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Odd-dimensional self-duality for non-Abelian tensor-multiplet in $D = 3 + 2$ as master theory of integrable-systems

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We present $N = 2$ supersymmetric non-Abelian duality-symmetry between a tensor multiplet and an extra vector multiplet in $D = 3 + 2$ dimensions. Our system has the Yang-Mills (YM) vector multiplet $({A_\mu}^I, \lambda^I)$, a tensor-multiplet $({B_{\mu\nu}}^I, \chi^I, \varphi^I)$, and an extra vector multiplet $({C_\mu}^I, \rho^I, \sigma^I)$. The index $_{I=1,2,\cdots,\dim G}$ is for the adjoint representation of a non-Abelian group G. The ${A_\mu}^I$ is the conventional YM gauge field, ${B_{\mu\nu}}^I$ is a *non-Abelian* tensor field, while φ^I and σ^I are scalar fields. The λ^I, χ^I and ρ^I are Majorana fermions in the **2** of $Sp(1)$. The ${B_{\mu\nu}}^I$ and ${C_\mu}^I$-fields have their respective field-strengths defined by ${G_{\mu\nu\rho}}^I \equiv +3D_{[\mu}{B_{\nu\rho]}}^I + 3f^{IJK}{F_{[\mu\nu}}^J {C_{\rho]}}^K$ and ${H_{\mu\nu}}^I \equiv +2D_{[\mu}{C_{\nu]}}^I + m{B_{\mu\nu}}^I + f^{IJK}\phi^J {H_{\mu\nu}}^K - f^{IJK}\sigma^J {F_{\mu\nu}}^K$. The duality relationship is ${H_{\mu\nu}}^I = (1/6)\epsilon_{\mu\nu}{}^{\rho\sigma\tau}{G_{\rho\sigma\tau}}^I - (1/2)f^{IJK}(\overline{\lambda}^J\gamma_{\mu\nu}\chi^K)$, with its super-partner relationships: $\varphi^I = -\sigma^I, \chi^I = -\rho^I$. Since ${H_{\sigma\tau}}^I$ contains $m{B_{\sigma\tau}}^I$ linearly, this is a 'massive' self-dual relationship. Interestingly, the closure of supersymmetries shows the intrinsic global scale symmetry: $\delta_\zeta({B_{\mu\nu}}^I, \chi^I, \varphi^I, {C_\mu}^I, \rho^I, \sigma^I) = +m\zeta({B_{\mu\nu}}^I, \chi^I, \varphi^I, {C_\mu}^I, \rho^I, \sigma^I)$. By certain dimensional-reduction scheme into $D = 2 + 2$, we show that self-dual supersymmetric tensor multiplet is generated. We deduce that our present theory in $D = 3 + 2$ can serve as the underlying 'Master Theory' of a similar system in $D = 2 + 2$.

1. Introduction

In our recent paper [1], we have presented a self-dual Yang-Mills vector multiplet (YMVM) in $D = 2 + 2$ space-time dimensions coupled to a self-dual tensor multiplet (TM) and an extra vector multiplet (EVM). The study of self-dual supersymmetric YM theories in $D = 2 + 2$ [2][3] was motivated by Atiyah's conjecture [4] that *all bosonic* integrable systems in dimensions $1 \le D \le 3$ are generated by the self-dual YM theory in $D = 2 + 2$. In other words, the self-dual YM theory in $D = 2 + 2$ is conjectured to be the 'Master Theory' of *all bosonic* integrable systems. It is then natural to expect that its *supersymmetrized* conjecture *i.e.*, the *supersymmetric* self-dual YM theories in $D = 2 + 2$ will generate *all supersymmetric* integrable systems in $1 \le D \le 3$. As such, it is natural to expect that our recent theory [1] generates supersymmetric integrable models in $1 \le D \le 3$, as well.

Additionally, expectations such as above compel one to explore even higher-dimensional 'Grand Master Theory' that generates self-dual supersymmetric theories in $D = 2 + 2$ [2][3][1][5]. If we are to consider such a theory, the space-time dimensions should have at least *two* time-coordinates, because of the overlap with theories in $D = 2 + 2$ [2][3][1]. In fact, in our recent paper [5], we have shown that our $N = 2$ supersymmetric theory in $D = 3 + 3$ actually generates $N = 1$ supersymmetric self-dual non-Abelian tensor multiplet in $D = 2 + 2$. In this present paper, we establish a supersymmetric theory with odd-dimensional duality in $D = 3 + 2$ generating self-dual supersymmetric tensor multiplet in $D = 2 + 2$ [1]. Even though $D = 3 + 3$ is higher dimensional than $D = 3 + 2$, our new result is that *odd-dimensional* self-dual theory can also generate $D = 2 + 2$ self-dual non-Abelian TM [1].

There are two new aspects in our present approach: (i) We need space-time with the signature $(+, +, +, -, -)$ instead of the conventional $(+, +, +, +, -)$ [6][7], due to the two time-coordinates needed. This is similar to our recent theory in $D = 3 + 3$ [5] instead of the

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conventional $D = 5 + 1$ [8][9]. (ii) There have been *no* works formulating supersymmetric odd-dimensional self-duality for non-Abelian tensor-multiplet in the past in $D = 3 + 2$.

Close considerations reveal that there is a good analogy between TMs in $D = 3 + 2$ and $D = 2 + 2$. The TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ in $D = 3 + 2$ with $4+4$ degrees of freedom (DOF) has formally the same field content as the TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ in $D = 2+2$ with $2+2$ DOF. In particular, the analog of the duality relationship[2] $G_{\mu\nu\rho}{}^I \stackrel{*}{=} +\epsilon_{\mu\nu\rho}{}^\sigma D_\sigma \sigma^I$ in $D = 2+2$ [3] is like $G_{\mu\nu\rho}{}^I = +(1/2)\epsilon_{\mu\nu\rho}{}^{\sigma\tau} H_{\sigma\tau}{}^I$ in $D = 3+2$. The latter is known as the '*odd-dimensional self-duality*' between a tensor field-strength and its mass-term [10][11], if $H_{\sigma\tau}{}^I$ contains the Stueckelberg type [12] mass-term $mB_{\mu\nu}{}^I$. A typical example for odd-dimensional self-duality is the 3D case [10][11][3]:

$$mA_\mu \stackrel{*}{=} +\frac{1}{2}\epsilon_\mu{}^{\rho\sigma} F_{\rho\sigma} \ . \tag{1.1}$$

Here the left-side is regarded as the 'mass'-term of the Abelian vector A_μ. The gauge *non*-invariant left-side can be improved by a gauge-invariant field-strength $H_\mu \equiv \partial_\mu \varphi + mA_\mu$ of a scalar φ. It is much like the Proca-Stueckelberg mechanism [12], where the field-redefinition can absorb the $\partial_\mu\varphi$, and one again goes back to (1.1). Even though general examples in odd dimensions for purely-bosonic and Abelian systems were given in [10], and non-Abelian cases are considered in 3D [11], still *no simultaneous accomplishment of supersymmetrization, $D = 3+2$-extension and non-Abelianization* was achieved thereafter, including [11].

Our objective is to find a 'Master Theory' more fundamental than self-dual supersymmetric systems in $D = 2 + 2$ [1]. The simplest choice is to consider $D = 3 + 2$ space-time. To this end, we also need to follow the formulations in [6], and make use of a VM, paying attention to the subtle difference between $D = 3 + 2$ and $D = 4 + 1$.

Another important point is the lack of action formulation for massive self-dualities in $4k + 1 = 5, 9, 13, \cdots$-dimensional space-time. For example, consider the 5D case. A 2nd-rank tensor $B_{\mu\nu}$ has its self-duality

$$mB_{\mu\nu}{}^I \stackrel{*}{=} +\frac{1}{6}\epsilon^{\mu\nu\rho\sigma\tau} G^{(0)}_{\mu\nu\rho} \ , \tag{1.2}$$

where $G^{(0)}_{\rho\sigma\tau} \equiv 3\partial_{[\rho} B_{\sigma\tau]}$. A candidate lagrangian is

$$\mathcal{L} = +\frac{1}{4}m^2(B_{\mu\nu})^2 + \frac{1}{24}m\,\epsilon^{\mu\nu\rho\sigma\tau} B_{\mu\nu} G^{(0)}_{\rho\sigma\tau} \ . \tag{1.3}$$

This lagrangian looks like yielding the self-duality (1.1) as its B-field equation. However, the 2nd term in (1.3) is actually *a total divergence* by itself, and will *not* yield the ϵG-term in (1.2). As the sign change by index-flipping shows, this total-divergence problem arises in odd-dimensions $4k + 1 = 5, 9, 14, \cdots$, but *not* in $D = 4k - 1 = 3, 7, \cdots$. In [10], this problem with general $D = 4k + 1$ was described as 'tachyon'-problem. This crucial difference between $D = 4k - 1$ and $D = 4k + 1$ was *not* discussed in [11]. Such a tachyonic problem is *not* the essential problem here, since we are dealing with $D = 3 + 2$ dimensions with two time coordinates. However, the total-divergence problem persists against the action principle in 5D. For this reason, we have to rely on an action-less formulation without field equations.

There is an additional problem with the fermionic mass-term that vanishes identically. This is because in $D = 3 + 2$, the flipping property of fermionic-bilinears gives the identically-vanishing mass-term like $(\overline{\psi}\psi) \equiv 0$, as the $n = 0$ case shown in (2.3a) later. For these reasons, we heavily rely on the field-equation analysis instead of lagrangian formulation.

Independent of odd-dimensional self-duality itself [10], 5D space-time dimension has additional interest, because of Randall-Sundrum type brane-world scenario [13], and its supersymmetrization [14]. Even though our present metric-signagure differs from $D = 4 + 1$, there are still potential interest to consider supersymmetric theories in $D = 3 + 2$. In fact, plural versions [15] of M-theories [16] were found in $D = 9 + 2$ and $D = 6 + 5$, yielding type-IIA string theories [17] in $D = 10 + 0$, $D = 9 + 1$, $D = 8 + 2$, $D = 6 + 4$ and $D = 5 + 5$, which are linked each other by duality-transformations.

In the present paper, we take the first step toward the non-Abelianization and supersymmetrization of the original odd-dimensional self-duality [10] into $D = 3 + 2$ space-time. After preliminary preparations in the next section, we present the total system in section 3. In section 4, we show that our system in $D = 3 + 2$ actually generates self-dual tensor multiplet in $D = 2 + 2$. The concluding remarks are given in section 5.

2. Preliminaries

We start with the preliminaries for clarity in presentation. First of all, the fermions in $D = 3+2$ are assigned with an additional doublet index of *e.g.,* $Sp(1)$, because of the required $4 + 4$ DOF. This has been well-known in the case of $D = 4 + 1$ [18][6], similar to $D = 3 + 2$.

The odd-dimensional self-duality (1.1) in 3D is generalized in 5D to the case of a second-rank tensor $B_{\mu\nu}{}^I$ as

$$H_{\mu\nu}{}^I \stackrel{*}{=} +\frac{1}{6}\epsilon_{\mu\nu}{}^{\rho\sigma\tau} G_{\rho\sigma\tau}{}^I \ , \tag{2.1}$$

where $H_{\mu\nu}{}^I \equiv 2D_{[\mu} C_{\nu]}{}^I + mB_{\mu\nu}{}^I$[4] and $G_{\mu\nu\rho}{}^I \equiv 3D_{[\mu} B_{\nu\rho]}{}^I + 3f^{IJK} F_{\mu\nu}{}^J C_{\rho]}{}^I$ are respectively the field-strengths of $C_\mu{}^I$ and $B_{\mu\nu}{}^I$. As in (1.1), the left side has now the 'mass' term of $B_{\mu\nu}{}^I$. Because of the *non*-Abelian tensor involved, we need to adopt the general formulation of tensor-hierarchy [19][20]. In other words, we need at least one tensor $B_{\mu\nu}{}^I$ and one extra vector $C_\mu{}^I$, in addition to a vector for YM gauge group. This fixes the 'candidate multiplets' in our system as the YMVM $(A_\mu{}^I, \lambda^I, \phi^I)$, a TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ and an extra vector-multiplet (EVM) $(C_\mu{}^I, \rho^I, \sigma^I)$. All of these multiplets are in the adjoint representation of the gauge group. These multiplets are parallel to the self-dual tensor multiplet in $D = 2 + 2$ [1]. Each of these multiplets has the $4 + 4$ DOF, *modulo* their adjoint indices.

[2] The symbol $\stackrel{*}{=}$ stands for a duality relationship or an equality that holds by the use of the duality, that is distinguished from usual algebraic equalities or field equations.
[3] Our convention in $D = 3+2$ is such as $(1/n!)\epsilon_{\mu_1\cdots\mu_{5-n}[n]}\epsilon^{[n]\nu_1\cdots\nu_{5-n}} = +(5-n)!\,\delta_{[\mu_1}{}^{\nu_1}\cdots\delta_{\mu_{5-n}]}{}^{\nu_{5-n}}$, and $(1/n!)\,\epsilon_{\mu_1\cdots\mu_{5-n}}{}^{[n]}\gamma_{[n]} = (-1)^{n(n-1)/2}\gamma_{\mu_1\cdots\mu_{5-n}}$. Here the symbol $[n]$ is for totally antisymmetric n-indices.
[4] The definitions of the H-field strength are temporarily simplified here for the sake of argument, but their final forms will be defined later by (3.4c) with the additional ϕH and σF-terms. Also in the supersymmetric case later as in (3.1), the right side of (2.1) will also get a fermionic bilinear term.

As has been mentioned, the duality (2.1) obstructs the action principle, because the lagrangian term of the type $H \wedge G$ has only a total divergence at the lowest order: $(1/24)\,\epsilon^{\mu\nu\rho\sigma\tau} H_{\mu\nu}{}^I G_{\rho\sigma\tau}{}^I \overset{\nabla}{=} \mathcal{O}(\Phi^3)$.[5] This feature is valid even with our covariantized field strengths such as $G_{\mu\nu\rho}{}^I$ and $H_{\mu\nu}{}^I$. For this reason, we have to rely only on field equations instead of the action principle.

Similar to $D = 4+1$ case [6][7], we need to assign the fermions to carry an additional **2**-index of, *e.g.,* $Sp(1)$. To be more specific, a Majorana fermion in $D = 3+2$ has 2 on-shell physical degrees of freedom, just in $D = 3+1$. Moreover, the right DOF for an $N = 2$ vector multiplet must be $4+4$, because a vector in $D = 3+2$ already has 3 (but *not* 2) DOF as in $D = 4+1$ [6]. For this reason, the DOF of a gaugino should be doubled to have 4 on-shell DOF, and a scalar must be also present in the multiplet. For this reason, the gaugino in our YMVM $(A_\mu{}^I, \lambda^I, \phi^I)$ should be in the **2** of $Sp(1)$. Accordingly, we introduce the 2×2 matrices of $Sp(1)$-generators t^i $_{(i=1,2,3)}$ [5]:

$$\left((\tau_1)_a{}^b\right) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \left((\tau_2)_a{}^b\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \left((\tau_3)_a{}^b\right) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{2.2a}$$

$$(\epsilon_{ab}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau^i \tau^j = -\delta^{ij} I + \epsilon^{ijk}\tau^k . \tag{2.2b}$$

The multiplication is performed in an $Sp(1)$-covariant way, *e.g.,* $(\tau^i\tau^j)_a{}^b \equiv (\tau^i)_a{}^c(\tau^j)_c{}^b$, $(\tau^I)_{ab} \equiv (\tau^i)_a{}^c\epsilon_{cb} = +(\tau^I)_{ba}$ $_{(a,b,\cdots=1,2)}$, *etc.*

Following [18], the flipping and hermiticity properties of scalar products of Majorana spinors ψ and χ in $D = 3+2$ are

$$(\overline{\psi}\gamma^{[n]}\chi) = -(-1)^{n(n-1)/2}(\overline{\chi}\gamma^{[n]}\psi), \qquad (\overline{\psi}\gamma^{[n]}\tau^i\chi) = (-1)^{n(n-1)/2}(\overline{\chi}\gamma^{[n]}\tau^i\psi), \tag{2.3a}$$

$$(\overline{\psi}\gamma^{[n]}\chi)^\dagger = +(\overline{\psi}\gamma^{[n]}\chi), \tag{2.3b}$$

where ψ and χ here carry the doublet-index $_{a=1,2}$ of $Sp(1)$. Accordingly, $(\overline{\psi}\gamma^{[n]}\tau^i\chi) \equiv (\tau^i)_a{}^b(\overline{\psi}^a\gamma^{[n]}\chi_b)$, *etc.* The importance of these doublet representations in the flipping property (2.3a) is reflected in the case of $n = 1$ in the translation parameter $\xi^\mu \equiv +2(\overline{\epsilon}_1\gamma^\mu\epsilon_2)$ in the commutator of two supersymmetry transformations. This is because the parameter $\xi^\mu \equiv +2(\overline{\epsilon}_1\gamma^\mu\epsilon_2) = -2(\overline{\epsilon}_2\gamma^\mu\epsilon_1)$ in $[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P(\xi^\mu)$ should be *antisymmetric* under $\epsilon_1 \leftrightarrow \epsilon_2$. If ψ and χ in (2.3a) *were singlets* instead of doublets of $Sp(1)$, then the flipping *would be* $(\overline{\psi}\gamma^\mu\chi) = +(\overline{\chi}\gamma^\mu\psi)$, leading to $\xi^\mu \equiv +2(\overline{\epsilon}_1\gamma^\mu\epsilon_2) = +2(\overline{\epsilon}_2\gamma^\mu\epsilon_1)$ *symmetric* in $\epsilon_1 \leftrightarrow \epsilon_2$, which *would not be* acceptable. Therefore, the *doubling* of the Majorana spinor as the **2** of $Sp(1)$ is needed with the contraction by the anti-symmetric metric $\epsilon^{ab} = -\epsilon^{ba}$ like $\xi^\mu \equiv +2(\overline{\epsilon}_1\gamma^\mu\epsilon_2) = +2(\overline{\epsilon}_1{}^a\gamma^\mu\epsilon_{2a}) = +2\epsilon^{ab}(\overline{\epsilon}_{1b}\gamma^\mu\epsilon_{2a}) = +2\epsilon^{ab}(\overline{\epsilon}_{2a}\gamma^\mu\epsilon_{1b}) = -2\epsilon^{ba}(\overline{\epsilon}_{2a}\gamma^\mu\epsilon_{1b}) = -2(\overline{\epsilon}_2{}^b\gamma^\mu\epsilon_{1b}) = -2(\overline{\epsilon}_2\gamma^\mu\epsilon_1)$.

The Fierz-rearrangement formula for Majorana spinors in the **2** of $Sp(1)$ is given by

$$\begin{aligned}(\overline{\psi}\chi)(\overline{\rho}\omega) = &-\tfrac{1}{8}(\overline{\psi}\omega)(\overline{\rho}\chi) + \tfrac{1}{8}(\overline{\psi}\tau^i\omega)(\overline{\rho}\tau^i\chi) - \tfrac{1}{8}(\overline{\psi}\gamma_\mu\psi\omega)(\overline{\rho}\gamma^\mu\chi) + \tfrac{1}{8}(\overline{\psi}\gamma_\mu\tau^i\omega)(\overline{\rho}\gamma^\mu\tau^i\chi) \\ &+ \tfrac{1}{16}(\overline{\psi}\gamma_{\mu\nu}\omega)(\overline{\rho}\gamma^{\mu\nu}\chi) - \tfrac{1}{16}(\overline{\psi}\gamma_{\mu\nu}\tau^i\omega)(\overline{\rho}\gamma^{\mu\nu}\tau^i\chi) .\end{aligned} \tag{2.4}$$

As is seen from our self-dual TM in $D = 2+2$ [1], the tensor-hierarchy with the Proca-Stueckelberg-type field-strength [12] necessitates the fermions to be *massive*. As will be shown below, in the usual $D = 4+1$, such a massive fermion leads to a *tachyonic* field equation. However, since our space-time is $D = 3+2$, it does *not* pose the 'tachyon problem' for massive fields.

To clarify this, consider the conventional $D = 4+1$ case [6][7]. According to [18], the hermiticity requires the presence of the imaginary unit 'i' in front of some fermionic bilinears. This result follows from the general analysis in [18]:

$$(\overline{\psi}\gamma^{[n]}\chi)^\dagger = -(-1)^n(\overline{\psi}\gamma^{[n]}\chi), \tag{2.5}$$

in contrast to (2.3b). For example, $n = 0$ needs 'i', while $n = 1$ does *not* need it to make each of $i(\overline{\psi}\chi)$ and $(\overline{\psi}\gamma^\mu\chi)$ *hermitian* [6]. Accordingly, the right field-equation for a massive Majorana fermion in $D = 4+1$ needs the imaginary unit 'i' as the relative coefficient ratio between the kinetic and the mass term[6]:

$$\partial\!\!\!/\psi \doteq im\psi \quad\Longrightarrow\quad \partial_\mu^2\psi \doteq +\partial\!\!\!/(im\psi) \doteq (im)(im)\psi = -m^2\psi \quad\Longrightarrow\quad \partial_\mu^2\psi + m^2\psi \doteq 0 . \tag{2.6}$$

In $D = 4+1$, such a Klein-Gordon equation has a *tachyonic* mass term. In contrast in our $D = 3+2$, we do *not* need to worry about this for the two reasons: (i) The signature of the metric $(+,+,+,-,-)$ implies that tachyonic field-equation will *not* matter. This resembles $D = 2+2$. (ii) Since the presence of the *massive* field equation is *inevitable* for tensor-hierarchy formulations [19][20] *via* generalized Stueckelberg mechanisms [12], $D = 3+2$ instead of $D = 4+1$ is the right space-time, where tensor-hierarchy formulation is realized.

3. Self-dual non-Abelian tensor-multiplet in D = 3 + 2

After these preliminaries, we are now ready to present the formulation of our total system. First, there are three multiplets in our system: YMVM $(A_\mu{}^I, \lambda^I, \phi^I)$, TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ and EVM $(C_\mu{}^I, \rho^I, \sigma^I)$. Second, due to the doubling of fermions needed, supersymmetry parameter should be assigned in the **2** of $Sp(1)$. In this sense, our supersymmetry is extended $N = 2$ supersymmetry. Third, our supersymmetric duality-relationship is

$$H_{\mu\nu}{}^I \overset{*}{=} +\tfrac{1}{6}\,\epsilon_{\mu\nu}{}^{\rho\sigma\tau} G_{\rho\sigma\tau}{}^I - \tfrac{1}{2} f^{IJK}(\overline{\lambda}^J\gamma_{\mu\nu}\chi^K), \tag{3.1}$$

where the last term is required by $N = 2$ supersymmetry. Our $N = 2$ supersymmetry requires its super-partner conditions

[5] We use the symbol $\overset{\nabla}{=}$ for an equality up to a total divergence, while $\mathcal{O}(\Phi^n)$ for the n-th order term in fundamental fields symbolized by Φ.

[6] The symbol $\doteq$ stands for a field equation, or an equality that holds by the use of field equations, that is distinguished from usual algebraic equalities or field equations.

$$\varphi^I \overset{*}{=} -\sigma^I , \qquad \chi^I \overset{*}{=} -\rho^I . \tag{3.2}$$

Our $N = 2$ supersymmetry transformation rule is fixed as

$$\delta_Q A_\mu{}^I = +(\overline{\epsilon}\gamma_\mu\lambda^I) , \tag{3.3a}$$
$$\delta_Q \lambda^I = +\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^I - (\gamma^\mu\epsilon)D_\mu\phi^I , \tag{3.3b}$$
$$\delta_Q \phi^I = +(\overline{\epsilon}\lambda^I) , \tag{3.3c}$$
$$\delta_Q B_{\mu\nu}{}^I = +(\epsilon\gamma_{\mu\nu}\chi^I) - 2f^{IJK}(\overline{\epsilon}\gamma_{[\mu}\lambda^J)C_{\nu]}{}^K \tag{3.3d}$$
$$\begin{aligned}\delta_Q \chi^I = &+\tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon)G_{\mu\nu\rho}{}^I - (\gamma^\mu\epsilon)D_\mu\varphi^I + m\epsilon\varphi^I - mf^{IJK}\epsilon\varphi^J\phi^K \\ &+\tfrac{1}{8}f^{IJK}(\gamma_{\mu\nu}\epsilon)(\overline{\lambda}^J\gamma^{\mu\nu}\chi^K) + \tfrac{1}{4}(\tau^i\epsilon)(\overline{\lambda}^J\tau^i\chi^K) - \tfrac{1}{4}(\tau^i\gamma_\mu\epsilon)(\overline{\lambda}^J\gamma^\mu\tau^i\chi^K) ,\end{aligned} \tag{3.3e}$$
$$\delta_Q \varphi^I = +(\overline{\epsilon}\chi^I) , \tag{3.3f}$$
$$\delta_Q C_\mu{}^I = +(\overline{\epsilon}\gamma_\mu\rho^I) - f^{IJK}\phi^J(\overline{\epsilon}\gamma_\mu\rho^K) + f^{IJK}\sigma^J(\overline{\epsilon}\gamma_\mu\lambda^K) , \tag{3.3g}$$
$$\begin{aligned}\delta_Q \rho^I = &+\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)\Big[H_{\mu\nu}{}^I + \tfrac{1}{2}f^{IJK}(\overline{\lambda}^J\gamma_{\mu\nu}\chi^K)\Big] + (\gamma^\mu\epsilon)D_\mu\varphi^I + m\epsilon\varphi^I + mf^{IJK}\epsilon\varphi^J\phi^K \\ &- \tfrac{1}{8}f^{IJK}(\gamma_{\mu\nu}\epsilon)(\overline{\lambda}^J\gamma^{\mu\nu}\chi^K) - \tfrac{1}{4}(\tau^i\epsilon)(\overline{\lambda}^J\tau^i\chi^K) + \tfrac{1}{4}(\gamma_\mu\tau^i\epsilon)(\overline{\lambda}^J\gamma^\mu\tau^i\chi^K) ,\end{aligned} \tag{3.3h}$$
$$\delta_Q \sigma^I = +(\overline{\epsilon}\rho^I) . \tag{3.3i}$$

These equations are valid up to $\mathcal{O}(\Phi^3)$.[7]

The field-strengths F, G and H of the potentials A, B and C are defined by

$$F_{\mu\nu}{}^I \equiv +\partial_\mu A_\nu{}^I - \partial_\nu A_\mu{}^I + mf^{IJK}A_\mu{}^J A_\nu{}^K , \tag{3.4a}$$
$$G_{\mu\nu\rho}{}^I \equiv +3D_{[\mu}B_{\nu\rho]}{}^I + 3f^{IJK}F_{[\mu\nu}{}^J C_{\rho]}{}^K , \tag{3.4b}$$
$$H_{\mu\nu}{}^I \equiv +2D_{[\mu}C_{\nu]}{}^I + mB_{\mu\nu}{}^I + f^{IJK}\phi^J H_{\mu\nu}{}^K - f^{IJK}\sigma^J F_{\mu\nu}{}^K . \tag{3.4c}$$

Note that the 3rd and 4th terms in (3.4c) are *not* required by tensor-hierarchy consistency [19][20], but by supersymmetry. They satisfy their proper Bianchi identities

$$D_{[\mu}F_{\nu\rho]}{}^I \equiv 0 , \tag{3.5a}$$
$$D_{[\mu}G_{\nu\rho\sigma]}{}^I \equiv +\tfrac{3}{2}f^{IJK}F_{[\mu\nu}{}^J H_{\rho\sigma]}{}^K , \tag{3.5b}$$
$$D_{[\mu}H_{\nu\rho]}{}^I \equiv +\tfrac{1}{2}mG_{\mu\nu\rho}{}^I - f^{IJK}H_{\mu\nu}{}^J D_{\rho]}\phi^K + f^{IJK}F_{[\mu\nu}{}^J D_{\rho]}\sigma^K + \tfrac{1}{3}f^{IJK}\phi^J G_{\mu\nu\rho}{}^K . \tag{3.5c}$$

Eqs. (3.5b,c) are valid up to $\mathcal{O}(\Phi^3)$-terms. The general variations of the field strengths F, G and H are

$$\delta F_{\mu\nu}{}^I = +2D_{[\mu}(\delta A_{\nu]}{}^I) , \tag{3.6a}$$
$$\delta G_{\mu\nu\rho}{}^I = 3D_{[\mu}(\widetilde{\delta}B_{\nu\rho]}{}^I) - 3f^{IJK}(\widetilde{\delta}C_{[\mu}{}^J)F_{\nu\rho]}{}^K + 3f^{IJK}(\delta A_{[\mu}{}^J)H_{\nu\rho]}{}^K \tag{3.6b}$$
$$\begin{aligned}\delta H_{\mu\nu}{}^I = &2D_{[\mu}(\widetilde{\delta}C_{\nu]}{}^I) + m(\widetilde{\delta}B_{\mu\nu}{}^I) - mf^{IJK}(\widetilde{\delta}B_{\mu\nu}{}^J)\phi^K \\ &+ f^{IJK}(\delta\phi^J)H_{\mu\nu}{}^K - 2f^{IJK}(\widetilde{\delta}C_{[\mu}{}^J)D_{\nu]}\phi^K \\ &- f^{IJK}(\delta\sigma^J)F_{\mu\nu}{}^K + 2f^{IJK}(\delta A_{[\mu}{}^J)D_{\nu]}\sigma^K ,\end{aligned} \tag{3.6c}$$
$$\widetilde{\delta}B_{\mu\nu}{}^I \equiv +\delta B_{\mu\nu}{}^I + 2f^{IJK}(\delta A_{[\mu}{}^J)C_{\nu]}{}^K , \tag{3.6d}$$
$$\widetilde{\delta}C_\mu{}^I \equiv +\delta C_\mu{}^I + f^{IJK}\phi^J(\widetilde{\delta}C_\mu{}^K) - f^{IJK}\sigma^J(\delta A_\mu{}^K) . \tag{3.6e}$$

In addition to the supersymmetric dualities (3.1) and (3.2), the dynamics of our system is determined by the field equations

$$\not{D}\lambda^I + mf^{IJK}\lambda^J\phi^K \doteq 0 , \tag{3.7a}$$
$$\begin{aligned}&\not{D}\chi^I + m\chi^I - \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\chi^J)F_{\mu\nu}{}^K + \tfrac{1}{2}f^{IJK}(\gamma^\mu\chi^J)D_\mu\phi^K - \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\lambda^J)H_{\mu\nu}{}^K \\ &\quad - \tfrac{1}{2}f^{IJK}(\gamma^\mu\lambda^J)D_\mu\varphi^K + \tfrac{1}{2}mf^{IJK}\lambda^J\varphi^K - mf^{IJK}\chi^J\phi^K \doteq 0 ,\end{aligned} \tag{3.7b}$$
$$D_\nu F_\mu{}^{\nu I} - mf^{IJK}\phi^J D_\mu\phi^K + \tfrac{1}{2}mf^{IJK}(\overline{\lambda}^J\gamma_\mu\lambda^K) \doteq 0 , \tag{3.7c}$$
$$D_\mu^2\phi^I - \tfrac{1}{2}mf^{IJK}(\overline{\lambda}^J\lambda^K) \doteq 0 , \tag{3.7d}$$
$$\begin{aligned}&D_\mu^2\varphi - m^2\varphi^I - \tfrac{1}{2}f^{IJK}F_{\mu\nu}{}^J H^{\mu\nu K} - f^{IJK}(D_\mu\phi^J)(D^\mu\varphi^K) \\ &\quad - mf^{IJK}(\overline{\lambda}^J\chi^K) - 2m^2f^{IJK}\phi^J\varphi^K \doteq 0 .\end{aligned} \tag{3.7e}$$

[7] Similarly, our field equations *e.g.*, (3.7) will be also valid up to $\mathcal{O}(\Phi^3)-$terms.

$$D_\rho G_{\mu\nu}{}^{\rho I} - mH_{\mu\nu}{}^I - mf^{IJK}(\overline{\lambda}^J\gamma_{\mu\nu}\chi^K)$$
$$- f^{IJK}G_{\mu\nu}{}^{\rho J}D_\rho\phi^K - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\tau}F_{\rho\sigma}{}^J D_\tau\varphi^K + f^{IJK}\phi^J H_{\mu\nu}{}^K$$
$$- f^{IJK}(\overline{\lambda}^J\gamma_{[\mu}D_{\nu]}\chi^K) + f^{IJK}(\overline{\chi}^J\gamma_{[\mu}D_{\nu]}\lambda^K) \doteq 0 \,, \tag{3.7f}$$
$$D_\nu H_\mu{}^{\nu I} - \frac{1}{2}f^{IJK}F_{\nu\rho}{}^J G_\mu{}^{\nu\rho K} - \frac{1}{2}mf^{IJK}(\overline{\lambda}^J\gamma_\mu\chi^K)$$
$$+ \frac{1}{2}f^{IJK}(\overline{\chi}^J D_\mu\lambda^K) - \frac{1}{2}f^{IJK}(\overline{\lambda}^J D_\mu\chi^K) \doteq 0 \,. \tag{3.7g}$$

These field equations are valid up to $\mathcal{O}(\Phi^3)$. Due to (3.2), *i.e.* the proportionalities $\sigma^I \doteq -\varphi^I$ as well as $\rho^I \doteq -\chi^I$, we skip the field equations of σ^I and ρ^I. Eq. (3.7f) and (3.7g) are the corollaries of the duality (3.1) by taking its divergences.

The mutual consistency of these field equations under supersymmetry can be confirmed as usual by their variations under supersymmetry. First, the mutual consistency between λ^I, $A_\mu{}^I$ and ϕ^I-field equations (3.7a,c,d) is easy to confirm, *e.g.*,

$$\delta_Q\left[+\not{D}\lambda^I + mf^{IJK}\lambda^J\phi^K\right]$$
$$= +(\gamma^\mu\epsilon)\left[-D_\nu F_\mu{}^{\nu I} - \frac{1}{2}mf^{IJK}(\overline{\lambda}^J\gamma_\mu\lambda^K) + mf^{IJK}\phi^J D_\mu\phi^K\right]$$
$$+\epsilon\left[-D_\mu^2\phi^I + \frac{1}{2}mf^{IJK}(\overline{\lambda}^J\lambda^K)\right] \doteq 0 \,, \tag{3.8}$$

upon the use of (3.7c,d). We can also take δ_Q of the $A_\mu{}^I$ and ϕ^I-field equations, whose details are skipped here.

Second, the χ-field equation (3.7b) is obtained by the δ_Q-variation of the duality relationship (3.1):

$$\delta_Q\left[+G_{\mu\nu\rho}{}^I - \frac{1}{2}\epsilon_{\mu\nu\rho}{}^{\sigma\tau}H_{\sigma\tau}{}^I + \frac{1}{2}f^{IJK}(\overline{\lambda}^J\gamma_{\mu\nu\rho}\chi^K)\right]$$
$$\overset{*}{=} +\overline{\epsilon}\gamma_{\mu\nu\rho}\Big[\not{D}\chi^I + m\chi^I - \frac{1}{4}f^{IJK}\gamma^{\sigma\tau}\chi^J F_{\sigma\tau}{}^K + \frac{1}{2}f^{IJK}(\gamma^\sigma\chi^J)D_\sigma\phi^K - \frac{1}{4}f^{IJK}(\gamma^{\sigma\tau}\lambda^J)H_{\sigma\tau}{}^K$$
$$- \frac{1}{2}f^{IJK}(\gamma^\sigma\chi^J)D_\sigma\varphi^K + \frac{1}{2}mf^{IJK}\lambda^J\varphi^K - mf^{IJK}\chi^J\phi^K\Big] \doteq 0 \quad (Q.E.D.) \tag{3.9}$$

As the 1st equality $\overset{*}{=}$ shows, we have used the duality (3.1), such as re-expressing the field-strength G by H. The last equality $\doteq$ holds upon the use of the χ-field equation (3.7b).

Third, the φ^I-field equation (3.7e) is obtained by the δ_Q-variation of the χ-field equation (3.7b):

$$\delta_Q\Big[+\not{D}\chi^I + m\chi^I - \frac{1}{4}f^{IJK}(\gamma^{\mu\nu}\chi^J)F_{\mu\nu}{}^K + \frac{1}{2}f^{IJK}(\gamma^\mu\chi^J)D_\mu\phi^K - \frac{1}{4}f^{IJK}(\gamma^{\mu\nu}\lambda^J)H_{\mu\nu}{}^K$$
$$- \frac{1}{2}f^{IJK}(\gamma^\mu\lambda^J)D_\mu\varphi^K + \frac{1}{2}mf^{IJK}\lambda^J\varphi^K - mf^{IJK}\chi^J\phi^K\Big]$$
$$\overset{*}{=} -\epsilon\Big[+D_\mu^2\varphi^I - m^2\varphi^I - \frac{1}{2}f^{IJK}F_{\mu\nu}{}^J H^{\mu\nu K}$$
$$- f^{IJK}(D_\mu\phi^J)(D^\mu\varphi^K) - 2m^2 f^{IJK}\phi^J\varphi^K - mf^{IJK}(\overline{\lambda}^J\chi^K)\Big] \doteq 0 \,. \tag{3.10}$$

In the 1st equality $\overset{*}{=}$, we have used only the duality (3.1), but *not* field equations in (3.7).

As in the general tensor-hierarchy formulations [19][20], the tensor B and the vector C have their proper tensorial transformations:

$$\delta_\beta B_{\mu\nu}{}^I = 2D_{[\mu}\beta_{\nu]}{}^I \,, \qquad \delta_\beta C_\mu{}^I = -m\beta_\mu{}^I \,, \tag{3.11a}$$
$$\delta_\gamma B_{\mu\nu}{}^I = +f^{IJK}\gamma^J F_{\mu\nu}{}^K \,, \qquad \delta_\gamma C_\mu{}^I = +D_\mu\gamma^I \,. \tag{3.11b}$$

It is straightforward to show the invariances of the G and H-field strengths: $\delta_\beta G_{\mu\nu\rho}{}^I = \delta_\beta H_{\mu\nu}{}^I = 0$, $\delta_\gamma G_{\mu\nu\rho}{}^I = \delta_\gamma H_{\mu\nu}{}^I = 0$.

As an additional confirmation, we give the closure of two supersymmetries:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_\xi + \delta_\alpha + \delta_\beta + \delta_\gamma + \delta_\zeta \,, \tag{3.12}$$

where δ_ξ and δ_α are respectively the translation and YM-gauge transformations. The δ_β, δ_γ and global scale transformation δ_ζ have the parameters:

$$\xi^\mu \equiv +2(\overline{\epsilon}_1\gamma^\mu\epsilon_2) \,, \qquad \alpha^I \equiv -\xi^\mu A_\mu{}^I \,, \tag{3.13a}$$
$$\beta_\mu{}^I \equiv -\xi^\nu B_{\nu\mu}{}^I - \xi_\mu\varphi^I + \zeta C_\mu{}^I \,, \qquad \gamma^I \equiv -\xi^\mu C_\mu{}^I - 2\zeta\varphi^I \,, \qquad \zeta \equiv +(\overline{\epsilon}_2\epsilon_1) \,, \tag{3.13b}$$
$$\delta_\xi Z^I \equiv +\xi^\mu\partial_\mu Z^I \,, \quad \delta_\alpha Z'^I = -mf^{IJK}\alpha^J Z'^K \,, \quad \delta_\alpha A_\mu{}^I = D_\mu\alpha^I \,, \quad \delta_\zeta Y^I = +m\zeta Y^I \,, \tag{3.13c}$$

where Z^I is *any* field in our system with the adjoint index I, Z'^I is any field except for $A_\mu{}^I$, while $Y^I \in \{B_{\mu\nu}{}^I, \chi^I, \varphi^I, C_\mu{}^I, \rho^I, \sigma^I\}$. The δ_β and δ_γ are the same as in (3.11). The YMVM fields are all intact under δ_ζ.

Note the existence of the intrinsic global scale transformations for our fields Y^I, which is a new feature of our present system that has *not* been encountered in the past in tensor-hierarchy formulations [19][20].

As additional important confirmation, we can see the divergences of (3.7f):

$$\begin{aligned}
0 \overset{?}{=} & D_\nu \Big[D_\rho G^{\mu\nu\rho\, I} - m H^{\mu\nu\, I} - m f^{IJK} (\overline{\lambda}^J \gamma^{\mu\nu} \chi^K) \\
& - f^{IJK} G^{\mu\nu\rho\, J} D_\rho \phi^K - \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma\tau} F_{\rho\sigma}{}^J D_\tau \varphi^K + f^{IJK} \phi^J H^{\mu\nu\, K} \\
& - f^{IJK} (\overline{\lambda}^J \gamma^{[\mu} D^{\nu]} \chi^K) + f^{IJK} (\overline{\chi}^J \gamma^{[\mu} D^{\nu]} \lambda^K) \Big] \\
\doteq & + \Big(\tfrac{1}{2} - 1 + \tfrac{1}{2} \Big) m f^{IJK} (\overline{\chi}^J D_\mu \lambda^K) + \Big(-\tfrac{1}{2} + 1 - \tfrac{1}{2} \Big) m f^{IJK} (\overline{\lambda}^J D_\mu \chi) \\
& + \Big(-\tfrac{1}{2} + 1 - \tfrac{1}{2} \Big) m^2 f^{IJK} (\overline{\lambda}^J \gamma_\mu \chi^K) = 0 \quad (Q.E.D.)
\end{aligned} \tag{3.14}$$

Here we have used (3.7g) and fermionic field equations (3.7a) and (3.7b). This result is also valid up to $\mathcal{O}(\Phi^3)$.

We have so far fixed our field equations (3.7) and our transformation rule (3.3) up to $\mathcal{O}(\Phi^3)$-terms. This principle is based on our past experience that any supersymmetric theory with lowest-order interactions will *not* encounter inconsistency at higher-orders. Also, fixing higher-order terms is beyond the scope of this Letter.

Before closing this section, we mention a natural question whether our present theory in $D = 3 + 2$ comes from higher-dimensional theories, such as in $D = 3 + 3$ [5]. Judged by the certain similarities of the field contents of these two theories, we *conjecture* that our $D = 3 + 3$ [5] will yield our present $D = 3 + 2$ theory by certain dimensional reductions. We have *not* performed its detailed computation, which is beyond the scope of this paper.

4. Reduction to self-dual TM in D = 2 + 2

The original motivation of our present theory is to formulate a self-dual theory in $D = 3 + 2$ as the possible fundamental 'Master Theory' of $D = 2 + 2$ self-dual theories [2][3][1]. To this end, it should be demonstrated that the self-dual tensor multiplet given in [1] is generated by dimensional-reduction and truncation. The resulting theory in $D = (2, 2)$ in this section will be what is called "Theory-I" in [5].[8]

Here we take advantage of the similarity between the tensor multiplets in $D = 3 + 2$ and $D = 2 + 2$. For example, the latter field contents: YMVM $(A_\mu{}^I, \lambda_-{}^I)$, TM $(K_{\mu\nu}{}^I, \zeta_+{}^I, M^I)$ and EVM $(N_\mu{}^I, \eta_-{}^I)$ are parallel to the former.[9]

The original supersymmetry transformation rule in $D = 3 + 2$ is[10]

$$\delta_Q \widehat{A}_{\hat\mu}{}^I = + (\overline{\widehat\epsilon} \widehat\gamma_{\hat\mu} \widehat\lambda^I) , \tag{4.1a}$$

$$\delta_Q \widehat\lambda^I = + \tfrac{1}{2} (\widehat\gamma^{\hat\mu\hat\nu} \widehat\epsilon) \widehat{F}_{\hat\mu\hat\nu}{}^I - (\widehat\gamma^{\hat\mu} \widehat\epsilon) \widehat{D}_{\hat\mu} \widehat\phi^I , \tag{4.1b}$$

$$\delta_Q \widehat\phi^I = + (\overline{\widehat\epsilon} \widehat\lambda^I) , \tag{4.1c}$$

$$\delta_Q \widehat{B}_{\hat\mu\hat\nu}{}^I = + (\widehat\epsilon \gamma_{\hat\mu\hat\nu} \widehat\chi^I) - 2 f^{IJK} (\overline{\widehat\epsilon} \widehat\gamma_{[\hat\mu} \widehat\lambda^J) \widehat{C}_{\hat\nu]}{}^K \tag{4.1d}$$

$$\begin{aligned}
\delta_Q \widehat\chi^I = & + \tfrac{1}{6} (\widehat\gamma^{\hat\mu\hat\nu\hat\rho} \widehat\epsilon) \widehat{G}_{\hat\mu\hat\nu\hat\rho}{}^I - (\widehat\gamma^{\hat\mu} \widehat\epsilon) \widehat{D}_{\hat\mu} \widehat\varphi^I + m \widehat\epsilon \widehat\varphi^I - m f^{IJK} \widehat\epsilon \widehat\varphi^J \widehat\phi^K \\
& + \tfrac{1}{8} f^{IJK} (\widehat\gamma_{\hat\mu\hat\nu} \widehat\epsilon) (\overline{\widehat\lambda}^J \widehat\gamma^{\hat\mu\hat\nu} \widehat\chi^K) + \tfrac{1}{4} (\tau^i \widehat\epsilon) (\overline{\widehat\lambda}^J \tau^i \widehat\chi^K) - \tfrac{1}{4} (\tau^i \widehat\gamma_{\hat\mu} \widehat\epsilon) (\overline{\widehat\lambda}^J \widehat\gamma^{\hat\mu} \tau^i \widehat\chi^K) ,
\end{aligned} \tag{4.1e}$$

$$\delta_Q \widehat\varphi^I = + (\overline{\widehat\epsilon} \widehat\chi^I) , \tag{4.1f}$$

$$\delta_Q \widehat{C}_{\hat\mu}{}^I = + (\overline{\widehat\epsilon} \widehat\gamma_{\hat\mu} \widehat\rho^I) - f^{IJK} \widehat\phi^J (\overline{\widehat\epsilon} \widehat\gamma_{\hat\mu} \widehat\rho^K) + f^{IJK} \widehat\sigma^J (\overline{\widehat\epsilon} \widehat\gamma_{\hat\mu} \widehat\lambda^K) . \tag{4.1g}$$

We have skipped $\delta_Q \widehat\rho^I$ and $\delta_Q \widehat\sigma^I$, because of $\widehat\rho^I \doteq - \widehat\chi^I$ and $\widehat\sigma^I \doteq - \widehat\varphi^I$.

We fix the signatures of our $D = 3 + 2$ metric as

$$d\widehat{s}^2 = +(dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2 + (dy)^2 , \tag{4.2}$$

regarding the $y \equiv x^5$-coordinate as the extra dimension from the $D = 2 + 2$ viewpoint. Our resulting field content YMVM, TM and EVM [1][5] in $D = 2 + 2$ in our dimensional-reduction rule are

$$\widehat{A}_{\hat\mu}{}^I = \begin{cases} \widehat{A}_\mu{}^I = A_\mu{}^I & (\hat\mu = \mu) , \\ \widehat{A}_5{}^I = 0 & (\hat\mu = 5) , \end{cases} \tag{4.3a}$$

$$\widehat{B}_{\hat\mu\hat\nu}{}^I = \begin{cases} \widehat{B}_{\mu\nu}{}^I = K_{\mu\nu}{}^I & (\hat\mu = \mu; \hat\nu = \nu) , \\ \widehat{B}_{\mu 5}{}^I = N_\mu{}^I & (\hat\mu = \mu; \hat\nu = 5) , \end{cases} \tag{4.3b}$$

$$\widehat{C}_{\hat\mu}{}^I = \begin{cases} \widehat{C}_\mu{}^I = N_\mu{}^I & (\hat\mu = \mu) , \\ \widehat{C}_5{}^I = -M^I & (\hat\mu = 5) , \end{cases} \tag{4.3c}$$

$$\widehat\phi^I = 0 , \qquad \widehat\varphi^I = -\widehat\sigma^I = M^I , \tag{4.3d}$$

[8] We conjecture that "Theory-II" will be also obtained, whose confirmation is beyond the scope of this paper.

[9] Here the subscripts $\pm$ are for the chiralities in $D = 2 + 2$. The fields $K_{\mu\nu}{}^I, \zeta_+{}^I, M^I, N_\mu{}^I$, and $\eta_-{}^I$ respectively correspond to $B_{\mu\nu}{}^I, \chi_+{}^I, \varphi^I, C_\mu{}^I$, and $\rho_-{}^I$ in [1].

[10] Only in this section, we use all fields and indices in $D = 3 + 2$ with *hats*, in order to distinguish them from $D = 2 + 2$ quantities *without hats*. Our convention in $D = 2 + 2$ is such as $(1/n!) \epsilon_{\mu_1 \cdots \mu_{4-n} [n]} \epsilon^{[n] \nu_1 \cdots \nu_{4-n}} = +(-1)^n (4 - n)! \delta_{[\mu_1}{}^{\nu_1} \cdots \delta_{\mu_{4-n}]}{}^{\nu_{4-n}}$, and $(1/n!) \epsilon_{\mu_1 \cdots \mu_{4-n}}{}^{[n]} \gamma_{[n]} = -(-1)^{(n-1)(n-2)/2} \gamma_5 \gamma_{[n]}$.

$$\widehat{\lambda}_a{}^I = \begin{cases} \widehat{\lambda}_{-1}{}^I = +\lambda_-{}^I \, , & \widehat{\lambda}_{+1}{}^I = 0 \quad (a=1) \, , \\ \widehat{\lambda}_{-2}{}^I = +\lambda_-{}^I \, , & \widehat{\lambda}_{+2}{}^I = 0 \quad (a=2) \, , \end{cases} \tag{4.3e}$$

$$\widehat{\chi}_a{}^I = \begin{cases} \widehat{\chi}_{+1}{}^I = +\zeta_+{}^I \, , & \widehat{\chi}_{-1}{}^I = -\eta_-{}^I - f^{IJK}\lambda_-{}^J M^K \quad (a=1) \, , \\ \widehat{\chi}_{+2}{}^I = +\zeta_+{}^I \, , & \widehat{\chi}_{-2}{}^I = -\eta_-{}^I - f^{IJK}\lambda_-{}^J M^K \quad (a=2) \, , \end{cases} \tag{4.3f}$$

$$\widehat{\rho}_a{}^I = - \, \widehat{\chi}_a{}^I \, , \tag{4.3g}$$

$$\widehat{\epsilon}_a = \begin{cases} \epsilon_{-1} = +\frac{1}{2}\epsilon_- \, , & \epsilon_{+1} = -\frac{1}{2}\epsilon_+ \quad (a=1) \, , \\ \epsilon_{-2} = +\frac{1}{2}\epsilon_- \, , & \epsilon_{+2} = +\frac{1}{2}\epsilon_+ \quad (a=2) \, . \end{cases} \tag{4.3h}$$

Each of $\lambda_-{}^I, \zeta_+{}^I, \eta_-{}^I$ and $\epsilon_\pm$ is Majorana-Weyl spinor in $D = 2+2$ [2][3] with its indicated chirality. We also require that

$$\partial_y \widehat{X}^I = 0 \, , \qquad \partial_y \widehat{Y}^I = +m\widehat{Y}^I \, , \tag{4.4}$$

where $\widehat{X}^I \in \{\widehat{A}_{\hat{\mu}}{}^I, \widehat{\lambda}^I, \widehat{\phi}^I\}$ and $\widehat{Y}^I \in \{\widehat{B}_{\hat{\mu}\hat{\nu}}{}^I, \widehat{\chi}^I, \widehat{\varphi}^I, \widehat{C}_{\hat{\mu}}{}^I, \widehat{\rho}^I, \widehat{\sigma}^I\}$.

This reduction-rule looks unconventional compared with generalized dimensional reduction [21], but its validity is supported by the following three viewpoints: First, eq. (4.4) is regarded as the dimensional reduction rule for non-compact isometry for the y-coordinate, corresponding to the δ_ζ in (3.13). To be more specific, we can assign the extra factor e^{my} to each of the fields in the TM and EVM, like $\widehat{B}_{\hat{\mu}\hat{\nu}}{}^I(x, y) = e^{my} B_{\hat{\mu}\hat{\nu}}{}^I(x)$, *etc.* Second, in the usual reduction only for one extra dimension, we allow only compact isometry with e^{imy} instead of the *non-compact* function e^{my}. This is because we have to exclude *tachyonic* mass-terms in the resulting lower-dimensional theory. Since our resulting theory is in $D = 2+2$, where *tachyonic* mass-term is *harmless*, we can allow such a function e^{my}. Third, as long as such a configuration with e^{my} satisfies the original field equations in $D = 3+2$, there should be *no* inconsistency with this prescription, as can be easily confirmed.

We also add the motive for such an unconventional dependence as (4.4): An intuitive reasoning is that our fermion χ^I in the TM $D = 3+2$ is essentially *massive*, as its field equation (3.7) shows, while in $D = 2+2$, the fermion $\zeta_+{}^I$ in the TM is *massless*, as its field equation (4.7c) below shows. This implies that the original mass-term for χ^I in $D = 3+2$ should be canceled to get the *massless* $\zeta_+{}^I$-field equation. In order to cancel such an unwanted mass-term by the γ_5-linear term in $\widehat{\not{D}}\widehat{\chi}^I = \not{D}\widehat{\chi}^I + \gamma^5\widehat{\partial}_y\widehat{\chi}^I$, we need the special y-dependence (4.4). Technical details will be given after (4.8).

Note that half of original minimal 8 supersymmetries in $D = 3+2$ is halved into 4 supersymmetries in $D = 2+2$ by our dimensional reduction rule such as (4.3h). These remaining 4 supersymmetries correspond to $N = (1, 1)$ supersymmetries in terms of Majorana-Weyl spinors with the two parameters ϵ_+ and ϵ_-.

After applying these to (4.1), we get the final $N = 1$ supersymmetry transformation rule in $D = 2+2$ [1]:

$$\delta_Q A_\mu{}^I = +(\overline{\epsilon}_+\gamma_\mu\lambda_-{}^I) \, , \tag{4.5a}$$

$$\delta_Q \lambda_-{}^I = +\frac{1}{4}(\gamma^{\mu\nu}\epsilon_-)F_{\mu\nu}{}^I \, , \tag{4.5b}$$

$$\delta_Q K_{\mu\nu}{}^I = +(\epsilon_+\gamma_{\mu\nu}\zeta_+{}^I) - 2f^{IJK}(\overline{\epsilon_+}\gamma_{[\mu|}\lambda_-{}^J)N_{|\nu]}{}^K \tag{4.5d}$$

$$\delta_Q \zeta_+{}^I = +\frac{1}{12}(\gamma^{\mu\nu\rho}\epsilon_-)L_{\mu\nu\rho}{}^I - \frac{1}{2}(\gamma^\mu\epsilon_-)D_\mu M^I + f^{IJK}e_+(\overline{\lambda}_-{}^J\eta_-{}^K) \, , \tag{4.5e}$$

$$\delta_Q M^I = +(\overline{\epsilon}_+\zeta_+{}^I) \, , \tag{4.5f}$$

$$\delta_Q N_\mu{}^I = +(\overline{\epsilon}_+\gamma_\mu\eta_-{}^I) \, , \tag{4.5g}$$

$$\delta_Q \eta_-{}^I = +\frac{1}{4}(\gamma^{\mu\nu}\epsilon_-)P_{\mu\nu}{}^I - m\epsilon_- M^I \, . \tag{4.5h}$$

These are nothing but the transformation rule in $D = 2+2$ given in [1] up to unessential re-scalings of fields. The field-strengths L and P are defined by [1][5]

$$L_{\mu\nu\rho}{}^I \equiv +3D_{[\mu}K_{\nu\rho]}{}^I - 3f^{IJK}N_{[\mu}{}^J F_{\nu\rho]}{}^K \, , \qquad P_{\mu\nu}{}^I \equiv +2D_{[\mu}N_{\nu]}{}^I + mK_{\mu\nu}{}^I \, . \tag{4.6}$$

Similarly, the dimensional-reduction rule (4.3) applied to our field equations (3.7) yields the field equations in $D = 2+2$:

$$\not{D}\lambda_-{}^I \doteq 0 \, , \tag{4.7a}$$

$$\not{D}\eta_-{}^I - 2m\zeta_+{}^I \doteq 0 \, , \tag{4.7b}$$

$$\not{D}\zeta_+{}^I - \frac{1}{4}f^{IJK}(\gamma^{\mu\nu}\lambda_-{}^J)P_{\mu\nu}{}^K + \frac{1}{4}f^{IJK}(\gamma^{\mu\nu}\eta_-{}^J)F_{\mu\nu}{}^K + mf^{IJK}\lambda_-{}^J M^K \doteq 0 \, , \tag{4.7c}$$

$$F_{\mu\nu}{}^I \overset{*}{=} +\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^I \, , \tag{4.7d}$$

$$L_{\mu\nu\rho}{}^I \overset{*}{=} -\epsilon_{\mu\nu\rho}{}^\sigma D_\sigma M^I \, , \qquad D_\mu M^I \overset{*}{=} +\frac{1}{6}\epsilon_\mu{}^{\rho\sigma\tau}L_{\rho\sigma\tau}{}^I \, , \tag{4.7e}$$

$$P_{\mu\nu}{}^I \overset{*}{=} +\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}P_{\rho\sigma}{}^I \, , \tag{4.7f}$$

$$D_\mu^2 M^I \doteq +\frac{1}{2}f^{IJK}F_{\mu\nu}{}^J H^{\mu\nu K} \, . \tag{4.7g}$$

These agree with the field equations in $D = 2 + 2$ [1][5], as desired.

To demonstrate computations in our dimensional-reduction, we show the example of $\delta_Q \zeta_+{}^I$:

$$\begin{aligned}\delta_Q \zeta_+{}^I =& \delta_Q \chi_{+1}{}^I = + \frac{1}{12}(\gamma^{\mu\nu\rho}\epsilon_-)L_{\mu\nu\rho}{}^I + \left[-\tfrac{1}{2}(\gamma^\mu \epsilon_-)D_\mu M^I + \cancel{\tfrac{1}{2} m\epsilon_+ M^I} \right] - \cancel{\tfrac{1}{2} m\epsilon_+ M^I} \\ &+ \frac{1}{8}\left[-4 f^{IJK}\epsilon_+(\overline{\lambda}_-{}^J \eta_-{}^K) \right] + \frac{1}{8}\left[-4 f^{IJK}\epsilon_+(\overline{\lambda}_-{}^J \eta_-{}^K) \right] \qquad (4.8a)\\ =& + \frac{1}{12}(\gamma^{\mu\nu\rho}\epsilon_-)L_{\mu\nu\rho}{}^I - \tfrac{1}{2}(\gamma^\nu\epsilon_-)D_\mu M^I - f^{IJK}\epsilon_+(\overline{\lambda}_-{}^J\eta_-{}^K)\ , \qquad (4.8b)\end{aligned}$$

in agreement with (4.5e). In (4.8a), in the cancellation of two $m\epsilon_+ M^I$-terms, the first one is from $(\gamma^5\epsilon_+)\partial_5 \widehat{\varphi} = +m\epsilon_+ M^I$ that cancels the original $m\epsilon_+ M^I$-term, yielding *no* linear $m\epsilon_+ M^I$-term in $\delta_Q \zeta_+{}^I$.

As we promised, the technical reason for the special y-dependence (4.4) is elucidated by the following example. If we apply our dimensional-reduction rule (4.3) and (4.4) to the negative-chirality component of the $\widehat{\chi}^I$-field equation (3.7b), we get

$$\begin{aligned}0 \overset{?}{=}& \Big[+ (\widehat{\not{\partial}} \widehat{\chi}_1{}^I)_{\_} + m \widehat{\chi}_{\_1}{}^I - \tfrac{1}{4} f^{IJK}(\widehat{\gamma}^{\hat\mu\hat\nu}\widehat{\chi}_1{}^J)_{\_}\widehat{F}_{\hat\mu\hat\nu}{}^K - \tfrac{1}{4} f^{IJK}(\widehat{\gamma}^{\hat\mu\hat\nu}\widehat{\lambda}_1{}^J)_{\_}\widehat{H}_{\hat\mu\hat\nu}{}^K \\ &- \tfrac{1}{2} f^{IJK}(\widehat{\gamma}^{\hat\mu}\widehat{\lambda}_1{}^J)_{\_}\widehat{D}_{\hat\mu}\widehat{\varphi}^K + \tfrac{1}{2} m f^{IJK}\widehat{\lambda}_{-1}{}^J\widehat{\varphi}^K - m f^{IJK}\widehat{\chi}_{-1}{}^J\widehat{\phi}^K \Big] + (1\to 2) \qquad (4.9a)\\ =& (+\not{D}\zeta_+{}^I - \cancel{m\eta_-{}^I}) + \cancel{m\eta_-{}^I} + \tfrac{1}{4} f^{IJK}(\gamma^{\mu\nu}\eta_-{}^J)F_{\mu\nu}{}^J \\ &- \tfrac{1}{4} f^{IJK}(\gamma^{\mu\nu}\lambda_-{}^J)P_{\mu\nu}{}^J + m f^{IJK}\lambda_-{}^J M^K \doteq 0 \quad (Q.E.D.). \qquad (4.9b)\end{aligned}$$

Here the subscript $_{\_}$ is for the negative-chirality in $D = 2 + 2$. In (4.9a), "$+(1\to 2)$" implies that the terms with the subscript $_{a=1}$ in the preceding brackets replaced by $_{a=2}$ should be added. The first $-m\eta_-{}^I$-term in (4.9b) is from $\gamma^5\partial_5\chi_{\_1}{}^I$, which cancels the original $+m\eta_{\_}{}^I$-term from the $\widehat{\chi}^I$-field equation.

Even though we do *not* give all details, there are many other consistency confirmations. For example, we can show that $\delta_Q(\chi_{\_1}{}^I - \chi_{\_2}{}^I) = 0$:

$$0 \overset{?}{=} \delta_Q(\chi_{\_1}{}^I - \chi_{\_2}{}^I) = -\frac{1}{12}(\gamma^{\mu\nu\rho}\epsilon_+)L_{\mu\nu\rho}{}^I + \tfrac{1}{2}(\gamma^\mu\epsilon_+)D_\mu M^I \overset{*}{=} 0\ . \qquad (4.10)$$

In the last equality, the duality relation (4.7e) has been used.

As the last remark, we mention what the most-important self-duality relations (3.1) and (3.2) yield after our dimensional reduction (4.3). First, the $[\mu\nu]$-component of (3.1) produces the self-duality (4.7f) for $P_{\mu\nu}{}^I$. Note that the FM-terms cancel between left and right sides, due to the F-duality (4.7d). Second, the $[5\mu]$-component of (3.1) yields the duality (4.7e) between L and DM. Note that the fermionic quadratic term disappears because of (4.3e) and (4.3f). As for (3.2), $\widehat{\varphi}^I \overset{*}{=} -\widehat{\sigma}^I$ has been already identified as the M^I-field in (4.3d), while $\widehat{\chi}^I \overset{*}{=} -\widehat{\rho}^I$ has been also given in (4.3g).

5. Summary and concluding remarks

In this Letter, we have established the $N = 2$ supersymmetric system of odd-dimensional self-duality between a TM and an EVM in $D = 3 + 2$. Our formulation is nothing but the generalization of Abelian 'odd-dimensional self-duality' [10] to $D = 3 + 2$.

We accomplished this formulation by combining three important concepts: (i) The odd-dimensional self-duality [10], (ii) The so-called tensor-hierarchy [19][20], and (iii) $N = 2$ Supersymmetry in $D = 3 + 2$. To our knowledge, there has been *no* formulation that realized both the 'non-Abelianization' and supersymmetrization of odd-dimensional self-duality [10] in 5D.

Our objective has been to establish a higher-dimensional 'Master Theory' more fundamental than self-dual supersymmetric YM theories in $D = 2 + 2$ [2][3][1], which in turn generate integrable systems in $1 \le D \le 3$, following the Atiyah's conjecture for purely-bosonic case [4]. In the present paper, we have achieved this objective by reaching our self-dual TM in $D = 2 + 2$ [1] upon a dimensional-reduction. In other words, our present $D = 3 + 2$ theory *does* serve as the 'Master Theory' for the self-dual TM theory in $D = 2 + 2$ [1]. Since the self-dual super-YM multiplet in $D = 2 + 2$ itself is already generated by our new theory in $D = 3 + 2$, it is established that our system also generates supersymmetric integrable systems in $1 \le D \le 3$. In other words, we are finding more and more fundamental theories for self-dual super-YM theories in $D = 2 + 2$ [2][3] in higher dimensions.

Our dimensional-reduction in section 4 is highly non-trivial, because the process should be consistent with supersymmetry both in $D = 3 + 2$ and $D = 2 + 2$. If the total system *consisted* only of *purely-bosonic* fields, the process *would be* much simpler. The dimensional-reduction rule in section 4 is consistent both with tensor-hierarchy formulation [19][20] and $N = 2$ supersymmetry.

As a by-product, we have also established the existence of global scale invariance (3.13c) inherent in our system, as the commutator of two supersymmetries yields. Our global scale symmetry acts on all fields in the TM and EVM with the common scaling-weight. To our knowledge, there was *no* such multiplets presented in the past with supersymmetric non-Abelian tensor-formulation.

Our dimensional-reduction rule is unconventional, because it does *not* follow the traditional dimensional-reduction, such as the simple dimensional reduction [21], but is based on *non-compact* isometry. The isometry is associated with the scale covariance (3.13c) of the fields in TM and EVM. The two time-coordinates in $D = 2 + 2$ also justifies the exponential factor e^{my} in (4.4) instead of the usual compact one e^{imy}, because we do *not* have to worry about possible tachyons in $D = 2 + 2$. Our non-trivial reduction rule also shows that our $D = 3 + 2$ theory can be regarded as the fundamental 'Master-Theory' in $D = 3 + 2$ generating the self-dual TM system in $D = 2 + 2$. Such a relationship has *never* been presented in the past.

We mention that our formulation is the *action-less* system, as explained in the Introduction. As described with (1.3), the main reason is that the possible lagrangian term like $\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}{}^I G_{\rho\sigma\tau}{}^I$ turns out to be a surface-term. Additionally, as was described at the end of section 2, an additional reason is that any mass-term of a Majorana-spinor is identically vanishing in $D = 3 + 2$, due to (2.3a). This gives a new motivation of studying analogous system in higher odd-dimensions.

Because of the signature $(+, +, +, −, −)$ of the metric, there arises *no* problem with tachyons, just as in the case of self-dual theories in $D = 2 + 2$ [2][3]. This situation is different from the conventional $D = 4 + 1$ case [6][7], in which the mass term of a fermion leads to tachyonic Klein-Gordon equation, as discussed with (2.6). This is because tensor-hierarchy formulations [19][20] generally result in *massive* fields *via* generalized Stueckelberg mechanisms [12], that necessitate massive fermions which in turn lead to tachyons. The very first paper [10] on odd-dimensional self-duality did *not* have enough motivation to generalize it to *non-Abelian supersymmetric* case, without the recent development in tensor-hierarchy [19][20] and supersymmetric self-duality in $D = 2 + 2$ [2][3].

Our formulation also gives a strong motivation to study odd-dimensional self-duality with *plural* time-coordinates, that has never been well-explored in the past. In [5], we presented a theory in $D = 3 + 3$ based on similar philosophy from possible a 'Master-Theory' viewpoint [5]. The theory in $D = 3 + 3$ [5] even higher than $D = 3 + 2$ is more fundamental than the latter, as the 'Grand-Master Theory'. However, the present result gives yet another new direction in $D = 3 + 2$ with *odd-dimensional self-duality*, that was originated in [10], but has never been well studied in the past for *non-Abelian and supersymmetric* cases in dimensions higher than 3D.

Once we have established a duality-symmetric system in $D = 3 + 2$, it remains to explore, if similar formulations exist in higher odd dimensions, that are even more fundamental than our present theory in $D = 3 + 2$, or previous self-dual supersymmetric theories in $D = 3 + 3$ [5], as well.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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