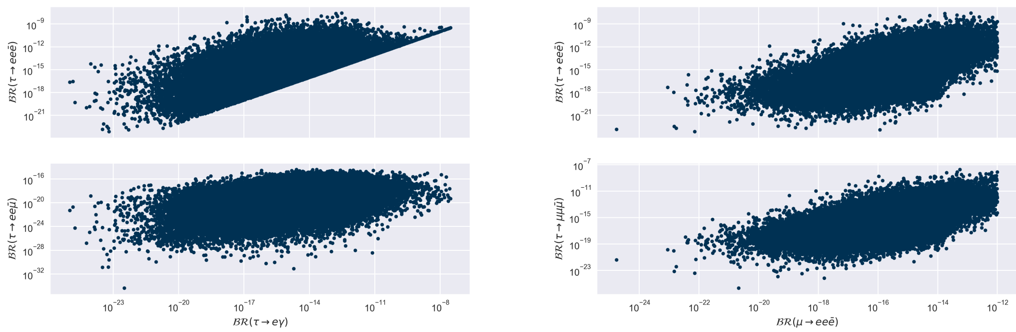
The case in which $z \to 1$ (argument omitted below) is also necessary, reading:

$$\begin{aligned}
d_0(x, y) &= -\Big[\frac{x \log[x]}{(1-x)^2 (x-y)} - \frac{y \log[y]}{(1-y)^2 (x-y)} + \frac{1}{(1-x)(1-y)}\Big], \\
\tilde{d}_0(x, y) &= -\Big[\frac{x^2 \log[x]}{(1-x)^2 (x-y)} - \frac{y^2 \log[y]}{(1-y)^2 (x-y)} + \frac{1}{(1-x)(1-y)}\Big].
\end{aligned} \tag{399}$$

B ADDITIONAL PLOTS

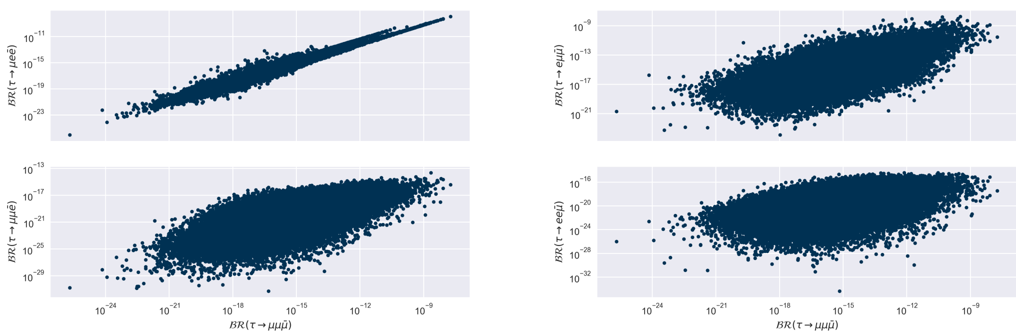
In this appendix we show some results that were not included in the main discussion, since the graphs are analogous to those already shown and do not provide new information.

In figure 30a we show the correlation of $\mathcal{BR}(\tau \to e\gamma)$ against other decays. As we expected there is not a testable correlation with the wrong sign decay $\mathcal{BR}(\tau \to ee\bar{\mu})$ but with the same flavor decay $\mathcal{BR}(\tau \to ee\bar{e})$ there is a correlation which is like the analogous processes with muons (see figure 19a). In figure 30b we show the behavior of the three same flavor decays, we see a small correlation in our results, that are basically the same for the cases of $\mathcal{BR}(\tau \to 3\ell)$, $\ell = e, \mu$ against $\mathcal{BR}(\mu \to ee\bar{e})$, although the decay width is essentially the same, flavor coefficients are different in all the cases.



(a) $\mathcal{BR}(\tau \to e\gamma)$ vs. $\mathcal{BR}(\tau \to ee\bar{e})$, $\mathcal{BR}(\tau \to ee\bar{\mu})$ **(b)** $\mathcal{BR}(\mu \to ee\bar{e})$ vs. $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$, $\mathcal{BR}(\tau \to ee\bar{e})$

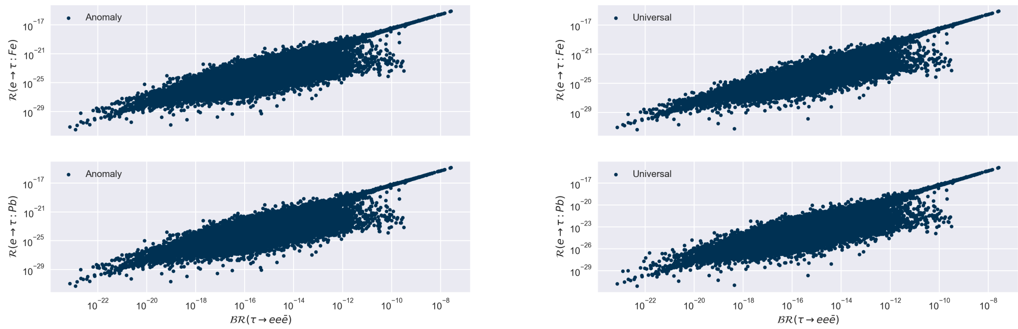
Figure 30: Scatter plots for $\ell \to \ell'\gamma$ and some $\ell \to 3\ell'$ decays.



(a) $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$ vs. $\mathcal{BR}(\tau \to \mu\mu\bar{e})$, $\mathcal{BR}(\tau \to \mu e\bar{e})$ **(b)** $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$ vs. $\mathcal{BR}(\tau \to ee\bar{\mu})$, $\mathcal{BR}(\tau \to e\mu\bar{\mu})$

Figure 31: Scatter plots for $\ell \to 3\ell'$ decays.

In figure 31 we show the analogous processes of figures 19b and 20a. In this plot we see the same behavior, where there is a big correlation between $\mathcal{BR}(\tau \to \mu\mu\bar{\mu})$ and $\mathcal{BR}(\tau \to \mu e\bar{e})$ and a small one with $\mathcal{BR}(\tau \to e\mu\bar{\mu})$, and the correlation with the wrong sign decays is quite small (as in the electron case), however, correlation is bigger in the case of the same flavor coefficients that in the case of different ones. In figures 32, 33 we show the analogous processes (with electrons) of figures 22, 23, in which we see a similar behavior that in the muon case but with a suppression of three orders of magnitude.



(a) $\mathcal{BR}(\tau \to ee\bar{e})$ vs. $\mathcal{R}(e \to \tau : \mathrm{Fe})$, $\mathcal{R}(e \to \tau : \mathrm{Pb})$ **(b)** $\mathcal{BR}(\tau \to ee\bar{e})$ vs. $\mathcal{R}(e \to \tau : \mathrm{Fe})$, $\mathcal{R}(e \to \tau : \mathrm{Pb})$

Figure 32: Scatter plots for $\ell \to \ell'$ nuclei conversion.

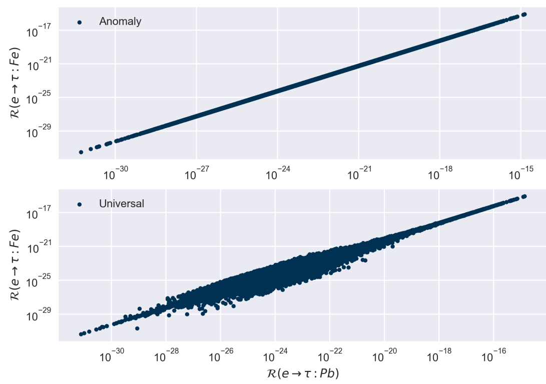


Figure 33: $\mathcal{R}(e \to \tau : \mathrm{Pb})$ vs. $\mathcal{R}(e \to \tau : \mathrm{Fe})$

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