# Quantum Reference Frame Transformations, Noncommutative Values of Observables, and Quantum Relativity

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**Abstract.** The subject of quantum reference frame transformations gets popular lately with some interesting new theoretical development partly for the reason that the physics involved is becoming experimentally accessible. The position of a position eigenstate when observed from an object with 'uncertainty' in position would be seen with 'uncertainty'. In fact, even the existence of entanglement is reference frame-dependent. We present an improved formulation of such a transformation and give a novel way to describe exactly by 'how much' the 'value of the position' has changed which fully encodes all information about the changes, including the 'uncertainty' and entanglement. That is an application of the notion of noncommutative values of physical quantities we introduced to understand the reality of quantum physics and beyond. Some implications on fundamental physics will also be discussed. In particular, we suggest thinking about quantum gravity as a theory of general quantum relativity, alleviating Penrose's notion of incompatibility of qauntum mechanics with the relativity principle.

## 1 Introduction

The question about the compatibility of quantum physics with the relativity principle is of fundamental importance. The Relativity principle is about the descriptions of physics from different frames of reference. Any understanding of the latter rests firstly on our model of spacetime. A traditional thinking about spacetime in quantum physics easily gives a case for the incompatibility. As Penrose has argued [1], there seems to be an absolute difference between a particle with a definitive position, as in a position eigenstate, and one that is in a nontrivial superposition of more than one such eigenstate. No reference frame transformation can reconcile the two pictures is the claim. We present here explicit illustrations of the contrary. The recently popular notion of quantum reference frame transformations [2] can be used to show that the nature of a state as superpositions of eigenstates (say for the position observables), its Heisenberg uncertainties of observables, and even the entanglement of the particle or object with other parts of the full composite system are reference frame-dependent. We present a novel way to describe exactly 'how much' the 'value of the position' has changed which fully encodes all information about the changes, including the 'uncertainty' and entanglement [3]. That is an application of the notion of noncommutative values of physical quantities we introduced [4] to understand the reality of quantum physics [5] and beyond. There is a quantum relativity principle, one about quantum reference frame transformations, that is the correct picture about the subject matter. It is about the proper way to think about spacetime within quantum mechanics. The physical picture of space has to come from the notion of all positions a particle can take. A classical picture modeled by real number geometry cannot serve the purpose well. The full information a particle in a specific state has about its position in space is definitely beyond the possible eigenvalue outcomes we can get from its projective measurements. Our noncommutative values of the position observables give a good mathematical description about that. The corresponding geometric picture [6, 7] is to be based on noncommutative geometry [8, 9, 10].

Our understanding of spacetime is one of the most fundamental issue in physics. Einstein taught us that gravitation is spacetime curvature and his gravitational theory is the theory of general relativity. Then, the theory of quantum gravity should be one of general quantum relativity. We sketch some of our recent results towards the direction at the last part of the presentation.

## 2 Quantum Reference Frame Transformations

Let us look directly at a quantum spatial translation as the position of a quantum particle $C$ relative to another quantum particle $B$. The spatial translation is a change of relative position coordinates as seen from an inertial (laboratory) frame $A$ (simply called particle $A$ here) to the relative position coordinates as seen from another particle $B$ as the new reference frame. It has been well-appreciated that it is a canonical transformation to be given explicitly as

$$\begin{aligned} \hat{x}_B^{(A)} &\longrightarrow -\hat{x}_A^{(B)} , \qquad \hat{p}_B^{(A)} \longrightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)}) , \\ \hat{x}_C^{(A)} &\longrightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)} , \qquad \hat{p}_C^{(A)} \longrightarrow \hat{p}_C^{(B)} . \end{aligned} \tag{1}$$

Here, notation like $\hat{x}_B^{(A)}$ denote the observable $\hat{x}$ of particle $B$ as observed from particle $A$ as the frame. The expressions has the position and momentum observables each as a singlequantum quantity. Generalization to the case that each is a three-vector of independent components would be straightforward. The transformation as a quantum spatial translation is easy to appreciate. The part of the position observables read as classical ones would be exactly what one has in a classical theory. The part of momentum observables is what is required to make the full transformation a canonical one, *i.e.* to have the Poisson bracket $\frac{1}{i\hbar}[\cdot,\cdot]$ or all $\hat{x}$-$\hat{p}$ commutators preserved. Implicitly, the thinking about quantum reference frame transformations has hidden in it an intuitive but formally not so trivial [7] picture of the position and momentum observables as (noncommutative) coordinates of the phase space for the quantum system. Quantum reference frame transformations are symmetry transformations of the latter.

The quantum spatial translation can be written in terms of a unitary operator

$$\hat{S}_x = \hat{\mathcal{P}}_{AB} e^{i\hat{x}_B^{(A)}\hat{p}_C^{(A)}} , \tag{2}$$

($\hbar = 1$), where $\hat{\mathcal{P}}_{AB}$ is a parity-swap that sends $|x\rangle_B \otimes |y\rangle_C$ to $|-x\rangle_A \otimes |y\rangle_C$, mapping from $\mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)}$, the Hilbert space for states of the composite system $BC$ as described from $A$, to $\mathcal{H}_A^{(B)} \otimes \mathcal{H}_C^{(B)}$, the Hilbert space for states of the composite system $AC$ as described from $B$. One can easily check that it gives exactly the above operator transformations from $\mathcal{O} \to \hat{S}_x \mathcal{O} \hat{S}_x^\dagger$ [2]. We have $e^{i\hat{x}_B^{(A)}\hat{p}_C^{(A)}}$ naively behaves as a translation in $\hat{x}_C^{(A)}$ by the 'parameter' $\hat{x}_B^{(A)}$ and as a translation in $\hat{p}_B^{(A)}$ by the 'parameter' $-\hat{p}_C^{(A)}$ and the subsequent action of $\hat{\mathcal{P}}_{AB}$ finishes the job.

We have given an alternative formulation of the transformation in terms of position eigenstates [3]. The formulation allows the transformation to be seen more directly as a symmetry transformation within the Hilbert space of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, with the initial and final frames of reference taken as the 'states' $|\mathbf{0}\rangle_A$ and $|\mathbf{0}\rangle_B$, *i.e.* the zero vectors. We introduce the use of the zero vector under the following considerations. A zero vector of course has no observable physical properties. Any operator acts on it trivially. That corresponds exactly to the idea that a frame of reference does not see itself as a dynamical object, and hence cannot have a state with any nontrivial observable properties. We emphasize that it is not enough that the description of a composite state of $BC$ as observed from $A$ has no nontrivial content for its own position, $\hat{x}_A^{(A)}$, even if only the quantum spatial translation is concerned. First of all, the idea of writing that state of $BC$ as observed from $A$ as a vector within $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ should be independent of which particular quantum reference frame transformation one may want to formulate. For example, it should not depend on if we want to consider the quantum spatial translation of a quantum momentum translation. In the picture of Eq.(1) as presented in Ref.[2], there is no $\hat{p}_A^{(A)}$ and $\hat{p}_B^{(B)}$ to considered as there is no $\hat{x}_A^{(A)}$ and $\hat{x}_B^{(B)}$. The zero position eigenstate of $A$ has nontrivial $\hat{p}_A^{(A)}$ with quantum fluctuations, and would lead to similar $\hat{p}_B^{(B)}$ after the quantum translation. Our formulation with the zero vector is free from that and gives consistent results. Of course whatever makes up the physical frame of reference would be observed as a usual object from another frame of reference. The spatial translation is then presented as the action of the unitary operator

$$\hat{U}_x = \hat{\mathcal{S}}_{AB}^W \hat{I}_A \otimes \int dx'dy' \, |-x'\rangle\langle x'|_B \otimes |y'-x'\rangle\langle y'|_C , \tag{3}$$

which takes a generic state

$$|\psi\rangle = |\mathbf{0}\rangle_{\!A} \otimes \int\! dxdy\; \psi(x,y)\, |x\rangle_{\!B} \otimes |y\rangle_{\!C}\;\;, \tag{4}$$

to

$$\begin{aligned} \hat{U}_x\,|\psi\rangle \quad &= \int\! dxdy\; \psi(x,y)\, |{-x}\rangle_{\!A} \otimes |\mathbf{0}\rangle_{\!B} \otimes |y-x\rangle_{\!C}\;\;, \\ &= \int\! dxdy\; \psi(x,y+x)\, |{-x}\rangle_{\!A} \otimes |\mathbf{0}\rangle_{\!B} \otimes |y\rangle_{\!C}\;\;. \end{aligned} \tag{5}$$

$\hat{\mathcal{S}}^W_{AB}$ is a simple swap sending $|z\rangle_{\!A} \otimes |x\rangle_{\!B} \otimes |y\rangle_{\!C}$ to $|x\rangle_{\!A} \otimes |z\rangle_{\!B} \otimes |y\rangle_{\!C}$. It can further be checked explicitly that

$$\begin{aligned} \hat{U}_x\!\int\! dz'dx'dy'\; x'\, |z'\rangle\langle z'|_{A} \otimes |x'\rangle\langle x'|_{B} \otimes |y'\rangle\langle y'|_{C}\, \hat{U}^\dagger_x &= \!\int\! dzdxdy\; (-x)\, |x\rangle\langle x|_{A} \otimes |z\rangle\langle z|_{B} \otimes |y\rangle\langle y|_{C}\,, \\ \hat{U}_x\!\int\! dz'dx'dy'\; y'\, |z'\rangle\langle z'|_{A} \otimes |x'\rangle\langle x'|_{B} \otimes |y'\rangle\langle y'|_{C}\, \hat{U}^\dagger_x &= \!\int\! dzdxdy\; (y-x)\, |x\rangle\langle x|_{A} \otimes |z\rangle\langle z|_{B} \otimes |y\rangle\langle y|_{C}\,, \end{aligned} \tag{6}$$

which are exactly $\hat{U}_x \hat{x}^{(A)}_B \hat{U}^\dagger_x = -\hat{x}^{(B)}_A$ and $\hat{U}_x \hat{x}^{(A)}_C \hat{U}^\dagger_x = \hat{x}^{(B)}_C - \hat{x}^{(B)}_A$. Now, we can write explicitly $\hat{x}^{(B)}_C - \hat{x}^{(A)}_C$ as $\hat{x}_C - \hat{U}_x \hat{x}_C \hat{U}^\dagger_x = \hat{x}_A = -\hat{U}_x \hat{x}_B \hat{U}^\dagger_x$, or $\hat{x}^{(B)}_C = \hat{U}_x \hat{x}_C \hat{U}^\dagger_x - \hat{U}_x \hat{x}_B \hat{U}^\dagger_x$ as the classical analog of $x'_C = x_C - x_B$. One can also check explicitly for the momentum observables. Unitary transformations on the Hilbert space are generally canonical transformations anyway. $\hat{U}_x$ disagrees with a naive form of $\hat{S}_x$ as an operator on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ but is the only consistent formulation for composition of such transformations.

Ref.[2] presents some very illustrative nice pictures of the effects of the transformation in its figure 3. We gives here explicit analytical expressions for the four cases in that figure. The results are to be used in our analysis below. They are, in terms of simplified notations and with the zero vectors for the frames omitted, as

$$\begin{aligned} &\text{(a)}: \quad |x_o\rangle \otimes \int\! dy\; \psi(y)\, |y\rangle \quad \longrightarrow \quad |{-x_o}\rangle \otimes \int\! dy\; \psi(y)\, |y-x_o\rangle\;\;; \\ &\text{(b)}: \quad \frac{1}{\sqrt{2}}(|x_1\rangle + |x_2\rangle) \otimes \int\! dy\; \psi(y)\, |y\rangle \\ &\qquad\qquad \longrightarrow \frac{1}{\sqrt{2}}\left( |{-x_1}\rangle \otimes \int\! dy\; \psi(y)\, |y-x_1\rangle + |{-x_2}\rangle \otimes \int\! dy\; \psi(y)\, |y-x_2\rangle \right)\;\;; \\ &\text{(c)}: \quad c\,|x_1, y_o + x_1\rangle + s\,|x_2, y_o + x_2\rangle \quad \longrightarrow \quad (c\,|{-x_1}\rangle + s\,|{-x_2}\rangle) \otimes |y_o\rangle\;\;; \\ &\text{(d)}: \quad \int\! dx\; \psi(x)\, |x, y_o + x\rangle \quad \longrightarrow \quad \int\! dx\; \psi(x)\, |{-x}\rangle \otimes |y_o\rangle\;\;; \end{aligned} \tag{7}$$

where $c \equiv \cos(\frac{\theta}{2})e^{\frac{-i\zeta}{2}}$ and $s \equiv \sin(\frac{\theta}{2})e^{\frac{i\zeta}{2}}$, $0 \le \theta \le \pi$, $0 \le \zeta < 2\pi$, used to write a generic linear combination of two states. Given the above presentation, the interpretation of the simplified notations should be unambiguous. Case (a) has as the initial state a product of position eigenstate for $B$ and a generic state for $C$ (together with $|\mathbf{0}\rangle_{\!A}$). The final state maintains being a product state as shown (involving $|{-x}\rangle_{\!A}$ and $|\mathbf{0}\rangle_{\!B}$).[1] Case (b) has the transformation of a product state to one with nontrivial entanglement (between $A$ and $C$). Initial state for case (c) rather generalized somewhat the one in the figure, as a not necessarily equal combinations of two perfectly correlated parts of products of position eigenstates of $B$, and $C$ with a fixed difference in eigenvalue. The translation to have $B$ as the reference frame gives the final state as a product with the part for $C$ as a simple eigenstate. The perfect correlation makes all the quantum fluctuations of $C$ unobservable from $B$. (d) is really just a more general form of (c) with the same basic feature. Note that (b) is much like the inverse of (c) or (d). The initial and final state of (b) for $|\psi\rangle = |y_o\rangle$ can be identified essentially with the final and initial state of (c) with $c = s = \frac{1}{\sqrt{2}}$, respectively.

[1] $C$, $A$, and $B$ in the figure (3 of Ref.[2]) correspond to $A$, $B$, and $C$ of our notation, respectively.

## 3 Description with Noncommutative Values of Observables

The generic formulation gives a quantum spatial translation of $C$ as $\hat{x}_C$ changed into $\hat{x}_C - \hat{x}_B$, with then both observables $\hat{x}_C$ and $\hat{x}_B$ as dynamic variables. In the classical case, we typically consider a simple translation of a coordinate by a fixed amount, like changing to $x' = x - a$. Taking that as a classical reference frame transformation, we can see $a$ as the $x$-coordinate value of the particle to serve as the new frame of reference. For such a classical reference frame transformation on a quantum system, we have for example

$$S_x = e^{ix_B \hat{p}_C} \tag{8}$$

to describe a translation of the position of particle $C$ generated by $\hat{p}_C$ by the real number value of $x_B = a$. When we put that into the form of going from frame $A$ to frame $B$ both as classical frames, the translation gives

$$x_B^{(A)} \longrightarrow -x_A^{(B)}, \qquad \hat{x}_C^{(A)} \longrightarrow \hat{x}_C^{(B)} - x_A^{(B)}, \tag{9}$$

with no changes to any momentum. Here, $x_B^{(A)}$ and $x_A^{(B)}$ can be seen as positions of $B$ and $A$ as observed, so long as they can be seen as classical objects. For later convenience, let us use $^i$ and $^f$ to denote the quantities before and after a transformation, as the initial (here frame $A$) and final (here frame $B$) expressions. The translation would then give

$$x_A^f = -x_B^i, \qquad \hat{x}_C^f = \hat{x}_C^i - x_B^i = \hat{x}_C^i - a . \tag{10}$$

The important conceptual notion is the noncommutative value of a quantum observable [4, 7] allows us to look at the exact analog for the quantum translation. It gives a rigorous way of seeing an individual definite quantum translation as a generalization of the classical one of translating by a fixed value of distance $a$, given by the initial position of a quantum particle $B$ for any specific quantum state $B$ has. That is to say, we have

$$\hat{x}_A^f = -[x_B]_\phi^i, \qquad \hat{x}_C^f = \hat{x}_C^i - [x_B]_\phi^i, \tag{11}$$

where $[\hat{x}_B]_\phi$ is that 'distance' translated not as the variable $\hat{x}_B$ but an explicit 'value' specific to the state $|\phi\rangle$ as the analog of the real number $a$ of the classical case. That distance 'value' obviously cannot be a single real number. The latter simply cannot encode the full quantum information about the position of $B$ at a fixed state $|\phi\rangle$ including quantum fluctuations and entanglement features it may have which are of key interest about quantum reference frame transformations as illustrated above. For Case (b), for example, the initial $B$ state is a superposition of two eigenstates of $\hat{x}_B^i$. That nontrivial nature, versus that of Case (a), is what gives the interesting entanglement results between $C$ and $A$ and that is independent of the initial state of $C$. With further specification of the state of $C$, one would have consistently $[\hat{x}_C]_{\phi'}^f = [\hat{x}_C]_\phi^i - [x_B]_\phi^i$ where we have $|\phi\rangle$ and $|\phi'\rangle$ as the initial and final state considered for the more general situation as states of the corresponding composite systems.

Let us elaborate more on the notion of the noncommutative value as a description of the full quantum information involved. One usually looks at the information a specific state carries about an observable through the results of the projective measurements. Yet, a single eigenvalue outcome carries hardly any useful information. The naive thinking is to take the expectation value from the measurement results, maybe also to note its Heisenberg uncertainty, which is the standard deviation of the distribution of the latter. The theory predicts, however, the full statistical distribution that can be checked up to any required precision without theoretical uncertainty. So, the full information is at least the full statistics. That can be expressed through the sequence of real numbers as all the moments of the distribution. That is a lot more information than what can be encoded in a single real number. The statistical distribution does not encode the full information involved. It has been appreciated, for example, that position and momentum distributions do not have enough information to determine the state [11]. The notion of a noncommutative value of a quantum observable we introduced earlier [4, 7], is exactly a concrete mathematical way to describe the full information. It generalized the mathematical idea of a state, on an algebra, as a functional from a physical point of view. Such a functional on a commutative algebra is an algebraic homomorphism, keeping the algebraic relationships among the observables in their values for the state. The classical observable algebra as a commutative algebra has, for each state, those real number functional values as the physical values. The homomorphic property is important. When our theory predicts any relation among the observables as dynamical variables, we have to check them through those values as experimentally determined. Exact verifications of those algebraic relations, up to the experimentally manageable precisions, rest on that homomorphic property. Let us denote a state by $\phi$ and write that evaluation map as

$$[\cdot]_\phi : f(x^i, p^i) \to I\!R$$

taking a classical observable $\beta$ as a function of the position and momentum observables to its real number value $[\beta]_\phi$. The homomorphic property

$$[a\beta + b\gamma]_\phi = a[\beta]_\phi + b[\gamma]_\phi \;, \qquad [\beta\gamma]_\phi = [\beta]_\phi[\gamma]_\phi \;, \tag{12}$$

here is obvious for the observables, $\beta$ and $\gamma$ as real-valued functions, $f_\beta$ and $f_\gamma$, on the phase space. For example, if our theory says the energy $E$, of a one-dimensional harmonic oscillator with $m = \frac{1}{2}$ and $k = 1$, is given by

$$E = p^2 + x^2 = pp + xx \;, \tag{13}$$

we need to have on any state $\phi$, say one with $x = 2$, $p = 3$,

$$[E]_\phi = [p^2]_\phi + [x^2]_\phi = [p]_\phi[p]_\phi + [x]_\phi[x]_\phi = 3\dot{3} + 2\dot{2} = 13 \;. \tag{14}$$

We spell out in details what should be trivial here only to bring home the point that most have not paid enough attention to. That is what should be the *required property of* the evaluation map on the observable algebra that a state is, from the mathematical point of view. The notion of noncommutative value is to look at a quantum state as a homomorphic map on the quantum observable algebra. The set of values as the image of such a map then has to be a noncommutative algebra. After all, real number is a set of mathematical symbols, as a commutative algebra, that has been used successfully to model the notion of values of physical quantities in classical physics. It fails to do the same for quantum physics. Then we should consider finding a new model for the job. Intuitive concepts are not classical. Quantum notions about concept is physics are not less intuitive. The corresponding classical notions are only what we are more familiar with. *The philosopher Quine called real numbers convenient fiction.* That is, of course, about the real number values of physical quantities, as he also stated that to be is to be the value of a variable. With quantum physics, real number values are no longer good enough. *We need the new convenient fiction, our noncommutative values or even noncommutative numbers.*

For a noncommutative algebra, the functionals as states are given by the having the values as the expectation values. The idea can be promoted to give a one-to-one homomorphism. We can this the full expectation value function as a function on the quantum phase space and look at the sequence of coefficients of its local Taylor series expansion on a state. The sequence is state specific, fully predicted by the theory, and all numbers in it can be determined experimentally at least in principle. Most importantly, there is a noncommutative product between two such sequences for two observables to retrieve the sequence for the product observable. With the latter product, the set of such sequences for form an algebraically isomorphic image of the observable algebra. In short, a state $|\phi\rangle$ defines a homomorphism $[\cdot]_\phi$ that takes an observable $\hat{\beta}$ to an element of the noncommutative algebra of its value $[\hat{\beta}]_\phi$. The latter as a noncommutative value can be represented by the sequence of Taylor coefficients. Yet, we can have much simplified representation of is essentially including only the first three coefficients due to specific mathematical properties of the expectation value functions [4].

### *3.1 Noncommutative Values of Observables*

Let us first give a representation of the noncommutative value of a quantum observable $\hat{\beta}$ on a given physical state. For the $f_{\hat{\beta}}(z_n, \bar{z}_n)$ function being the expectation value function of Hermitian operator $\hat{\beta}$, we have

$$V_{\hat{\beta}_n} = \partial_n f_{\hat{\beta}} = -f_{\hat{\beta}}\bar{z}_n + \sum_m \bar{z}_m (\hat{\beta})^m_n \;, \tag{15}$$

where $(\hat{\beta})^m_n$ are the matrix element $\left\langle m|\hat{\beta}|n \right\rangle$ over an orthonormal basis $\langle m|n\rangle = \delta^m_n$, and $z^n$ the complex coordinates of a normalized state $|\phi\rangle = \sum_n z^n |n\rangle$, $n$ runs over the dimension of the Hilbert space for the system under consideration. The set of $z^n$ also serves as the homogeneous coordinates of the projective

Hilbert space as a Kähler manifold [12]. One can check that

$$\begin{aligned} f_{\hat\beta\hat\gamma} &= f_{\hat\beta} f_{\hat\gamma} + \sum_n V_{\hat\beta_n} V_{\hat\gamma_{\bar n}} \ , \\ (\hat\beta\hat\gamma)^m_n &= \sum_l (\hat\beta)^m_l (\hat\gamma)^l_n \ , \\ V_{\hat\beta\hat\gamma_n} &= -f_{\hat\beta\hat\gamma}\bar z_n + \sum_m \bar z_m (\hat\beta\hat\gamma)^m_n \ , \end{aligned} \tag{16}$$

where $V_{\hat\gamma_{\bar n}} = \partial_{\bar n} f_{\hat\gamma}$ is just the complex conjugate of $V_{\hat\gamma_n}$ for any (Hermitian) operator $\hat\gamma$. We can take the noncommutative/quantum value $[\hat\beta]_\phi$ as represented by the sequence and complex number values of the quantities $\{f_{\hat\beta}, V_{\hat\beta_n}, (\hat\beta)^m_n\}$, evaluated on the state. The noncommutative value of an observable as the product $\hat\beta\hat\gamma$ is then the noncommutative product for two noncommutative values, *i.e.* $[\hat\beta\hat\gamma]_\phi = [\hat\beta]_\phi \star_\kappa [\hat\gamma]_\phi$, with elements of the sequence as given by the equations above. The equation gives the explicit definition of the noncommutative (Kähler) product $\star_\kappa$,derived from the notion of such a product between expectation value functions first introduced in Ref.[13]. For any specific state, the map from the observable algebra to the noncommutative values, taken as a noncommutative algebra with the product as given is obviously a homomorphism, maintaining the algebraic relation among the observables in their values. In particular, for $\hat\beta = \sum \lambda_m |m\rangle\langle m|$ at $|n\rangle$, we have

$$f_{\hat\beta} = \lambda_n \ , \quad V_{\hat\beta_m} = \bar z_m(\lambda_m - f) = 0 \ , \quad (\hat\beta)^m_l = \delta^m_l \lambda_m \ .$$

So, an eigenstate of an observable always has all corresponding $V_{\hat\beta_n}$ being zero, and degenerate eigenstates for an observable have identical noncommutative values. Moreover, $\hat\beta = r\hat I$ gives $f = r$, have the noncommutative value behaving essentially as a commutative classical real number value. Note that the matrix element $(\hat\beta)^m_n$ can be expressed in terms of $f_{\hat\beta}$, $V_{\hat\beta_n}$ and $\tilde k_{\hat\beta_{\bar m n}} \equiv \partial_n \partial_{\bar m} f_{\hat\beta}$ [4], hence the full sequence for the noncommutative value can be obtained from a given expectation value function $f_{\hat\beta}$ on the projective Hilbert space without knowing *a priori* the explicit operator form of the $\hat\beta$. In fact, one can check if a function $f(z_n, \bar z_n)$ is indeed an $f_{\hat\beta}$ without knowing $\hat\beta$ [7, 13]. Moreover, the classical value $r$ as a constant noncommutative value has also $\tilde k_{\hat\beta_{\bar m n}} = 0$. The particular representation of the noncommutative value, which is really a single quantity as an element in a noncommutative algebra, is chosen as the optimal one for an easy and more transparent illustration of the theoretical issue address in this presentation. The sequence of complex numbers representing a $[\hat\beta]_\phi$ has three parts. The $V_{\hat\beta_n}$ part is the key focus here. It gives important information about how much the state differs from an eigenstate, hence the quantum nature of the quantity $[\hat\beta]_\phi$. For example, the Heisenberg uncertainty characterizing the spread of the eigenvalue results from projective measurements about the expectation value is given by

$$(\Delta\beta)^2_\phi = f_{\hat\beta^2} - f^2_{\hat\beta} = \sum_n |V_{\hat\beta_n}|^2 \ . \tag{17}$$

For a convenient analysis of the noncommutative value for the position operator on states as described by wavefunction, we need the form of the noncommutative value Schrödinger representation. Note that the wavefunction $\phi(x)$ is really a collection of infinite numbers of complex number coordinates as $\langle x|\phi\rangle$, one for each eigenstate $|x\rangle$ for the value of $x$, $-\infty \le x \le \infty$. A complex function can be seen as a collection of complex numbers (functional values) one at each point of $x$. Then, the matrix elements $(\hat\beta)^{x'}_x = \langle x'|\hat\beta|x\rangle$ are to be expressed together as a two-variable function; for example, $(\hat x)^{x'}_x = x\delta(x'-x)$. The coordinate derivatives corresponding to $V_{\hat\beta_n}$ may then be expressed together as a function which is the functional derivative $\delta_\phi f_{\hat\beta}$. That is, we have $[\hat\beta]_\phi = \{f_{\hat\beta}, \delta_\phi f_{\hat\beta}, (\hat\beta)^{x'}_x\}$ as the noncommutative value. From

$$f_{\hat x}(\phi) = \frac{\int dx\, \bar\phi(x) x \phi(x)}{\int dx\, \bar\phi(x)\phi(x)} \tag{18}$$

taken as a functional of the (normalized) wavefunction, we have the set of infinite coordinate derivatives can be expressed as the functional derivative

$$V_{\hat x}(x) = \delta_\phi f_{\hat x}(\phi) \equiv \frac{\delta f_{\hat x}}{\delta\phi}(\phi) = \bar\phi(x)(x - x_o) \ , \tag{19}$$

where $x_o$ here denotes the expectation value of $f_{\hat{x}}$ evaluated for the fixed $\phi(x)$. There is one value of $V_{\hat{x}}(x)$ at each $x$ value matching to the coordinate value of $\phi(x)$. For the momentum observable, we have

$$f_{\hat{p}}(\phi) = \frac{\int dx\ \bar{\phi}(x)(-i\partial_x)\phi(x)}{\int dx\ \bar{\phi}(x)\phi(x)} = \frac{\int dx\ [i\partial_x\bar{\phi}(x)]\phi(x)}{\int dx\ \bar{\phi}(x)\phi(x)}\ , \tag{20}$$

which gives

$$V_{\hat{p}}(x) = \delta_\phi f_{\hat{p}}(\phi) = (i\partial_x - p_o)\bar{\phi}(x)\ , \tag{21}$$

where $p_o$ again denotes the expectation value. One can check that we have [3]

$$\begin{aligned} f_{\hat{x}\hat{p}} &= -i\int dx\ \bar{\phi}(x)x\partial_x\phi(x) = f_{\hat{x}}f_{\hat{p}} + \int dx\ V_{\hat{x}}(x)\bar{V}_{\hat{p}}(x)\ , \\ f_{\hat{p}\hat{x}} &= -i\int dx\ \bar{\phi}(x)[x\partial_x\phi(x) + \phi(x)] = f_{\hat{p}}f_{\hat{x}} + \int dx\ V_{\hat{p}}(x)\bar{V}_{\hat{x}}(x)\ , \end{aligned} \tag{22}$$

and

$$f_{\hat{x}\hat{p}} - f_{\hat{p}\hat{x}} = i\ , \qquad V_{\hat{x}\hat{p}}(x) - V_{\hat{p}\hat{x}}(x) = 0\ . \tag{23}$$

*3.2 The Quantum Amount Transformed*

With the results above, we can move on to illustrate the change in noncommutative values of the position observables in the quantum spatial translations. Let us do that for Case (d). The first thing to note is that an expectation value function $f_{\hat{\beta}}(\phi)$ is of course invariant under any unitary transformation. As the terms in the sequence representing the corresponding noncommutative value are all fixed by the values of the derivatives of $f_{\hat{\beta}}(\phi)$ for a physical state, the whole noncommutative value should be invariant. For the quantum spatial translation with $\hat{x}_C \to \hat{x}_C - \hat{x}_A$, the operator $\hat{x}_C$ before and after the transformation are different operators on the same Hilbert space, as position operator formulated on differently defined position eigenstate basis which gives the easily appreciable picture of the translation, as $|y\rangle \to |y-x\rangle$. The noncommutative values of the position observable of $C$ for any physical state changes. That is exactly like the translation (of reference frame) in classical physics $x_C \to x_C - x_A$, the explicit operators describing the same position of $C$ are two different (quantum) position coordinate observables and they have different values. However, there is a further subtlety as the $\delta_\phi f_{\hat{\beta}}(\phi)$ and $(\hat{\beta})_x^{x'}$ terms have values which depend on the choice of basis of the Hilbert space. We can only compare two noncommutative values explicitly in the sequence representations when the latter has the $\delta_\phi f_{\hat{\beta}}(\phi)$ and $(\hat{\beta})_x^{x'}$ terms expressed in the same basis. Say, we have to compare the initial and final value of $x_C$ through expressing both noncommutative values in either the eigenstate basis before or after the transformation. We are only going to present the key results for Case (d) here. Full results for all the cases above are available in Ref.[3].

(d) : $\quad |\phi\rangle = \int dx\ \psi(x)\,|x, y_o + x\rangle \quad \longrightarrow \quad |\phi'\rangle = \hat{S}_x\,|\phi\rangle = \int dx\ \psi(x)\,|{-x}\rangle \otimes |y_o\rangle$

The initial state wavefunction is $\phi(x,y) = \psi(x)\delta(y-x-y_o)$. We have

$$\begin{aligned} \delta_\phi f^i_{\hat{x}_B} &= (x - x_o)\bar{\psi}(x)\delta(y-x-y_o)\ , \\ \delta_\phi f^i_{\hat{x}_C} &= (y - y_o - x_o)\bar{\psi}(x)\delta(y-x-y_o) = (x-x_o)\bar{\psi}(x)\delta(y-x-y_o)\ , \end{aligned}$$

where $x_o$, again, denotes the value of $f_{\hat{x}_B}$, and the value of $f_{\hat{x}_C}$ is then $y_o + x_o$. The nature of the results not as products of a function of $x$ and another of $y$ is the signature of the nontrivial entanglement here seen in the noncommutative values of the observables. In addition, the equality of the two is the signature of their perfect correlation. The final state wavefunction is $\phi'(x',y') = \psi(-x')\delta(y'-y_o)$, with

$$\begin{aligned} \delta_{\phi'} f^f_{\hat{x}_A} &= (x' + x_o)\bar{\psi}(-x')\delta(y'-y_o)\ , \\ \delta_{\phi'} f^f_{\hat{x}_C} &= (y' - y_o)\bar{\psi}(-x')\delta(y'-y_o) = 0\ , \end{aligned}$$

checking out $[\hat{x}_A]^f_{\phi'} = [-\hat{x}_B]^i_\phi$ and $[\hat{x}_C - \hat{x}_A]^f_{\phi'} = [\hat{x}_C]^i_\phi$. The result of $[\hat{x}_C]^f_{\phi'} = [\hat{x}_C]^i_\phi - [\hat{x}_B]^i_\phi$ has zero $\delta_{\phi'} f^f_{\hat{x}_C}$ from the cancellation $\delta_\phi f^i_{\hat{x}_C} - \delta_\phi f^i_{\hat{x}_B}$. The perfect correlation between the observables leads to their difference bearing zero uncertainty, as a result of the cancellation of the uncertainties. We can also read the transformation in the reverse, taking the final product state of $\phi(x',y')$ given in the reference frame of $B$ as the initial, which would be then expressed as the entangled state of $\phi(x,y)$ in the reference frame of $A$ upon the quantum spatial translation. The difference between the $[\hat{x}_C]^f_{\phi'}$ and $[\hat{x}_C]^i_\phi$ above as $-[\hat{x}_B]^i_\phi = [\hat{x}_A]^f_{\phi'}$, reads as a function of $x'$ and $y'$, as the Hilbert space coordinates in the position eigenstate basis in the frame of $B$, shows no entanglement as $\phi(x',y')$ and $\delta_\phi f^i_{\hat{x}_B}$ or $\delta_{\phi'} f^f_{\hat{x}_A}$ factorize into a product of functions of $x'$ and $y'$. This inverse transformation picture essentially illustrates the key features of case (b), *i.e.* of turning a product state into one with entanglement. While the difference in $[\hat{x}_C]^f_{\phi'}$ and $[\hat{x}_C]^i_\phi$ in case (a) above has factorizable expressions in terms of $x$-$y$ or $x'$-$y'$, here the result has a factorizable expression only in terms of $x'$-$y'$. Note that though $[\hat{x}_C]^f_{\phi'}$ has a nonfactorizable expression in terms of $x$-$y$, we cannot say that the latter show entanglement. $x$ is about the eigenstate or eigenvalue of $B$ which has no meaning with $B$ as the reference frame. One can further check that $[\hat{p}_C]^f_{\phi'} = [\hat{p}_C]^i_\phi$ and $[\hat{p}_A]^f_{\phi'} = -[\hat{p}_A]^i_\phi - [\hat{p}_C]^i_\phi$.

### *3.3 Qubit Systems and Noncommutative Numbers*

We want to go beyond our discussion of the noncommutative values of the quantum observables in relation to the picture of quantum spatial translations as the focus of the presentation here. In Ref.[3], formulation of an example of quantum reference transformations in a qubit system is also presented, together with the analysis of the changes in noncommutative values of cases of states as the parallel of those for the quantum translations above. We say a couple of words about that here to give readers an idea about the kind of quantum transformations.

The transformation considered is the parallel of the translation by $\hat{x}_B$ above taken up by the observable $\hat{\sigma}_{3_B}$ (as observed from $A$). For each qubit, we have only two base vectors, $|0\rangle$ and $|1\rangle$ as eigenstates of $\hat{\sigma}_3$ with eigenvalues plus and minus 1. The analog of $|-x'\rangle\langle x'|_B$ is clearly $|0\rangle\langle 1|_B$ and $|1\rangle\langle 0|_B$ flipping the sign of the eigenvalues, hence taking

$$\hat{\sigma}_{3_B} \to -\hat{\sigma}_{3_A} \;, \quad : \quad |0\rangle_B \to |1\rangle_A \;, \qquad |1\rangle_B \to |0\rangle_A \;, \tag{24}$$

However, $\hat{\sigma}_{3_C} \to \hat{\sigma}_{3_C} - \hat{\sigma}_{3_B} d$, for the parallel of $|y'-x'\rangle\langle y'|_C$, is impossible as the observables have no eigenstates wiith 0 and $\pm 2$ as eigenvalues. A sensible choice is to have the unitary transformation based on $|0\rangle_C \to \frac{1}{\sqrt{2}}(|0\rangle_C + |1\rangle_C)$. Explicitly, in the simplified notation, the transformation is given by

$$\begin{aligned}
|\mathbf{0}\rangle_A \otimes |00\rangle &\longrightarrow |\mathbf{0}\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \;, \\
|\mathbf{0}\rangle_A \otimes |01\rangle &\longrightarrow |\mathbf{0}\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \;, \\
|\mathbf{0}\rangle_A \otimes |10\rangle &\longrightarrow |\mathbf{0}\rangle_B \otimes \frac{1}{\sqrt{2}}(|01\rangle - |00\rangle) \;, \\
|\mathbf{0}\rangle_A \otimes |11\rangle &\longrightarrow |\mathbf{0}\rangle_B \otimes \frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) \;.
\end{aligned} \tag{25}$$

On the basic operators, it gives

$$\begin{aligned}
\hat{\sigma}_{1_B}, \hat{\sigma}_{2_B}, \hat{\sigma}_{3_B} &\longrightarrow \hat{\sigma}_{2_A}\hat{\sigma}_{2_C}, \hat{\sigma}_{1_A}\hat{\sigma}_{2_C}, -\hat{\sigma}_{3_A} \;, \\
\hat{\sigma}_{1_C}, \hat{\sigma}_{2_C}, \hat{\sigma}_{3_C} &\longrightarrow -\hat{\sigma}_{3_A}\hat{\sigma}_{3_C}, -\hat{\sigma}_{2_C}, -\hat{\sigma}_{3_A}\hat{\sigma}_{1_C} \;.
\end{aligned} \tag{26}$$

The transformation of course preserves the commutation relations among the operators. On the four-dimensional two-qubit composite Hilbert spaces, we have a noncommutative value with the key part to analyzed as

$$[\sigma]_\phi = \{f_\sigma, V_{\sigma_{00}}, V_{\sigma_{01}}, V_{\sigma_{10}}, V_{\sigma_{11}}\} \;. \tag{27}$$

The mathematical form of the noncommutative values of quantum observables for a system with a Hilbert space of a small dimension is certainly much simpler and may be easier to appreciate.

Let us also sketch a different, probably conceptually and practically more interesting representation of an algebra of the noncommutative values, for the simplest quantum system, a qubit. It is conceptually more interesting because it can be seen as an algebra of noncommutative numbers [16]. As such, it may stand a better chance to be directly determined experimentally as a single piece of quantum information, though doing anything of the kind requires completely revolutionary thinking about how we use and calibrate our apparatus.

We start with some interesting quotes from Dirac. Dirac's founding contribution to the theory of quantum mechanics is to give a full abstract quantum observable picture of Heisenberg's basic idea of taking the observables beyond the notion of real number-valued variables while keeping our physics picture about the dynamical theory. Dirac introduced the quantum Poisson bracket

$$\{\cdot,\cdot\} = \frac{1}{i\hbar}[\cdot,\cdot]$$

in terms of the algebraic commutator for the Hamiltonian formulation, keeping the position and momentum observables as canonical coordinate variables of the phase space, *i.e*

$$\{\hat{x}^i, \hat{p}_j\} = \frac{1}{i\hbar}[\hat{x}^i, \hat{p}_j] = \delta^i_j \ . \tag{28}$$

At that point, of course, nobody has any idea about the geometry of that phase space, or maybe the space as coordinated by $\hat{x}^i$. They have already given up the classical geometric picture of the physical space though. The modern noncommutative geometry is the natural candidate for that. The resurrection of the Newtonian space picture along Schrödinger's theory of wave mechanics is a key source of confusion, which also led to the Bohr-Born picture with all its mysterious implications, though Schrödinger himself soon gave up the idea of his wavefunction being about a physical wave. That was not the idea behind the theory of Heisenberg and Dirac. We see even the picture of matrix mechanics from Born and Jordan is quite a compromise of the more revolutionary idea of the couple. Note that formal equivalence of two theories certainly does not reconcile the different conceptual interpretations. Dirac introduced the term $q$-number variables for the quantum observables as a new kind of numbers, essentially our new convenient fiction. We quote:

*To distinguish the two kinds of numbers, we shall call the quantum variables q-numbers and the numbers of classical mathematics which satisfy the commutative law c-numbers, while the word number alone will be used to denote either a q-number or a c-number."* [14]

*Owing to the fact that we count the time as a c-number, we are allowed to use the notion of the value of the dynamical variable at any instance of time. This value is a q-number, capable of being represented by a generalized 'matrix', . . .* [15]

A difficulty then is that, as Dirac also stated,

*At present one can form no picture of what a q-number is like.* [14]

And he has never given up an answer. His $q$-numbers we would like to call noncommutative numbers. Well, if the observables are taken as operators, or matrices, on the Hilbert space, it is difficult to think about how they can have different values for different states. Moreover, with the (projective) Hilbert space as the quantum phase space, there seems to be no need to find a new geometry for the theory. One needs to reconcile the Hilbert space picture with Dirac's intuitive idea of the phase space with the observables as coordinates [7].

Our notion of the noncommutative values discussed above fits well into Dirac's idea of the $q$-number values, except for the fact that we have not been able to express the form of the Kähler product in a state-independent manner. For the $q$-numbers as numbers, one needs a general product rule. The set of noncommutative values of observables for a specific state is an isomorphic image of the observable algebra. To have those values directly taken as numbers with a noncommutative product, we need to have the algebra of the values from all different states embedded into a single algebra.

The notion of $q$-number, or noncommutative number, is a natural one, even from the point of view of extensions of the notion of numbers in the history of mathematics. It starts with natural numbers. One can see it as the effort to find solutions to algebraic equations that led to the extensions. When no solution to an algebraic equation exists, the solution would be invented as a new kind of number that extends the system. Here is a list of simple equations that could be seen as each leading to the introduction of the new kind of number given:

$$\begin{array}{lcl} x+2=0 & \longrightarrow & \text{negative numbers} \\ 2x-1=0 & \longrightarrow & \text{rational numbers} \\ x^2-2=0 & \longrightarrow & \text{real numbers} \\ x^2+1=0 & \longrightarrow & \text{complex numbers} \end{array}$$

Going to equations with more than one variable, for example, we have

$$xy - x - i = 0 \qquad \longrightarrow \qquad (x,y) = \{(i,2), (\frac{1}{i-1}, -i), \dots\} \ .$$

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But then we can have

$$xy - yx - 1 = 0 \qquad \longrightarrow \qquad \text{noncommutative numbers.}$$

With the equation for the (quantum) variables

$$\hat{x}\hat{p} - \hat{p}\hat{x} - i(\hbar) = 0$$

there is no solution for it within real or complex numbers. That should not mean we do not have definite real number values for the variables but we need a new kind of numbers to give the solutions. That is the noncommutative/$q$-numbers, and the wisdom of Dirac.

Inspired by a study of quantum information flows in qubit systems from the Heisenberg picture by Deutsch and Hayden[17], we have introduced another formulation of the noncommutative values of observables, called the DH-matrix values [18]. They are just matrices, hence mathematically the natural candidates of numbers with a noncommutative product. Carefully analysis of the idea and a presentation of a full conventional scheme for the consistent assignment of the DH-matrix values to all observables with any state are given in Ref.[16]. In summary, the usual representation of observables as specific matrices is indeed a representation in terms of the DH-matrix values for a chosen reference states. Any other state can be given in terms of a unitary transformation $U_\phi$ acting on the latter. Each state $|\phi\rangle$ may then be seen as described by $U_\phi$ which maps the observable to its value. In the explicit example for the case of a single qubit, an observable is a dynamical variable as an abstract operator on the two-dimensional Hilbert space, $\hat{s}^i$. Its values for the state $|\phi\rangle$ is conventionally given by

$$[\hat{s}^i]_\phi = U_\phi^{-1} \sigma^i U_\phi \tag{29}$$

where

$$U_\phi = \begin{pmatrix} c & -\bar{s} \\ s & \bar{c} \end{pmatrix}, \qquad |\phi\rangle = U_\phi\,|0\rangle = c\,|0\rangle + s\,|1\rangle \ , \tag{30}$$

with $c \equiv \cos(\frac{\theta}{2})e^{\frac{-i\psi}{2}}$, $s \equiv \sin(\frac{\theta}{2})e^{\frac{i\psi}{2}}$, $0 \le \theta \le \pi$, $0 \le \psi < 2\pi$, $|0\rangle$ being the reference state and the Pauli matrix $\sigma^i$ itself assigned value. Note that vales physical quantities are only meaningful as relative to one another. The absolute or exact value of one is only conventionally taken, through the introduction of the units and beyond. $[\hat{s}^i]_\phi$ with $|\phi\rangle$ specified is a value, as a fixed quantity. Taken the state with the parameter $c$ and $s$ as variables, we have $\hat{s}^i$ with its nature as a dynamical variable restore. In that sense, quantum mechanics is not matrix mechanics, but matrix-valued mechanics.

## 4 From Quantum Relativity to Quantum Gravity

Our noncommutative geometric perspective sees quantum mechanics as a theory of particle dynamics in quantum spacetime, naturally with quantum frames of reference. The perspective and its emphasis on the position coordinate observables as physical coordinates of the geometry have been adopted to look at some problems related to gravitation, having the classical metric promoted to a quantum observable. Firstly, we have studied the Equivalence Principle and established the version about a particle as the exact analog of the classical case, with quantum geodesic equations [19]. In classical Hamiltonian dynamics, the particle phase space as the cotangent bundle of its position space has a symplectic geometry independent of the existence of the Riemannian metric. For any general position coordinates $x^a$ taken, one has the conjugate momentum as a cotangent vector and the canonical conditions of the Poisson bracket. The same can be checked to hold for the quantum case. We have, generally,

$$\{\hat{x}^a, \hat{p}_b\} = \delta^a_b \ , \qquad \{\hat{x}^a, \hat{x}^b\} = 0 = \{, \hat{p}_b\} \ . \tag{31}$$

The Hamiltonian vector fields define the coordinate partial derivatives as derivations:

$$\partial_{\hat{x}^a} \equiv \{\cdot, \hat{p}_a\} \ , \qquad \partial_{\hat{p}_a} \equiv -\{\cdot, \hat{x}^a\} \ . \tag{32}$$

The quantum geodesic equations in a classical or quantum Rindler frame [20] the instantaneous free-falling frame maintains the simple form of

$$\frac{d^2\hat{x}^\mu}{ds^2} + \frac{d\hat{x}^\nu}{ds}\Gamma^\mu_{\nu\sigma}(\hat{x})\frac{d\hat{x}^\sigma}{ds} = 0 \ . \tag{33}$$

The general form of the quantum geodesic equations as equations of free particle motion, with the invariant Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}_A \hat{g}^{Ab}\hat{p}_b \equiv= \frac{1}{2m}\hat{p}^b\hat{p}_b \ , \tag{34}$$

obtained have a more complicated form [21]. Yet, they are noncommutative generalizations of the classical ones and particle mass-independent. That forms the base for a theory of quantum mechanics in curved spacetime with all the appealing features the conventional approach missed [22]. A Schrödinger wave-function representation, what the latter assumed, is indeed generally not admissible. Moreover, one has a coordinate system-based, or metric-dependent, notion of Hermiticity for the momentum observables.

The above paragraphs sketch some of the results related to gravitation. All is hinged on the noncommutative geometric perspective. Noncommutative geometries are certainly not real number geometries. The position coordinate observables as quantum observables are $q$-number variables taking $q$-number values. The conjugated momentum observables are intrinsically bounded to the position ones and the full phase space is a more proper picture of quantum spacetime [7]. The phase space for standard quantum mechanics, with Cartesian $\hat{x}^i$, may well be considered the noncommutative geometric notion of 'Euclidean' geometry. The geometry they by coordinated any $\hat{x}^a$, and $\hat{p}_a$, is then a $q$-number geometry. Quantum physics is $q$-number physics, as Dirac observed. Quantum reference frame transformations changing the $q$-number values coordinate description of it are what is naturally behind a Quantum Relativity Principle. Quantum gravity is then a theory of General Quantum Relativity. That is a comprehensive picture we aim at establishing in details in the future.

Perhaps we can provides a few words about the relation of our approach may have to existing theories of quantum gravity. First of all, it is important to note that there is no such theory with its basic approach more or less established to be correct. We have rather contending candidates work on quite different starting points with different focuses [23]. Yet, most of them have in common the adopting of a classical, real number, geometric picture of spacetime and implement some quantization procedures to a classical theory. Our approach, however, starts with a quantum picture of spacetime following and fully implementing the otherwise naive Heisenberg-Dirac idea on taking physical quantities as quantum ones. Instead of looking at some quantum field theory on classical geometry, we want to consider gravitational field of quantum spacetime but still have to work towards that. De Witt was an important pioneer of the traditional approach [24]. We have outlined the different results on the step before that, as formulating quantum mechanics in curved spacetime, from our perspective preserving more of the conceptual notions that work in classical physics at the quantum level. Conceptually, our more general geometric background can accommodate as classical approximations the ones adopted in the traditional approach. We can avoid most of the difficulties and controversal issues of the different quantization procedures, but need to learn to deal with the noncommutative quantum quantities. At some levels of approximation, our approach would be expected to yield results consistent with the traditional one.

Further development of various aspects of the mathematical theory of $q$-numbers and their extensions would be interesting, and relevant to physics. One may have to look into, for example, an extension of algebra with $q$-numbers taking up the role of the scalars. Quantum information may be seen as $q$-number information and we could have $q$-number technology.

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