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Localization of N=2 Field Theories: a Twisted Path Across Four Manifolds

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Abstract

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Supersymmetric quantum field theories provide a framework where certain physical observables can be computed exactly. In those cases, one not only has control over perturbative contributions but also over non-perturbative contributions. In this thesis the main focus are N=2 supersymmetric quantum field theories on compact manifolds with U(1)xU(1) isometry and a Killing vector with isolated fixed points.

In Part I, focusing on pure gauge theories, it is explained how equivariant Donaldson-Witten theory and a certain class of non-topological theories, related to the well-known result of Pestun on the four-dimensional sphere, can be described as two instances of an underlying framework. Employing this formalism, a general formula for the partition functions has been proposed which is valid both for equivariant Donaldson-Witten and Pestun-like theories. On top of perturbative contributions, the partition functions get contributions from instantons and fluxes.

In Part II, the results appearing in the papers attached to this thesis are presented. First, a formal treatment of the perturbative part is discussed. Then, the dependence on flux of the partition function is studied and it is shown how Donaldson-Witten and Pestun-like theories arise from a unique five-dimensional theory, after dimensional reduction. Finally, matter coupled to gauge fields are included in the framework above.

Keywords: Supersymmmetry, gauge theories, localization

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I R. Mauch and L. Ruggeri, *Index of the Transversally Elliptic Complex from* $\mathcal{N}=2$ *Localization in Four Dimensions*, [arxiv: 2112.10658]

II J. Lundin and L.Ruggeri, *SYM on Quotients of Spheres and Complex Projective Spaces*, JHEP 03 (2022) 204, [arxiv: 2110.13065]

III G. Festuccia, A. Gorantis, A. Pitteli, K. Polydorou and L. Ruggeri, *Cohomological localization of* $\mathcal{N}=2$ *gauge theories with matter*, JHEP 09 (2020) 133, [arxiv: 2005.12944]

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Contents

1. Introduction

Quantum field theory (QFT) of gauge fields coupled to matter has had tremendous success in describing three (electromagnetic, weak, strong) of the four fundamental interactions and incorporating them in the Standard Model of particle physics. The success of QFT has a natural explanation as it is a framework combining special relativity and quantum mechanics. The study of its supersymmetric version (SQFT) started in the early seventies [1–4] and soon after a classification of super Yang-Mills (SYM) theories in different dimensions appeared [5]. These theories have been later extensively considered because of string theory considerations and as options for models beyond the standard model. The extra structure given by supersymmetry enables to (i) constrain protected physical observables under deformations of the SQFT and, in some cases, (ii) large cancellations due to supersymmetry occur and quantities can be exactly determined.

To introduce these two concepts, we will review one of the simplest, and earliest, example. This is the Witten index [6], a topological quantity counting the difference between bosonic and fermionic zero energy states in a supersymmetric theory and which can be used to establish whether the ground state of a theory preserves supersymmetry. The motivation is that the ground state of a theory is supersymmetric if and only if its energy vanishes exactly, thus a non-zero Witten index determines that the ground state is supersymmetric. Let us then take a supersymmetric theory in a d-dimensional Euclidean finite volume V with periodic boundary conditions both for bosons and fermions and a Hilbert space $\mathscr{H}$. The periodic boundary conditions, forced by supersymmetry, ensure that the finite volume is effectively a torus $T^{d-1} \times S^1_\beta$, where β is the circumference of the Euclidean time circle.

Let us assume, for simplicity, that the theory possesses a single Hermitian supercharge Q, whose superalgebra, restricted to the subset of states with zero momentum[1] $\mathbf{P} = 0$, is given by:

$$Q^2 = H, \tag{1.1}$$

where H is the Hamiltonian of the theory and we have chosen properly a possible additive constant in H. If we define $|\Omega\rangle$ the ground state of the theory, we find:

$$\langle\Omega| H |\Omega\rangle = \langle\Omega| Q^2 |\Omega\rangle = ||Q|\Omega\rangle||^2. \tag{1.2}$$

[1] Without loss of generality, we can consider massive and massless particles in the rest frame where, respectively, $P_\mu = (E, 0, ..., 0)$ and $P_\mu = (E, 0, ..., 0, E)$. Hence zero energy states need to have $\mathbf{P} = 0$.

Thus the ground state is supersymmetric, $Q|\Omega\rangle = 0$, if and only if its energy is zero.

The Witten index is defined as:

$$I_{Witten} = \mathrm{Tr}(-1)^F e^{-\beta H} = \mathrm{Tr}\, e^{2\pi i J_z} e^{-\beta H}. \tag{1.3}$$

The operator $(-1)^F$ acts on bosonic and fermionic states as $e^{2\pi i J_z}|b\rangle = +|b\rangle$, $e^{2\pi i J_z}|f\rangle = -|f\rangle$. For a non-supersymmetric theory (1.3) could get contributions from an infinite number of states. However, the claim is that the Witten index only counts the difference between the number of bosonic and fermionic zero energy states and thus it can be used to determine whether a ground state is supersymmetric. The action of Q on bosonic and fermionic states is:

$$Q|b\rangle = \sqrt{E}\,|f\rangle, \qquad Q|f\rangle = \sqrt{E}\,|b\rangle. \tag{1.4}$$

Hence, as $[Q,H]=0$, all states come in pairs with the same energy, except for those with vanishing energy. The latter states then form a small multiplet with half (one) the dimension of the multiplets above the ground state[2]. Thus, large multiplets give vanishing contribution to the Witten index (1.3) which then only counts:

$$I_{Witten} = \mathrm{Tr}(-1)^F e^{-\beta H} = n_B - n_F, \tag{1.5}$$

where n_B, n_F are the number of bosonic and fermionic zero energy states, respectively. This example shows how large cancellations occur in supersymmetric theories making the computation of some observables much easier (ii). Moreover, a non-zero Witten index means that supersymmetry is unbroken as there are zero energy states. Instead, using the Witten index to prove that the ground state breaks supersymmetry is harder, as a zero Witten index might be due to either $n_B = n_F \neq 0$ or $n_B = n_F = 0$.

If one is able to compute the Witten index in a certain region in the parameter space of the theory, a natural question is to ask how it changes under deformations. It turns out that the pairing (1.4) also constrains the behaviour of the Witten index under deformations of the theory. A large multiplet can generically change its energy under a deformation and even acquire zero energy for some particular values of the parameters. If this happens, the large multiplet splits into two small multiplets of opposite eigenvalue under $(-1)^F$, therefore not contributing to the Witten index. Vice versa, a bosonic small multiplet can get non-zero energy under a deformation only together with a fermionic small multiplet, so that they can form a non-zero energy large multiplet. An example of these deformations can be obtained taking the large volume limit $V \to \infty$. If the energy of a state is zero, its large V limit is again zero[3]. Hence, under certain deformations, the Witten index is left unchanged.

[2] Small supersymmetric multiplets are usually named BPS multiplets, after Bogomol'nyi [7], Prasad and Sommerfield [8].

[3] Note however that the converse is not true, so supersymmetry can be broken at finite V and restored in the infinite volume limit.

We see then how observables can be protected under deformations because of the extra constraints given by supersymmetry (i).

The quantity (1.3) we are considering is strictly related to the mathematical concept of index of an operator. To show this, it is enough to split the Hilbert space into its bosonic and fermionic components $\mathscr{H} = \mathscr{H}_B \oplus \mathscr{H}_F$. With respect to this decomposition the supercharge Q acts as:

$$Q = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}, \tag{1.6}$$

where $M^\dagger$ is the adjoint of M as Q is Hermitian. Zero energy states in $\mathscr{H}$ have zero eigenvalue under $H|\psi\rangle = Q^2|\psi\rangle$ and thus also under Q. Therefore, bosonic small multiplets in $\mathscr{H}_B$ satisfy $M|b\rangle = 0$ while fermionic ones $M^\dagger|f\rangle = 0$. Hence, the Witten index is given by:

$$I_{Witten} = \ker M - \ker M^\dagger, \tag{1.7}$$

which is the definition of the index of an operator [9]. We will encounter again the interplay between supersymmetric theories and indices of (transversally) elliptic operators later in the thesis.

The properties (i)-(ii) discussed above for the Witten index are a common feature of supersymmetric theories. Supersymmetric localization relies heavily on these features and it has been employed, over the years, to compute exactly protected observables in supersymmetric theories of increasing complexity. The main idea is to deform the theory, without affecting the value of certain protected observables, to a region where exact computations are simpler. For example, in some cases, observables are protected under the RG flow connecting a strongly coupled region to a weakly coupled one. Because of this independence, it is possible to employ weakly coupled computations to describe observables in a strongly coupled region. Moreover, supersymmetry makes these computations much more tractable, giving a certain control also over non perturbative contributions. For reviews of supersymmetric localization, see [10, 11].

In Part I of this thesis we introduce the localization technique applied to $\mathscr{N} = 2$ super Yang-Mills theories, reviewing some of the major advances in the field. The earliest localization computation has been employed by Witten in relation to Morse theory [12] and for the topological sigma model in two dimensions [13]. As we want to focus on gauge theories, the first example we consider in chapter 2 is that of topologically twisted $\mathscr{N} = 2$ SYM [14]. Later results include the seminal paper by Nekrasov [15], which we introduce in chapter 3, computing the $\mathscr{N} = 2$ SYM partition function as an integral over the moduli space of instantons in the Ω-background $\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}$. Nekrasov's computation is performed in the UV weakly coupled region and it is shown to match, in a certain limit, the Seiberg-Witten prepotential [16] computed in the IR strongly coupled region.

Relying on [15], Pestun computed the full partition function for $\mathscr{N} = 2$ SYM on S^4 [17]. To introduce this result we employ, in chapter 4, a modern approach proposed in [18, 19] which includes both Pestun's theory and topologically twisted SYM as two particular instances of a unique framework, also applicable to a more general class of four-dimensional compact manifolds.

The goal of the first part is to introduce some background material needed to understand the works presented in Part II. First, in chapter 5, following Paper I, we provide a formal treatment of the perturbative contribution of the partition functions considered in the framework of [18, 19]. Then, in chapter 6, we discuss how fluxes enter these partition functions, focusing on the case of $\mathbb{CP}^2$ studied in Paper II. Finally, chapter 7 is based on Paper III, where it is shown how to include $\mathscr{N} = 2$ hypermultiplets coupled to gauge theories.

Part I:
Background

2. Topological Twisting

The work of Witten [14] for the topological subsector of twisted $\mathcal{N}=2$ SYM in $d=4$ has been a first step towards computing exact observables in SQFTs, including non-perturbative contributions. Supersymmetry is present through a scalar supercharge, acting like a BRST operator. The idea is to deform the theory to a weakly coupled region employing the independence of supersymmetric, or BRST closed, observables under a rescaling of the Yang-Mills coupling constant g^2_{YM} (i). In this region, only small perturbations around classical solutions contribute to the path integral, which then simplifies to a sum over instanton sectors, reducing the complexity of computing observables (ii). Remarkably, the work of Witten gives a physical interpretation to Donaldson's study of smooth four manifolds [20] employing the moduli space of anti-self-dual field strengths. Thus, topologically twisted $\mathcal{N}=2$ will often be denoted Donaldson-Witten (DW) theory.

2.1 Action

Following [14], we take $d=4$ Euclidean space with $SU(2)_L \times_{\mathbb{Z}_2} SU(2)_R$ rotation group and an $\mathcal{N}=2$ vector multiplet with gauge group G, whose on-shell field content is that of a gauge boson A_μ, scalars ϕ, φ and gauginos $\lambda_{i\alpha}, \bar{\lambda}^i_{\dot{\alpha}}$[1]. Both the scalars and the gauginos transform in the adjoint of the gauge group. The theory also has internal symmetries $SU(2)_I \times U(1)_U$ under which $\lambda_{i\alpha}$ and $\bar{\lambda}^i_{\dot{\alpha}}$ transform in the fundamental representation of $SU(2)_I$ and with ± 1 charge for the $U(1)_U$ rotations.

On top of the bosonic symmetries we also want to define a fermionic symmetry. In flat space it is possible to write an $\mathcal{N}=2$ Lagrangian invariant under supersymmetry transformations generated by $\delta = Q^i \varepsilon_i$, where ε_i is a constant Killing spinor[2] solving:

$$\partial_\mu \varepsilon_i = 0. \tag{2.1}$$

As we will be interested in generic four-manifolds, we need to replace the derivatives with covariant ones. However, covariantly constant spinors do

[1] Here we are employing the standard notation of denoting $\alpha, \dot{\alpha}$ indices of $SU(2)_L$ and $SU(2)_R$ respectively.

[2] The Killing spinor will be, in general, a linear combination of two spinors of opposite chirality, thus it transforms under both $SU(2)_L$ and $SU(2)_R$. However we will omit spinor indices.

not exist on generic four manifolds and writing down supersymmetric Lagrangians is often quite complicated[3]. To tackle this issue in general one has to couple SYM to a rigid supergravity background, a technique pioneered in [22].

The trick employed by Witten to circumvent this problem is to introduce a twisting of the isometry group of flat space, obtained taking the diagonal sum $SU(2)'_R \subset SU(2)_R \times SU(2)_I$. The gauginos $\lambda_{i\alpha}, \bar{\lambda}^i_{\dot{\alpha}}$ transform, under $SU(2)_L \times SU(2)'_R \times U(1)_U$, as a one-form Ψ_μ, a self-dual two-form $\chi_{\mu\nu}$ and a scalar η:

$$(1/2,1/2)^1 \oplus (0,1)^{-1} \oplus (0,0)^{-1}, \tag{2.2}$$

where the exponents label the $U(1)_U$ charges. The same splitting occurs for the Killing spinor ε_i. If we now keep only the constant anti-commuting scalar component, which we denote ε, we have that ε solves:

$$\partial_\mu \varepsilon = 0, \tag{2.3}$$

on every four manifold[4]. We can use this result to define a scalar fermionic generator Q, such that $Q^2 = 0$, reminiscent of a BRST-like operator, defining physical states as equivalence classes of Q-closed states modulo Q-exact ones. Supersymmetry transformations with constant parameter ε are:

$$\begin{aligned} &\delta A_\mu = i\varepsilon\Psi_\mu, \quad \delta\Psi_\mu = -\varepsilon D_\mu\phi, \quad \delta\phi = 0, \\ &\delta\varphi = 2i\varepsilon\eta, \quad \delta\eta = \frac{1}{2}\varepsilon[\phi,\varphi], \\ &\delta\chi_{\mu\nu} = \varepsilon(F_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}), \end{aligned} \tag{2.4}$$

defining the field strength as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A supersymmetric Lagrangian can be written as follows:

$$\begin{aligned} \mathscr{L} = +\,\mathrm{Tr}\Bigg[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi D_\mu D^\mu \varphi - i\eta D_\mu \Psi^\mu + iD_\mu \Psi_\nu \chi^{\mu\nu} + \\ &- \frac{i}{8}\phi[\chi_{\mu\nu},\chi^{\mu\nu}] - \frac{i}{2}\lambda[\Psi_\mu,\Psi^\mu] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\varphi]^2 \Bigg]. \end{aligned} \tag{2.5}$$

It is possible to check that the Lagrangian is supersymmetric under (2.4) on any orientable Riemannian four-manifold, not only flat space. In non-flat backgrounds the Riemann tensor can appear as the commutator of covariant derivatives in the variation of $\delta\mathscr{L}$. However such commutator appears only acting on scalars $\sim [D_\mu, D_\nu]\phi$, thus not creating any problem.

[3]For an early attempt on spheres of diverse dimensions, see [21].

[4]From a rigid supergravity point of view, the twisting is obtained turning on a background field in the $SU(2)_I$ R-symmetry, chosen such that it cancels the contribution of the spin connection in the covariant derivative.

2.2 Stress-energy tensor

We now show that the stress energy tensor $T_{\mu\nu}$ can be written as a BRST anti-commutator, a key ingredient in defining Witten-type topological field theories (TFTs). The definition of $T_{\mu\nu}$ follows from the variation of the action $S = \int_M \sqrt{g}\mathscr{L}$ under an infinitesimal metric transformation[5]:

$$\delta S = \frac{1}{2}\int_M \sqrt{g}\delta g^{\mu\nu} T_{\mu\nu}. \tag{2.6}$$

The stress-energy tensor turns out to be a complicated function of the vector multiplet fields. As expected, on the equations of motion, one finds that $T_{\mu\nu}$ is conserved:

$$D_\mu T^{\mu\nu} = 0. \tag{2.7}$$

Moreover the trace of the stress-energy tensor is given by:

$$g^{\mu\nu} T_{\mu\nu} = D_\mu R^\mu, \tag{2.8}$$

where:

$$R^\mu = \mathrm{Tr}[\varphi D^\mu \phi - 2i\eta \Psi^\mu]. \tag{2.9}$$

Because of this, under a generic conformal transformation $\delta g^{\mu\nu} = w(x) g^{\mu\nu}$, the variation of the action (2.6) does not vanish:

$$\delta S = \frac{1}{2}\int_M \sqrt{g} w(x) g^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\int_M \sqrt{g} w(x) D_\mu R^\mu \neq 0. \tag{2.10}$$

However, for a constant function w, one finds $\delta S = 0$, and the action is invariant under a global rescaling of the metric. Similarly, taking the manifold to be flat space and transforming the coordinates as $\delta x^\mu = w x^\mu$, one finds, because of (2.8), that the corresponding current is conserved:

$$D_\mu S^\mu = 0, \qquad S^\mu = T^{\mu\nu} x_\nu - R^\mu. \tag{2.11}$$

The insight of Witten was to realize that the stress-energy tensor, even if it does not vanish as in Schwarz-type TQFTs[6], can be written as a BRST anti-commutator:

$$T_{\mu\nu} = \{Q, \lambda_{\mu\nu}\}, \tag{2.12}$$

where:

$$\begin{aligned}\lambda_{\mu\nu} =& \frac{1}{2}\mathrm{Tr}(F_{\mu\rho}\chi_\nu{}^\rho + F_{\nu\rho}\chi_\mu{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma}\chi^{\rho\sigma}) + \\ &+ \frac{1}{2}\mathrm{Tr}(\Psi_\mu D_\nu \varphi + \psi_\nu D_\mu \varphi - g_{\mu\nu}\psi_\rho D^\rho \varphi) + \\ &+ \frac{1}{4} g_{\mu\nu}\mathrm{Tr}(\eta[\phi, \varphi]).\end{aligned} \tag{2.13}$$

[5] When computing variations under a change in the metric one has to be careful that the self-dual condition on $\chi_{\mu\nu}$ is preserved

[6] Chern-Simons theory is an example of Schwarz-type TQFT where the metric does not appear anywhere in the theory [23].

Equation (2.12) will prove crucial in showing that the partition function is a topological invariant.

Similar considerations hold also for the Lagrangian (2.5):

$$\{Q,V\} = \mathscr{L}', \tag{2.14}$$

where:

$$\mathscr{L}' = \mathscr{L} + \frac{1}{4}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2.15}$$

and:

$$V = \frac{1}{4}\mathrm{Tr}F_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\Psi_\mu D^\mu \varphi - \frac{1}{4}\mathrm{Tr}(\eta[\phi,\varphi]). \tag{2.16}$$

In deriving (2.14) one needs to use the equations of motion for $\chi_{\mu\nu}$. Moreover, the term added in $\mathscr{L}$ is a topological invariant labeling instanton sectors by their charge $k \in \mathbb{Z}$. We denote S' the action computed using $\mathscr{L}'$. As the term added is topological, the considerations above regarding infinitesimal transformations are left untouched.

2.3 Partition function

This section is aimed at showing that the partition function is a topological invariant due to the stress-energy tensor being a BRST anti-commutator. Moreover, the partition function computes a certain class of Donaldson invariants of smooth orientable four manifolds [20].

The expectation value of a generic operator $\mathscr{O}$ is given by the following path integral:

$$Z(\mathscr{O}) = \int (\mathscr{D}X) e^{-\frac{1}{g_{YM}^2}S'[X]} \cdot \mathscr{O}. \tag{2.17}$$

Here, the action is determined by the Lagrangian (2.15) and the integration measure $(\mathscr{D}X)$ is defined over the fields of the vector multiplet $(A,\phi,\varphi,\eta,\Psi,\chi)$.

The integral (2.17) does not depend on the supersymmetry parameter ε as the integration measure is supersymmetric. Thus:

$$Z(\mathscr{O}) = Z_\varepsilon(\mathscr{O}) = \int (\mathscr{D}X) \exp(\varepsilon Q) \cdot e^{-\frac{1}{g_{YM}^2}S'[X]} \cdot \mathscr{O}. \tag{2.18}$$

Moreover, expanding $\exp(\varepsilon Q)$ and using that the Lagrangian is supersymmetric, we find:

$$Z_\varepsilon(\mathscr{O}) = \int (\mathscr{D}X) e^{-\frac{1}{g_{YM}^2}S'[X]} (\mathscr{O} + \varepsilon\{Q,\mathscr{O}\}). \tag{2.19}$$

Therefore we conclude:

$$\int (\mathscr{D}X) e^{-\frac{1}{g_{YM}^2}S'[X]} \cdot \{Q,\mathscr{O}\} = 0. \tag{2.20}$$

The statement that the expectation value of a BRST anti-commutator vanishes can be used to show that the partition function is a topological invariant. Under a change in the metric, the variation of the partition function can be implemented by inserting an operator (2.6):

$$-\frac{1}{g_{YM}^2}\delta S = -\frac{1}{2g_{YM}^2}\int_M \sqrt{g}\delta g^{\mu\nu}T_{\mu\nu} = -\frac{1}{2g_{YM}^2}\left\{Q,\int_M \sqrt{g}\delta g^{\mu\nu}\lambda_{\mu\nu}\right\}. \tag{2.21}$$

From this we conclude[7]:

$$\delta Z = \int (\mathscr{D}X) e^{-\frac{1}{g_{YM}^2}S'[X]}\cdot\left(-\frac{1}{g_{YM}^2}\delta S'\right) = 0. \tag{2.22}$$

Similarly, varying the partition function with respect to the Yang-Mills coupling is obtained inserting the BRST anti-commutator $\{Q,V\}$. Hence, the partition function is also independent of g_{YM}^2, as long as $g_{YM}^2 \neq 0$, similarly to what we have discussed above for the Witten index.

To compute the partition function we will use its independence of the Yang-Mills coupling and take the limit $g_{YM}^2 \to 0$, where the theory is weakly coupled and the path integral is dominated by the classical minima of the action. These are classified considering the action for the field strength $F_{\mu\nu}$:

$$\frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + F^{\mu\nu}\tilde{F}_{\mu\nu}\right) = \frac{1}{8}\mathrm{Tr}\left(F_{\mu\nu}+\tilde{F}_{\mu\nu})(F^{\mu\nu}+\tilde{F}^{\mu\nu}\right), \tag{2.23}$$

which is positive semi-definite and vanishes if and only if:

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}. \tag{2.24}$$

While strictly speaking these are anti-self-dual anti-instantons, the author of [14] calls them instantons as "it would be tiresome to call them anti-instantons" and we will follow this convention. Substituting for (2.24) and setting ϕ,φ,η, ψ,χ to zero in (2.4), one finds that these configurations are supersymmetric. As the Lagrangian $\mathscr{L}'$ is Q-exact (2.14) and positive definite, it is consistent that configurations minimizing the action are supersymmetric.

On manifolds where non-trivial solutions to (2.24) exist, the instantons can have a moduli spaces of solutions $\mathscr{M}$ of dimensions different than zero. We postpone till later the treatment of these cases where, due to the existence of fermionic zero modes, one has to modify the observables under consideration. Thus, we assume that gauge bundle over the manifold is such that the dimension of the instanton moduli space vanishes and that the instantons are discrete and isolated solutions. Hence, for an arbitrary instanton sector and very small g_{YM}^2, we can expand the Lagrangian (2.15) at quadratic order in the fields $\Phi = (A_\mu,\phi,\varphi)$ and $\Omega = (\eta,\Psi_\mu,\chi_{\mu\nu})$, and find:

$$\mathscr{L}'_{(2)} = \int_M \sqrt{g}(\Phi\Delta_B\Phi + i\Omega D_F\Omega). \tag{2.25}$$

[7] Here we assume that the measure $(\mathscr{D}X)$ is invariant under a transformation of the metric.

Here, Δ_B and D_F are, respectively, second and first order operators. A property of Δ_B, is that, replacing the highest order derivatives in Δ_B with vector fields ξ, we can find a polynomial $\sigma_\xi(\Delta_B)$, the principal symbol of Δ_B, such that $\sigma_\xi(\Delta_B)$ is an isomorphism for non-zero ξ. This, by definition, shows that Δ_B is an elliptic operator.

The partition function, at an arbitrary instanton sector, is then given by bosonic and fermionic Gaussian integrals:

$$Z_k = \int_M (\mathscr{D}X) e^{-\frac{1}{g_{YM}^2}\mathscr{L}'_{(2)}} = \frac{\mathrm{Pf}(D_F)}{\sqrt{\det(\Delta_B)}}. \tag{2.26}$$

We have been able to reduce to a Gaussian integral a complicated integral over the infinite-dimensional configuration space of the fields in the vector multiplet. In doing so, we have employed the independence of the partition function under a change in g_{YM}^2, due to the Q-exactness of the Lagrangian $\mathscr{L}'$.

In determining the partition function Z_k we are helped again by supersymmetry. We have shown previously that the supersymmetry transformations vanish on $F_{\mu\nu}$ satisfying (2.24), with all other fields in the (twisted) vector multiplet vanishing. Therefore for every eigenvalue λ of $D_F\Omega = \lambda\Omega$, there exists an eigenvalue of a bosonic field $\Delta_B\Phi = \lambda^2\Phi$. This observation hugely simplifies the ratio of determinants and we find:

$$Z_k = \pm \prod_i \frac{\lambda_i}{\sqrt{|\lambda_i|^2}}. \tag{2.27}$$

The overall uncertainty in the sign is due to a choice of orientation on the manifold.

Ignoring the overall minus sign, we can compute the full partition function of the topological subsector of $\mathscr{N} = 2$ twisted SYM as a sum over instanton sectors:

$$Z = \sum_k Z_k = \sum_k (-1)^{n_k}. \tag{2.28}$$

As we have shown above the partition function is a topological invariant and it can be shown to match a particular class of Donaldson invariants, defined for gauge bundles over M with vanishing dimension of the moduli space of instantons.

2.4 Non vanishing moduli space

So far we have considered zero-dimensional instanton moduli spaces. Let us now assume that the space is not trivial: $d(\mathscr{M}) \neq 0$. Thus, around each anti-self-dual connection there exist $d(\mathscr{M})$ flat directions such that, if A is an instanton gauge field, there will exist deformations $A + \delta A$ which will again solve the anti-self-dual condition (2.24):

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu + \varepsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma = 0. \tag{2.29}$$

In order to obtain physically inequivalent configurations, we impose that δA is not obtained only through a gauge transformation. Hence, we demand the following gauge condition:

$$D_\mu \delta A^\mu = 0. \tag{2.30}$$

Because of supersymmetry (2.4), we expect to have also fermionic zero-modes. Indeed, solving for χ and η, one finds:

$$\begin{aligned} &D_\mu \Psi_\nu - D_\nu \Psi_\mu + \varepsilon_{\mu\nu\rho\sigma} D^\rho \Psi^\sigma = 0, \\ &D_\mu \Psi^\mu = 0. \end{aligned} \tag{2.31}$$

But these are exactly the equations above which we assumed to have $d(\mathscr{M})$ solutions. Finally, the index theorem [9] says that the number of Ψ zero modes minus the zero modes of (η, χ) is exactly $d(\mathscr{M})$. Therefore there cannot be any zero mode for (η, χ).

If we were to compute a partition function as above, we would find that it vanishes due to the presence of fermionic zero modes. The reason is that, while the Lagrangian (2.5) is $U(1)_U$ symmetric, this is not the case for the integration measure $(\mathscr{D}X)$, which transforms with $-d(\mathscr{M})$ weight. The lack of invariance happens as only zero modes of Ψ are present and these have $+1$ charge under the $U(1)_U$. However, there is a straightforward method to absorb the zero modes and define a meaningful observable:

$$Z(\mathscr{O}) = \int (\mathscr{D}X) e^{-\frac{\mathscr{L}'}{g^2_{YM}}} \cdot \mathscr{O}, \tag{2.32}$$

where the operator $\mathscr{O}$ is chosen so that it has a $U(1)_U$ charge equal to $d(\mathscr{M})$. One can check that (2.32) is a non-trivial topological invariant if:

$$\begin{aligned} &\{Q, \mathscr{O}\} = 0 \quad \text{and} \quad \mathscr{O} \neq \{Q, \rho\}, \\ &\delta_g \mathscr{O} = \{Q, \rho'\}, \end{aligned} \tag{2.33}$$

where the condition on the second line is the variation of $\mathscr{O}$ under a change in metric. Operators satisfying these conditions are gauge invariant polynomials in the scalar field ϕ, as $\mathrm{Tr}\phi^2$, $\mathrm{Tr}\phi^4$ and higher even powers in ϕ. The amount of independent operators depend on the gauge group under consideration.

As an example, we consider $SU(2)$ gauge group and a manifold M such that $d(\mathscr{M}) = 4k$. The rank of $SU(2)$ is one and thus the only independent operator inserted at a point $p \in M$ is:

$$W|_p = \frac{1}{2} \mathrm{Tr}\phi^2|_p, \tag{2.34}$$

whose charge under $U(1)_U$ is four. Therefore, we can define topological invariant correlators as follows:

$$Z(k) = \int (\mathscr{D}X) e^{-\frac{\mathscr{L}'}{g^2_{YM}}} \cdot \prod_{i=1}^{k} W|_{p_i}. \tag{2.35}$$

Following a similar logic one can define topological invariants for any value of $d(\mathscr{M})$.

3. Nekrasov Partition Function

The derivation of Seiberg and Witten (SW) [16] of the low-energy exact prepotential of $\mathcal{N}=2$ $SU(2)$ SYM relied on the Kähler structure of the moduli space of vacua and on a version of Montonen-Olive [24] duality for $\mathcal{N}=2$ theories. In the following years there were many attempts to derive the SW prepotential in the UV weakly coupled region of an $SU(N)$ gauge theory. In this region the prepotential can be expanded as a sum of perturbative and non-perturbative contributions:

$$\mathscr{F}=\frac{N}{\pi i}\sum_{m=1}^{N-1}a_m+\frac{i}{4\pi}\sum_{1<i<j}^{N-1}(a_i-a_j)^2\log\frac{(a_i-a_j)^2}{\Lambda^2}+\sum_{k=1}^{\infty}\frac{\Lambda^{2Nk}}{2k\pi i}\mathscr{F}^k(a). \quad (3.1)$$

Here $a=(a_1,...,a_{N-1})$ is the vacuum expectation value of the scalar in the vector multiplet and Λ is a renormalization invariant dimensionful parameter. Also, the first two terms account for classical and one-loop contributions while the last term is a sum over non-trivial instantonic sectors.

As discussed in the previous chapter for the topological subsector of $\mathcal{N}=2$ SYM, instantons, in general, have a moduli space of solutions. The ADHM (Atiyah, Drinfeld, Hitchin and Manin) [25] construction described the moduli space of instantons $\mathscr{M}_{N,k}$ as a quotient of a hyperkähler manifold. One can then use supersymmetry to reduce the path integral over the infinite-dimensional field configurations space to an integral over the moduli space of instantons, similarly as for topologically twisted $\mathcal{N}=2$ SYM. Early attempts at computing (3.1) were semi-classical computations around a fixed instanton background. In the simplest cases they could successfully match some coefficients $\mathscr{F}^k(a)$ in (3.1). However, these techniques become less efficient for high instanton number or high rank of the gauge group, when the instanton moduli space becomes extremely complicated. A comprehensive review for this approach can be found in [26].

The crucial insight by Nekrasov [15, 27] was to modify Witten's localization computation [14] using also the the one-form supercharge G_μ, appearing after the twisting of the isometry group. The deformation is such that it reduces the integration domain in the path integral to the fixed points of $\mathscr{M}_{N,k}$ under the $U(1)^2$ Cartan of the $SO(4)$ isometry group[1]. This hugely simplifies the computation and the partition function, in a particular limit, shows exact agreement with the low-energy prepotential $\mathscr{F}$ (3.1).

[1]The fixed points are in general not isolated and one has to consider a non-commutative deformation to find a discrete set of fixed points.

A complete review of the derivation of Nekrasov partition function goes beyond the scope of the present work. Thus, in this chapter, we only review some aspects which are more important for the rest of thesis. The Nekrasov partition functions will then be used as building block for partition functions of $\mathscr{N}=2$ SYM theories on compact manifolds [18, 19] which will be considered in the next chapter.

3.1 ADHM construction

In this section, rather than following the original work [25], we will take the opposite approach [28] and show how ADHM equations arise from solutions of the anti-self-dual condition (2.24). The first step will consist in formulating $F=-\star F$, a second order partial differential equation, into the equivalent problem of solving a first-order Dirac equation, after identifying the spinor bundle with that of holomorphic differential forms. Second, we will find solutions of the Dirac equation which will be used, eventually, to describe the moduli space of instantons as algebraic equations describing an hyperkähler manifold.

Let us start considering the $SU(N)$ YM action on $\mathbb{R}^4 \cong \mathbb{C}^2$, with metric $ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$:

$$\begin{aligned}\mathscr{L}_{SYM} =& \frac{1}{4g_{YM}^2}\int \mathrm{Tr} F\wedge\star F + \frac{i\theta}{8\pi}\int \mathrm{Tr} F\wedge F \\ =& \frac{1}{8g_{YM}^2}\int \mathrm{Tr}(F+\star F)^2 + i\frac{\tau}{4\pi}\int \mathrm{Tr} F\wedge F,\end{aligned} \tag{3.2}$$

where we have defined the complexified gauge coupling

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}. \tag{3.3}$$

The first term is minimized by anti-self-dual[2] field strengths $F=-\star F$ while the second one is a topological invariant proportional to the instanton charge $k\in\mathbb{Z}$. Similar considerations hold for the self-dual two-form $F=+\star F$:

$$\mathscr{L}_{SYM} = \frac{1}{8g_{YM}^2}\int \mathrm{Tr}(F-\star F)^2 + i\frac{\bar{\tau}}{4\pi}\int \mathrm{Tr} F\wedge F. \tag{3.4}$$

To solve the anti-self-dual condition, it is useful to introduce the gauge covariant Dolbeault operator, the analog of the de Rahm cohomology for complex manifolds:

$$\begin{aligned}\bar{\partial}_A &: \Omega^{0,i} \to \Omega^{0,i+1}, \\ \bar{\partial}_A^\dagger &: \Omega^{0,i} \to \Omega^{0,i-1}.\end{aligned} \tag{3.5}$$

[2]Following the conventions of the previous chapter we label instantons by anti-self-dual field strengths.

where the adjoint operator $\bar{\partial}^\dagger$ is defined, for arbitrary $\alpha, \beta \in \Omega^{j,i}$, as:

$$(\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger \alpha, \beta), \tag{3.6}$$

and $(\cdot\,, \cdot)$ is the inner product.

We now look at solutions of the anti-self-dual condition at arbitrary k:

$$F^+ \equiv F + \star F = 0. \tag{3.7}$$

This condition is worth three real equations and, written in terms of complex geometry, it can be expressed as:

$$F^+ = 0 \iff \begin{cases} F^{0,2} = 0 \\ F^{1,1}_\omega = 0 \end{cases} \tag{3.8}$$

The first condition on the right is equivalent to solving the cohomology problem $\bar{\partial}_A^2 = 0$. Therefore, we look at the space of anti-holomorphic differential forms. This can be identified, after choosing a spin structure, with a spinor bundle on $\mathbb{C}^2$ of the same dimension:

$$s_+ \cong \Omega^{0,0} \oplus \Omega^{0,2}, \qquad s_- \cong \Omega^{0,1}, \tag{3.9}$$

where $s_\pm$ are spinor representations of the $SO(4)$ Lorentz group of opposite chirality. Thus, we need to solve:

$$\bar{\partial}_A \tilde{\psi} = 0 \quad \text{ up to exact forms,} \quad \tilde{\psi} \sim \tilde{\psi} + \bar{\partial}_A \chi \tag{3.10}$$

where $\tilde{\psi} \in L^2(\Omega^{0,i} \otimes E)$, $\chi \in L^2(\Omega^{0,i-1} \otimes E)$ and $E \cong \mathbb{C}^N$ is a rank N complex vector bundle on $\mathbb{R}^4$ representing the fundamental representation of the gauge group.

The latter condition in (3.8), $F^{1,1}_\omega = 0$, requires the component of the curvature along the Kähler form:

$$\omega = -\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \tag{3.11}$$

to vanish. While $F^{0,2}$ is invariant under complexified gauge transformation, this is not the case for $F^{1,1}_\omega$. Thus, instead of imposing $F^{0,2} = F^{1,1}_\omega = 0$ and dividing by real gauge transformation, we can equivalently require $F^{0,2} = 0$ quotienting by complex gauge transformations. Thus, we set a gauge condition by imposing:

$$\bar{\partial}_A^\dagger \tilde{\psi} = 0. \tag{3.12}$$

Therefore, combining (3.10) and (3.12), we need to find solutions of:

$$(\bar{\partial}_A \oplus \bar{\partial}_A^\dagger)\tilde{\psi} = 0, \tag{3.13}$$

but this is just the Dirac equation:

$$\not{D}_A \equiv (\bar{\partial}_A \oplus \bar{\partial}_A^\dagger) : s_\pm \to s_\mp. \tag{3.14}$$

Thus, as stated earlier, we have shown how solving the anti-self-dual condition, a second-order partial differential equation, has reduced to solving a first-order Dirac equation.

The second part of this section is devoted to describing solutions of the Dirac equation (3.13). We start looking at s_+, and thus at $\eta \in L^2(\Omega^{0,0} \otimes E)$ and $\chi \in L^2(\Omega^{0,2} \otimes E)$. Using the definition of the Dolbeault operators (3.5), we find:

$$(\bar{\partial}_A \oplus \bar{\partial}_A^*)(\eta \oplus \chi) = \bar{\partial}_A \eta + \bar{\partial}_A^* \chi = 0, \tag{3.15}$$

which can be rewritten, showing explicitly the components, as:

$$\begin{cases} D_{\bar{1}}\eta + D_2\chi = 0 \\ D_{\bar{2}}\eta - D_1\chi = 0. \end{cases} \tag{3.16}$$

Acting on the first equation with D_1 and on the second with D_2, and using that $[D_1, D_2]$ commute, one finds:

$$(D_1 D_{\bar{1}} + D_2 D_{\bar{2}})\eta = 0. \tag{3.17}$$

With a further rewriting, we find that η needs to be covariantly harmonic:

$$\frac{1}{2}(\{D_1, D_{\bar{1}}\} + \{D_2, D_{\bar{2}}\})\eta = 0. \tag{3.18}$$

However, on flat space the only normalizable solution is the vanishing solution. Then, also $\chi = 0$ and in general for positive chirality spinors there is no solution in an instanton background.

Employing the same argument presented below (2.31), instead of solving explicitly the Dirac equation (3.13) also for negative chirality spinors s_-, we can apply the index theorem [9] to $\not{D}_A$:

$$\text{ind}\, \not{D}_A = \dim \ker_- \not{D}_A - \dim \ker_+ \not{D}_A = k, \tag{3.19}$$

where $\dim \ker_\pm$ stands for the kernel on $L^2(s_\pm \otimes E)$-normalizable forms. Thus, we have a k-dimensional space of solutions of negative chirality, which we label by K. The space is spanned by forms $\psi \in (\Omega^{0,1} \otimes E)$ solving:

$$\bar{\partial}_A \psi = 0, \quad \bar{\partial}_A^* \psi = 0, \tag{3.20}$$

where we recall that the first condition is equivalent to the cohomology problem $\bar{\partial}_A^2 = 0$ while the second is a choice of gauge.

Finally, we need to properly describe solutions of (3.20). For each $\psi \in K$ we define a four-vector multiplying ψ by the coordinate functions $(z_1, z_2, \bar{z}_1, \bar{z}_2)$. Thus, we will obtain elements $(z_i \Psi) \in (\Omega^{0,1} \otimes E)$ not necessarily solving the Dirac equation. The space of all L^2-normalizable $(0,1)$-forms, however, can be decomposed into the kernel of the Dirac operator, K, and its orthogonal component with respect to the L^2-norm. The projector acts as $\Pi(z_i,\psi) \in K$ such that $\not{D}_A \Pi(z_i,\psi) = 0$[3]. We can then define $(B_1, B_2, B_2^\dagger, B_2^\dagger) : K \to K$:

$$
\begin{aligned}
B_i \psi &\equiv \Pi(z_i,\psi), \\
B_i^\dagger \psi &\equiv \Pi(\bar{z}_i,\psi).
\end{aligned}
\tag{3.21}
$$

An important point is that multiplying solutions of Dirac equation (3.20) by coordinate functions $(z_i \Psi)$ is an operation that commutes. However, this is not the case anymore, when projecting onto the space of solution of Dirac equation to define the matrices $B_1, B_2, B_1^\dagger, B_2^\dagger$.

We also need to look at the asymptotics of the solution at large $r^2 = |z_1|^2 + |z_2|^2$, where the gauge field approaches the pure gauge $A \to g^{-1}dg$ and thus the Dirac operator becomes the flat Dirac operator. Writing again in components, we need to solve (3.13):

$$
\begin{aligned}
D_{\bar{1}}^{flat} \psi_{\bar{2}} - D_{\bar{2}}^{flat} \psi_{\bar{1}} &= 0 \\
D_1^{flat} \psi_{\bar{1}} + D_2^{flat} \psi_{\bar{2}} &= 0.
\end{aligned}
\tag{3.22}
$$

To solve the second equation we consider a solution of the form $\psi_{\bar{\alpha}} = D_{\bar{\alpha}}^{flat} \chi$. Inserting this solution in the first equation we find that χ needs to solve the Laplace equation:

$$
\Delta_A \chi = 0, \tag{3.23}
$$

and thus χ needs to be a harmonic function[4]. Then, for $r \to \infty$:

$$
\psi_{\bar{\alpha}} \sim D_{\bar{\alpha}} \frac{1}{r^2} I^\dagger - \varepsilon_{\bar{\alpha}\bar{\beta}} g^{\gamma\bar{\beta}} D_\gamma \left(\frac{1}{r^2} J \right), \tag{3.24}
$$

where I, J are, respectively, $N \times k$ and $k \times N$ matrices:

$$
\begin{aligned}
I &: E \to K, \\
J &: K \to E,
\end{aligned}
\tag{3.25}
$$

and $E \cong C^N$ on flat space.

The construction of the solution leads to an algebraic structure on the vector space K, given by (3.21) and (3.25). It was shown in [29] that:

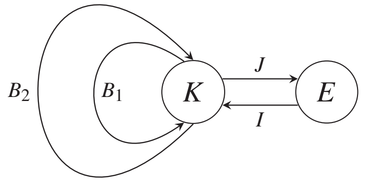
$$
\begin{aligned}
&[B_1, B_2] + IJ = 0 \\
&[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger = 0,
\end{aligned}
\tag{3.26}
$$

up to a $U(k)$ change of basis in the vector space K. These are the well-known ADHM equations describing a finite-dimensional hyperkähler manifold being

[3] The explicit projector, introducing the Laplacian operator $\not{D}_A \not{D}_A^\dagger \equiv \Delta_A$, is written as:$\Pi = 1 - \not{D}_A^\dagger \frac{1}{\Delta_A} \not{D}_A$.

[4] We stress that solving $\Delta_A \chi = 0$ only makes sense locally at infinity as we have shown above that, globally, χ needs to be zero.

the moduli space of anti-self-dual instantons. This can be written using a quiver diagram:



Above, we have solved the Dirac equations for spinors in the fundamental representations of the gauge group, however, we will mostly be interested in adjoint-valued spinors. In this case the dimension of the moduli space of instantons matches the degrees of freedom of the ADHM construction. These are four $k \times k$ matrices $(B_1, B_2, B_1^\dagger, B_2^\dagger)$ and two complex $N \times k$ matrices (I, J). Moreover, one has to impose three equations (3.26) and divide by $U(k)$ gauge transformations. Finally one finds that the dimension of the moduli space of instantons is $4Nk$.

The precise definition of the ADHM data goes as follows: take two complex vector spaces E, K of dimension respectively k and N and consider the following space:

$$\mathbb{X} = (K^* \otimes K) \oplus (E^* \otimes K) \oplus (K^* \otimes E). \tag{3.27}$$

An element of $\mathbb{X}$ is specified by a quadruple:

$$B_{1,2} \in \mathrm{End}(K), \quad I \in \mathrm{H}om(E, K), \quad J \in \mathrm{H}om(K, E). \tag{3.28}$$

We also need to define an anti-linear involution:

$$Y : (B_1, B_2, I, J) \to (B_2^\dagger, -B_1^\dagger, J^\dagger, -I^\dagger). \tag{3.29}$$

Hence, $\mathbb{X}$ is both hyperkähler and flat.

The action of the groups $U(k)$ and $SU(N)$ on $\mathbb{X}$ is naturally deduced from their action on K, E and preserves the hyperkähler structure. We define the three Hamiltonians (*i.e.* moment maps) generating the group action on $\mathbb{X}$:

$$\begin{aligned} \mu_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ. \end{aligned} \tag{3.30}$$

Then the ADHM equations (3.26) can be written as:

$$\mu_{\mathbb{R}} = 0, \qquad \mu_{\mathbb{C}} = 0. \tag{3.31}$$

Defining the inclusion $\iota : \mathbb{R} \cong u(1)^* \to u(k)^*$, the hyperkähler quotient of $\mathbb{X}$ is:

$$\mathbb{X}///U(k) \equiv \mu^{-1}\iota(\zeta^i)/U(k). \tag{3.32}$$

Finally, imposing the ADHM equations, one finds the moduli space of instantons:

$$\mathscr{M}_{N,k} \equiv \mu^{-1}\iota(0)/U(k), \tag{3.33}$$

whose dimension, as discussed above, is given by:

$$\dim \mathscr{M}_{N,k} = 4kN. \tag{3.34}$$

The original construction [25] consists in showing, given a set of solutions of (3.31), how to construct an anti-self-dual two-form. Here, we took the opposite route of motivating the ADHM equation starting with an instanton configuration.

3.2 Omega-background

Along the lines of Witten's [14] localization computation for the topological subsector of twisted $\mathscr{N} = 2$ SYM, discussed in the previous section, one can show how the infinite-dimensional path integral over the field configurations can be reduced to an integral over the finite-dimensional moduli space of instantons $\mathscr{M}_{N,k}$. The trick employed by Nekrasov [15] is to introduce the action of the flat space $SO(4)$ isometries on the instanton moduli space, through the Ω-background. This further localizes the path integral to a sum of contributions over the fixed point of $\mathscr{M}_{N,k}$ under the $T^2 \subset SO(4)$ rotations.

The Ω-background can be understood as the reduction to four dimensions of the following five-dimensional metric:

$$ds^2 = (dx_\mu + \mathscr{A}_\mu dx_5)^2 + dx_5^2. \tag{3.35}$$

The connection $\mathscr{A}_\mu$ is independent of the four-dimensional space and gives rise to an anti-self-dual field strength $\mathscr{F}_{\mu\nu}$. The metric (3.35) describes a $\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}$ bundle over S^1 with the following identification:

$$(z_1, z_2, 0) \sim (e^{i\beta\varepsilon_1}z_1, e^{i\beta\varepsilon_2}z_2, \beta), \tag{3.36}$$

where β is the circumference of the circle in the fifth dimension. The idea is then to study the reduction of a five-dimensional $\mathscr{N} = 1$ vector multiplet to an $\mathscr{N} = 2$ vector multiplet on $\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}$. We set the integral along x^5 of the gauge field at infinity to be:

$$\mathrm{diag}(e^{i\beta a_1}, \ldots, e^{i\beta a_{N-1}}) \in U(1)^{N-1} \tag{3.37}$$

Effectively, what we are doing in the four-dimensional set-up is to deform the scalar supercharge employed by Witten (2.3) by also considering the one-form supercharge G_μ:

$$\tilde{Q} = Q + \Omega_{\mu\nu} x^\nu G^\mu, \tag{3.38}$$

with $\Omega_{12} = \Omega_{21} = \varepsilon_1$ and $\Omega_{34} = \Omega_{43} = \varepsilon_2$. The deformation is related to $\mathscr{A}_\mu$ as follows:

$$\mathscr{A}_\mu = \Omega_{\mu\nu} x^\nu. \tag{3.39}$$

The Killing vector generating the $SO(4)$ transformation is:

$$v = i\varepsilon_1 \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) + i\varepsilon_2 \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right), \tag{3.40}$$

with $\varepsilon_{1,2} \in \mathbb{C}$. Note, however, that the isometries of flat space are generated by real $\varepsilon_{1,2}$.

With respect to the deformed supercharge, the supersymmetry transformations (2.4), become:

$$\begin{aligned}
&\delta A = i\Psi, && \delta\Psi = \iota_v F + i d_A \phi, \\
&\delta\varphi = i\eta, && \delta\eta = \mathscr{L}_v^A \varphi - [\phi, \varphi], \\
&\delta\chi = H, && \delta H = i\mathscr{L}_v^A \chi - i[\phi, \chi], \\
&\delta\phi = \iota_v \Psi,
\end{aligned} \tag{3.41}$$

Here, we have defined the covariant Lie derivative as:

$$\mathscr{L}_v^A = d_A \iota_v + \iota_v d_A = \mathscr{L}_v - i[\iota_v A,], \tag{3.42}$$

where ι_v is the contraction of v and an arbitrary differential form $\omega^{(n)}$:

$$(\iota_v \omega)^{(n-1)}_{\mu_1 \dots \mu_{n-1}} = v^\nu \omega^{(n)}_{\nu\mu_1 \dots \mu_{n-1}}. \tag{3.43}$$

Moreover, one can check that the square of a supersymmetry transformation gives:

$$\delta^2 = i\mathscr{L}_v - G_{\phi + i\iota_v A}, \tag{3.44}$$

where:

$$G_\varepsilon A = d_A \varepsilon, \tag{3.45}$$

and:

$$G_\varepsilon \bullet = i[\varepsilon, \bullet], \tag{3.46}$$

for all fields transforming in the adjoint.

With respect to the new supercharge $\tilde{Q}$, the topological observables computed by Witten are no longer invariant, except for those inserted at the origin of $\mathbb{R}^4$. However we can define new observables. The starting point is the five-dimensional supersymmetric partition function on $\mathbb{C}_{\varepsilon_1,\varepsilon_2} \times S^1$ in (3.35):

$$Z_{5d}^{Nek}(\beta, \varepsilon_{1,2}; a_{1,\dots,N-1}) = \mathrm{Tr}_{\mathscr{H}} (-1)^F e^{i\beta(\varepsilon_1 J_1 + \varepsilon_2 J_2 + \sum_{s=1}^{N-1} a_s Q_s)}. \tag{3.47}$$

This expression is a refined version of the Witten index obtained introducing the generators for the spatial rotations $J_{1,2}$ and the charges $Q_{1,\dots,N-1}$. The trace is taken over the Hilbert space on $\mathbb{C}_{\varepsilon_1,\varepsilon_2}$. As above one can use supersymmetry to restrict the Hilbert space only to the action-minimizing instantons. Then, we need to compute:

$$Z_{5d}^{Nek}(\beta,\varepsilon_{1,2};a_{1,\dots,N-1}) = \sum_{n\geq 0} e^{-\frac{8\pi^2 n\beta}{g_{5,YM}^2}} \mathrm{Tr}_{\mathscr{H}_n}(-1)^F e^{i\beta(\varepsilon_1 J_1+\varepsilon_2 J_2+\sum_{s=1}^{N-1} a_s Q_s)}. \tag{3.48}$$

We have defined $g_{5,YM}$ the five-dimensional Yang-Mills coupling constant. The Hilbert space $\mathscr{H}_n$ coincides with the instanton moduli space $\mathscr{M}_{N,k}$. The $U(1)^2 \subset SO(4)$ transformations generated deforming the supercharge $\tilde{Q}$ as in (3.38) can be naturally extended to act on $\mathscr{M}_{N,k}$ and similarly for the $SU(N)$ gauge transformations.

Now, if the fixed point of $\mathscr{M}_{N,k}$ under $U(1)^{N+1} \subset T^2 \times SU(N)$ were isolated, we could apply the fixed points theorem [30] and find:

$$Z_{5d}^{Nek}(\beta,\varepsilon_{1,2};a_{1,\dots,N-1}) = \sum_{n\geq 0} e^{-\frac{8\pi^2 n\beta}{g_{5,YM}^2}} \sum_p \prod_{t=1}^{4kN} \frac{1}{1-e^{i\beta v_t(p)}}, \tag{3.49}$$

where v_t are linear combinations of ε_1, ε_2 and $a_1,\dots,a_{N-1}$. Moreover, identifying the characters under T^2 with a vector space, we can write:

$$T\mathscr{M}_{N,k}|_p = \sum_{t=1}^{4kN} e^{i\beta v_t(p)}. \tag{3.50}$$

To take the four-dimensional limit we need to shrink the radius of the circle in the fifth direction taking the limit $\beta \to 0$ while keeping the parameters $\varepsilon_1,\varepsilon_2,a_i$ of $T^2\times SU(N)$ fixed. A term at the n-th instanton sector in (3.49) has a factor $\sim \beta^{-4kN}$. To find a meaningful limit $\beta \to 0$ we take the classical contribution to be:

$$e^{-\frac{8\pi^2\beta}{g_{5,YM}^2}} = (-i\beta)^{4k} q. \tag{3.51}$$

Hence, if we keep fixed q while $\beta \to 0$, we find:

$$Z^{Nek}(\varepsilon_{1,2};a_{1,\dots,N-1}) = \sum_{n\geq 0} q^n \sum_p \prod_{t=1}^{4kN} \frac{1}{v_t(p)}. \tag{3.52}$$

The four-dimensional coupling constant is related to $g_{5,YM}$ as follows:

$$\frac{8\pi^2}{g_{4,YM}^2} = \frac{8\pi^2\beta}{g_{5,YM}^2}. \tag{3.53}$$

However, it turns out that the tangent space is not well defined at the fixed points of $\mathscr{M}_{N,k}$. To remove this problem we need to consider a non-commutative deformation of space-time [31–33]:

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}, \tag{3.54}$$

where x^{μ} are coordinates on $\mathbb{R}^4$ and $\theta^{\mu\nu}$ a real anti-symmetric matrix. Technically, this is equivalent to consider the non-zero level of the moment map (3.30):

$$\mu_{\mathbb{R}} = \zeta_{\mathbb{R}}\mathbf{1} \sim (\theta_{1\bar{1}} - \theta_{2\bar{2}})\mathbf{1}. \tag{3.55}$$

We denote the corresponding instanton moduli space $\mathscr{M}^{nc}_{N,k}$. With this modification it is possible to make sense of the above expressions (3.49) and (3.52). The result is independent of the value of $\zeta_{\mathbb{R}}$ and thus we can consider $\zeta_{\mathbb{R}} = 0$ where, however, there is no interpretation in terms of contributions coming from different fixed points.

To actually describe (3.50), we need to introduce the T^2-action on $\mathbb{X}$ (3.27). Following [34]:

$$\mathbb{X}' = (T_1^{\otimes -1} \oplus T_2^{\otimes -1}) \otimes (K^* \otimes K) \oplus (E^* \otimes K) \oplus (T_1^{\otimes -1} \otimes T_2^{\otimes -1} \otimes K^* \otimes E), \tag{3.56}$$

where T_i is a one-dimensional space on which the generators for the spatial rotations J_i have eigenvalue $+1$. As above, we identify a vector space with its character and rewrite $\mathbb{X}'$ as:

$$\mathbb{X}' = (e^{-i\beta\varepsilon_1} + e^{-i\beta\varepsilon_2})(K^*K) + (E^*K) + (e^{-i\beta(\varepsilon_1+\varepsilon_2)})(K^*E). \tag{3.57}$$

Then, the fixed points of $U(1)^{1+N}$ on $\mathscr{M}^{nc}_{N,k}$ have been classified in [35] and are labeled by $N-1$ Young diagrams:

$$Y = (Y_1, \ldots, Y_{N-1}), \quad \text{such that the number of boxes} \quad |Y| = k. \tag{3.58}$$

Denoting $(i, j) \in Y$ the position in a Young diagram for a fixed point p, the action on K and E of T^2 and $U(1)^{N-1}$ is:

$$\begin{aligned} K_p &= \sum_{s=1}^{N-1} \sum_{(i,j)\in Y_s} e^{i\beta(a_s+(1-i)\varepsilon_1+(1-j)\varepsilon_2)}, \\ E_p &= \sum_{s=1}^{N-1} e^{i\beta a_s}. \end{aligned} \tag{3.59}$$

Therefore, one can read $v(p)_t$ entering (3.52) from:

$$T\mathscr{M}^{nc}_{N,k}|_p = E_p^* K_p + e^{i\beta(\varepsilon_1+\varepsilon_2)} K_p^* E_p - (1-e^{i\beta\varepsilon_1})(1-e^{i\beta\varepsilon_2}) K_p K_p^*. \tag{3.60}$$

Finally, one can check that Nekrasov partition function $Z^{Nek}(\varepsilon_{1,2}; a_{1,\ldots,N-1})$ for anti-self-dual instantons reproduces the SW prepotential (3.1):

$$\mathscr{F} = \lim_{\varepsilon_{1,2}\to 0} \varepsilon_1\varepsilon_2 \log Z^{Nek}(\varepsilon_{1,2}; a_{1,\ldots,N-1}), \tag{3.61}$$

This limit can be understood as follows: the partition functions on the Ω-background is finite as the deformation makes the volume:

$$V = \frac{1}{\varepsilon_1 \varepsilon_2}, \tag{3.62}$$

finite. Clearly, V diverges for $\varepsilon_1, \varepsilon_2 \to 0$ and so does the free energy:

$$F = -\log Z^{Nek}(\varepsilon_{1,2}; a_{1,\dots,N-1}). \tag{3.63}$$

However, the SW prepotential coincides with the specific free energy F/V which does not diverge.

So far, we have only considered pure gauge theories. The inclusion of matter in the Ω-background, both in the fundamental and in the adjoint representation of the gauge group, has been studied in [15, 27] and it will be discussed in chapter 7.

4. Pestunization

On S^4, a non-topological theory with the field content of an $\mathcal{N}=2$ SYM theory has been studied in the seminal work of Pestun [17]. The result of Pestun shows how the partition function on S^4 can be computed gluing an instanton and an anti-instanton Nekrasov partition function, respectively at the north and south pole of S^4, where the theory on the sphere is identified with the theory on the Ω-background discussed in chapter 3. On the other hand, an equivariant version of Witten's topologically twisted SYM is found placing instantons at both north and south poles.

The localization computation performed in [17] employs only a single $U(1)$ contained in the $SO(5)$ isometry group. This corresponds to setting $\varepsilon_1=\varepsilon_2$ in the Ω-background. Later, in [36], the entire $U(1)^2$ Cartan of the isometry group was employed. Their result can be written as follows:

$$Z_{S^4}=\int_h dae^{-S_{cl}}\left|Z^{inst}_{\varepsilon_1,\varepsilon_2}(ia,q)\right|^2, \tag{4.1}$$

where, for convenience, we included the perturbative contribution in (3.52) and define:

$$Z^{inst}_{\varepsilon_1^i,\varepsilon_2^i}\equiv Z^{pert}\cdot Z^{Nek}. \tag{4.2}$$

It is important, in Pestun's result, that $\varepsilon_1,\varepsilon_2\in\mathbb{R}$. Moreover, the expectation value of certain supersymmetric Wilson loops operators are also computed in [17]. In recent years there has been extensive progress towards localizing SQFTs living on manifolds in different dimensions and with different amounts of supercharges. For a review see [11].

The work of Pestun is also crucial in deriving the AGT (Alday, Gaiotto, Tachikawa) correspondence [37] between a class of $d=4$ $\mathcal{N}=2$ superconformal field theories (SCFTs) [38] and Liouville field theory on punctured Riemann surfaces. The correspondence is understood, geometrically, starting from $d=6$ $(2,0)$ theories of type A_1 and reducing on a punctured Riemann surface, with the simplest example being that of reducing on a two-torus with modular parameter τ, which leads to $\mathcal{N}=4$ $SU(2)$ gauge theory. The $SL(2,\mathbb{Z})$ transformations of the modular parameter give rise to S-duality transformations of the $\mathcal{N}=4$ theory.

Instead of reviewing Pestun's work on S^4, we conclude the introductory part focusing on more general four-dimensional $\mathcal{N}=2$ vector multiplets on a compact manifold M admitting a T^2-isometry and a Killing vector with a discrete set of fixed points. The framework developed in [18] consists in defining a modified notion of (anti-)self-duality which allows for flips between self-dual two-forms to anti-self-dual ones at different fixed points. In general, then, one can consider an arbitrary choice of field strength F approaching, at each fixed point, either instantons or anti-instantons. Hence, the full partition function on M is conjectured to be obtained patching Nekrasov instanton or anti-instanton partition functions (3.52) at each fixed point:

$$Z_M = \sum_{k_i} \int_h da e^{-S_{cl}} \prod_{i=1}^{p} Z^{inst}_{\varepsilon_1^i,\varepsilon_2^i}(ia + k_i(\varepsilon_1^i,\varepsilon_2^i), q) \prod_{i=p+1}^{l} Z^{anti-inst}_{\varepsilon_1^i,\varepsilon_2^i}(ia + k_i(\varepsilon_1^i,\varepsilon_2^i), \bar{q}), \tag{4.3}$$

where the parameters $\varepsilon_1^i, \varepsilon_2^i \in \mathbb{C}$ determine the T^2-action at each fixed point, as given by the Killing vector v (3.40). We have also introduced a sum over possible flux contributions k_i which will be discussed in chapter 6. As discussed above, equivariant DW and Pestun's theory can be obtained from (4.3) setting $p = l = 2$ in the first case and $2p = l = 2$ in the latter one. In particular, to match with (4.1), one needs to assume $\varepsilon_1^i, \varepsilon_2^i \in \mathbb{R}$. Only with this assumption it is true that:

$$Z^{anti-inst}_{\varepsilon_1^i,\varepsilon_2^i}(ia, \bar{q}) = \overline{Z^{inst}_{\varepsilon_1^i,\varepsilon_2^i}(ia, q)}. \tag{4.4}$$

The framework developed in [18, 19] is quite powerful as it allows to compute partition functions with different distributions of (anti-)instantons at the different fixed point of a generic four manifold. In this chapter we will review this construction highlighting some of the open problems, which will then be tackled in Part II.

4.1 Projector

We consider the manifold M and its open cover $M = \cup_{i=1}^{l} U_i$ such that each set contains at most one fixed point of the T^2-action. The idea is to define a generalized projector on the space of two-forms Ω^2 which approaches, at each fixed point, the standard (anti-)self-dual projector $P^{\pm} = (1 \pm \star)$. Now, we consider the intersection $U_i \cap U_j$ which, by definition, does not contain any fixed point. Also U_i, U_j contain fixed points where the two-forms approach, respectively, anti-self-dual Ω^{2+} and self-dual two-forms Ω^{2-}. We will denote the former plus fixed points and the latter minus fixed point. To define a projector on the entire M, we need to find a map $m_{ij} : \Omega^{2+} \to \Omega^{2-}$:

$$m_{ij} : \; B \to -B + \frac{2}{\iota_v \kappa} \kappa \wedge \iota_v B. \tag{4.5}$$

Here we have defined the one-form κ as $\kappa = g(v)$ and $\iota_v \kappa = g(v,v) = ||v||^2$. In order to glue patches containing fixed points of the same kind one uses the identity map.

Once it is understood how to glue $\Omega^{2\pm}$ bundles, we need to construct the projector. Away from the fixed points, m^2 acts like the identity on Ω^2 and, assuming that $\alpha^2+\beta^2=1$, one finds:

$$(\alpha\star+\beta m)^2=1. \tag{4.6}$$

Employing this condition, and introducing $\alpha=\cos 2\rho$, $\beta=\sin 2\rho$, the most general projector constructed using $\star$ and m is given by:

$$P^+_\omega=\frac{1}{2}(1+\cos 2\rho\star+\sin 2\rho m). \tag{4.7}$$

We also need to impose that $2\rho|_{v=0}=0,\pi$ which ensures that one recovers the standard projector $P^\pm=\frac{1}{2}(1\pm\star)$ at the zeroes of v. Moreover, with a further change of variables:

$$1-\sin 2\rho=\frac{2}{1+\cos^2\omega}, \tag{4.8}$$

we rewrite (4.7) as:

$$P^+_\omega=\frac{1}{1+\cos^2\omega}\left(1+\cos\omega-\sin^2\omega\frac{\kappa\wedge\iota_v}{\iota_v\kappa}\right). \tag{4.9}$$

The function ω is such that $\omega=0$ at plus fixed points and $\omega=\pi$ at minus ones. Substituting $\omega=0,\pi$ one recovers the standard projector $P^\pm=\frac{1}{2}(1\pm\star)$. However, in order for the projector to be well defined at the fixed points of v, we need that $\sin^2\omega$ goes to zero at least as $\iota_v\kappa=||v||^2$. Finally, field strengths satisfying $P^+_\omega F=0$ are denoted flip instantons. We can thus use this projector to define a generalized decomposition of two-forms as:

$$\Omega^2=P^+_\omega\Omega^2\oplus P^-_\omega\Omega^2, \tag{4.10}$$

where $P^-_\omega=1-P^+_\omega$.

4.2 Cohomological theory

We can use the projector defined above to explain the difference between equivariant DW theories and Pestun-like theories[1] on a generic compact manifold M with isolated fixed points under T^2. Linearization of topologically twisted SYM is related to an elliptic complex:

$$\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{P^+d}P^+\Omega^2(M)=\Omega^{2+}(M), \tag{4.11}$$

where d is the de Rham differential and $P^\pm$ the (anti-)self-dual projector $P^\pm=\frac{1}{2}(1\pm\star)$. We denote this complex $(E^\bullet,P^+d)$. The cohomology $\Omega^1(M)$ represents small deformations δA of the gauge connection in the kernel of P^+d,

[1] We define Pestun-like theories the cases where not all fixed points are of the same kind.

that is such that the anti-self-dual condition is maintained. These deformations, as in chapter 2, need to be taken modulo gauge transformations, that is up to elements in the image of the first map d. The ellipticity of the problem is related to the fact that the moduli space of instantons is finite-dimensional.

On manifolds with a T^2-isometry it possible to consider a more general complex, defined using the projector (4.9) introduced above:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \oplus \Omega^0(M) \xrightarrow{\tilde{D}} P_\omega^+ \Omega^2(M) \oplus \Omega^0(M), \tag{4.12}$$

where:

$$\tilde{D} = \begin{pmatrix} P_\omega^+ d & P_\omega^+ \iota_v \star d \\ d^\dagger \iota_v d & -d^\dagger d \cos\omega \end{pmatrix}. \tag{4.13}$$

We denote the complex (4.12) as $(E^\bullet, \tilde{D})$. Flips between instantons and anti-instantons at different fixed points is what characterizes Pestun-like theories.

For equivariant DW theory the supersymmetry transformations are those in (4.14):

$$\begin{aligned} \delta A &= i\Psi, & \delta\Psi &= \iota_v F + i d_A \phi, \\ \delta\varphi &= i\eta, & \delta\eta &= \mathscr{L}_v^A \varphi - [\phi,\varphi], \\ \delta\chi &= H, & \delta H &= i\mathscr{L}_v^A \chi - i[\phi,\chi], \\ \delta\phi &= \iota_v \Psi, \end{aligned} \tag{4.14}$$

In particular both χ and H are anti-self-dual two-forms. When considering the complex $(E^\bullet, \tilde{D})$ (4.12), associated to a manifold with a generic distribution of plus/minus fixed points, the two-forms need to satisfy:

$$P_\omega^+ \chi = \chi, \qquad P_\omega^+ H = H. \tag{4.15}$$

Moreover, the Lie derivative $\mathscr{L}_v$ preserves the decomposition of two-forms in $P_\omega^+\Omega^2 \oplus P_\omega^-\Omega^2$. Therefore, the same supersymmetry transformations (4.14) hold also in this case.

The observables of the theory are generalizations of observables of equivariant DW theory. One can check that:

$$\delta(\phi + \Psi + F) = (i d_A + \iota_v)(\phi + \Psi + F), \tag{4.16}$$

thus:

$$\delta \mathrm{Tr}(\phi + \Psi + F)^k = (i d_A + \iota_v) \mathrm{Tr}(\phi + \Psi + F)^k. \tag{4.17}$$

If considering a form Ω such that $\delta\Omega = 0$, then also:

$$\delta \int_M \Omega \wedge \mathrm{Tr}(\phi + \Psi + F)^k = 0. \tag{4.18}$$

These will be the observables computed and do not depend on a δ-exact deformations of Ω:

$$\Omega \to \Omega + (id + \iota_v)(\ldots). \tag{4.19}$$

In particular, we show now how the SYM Lagrangian can be recast in this form. The SYM Lagrangian on a generic four-manifold can be found through a rigid supergravity approach [22] and its more generic form can be found in [18]. As we need something quadratic in F for the Lagrangian, we look at:

$$\begin{aligned}\mathscr{O} &= \int_M (\Omega_0+\Omega_2+\Omega_4)\wedge \mathrm{Tr}(\phi+\Psi+F)^2 \\ &= \int_M (\mathrm{Tr}(\phi^2)\Omega_4 + 2\Omega_2\wedge \mathrm{Tr}(\phi F)+\Omega_0 \mathrm{Tr}(F^2)+\Omega_2\wedge \mathrm{Tr}(\Psi^2)).\end{aligned} \tag{4.20}$$

The computation for generic four manifolds can be found in [18], here we focus on S^4 with Killing vector $v=\partial_\alpha+\partial_\beta$. One can check that taking:

$$\begin{aligned}\Omega_0 &= \cos\theta, \\ \Omega_2 &= -i(\sin\theta d\theta\wedge(xd\alpha+(x-1)d\beta)+\frac{i}{2}\cos\theta\sin^2\theta dx\wedge(d\alpha+d\beta), \\ \Omega_4 &= \frac{3}{2}\sin^3\theta d\theta\wedge dx\wedge d\alpha\wedge d\beta = 3\mathrm{Vol}_{S^4},\end{aligned} \tag{4.21}$$

reproduces Pestun's action on S^4 [17]:

$$S=\frac{1}{g_{YM}^2}\int_{S^4}(\cos\theta+\Omega_2+3\mathrm{Vol}_{S^4})\mathrm{Tr}(\phi+\Psi+F)^2+\delta(\ldots), \tag{4.22}$$

up to a term $\sim F\wedge F$. We also observe that the top component Ω^4 is the volume form on S^4 and more generally $\Omega=\Omega_0+\Omega_2+\Omega_4$ is the equivariant extension of the volume form on S^4.

4.3 BPS locus

We have described how supersymmetry behaves in this more general set-up which includes, in a unique framework, both equivariant DW and Pestun-like theories on more generic four manifolds and, in the last section, the observables we are interested in computing. This section is devoted to study the reduced integration locus, obtained deforming the path integral inserting a δ-exact term with coefficient t. Taking the limit $t\to\infty$ we restrict the integration domain in the path integral to configurations minimizing the deformation, that is supersymmetric configurations.

The localization locus is defined by the intersection of imposing reality conditions on the fields and solving the supersymmetry transformations. The reality conditions are needed to ensure that the localization action is positive definite and gives a well-defined Gaussian integration. Focusing on the bosonic fields we have that F is Hermitian and the scalars:

$$\varphi\in\mathbb{R},\qquad \tilde{\phi}=\phi+\frac{i}{2}(s-\tilde{s})\varphi\in\mathbb{R}. \tag{4.23}$$

Similarly, H is chosen such that the Gaussian integral converges.

The localization term considered is:

$$\mathscr{L}_{loc}\mathrm{Vol}_M = \delta\mathrm{Tr}\Big[2(2\Omega - H)\wedge\star\chi + (\iota_v F - i d_A(\phi + i(s-\tilde{s})\varphi))\wedge\star\Psi + \\ +(\iota_v d_A\varphi + [\phi,\varphi])\eta)\Big]. \tag{4.24}$$

The adjoint valued two-form $\Omega(\Phi)$ depends on the bosonic fields in the particular theory under consideration, with the requirement that:

$$\begin{aligned} &\mathscr{L}_v\Omega(\Phi) = \Omega(\mathscr{L}_v\Phi), \\ &\Omega(\Phi) \text{ real on the integration contour.} \end{aligned} \tag{4.25}$$

Focusing on the bosonic terms:

$$\begin{aligned} \mathscr{L}^{bos}_{loc}\mathrm{Vol}_M = \mathrm{Tr}\Bigg[&\frac{1}{4}(2\iota_v F + d_A((s-\tilde{s})\varphi))\wedge\star(2\iota_v F + d_A((s-\tilde{s})\varphi)) + \\ &+ d_A\tilde{\phi}\wedge\star d_A\tilde{\phi} + \mathrm{Vol}_M((\iota_v d_A\varphi)^2 - [\tilde{\phi},\varphi]^2) + \\ &+ 2P^+_\omega\Omega\wedge\star\Omega - 2P^+_\omega(H-\Omega)\wedge\star(H-\Omega)\Bigg]. \end{aligned} \tag{4.26}$$

Hence, asking for vanishing supersymmetry transformations gives:

$$\begin{aligned} &[\tilde{\phi},\varphi] = 0, \quad \iota_v d_A\varphi = 0, \quad d_A\tilde{\phi} = 0, \\ &2\iota_v F + d_A((s-\tilde{s})\varphi) = 0, \quad P^+_\omega\Omega = 0. \end{aligned} \tag{4.27}$$

The first line is solved imposing:

$$\varphi = \mathrm{diag}(\varphi^a), \qquad \tilde{\phi} = \mathrm{diag}(\tilde{\phi}^a), \tag{4.28}$$

where a runs over the element in the $U(1)^{N-1}$ Cartan of the gauge group. Moreover, at the fixed points of M, the projector P^+_ω reduces to the standard (anti-)self-dual projector and we need to consider point-like instantons or anti-instantons respectively for $\tilde{s} = 0$ or $s = 0$. One can also find solutions for the field strength with flux, but these will be considered in chapter 6.

Once the localization locus is found, the next step is to compute the one-loop superdeterminant around these configurations. The result will split into contributions coming from each fixed point which then need to be glued consistently. The technique developed in [18, 19] motivates the computation only for certain distribution of $\pm$ fixed points on certain manifolds which can be obtained dimensionally reducing Sasaki-Einstein manifolds in $d = 5$. Thus, we leave the explicit computations for chapter 5, where we will develop a way to patch contributions which applies to generic four manifolds admitting a Killing vector with isolated fixed points.

Part II:
Results

5. Perturbative partition function

In Part I we have presented an expression (4.3) for the full partition function on certain compact manifolds M with a T^2-isometry. In this chapter, whose content is based on Paper I, we provide a method to compute the perturbative part of (4.3), entering through (4.2).

We concluded the previous chapter with the study of the BPS locus of field configurations where the localization action (4.24) vanishes. Using localization, the one-loop contribution is thus given by a Gaussian integral around these configurations as in the simpler case of (non-equivariant) DW theory studied in chapter 2. Hence, focusing on the trivial instanton sector, we will show how the integral over bosonic and fermionic fields gives rise to a superdeterminant. This can be equivalently computed by the equivariant index of either an elliptic or a transversally elliptic operator, respectively for equivariant DW and Pestun-like theories. The study of these operators on compact manifolds has been developed in [30].

Elliptic operators have been defined below (2.25) as operators whose principal symbol $\sigma_\xi(\cdot)$, obtained by replacing the highest order derivatives with vector fields ξ, is invertible for non-zero ξ. Moreover, elliptic operators on compact manifolds have finite-dimensional kernel and cokernel. Thus, to compute the equivariant index of an elliptic operator it is enough to employ the Atiyah-Bott formula [39]. If we take the elliptic complex $(E^\bullet, P^+d)$ (4.11) associated to equivariant DW theory, we find:

$$\text{index}\,(E^\bullet, P^+d) = \sum_p \frac{\chi_{eq}(\Omega^\bullet_p)}{\det(1-df)}, \tag{5.1}$$

where $\chi_{eq}(\Omega^\bullet_p)$ is the equivariant Euler character of the fiber $\Omega^\bullet_p$ and the diffeomorphism induced by the T^2-action on M is labeled by the map $f : M \to M$. Thus, the index is a sum over finite-dimensional contributions arising from the fixed points of the manifold under a T^2-action. Examples of this procedure, related to equivariant DW, can be found in [18].

Transversally elliptic operators, instead, are operators which are elliptic only in directions transversal to the orbits of T^2. Hence, the cohomologies turn out to be infinite-dimensional and using (5.1) gives infinite power series at each fixed point. The problem, from a physical perspective, is to regularize these contributions so that, once patched together on M, they give the correct perturbative partition function. From a mathematical point of view, instead, we need to compute the equivariant index of a transversally elliptic operator. Such computations have been performed on S^4 for $\mathcal{N}=2$ theories [17, 36] and, more in general, for certain distributions of $\pm$ fixed points on four manifolds which can be uplifted to five-dimensional $\mathcal{N}=1$ theories on Sasaki-Einstein manifolds [18, 19]. Here, we extend the latter result by developing an approach applicable to any four-dimensional compact manifold with a T^2-action and isolated fixed points, following the study of the equivariant index of transversally elliptic operators in [30].

5.1 Complex from localization

In this section we show the relation between the perturbative partition function of Pestun-like theories on M and the equivariant index of the associated complex. Moreover, we introduce the objects which will be required, later, to compute the equivariant index of a transversally elliptic operator. In particular, we need the symbol of the transversally elliptic complex associated to Pestun-like theories.

Let us start from the supersymmetry transformations (4.14), which can be written schematically as:

$$\begin{aligned} Q\phi &= \psi, \qquad & Q\psi &= \mathscr{L}_v^A \phi, \\ Q\tilde{\psi} &= \tilde{\phi}, \qquad & Q\tilde{\phi} &= \mathscr{L}_v^A \tilde{\psi}. \end{aligned} \tag{5.2}$$

where $\phi=(A,\phi,\varphi)$, $\tilde{\phi}=H$ and $\psi=(\Psi,\eta)$, $\tilde{\psi}=\chi$ label, respectively, even and odd fields. The superdeterminant arises from expanding the Q-exact action (4.24) around the BPS configurations (4.28) at quadratic order:

$$QV^{(2)} = Q\left(\langle\psi,\mathscr{L}_v^A\phi\rangle + \langle\tilde{\psi},\tilde{D}\phi\rangle + \langle\tilde{\psi},\tilde{\phi}\rangle\right). \tag{5.3}$$

Here, $\tilde{D}$ is the differential operator given in (4.13). Integrating (5.3) gives rise to a ratio of one-loop determinants of fermionic and bosonic contributions:

$$\frac{\det^{1/2}|_{\mathrm{coker}\,\tilde{D}}\mathscr{L}}{\det^{1/2}|_{\ker\,\tilde{D}}\mathscr{L}} = \mathrm{sdet}^{1/2}|_{H^\bullet(\tilde{D})}\mathscr{L}, \tag{5.4}$$

where $H^\bullet(\tilde{D})$ is the cohomology of the operator $\tilde{D}$ mapping ϕ to $\tilde{\psi}$. We stated above that these cohomologies are infinite-dimensional for transversally elliptic operators, unlike elliptic operators. However, a property of transversally elliptic operators is that both the kernel and the cokernel can be decomposed into irreducible representations of T^2, labeled by α, and appearing with finite multiplicities m_α. Hence, we find:

$$\mathrm{sdet}^{\frac{1}{2}}\mathscr{L} = \prod_\alpha w_\alpha(\varepsilon)^{\frac{1}{2}(m_\alpha^{\mathrm{coker}} - m_\alpha^{\mathrm{ker}})}, \tag{5.5}$$

where $w_\alpha(\varepsilon)$ denote the weights of the representation depending on the equivariant parameters. This can be translated into the equivariant index of $\tilde{D}$:

$$\text{index}\ \tilde{D} = \sum_\alpha (m_\alpha^{\text{coker}} - m_\alpha^{\text{ker}}) e^{w_\alpha}, \tag{5.6}$$

where now e^{w_α} is the character of the corresponding representation.

As discussed in chapter 4, the complex associated to Pestun-like theories is given by $(E^\bullet, \tilde{D})$:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \oplus \Omega^0(M) \xrightarrow{\tilde{D}} P_\omega^+ \Omega^2(M) \oplus \Omega^0(M). \tag{5.7}$$

An equivalent complex is found by folding:

$$\Omega^1(M) \oplus \Omega^0(M) \xrightarrow{\eth} P_\omega^+ \Omega^2(M) \oplus \Omega^0(M) \oplus \Omega^0(M). \tag{5.8}$$

The differential operator $\eth$ is given by[1]:

$$\eth = \begin{pmatrix} P_\omega^+ \mathrm{d} & P_\omega^+ \iota_v \star \mathrm{d} \\ \mathrm{d}^\dagger \iota_v \mathrm{d} & \mathrm{d}^\dagger \mathrm{d} \cos\omega \\ \mathrm{d}^\dagger & 0 \end{pmatrix}, \tag{5.9}$$

Notice that here we have been neglecting the gauge part of the index. It contributes as follows:

$$\text{index}(\eth \otimes 1) = \text{index}\ \eth \cdot \chi_{Ad}, \tag{5.10}$$

where χ_{Ad} is the character for the adjoint representation of G. In the rest of the chapter we will focus on the part arising from the T^2-action only.

Hence, let us consider the complex (5.8). At the fixed points of the Killing vector v the complex splits into the part corresponding to the (anti)-self-dual (ASD/SD) part and the scalar Laplacian:

$$\left(\Omega^1(M) \oplus \Omega^0(M) \xrightarrow{d^\pm \oplus d^\dagger} \Omega^{2+}(M) \oplus \Omega^0(M)\right) \oplus \left(\Omega^0(M) \xrightarrow{\Delta} \Omega^0(M)\right). \tag{5.11}$$

Thus, the original complex approaches either an ASD complex $(\Omega^\bullet, d^+)$ or an SD one $(\Omega^\bullet, d^-)$.

Computing the index requires the knowledge of the symbol of a complex. Considering the complex in (5.8), its symbol complex denoted by $\sigma(\eth)$ is given by:

$$\pi^* \left(\Lambda^1(M) \oplus \Lambda^0(M)\right) \xrightarrow{\sigma(\eth)} \pi^* \left(P_\omega^+ \Lambda^2(M) \oplus \Lambda^0(M) \oplus \Lambda^0(M)\right), \tag{5.12}$$

where $\Lambda^i(M) = \Lambda^i T^*M$ and $\pi : T^*M \to M$ is the projection.

[1] Notice that index $(E^\bullet, \tilde{D}) = -$index $(E^\bullet, \eth)$ as the complex (5.8) starts at level one.

It is useful to describe the symbol as an element of the equivariant K-group K_{T^2} over the cotangent bundle[2] TM, which is defined as:

$$K_{T^2}(TM) \equiv C^n_{T^2}(M)/C^n_{\emptyset,T^2}(M), \tag{5.13}$$

where $C^n_{T^2}(M)$ is the set of complexes, up to homotopy, of length n, with compact support[3] and respecting the T^2-action. Instead, $C^n_{\emptyset,T^2}(M)$ labels those with empty support. For a proof of this statement see [40, 41] or appendix B of Paper I. We can now exploit the relation of the analytic index($\eth$) to what is called the topological index of $\eth$ [30]:

$$\mathrm{ind}_{T^2}[\sigma(\eth)] = \mathrm{index}(\eth), \tag{5.14}$$

where, for an $R(T^2)$-module homomorphism:

$$\mathrm{ind}_{T^2}:\ K_{T^2}(TM) \to \mathscr{D}'(T^2). \tag{5.15}$$

Here, $R(T^2)$ denotes the representation ring of T^2 and, unlike for an elliptic complex, the image of the equivariant index of a transversally elliptic complex is not a regular function but rather an element of the space of distributions $\mathscr{D}'(T^2)$ over the space of test functions on T^2.

Explicitly, the map (5.12) restricted to the fiber over $(x,\xi) \in M \times T_xM$ is given by:

$$\begin{aligned}\sigma(\eth)|_{(x,\xi)} : (a,\phi) \longrightarrow \Big(&P^+_\omega[\xi \wedge a + \star(\xi \wedge \kappa)\phi],\\ &\|\xi\|^2 \iota_v a - \xi_v\langle\xi,a\rangle - \|\xi\|^2\cos\omega\ \phi, -\langle\xi,a\rangle\Big),\end{aligned} \tag{5.16}$$

where $a \in \Lambda^1(M)$, $\phi \in \Lambda^0(M)$ and we defined $\xi_v \equiv \langle\xi,v\rangle$. Notice that the support of $\sigma(\eth)$ is the zero section over M and, at $\cos\omega = 0$, also $\xi \neq 0$ in the subspace of the tangent space that is along v.

Finally, one can show that the scalar Laplacian part of the symbol complex splits globally:

$$\mathrm{ind}_{T^2}[\sigma(\eth)] = \mathrm{ind}_{T^2}[\sigma_\omega] + \mathrm{ind}_{T^2}[\sigma(\Delta)]. \tag{5.17}$$

It is shown in Paper I that $\mathrm{ind}_{T^2}[\sigma(\Delta)] = 0$ and therefore we focus on the symbol σ_ω which approaches $\sigma(d^\pm)$ at the ASD/SD fixed points (5.11). This is the object we need to compute and which can be used to obtain the perturbative part of the partition function on M, as in (5.5).

[2]Using the metric, we identify tangent and cotangent bundles over M in the following.

[3]The support of a complex is defined as the points $x \in M$ where $[\sigma_\omega]|_x$ fails to be exact.

5.2 Trivialization

To compute (5.17) one could follow [42, 43], however, we find it more convenient to follow the approach of [30] focusing on σ_ω. The idea behind the trivialization of the symbol (5.16), is to deform it in order to reduce its support to the zero section at the set of fixed points, where the complex is elliptic. This is achieved by deforming the symbol in the direction of the Killing vector v, or against it. The choice of sign will follow the distribution of ASD/SD complexes at the fixed points. From a physical perspective, the direction of v will instruct us on how to regularize the infinite power series arising using the Atiyah-Bott formula (5.1) at each fixed point. Explicit examples will be shown in the next section.

The first step in the construction is to isolate the fixed point contributions in σ_ω. For this purpose, we can define a filtration, introducing $M_i = \{x \in M | \dim H_x \geq i\}$ where H_x is the stabilizer set of x:

$$M = M_0 \supset M_1 \supset M_2 \supset M_3 = \emptyset. \tag{5.18}$$

Notice that M_2 is the set of the fixed points of M under the T^2-action. It was shown in [30] that for this filtration there exists a homomorphism θ_i and split short exact sequences[4]:

$$K_{T^2}(T_{T^2}(M-M_i)) \to K_{T^2}(T_{T^2}(M-M_{i+1})) \overset{\theta_i}{\underset{}{\overleftarrow{\longrightarrow}}} K_{T^2}(T_{T^2}X|_{M_i-M_{i+1}})). \tag{5.19}$$

From this, we find recursively:

$$K_{T^2}(T_{T^2}M) = \bigoplus_{i=0}^{2} \theta_i K_{T^2}(T_{T^2}M|_{M_i-M_{i+1}}). \tag{5.20}$$

The crucial observation [30] is that, using the Killing vector v, we can define a new symbol homotopic to σ_ω by trivializing it away from the zero section at the fixed points M_2. The new symbol has only support at M_2, where the symbol complex is elliptic, and gets contributions only from the fixed points:

$$[\sigma_\omega] = [0] + \theta_2[\sigma_\omega|_{M_2}] \in K_{T^2}(T_H(M-M_2)) \oplus \theta_2 K_{T^2}(TM|_{M_2}). \tag{5.21}$$

Therefore, the symbol complex $[\sigma_\omega]$ is completely determined by the homomorphism θ_2 which, effectively, extends $[\sigma_\omega|_{M_2}] \in K_{T^2}(TM|_{M_2})$ to an element of $K_{T^2}(T_{T^2}M)$. In particular, when extending $[\sigma_\omega|_{M_2}]$ to a neighbour $U \supset M_2$, and restricting to $T_H U$, the symbol fails to have a compact support as it has support on the zero section over U. However, on $U - M_2$ it is possible to push the support away from the zero section, either along or against the Killing vector field v. Asking for continuity of the deformed symbol complex, it is

[4] An exact sequence is a chain complex where the image of one morphism actually equals the kernel of the next.

possible to argue that the choice of push follows the distribution of ASD/SD complexes. More details on the trivialization can be found in [30] and its application to the complex under consideration in Paper I.

Here, instead, we present a simpler example [30], that is a $U(1)$-action, with weight 1, on the one-point compactification of the complex plane, $\mathbb{C} \cup \{\mathrm{pt}\} = S^2$. We label E^0 the trivial complex line bundle on the sphere. We denote ε the coordinates on $\mathfrak{u}(1)$ and $t = e^{i\varepsilon}$ the coordinates on $U(1)$. Then, we can define $E^1 = E^0 \otimes t$. We consider the $U(1)$-invariant operator D:

$$D : \mathscr{D}(S^2, E_0) \to \mathscr{D}(S^2, E_1), \tag{5.22}$$

defined by:

$$D = \begin{cases} \frac{\partial}{\partial \bar{z}} = \frac{1}{2} e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) & \text{near the north pole } z = 0,\ z = re^{i\theta}, \\ -\frac{\partial}{\partial \omega} = \frac{r^2}{2} e^{i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) & \text{near the south pole } z = \infty,\ \omega = \frac{1}{z}, \\ \frac{1}{2} \phi(r) e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i\psi(r)}{r} \frac{\partial}{\partial \theta} \right) & \text{otherwise.} \end{cases} \tag{5.23}$$

Here:

$$\begin{aligned} \phi(r) &= 1 \text{ for } r \ll 1, & \phi(r) &= r^2 \text{ for } r \gg 1, \\ \psi(r) &= 1 \text{ for } r \ll 1, & \psi(r) &= -1 \text{ for } r \gg 1. \end{aligned} \tag{5.24}$$

Also, $\phi(r)$ never vanishes while $\psi(r)$, without loss of generality, can be chosen to vanish at the equator $r = 1$. The symbol $\sigma_\xi(D)$, as in (5.16), acts as a linear map once restricted to $(x, \xi) \in S^2 \times T_x S^2$:

$$\sigma(D)|_{(x,\xi)} : \lambda \longrightarrow \frac{1}{2} \phi(r) e^{i\theta} \left(\xi_r + \frac{i\psi(r)}{r} \xi_\theta \right) \lambda \tag{5.25}$$

where $\lambda \in \Lambda^0(S^2)$. Thus, the operator D fails to be elliptic at the equator where $\frac{\partial}{\partial \theta}$ has vanishing coefficient and the symbol $\sigma(D)$ fails to be invertible, not only for the zero section $\xi_r = \xi_\theta = 0$, but also for non-zero $\xi_\theta \in T^*\mathbb{C}$. However it is elliptic in directions transversal to the $U(1)$-action, that is along r.

Assuming that (5.21) holds, we just focus on[5] $\theta_1[\sigma(D)|_{M_1}]$, where M_1 is the set of fixed points. The operator D approaches $\bar{\partial}$ near $z = 0$ and $-\bar{\partial}$ near $z = \infty$[6]. If U is an open neighborhood around the two poles, the push away

[5] As there is a single $U(1)$, the set of the fixed points obtained through the filtration (5.20) is denoted M_1.

[6] In terms of the coordinate $\bar{\omega}$.

from the zero section on $U - M_1$ can be achieved by introducing a new symbol homotopic to the undeformed one as follows:

$$\tilde{\sigma}|_{(x,\xi)} = \sigma|_{(x,\xi \pm ve^{-\xi^2})}, \tag{5.26}$$

where (x,ξ) are coordinates on TU. The crucial observation is that $\tilde{\sigma}(D)$ is an isomorphism outside of the zero section once the tangent space is restricted to the one transversal to $U(1)$, $T_{U(1)}U$. Moreover, if we define $\xi_\pm \equiv \xi \pm ve^{-}\xi^2$ and impose continuity of the deformed symbol at the equator, we find:

$$\sigma\left(\frac{\partial}{\partial \bar{z}}\right)(\xi_+) = \sigma\left(-\frac{\partial}{\partial \omega}\right)(\xi_-). \tag{5.27}$$

Therefore, we can trivialize D at the equator by trivializing $\frac{\partial}{\partial \bar{z}}$ in the positive direction and $-\frac{\partial}{\partial \omega}$ in the negative direction. Thus, we have restricted the support of $\sigma(D)$ to the set of fixed points and we can write:

$$[\sigma(D)] = [\sigma(D)]_0 + [\sigma(D)]_\infty. \tag{5.28}$$

Moreover, $[\sigma(D)]$ defines an element in:

$$K_{U(1)}(T_{U(1)}(S^2 - S^1)) = K_{U(1)}(T_{U(1)}(B_0 - S^1)) \oplus K_{U(1)}(T_{U(1)}(B_\infty - S^1)), \tag{5.29}$$

where B_0, B_∞ are the northern and southern hemispheres whose boundary is the equator of S^2.

The first contribution gives:

$$[\sigma(D)]_0 = [\bar{\partial}^+], \tag{5.30}$$

where the exponent labels the choice of deformation in (5.26). The second contribution is associated to an operator:

$$-\partial^- : \Lambda^0(T\mathbb{C}) \to \Lambda^1(T\mathbb{C}). \tag{5.31}$$

Here, we have been using ω as coordinate, hence $U(1)$ acts by the representation t^{-1}. Instead, if we take the $\bar{\omega}$ coordinate, $U(1)$ acts by t, and we find:

$$-\bar{\partial}^- : \Lambda^0(\bar{T}\mathbb{C}) \to \Lambda^1(\bar{T}\mathbb{C}), \tag{5.32}$$

Then the index is given by [30]:

$$\mathrm{ind}(D) = \mathrm{ind}[\bar{\partial}^+] + \mathrm{ind}[\bar{\partial}^-]. \tag{5.33}$$

Again, the $\pm$ subscripts are associated to the direction of the vector v appearing in (5.27). This prescription tells us how to associate, at each fixed point on S^2, a distribution. In particular (in our case $\alpha = 1$):

$$\begin{aligned} \mathrm{ind}[\bar{\partial}^+] &= \left(\frac{1}{1-t^{-\alpha}}\right)^+ = -t^\alpha - t^{2\alpha} - \dots\,, \\ \mathrm{ind}[\bar{\partial}^-] &= \left(\frac{1}{1-t^{-\alpha}}\right)^- = 1 + t^{-\alpha} + t^{-2\alpha} - \dots\,, \end{aligned} \tag{5.34}$$

where $(\cdot)^{\pm}$ denotes the Laurent expansions at $t=0$ and $t=\infty$, respectively. We will also need later the expansions for negative weights:

$$
\begin{aligned}
\operatorname{ind}[\bar{\partial}^{+}] &= \left(\frac{1}{1-t^{\alpha}}\right)^{+} = 1+t^{\alpha}+t^{2\alpha}-\dots\,, \\
\operatorname{ind}[\bar{\partial}^{-}] &= \left(\frac{1}{1-t^{\alpha}}\right)^{-} = -t^{-\alpha}-t^{-2\alpha}-\dots\,.
\end{aligned}
\tag{5.35}
$$

A similar computation for the symbol complex $\sigma(\eth)$ can be found in Paper I. Here we only show the final result. Let us then consider a patch $U_l \simeq \mathbb{C}^2$ around a fixed point $F_l \in M_2$. We pick complex coordinates (z_1,z_2) and let $\varepsilon_1,\varepsilon_2$ be coordinates on $\mathfrak{t}^2$ such that $t_1=\exp(\mathrm{i}\varepsilon_1), t_2=\exp(\mathrm{i}\varepsilon_2)$ are coordinates on T^2. Then, the infinitesimal weights $\alpha_1^{(l)},\alpha_2^{(l)}$ act on $(z_1,z_2)\in\mathbb{C}^2$ as $\alpha_i^{(l)} = \sum_{j=1}^{2}\alpha_{ij}^{(l)}\varepsilon_j$ with $\alpha_{ij}^{(l)}\in\mathbb{Z}$ for $i,j=1,2$. Then we get:

$$
\begin{aligned}
\operatorname{index}\eth =&\operatorname{ind}\theta_2[\sigma_\omega|_Y] = \\
=&\sum_{l\in M_2}\left(1+\prod_{i=1}^{2}t^{-\alpha_i^{(l)}}\right)\prod_{k=1}^{2}\left(\frac{1}{1-t^{-\alpha_k^{(l)}}}\right)^{s_l},
\end{aligned}
\tag{5.36}
$$

where:

$$
t^{\alpha_i^{(l)}} := \prod_{j=1}^{2} t_j^{\alpha_{ij}^{(l)}}\,,
\tag{5.37}
$$

and s_l is $+$ for ASD fixed points and $-$ for SD ones. Moreover, when considering an SD fixed point one also has to flip the local infinitesimal weights as $\alpha_1\to-\alpha_1$. This is equivalent to a change in the complex structure $(z_1,z_2)\to(\bar{z}_1,z_2)$ which triggers the isomorphism $(\Omega^\bullet,d^+)\cong(\Omega^\bullet,d^-)$.

We see how the equivariant index, and thus the perturbative partition function, of Pestun-like theories is computed employing the Atiyah-Bott formula (5.1) for the local, and elliptic, contributions at each fixed point, and regularizing the infinite power series according to the distribution of ASD/SD fixed points.

5.3 Examples

We can use the equivariant index (5.36) to compute the one-loop superdeterminant as in (5.5)-(5.6) and we now show some explicit computations. The S^4 example, both for DW and Pestun [17], is treated in full details. For the other cases we only show the results.

S^4

We describe S^4 as a quaternion projective space $\mathbb{HP}^1$:

$$
[q_1,q_2]\sim[q_1\tilde{q},q_2\tilde{q}],\ \text{where } q_1,q_2\in\mathbb{H} \text{ and } \tilde{q}\in\mathbb{H}^{\times}.
\tag{5.38}
$$

We can also define inhomogenous coordinates:

$$\begin{aligned} &\text{northern hemisphere: } q = q_1 q_2^{-1} = z_1 + jz_2, \quad q \in \mathbb{H}^\times, \\ &\text{southern hemisphere: } q^{-1} = \frac{1}{|q|^2}(\bar{z}_1 - jz_2), \end{aligned} \tag{5.39}$$

where $z_1, z_2 \in \mathbb{C}$. The infinitesimal weights can be read from the action of $U(1)^2$:

$$q_1 \to t_1 q_1, \qquad q_2 \to t_2 q_2, \tag{5.40}$$

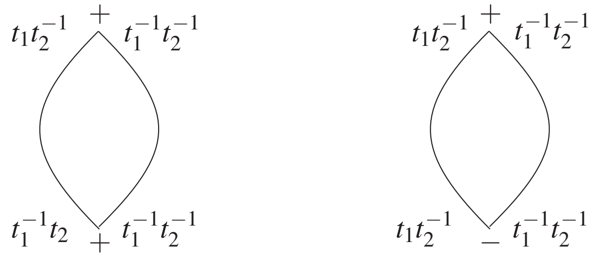
or equivalently:

$$z_1 \to t_1 t_2^{-1} z_1, \qquad z_2 \to t_1^{-1} t_2^{-1} z_2. \tag{5.41}$$

Thus, the infinitesimal weights of the T^2-action are:

$$(a_{ij}) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}. \tag{5.42}$$

Similarly, one finds $\bar{z}_1 \to t_1^{-1} t_2 \bar{z}_1$. Thus, taking into account the flip of the complex structure for SD fixed points, equivariant DW and Pestun [17] correspond to the following [7] "Delzant polygons":



As stated above, the deformation of symbol complex follows the distribution of ASD/SD fixed points. Hence, we find, respectively for DW and Pestun:

$$\begin{aligned} \text{index}\eth_1 =& (1+t_2^{-2}) \left(\frac{1}{1-t_1 t_2^{-1}}\right)^+ \left(\frac{1}{1-t_1^{-1}t_2^{-1}}\right)^+ \\ &+ (1+t_1^{-2}) \left(\frac{1}{1-t_1^{-1} t_2}\right)^+ \left(\frac{1}{1-t_1^{-1}t_2^{-1}}\right)^+ \\ =& 1. \end{aligned} \tag{5.43}$$

$$\begin{aligned} \text{index}\eth_2 =& (1+t_2^{-2}) \left(\frac{1}{1-t_1 t_2^{-1}}\right)^+ \left(\frac{1}{1-t_1^{-1}t_2^{-1}}\right)^+ \\ &+ (1+t_2^{-2}) \left(\frac{1}{1-t_1 t_2^{-1}}\right)^- \left(\frac{1}{1-t_1^{-1}t_2^{-1}}\right)^-. \end{aligned} \tag{5.44}$$

[7] Equivalent DW corresponds to two ASD fixed points while Pestun to a flip from ASD to SD fixed point

The first complex, being elliptic, is a regular function. Instead, the second case corresponds to a transversally elliptic complex whose equivariant index is an infinite power series in $(t_1 t_2^{-1})^{n_1}$ and $(t_1^{-1} t_2^{-1})^{n_2}$, where each term appears with a finite multiplicity. We thus plot the multiplicities in the (n_1, n_2) plane in Figure 5.1.

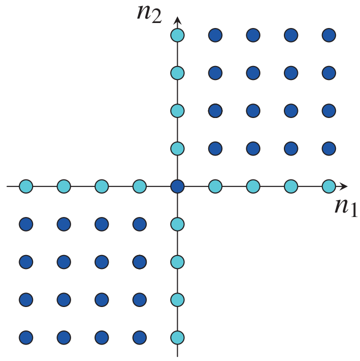


Figure 5.1. The plot shows the exponents of the weights in (5.44). Light blue points have multiplicity one, blue points have multiplicity two.

$\mathbb{CP}^2$

In the complex projective space:

$$[z_1, z_2] \sim [z_1 \tilde{z}, z_2 \tilde{z}], \text{ where } z_1, z_2 \in \mathbb{C} \text{ and } \tilde{z} \in \mathbb{C}^\times, \tag{5.45}$$

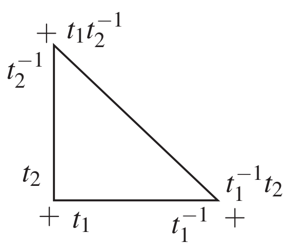
we consider a patch U_1, $z_1 \neq 0$, with inhomogenous coordinates $(z_2/z_1, z_3/z_1)$ on whom T^2 acts as:

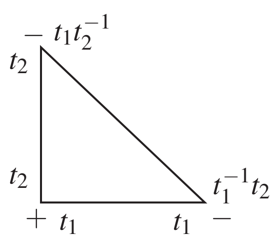
$$\frac{z_2}{z_1} \to t_1 \frac{z_2}{z_1}, \qquad \frac{z_3}{z_1} \to t_2 \frac{z_3}{z_1}. \tag{5.46}$$

Thus the infinitesimal weights become:

$$(a_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.47}$$

Once this is determined, the action of T^2 on the other two patches around the other two fixed points follows. Again, we can study the elliptic complex with three ASD fixed points and a transversally elliptic one with one ASD fixed point and two SD fixed points:





The first complex is elliptic and one can find, as for (5.43):

$$\text{index}\ \eth_1 = 2. \tag{5.48}$$

The second case, corresponding to a transversally elliptic complex, is shown in Figure 5.2.

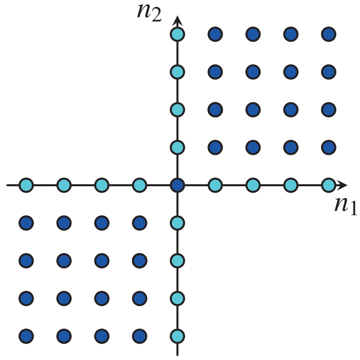


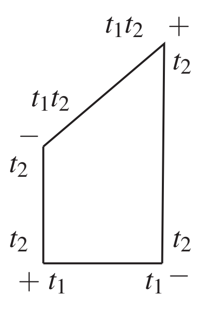
Figure 5.2. This plots shows the multiplicities of the transversally elliptic complex on $\mathbb{CP}^2$. Light blue points have multiplicity one, blue points have multiplicity two.

Hirzebruch surface $\mathbb{F}^1$

Finally, we consider a particular complex on $\mathbb{F}^1$ which cannot be obtained with the regularization procedure of [18, 19], that is it does not arise from a five-dimensional $\mathscr{N}=1$ theory on a Sasaki-Einstein manifold. The Hirzebruch surface, which has four fixed points under the T^2-action, is defined as the equivalence class $(z_1, z_2; u_1, u_2) \sim (z_1', z_2'; u_1', u_2')$ where two elements are equivalent if:

$$\exists \lambda, \mu \in \mathbb{C}^\times : (z_1', z_2'; u_1', u_2') = (\lambda z_1, \lambda z_2; \lambda\mu u_1, \mu u_2). \tag{5.49}$$

The T^2-action on z_1, z_2 is $z_1 \to t_1 z_1$, $u_1 \to t_2 u_1$. We consider an alternating distribution of ASD/SD fixed points with the following Delzant polygon:



As for the two previous examples we plot the multiplicities in Figure 5.3.

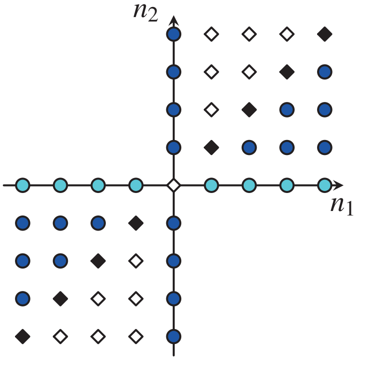


Figure 5.3. We show the multiplicities of the complex under considearation. Light blue points have multiplicity one, blue points multiplicity two, black squares multiplicity three and white squares multiplicity four.

6. Fluxes

In the previous chapter we computed the equivariant index of a generic distribution of ASD/SD complexes at the isolated fixed points of a manifold M under a T^2-action. The approach we took, studied in [18, 19], shows how different distributions originate from a choice of projector (4.9) approaching either $P^{\pm} = \frac{1}{2}(1 \pm \star)$ at each fixed point. However, when studying BPS configurations in chapter 4, we omitted field strengths with non-trivial flux on two-cycles, labeled by elements of $H^2(M,\mathbb{Z})$ and which enter the proposed result for the full partition function (4.3).

In this chapter we extend the framework presented earlier in two directions:

- We compute, at all flux sectors, the perturbative partition function on $\mathbb{CP}^2$ for the two complexes considered in chapter 5. These correspond to an elliptic complex and a transversally elliptic one. The flux dependence in the perturbative partition function is via a shift of the Coulomb branch parameter σ_0, as proposed on non-compact manifolds in [44] and for compact manifolds in [18].
- We discuss how different theories in four dimensions stem from a unique five-dimensional $\mathcal{N} = 1$ theory on a Sasaki-Eistein manifold. This was suggested in [19] and shown, at the trivial instanton sector, for the reduction along the Hopf fiber $S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$ in Paper II[1].

The dimensional reduction could be performed shrinking the radius of the Hopf circle. Instead, in Paper II, it is introduced a $\mathbb{Z}_p$ quotient acting on the fiber. At finite p the quotient defines a higher-dimensional generalization of lens spaces supporting non trivial flat connections. At large p, the manifold effectively resembles $\mathbb{CP}^2$ and the flat connections give rise to flux on the base manifold. Crucially, there are two inequivalent choices of fiber with respect to the five-dimensional Killing vector. The two choices of reduction will turn into the two different complexes on $\mathbb{CP}^2$.

6.1 Geometry of the five-sphere

The five-dimensional sphere is an example of Sasaki-Einstein manifolds (S,g), for a review see [47]. Here, we will only need some basic facts. These manifolds are defined in relation to their metric cone:

$$C(S) = \mathbb{R}_{>0} \times S, \qquad \bar{g} = dr^2 + r^2 g, \tag{6.1}$$

[1] In Paper II it is also discussed the $S^1 \hookrightarrow S^3 \to \mathbb{CP}^1$ case which is shown to match the known results [45, 46].

being Kähler and Ricci-flat. For example, the Kähler cone of odd-dimensional spheres is $\mathbb{C}^n \setminus 0$, equipped with its flat metric. Moreover, Sasaki-Einstein manifolds admit a characteristic vector field, called the Reeb vector:

$$\xi = J(r\partial_r), \tag{6.2}$$

where J is the complex structure on the metric cone $(C(S), \bar{g})$. This vector field can be shown to be Killing $\mathscr{L}_\xi \bar{g} = 0$ and with square length $\bar{g}(\xi, \xi) = r^2$. We thus consider its restriction to the Sasaki-Einstein manifold, through the inclusion $S = \{r = 1\} \times S$, where ξ has unit length. Its integral curves are geodesics and can be used to define the Reeb foliation $\mathscr{F}_\xi$ whose leaf space inherits a Kähler metric. We are interested in the regular case where the orbits are all closed, and thus the Reeb vector field integrates to an isometric, and free, $U(1)$-action on (S, g). In this case the leaf space $Z = S/\mathscr{F}_\xi = S/U(1)$ is a compact manifold. Hence, a regular Sasaki-Einstein manifold can be seen as a total space of a principal $U(1)$ bundle over a Kähler-Einstein manifold, $\mathbb{CP}^2$ in the case of S^5.

Besides this, simply-connected regular Sasaki-Einstein manifolds always admit a $\mathbb{Z}_p \subset U(1)$ quotient. The resulting manifold is a regular Sasaki-Einstein manifold with $\pi_1(S/\mathbb{Z}_p) = \mathbb{Z}_p$.

While in the rest of the chapter we will focus on S^5, the procedure described can be extended to more general cases, as the $Y^{p,q}$ spaces [48]. The idea is to dimensionally reduce S^5 onto the leaf space $\mathbb{CP}^2$, employing two different fibers with respect to a direction fixed by the Killing vector. The starting point in Paper II is an $\mathscr{N} = 1$ vector multiplet on S^5 embedded in $\mathbb{C}^3$ with the metric:

$$ds^2_{S^5} = \sum_{i=1}^{3} |dz_i|^2. \tag{6.3}$$

This can be rewritten as the Hopf metric:

$$ds^2_{S^5} = ds^2_{\mathbb{CP}^2} + (d\alpha + V)^2, \tag{6.4}$$

where the metric on $\mathbb{CP}^2$ is the Fubini-Study one, α is the coordinate along the Hopf fiber and V is a connection one-form. At this point, each choice of fiber is equivalent to perform dimensional reduction as they are related by an $SO(6)$ rotation. However, a choice of supercharge will determine a preferred direction through the Killing vector v, which is also Reeb. To describe this choice we introduce the vector fields e_i, $i = 1, \ldots, 3$ for the action $z_i \to e^{i\alpha_i} z_i$ of the $U(1)^3$ Cartan of the isometry group. Hence:

$$v = e_1 + e_2 + e_3. \tag{6.5}$$

With respect to v we find two different choices of fiber:

$$\text{top:} \quad x^{top} = +e_1 + \ldots + e_r, \tag{6.6}$$

$$\text{ex:} \quad x^{ex} = -e_1 + \ldots + e_r. \tag{6.7}$$

Here, we introduced the notation "top", for topologically twisted theories, and "ex" for Pestun-like theories. In the next section we will show how these two choices give rise two two different theories on the base manifold. Notice already that x^{top} is along the Killing vector and thus, the reduced theory will have a supercharge squaring to zero, as in chapter 2. This, instead, is not the case for the choice x^{ex} which will have fermionic generators squaring to the isometry of the base manifold.

We conclude this section considering the action of squashing and quotienting on five-spheres. Starting with the former, we define squashing parameters:

$$\omega \equiv (\omega_1, \omega_2, \omega_3), \qquad \omega_i = 1 + a_i \in \mathbb{R}, \tag{6.8}$$

where $a_1 = a_2 = a_3 = 0$ corresponds to the unsquashed limit of S^5 and each squashing parameter deforms a single $\mathbb{C}$-plane inside $\mathbb{C}^3$. Thus, the sphere, which is embedded in $\mathbb{C}^3$ (6.3), is also squashed. The Killing vector (6.5) becomes:

$$v = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3. \tag{6.9}$$

The important point is that the squashing is set to act only on the base manifold, leaving the fiber untouched. To achieve this one sets:

$$\text{top:} \quad + a_1 + a_2 + \ldots + a_r = 0, \tag{6.10}$$
$$\text{ex:} \quad - a_1 + a_2 + \ldots + a_r = 0. \tag{6.11}$$

Hence, the reduction can still be performed along the fibers (6.6). In the topologically twisted case we will show how a squashing acting on the base gives rise to equivariant DW theory.

The dimensional reduction we are going to perform is achieved through quotienting by a freely-acting $\mathbb{Z}_p$ acting on the fibers x^{top} or x^{ex}:

$$(z_1, z_2, \ldots, z_r) \rightarrow (z_1 e^{\pm 2\pi i/p}, z_2 e^{+2\pi i/p}, \ldots, z_r e^{+2\pi i/p}). \tag{6.12}$$

The sign in the first factor is, respectively, for an action along x^{top} or x^{ex}. The quotient defines a higher-dimensional generalization of lens space:

$$L^5(p, \pm 1) \equiv S^5/\mathbb{Z}_p, \tag{6.13}$$

and it induces a change in the topology of the manifold, which is now not simply connected, and thus:

$$\pi_1(L^5(p, \pm 1)) \cong \mathbb{Z}_p. \tag{6.14}$$

Because of this, there exist p topologically inequivalent complex line bundles labeled by flat connections:

$$A = \text{diag}\,(A_p^{m_1}, \ldots, A_p^{m_{N-1}}), \tag{6.15}$$

where $0 \leq m_i < p$ and $i = 1, \ldots, N-1$ counts the element in the Cartan of the gauge group $SU(N)$.

6.2 Perturbative partition function on S^5

A localization computation on S^5 has first been performed in [49–51] and it has been later generalized to toric Sasaki-Einstein manifolds, see [52] for a review. The field content of an $\mathcal{N}=1$ vector multiplet consists in a gauge boson A_μ, a real scalar σ, gauginos λ_i, $\bar{\lambda}^i$ and an auxiliary scalar D_{ij}, where λ_i and D_{ij} are, respectively, a doublet and a triplet of $SU(2)_I$. Similarly to four dimensions, it is possible to recast the fermionic fields in cohomological variables Ψ_μ and $\chi_{\mu\nu}$, such that they do not transform under the R-symmetry. It is important to notice that the reduction onto $\mathbb{CP}^2$, which is not a spin manifold, is possible as the cohomological variables are forms and not spinors.

The BPS locus for these theories is given by a covariantly constant scalar σ_0 and contact instantons at the three fixed fibers of S^5:

$$F_H^+ = 0, \quad \iota_v F = 0 \Rightarrow \text{Contact instanton: } \star F = -\kappa \wedge F. \tag{6.16}$$

An heuristic treatment of the instanton part and its reduction to $\mathbb{CP}^2$ can be found in [19]. Here, we will focus on the perturbative partition function[2] on S^5, given by:

$$\begin{aligned} Z_{S^5}^{pert} = \prod_{\alpha\in roots} \prod_{n_1,n_2,n_3\geq 0} &\Bigg(n_1\omega_1 + n_2\omega_2 + n_3\omega_3 + i\alpha(\sigma_0) \Bigg) \\ \prod_{n_1,n_2,n_3\geq 1} &\Bigg(n_1\omega_1 + n_2\omega_2 + n_3\omega_3 + i\alpha(\sigma_0) \Bigg). \end{aligned} \tag{6.17}$$

The three positive integers n_1, n_2, n_3 in (6.17) are eigenvalues, under the $U(1)^3$ rotation, of the modes contributing to the partition function after cancellations due to supersymmetry.

An elegant way to express (6.17) is through multiple gamma functions [54]:

$$\Gamma_r(z,\omega) = \prod_{n\geq 0} (n\cdot\omega + z), \tag{6.18}$$

where $\omega = (\omega_1, ..., \omega_r)$, and multiple sine functions [55, 56]:

$$S_r(z,\omega) = \Gamma_r(z,\omega)^{-1}\Gamma_r(\omega_{tot} - z, \omega)^{(-1)^r}, \tag{6.19}$$

where $\omega_{tot} = \omega_1 + ... + \omega_r$. Hence:

$$Z_{S^5}^{pert} = \prod_{\alpha\in \text{roots}} S_3(-i\alpha(\sigma_0), \omega). \tag{6.20}$$

Multiple sine functions enjoy factorization properties [57].

[2] The full partition function also has a classical contribution whose reduction to $\mathbb{CP}^2$ is briefly discussed in Paper II, along the lines of [53].

Moreover, for ease of notation, we will omit from the next formulae the product in (6.17) over $n_1, n_2, n_3 \geq 1$ and the product over the roots. However, to obtain the correct perturbative partition function, both contributions need to be restored.

We introduce equivariant parameters ε_1, ε_2, related to the squashing parameters as:

$$\begin{aligned} \text{top:} \quad & \varepsilon_1^{top} = \omega_2 - \omega_1, \quad \varepsilon_2^{top} = \omega_3 - \omega_1, \\ \text{ex:} \quad & \varepsilon_1^{ex} = \omega_2 + \omega_1, \quad \varepsilon_2^{ex} = \omega_3 + \omega_1. \end{aligned} \tag{6.21}$$

The unsquashed limit is $\varepsilon_1^{top} = \varepsilon_2^{top} = 0$ and $\varepsilon_1^{ex} = \varepsilon_2^{ex} = 2$. We will often denote the equivariant parameters ε_1, ε_2 for both cases, however, they need to be intended as defined above.

A useful rewriting is in terms of the quantum number for rotations along the fibers (6.6):

$$t_{top} = +n_1 + n_2 + n_3, \tag{6.22}$$

$$t_{ex} = -n_1 + n_2 + n_3. \tag{6.23}$$

Also in this case we will denote both quantum numbers as t. Hence, we find:

$$\begin{aligned} Z_{S^5}^{pert,top} &= \prod_{t \geq n_2+n_3} \prod_{n_2,n_3 \geq 0} \Big(\omega_1 t + (\omega_2 - \omega_1) n_2 + (\omega_3 - \omega_1) n_3 + i\alpha(\sigma_0) \Big) \\ &= \prod_{t \geq n_2+n_3} \prod_{n_2,n_3 \geq 0} \left(\varepsilon_1 n_2 + \varepsilon_2 n_3 + i\alpha(\sigma_0) + \left(1 - \frac{\varepsilon_1 + \varepsilon_2}{3}\right) t \right). \end{aligned} \tag{6.24}$$

$$\begin{aligned} Z_{S^5}^{pert,ex} &= \prod_{t \leq n_2+n_3} \prod_{n_2,n_3 \geq 0} \Big(-\omega_1 t + (\omega_2 + \omega_1) n_2 + (\omega_3 + \omega_1) n_3 + i\alpha(\sigma_0) \Big) \\ &= \prod_{t \leq n_2+n_3} \prod_{n_2,n_3 \geq 0} \left(\varepsilon_1 n_2 + \varepsilon_2 n_3 + i\alpha(\sigma_0) + \left(1 - \frac{\varepsilon_1 + \varepsilon_2}{3}\right) t \right). \end{aligned} \tag{6.25}$$

At this point the two rewriting are equivalent to (6.17), however, their reduction to $\mathbb{CP}^2$ will give different results, as we are going to show momentarily. Note that the bounds on t are different in the two cases: at fixed t a finite number of n_2, n_3 contributes to $Z_{S^5}^{pert,top}$ while, for $Z_{S^5}^{pert,ex}$, an infinite number of n_2, n_3 contributes. Finally, these expression can be factorized in contributions coming from the three fixed fibers of S^5, employing the factorization properties of the triple sine function [57]. The precise regularization of each contribution is the higher-dimensional equivalent of the computation described in chapter 5.

6.3 Reduction to $\mathbb{CP}^2$

As discussed above, we will perform the dimensional reduction introducing a $\mathbb{Z}_p$ quotient along either x^{top} or x^{ex}. On the resulting manifolds, $L^5(p,\pm 1)$, the partition function localizes to a set of inequivalent flat connections (6.15):

$$Z_{L^5(p,\pm1)} = \sum_{[\mathfrak{m}]} \int d\sigma_0 e^{-S_{cl}} Z^{pert}_{L^5(p,\pm1)}(\sigma_0,\mathfrak{m}) Z^{non-pert}_{L^5(p,\pm1)}(\sigma_0,\mathfrak{m}), \tag{6.26}$$

where the sum is over the winding numbers $\mathfrak{m} = \mathrm{diag}\,(m_1,...,m_{N-1})$ of the flat connections. The crucial observation is that the quotient introduces also a projection conditions for the quantum number for rotations along the fiber:

$$\begin{aligned} \text{top:} \quad & t_{top} = +n_1 + n_2 + n_3 = \alpha(\mathfrak{m}) \bmod p, \\ \text{ex:} \quad & t_{ex} = -n_1 + n_2 + n_3 = \alpha(\mathfrak{m}) \bmod p. \end{aligned} \tag{6.27}$$

If we now take the large p limit, we find that we can simply set:

$$\begin{aligned} \text{top:} \quad & t = \alpha(\mathfrak{m}), \\ \text{ex:} \quad & t = \alpha(\mathfrak{m}). \end{aligned} \tag{6.28}$$

Substituting this into (6.24)-(6.25), we find the all-flux partition function on $\mathbb{CP}^2$ for equivariant DW and Pestun-like theories:

$$Z^{pert,top}_{\mathbb{CP}^2} = \prod_{\alpha(\mathfrak{m})\geq n_2+n_3} \prod_{n_2,n_3\geq 0} \left(\varepsilon_1 n_2 + \varepsilon_2 n_3 + i\alpha(\sigma_0) + \left(1 - \frac{\varepsilon_1+\varepsilon_2}{3}\right)\alpha(\mathfrak{m}) \right), \tag{6.29}$$

$$Z^{pert,ex}_{\mathbb{CP}^2} = \prod_{\alpha(\mathfrak{m})\leq n_2+n_3} \prod_{n_2,n_3\geq 0} \left(\varepsilon_1 n_2 + \varepsilon_2 n_3 + i\alpha(\sigma_0) + \left(\frac{1}{3} - \frac{\varepsilon_1+\varepsilon_2}{3}\right)\alpha(\mathfrak{m}) \right), \tag{6.30}$$

where the full partition function on $\mathbb{CP}^2$ is given as a sum over flux sectors:

$$Z_{\mathbb{CP}^2} = \sum_{[\mathfrak{m}]} \int d\sigma_0 e^{-S_{cl}} Z^{pert}_{\mathbb{CP}^2}(\sigma_0,\mathfrak{m}) Z^{non-pert}_{\mathbb{CP}^2}(\sigma_0,\mathfrak{m}). \tag{6.31}$$

Hence, we see how flat connections on lens spaces give rise to fluxes on the base manifold. Moreover, at each flux sector, there is a finite amount of n_2, n_3 contributing to the partition function, as expected for an elliptic complex. Vice versa, we find an infinite amount of n_2, n_3 at each flux sector for the Pestun-like theory, corresponding to a transversally elliptic complex. As proposed in [18, 44] the dependence on fluxes in (6.29)-(6.30) enters through a squashing dependent shift of the Coulomb branch parameter σ_0.

A consistency check of these results is that the trivial flux sector of (6.29)-(6.30) gives the result derived in chapter 5. At non-trivial flux, the eigenvalues come from a slice of the original cone $(n_1,n_2,n_3) \geq 0$. The slices are performed along the direction determined by the fibers x^{top} or x^{ex}. Examples of these slicings can be found in Figure 6.1.

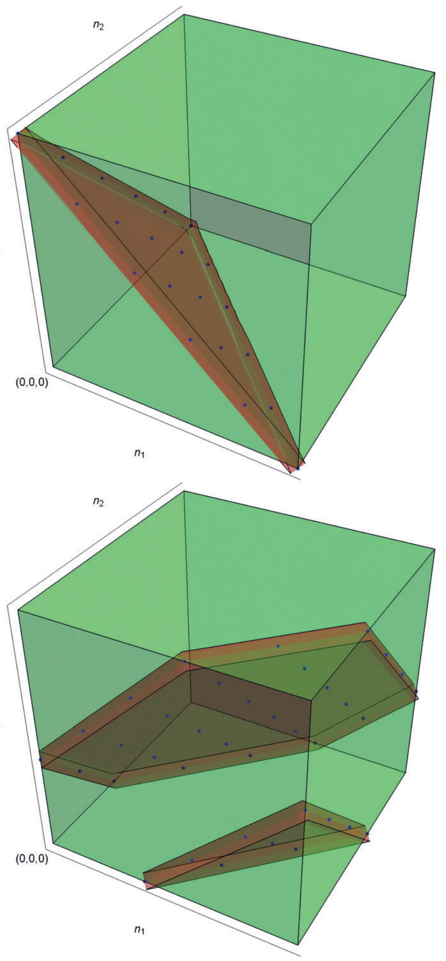


Figure 6.1. The plots show slices at different values of $t = \alpha(\mathfrak{m})$ of the octant in $\mathbb{R}^3$ spanned by positive (n_1, n_2, n_3). Each plane is determined by $n_1 = \mp(n_2 + n_3 - t)$, where the orientation is related to the choice of fiber used to reduce and it determines the eigenvalues contributing at the each flux sector.

As stated above, the same procedure can be performed starting from an $\mathscr{N} = 2$ vector multiplet on S^3 [58, 59] to achieve two different $\mathscr{N} = (2,2)$ theories on the base manifold $\mathbb{CP}^1 \cong S^2$. Lens spaces can be introduced as in [60] and the results, at large quotienting, show agreement with known results on S^2 [45, 46].

7. Hypermultiplet

In this final chapter, following Paper III, we will include matter coupled to the gauge fields in the framework developed in [18, 19] and presented in chapter 4. Writing the content of a four-dimensional $\mathcal{N}=2$ hypermultiplet in terms of cohomological variables gives rise to spinors, unlike for the vector multiplet. Hence, it is generically needed for the manifold under consideration to be spin[1]. For example, $\mathbb{CP}^2$, considered previously, does not admit a spin structure. Moreover the projector P^+_ω (4.9) acting on flipping ASD/SD two-forms comes together with a projector acting on chiral spinors which are allowed to flip chirality at different fixed points. Similarly, as for the vector multiplet, equivariant DW theory on S^4 is obtained with left-handed spinors at both poles while Pestun's theory stems from a flip of the chirality between the two poles.

7.1 Cohomological variables

We start by introducing the field content of an $\mathcal{N}=2$ hypermultiplet in $d=4$, as in chapter 2, and then switch to cohomological variables. Hence, let us consider a single[2] hypermultiplet containing a scalar q_i, transforming in the fundamental of $SU(2)_I$, spinors ψ_α, $\tilde{\psi}_{\dot\alpha}$ and auxiliary fields $F_{\hat{i}}$, in the fundamental of an $SU(2)_{\hat{I}}$-symmetry[3]. Here, the $SU(2)_{\hat{I}}$ bundle is in general distinct from the $SU(2)_I$ bundle and it is required to write off-shell transformations. More details can be found in Paper III and Appendix A. The fields are then expressed in terms of cohomological variables as follows:

$$q=\begin{pmatrix} \zeta^i q_i \\ \bar{\chi}^i q_i \end{pmatrix}, \qquad c=-\frac{1}{4}\begin{pmatrix} s\psi - v^\mu \sigma_\mu \bar{\psi} \\ \tilde{s}\bar{\psi}+v^\mu \bar{\sigma}_\mu \psi, \end{pmatrix}, \tag{7.1}$$

and:

$$b=\frac{1}{4}\begin{pmatrix} \tilde{s}\psi - v^\mu \sigma_\mu \bar{\psi} \\ -s\bar{\psi}+v^\mu \bar{\sigma}_\mu \psi \end{pmatrix}, \qquad h=\frac{s+\tilde{s}}{2}\begin{pmatrix} \hat{\zeta}^i \\ \hat{\bar{\chi}}^i \end{pmatrix} F_{\hat{i}}+(\ldots)q, \tag{7.2}$$

[1] Discussion on spin^c structures can be found, for example, in Paper III.

[2] Generalization to n hypermultiplets transforming in the fundamental of $Sp(n)$ is straightforward.

[3] Reality conditions for the bosons are: $(q_i)^*=q^i$ and $(F_{\hat{i}})^*=F^{\hat{i}}$.

where v is the Killing vector and we refer again to Appendix A for definitions of s, $\tilde{s}$ and of the Killing spinors $\hat{\zeta}^i$, $\hat{\bar{\chi}}^i$, ζ^i, $\bar{\chi}^i$. Plus fixed points (ASD) are such that $\tilde{s} = 0$ and $\hat{\bar{\chi}}^i = 0$ while, vice versa, at minus fixed points (SD) $s = 0$ and $\hat{\zeta}^i = 0$. Moreover, the dots in the definition of h in (7.2) stand for a dependence on the supergravity background fields, as discussed in Paper III. It can be shown that these maps have a smooth inverse. Hence, the cohomological fields for an hypermultiplet turn out to be Grassmann-even fermions q, h and Grassmann-odd fermions b, c. Moreover, imposing reality conditions on q_i, $F_{\hat{i}}$ and the Killing spinors, one finds:

$$\begin{aligned} (h)^* &= -\bar{h} - (\ldots)\bar{q}, \\ (q)^* &= \bar{q}, \end{aligned} \tag{7.3}$$

where the term in the dots is computed using reality conditions on the supergravity background fields, see Paper III.

In terms of cohomological variables, the supersymmetry transformations are given by:

$$\begin{aligned} \delta q &= c, \qquad & \delta c &= (i\mathscr{L}_v - \mathscr{G}_\Phi)q, \\ \delta b &= ih, \qquad & \delta h &= (\mathscr{L}_v + i\mathscr{G}_\Phi)b, \end{aligned} \tag{7.4}$$

where $\mathscr{G}_\Phi$ acts on the variables according to their representation. For example:

$$\mathscr{G}_\Phi\, q = i[i(\iota_v A) + \phi]q. \tag{7.5}$$

Thus, δ acts on the cohomological fields as:

$$\delta^2 = i\mathscr{L}_v - \mathscr{G}_\Phi, \tag{7.6}$$

and here no R-symmetry transformations appears, due to the definitions (7.1)-(7.2).

Moreover, the $\mathscr{N} = 2$ Lagrangian is found by taking the variation of a fermionic potential:

$$\mathscr{L}^{hyper} = \delta V_G, \tag{7.7}$$

whose explicit expression can be found in Paper III.

7.2 Projector

Counting the degrees of freedom of the cohomological variables (q, c, b, h) one finds that these exceed those of the original content of an $\mathscr{N} = 2$ hypermultiplet. Thus, one expects that the cohomological variables need to satisfy certain conditions. Therefore, the idea is to define flipping projectors, as for P^+_ω and two-forms, but for Dirac spinors on a four manifold M with a T^2-isometry and a Killing vector with isolated fixed points. Such projector would have to approach the left/right-handed projectors:

$$L = \frac{1}{2}(1+\gamma_5), \qquad R = \frac{1}{2}(1-\gamma_5), \tag{7.8}$$

respectively at plus/minus fixed points where $s = 0$ or $\tilde{s} = 0$. Hence, one finds:

$$Z_+ = \frac{1}{2}\left(1 + \frac{s-\tilde{s}}{s+\tilde{s}}\gamma_5 - \frac{2}{s+\tilde{s}}v^\mu\gamma_5\gamma_\mu\right), \tag{7.9}$$

whose image comprises spinors which are in the image of L when $s = 0$ and in the image of R when $\tilde{s} = 0$. Because of the definition of the Killing spinors (A.3)-(A.4), one finds:

$$Z_+ \begin{pmatrix} \zeta^i \\ \bar{\chi}^i \end{pmatrix} = \begin{pmatrix} \zeta^i \\ \bar{\chi}^i \end{pmatrix}, \tag{7.10}$$

which shows the strict relation between supersymmetry and the projector (7.9), similarly as for the projector P^+_ω of chapter 4.

Perhaps the most intuitive way to understand this projector is in its relation with P^+_ω. Let us consider Dirac spinors $\Psi_{1,2} = Z_+\Psi_{1,2}$. Then, one can construct a two-form as follows[4]:

$$\Lambda_{\mu\nu} = \bar{\Psi}_2\gamma_{\mu\nu}\Psi_1 = \psi_2\sigma_{\mu\nu}\psi_1 + \tilde{\psi}_2\bar{\sigma}_{\mu\nu}\tilde{\psi}_1. \tag{7.11}$$

One can check that this two-form satisfies:

$$P^+_\omega\Lambda = \Lambda. \tag{7.12}$$

Also it possible to construct related projectors as follows:

$$Z_- = 1 - Z_+, \qquad \tilde{Z}_+ = \gamma_5 Z_+ \gamma_5, \qquad \tilde{Z}_- = 1 - \tilde{Z}_+. \tag{7.13}$$

Moreover, due to (7.10), one finds that q and c are in the image of Z_+ projector:

$$Z_+ q = q, \qquad Z_+ c = c, \tag{7.14}$$

while b and h are in the image of $\tilde{Z}_-$:

$$\tilde{Z}_- b = b, \qquad \tilde{Z}_- h = h. \tag{7.15}$$

The action of supersymmetry (7.4) commutes with the projectors as the vector appearing in Z_+ (7.9) is the Killing vector.

[4]The spinors ψ_α and $\tilde{\psi}^{\dot{\alpha}}$ are the two components of the Dirac spinor. Check appendix A of Paper III for more extensive treatment of conventions.

7.3 One-loop determinant

The BPS locus of an $\mathcal{N}=2$ hypermultiplet is found imposing supersymmetry transformations (7.4) and reality conditions (7.3):

$$b = h = c = h = 0. \tag{7.16}$$

To localize the partition function we use:

$$\begin{aligned} V_{loc}^{hyper} =& \frac{8}{(s+\tilde{s})^2}[(\delta b)^* b + (\delta c)^* c] \\ =& \frac{8}{(s+\tilde{s})^2}\begin{pmatrix} \bar{\delta q}, \ \bar{q} \end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} q \\ \delta b \end{pmatrix}. \end{aligned} \tag{7.17}$$

as deformation term. The relevant operator is D_{10}, which fails to be elliptic at points where $\tilde{s} = s$, but it turns out to be transversally elliptic with respect to the Killing vector v. Moreover, the symbol complex $\sigma(D_{10})$ approaches either $\bar{\partial}$ or ∂ respectively at left/right chirality fixed points. Unlike the vector multiplet, there is no contribution of the complex conjugate operator.

The one-loop determinant can be computed using the formalism introduced in chapter 5, that is to use to Atiyah-Bott formula (5.1) at each fixed point being careful of patching properly all contributions. We assume the patching can be carried along the lines of chapter 5, even if a formal treatment of this would require a trivialization of the symbol complex of the hypermultiplet.

Therefore, at plus fixed points, we need to compute:

$$\begin{aligned} &\det(1-df) = \frac{1}{(1-t_1)(1-t_2)}, \\ &\chi_{eq}(E_{p_i}^{\bullet}) = \sqrt{t_1 t_2}. \end{aligned} \tag{7.18}$$

Combining these contributions, one finds:

$$\operatorname{index}(D_{10})|_{pluspoint} = \frac{\sqrt{t_1 t_2}}{(1-t_1)(1-t_2)}. \tag{7.19}$$

Similar expressions can be found at minus fixed points, being careful of performing the same flip of infinitesimal weight $\alpha_1 \to -\alpha_1$, as in chapter 5.

Hence, for a generic distribution of plus/minus fixed points, we find:

$$\operatorname{index} D_{10} = \sum_{l \in M_2} \prod_{k=1}^{2} \left(\frac{\sqrt{t^{-\alpha_k^{(l)}}}}{1 - t^{-\alpha_k^{(l)}}} \right)^{s_l}. \tag{7.20}$$

This expression, which holds for the perturbative partition function of an $\mathcal{N}=2$ hypermultiplet, is the equivalent of (5.36) for an $\mathcal{N}=2$ vector multiplet.

As an example, we can compute the equivariant index of a Pestun-like theory on S^4 [17, 36]. The weights can be read from chapter 5, hence one finds:

$$\begin{aligned} \mathrm{index} D_{10} = & t_2^{-1} \left(\frac{1}{1 - t_1 t_2^{-1}} \right)^+ \left(\frac{1}{1 - t_1^{-1} t_2^{-1}} \right)^+ \\ & + t_2^{-1} \left(\frac{1}{1 - t_1 t_2^{-1}} \right)^- \left(\frac{1}{1 - t_1^{-1} t_2^{-1}} \right)^- . \end{aligned} \tag{7.21}$$

Appendices

A. Supersymmetry on curved manifolds

The Killing spinors associated to the supersymmetry transformations are a pair of spinors ζ^i_α and $\bar{\chi}^{\dot{\alpha}}_i$ which satisfy:

$$\begin{aligned} &(\zeta_{i\alpha})^* = \zeta^{i\alpha}, \qquad \zeta^i\zeta_i = \frac{s}{2}, \\ &(\bar{\chi}^{\dot{\alpha}}_i)^* = \bar{\chi}^i_{\dot{\alpha}}, \qquad \bar{\chi}^i\bar{\chi}_i = \frac{\tilde{s}}{2}, \\ &\bar{\chi}^i\bar{\sigma}^\mu\zeta_i = \frac{1}{2}v^\mu. \end{aligned} \tag{A.1}$$

Notice then that the square norm of the Killing vector is:

$$||v||^2 = s\tilde{s}, \tag{A.2}$$

so that either s or $\tilde{s}$ vanishes at the fixed points. In particular, we label plus fixed points those where $s = 0$ and minus ones where $\tilde{s} = 0$.

We consider an open cover of M, $\cup_{i=l}U_l = M$, where each patch is equipped with a local frame e^a_l and each fixed point is contained in a single U_l. The frames, in the overlap $U_l \cap U_k$, are related by an $SU(2)_L \times_{\mathbb{Z}_2} SU(2)_R$ transformation. A Killing spinor for topological twisting, discussed in chapter 2, is found identifying $SU(2)_I$ and $SU(2)_R$. It is globally defined and its expression, in each patch, is $(\zeta_t)^i_\alpha = \delta^i_\alpha$.

In patches were s does not vanish, we can find spinors satisfying (A.1):

$$\zeta^i = \frac{\sqrt{s}}{2}\zeta^i_t, \quad \bar{\chi}_i = \frac{1}{s}v^\mu\bar{\sigma}_\mu\zeta_i. \tag{A.3}$$

Similarly, on patches where $\tilde{s} = 0$, solutions of (A.1) are given by:

$$\hat{\bar{\chi}}_i = -i\frac{\sqrt{\tilde{s}}}{2}\delta_i, \quad \hat{\zeta}_i = -\frac{1}{\tilde{s}}v^\mu\sigma_\mu\hat{\bar{\chi}}_i. \tag{A.4}$$

In an overlap between two patches containing $\pm$ fixed points, solutions of (A.3) and (A.4) are related by an $SU(2)_I$ transformation:

$$\bar{\chi} = U_i^{\ j}\hat{\bar{\chi}}_j, \quad \zeta_i = U_i^{\ j}\hat{\zeta}_j, \quad U_i^{\ j} = i\frac{v^\mu}{||v||}\sigma_{\mu i}^{\ \ j}. \tag{A.5}$$

Surrounding a fixed point with $s = 0$ with a small three-sphere Σ, the map $U_i^{\ j}$ from Σ to $SU(2)_I$ is non-singular and of degree one. Hence, moving between patches with different fixed points, the Killing spinor first transforms under the $SU(2)_I$ transformation (A.5), and second under the transformation associated to topological twisting. Spinors constructed as above are smooth on M. Moreover, it is shown in [18], that they solve the Killing spinor equations arising from the appropriate rigid supergravity background on M.

Finally, in order to write $\mathcal{N}=2$ off-shell transformations, we need to add auxiliary spinors $\hat{\zeta}^i_\alpha$ and $\hat{\bar{\chi}}_i^{\dot\alpha}$ satisfying:

$$
\begin{aligned}
&(\zeta^\alpha_{\hat{i}})^* = \zeta^{\hat{i}}_\alpha, \qquad && \zeta_{\hat{i}}\zeta^{\hat{i}} = \frac{\tilde{s}}{2}, \\
&(\bar{\chi}^{\hat{i}\dot\alpha})^* = \bar{\chi}_{\hat{i}\dot\alpha}, \qquad && \bar{\chi}_{\hat{i}}\bar{\chi}^{\hat{i}} = \frac{s}{2}, \\
&\bar{\chi}^{\hat{i}}\bar{\sigma}^\mu\zeta_{\hat{i}} = -\frac{1}{2}v^\mu, && \\
&\zeta_i\zeta_{\hat{i}} - \chi_i\chi_{\hat{i}} = 0. &&
\end{aligned}
\tag{A.6}
$$

Moreover, they transform under $SU(2)_L \times_{\mathbb{Z}_2} SU(2)_{\hat{I}}$ and $SU(2)_R \times_{\mathbb{Z}_2} SU(2)_{\hat{I}}$. In the general the bundles $SU(2)_I$ and $SU(2)_{\hat{I}}$ are not identified. More details can be found in Paper III.

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I feel it has already been said that life as a PhD student is full of ups and downs, so let me get straight to the point. I would like to thank my supervisor, Guido Festuccia, for the unlimited patience he has had with me and for the uncountable discussions we had about physics. Even more during the writing of this thesis, I have appreciated both his scientific advice and his human support. Moreover, I would like to thank Joe Minahan, for always providing useful advice, and Agnese Bissi, for being so friendly and for making my teaching experience a far better one than I could ever expect.

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Svensk Sammanfattning

Uppkomsten av kvantfältsteori under 1920-talet, en teori som kombinerar kvantmekanik och speciell relativitetsteori, tillät fysiker under andra hälften av seklet att ena tre (elektromagnetism och svag samt stark växelverkan) av de fyra fundamentala krafterna i partikelfysikens Standardmodell. Sedan dess har den stora utmaningen för teoretiska fysiker varit att också inkludera gravitation, och därmed allmän relativitetsteori, tillsammans med de tre andra krafterna i ett och samma ramverk.

Det är välkänt att om gravitation behandlas som en kvantfältsteori leder detta till svårhanterliga divergenser. En möjlig lösning kom dock från strängteori, en teori först utvecklade i slutet av 1960-talet och början på 1970-talet som en, icke-framgångsrik, modell för hadroner. Huvudiden bakom strängteori är att ersätta punktpartiklar med vibrerande öppna och slutna strängar. Det är viktigt att notera att strängteori har en naturlig lågenergigräns där den karakteristiska längdskalan av en sträng är liten och, effektivt, liknar en partikel vilken kan beskrivas av en approximativ kvantfältsteori. Därmed måste strängteori ses som en högenergikomplettering av kvantfältsteori. Dessutom beskriver en av strängvibrationerna i punktpartikelgränsen en masslös spin 2 partikel: gravitonen. ännu mer intressant är att strängens karakteristiska längdskala ger ett enkelt sätt att bota de tidigare nämnda divergenser som uppkommer när gravitation naivt beskrivs som en kvantfältsteori.

Två ytterligare ingredienser i strängteori är extra dimensioner samt supersymmetri, en symmetri som relaterar bosoniska och fermioniska frihetsgrader. I synnerhet lever en konsistent teori med supersymmetriska strängar i tio dimensioner vilket leder till den naturliga frågan hur vi bör hantera de sex extra dimensioner vi inte upplever i vår vardag. Den enklaste lösning är att tänka sig att dessa extra dimensioner är så små att de har en negligerbar påverkan på den fyra-dimensionella rymdtid vi upplever i vårt vardagliga liv.

Denna avhandling behandlar dock främst den andra aspekten av strängteori: supersymmetri. Uppenbarligen är supersymmetri bruten vid de energiskalor vi möter i vår vardag och även vid de energinivåer som kan nås i partikelacceleratorer som LHC. Oavsett antags det vanligtvis att den energinivå vid vilken supersymmetri bryts är lägre än den energinivå vid vilken strängar ej längre kan approximeras med punktpartiklar. Det är därmed av yttersta vikt att förstå supersymmetriska kvantfältsteorier. Utöver motivation från strängteori är supersymmetriska kvantfältsteorier också av intresse på grund av deras fundamentala natur. Supersymmetri ger mer kontroll över fysiska observabler och har därmed också varit till hjälp för att förstå aspekter av vanliga kvantfältsteorier.

Den extra struktur som ges av supersymmetri gör det möjligt att studera skyddade fysiska observabler under deformationer av den underliggande supersymmetriska kvantfältsteorin och i vissa fall kancellerar bidrag från olika sektorer varandra på grund av supersymmetri och kvantiteter kan bestämmas exakt. Detta är avhandlingens huvudämne. Mer specifikt behandlas exakta beräkningar för partitionsfunktioner för $\mathscr{N} = 2$ supersymmetriska kvantfältteorier på en klass av fyradimensionella mångfalder som tillåter en T^2 isometri och isolerade fixpunkter.

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