THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Supergravity with Eight Supercharges: Exploring Higher-Derivative Corrections and Partial Supersymmetry Breaking

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BS-MS Dual Degree

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A thesis submitted for the degree of Doctor of Philosophy at The University of Queensland in 2024

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Abstract

In the quest to unite quantum mechanics and Einstein's theory of gravity, two leading candidates have emerged: string theory and supergravity. This thesis explores various aspects of supergravity theories with eight real supersymmetry generators (supercharges), focusing on $4D$ $N = 2$, $5D$ $N = 1$, and $6D$ $N = (1,0)$ supergravity theories. Here, N refers to the number of copies of supersymmetry. The research investigates quantum corrections in the form of higher-derivative deformations, as well as the mechanisms of partial supersymmetry breaking within supersymmetric gravity, as predicted by string theory. Additionally, the thesis aims to construct new multiplets of conformal supergravity and demonstrates how these multiplets facilitate the analysis of the on-shell supergravity action.

In this thesis, we review minimal (N=1) off-shell two-derivative gauged supergravity in five dimensions and investigate three independent four-derivative superspace invariant terms, with the aim to improve our understanding of higher-derivative corrections and their consequences on the AdS/CFT correspondence. These invariants give rise to various locally supersymmetric extensions of fundamental gravitational terms, such as the Einstein-Hilbert term (R), a cosmological constant (Λ), a Riemann tensor squared ($R_{abcd}R^{abcd}$), a Ricci tensor squared ($R_{ab}R^{ab}$), and a scalar curvature squared (R^2). The study employs off-shell techniques and symbolic computational tools, such as Cadabra, to derive, for the first time, the component fields' actions and primary equations of motion for the three gauged curvature-squared invariant terms in standard Weyl multiplet background and uplift them in vector-dilaton Weyl multiplet background. The gauged aspect is crucial for obtaining supersymmetric AdS solutions, which play an important role in holography. This facilitated the computation of the Weyl anomaly coefficient for the dual conformal field theory through holographic renormalization. This computation serves as a precision test of the AdS/CFT correspondence at next-to-leading order, providing critical insights into the correspondence's validity and applications.

Additionally, we also explore alternate dilaton Weyl multiplets of conformal supergravity. Our analysis reveals that the vector-dilaton Weyl multiplet significantly simplifies calculations when dealing with curvature-squared invariant terms and streamlines the construction of these terms in the on-shell supergravity action. This thesis introduces the hyper-dilaton Weyl multiplet, a novel representation of the conformal supergravity multiplet for the aforementioned supergravity theories with eight supercharges. We successfully coupled the hyper-dilaton Weyl multiplet to an off-shell vector multiplet compensator and employed superconformal techniques to reproduce the established $4D$ $N = 2$ supergravity action and extend it to the $5D$ $N = 1$ theory. Furthermore, the hyper-dilaton Weyl multiplet enabled us to construct an off-shell model demonstrating partial supersymmetry breaking in Minkowski spacetime. This was achieved by coupling the hyper-dilaton Weyl multiplet to deformed off-shell vector and tensor multiplets.

We present a general off-shell gauged $4D$ $N = 2$ supergravity model, deformed by an arbitrary number of vector and tensor multiplets, and derive the primary equations of motion for this general theory. A detailed analysis of a specific $SU(1,1)/U(1)$ model, engineered fully off-shell, is provided, showcasing on-shell partial supersymmetry breaking in Minkowski spacetime. These findings offer a

robust foundation for further exploration of supergravity theories and their applications in high-energy physics.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

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Publications included in this thesis

The articles listed below were published during my candidature and are included in the thesis:

1. Gregory Gold, Jessica Hutomo, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *Components of curvature-squared invariants of minimal supergravity in five dimensions,* JHEP 07 (2024) 221 [2311.00679] [1].

2. Gregory Gold, Jessica Hutomo, **Saurish Khandelwal**, Mehmet Ozkan, Yi Pang, Gabriele Tartaglino-Mazzucchelli, *All Gauged Curvature Squared Supergravities in Five Dimensions*, *Phys. Rev. Lett.* **131** (2023) 25, 251603 [2309.07637] [2].

3. Jessica Hutomo, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, Jesse Woods, *Hyper-dilaton Weyl multiplets of 5D and 6D minimal conformal supergravity*, *Phys. Rev. D* 107 (2023) 4, 046009 [2209.05748] [3].

4. Gregory Gold, **Saurish Khandelwal**, William Kitchin, Gabriele Tartaglino-Mazzucchelli, *Hyper-dilaton Weyl multiplet of* 4*D,* $\mathcal{N} = 2$ *conformal supergravity*, JHEP 09 (2022) 016 [2203.12203] [4].

In preparation manuscripts included in this thesis

The article listed below is in preparation for publication and is included in the thesis:

1. Gregory Gold, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *On 4D,* $\mathcal{N} = 2$ *deformed vector multiplets and partial supersymmetry breaking in off-shell supergravity* [5].

Other publications during candidature

The articles listed below were submitted for publication during my candidature and are not included in the thesis:

1. Gregory Gold, Jessica Hutomo, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *Curvature-squared invariants of minimal five-dimensional supergravity from superspace*, *Phys. Rev. D* 107 (2023) 10, 106013 [2302.14295] [6].

2. Gregory Gold, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *Supergravity Component Reduction with Computer Algebra*, Submitted for referee publication in the MATRIX annals book series by Springer as a contribution to the proceedings of the MATRIX Workshop "New Deformations of Quantum Field and Gravity Theories," 21 Jan – 2 Feb 2024, [2406.19687] [7,8].

Contributions by others to the thesis

All contributions by others are given in each individual chapter.

Statement of parts of the thesis submitted to qualify for the award of another degree

No works submitted towards another degree have been included in this thesis.

Research involving human or animal subjects

No animal or human subjects were involved in this research.

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my supervisors, Dr. Gabriele Tartaglino-Mazzucchelli and Prof. Jorgen Rasmussen, for their unwavering guidance, support, and encouragement throughout my candidature. Their invaluable insights and expertise were instrumental in the successful completion of my thesis. I am especially thankful to Dr. Gabriele Tartaglino-Mazzucchelli for his tremendous efforts that facilitated my journey to Australia during the challenging times of COVID-19.

I also extend my gratitude to all my collaborators and colleagues, especially Greg, Liam, Christian, and Jessica, for the engaging discussions we have had over the last few years. Their intellectual contributions have greatly enriched my research experience.

I would like to take this opportunity to express my sincere appreciation towards Parita, for her patience and compassion throughout this academic journey. Her presence has been a constant source of stability for me. I am deeply grateful for her love and companionship.

I am grateful to all my friends in Brisbane: Harshit, Nandani, Greg, Willow, Liam, Christian, Siddharth, Cian, Sayantan, Nibedita, Ingita, Tavshabad, and Pratyush. The conversations we have had about our Ph.D. experiences and beyond were incredibly helpful and have made this journey much easier. Their friendship and support have been a source of strength and joy.

I am also indebted to many other individuals for their support over the years. Although it is not possible to mention everyone here, I want to thank all of them who have helped me grow.

Finally, I would like to thank my family, especially my parents, who have always unconditionally supported me and my decisions. Their strong belief in my abilities and constant encouragement has been the anchor in my journey.

Financial support

This research was supported by an Australian Government Research Training Program Scholarship along with a top-up from the Australian Research Council Future Fellowship FT180100353 awarded to Dr. Gabriele Tartaglino Mazzucchelli.

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Chapter 1

Introduction

The Standard Model (SM) has been remarkably successful in describing three of the four fundamental forces, the strong nuclear force, the weak nuclear force, and the electromagnetic force, and accurately accounting for the properties of elementary particles. However, it has several limitations that suggest the need for a more comprehensive theoretical framework. The SM fails to explain phenomena such as dark matter, the origin of neutrino masses, and the matter-antimatter asymmetry in the universe, all of which have been experimentally confirmed. Moreover, the SM cannot incorporate gravity, the fourth fundamental force described by Einstein's theory of general relativity, making it an incomplete description of the fundamental interactions. Additionally, it struggles to address the hierarchy problem, which refers to the large discrepancy between the theoretical Planck scale (approximately 10^{19} GeV) and the observed electroweak scale (around 100 GeV) values of the Higgs boson mass.

Symmetries play a fundamental role in physics. They are essential in describing particles and the interactions between them. The SM describes a universe invariant under the global Poincaré algebra, which are the symmetries of special relativity, and an internal gauge symmetry Lie group that accounts for the three fundamental forces: the strong nuclear force, the weak nuclear force, and the electromagnetic force. According to the Coleman-Mandula theorem [9], under some reasonable assumptions for a quantum field theory (QFT) with *massive fields*, the only possible Lie group symmetry is the direct product of the Poincaré group and an internal compact Lie group of Lorentz scalars. In other words, all generators of the Poincaré group will commute with the generators of internal symmetries, e.g., gauge symmetries in the SM. Supersymmetry introduces new symmetries by relaxing one assumption of the Coleman-Mandula theorem. It extends the Poincaré algebra by allowing for anticommutators in addition to commutators, resulting in a graded Lie algebra structure. The Haag, Lopuszanski-Sohnius (HLS) [10] theorem proves that the only graded Lie algebra consistent with QFT with massive fields is the so-called supersymmetry algebra.

Thus supersymmetry is a theoretical framework in particle physics that extends the SM by introducing a new type of symmetry between two fundamental classes of particles: bosons, which have integer spin, and fermions, which have half-integer spin. This symmetry was first proposed in the early 1970s by Golfand and Likhtman [11], followed by Volkov and Akulov [12, 13] in the context of non-linear realization of supersymmetry, and by Wess and Zumino [14, 15] as a symmetry of the worldsheet in the context of string theory. In its early days, it gained attention as a potential solution to the hierarchy problem [16, 17] and provided a potential dark matter candidate in the form of a supersymmetric particle [18, 19]. However, despite extensive experimental efforts, such as those at the Large Hadron Collider (LHC) and in dark matter detection experiments, no evidence of supersymmetric particles has been found so far. This lack of detection has pushed the energy scale at which supersymmetry might be realized to higher values, making it less accessible to current experiments. Despite this, supersymmetry remains an important theoretical framework, particularly due to its essential role in string theory, where it is necessary to ensure the mathematical consistency of the theory. Recently, it has been suggested that supersymmetry could be crucial for the AdS/CFT correspondence [20] that connects a QFT to a gravity theory. Additionally, supersymmetry offers a framework for unifying gravity with the other fundamental forces. When supersymmetry is gauged by promoting the global supersymmetry transformations to local transformations that depend on the spacetime coordinates, it leads to supergravity, a theory that integrates general relativity with supersymmetry [14, 15, 21]. Supergravity serves as a low-energy effective approximation of string theory. Its role in bridging string theory with low-energy physics makes it an indispensable tool in exploring the nature of quantum gravity and the unification of the fundamental interactions.

Beyond theoretical limitations, the SM also poses challenges in mathematical physics, particularly in establishing a rigorous framework for QFT and its extensions to quantum gravity. Conformal field theories (CFTs) have been instrumental in these efforts, in two as well as higher dimensions. The Coleman-Mandula theorem again dictates that in a QFT with only *massless fields*, the maximal possible spacetime symmetry is the conformal algebra, and its supersymmetric extension is the superconformal algebra, as given by the HLS theorem. This superconformal algebra is useful in systematically studying (matter-coupled) supergravity through superconformal methods like superconformal tensor calculus [22–24] and conformal superspaces [25–33]. See also the reviews on superconformal tensor calculus [34–38] and conformal superspace [39, 40]. In [29, 32, 41], the connection between the two approaches has been discussed in detail.

Historically, supergravity theories were constructed by writing a super-Poincaré invariant action composed of fields that make up on-shell irreducible representations of the supersymmetry algebra [21]. This procedure, however, leads to the realization of supersymmetry only on-shell, that is the symmetry algebra closes by using equations of motion. Additionally, this approach requires one to make an educated guess for an action invariant under super-Poincaré transformations, making it cumbersome and unsystematic, especially for matter-coupled supergravity theories.

The two methods, superconformal tensor calculus and conformal superspaces provide a more systematic approach to studying supergravity. In these methods, theories with super-Poincaré symmetry are derived as theories with broken superconformal symmetry. As a result of a larger superconformal symmetry group, various aspects of the theory gain a clearer picture. For example, they lead to off-shell approaches to supergravity, where auxiliary fields are introduced to obtain off-shell representations of the supersymmetry algebra with model-independent transformation rules. This is in contrast with the traditional on-shell approach. These techniques are discussed in detail in Chapter 2.

In this thesis, we focus on theories with 8 supersymmetry generators, specifically $4D$ $N=2$, $5D$ $N=1$, and $6D$ $N=(1,0)$. Here, N refers to the number of copies of supersymmetry. For example, in $4D$, each spinor has 4 real degrees of freedom, so $4D$ $N=2$ has two copies of the spinor generator, resulting in 8 real supersymmetry generators. These theories hold significant importance in theoretical physics for several reasons. Firstly, they have the highest number of supersymmetry generators that still allow for interesting geometric structures. Unlike theories with higher (≥ 16) supersymmetry generators, where geometry gets completely determined by the number of matter multiplets, these theories include functions of matter multiplets that allow for the deformation of the manifold's metric, leading to 'special Kähler manifolds' geometry [42]. Although theories with lesser supersymmetries (≤ 4) allow for more interesting geometries to appear, such as 'general Kähler manifolds', those with 8 supersymmetries impose just enough constraints on interactions to allow for exact solutions in certain cases. For instance, 8 supercharges were necessary to find a solution to the Seiberg-Witten model [43, 44]. Secondly, these theories are excellent candidates for exploring string theory, black hole physics, and dualities, like the AdS/CFT correspondence.

Since its discovery, the AdS/CFT correspondence has been at the forefront of theoretical physics. Supergravity, the leading-order (two-derivative) low-energy effective field theory description of string theory, has played a crucial role in improving our understanding of this correspondence [45–49]. However, string theory predicts an infinite series of higher-derivative corrections to supergravity, making it important for holography to consider higher-derivative corrections to assess the validity of the correspondence. In particular, four-derivative curvature-squared invariant terms are needed for the next-to-leading order precision test of the correspondence. These terms also play a crucial role in matching the microscopic and macroscopic descriptions of the black hole entropy [50–52]. The microscopic entropy is related to quantum gravity theories and involves counting the quantum states of the system, whereas the macroscopic entropy is derived from general relativity and is related to the area of the black hole event horizon. The black hole entropy calculations can be extended by Wald's entropy formalism beyond Einstein's theory to a wide range of diffeomorphism-invariant theories, including those with higher-derivative terms [53, 54]. By utilizing the on-shell action for general gravity theories, this method allows us to systematically analyze entropy with higher-curvature corrections and general gravitational interactions, providing a robust foundation for exploring black hole thermodynamics in generalized gravity theories.

Historically, dealing with these higher-derivative terms was very challenging as each additional derivative rapidly increases the number of terms in the action and traditional on-shell methods become inefficient. With the off-shell techniques, technical challenges have been addressed and since then significant progress has been made in the direction of higher derivative supergravity [28, 29, 31, 32, 51, 55–86]. Despite this progress, the computational challenges arising from the large number of terms in the action remain inadequately addressed. Consequently, even the classification of four-derivative correction terms in supergravity, which are only the next-to-leading terms, remains incomplete. However, recent advancements in the off-shell techniques, which allow supersymmetry to be studied in a model-independent way, can be implemented algorithmically with the help of computer algebra programs. In 2007, Kasper Peeters developed a symbolic computer algebra program called *Cadabra* that is suited for performing field theoretical calculations [87, 88]. We use Cadabra to address a long-standing problem in 5*D* gauged supergravity, specifically computing all three four-derivative curvature-squared invariant terms. Besides developing the algorithm to implement the off-shell techniques, we further optimize several core algorithms along the way. All these algorithms can be found on the GitHub repository [8]. This helps us compute all three curvature-squared invariant terms for 5*D* minimal gauged supergravity [1, 6] in a *standard Weyl multiplet background*, an off-shell conformal supergravity multiplet. Here, gauged supergravity refers to supergravity theories where Anti-de Sitter (AdS) solutions exist, as opposed to the ungauged supergravity that can only accommodate asymptotically flat solutions.

The conformal supergravity multiplets form an off-shell representation of the local superconformal algebra and are essential for constructing supergravity theory. They include the vielbein and its supersymmetric partners as their field content. It has been known since 1986 that variants of the Weyl multiplets, other than the standard Weyl multiplet, exist. This was shown by Bergshoeff et al. in [89], when they constructed a tensor-dilaton Weyl multiplet for minimal 6*D* supergravity. This was achieved by reinterpreting the on-shell equations of motion of the tensor multiplet as algebraic equations, which allowed some auxiliary fields of the standard Weyl multiplet to be replaced by fields of the tensor multiplet. Since then, progress in this direction has been slow. It took decades to construct the vector-dilaton Weyl multiplet for minimal 5*D* [90] and 4*D* $N = 2$ [91, 92] conformal supergravity. These variants of dilaton Weyl multiplet made it possible to construct all curvature-squared invariants for 5*D* and 6*D* ungauged supergravity [28, 29, 31, 32, 51, 55–57, 59–72]. We extend this by uplifting the three curvature-squared invariants that we construct in [1, 6] for minimal 5*D* gauged supergravity to the vector-dilaton Weyl multiplet background in [2]. As the gauged aspect leads to a supersymmetric Anti-de-Sitter (AdS) solution on the gravity side, we perform a next-to-leading order precision test of the AdS/CFT correspondence by computing the Weyl anomaly coefficient of the dual CFT [2].

In the ungauged limit, the on-shell theory only receives corrections from one out of the three invariant terms, generalizing the non-renormalization theorem of 4*D* $N = 2$ ungauged supergravity theory [76] for minimal 5*D* ungauged supergravity. According to this theorem, certain classes of actions and their first derivatives with respect to fields or coupling constants must vanish in a fully supersymmetric background, implying they will not contribute to BPS black hole entropy or to the field equations in supersymmetric configurations. In other words, physics remains independent of these actions.

We extend the algorithms for use in 4*D* $N = 2$ supergravity, and they have also been made available on the GitHub repository [8]. These computer algebra approaches are highly effective and can be used to investigate various unexplored territories in higher derivative supergravity theories, such as to classify matter-coupled supergravity theories up to four-derivative curvature-squared invariant terms and beyond.

As these variants of the Weyl multiplet present a significant opportunity to construct new conformal supergravity theories and their applications in exploring higher-derivative correction terms, this makes it important to look for new multiplets of conformal supergravity. In [4], we develop a new hyper-dilaton Weyl multiplet for $4D$ $N = 2$ conformal supergravity and further extend this construction to minimal $5D$ and $6D$ conformal supergravity [3]. This was obtained by coupling an on-shell hypermultiplet to a standard Weyl multiplet and then reinterpreting the on-shell equations of motion as algebraic equations to replace auxiliary fields of the standard Weyl multiplet with fields from the hypermultiplet. These variant conformal supergravity multiplets allow us to explore new off-shell supergravity models and open up avenues for research in supergravity.

Since supersymmetry has not yet been observed, it is crucial to study the conditions under which supersymmetry can be spontaneously broken and examined in a model-independent manner. One key condition for supersymmetry breaking is that the theory must have a fermionic field that transforms nonlinearly under supersymmetry, and these transformations cannot come from the background values of auxiliary fields [93–95]. In string theory, compactifications naturally give rise to scenarios where supersymmetry is partially broken, meaning that only part of the initial supersymmetry is broken while others remain intact [12, 13, 17, 96–110]. This motivates the investigation of partial supersymmetry breaking models in supergravity, which has remained an elusive subject for decades, with theorems and counter theorems produced on this subject [93, 101, 111].

Despite its significance, partial supersymmetry breaking in supergravity has been engineered only through on-shell models [112–115]. As mentioned earlier, it is often desirable to study supergravity theory in an off-shell setting. One reason for the existence of only on-shell partial supersymmetry breaking models can be traced to the choice of the conformal gravity multiplet (Weyl multiplet) used so far. Previously, only the standard Weyl multiplet was employed, which requires either central charges or multiplets with an infinite number of auxiliary fields to potentially lead to off-shell partial supersymmetry breaking [29, 116–133]. These requirements are overcome with the new hyper-dilaton Weyl multiplet of conformal supergravity [4], enabling us to construct off-shell models of partial supersymmetry breaking in a systematic and simple manner.

The rest of the thesis is organized as follows:

Chapter 2 introduces the essential background theory and mathematical techniques necessary for this thesis. We begin by exploring supersymmetry and its local counterpart, supergravity. Following this, we will delve into supersymmetric multiplets and superfields in superspace, which provide equivalent representations of the underlying supersymmetry algebra. We will then proceed to discuss superconformal methods, which offer off-shell techniques to study supergravity theories and are central to this thesis.

In Chapter 3, we explore the higher-derivative correction to supergravity. We start by reviewing the definition of minimal off-shell two-derivative gauged supergravity in five dimensions. We further present the three independent four-derivative curvature-squared invariant terms for gauged supergravity in a standard Weyl multiplet background. We then present the derivation of the superconformal primary equations of motion of all the curvature-squared terms for the minimal $5D$ gauged off-shell supergravity based on the standard Weyl multiplet. Finally, we analyze all the descendants of these equations of motion.

In Chapter 4, we analyze the on-shell $5D$ minimal gauged supergravity theory deformed by the three curvature-squared invariant terms found in Chapter 3. This analysis is conducted in the vector-dilaton Weyl multiplet background. Contrary to expectations, we will see that the on-shell theory gets deformed by all three invariants. Given that the on-shell theory has an AdS_5 solution, we use these results to compute the Weyl anomaly coefficient of the dual CFT [2] and verify it against existing literature [134, 135].

In Chapter 5, we discuss the dilaton Weyl multiplet of conformal gravity and construct a new multiplet of $4D$ $N = 2$ conformal supergravity, the hyper-dilaton Weyl multiplet. These multiplets allow us to construct $4D$ $N = 2$ Poincaré supergravity by coupling the new hyper-dilaton Weyl multiplet to an off-shell compensating vector multiplet. Using this hyper-dilaton Weyl multiplet, we explore partial supersymmetry breaking in extended supergravity theory. Chapter 6 extends the construction of the hyper-dilaton Weyl multiplet to other conformal supergravity theories with 8 supercharges, specifically $5D$ $N = 1$ and $6D$ $N = (1,0)$. In the five-dimensional case, we also construct a minimal $5D$ Poincaré supergravity by coupling the new hyper-dilaton Weyl multiplet to an off-shell compensating vector multiplet.

Using these new conformal supergravity multiplets, in Chapter 7, we engineer an off-shell model that exhibits partial supersymmetry breaking from $N = 2$ to $N = 1$ in $4D$ supergravity with a zero cosmological constant. This is based on the hyper-dilaton Weyl multiplet introduced in Chapter 5, coupled to the SU(1,1)/U(1) special-Kähler sigma model in a symplectic frame admitting a holomorphic prepotential, with one compensating and one physical vector multiplet, the latter magnetically deformed. The chapter concludes by analyzing the on-shell theory. As a result of this breaking, some fermions in the theory gain mass while others remain massless, indicating that only part of the supersymmetry is broken.

In Chapter 8, we discuss the outcome of our work and highlight how they can be used in the study of higher-derivative supergravity, gauge/gravity duality, and partial supersymmetry breaking. We further comment on how this work can be generalized for various matter-coupled supergravity.

Chapter 2

Background theory

In this chapter, we will introduce the necessary background theory pertinent to this thesis, including the two superconformal methods for studying supergravity theory: the conformal superspace and the superconformal tensor calculus. Readers already familiar with these approaches can skip this chapter and proceed directly to Chapter 3. To avoid redundancy amongst the papers incorporated as Chapters 3, 4, 5, 6, and 7 in this thesis, some background content from those papers [1–4, 6] has been included in this chapter, with the necessary modifications made to those papers.

2.1 Supersymmetry & Supergravity

The standard model of particle physics describes a universe invariant under the global Poincaré algebra coupled to an internal Lie group. Supersymmetry is a theoretical framework that extends the standard model by introducing a new type of fermionic symmetry between bosons and fermions. This adds to the Poincaré algebra a set of new spinor generators, Q_r. These generators, known as supercharges, transform bosonic fields into fermionic fields and vice versa, thus uniting them and viewing them as a single object: a supersymmetric multiplet.

The anti-commutator of two supercharges generates a translation (P_a) in spacetime, as expressed in the following relation:

$$\{Q_r, Q_s\} \propto (\sigma^a)_{rs} P_a \ . \tag{2.1.1}$$

Here σ^a are the matrices that generate the Clifford algebra $C(D-1,1)$, where D is the dimension of spacetime. The precise representation of the supercharge Q_r depends on the dimension D of the spacetime, as the spinor representation changes with the dimension. Chapter 3 of [37] provides a detailed summary of spinor representations in different dimensions. For example, in 4 dimensions, the supercharges are represented by Majorana spinors $(Q_\alpha, \bar{Q}_{\dot{\alpha}})$, and when combined with the Poincaré algebra, it forms a closed supersymmetry algebra:

$$
\begin{aligned}
&[M^{ab},P^c] = 2\eta^{c[a}P^{b]} \ , \qquad [M^{ab},M^{cd}] = 2\eta^{c[a}M^{b]d} - 2\eta^{d[a}M^{b]c}, \\
&\{Q_\alpha,\bar{Q}_{\dot\beta}\} = -2i(\sigma^a)_{\alpha\dot\beta}P_a, \\
&[M^{ab},Q_\alpha] = (\sigma^{ab})_\alpha{}^\beta Q_\beta, \qquad [M^{ab},\bar{Q}_{\dot\alpha}] = (\tilde\sigma^{ab})_{\dot\alpha\dot\beta}\bar{Q}^{\dot\beta},
\end{aligned} \tag{2.1.2}
$$

where all other commutators vanish and the spinor indices $\alpha,\dot\alpha = 1,2$. Therefore, we have four supersymmetry generators $Q_\alpha, Q_{\dot\alpha}$, which form a single spinor representation in $4D$. This is the $4D$, $N = 1$ supersymmetry algebra. In principle, there can be multiple copies of supersymmetry generators; for instance, N-extended $4D$ supersymmetry will have $4N$ supersymmetry generators.

Supersymmetry, like most other symmetries in physics, can be global or local. Under a global symmetry, the transformation remains the same across all points in spacetime, whereas a local symmetry transformation can vary from point to point. Extending a global symmetry to a local one is known as “gauging,” where a gauge field (connection) is introduced for each symmetry, propagating the information of these transformations from one spacetime point to another. This local symmetry is called gauged symmetry.

In Einstein’s theory of general relativity, we have a gauge field associated with each gauged symmetry generator of the Poincaré algebra: the vielbein $e_m{}^a$ for translations and the spin connection $\omega_m{}^{ab}$ for Lorentz transformations. The index m denotes the curved space index, while a is the local Lorentz flat spacetime index. A tensor Ψ is said to be covariant if it transforms under local Lorentz transformations without any derivative acting on the transformation parameter $\lambda^{ab}(x)$, according to:

$$\delta\Psi = \lambda^{ab}(x)M_{ab}\Psi. \tag{2.1.3}$$

Now, consider taking a derivative of this covariant tensor Ψ. Since $\lambda^{ab}(x)$ is a local parameter, it’s clear that $\partial_a\Psi$ does not transform covariantly under local Lorentz transformations. To ensure that derivatives of covariant tensors also transform covariantly, we must introduce a covariant derivative. The covariant derivative is defined as:

$$D_a = e_a{}^m D_m = e_a{}^m\left(\partial_m - \frac{1}{2}\omega_m{}^{cd}M_{cd}\right) . \tag{2.1.4}$$

With this definition, one can show that the tensor $\nabla_a\Psi$ transforms covariantly, without any derivatives acting on the local Lorentz transformation parameter $\lambda^{ab}(x)$. The covariant derivative ensures that tensors are parallelly transported along curves in spacetime, preserving their transformation properties under both general coordinate transformations and local Lorentz transformations. The covariant derivatives satisfy the algebra

$$[D_a,D_b] \ = \ -R(P)_{ab}{}^c D_c - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} \ . \tag{2.1.5}$$

where $R(P)^{ab}{}_c$ and $R(M)^{ab}{}_{cd}$ are the curvature tensors associated with the translation and Lorentz generators of the Poincaré algebra. In principle, the spin connection can be treated as an independent gauge field, however, it gets integrated out using its field equations, or equivalently, it can be fixed from the start by imposing the torsion-free constraint:

$$R(P)_{ab}{}^{c} = 0. \tag{2.1.6}$$

Such a constraint is called a 'conventional' constraint, and it can be used to express the torsion-free spin connection as a composite field in terms of the vielbein, as follows:

$$\omega(e)_{abc} = -\frac{1}{2}(\mathcal{C}_{abc} + \mathcal{C}_{cab} - \mathcal{C}_{bca}) \tag{2.1.7}$$

where $\mathcal{C}_{ab}{}^{c} = e_a{}^m e_b{}^n \mathcal{C}_{mn}{}^{c}$ are coefficients given by $\mathcal{C}_{mn}{}^{c} := 2\,\partial_{[m} e_{n]}{}^{c}$. Thus, in general relativity, only the vielbein is an independent field.

When supersymmetry is gauged, the resulting theory naturally contains gauge fields for supersymmetry transformations, called the gravitini, and for spacetime translations, the vielbein, due to the underlying supersymmetry algebra (2.1.1). This leads to an extension of general relativity, known as supergravity, where bosonic fields like the vielbein are accompanied by a fermionic superpartner, the gravitini. The vielbein and its superpartners form a multiplet, a representation of the supersymmetry algebra.

In supersymmetric theories, elementary particles are generally organized into these multiplets, which contain bosonic and fermionic particles as superpartners. This unique structure allows for the cancellation of quadratic divergences, addresses the hierarchy problem, and provides a potential dark matter candidate in the form of the supersymmetric particle. We will explore the details of these supersymmetric multiplets in the next subsection.

2.1.1 Supersymmetric multiplets

A supersymmetric multiplet is a collection of fields that transform into one another under supersymmetry transformations. The following operator defines these transformations:

$$\delta_\xi = \frac{1}{2}\xi Q + \frac{1}{2}\bar{\xi}\bar{Q}\,, \tag{2.1.8}$$

where $\xi, \bar{\xi}$ is the supersymmetry transformation parameter. As a consequence of the underlying supersymmetry algebra (2.1.1), these local supersymmetry transformations close as follows :

$$[\delta_\xi, \delta_\eta] = (\xi\sigma^a\bar{\eta} - \eta\sigma^a\bar{\xi})P_a\,. \tag{2.1.9}$$

The supersymmetric multiplets form representations of the supersymmetry algebra and can be classified as either on-shell or off-shell. On-shell multiplets contain the minimal number of physical fields required to represent a given supersymmetry representation, with the fields satisfying their respective equations of motion. Off-shell multiplets, on the other hand, include additional auxiliary fields that do not have dynamics of their own but serve to close the supersymmetry algebra without imposing the equations of motion.

To understand the distinction between on-shell and off-shell multiplets, let's consider the example of the $4D$ $N=1$ chiral multiplet. This multiplet consists of a complex scalar field ϕ, and a Weyl fermion ψ in the on-shell representation. The supersymmetry transformations for this multiplet are given by:

$$\delta_\xi \phi = \sqrt{2}\, \xi \psi \ , \quad \delta_\xi \psi = \sqrt{2}\, i\sigma^a \bar{\xi} \partial_a \phi \ . \tag{2.1.10}$$

In the on-shell formulation, the fields ϕ and ψ must satisfy the following equations of motion:

$$\partial^a \partial_a \phi = 0 \ , \quad i\bar{\sigma}^a \partial_a \psi = 0 \ . \tag{2.1.11}$$

These on-shell equations of motion are necessary to close the supersymmetry algebra, as expressed in 2.1.9. As all the fields involved are physical, the on-shell representation provides a more concise description of the supersymmetry algebra, but the construction of supersymmetric actions is more involved, as one must ensure the closure of the algebra on-shell.

On the other hand, the off-shell chiral multiplet includes an additional complex auxiliary field F, which does not have its own dynamics but serves to close the supersymmetry algebra without imposing the equations of motion on the physical fields ϕ and ψ. The supersymmetry transformations for the off-shell chiral multiplet are:

$$\delta_\xi \phi = \sqrt{2}\, \xi \psi \ , \quad \delta_\xi \psi = \sqrt{2}\, i\sigma^a \bar{\xi} \partial_a \phi + \sqrt{2}\, \xi F \ , \quad \delta_\xi F = \sqrt{2}\, i\bar{\xi} \bar{\sigma}^a \partial_a \psi \ . \tag{2.1.12}$$

The advantage of off-shell multiplets is that they provide a more convenient framework for constructing supersymmetric Lagrangians and simplify calculations, as the auxiliary fields allow the supersymmetry transformations to close without restricting the fields to satisfy their equations of motion. This off-shell formulation is useful in the context of supergravity, where the auxiliary fields play a crucial role in the supersymmetric completion of the usual Einstein-Hilbert scalar curvature term (general relativity) and higher derivative curvature square correction terms. This off-shell approach to supergravity will be used throughout the thesis.

Finally, we distinguish between the multiplet containing the graviton and its supersymmetric partner and those containing matter fields. The multiplet that includes the graviton is called the supergravity multiplet, while the others with matter fields are called matter multiplet.

2.1.2 Superspace and superfields

Having a formulation where symmetries are manifestly realized is advantageous as it ensures that the symmetry is preserved at all stages. To construct a theory where supersymmetry is manifest, one needs to extend the Minkowski spacetime by including fermionic (anticommuting) coordinates, thereby creating a new mathematical space called superspace. This is analogous to electrodynamics where Lorentz invariance is made manifest by extending the 3D Euclidean space to include the time direction, creating the $4D$ Minkowski spacetime. This superspace approach with its manifest supersymmetry, was first introduced by Salam and Strathdee [136].

Superspace is a generalization of the ordinary spacetime by incorporating fermionic degrees of freedom. The idea behind superspace is to expand the standard bosonic spacetime $\mathbb{R}^D$ by introducing one additional fermionic coordinate for each supersymmetry generator, creating a superspace $\mathbb{R}^{D|\mathcal{T}}$. Here, D denotes the number of bosonic dimensions and $\mathcal{T}$ denotes the number of fermionic dimensions. As the structure of spinor representations depends on the dimension of spacetime, $\mathcal{T}$ depends on the bosonic dimension D of the spacetime and the number of copies of supersymmetry.

Superspace is parametrized by local bosonic coordinates x^m and fermionic (Grassmann) coordinates θ^i, collectively denoted as $z^M = (x^m, \theta^i)$, where $m = 0, 1, \ldots, D-1$ and $i = 1, 2, \ldots, \mathcal{T}$. The fermionic coordinates θ^i satisfy the anti-commutation relation $\{\theta^i, \theta^j\} = 0$ due to their Grassmann-odd nature.

In the superspace formulation, the representations of the supersymmetry algebra can be compactly expressed using special objects called superfields. A superfield $\Phi(z)$ is an operator-valued function defined on this superspace. Since θ is an anti-commuting Grassmann coordinate that satisfies $\theta^2 = 0$, any superfield $\Phi(z)$ can be expanded in a finite power series in θ_i:

$$\Phi(z) = \phi(x) + \prod_{k=1}^{N} \theta^{i_1} \ldots \theta^{i_k} \phi_{i_1 \ldots i_k}(x) \tag{2.1.13}$$

where $\phi(x)$, $\phi_i(x)$, $\phi_{ij}(x)$, and so on are the local component fields over Minkowski space that make up a supersymmetric multiplet. Thus a superfield is a short way to denote a finite multiplet of fields.

The connection between a superfield and a multiplet can be made by applying the supersymmetry generators Q_i to the superfield and then taking the $\theta = 0$ projection. This allows one to extract the component fields of the supermultiplet from the superfield. Specifically, the component fields are given by:

$$\phi = \Phi|_{\theta=0} \ , \quad \phi_i = Q_i \Phi|_{\theta=0} \ , \quad \phi_{ij} = Q_i Q_j \Phi|_{\theta=0} \ . \tag{2.1.14}$$

Thus, the superfield and the multiplet are two equivalent representations of the supersymmetry algebra, and the component fields of the multiplet can be obtained by applying the appropriate number of supersymmetry derivatives to the superfield. This connection between superfields and component fields forms the basis for establishing a connection between the two superconformal methods described in the next section, where the superfields from the conformal superspace get mapped to the component fields of the superconformal tensor calculus via projection.

2.2 Superconformal methods

Superconformal methods are powerful tools for studying Poincaré and matter-coupled Poincaré supergravity, where the gravitational sector is coupled to matter fields. Although we aim to formulate theories that respect the Poincaré symmetry, which are the symmetries of our observable universe, we begin with a larger symmetry group, the superconformal group. While this approach might seem like a detour from the desired Poincaré symmetry, this larger symmetry provides better control over the system and provides a deeper understanding of the final Poincaré supergravity theory.

The superconformal method starts by formulating gauge theories of the superconformal algebra. These theories include extra fields that are important in the formulation but are later eliminated to obtain the gauge theories of the Poincaré supersymmetry subalgebra. Some of these fields are eliminated using curvature constraints, similar to the constraints in ordinary general relativity, which express the Lorentz connection in terms of the vielbein. Other unwanted fields, known as compensating fields, are eliminated through gauge fixing, a process that removes the extra symmetries. The resulting theories are the desired matter-coupled supergravity theories with local Poincaré supersymmetry, while the other symmetries are removed and no longer visible in the final result.

A good starting point to understand superconformal methods is the gauge equivalence principle. To illustrate this concept, we consider an example where Poincaré gravity is constructed from a conformal gravity theory invariant under a larger conformal symmetry group.

2.2.1 Gauge Equivalence Principle

Conformal gravity is a theory with the conformal group as its local symmetry group. The symmetries of the conformal group consist of translation, Lorentz, dilatation, and special conformal, generated by P_a, M_{ab}, $\mathbb{D}$, and K_a respectively. The gauge fields for the conformal algebra are $e_m{}^a$, $\omega_m{}^{ab}$, b_m, and $\mathfrak{f}_m{}^a$. The following table lists the generators, the corresponding gauge fields, and the gauge parameters.

Generator	P_a	M_{ab}	$\mathbb{D}$	K_a
Gauge Fields	$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$
Parameters	ξ^a	λ^{ab}	$\lambda_{\mathbb{D}}$	λ_K^a

Table 2.1: Generators, their corresponding gauge fields, and parameters of the gauged conformal algebra.

The local conformal transformation except for local translation (covariant general coordinate transformations) is defined by the following operator

$$\delta = \frac{1}{2}\lambda^{ab}M_{ab} + \lambda_{\mathbb{D}}\mathbb{D} + \lambda_K^a K_a \ . \tag{2.2.1}$$

The local conformal transformation of the gauge connection fields is given by

$$\begin{aligned}
\delta e_m{}^a &= -\lambda_{\mathbb{D}} e_m{}^a + \lambda^a{}_b e_m{}^b \ , && \text{(2.2.2a)}\\
\delta b_m &= \partial_m \lambda_{\mathbb{D}} - 2\lambda_{mK} \ , && \text{(2.2.2b)}\\
\delta \omega_m{}^{ab} &= \partial_m \lambda^{ab} + 2\omega_{mc}{}^{[a}\lambda^{b]c} - 4\lambda_K^{[a} e_m{}^{b]} \ , && \text{(2.2.2c)}\\
\delta \mathfrak{f}_m{}^a &= \partial_m \lambda_K^a - b_m \lambda_K^a + \omega_m{}^{ab}\lambda_{bK} - \lambda^{ab}\mathfrak{f}_{mb} + \lambda_{\mathbb{D}}\mathfrak{f}_m{}^a \ . && \text{(2.2.2d)}
\end{aligned}$$

However, similar to the case of general relativity, not all the gauge fields here describe new physical degrees of freedom. Specifically, the spin connection and the special conformal connection can be determined by imposing the following conventional constraints on the curvature.

$$R(P)_{ab}{}^c = 0 \ , \qquad R(M)_{ab}{}^{cb} = 0 \ . \tag{2.2.3}$$

These constraints can be algebraically solved to give:

$$\omega_{abc} = \omega(e)_{abc} - 2\eta_{a[b}b_{c]} \ , \tag{2.2.4a}$$

$$\mathfrak{f}_a{}^b = -\frac{1}{2(D-2)}R(\omega)_{ac}{}^{bc} + \frac{1}{4(D-2)(D-1)}\delta_a{}^b R(\omega)_{cd}{}^{cd} \ , \tag{2.2.4b}$$

where $\omega(e)_{abc}$ is the torsion-free spin connection given by (2.1.7) and the curvature is given by:

$$R_{ab}{}^{cd}(\omega) \ = \ e_a{}^m e_b{}^n \Big(2\partial_{[m}\omega_{n]}{}^{cd} - 2\omega_{[m}{}^{ce}\omega_{n]e}{}^d\Big) \ . \tag{2.2.5a}$$

Using a scalar field ϕ that transforms under local conformal transformation as,

$$\delta\phi \ = \ \lambda_{\mathbb{D}}\phi \ , \tag{2.2.6}$$

one can write down the following action which is invariant under local conformal transformation

$$S = \int d^4x\, e\, \phi \nabla^a \nabla_a \phi \ . \tag{2.2.7}$$

Where we have introduced the conformal covariant derivative (∇_a)

$$\nabla_a = e_a{}^m \partial_m - \frac{1}{2}\omega_a{}^{bc}M_{bc} - b_a\mathbb{D} - \mathfrak{f}_a{}^b K_b \ , \tag{2.2.8}$$

to ensure that the action transforms covariantly under conformal transformations. We choose the gauge $\phi = 1$ and $b_m = 0$ to fix dilatation and special conformal symmetry. To preserve these gauge conditions, the transformations (2.2.6) and (2.2.2b) imply that $\lambda_{\mathbb{D}} \equiv 0$ and $\lambda_m^K \equiv 0$, thus breaking the dilatation and special conformal symmetry. As the matter field ϕ is required to compensate for the dilatation symmetry, it is known as the compensator field. The gauge fixed action becomes

$$S = \int d^4x\, e\, \phi \nabla^a \nabla_a \phi \xrightarrow[b_m = 0]{\text{Gauge fix } \phi=1} S \propto \int d^4x\, e R \ , \tag{2.2.9}$$

which is the usual Einstein-Hilbert action for gravity and is invariant under the leftover local translations and Lorentz transformation. Thus we have constructed a Poincaré gravity theory as a broken conformal gravity theory.

We are now ready to generalize the gauge equivalence principle to construct the action for supergravity. We start with superconformal gravity, which by definition is a theory with the superconformal group as its local symmetry group. The symmetries of the superconformal group consist of translation, Lorentz, Supersymmetry, dilatation, R-symmetry, S-supersymmetry, and special conformal, generated by P_a, M_{ab}, Q, $\mathbb{D}$, J_R, S, and K_a respectively. Among these, R-symmetry is an internal symmetry that commutes with Lorentz transformations and translations but not with the supersymmetry generators, and it organizes the supersymmetry generators into representations of the symmetry group. S-supersymmetry, which can be thought of as the supersymmetric counterpart of special conformal transformations, complements the ordinary supersymmetry transformations. Both these symmetries are required to close the superconformal algebra. Through gauging, we can promote these symmetries to become local. A direct approach that has been used since the 1990s is to gauge these symmetries in ordinary spacetime, leading to the Superconformal tensor calculus methods, also known as the component approach to supergravity.

A newer approach is to gauge the superconformal symmetries in superspace, thereby creating a conformal superspace formulation. This conformal superspace method is particularly advantageous because it manifestly preserves superconformal covariance. However, it has some limitations, such as it lacks a prepotential formulation for the supercurrent multiplet in certain supergravity theories. Therefore, it is essential to have both the superconformal tensor calculus and the conformal superspace techniques available.

In the following subsections, we provide a broad overview of both the methods for general supergravity theories followed by some specific examples and demonstrate their equivalence.

2.2.2 Superconformal tensor calculus

As previously mentioned, the superconformal method begins by formulating gauge theories of the superconformal algebra. In the superconformal tensor calculus approach to supergravity, we construct superconformal gravity by gauging the superconformal algebra in ordinary spacetime. The following table lists the generators of the superconformal algebra, along with their corresponding gauge fields and parameters:

Generators	P_a	M_{ab}	$\mathbb{D}$	K_a	Q	S	J_R
Gauge Fields	$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	ψ_m	ϕ_m	$\phi_m{}^R$
Parameters	ξ^a	λ^{ab}	$\lambda_{\mathbb{D}}$	Λ_K^a	ξ	η	λ_R

Table 2.2: Generators, their corresponding gauge fields, and parameters of the gauged superconformal algebra.

Here, P_a, M_{ab}, $\mathbb{D}$, and K_a are the generators of the conformal group, while Q and S are the supersymmetry generators. The inclusion of the S supersymmetry generator is necessary to close the algebra. J_R is the R-symmetry generator. The precise representation of Q, S, and J_R generator depends on the dimension of the spacetime. The collection of gauge fields associated with gauging the superconformal algebra includes the vielbein and its supersymmetric partner, thus forming a supersymmetric multiplet of superconformal gravity. The gauge connections then describe the covariant superconformal derivative:

$$\nabla_a = e_a{}^m\nabla_m = \quad e_a{}^m\Big(\partial_m - \tfrac{1}{2}\psi_m Q - \tfrac{1}{2}\omega_m{}^{cd}M_{cd} - \phi_m{}^R J_R - b_m\mathbb{D} - \tfrac{1}{2}\phi_m S - \mathfrak{f}_m{}^c K_c\Big)\ . \quad (2.2.10)$$

Just like general relativity not all of these gauge fields describe new physical degrees of freedom. The spin connection, the $S-$ supersymmetry connections, and the special conformal connection are all composite connections and can be algebraically determined by imposing the conventional constraints on the curvature.

It is easy to see that this gauge connection multiplet has an unequal number of bosonic and fermionic degrees of freedom, which is not allowed in a supersymmetric theory. We will demonstrate an explicit mismatch in the counting of degrees of freedom when considering some particular examples in the coming sections. One way to resolve this mismatch is by imposing equations of motion to eliminate some of the gauge freedoms, leading to an on-shell gauging of the superconformal group. However, the obvious disadvantage of this type of gauging is that the action becomes constrained, as it must reproduce these equations of motion. Another way to tackle this mismatch is by introducing auxiliary degrees of freedom, which balance out the mismatch and allow the algebra of the transformations to close off-shell. These new fields are called auxiliary because they do not have a kinetic term contribution to the action and, therefore, are non-propagating. This approach results in an off-shell gauging of the superconformal group, and its representation forms an off-shell conformal supergravity multiplet, known as the Weyl multiplet. One particular such representation is known as the standard Weyl mulitplet.

Finally, before we move on to specific superconformal gravity theories, let us discuss how Poincaré supergravity can be constructed as a broken superconformal gravity theory. Recall that to build Poincaré gravity, we coupled the conformal gravity theory with a compensator field to gauge fix the dilatation symmetry. Similarly, to construct Poincaré supergravity, we couple the standard Weyl multiplet to compensating matter multiplets. Assuming these multiplets have enough degrees of freedom, they are used to gauge fix the dilatation, $S-$supersymmetry, and $R-$symmetries. The special conformal symmetry is always fixed by setting the dilatation gauge field $b_m = 0$. This leads to an off-shell Poincaré supergravity theory which after the elimination of auxiliary fields results in on-shell Poincaré supergravity.

We will now explore specific examples of gauging the superconformal algebra in spacetime, focusing on $4D$ $N = 2$, $5D$ $N = 1$, and $6D$ $N = (1,0)$ supergravity theories that have eight real supercharges.

$4D$ $N = 2$ **standard Weyl multiplet**

The standard Weyl multiplet of $4D$ $N = 2$ conformal supergravity is associated with the local off-shell gauging in space-time of the superconformal group $\mathrm{SU}(2,2|2)$ [137], see also [138–141] and [37, 38] for reviews. Our notations and conventions follow those in [36] and are summarized in appendix A. The following table lists the generators of the superconformal group $\mathrm{SU}(2,2|2)$, their corresponding gauge fields, and parameters.

Generators	P_a	M_{ab}	$\mathbb{D}$	K_a	$Q^i_\alpha, \bar{Q}^{\dot\alpha}_i$	$S^i_\alpha, \bar{S}^{\dot\alpha}_i$	Y	J_{ij}
Gauge Fields	$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\psi_m{}^\alpha_i, \bar\psi_m{}^i_{\dot\alpha}$	$\phi_m{}^\alpha_i, \bar\phi_m{}^i_{\dot\alpha}$	A_m	$\phi_m{}^{ij}$
Parameters	ξ^a	λ^{ab}	$\lambda_{\mathbb{D}}$	λ^a_K	$\xi^\alpha_i, \bar\xi^i_{\dot\alpha}$	$\eta^\alpha_i, \bar\eta^i_{\dot\alpha}$	λ_Y	λ^{ij}

Table 2.3: Generators of the $4D$ $N = 2$ superconformal algebra, their corresponding gauge fields, and parameters.

The gauge connections then describe the locally superconformal covariant derivative:

$$\begin{aligned}\nabla_a = e_a{}^m \nabla_m = e_a{}^m \Big(&\partial_m - \frac{1}{2}\psi_m{}^\alpha_i Q^i_\alpha - \frac{1}{2}\bar\psi_m{}^i_{\dot\alpha}\bar{Q}^{\dot\alpha}_i - \frac{1}{2}\omega_m{}^{cd}M_{cd} - \phi_m{}^{ij}J_{ij} \\ &- b_m\mathbb{D} - \mathrm{i}A_m Y - \frac{1}{2}\phi_m{}^\alpha_i S^i_\alpha - \frac{1}{2}\bar\phi_m{}^i_{\dot\alpha}\bar{S}^{\dot\alpha}_i - \mathfrak{f}_m{}^c K_c \Big) \ . \end{aligned} \tag{2.2.11}$$

The covariant derivatives satisfy the algebra

$$\begin{aligned}[\nabla_a, \nabla_b] = &-R(P)_{ab}{}^c\nabla_c - R(Q)_{ab}{}^\alpha_i Q^i_\alpha - R(\bar{Q})_{ab}{}^i_{\dot\alpha}\bar{Q}^{\dot\alpha}_i - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} - R(\mathbb{D})_{ab}\mathbb{D} \\ &-\mathrm{i}R(Y)_{ab}Y - R(J)_{ab}{}^{kl}J_{kl} - R(S)_{ab}{}^i_\alpha S^\alpha_i - R(\bar{S})_{ab}{}^{\dot\alpha}_i\bar{S}^i_{\dot\alpha} - R(K)_{abc}K^c \ . \end{aligned} \tag{2.2.12}$$

As previously mentioned, just like general relativity not all of these gauge fields describe new physical degrees of freedom. The only independent gauge connections now consist of the vielbein $e_m{}^a$; a dilatation connection b_m; the gravitino $(\psi_m{}_i^\alpha, \bar{\psi}_m{}_{\dot{\alpha}}^i)$, associated with the gauging of Q-supersymmetry; a $\mathrm{U}(1)_R$ gauge field A_m; and $\mathrm{SU}(2)_R$ gauge fields $\phi_m{}^{ij} = \phi_m{}^{ji}$. By a simple counting argument, see table 2.4, one can see that these independent gauge connections contribute $17+16$ degrees of freedom to the multiplet. Having an unequal number of bosonic and fermionic degrees of freedom is not allowed in a supersymmetric theory. To resolve this issue, together with the independent gauge connections, the standard Weyl multiplet includes a set of covariant matter fields that are necessary to close the local superconformal algebra off-shell: an anti-symmetric real tensor $W_{ab} = W_{ab}^+ + W_{ab}^-$, which decomposes into its imaginary-(anti-)self-dual components $W_{ab}^\pm$; a real scalar field D; and fermions $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i})$. Together with these matter fields, the gauge connections make a 24+24 standard Weyl multiplet of conformal gravity. Table 2.4 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields.

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\phi_m{}^{ij}$	A_m	$\psi_m{}_i^\alpha, \bar{\psi}_m{}_{\dot{\alpha}}^i$	$\phi_m{}_i^\alpha, \bar{\phi}_m{}_{\dot{\alpha}}^i$	W_{ab}	$\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i}$	D
$16B$	0	$4B$	0	$12B$	$4B$	$32F$	0	$6B$	$8F$	$1B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Y	$Q_\alpha^i, \bar{Q}_i^{\dot{\alpha}}$	$S_\alpha^i, \bar{S}_i^{\dot{\alpha}}$			
$-4B$	$-6B$	$-1B$	$-4B$	$-3B$	$-1B$	$-8F$	$-8F$			
Result: $17+16$ dof								$7+8$ dof		

Table 2.4: Degrees of freedom and symmetries of the $4D$ $N=2$ standard Weyl multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components of these fields – composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. Row five gives the resulting number of degrees of freedom.

It is also useful to list the non-trivial conjugation properties

$$(\psi_m{}_i^\alpha)^* = \bar{\psi}_m{}^{\dot{\alpha} i} \ , \quad (\phi_m{}_\alpha^i)^* = \bar{\phi}_{m \dot{\alpha} i} \ , \quad (\phi_m{}^{ij})^* = \phi_{m ij} \ , \quad (\Sigma^{\alpha i})^* = \bar{\Sigma}_i^{\dot{\alpha}} \ , \tag{2.2.13a}$$

$$(R(Q)_{ab}{}_i^\alpha)^* = R(\bar{Q})_{ab}{}^{\dot{\alpha} i} \ , \quad (R(S)_{ab}{}_\alpha^i)^* = R(\bar{S})_{ab \dot{\alpha} i} \ , \quad (R(J)_{ab}{}^{kl})^* = R(J)_{abkl} \ , \tag{2.2.13b}$$

while all the other fields and curvatures are real.

The spin connection, the $S-$ supersymmetry connections, and the special conformal connection are all composite connections and can be algebraically determined by imposing the following conventional constraints on the curvature. There is a large freedom in the choice of conventional constraints, here we adopt the conventional constraints used in [26] adapted to our conventions. They are given by

$$R(P)_{ab}{}^c = 0 \ , \tag{2.2.14a}$$

$$R(Q)_{ab\,j}\sigma^b = -\frac{3}{4}\Sigma_j \sigma_a \ , \qquad R(\bar{Q})_{ab}{}^j \tilde{\sigma}^b = \frac{3}{4}\bar{\Sigma}^j \tilde{\sigma}_a \ , \tag{2.2.14b}$$

$$R(M)^c{}_{acb} = R(\mathbb{D})_{ab} + 3\eta_{ab} D - \eta^{cd} W_{ac}^- W_{bd}^+ \ , \tag{2.2.14c}$$

where the spinor indices are suppressed and follow the convention defined in eq. (A.1.2): undotted indices obey the up-down contraction convention, while dotted indices obey the down-up contraction convention.

The composite Lorentz and S-supersymmetry connections are respectively

$$\omega_{abc} = \omega(e)_{abc} - 2\eta_{a[b}b_{c]} - \frac{\mathrm{i}}{2}\left(\psi_{aj}\sigma_{[b}\bar{\psi}_{c]}{}^{j} + \psi_{[bj}\sigma_{c]}\bar{\psi}_{a}{}^{j} + \psi_{[bj}\sigma_{|a|}\bar{\psi}_{c]}{}^{j}\right) , \qquad (2.2.15)$$

and

$$\begin{aligned}
\phi_{m}{}_{\beta}^{j} &= \frac{\mathrm{i}}{4}\left(\sigma^{bc}\sigma_{m} - \frac{1}{3}\sigma_{m}\tilde{\sigma}^{bc}\right)_{\beta\dot{\beta}}\overline{\Psi}_{bc}{}^{\dot{\beta}j} + \frac{1}{3}W^{-}_{mb}\psi^{bj}_{\ \beta} - \frac{1}{3}W^{-}_{mb}(\sigma^{bc}\psi_{c}{}^{j})_{\beta} + \frac{\mathrm{i}}{4}(\sigma_{m}\bar{\Sigma}^{j})_{\beta} , && (2.2.16a)\\
\bar{\phi}_{m}{}^{\dot{\beta}}_{j} &= \frac{\mathrm{i}}{4}\left(\tilde{\sigma}^{bc}\tilde{\sigma}_{m} - \frac{1}{3}\tilde{\sigma}_{m}\sigma^{bc}\right)^{\dot{\beta}\beta}\Psi_{bc\beta j} - \frac{1}{3}W^{+}_{mb}\bar{\psi}^{b\dot{\beta}}_{\ j} + \frac{1}{3}W^{+}_{mb}(\tilde{\sigma}^{bc}\bar{\psi}_{cj})^{\dot{\beta}} - \frac{\mathrm{i}}{4}(\tilde{\sigma}_{m}\Sigma_{j})^{\dot{\beta}} . && (2.2.16b)
\end{aligned}$$

The field $\omega(e)_{abc}$ is the torsion-free Lorentz connection given by (2.1.7), while the fields $\left(\Psi_{ab}{}_{k}^{\gamma}, \overline{\Psi}_{ab}{}^{k}_{\dot{\gamma}}\right)$ are the gravitini field strengths

$$\Psi_{ab}{}_{k}^{\gamma} = 2e_{a}{}^{m}e_{b}{}^{n}\mathcal{D}_{[m}\psi_{n]}{}_{k}^{\gamma} , \qquad \overline{\Psi}_{ab}{}^{k}_{\dot{\gamma}} = 2e_{a}{}^{m}e_{b}{}^{n}\mathcal{D}_{[m}\bar{\psi}_{n]}{}^{k}_{\dot{\gamma}} , \qquad (2.2.17)$$

where we have introduced the spin, dilatation, and $R-$symmetry covariant derivative $\mathcal{D}_{a}$

$$\mathcal{D}_{a} = e_{a}{}^{m}\mathcal{D}_{m} = e_{a}{}^{m}\left(\partial_{m} - \frac{1}{2}\omega_{m}{}^{cd}M_{cd} - \phi_{m}{}^{ij}J_{ij} - \mathrm{i}A_{m}Y - b_{m}\mathbb{D}\right) . \qquad (2.2.18)$$

Note that $\left(R(Q)_{ab}{}_{k}^{\gamma}, R(\bar{Q})_{ab}{}^{k}_{\dot{\gamma}}\right)$ are the Q-supersymmetry curvatures and satisfy

$$\begin{aligned}
R(Q)_{ab}{}_{k}^{\gamma} &= \frac{1}{2}\Psi_{ab}{}_{k}^{\gamma} - \mathrm{i}(\bar{\phi}_{[ak}\tilde{\sigma}_{b]})^{\gamma} + \frac{\mathrm{i}}{4}(\bar{\psi}_{[ak}\tilde{\sigma}_{b]}\sigma^{cd})^{\gamma}W^{+}_{cd} , && (2.2.19a)\\
R(\bar{Q})_{ab}{}^{k}_{\dot{\gamma}} &= \frac{1}{2}\overline{\Psi}_{ab}{}^{k}_{\dot{\gamma}} - \mathrm{i}(\phi_{[a}{}^{k}\sigma_{b]})_{\dot{\gamma}} - \frac{\mathrm{i}}{4}(\psi_{[a}{}^{k}\sigma_{b]}\tilde{\sigma}^{cd})_{\dot{\gamma}}W^{-}_{cd} , && (2.2.19b)
\end{aligned}$$

while $R(Y)_{ab}$ and $R(J)_{ab}{}^{kl}$ are

$$\begin{aligned}
R(Y)_{ab} &= 2e_{a}{}^{m}e_{b}{}^{n}\partial_{[m}A_{n]} - \frac{\mathrm{i}}{2}\psi_{[aj}\phi_{b]}{}^{j} + \frac{\mathrm{i}}{2}\bar{\psi}_{[a}{}^{j}\bar{\phi}_{b]j} + \frac{3}{8}\psi_{[aj}\sigma_{b]}\bar{\Sigma}^{j} + \frac{3}{8}\bar{\psi}_{[a}{}^{j}\tilde{\sigma}_{b]}\Sigma_{j} , && (2.2.20a)\\
R(J)_{ab}{}^{kl} &= 2e_{a}{}^{m}e_{b}{}^{n}\partial_{[m}\phi_{n]}{}^{kl} - 2\phi_{a}{}^{(k}{}_{p}\phi_{b}{}^{l)p} + 2\psi_{[a}{}^{(k}\phi_{b]}{}^{l)} - 2\bar{\psi}_{[a}{}^{(k}\bar{\phi}_{b]}{}^{l)}\\
&\quad - \frac{3\mathrm{i}}{2}\psi_{[a}{}^{(k}\sigma_{b]}\bar{\Sigma}^{l)} - \frac{3\mathrm{i}}{2}\bar{\psi}_{[a}{}^{(k}\tilde{\sigma}_{b]}\Sigma^{l)} . && (2.2.20b)
\end{aligned}$$

Note that, here we have provided only the relevant curvatures, however, for more details on superconformal curvatures, we refer the reader to [26, 41, 137]. Although, we do not present the expression for the composite special conformal connection $\mathfrak{f}_{ma}$, the trace is given by

$$\begin{aligned}
\mathfrak{f}_{a}{}^{a} = e_{a}{}^{m}\mathfrak{f}_{m}{}^{a} &= -\frac{1}{12}R + D - \frac{1}{24}\varepsilon^{mnpq}(\bar{\psi}^{j}_{m}\tilde{\sigma}_{n}\mathcal{D}_{p}\psi_{qj}) + \frac{1}{24}\varepsilon^{mnpq}(\psi_{mj}\sigma_{n}\mathcal{D}_{p}\bar{\psi}_{q}{}^{j})\\
&\quad - \frac{\mathrm{i}}{8}\psi_{aj}\sigma^{a}\bar{\Sigma}^{j} + \frac{\mathrm{i}}{8}\bar{\psi}_{a}{}^{j}\tilde{\sigma}^{a}\Sigma_{j} - \frac{1}{12}W^{ab+}(\bar{\psi}_{a}{}^{j}\bar{\psi}_{bj}) + \frac{1}{12}W^{ab-}(\psi_{aj}\psi_{b}{}^{j}) && (2.2.21)
\end{aligned}$$

where $R = e_{a}{}^{m}e_{b}{}^{n}R_{mn}{}^{ab}$ is the scalar curvature constructed from the Lorentz curvature

$$R_{mn}{}^{cd} = 2\partial_{[m}\omega_{n]}{}^{cd} - 2\omega_{[m}{}^{ce}\omega_{n]e}{}^{d} . \qquad (2.2.22)$$

Also, recall that the spin connection $\omega_{m}{}^{cd}$ is a composite field of the vielbein, the gravitini, and the dilatation connection, eq. (2.2.15).

In presenting the multiplet we restrict to all local superconformal transformations except local translations (covariant general coordinate transformations). Such transformations are identified by δ and defined by the following operator

$$\delta = \xi_i^{\alpha} Q_{\alpha}^i + \bar{\xi}^i_{\dot{\alpha}} \bar{Q}_i^{\dot{\alpha}} + \frac{1}{2}\lambda^{ab} M_{ab} + \lambda^{ij} J_{ij} + \lambda_{\mathbb{D}}\mathbb{D} + \mathrm{i}\lambda_Y Y + \lambda_a K^a + \eta^i_{\alpha} S_i^{\alpha} + \bar{\eta}_i^{\dot{\alpha}} \bar{S}^i_{\dot{\alpha}} \ . \tag{2.2.23}$$

The local superconformal transformation of the fundamental fields of the standard Weyl multiplet are then given by

$$\begin{aligned}
\delta e_m{}^a &= \mathrm{i}\,\xi_i \sigma^a \bar{\psi}_m{}^i + \mathrm{i}\,\bar{\xi}^i \tilde{\sigma}^a \psi_{mi} - \lambda_{\mathbb{D}} e_m{}^a + \lambda^a{}_b e_m{}^b \ , && (2.2.24a)\\
\delta \psi_{mi}^{\alpha} &= \Big(2\partial_m \xi_i^{\alpha} + \omega_m{}^{ab}(\xi_i\sigma_{ab})^{\alpha} + 2\phi_{mi}{}^j \xi_j^{\alpha} + 2\mathrm{i}A_m \xi_i^{\alpha} + b_m \xi_i^{\alpha}\Big) - \frac{\mathrm{i}}{2}(\bar{\xi}_i \tilde{\sigma}_m \sigma^{cd})^{\alpha} W_{cd}^+ \\
&\quad -\frac{1}{2}\lambda^{ab}(\psi_{mi}\sigma_{ab})^{\alpha} - \lambda_i{}^j\, \psi_{mj}^{\alpha} - \mathrm{i}\lambda_Y\, \psi_{mi}^{\alpha} - \frac{1}{2}\lambda_{\mathbb{D}}\, \psi_{mi}^{\alpha} + 2\mathrm{i}(\bar{\eta}_i \tilde{\sigma}_m)^{\alpha} \ , && (2.2.24b)\\
\delta \bar{\psi}_m{}^i_{\dot{\alpha}} &= \Big(2\partial_m \bar{\xi}^i_{\dot{\alpha}} + \omega_m{}^{ab}(\bar{\xi}^i \tilde{\sigma}_{ab})_{\dot{\alpha}} - 2\phi_m{}^i{}_j \bar{\xi}^j_{\dot{\alpha}} - 2\mathrm{i}A_m \bar{\xi}^i_{\dot{\alpha}} + b_m \bar{\xi}^i_{\dot{\alpha}}\Big) + \frac{\mathrm{i}}{2}(\xi^i \sigma_m \tilde{\sigma}^{cd})_{\dot{\alpha}} W_{cd}^- \\
&\quad -\frac{1}{2}\lambda^{ab}(\bar{\psi}_m{}^i \tilde{\sigma}_{ab})_{\dot{\alpha}} + \lambda^i{}_j\, \bar{\psi}_m{}^j_{\dot{\alpha}} + \mathrm{i}\lambda_Y\, \bar{\psi}_m{}^i_{\dot{\alpha}} - \frac{1}{2}\lambda^{\mathbb{D}}\, \bar{\psi}_m{}^i_{\dot{\alpha}} + 2\mathrm{i}(\eta^i \sigma_m)_{\dot{\alpha}} \ , && (2.2.24c)\\
\delta \phi_m{}^{ij} &= \Big(\partial_m \lambda^{ij} - 2\phi_m{}^{(i}{}_k \lambda^{j)k}\Big) + \frac{3\mathrm{i}}{2}\,\xi^{(i}\sigma_m \bar{\Sigma}^{j)} + \frac{3\mathrm{i}}{2}\,\bar{\xi}^{(i}\tilde{\sigma}_m \Sigma^{j)} - \phi_m{}^{(i}\xi^{j)} + \bar{\phi}_m{}^{(i}\bar{\xi}^{j)} \\
&\quad +2\psi_m{}^{(i}\eta^{j)} - 2\bar{\psi}_m{}^{(i}\bar{\eta}^{j)} \ , && (2.2.24d)\\
\delta A_m &= \partial_m \lambda_Y - \frac{3}{8}\,\xi_i \sigma_m \bar{\Sigma}^i - \frac{3}{8}\,\bar{\xi}^i \tilde{\sigma}_m \Sigma_i + \frac{\mathrm{i}}{2}\,\xi_i \phi_m{}^i - \frac{\mathrm{i}}{2}\,\bar{\xi}^i \bar{\phi}_{mi} - \frac{\mathrm{i}}{2}\,\psi_{mi}\eta^i + \frac{\mathrm{i}}{2}\,\bar{\psi}_m{}^i \bar{\eta}_i \ , && (2.2.24e)\\
\delta b_m &= \partial_m \lambda_{\mathbb{D}} - \frac{3\mathrm{i}}{4}\,\xi_i \sigma_m \bar{\Sigma}^i + \frac{3\mathrm{i}}{4}\,\bar{\xi}^i \tilde{\sigma}_m \Sigma_i + \xi_i \phi_m{}^i + \bar{\xi}^i \bar{\phi}_{mi} - \psi_{mi}\eta^i - \bar{\psi}_m{}^i \bar{\eta}_i - 2\lambda_{mK} \ , && (2.2.24f)\\
\delta W_{ab} &= -4\xi_k R(Q)_{ab}{}^k - 4\bar{\xi}^k R(\bar{Q})_{abk} - 2\lambda_{[a}{}^c W_{b]c} + \lambda_{\mathbb{D}} W_{ab} - 2\mathrm{i}\lambda_Y W_{ab}^+ + 2\mathrm{i}\lambda_Y W_{ab}^- \ , && (2.2.24g)\\
\delta D &= -\mathrm{i}\xi^k \sigma^a \nabla_a \bar{\Sigma}_k - \mathrm{i}\bar{\xi}_k \tilde{\sigma}^a \nabla_a \Sigma^k + 2\lambda_{\mathbb{D}} D \ , && (2.2.24h)\\
\delta \Sigma^{\alpha i} &= \xi^{\alpha i} D + \frac{4\mathrm{i}}{3}(\xi^i \sigma^{ab})^{\alpha} R(Y)_{ab} + \frac{2}{3}(\xi_j \sigma^{ab})^{\alpha} R(J)_{ab}{}^{ji} - \frac{\mathrm{i}}{3}(\bar{\xi}^i \tilde{\sigma}^a \sigma^{cd})^{\alpha} \nabla_a W_{cd}^+ \\
&\quad -\frac{1}{2}\lambda^{ab}(\Sigma^i \sigma_{ab})^{\alpha} + \lambda^i{}_j \Sigma^{\alpha j} + \frac{3}{2}\lambda_{\mathbb{D}} \Sigma^{\alpha i} - \mathrm{i}\lambda_Y \Sigma^{\alpha i} + \frac{2}{3}(\eta^i \sigma^{cd})^{\alpha} W_{cd}^+ \ , && (2.2.24i)\\
\delta \bar{\Sigma}_{\dot{\alpha} i} &= -\bar{\xi}_{\dot{\alpha} i} D + \frac{4\mathrm{i}}{3}(\bar{\xi}_i \tilde{\sigma}^{ab})_{\dot{\alpha}} R(Y)_{ab} + \frac{2}{3}(\bar{\xi}^j \tilde{\sigma}^{ab})_{\dot{\alpha}} R(J)_{abji} - \frac{\mathrm{i}}{3}(\xi_i \sigma^a \tilde{\sigma}^{cd})_{\dot{\alpha}} \nabla_a W_{cd}^- \\
&\quad -\frac{1}{2}\lambda^{ab}(\bar{\Sigma}_i \tilde{\sigma}_{ab})_{\dot{\alpha}} - \lambda_i{}^j \bar{\Sigma}_{\dot{\alpha} j} + \frac{3}{2}\lambda_{\mathbb{D}} \bar{\Sigma}_{\dot{\alpha} i} + \mathrm{i}\lambda_Y \bar{\Sigma}_{\dot{\alpha} i} + \frac{2}{3}(\bar{\eta}_i \tilde{\sigma}^{cd})_{\dot{\alpha}} W_{cd}^- \ , && (2.2.24j)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a W_{bc} &= \mathcal{D}_a W_{bc} + 2\psi_{ak} R(Q)_{bc}{}^k + 2\bar{\psi}_a{}^k R(\bar{Q})_{bck} \ , && (2.2.25a)\\
\nabla_a \Sigma^{\alpha i} &= \mathcal{D}_a \Sigma^{\alpha i} - \frac{1}{2}\psi_a{}^{\alpha i} D - \frac{2\mathrm{i}}{3}(\psi_a{}^i \sigma^{cd})^{\alpha} R(Y)_{cd} - \frac{1}{3}(\psi_{aj} \sigma^{cd})^{\alpha} R(J)_{cd}{}^{ji} \\
&\quad +\frac{\mathrm{i}}{6}(\bar{\psi}_a{}^i \tilde{\sigma}^b \sigma^{cd})^{\alpha} \nabla_b W_{cd}^+ + (\phi_a{}^i \sigma^{cd})^{\alpha}\, W_{cd}^+ \ , && (2.2.25b)\\
\nabla_a \bar{\Sigma}_{\dot{\alpha} i} &= \mathcal{D}_a \bar{\Sigma}_{\dot{\alpha} i} + \frac{1}{2}\bar{\psi}_{a\dot{\alpha} i} D - \frac{2\mathrm{i}}{3}(\bar{\psi}_{ai} \tilde{\sigma}^{cd})_{\dot{\alpha}} R(Y)_{cd} - \frac{1}{3}(\bar{\psi}_a{}^j \tilde{\sigma}^{cd})_{\dot{\alpha}} R(J)_{cdji} \\
&\quad +\frac{\mathrm{i}}{6}(\psi_{ai} \sigma^b \tilde{\sigma}^{cd})_{\dot{\alpha}} \nabla_b W_{cd}^- + (\bar{\phi}_{ai} \tilde{\sigma}^{cd})_{\dot{\alpha}}\, W_{cd}^- \ , && (2.2.25c)
\end{aligned}$$

where the spinor indices are suppressed following the convention defined in eq. (A.1.2).

We stress that the transformations (2.2.24) form an algebra that closes off-shell on a local extension of SU$(2,2|2)$. We will not need the explicit form of the algebra here, though it can be straightforwardly derived from results of [137] and [26, 41]. To conclude this subsection, for convenience, we include Table 2.5 which summarises the non-trivial chiral and dilatation weights of the fields and local gauge parameters of the standard Weyl multiplet.

	$e_m{}^a$	ψ_{mi}, ξ_i	$\bar\psi_m{}^i, \bar\xi^i$	$\phi_m{}^i, \eta^i$	$\bar\phi_{mi}, \bar\eta_i$	$\mathfrak{f}_{mc}$	W^+_{ab}	W^-_{ab}	Σ^i	$\bar\Sigma_i$	D
$\mathbb{D}$	-1	$-1/2$	$-1/2$	$1/2$	$1/2$	1	1	1	$3/2$	$3/2$	2
Y	0	-1	1	1	-1	0	-2	2	-1	1	0

Table 2.5: Summary of the non-trivial dilatation and chiral weights in the standard Weyl multiplet.

$5D$ $N=1$ standard Weyl multiplet

The standard Weyl multiplet of $5D$, $N=1$ conformal supergravity is associated with the local off-shell gauging in space-time of the superconformal group $\mathrm{F}^2(4)$. Our notations and conventions follow those in [3, 29], see also appendix B. The following table lists the generators of the superconformal group $\mathrm{F}^2(4)$, their corresponding gauge fields, and parameters.

Generators	P_a	M_{ab}	$\mathbb{D}$	K_a	Q^i_α	S^i_α	J_{ij}
Gauge Fields	$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\psi_{m}{}^\alpha_i$	$\phi_{m}{}^\alpha_i$	$\phi_m{}^{ij}$
Parameters	ξ^a	λ^{ab}	$\lambda_{\mathbb{D}}$	λ^a_K	ξ^α_i	η^α_i	λ^{ij}

Table 2.6: Generators of the $5D$ $N=1$ superconformal algebra, their corresponding gauge fields, and parameters.

The gauge connections then describe the locally superconformal covariant derivative:

$$\nabla_a = e_a{}^m\nabla_m = e_a{}^m\Big(\partial_m - \frac{1}{2}\psi_{m}{}^\alpha_i Q^i_\alpha - \frac{1}{2}\omega_m{}^{cd}M_{cd} - \phi_m{}^{ij}J_{ij} - b_m\mathbb{D} - \frac{1}{2}\phi_{m}{}^\alpha_i S^i_\alpha - \mathfrak{f}_m{}^c K_c\ .\Big) \qquad (2.2.26)$$

The covariant derivatives satisfy the algebra

$$\begin{aligned}[\nabla_a, \nabla_b] \;=\; & -R(P)_{ab}{}^c\nabla_c - R(Q)_{ab}{}^\alpha_i Q^i_\alpha - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} - R(\mathbb{D})_{ab}\mathbb{D} \\ & -R(J)_{ab}{}^{kl}J_{kl} - R(S)_{ab}{}^i_\alpha S^\alpha_i - R(K)_{abc}K^c\ . \end{aligned} \qquad (2.2.27)$$

Similar to the previous case the off-shell gauging of the $5D$ $N=1$ superconformal algebra leads to a $32+32$ standard Weyl multiplet. The following table 2.7 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields.

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\phi_m{}^{ij}$	ψ_{mi}	$\phi_m{}^i$	W_{ab}	χ^i	D
$25B$	0	$5B$	0	$15B$	$40F$	0	$10B$	$8F$	$1B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Q	S			
$-5B$	$-10B$	$-1B$	$-5B$	$-3B$	$-8F$	$-8F$			
Result: $21+24$ dof							$11+8$ dof		

Table 2.7: Degrees of freedom and symmetries of the $5D$ $N=1$ standard Weyl multiplet.

The component curvatures turn out to obey "traceless" conventional constraints [29]

$$R(P)_{ab}{}^c = 0 \,, \qquad (\Gamma^a)_\alpha{}^\beta R(Q)_{ab}{}^i_\beta = 0 \,, \qquad R(M)_{ab}{}^{cb} = 0 \,, \tag{2.2.28}$$

which allow us to solve for the composite connections as follows:

$$\omega_{abc} = \omega(e)_{abc} + \frac{\mathrm{i}}{4}(\psi_{ak}\Gamma_c\psi_b{}^k + \psi_{ck}\Gamma_b\psi_a{}^k - \psi_{bk}\Gamma_a\psi_c{}^k) + 2b_{[b}\eta_{c]a} \,, \tag{2.2.29a}$$

$$\mathrm{i}\,\phi_m{}^i = \frac{2}{3}(\Gamma^{[p}\delta_m{}^{q]} + \frac{1}{4}\Gamma_m\Sigma^{pq})\Big(\mathcal{D}_{[p}\psi_{q]}{}^i + \frac{1}{8}W_{cd}\big(3\Sigma^{cd}\Gamma_{[p}\psi_{q]}{}^i - \Gamma_{[p}\Sigma^{cd}\psi_{q]}{}^i\big)\Big) \,, \tag{2.2.29b}$$

$$\begin{aligned}
\mathfrak{f}_a{}^b = &-\frac{1}{6}R(\omega)_{ac}{}^{bc} + \frac{1}{48}\delta_a{}^b R(\omega)_{cd}{}^{cd} - \frac{\mathrm{i}}{6}\psi_{cj}\Gamma^{[b}R(Q)_a{}^{c]j} - \frac{\mathrm{i}}{12}\psi_{cj}\Gamma_a R(Q)^{bcj} \\
&+ \frac{1}{3}\psi_{[aj}\Sigma^{bd}\phi_{d]}{}^j - \frac{1}{24}\delta_a{}^b(\psi_{cj}\Sigma^{cd}\phi_d{}^j) - \frac{2\mathrm{i}}{3}(\psi_{aj}\Gamma^b\chi^j) \\
&- \frac{\mathrm{i}}{12}\psi_{aj}\psi_c{}^j W^{bc} + \frac{\mathrm{i}}{24}(\psi_{aj}\Gamma_e\psi_d{}^j)\tilde{W}^{bde} \\
&+ \frac{\mathrm{i}}{192}\delta_a{}^b\Big(2(\psi_{cj}\psi_d{}^j)W^{cd} - (\psi_{cj}\Gamma_e\psi_d{}^j)\tilde{W}^{cde}\Big) \,,
\end{aligned} \tag{2.2.29c}$$

where $\omega(e)_{abc}$ is the torsion-free spin connection. The spinor indices are suppressed, and in 5D, we follow the up-down contraction convention for spinor indices. For example, $\psi_{ak}\Gamma_c\psi_b{}^k := \psi^\alpha_{ak}(\Gamma_c)_\alpha{}^\beta\psi_b{}^k_\beta$. We have also defined

$$\tilde{W}_{abc} = \frac{1}{2}\varepsilon_{abcde}W^{de} \,. \tag{2.2.30}$$

Note that the curvatures now satisfy

$$R(P)_{ab}{}^c = 2\,e_a{}^m e_b{}^n \mathcal{D}_{[m}e_{n]}{}^c - \frac{\mathrm{i}}{2}\psi_{aj}\Gamma^c\psi_b{}^j \,, \tag{2.2.31a}$$

$$\begin{aligned}
R(Q)_{ab}{}^i_\alpha = &\, e_a{}^m e_b{}^n \mathcal{D}_{[m}\psi_{n]}{}^i_\alpha + \mathrm{i}(\Gamma_{[a}\phi_{b]}{}^i)_\alpha \\
&+ \frac{1}{8}W_{cd}\Big(3(\Sigma^{cd}\Gamma_{[a})_\alpha{}^\beta - (\Gamma_{[a}\Sigma^{cd})_\alpha{}^\beta\Big)\psi_{b]}{}^i_\beta \,,
\end{aligned} \tag{2.2.31b}$$

$$\begin{aligned}
R(M)_{ab}{}^{cd} = &\, R(\omega)_{ab}{}^{cd} + 8\,\delta_{[a}{}^{[c}\mathfrak{f}_{b]}{}^{d]} - 2\,\psi_{[aj}\Sigma^{cd}\phi_{b]}{}^j \\
&+ \frac{16\mathrm{i}}{3}\delta_{[a}{}^{[c}\psi_{b]i}\Gamma^{d]}\chi^i - \mathrm{i}\psi_{[ai}\Big(\Gamma_{b]}R(Q)^{cdi} + 2\Gamma^{[c}R(Q)_{b]}{}^{d]i}\Big) \\
&+ \frac{\mathrm{i}}{2}\psi_{aj}\psi_b{}^j W^{cd} - \frac{\mathrm{i}}{4}(\psi_{aj}\Gamma_e\psi_b{}^j)\tilde{W}^{cde} \,,
\end{aligned} \tag{2.2.31c}$$

$$R(J)_{ab}{}^{ij} = R(\phi)_{ab}{}^{ij} - 3\,\psi_{[a}^{(i}\phi_{b]}{}^{j)} - 8\mathrm{i}\,\psi_{[a}^{(i}\Gamma_{b]}\chi^{j)} \,, \tag{2.2.31d}$$

$$R(\mathbb{D})_{ab} = 2\,e_a{}^m e_b{}^n\partial_{[m}b_{n]} + 4\,\mathfrak{f}_{[ab]} + \psi_{[aj}\phi_{b]}{}^j + \frac{8\mathrm{i}}{3}\,\psi_{[aj}\Gamma_{b]}\chi^j \,. \tag{2.2.31e}$$

In the above we have introduced the spin, dilatation, and $SU(2)_R$ covariant derivative

$$\mathcal{D}_m = \partial_m - \frac{1}{2}\omega_m{}^{bc}M_{bc} - b_m\mathbb{D} - \phi_m{}^{ij}J_{ij} \,, \qquad \mathcal{D}_a = e_a{}^m\mathcal{D}_m \,, \tag{2.2.32}$$

along with the curvatures

$$R(\omega)_{ab}{}^{cd} := 2e_a{}^m e_b{}^n\Big(\partial_{[m}\omega_{n]}{}^{cd} - 2\omega_{[m}{}^{ce}\omega_{n]e}{}^d\Big) \,, \tag{2.2.33a}$$

$$R(\phi)_{ab}{}^{ij} := 2\,e_a{}^m e_b{}^n\Big(\partial_{[m}\phi_{n]}{}^{ij} + \phi_{[m}{}^{k(i}\phi^{j)}{}_{n]k}\Big) \,. \tag{2.2.33b}$$

The curvature $R(\mathbb{D})_{ab}$ now vanishes due to eqs. (2.2.29). We stress that in the traceless frame $\phi_m{}^i$ has no dependence upon the matter field χ^i and $\mathfrak{f}_a{}^b$ has no dependence upon D. This choice minimises the dependence of the covariant derivatives upon some matter "auxiliary" fields and will simplify part of the analysis in the coming sections.

The local superconformal transformations except for local translations (covariant general coordinate transformations) are defined by the following operator

$$\delta = \xi_i^\alpha Q_\alpha^i + \frac{1}{2}\lambda^{ab}M_{ab} + \lambda^{ij}J_{ij} + \lambda_{\mathbb{D}}\mathbb{D} + \lambda^a K_a + \eta^{\alpha i}S_{\alpha i} \ . \tag{2.2.34}$$

The local superconformal transformations of the independent connection fields of the standard Weyl multiplet are given by [29]

$$\begin{aligned}
\delta e_m{}^a &= \mathrm{i}\,(\xi_i\Gamma^a\psi_m{}^i) - \lambda_{\mathbb{D}}e_m{}^a + \lambda^a{}_b e_m{}^b \ , && (2.2.35\text{a})\\
\delta\psi_m{}^i_\alpha &= 2\mathcal{D}_m\xi_\alpha^i - \frac{1}{4}W_{cd}\Big((\Gamma_m\Sigma^{cd})_\alpha{}^\beta - 3(\Sigma^{cd}\Gamma_m)_\alpha{}^\beta\Big)\xi_\beta^i + 2\mathrm{i}\,(\Gamma_m\eta^i)_\alpha \\
&\quad + \frac{1}{2}\lambda^{ab}(\Sigma_{ab}\psi_m{}^i)_\alpha + \lambda^i{}_j\,\psi_m{}^j_\alpha - \frac{1}{2}\lambda_{\mathbb{D}}\,\psi_m{}^i_\alpha \ , && (2.2.35\text{b})\\
\delta\phi_m{}^{ij} &= \partial_m\lambda^{ij} - 2\phi_m{}^{(i}{}_k\lambda^{j)k} + 3\xi^{(i}\phi_m{}^{j)} - 3\eta^{(i}\psi_m{}^{j)} + 8\mathrm{i}\xi^{(i}\Gamma_m\chi^{j)} \ , && (2.2.35\text{c})\\
\delta b_m &= \partial_m\lambda_{\mathbb{D}} - \frac{8\mathrm{i}}{3}\,\xi_i\Gamma_m\chi^i - \xi_i\phi_m{}^i - \psi_m{}^i\eta_i - 2\lambda_{mK} \ . && (2.2.35\text{d})
\end{aligned}$$

In like fashion, one can derive the transformations $\delta\omega_m{}^{ab}, \delta\phi_{m\alpha}{}^i$, and $\delta\mathfrak{f}_{ma}$, which we omit since these fields are composite. For the covariant matter fields, the transformations are given by [29]

$$\begin{aligned}
\delta W_{ab} &= 2\mathrm{i}\,\xi_i R(Q)_{ab}{}^i - \frac{32\mathrm{i}}{3}\xi_i\Sigma_{ab}\chi^i - 2\lambda_{[a}{}^c W_{b]c} + \lambda_{\mathbb{D}}W_{ab} \ , && (2.2.35\text{e})\\
\delta\chi^{\alpha i} &= \frac{1}{2}\xi^{\alpha i}D - \frac{1}{16}(\xi_j\Sigma^{ab})^\alpha R(J)_{ab}{}^{ij} - \frac{3}{128}(\nabla_a W_{bc})\Big(3(\xi^i\Gamma^a\Sigma^{bc})^\alpha + (\xi^i\Sigma^{bc}\Gamma^a)^\alpha\Big) \\
&\quad + \frac{3}{256}W_{ab}W_{cd}\varepsilon^{abcde}(\xi^i\Gamma_e)^\alpha + \frac{3\mathrm{i}}{16}(\eta^i\Sigma^{ab})^\alpha W_{ab} \\
&\quad - \frac{1}{2}\lambda^{ab}(\chi^i\Sigma_{ab})^\alpha + \lambda^i{}_j\chi^{\alpha j} + \frac{3}{2}\lambda_{\mathbb{D}}\chi^{\alpha i} \ , && (2.2.35\text{f})\\
\delta D &= 2\mathrm{i}\,\xi_i\Gamma^a\nabla_a\chi^i + \mathrm{i}W_{ab}(\xi_i\Sigma^{ab}\chi^i) + 2\eta_i\chi^i + 2\lambda_{\mathbb{D}}D \ , && (2.2.35\text{g})
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a W_{bc} &= \mathcal{D}_a W_{bc} - \mathrm{i}\psi_{ai}R(Q)_{bc}{}^i + \frac{16\mathrm{i}}{3}\psi_{ai}\Sigma_{bc}\chi^i \ , && (2.2.36\text{a})\\
\nabla_a\chi^{\alpha i} &= \mathcal{D}_a\chi^{\alpha i} - \frac{1}{4}\psi_a{}^{\alpha i}D - \frac{3\mathrm{i}}{32}(\phi_a{}^i\Sigma^{bc})^\alpha W_{bc} + \frac{1}{32}(\psi_{aj}\Sigma^{bc})^\alpha R(J)_{bc}{}^{ij} \\
&\quad + \frac{3}{256}(\nabla_b W_{cd})\Big(3(\psi_a{}^i\Gamma^b\Sigma^{cd})^\alpha + (\psi_a{}^i\Sigma^{cd}\Gamma^b)^\alpha\Big) - \frac{3}{512}W_{bc}W_{de}\varepsilon^{bcdef}(\psi_a{}^i\Gamma_f)^\alpha && (2.2.36\text{b})
\end{aligned}$$

We stress that the transformations (2.2.35) form an algebra that closes off-shell on a local extension of $\mathrm{F}^2(4)$. We will not need the explicit form of the algebra here, though it can be straightforwardly derived from results of [137] and [26, 41]. To conclude this subsection, for convenience, we include Table 2.8 which summarises the non-trivial dilatation weights of the fields and local gauge parameters of the standard Weyl multiplet.

	$e_m{}^a$	ψ_{mi}, ξ_i	$\phi_m{}^i$, η^i	$\mathfrak{f}_{mc}$	W_{ab}	χ^i	D
$\mathbb{D}$	-1	$-1/2$	$1/2$	1	1	$3/2$	2

Table 2.8: Summary of the non-trivial dilatation and chiral weights in the standard Weyl multiplet.

6D $N=(1,0)$ standard Weyl multiplet

The standard Weyl multiplet of 6D $N=(1,0)$ conformal supergravity [89] contains $40+40$ physical components, and is associated with the gauging of the superconformal algebra $\mathrm{O}Sp(6,2|1)$. Our notations and conventions follow those in [69], see also appendix C. The following table lists the generators of the superconformal group $\mathrm{O}Sp(6,2|1)$, their corresponding gauge fields, and parameters.

Generators	P_a	M_{ab}	D	K_a	Q^i_α	S^i_α	J_{ij}
Gauge Fields	$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\psi_m{}^\alpha_i$	$\phi_m{}^\alpha_i$	$\phi_m{}^{ij}$
Parameters	ξ^a	λ^{ab}	$\lambda_{\mathbb{D}}$	λ_K^a	ξ_i^α	η_i^α	λ^{ij}

Table 2.9: Generators of the 5D $N=1$ superconformal algebra, their corresponding gauge fields, and parameters.

The component gauge connections can now be used to define the locally superconformal covariant derivative ∇_a

$$\nabla_a = e_a{}^m\nabla_m = e_a{}^m\big(\partial_m - \frac{1}{2}\psi_m{}^\alpha_i\nabla^i_\alpha - \frac{1}{2}\omega_m{}^{cd}M_{cd} - \phi_m{}^{ij}J_{ij} - b_m\mathbb{D} - \frac{1}{2}\phi_m{}^i_\alpha S^\alpha_i - \mathfrak{f}_{ma}K^a\big)\ . \quad (2.2.37)$$

This satisfies the algebra

$$\begin{aligned}[\nabla_a,\nabla_b] \;=\;& -R(P)_{ab}{}^c\nabla_c - R(Q)_{ab}{}^\alpha_i Q^i_\alpha - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} - R(J)_{ab}{}^{kl}J_{kl}\\ & -R(\mathbb{D})_{ab}\mathbb{D} - R(S)_{ab}{}^i_\gamma S^\gamma_i - R(K)_{ab}{}^c K_c\ . \end{aligned} \quad (2.2.38)$$

Similar to the previous case the off-shell gauging of the 6D $N=(1,0)$ superconformal algebra leads to a $40+40$ standard Weyl multiplet. The following table 2.10 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields.

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\phi_m{}^{ij}$	ψ_{mi}	$\phi_m{}^i$	T^-_{abc}	χ^i	D
36	0	$6B$	0	$18B$	$48F$	0	$10B$	$8F$	$1B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Q	S			
$-6B$	$-15B$	$-1B$	$-6B$	$-3B$	$-8F$	$-8F$			
Result: $29+32$ dof							$11+8$ dof		

Table 2.10: Degrees of freedom and symmetries of the 6D $N=(1,0)$ standard Weyl multiplet.

The component curvatures turn out to obey "traceless" conventional constraints [32]

$$R(P)_{ab}{}^c = 0\ , \qquad \gamma^b R(Q)_{abi} = 0\ , \qquad R(M)_{ab}{}^{cb} = 0\ . \quad (2.2.39)$$

The constraints (2.2.39) can be solved for the composite connections as follows:

$$\omega_{abc} \;=\; \omega(e)_{abc} - 2\eta_{a[b}b_{c]} - \frac{\mathrm{i}}{4}\psi_b{}^k\gamma_a\psi_{ck} - \frac{\mathrm{i}}{2}\psi_a{}^k\gamma_{[b}\psi_{c]k}\ , \quad (2.2.40a)$$

$$\begin{aligned}
\phi_m{}^k &= \frac{\mathrm{i}}{16}\Big(\gamma^{bc}\gamma_m - \frac{3}{5}\gamma_m\tilde{\gamma}^{bc}\Big)\Big(\Psi_{bc}{}^k + \frac{1}{12}T^-_{def}\tilde{\gamma}^{def}\gamma_{[b}\psi_{c]}{}^k\Big) , && (2.2.40b)\\
\mathfrak{f}_a{}^b &= -\frac{1}{8}R_a{}^b(\omega) + \frac{1}{80}\delta_a^b R(\omega) + \frac{1}{8}\psi_{[a\,j}\gamma^{bc}\phi_{c]}{}^j - \frac{1}{80}\delta_a^b\psi_{c\,j}\gamma^{cd}\phi_d{}^j\\
&\quad + \frac{\mathrm{i}}{16}\psi_{c\,j}\gamma_a R(Q)^{bc\,j} + \frac{\mathrm{i}}{8}\psi_{c\,j}\gamma^{[b}R(Q)_a{}^{c]\,j} + \frac{\mathrm{i}}{60}\psi_{a\,j}\gamma^b\chi^j\\
&\quad + \frac{\mathrm{i}}{16}\psi_a{}^j\gamma_c\psi_{d\,j}T^{-bcd} - \frac{\mathrm{i}}{160}\delta_a^b\psi_c{}^j\gamma_d\psi_{e\,j}T^{-cde} . && (2.2.40c)
\end{aligned}$$

where $R(\omega) := R_{ab}{}^{ab}(\omega)$ is the scalar curvature and $R_a{}^b(\omega) := R_{ac}{}^{bc}(\omega)$ is the Ricci curvature. The spinor indices are suppressed, and obey the contraction convention as defined in [69]. The field $\omega(e)_{abc}$ is the torsion-free Lorentz connection, while the field $\Psi_{ab}{}^{\gamma}_k$ is the gravitini field strengths

$$\Psi_{ab}{}^{\gamma}_k = 2e_a{}^m e_b{}^n \mathcal{D}_{[m}\psi_{n]}{}^{\gamma}_k , \qquad (2.2.41)$$

where $\mathcal{D}_m$ is the spin, dilatation, and $SU(2)_R$ covariant derivative

$$\mathcal{D}_m = \partial_m - \frac{1}{2}\omega_m{}^{bc}M_{bc} - b_m\mathbb{D} - \phi_m{}^{ij}J_{ij} , \quad \mathcal{D}_a = e_a{}^m\mathcal{D}_m , \qquad (2.2.42)$$

It is important to emphasise that in the traceless frame the composite connection $\phi_m{}^k$ does neither depend on χ and nor on D, however, the composite connection $\mathfrak{f}_a{}^b$ has a dependence on χ.

Note that the curvatures now satisfy

$$\begin{aligned}
R(P)_{ab}{}^c &= 2e_a{}^m e_b{}^n\mathcal{D}_{[m}e_{n]}{}^c + \frac{\mathrm{i}}{2}\psi_{[a\,j}\gamma^c\psi_{b]}{}^j , && (2.2.43a)\\
R(Q)_{abk} &= \frac{1}{2}\Psi_{abk} + \mathrm{i}\tilde{\gamma}_{[a}\phi_{b]k} + \frac{1}{24}T^-_{cde}\tilde{\gamma}^{cde}\gamma_{[a}\psi_{b]k} , && (2.2.43b)\\
R(\mathbb{D})_{ab} &= 2e_a{}^m e_b{}^n\partial_{[m}b_{n]} + 4\mathfrak{f}_{[ab]} + \psi_{[a}{}^i\phi_{b]i} + \frac{\mathrm{i}}{15}\psi_{[a}{}^j\gamma_{b]}\chi_j , && (2.2.43c)\\
R(M)_{ab}{}^{cd} &= R_{ab}{}^{cd}(\omega) + 8\delta^{[c}_{[a}\mathfrak{f}_{b]}{}^{d]} + \mathrm{i}\psi_{[a\,j}\gamma_{b]}R(Q)^{cd\,j} + 2\mathrm{i}\psi_{[a\,j}\gamma^{[c}R(Q)_{b]}{}^{d]\,j}\\
&\quad - \psi_{[a\,j}\gamma^{cd}\phi_{b]}{}^j - \frac{2\mathrm{i}}{15}\delta^{[c}_{[a}\psi_{b]\,j}\gamma^{d]}\chi^j + \frac{\mathrm{i}}{2}\psi_{[a}{}^j\gamma^e\psi_{b]\,j}T_e{}^{-cd} , && (2.2.43d)\\
R(J)_{ab}{}^{kl} &= R_{ab}{}^{kl}(\phi) + 4\psi_{[a}{}^{(k}\phi_{b]}{}^{l)} + \frac{4\mathrm{i}}{15}\psi_{[a}{}^{(k}\gamma_{b]}\chi^{l)} , && (2.2.43e)
\end{aligned}$$

where the curvature

$$\begin{aligned}
R_{ab}{}^{cd} &:= R_{ab}{}^{cd}(\omega) = e_a{}^m e_b{}^n\Big(2\partial_{[m}\omega_{n]}{}^{cd} - 2\omega_{[m}{}^{ce}\omega_{n]e}{}^d\Big) , && (2.2.44a)\\
R_{ab}{}^{kl} &:= R_{ab}{}^{kl}(\phi) = e_a{}^m e_b{}^n\Big(2\partial_{[m}\phi_{n]}{}^{kl} + 2\phi_{[m}{}^{p(k}\phi_{n]p}{}^{l)}\Big) . && (2.2.44b)
\end{aligned}$$

In components, the local superconformal transformations, except covariant general coordinate transformations, are identified by the following operator δ

$$\delta = \xi_i^\alpha Q_\alpha^i + \frac{1}{2}\lambda^{ab}M_{ab} + \lambda^{ij}J_{ij} + \lambda_{\mathbb{D}}\mathbb{D} + \lambda_a K^a + \eta^i_\alpha S^\alpha_i . \qquad (2.2.45)$$

The local superconformal transformations of the independent fields of the standard Weyl multiplet are given by

$$\delta e_m{}^a = -\mathrm{i}\,\xi_i\gamma^a\psi_m{}^i + \lambda^a{}_b e_m{}^b - \lambda_{\mathbb{D}}e_m{}^a , \qquad (2.2.46a)$$

$$
\begin{aligned}
\delta\psi_m{}_i^\alpha &= 2\mathcal{D}_m\xi_i^\alpha - \frac{1}{12}(\xi_i\gamma_m\tilde{\gamma}^{abc})^\alpha T^-_{abc} + \frac{1}{4}\lambda^{ab}(\psi_{mi}\gamma_{ab})^\alpha \\
&\quad -\lambda_i{}^j\,\psi_m{}_j^\alpha - \frac{1}{2}\lambda_{\mathbb{D}}\,\psi_m{}_i^\alpha - 2\mathrm{i}(\eta_i\tilde{\gamma}_m)^\alpha\,, && (2.2.46b)\\
\delta\phi_m{}^{kl} &= -4\xi^{(k}\phi_m{}^{l)} - \frac{4\mathrm{i}}{15}\xi^{(k}\gamma_m\chi^{l)} + \partial_m\lambda^{kl} - 2\phi_m{}^{(k}{}_i\lambda^{l)i} - 4\eta^{(k}\psi_m{}^{l)}\,, && (2.2.46c)\\
\delta b_m &= \frac{\mathrm{i}}{15}\,\xi_i\gamma_m\chi^i + \xi_i\phi_m{}^i + \partial_m\lambda_{\mathbb{D}} - \psi_m{}^i\eta_i - 2\lambda_{mK}\,, && (2.2.46d)\\
\delta T^-_{abc} &= -\frac{\mathrm{i}}{8}\xi^k\gamma^{ef}\gamma_{abc}R(Q)_{efk} - \frac{2\mathrm{i}}{15}\xi_i\gamma_{abc}\chi^i - 3\lambda^e{}_{[a}T^-_{bc]e} + \lambda_{\mathbb{D}}T^-_{abc}\,, && (2.2.46e)\\
\delta\chi^{\alpha i} &= \frac{1}{2}\xi^{\alpha i}D + \frac{3}{4}R(J)_{ab}{}^{ij}(\xi_j\gamma^{ab})^\alpha - \frac{1}{4}(\xi^i\gamma^a\tilde{\gamma}^{bcd})^\alpha\nabla_a T^-_{bcd} + \frac{1}{4}\lambda^{cd}(\chi^i\gamma_{cd})^\alpha \\
&\quad +\lambda^i{}_j\chi^{\alpha j} + \frac{3}{2}\lambda_{\mathbb{D}}\chi^{\alpha i} + \mathrm{i}(\eta^i\tilde{\gamma}^{abc})^\alpha T^-_{abc}\,, && (2.2.46f)\\
\delta D &= -2\mathrm{i}\,\xi_i\gamma^a\nabla_a\chi^i + 2\lambda_{\mathbb{D}}D - 4\chi^i\eta_i\,, && (2.2.46g)
\end{aligned}
$$

where

$$
\begin{aligned}
\nabla_d T^-_{abc} &= \mathcal{D}_a T^-_{abc} + \frac{\mathrm{i}}{15}\psi_{dk}\gamma_{abc}\chi^k + \frac{\mathrm{i}}{2}\psi_{dk}(\gamma_{abc})_{\alpha\beta}X^{k\alpha\beta}\,, && (2.2.47a)\\
\nabla_a\chi^{\beta j} &= \mathcal{D}_a\chi^{\beta j} - \frac{3}{8}(\psi_{ai}\gamma^{bc})^\beta R(J)_{bc}{}^{ij} - \frac{1}{8}(\psi_a{}^j\gamma^e\tilde{\gamma}^{bcd})^\beta\nabla_e T^-_{bcd} \\
&\quad +\frac{1}{4}\psi_a{}^{\beta j}D + \frac{\mathrm{i}}{2}T^-_{bcd}(\phi_a{}^j\tilde{\gamma}^{bcd})^\beta\,, && (2.2.47b)
\end{aligned}
$$

and we have defined $X^{k\alpha\beta}_\gamma := -\frac{1}{8}(\gamma^{ab})_\gamma{}^\alpha R(Q)_{ab}{}^{\beta k}$. Note that the transformations (2.2.46) form an algebra that closes off-shell on a local extension of $\mathrm{O}Sp(6,2|1)$. To conclude this subsection, for convenience, we include Table 2.11 which summarises the non-trivial dilatation weights of the fields and local gauge parameters of the standard Weyl multiplet.

	$e_m{}^a$	ψ_{mi}, ξ_i	$\phi_m{}^i$, η^i	$\mathfrak{f}_{mc}$	T^-_{abc}	χ^i	D
$\mathbb{D}$	-1	$-1/2$	$1/2$	1	1	$3/2$	2

Table 2.11: Summary of the non-trivial dilatation weights in the standard Weyl multiplet.

2.2.3 Conformal superspace

Conformal superspace is conceptually similar to superconformal tensor calculus (see [37] for a review and references) and is associated with the off-shell gauging of the superconformal algebra in superspace. This approach combines both component and superfield approaches in supergravity, providing compact and simple building blocks for locally supersymmetric invariants [25, 39, 40]. Despite its advantages, converting these invariants to component forms remains crucial for many applications. For example, analyzing the physical sector of a matter-coupled Poincaré supergravity constructed in conformal superspace involves generating locally superconformal actions in superspace, reducing them to component multiplets, removing non-physical symmetries through gauge fixing, and finally integrating out auxiliary fields. This process allows to derive physically relevant results from the initial superspace formulation.

By definition, conformal superspace is a supermanifold $\mathcal{M}^{D|\cdots}$ having the superconformal group as its local structure group. To implement the gauging, one introduces covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha)$, which have the following form:

$$\nabla_A = E_A - \omega_A{}^{\underline{b}} X_{\underline{b}} \;=\; E_A - \frac{1}{2}\Omega_A{}^{ab} M_{ab} - \Phi_A{}^R J_R - B_A \mathbb{D} - \mathfrak{F}_A{}^B K_B \,, \tag{2.2.48a}$$

$$\;=\; E_A - \frac{1}{2}\Omega_A{}^{ab} M_{ab} - \Phi_A{}^R J_R - B_A \mathbb{D} - \mathfrak{F}_A{}^\alpha S_\alpha - \mathfrak{F}_A{}^a K_a \,. \tag{2.2.48b}$$

Here $E_A = E_A{}^M \partial_M$ is the inverse super-vielbein, M_{ab} are the Lorentz generators, J_R are generators of the R-symmetry group, $\mathbb{D}$ is the dilatation generator, and $K_A = (K_a, S_\alpha)$ are the special superconformal generators. The super-vielbein one-form is $E^A = \mathrm{d}z^M E_M{}^A$ with $E_M{}^A E_A{}^N = \delta_M^N$ and $E_A{}^M E_M{}^B = \delta_A^B$. Associated with each generator $X_{\underline{a}} = (M_{ab}, J_R, \mathbb{D}, S_\alpha, K_a)$ is the connection super one-form $\omega^{\underline{a}} = (\Omega^{ab}, \Phi^R, B, \mathfrak{F}^\alpha, \mathfrak{F}^a) = \mathrm{d}z^M \omega_M{}^{\underline{a}} = E^A \omega_A{}^{\underline{a}}$ and gauge parameter $g^{\underline{a}} = (\lambda^{ab}, \lambda^R, \lambda_{\mathbb{D}}, \eta, \lambda^a)$. The local gauge transformation is given by the operator:

$$\delta = g^{\underline{a}} X_{\underline{a}} = \frac{1}{2}\lambda^{ab} M_{ab} + \lambda^R J_R + \lambda_{\mathbb{D}} \mathbb{D} + \lambda_a K^a + \eta_\alpha S^\alpha \,. \tag{2.2.49}$$

Note that as a consequence of gauging the algebra in superspace, we had to uplift both the translation generator and the supersymmetry generator to covariant derivative. The reason behind this is as follows: A *covariant* superfield Ψ is defined by the property that it transforms under gauge transformations $\mathcal{H}$ without any derivative on the parameter $g^{\underline{a}}$,

$$\delta_{\mathcal{H}} \Psi = g^{\underline{a}} X_{\underline{a}} \Psi. \tag{2.2.50}$$

Because the parameter $g^{\underline{a}}$ is a local superfield, both $\partial_a \Psi$ and $Q_\alpha \Psi$ do not transform covariantly. We must introduce instead the *covariant derivative*

$$\nabla_A \Psi \equiv E_A{}^M \partial_M \Psi - E_A{}^M \omega_M{}^{\underline{a}} X_{\underline{a}} \Psi. \tag{2.2.51}$$

The superfield $\nabla_A \Psi$ is also covariant; one can show that it transforms as

$$\delta_{\mathcal{H}}(\nabla_A \Psi) = g^{\underline{b}} \nabla_A X_{\underline{b}} \Psi - g^{\underline{b}} f_{\underline{b}A}{}^C \nabla_C \Psi - g^{\underline{b}} f_{\underline{b}A}{}^{\underline{c}} X_{\underline{c}} \Psi \tag{2.2.52}$$

without any derivatives of g.

The algebra of covariant derivatives

$$[\nabla_A, \nabla_B\} = -\mathscr{T}_{AB}{}^C \nabla_C - \frac{1}{2}\mathscr{R}(M)_{AB}{}^{cd} M_{cd} - \mathscr{R}(J)_{AB}{}^R J_R - \mathscr{R}(\mathbb{D})_{AB} \mathbb{D} - \mathscr{R}(S)_{AB}{}^{\gamma k} S_{\gamma k} - \mathscr{R}(K)_{AB}{}^c K_c \,, \tag{2.2.53}$$

is constrained to be expressed in terms of a single primary superfield, the super-Weyl tensor (or by Cotton tensor in $3d$). In equation (2.2.53), $\mathscr{T}_{AB}{}^C$ represents the torsion, while $\mathscr{R}(M)_{AB}{}^{cd}$, $\mathscr{R}(J)_{AB}{}^R$, $\mathscr{R}(\mathbb{D})_{AB}$, $\mathscr{R}(S)_{AB}{}^{\gamma k}$, and $\mathscr{R}(K)_{AB}{}^c$ are the curvatures associated with Lorentz, R-symmetry, dilatation, S-supersymmetry, and special conformal boosts, respectively. These curvatures can all be expressed in terms of the super-Weyl tensor.

Conformal Superspace	Super-Weyl Tensor Superfield	Dimension
$4D, N=1$	$W_{\alpha\beta\gamma}$	$3/2$
$4D, N=2$	$W_{\alpha\beta}$	1
$4D, N=3$	W_{α}	$1/2$
$3D, N=1$	$W_{\alpha\beta\gamma}$	$5/2$
$3D, N=2$	$W_{\alpha\beta}$	2
$5D, N=1$	$W_{\alpha\beta}$	1
$6D, N=(1,0)$	$W_{\alpha\beta}$	1

Table 2.12: Summary of known conformal superspaces, their corresponding super-Weyl tensor superfields, and the conformal dimensions of these super-Weyl tensors .

By projecting, it is easy to transition between the conformal superspace approach and the superconformal tensor calculus, as the component and superspace approaches are equivalent. This connection between the two approaches can be seen by identifying the various component fields of the standard Weyl multiplet from superspace. The one-form connections are related to the lowest component of the corresponding superform connection. For example, $e^a = \mathrm{d}x^m e_m{}^a = E^a\|$ and $\psi^i_\alpha = \mathrm{d}x^m \psi_m{}^i_\alpha = 2E^i_\alpha\|$, where the double-bar denotes setting $\theta = \mathrm{d}\theta = 0$. The other connections are mapped similarly. The auxiliary matter fields introduced by hand in the superconformal tensor calculus correspond to the super-Weyl tensor and its descendants in the superspace approach. However, the super-Weyl tensor also contains additional descendants that do not have counterparts in the superconformal tensor calculus. One way to eliminate these extra fields is by imposing curvature constraints, which are the same constraints that lead to the elimination of spin connections, supersymmetry connections, and special conformal connections.

The necessity of these curvature constraints is further highlighted by the requirement that conformal superspace must be reduced to Poincaré superspace upon breaking conformal symmetry. Without these constraints, the number of independent fields would include both the one-form connection superfields and the fields of the super-Weyl tensor, resulting in an excessive number of degrees of freedom. The following table 2.12 summarizes all known conformal superspaces [25–33], their corresponding super-Weyl tensors, and the conformal dimensions of these super-Weyl tensors. Similar conformal space construction exists for 3D N-extended [30] and 2D (p,q)-extended [142].

$4D$ $N=2$ **conformal superspace**

The $4D$ $N=2$ conformal superspace $\mathcal{M}^{4|8}$ is parametrised by local bosonic (x^m) and fermionic $(\theta^\mu_i, \bar{\theta}^i_{\dot{\mu}})$ coordinates $z^M = (x^m, \theta^\mu_i, \bar{\theta}^i_{\dot{\mu}})$, where $m = 0,1,2,3$, $\mu = 1,2$, $\dot{\mu} = 1,2$, and $i = 1,2$. The Grassmann variables θ^μ_i and $\bar{\theta}^i_{\dot{\mu}}$ are related to each other by complex conjugation: $\overline{\theta^\mu_i} = \bar{\theta}^{\dot{\mu}i}$. To gauge the superconformal algebra in superspace, one introduces covariant derivatives $\nabla_A = (\nabla_a, \nabla^i_\alpha, \bar{\nabla}^{\dot{\alpha}}_i)$ which have the form

$$\nabla_A = E_A - \omega_A{}^{\underline{b}} X_{\underline{b}} \;=\; E_A - \frac{1}{2}\Omega_A{}^{ab} M_{ab} - \mathrm{i}\Phi_A Y - \Phi_A{}^{ij} J_{ij} - B_A \mathbb{D} - \mathfrak{F}_A{}^B K_B \,. \tag{2.2.54}$$

Here $E_A = E_A{}^M \partial_M$ is the inverse super-vielbein, with $\partial_M = \partial/\partial z^M$, M_{ab} are the Lorentz generators, Y is the generator of the chiral rotation group $\mathrm{U}(1)_R$, J_{ij} are generators of the $\mathrm{S}U(2)_R$ R-symmetry group, $\mathbb{D}$ is the dilatation generator, and $K_A = (K_a, S_i^\alpha, \bar{S}^i_{\dot\alpha})$ are the special superconformal generators[1]. The super-vielbein one-form is $E^A = \mathrm{d}z^M E_M{}^A$ with $E_M{}^A E_A{}^N = \delta_M^N$ and $E_A{}^M E_M{}^B = \delta_A^B$. Associated with each generator $X_{\underline{a}} = (M_{ab}, Y, J_{ij}, \mathbb{D}, K^a, S_i^\alpha, \bar{S}^i_{\dot\alpha})$ is the connection super one-form $\omega^{\underline{a}} = (\Omega^{ab}, \Phi, \Phi^{ij}, B, \mathfrak{F}_A) = (\Omega^{ab}, \Phi, \Phi^{ij}, B, \mathfrak{F}_a, \mathfrak{F}^i_\alpha, \bar{\mathfrak{F}}_i^{\dot\alpha}) = \mathrm{d}z^M \omega_M{}^{\underline{a}} = E^A \omega_A{}^{\underline{a}}$. The conventions we use here differ in numerous ways from those used originally in [26]. For the most part, they follow the conventions of [36] and [127, 128] and are summarized in appendix A.

The algebra of covariant derivatives

$$\begin{aligned}[\nabla_A, \nabla_B\} = &-\mathscr{T}_{AB}{}^C \nabla_C - \frac{1}{2} \mathscr{R}(M)_{AB}{}^{cd} M_{cd} - \mathscr{R}(J)_{AB}{}^{kl} J_{kl} - \mathscr{R}(\mathbb{D})_{AB} \mathbb{D} \\ &- \mathrm{i} \mathscr{R}(Y)_{AB} \mathbb{Y} - \mathscr{R}(S)_{AB}{}^i{}_\alpha S_i^\alpha - \mathscr{R}(\bar{S})_{ABi}{}^{\dot\alpha} S^i_{\dot\alpha} - \mathscr{R}(K)_{AB}{}^c K_c \ ,\end{aligned} \tag{2.2.55}$$

is constrained to be expressed in terms of a single primary superfield, the super-Weyl tensor $(W_{\alpha\beta}, \overline{W}^{\dot\alpha\dot\beta})$, which has the following properties

$$W_{\alpha\beta} = W_{\beta\alpha} \ , \quad K_A W_{\alpha\beta} = 0 \ , \quad \mathbb{D} W_{\alpha\beta} = W_{\alpha\beta} \ , \quad Y W_{\alpha\beta} = -2 W_{\alpha\beta} \ , \tag{2.2.56}$$

$$\overline{W}^{\dot\alpha\dot\beta} = \overline{W}^{\dot\beta\dot\alpha} \ , \quad K_A \overline{W}^{\dot\alpha\dot\beta} = 0 \ , \quad \mathbb{D} \overline{W}^{\dot\alpha\dot\beta} = \overline{W}^{\dot\alpha\dot\beta} \ , \quad Y \overline{W}^{\dot\alpha\dot\beta} = 2 \overline{W}^{\dot\alpha\dot\beta} \ . \tag{2.2.57}$$

and obey the additional constraints

$$\bar\nabla_i^{\dot\alpha} W_{\beta\gamma} = 0 \ , \qquad \nabla_{\alpha\beta} W^{\alpha\beta} = \bar\nabla^{\dot\alpha\dot\beta} \overline{W}_{\dot\alpha\dot\beta} \ , \tag{2.2.58}$$

where we introduce the notation

$$\nabla_{\alpha\beta} := \nabla^k_{(\alpha} \nabla_{\beta)k} \ , \quad \bar\nabla^{\dot\alpha\dot\beta} := \bar\nabla_k^{(\dot\alpha} \bar\nabla^{\dot\beta)k} \ . \tag{2.2.59}$$

In (2.2.55), $\mathscr{T}_{AB}{}^C$ is the torsion, while $\mathscr{R}(M)_{AB}{}^{cd}$, $\mathscr{R}(J)_{AB}{}^{kl}$, $\mathscr{R}(\mathbb{D})_{AB}$, $\mathscr{R}(Y)_{AB}$, $\mathscr{R}(S)_{AB}{}^i{}_\alpha$, $\mathscr{R}(\bar{S})_{ABi}{}^{\dot\alpha}$, and $\mathscr{R}(K)_{AB}{}^c$ are the curvatures associated with Lorentz, $SU(2)_R$, dilatation, $U(1)_R$, S-supersymmetry, and special conformal boosts, respectively. The relevant algebra of covariant derivatives (2.2.55) (including the explicit expressions for the torsion and curvature tensors) are given in Appendix A.

Let us introduce the dimension-3/2 superfields

$$W_{\alpha\beta\gamma}{}^k := \nabla^k_{(\alpha} W_{\beta\gamma)} \ , \quad \Sigma^{\alpha i} := \frac{1}{3} \nabla^i_\beta W^{\alpha\beta} \ , \tag{2.2.60a}$$

$$\overline{W}^{\dot\alpha\dot\beta\dot\gamma}{}_k := \bar\nabla_k^{(\dot\alpha} \overline{W}^{\dot\beta\dot\gamma)} \ , \quad \bar\Sigma_{\dot\alpha i} := -\frac{1}{3} \bar\nabla_i^{\dot\beta} \overline{W}_{\dot\alpha\dot\beta} \ , \tag{2.2.60b}$$

dimension-2 descendant superfields

$$W_{\alpha\beta\gamma\delta} := \nabla^k_{(\alpha} W_{\beta\gamma\delta)k} \ , \quad \Sigma_{\alpha\beta}{}^{ij} := \nabla^{(i}_{(\alpha} \Sigma^{j)}_{\beta)} \ , \quad \Sigma_{\alpha\beta} := \nabla^i_{(\alpha} \Sigma_{\beta)i} \ , \tag{2.2.60c}$$

$$\overline{W}^{\dot\alpha\dot\beta\dot\gamma\dot\delta} := \bar\nabla_k^{(\dot\alpha} \overline{W}^{\dot\beta\dot\gamma\dot\delta)k} \ , \quad \bar\Sigma^{\dot\alpha\dot\beta}{}_{ij} := \bar\nabla^{(\dot\alpha}_{(i} \bar\Sigma^{\dot\beta)}_{j)} \ , \quad \bar\Sigma^{\dot\alpha\dot\beta} := \nabla_i^{(\dot\alpha} \bar\Sigma^{\dot\beta)i} \ , \tag{2.2.60d}$$

[1]Following common usage, we will refer to K^a as the special conformal generator and S_i^α as the S-supersymmetry generator.

$$D := \frac{1}{12}\nabla^{\alpha\beta}W_{\alpha\beta} = \frac{1}{12}\bar{\nabla}_{\dot{\alpha}\dot{\beta}}\overline{W}^{\dot{\alpha}\dot{\beta}} = \frac{1}{4}\nabla^k_\alpha\Sigma^\alpha_k = -\frac{1}{4}\bar{\nabla}^{\dot{\alpha}}_k\bar{\Sigma}^k_{\dot{\alpha}} , \tag{2.2.60e}$$

and the following dimension-5/2 descendant superfields

$$\Sigma_{\alpha\beta\gamma}{}^k = \nabla^k_{(\alpha}\Sigma_{\beta\gamma)} , \quad \bar{\Sigma}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_k = \bar{\nabla}^{(\dot{\alpha}}_k\Sigma^{\dot{\beta}\dot{\gamma})} . \tag{2.2.60f}$$

It can be checked that only the superfields (2.2.60) and their vector derivatives appear upon taking successive spinor derivatives of $W_{\alpha\beta}$. The independent descendant superfields of $(W_{\alpha\beta}, W^{\dot{\alpha}\dot{\beta}})$ are all annihilated by K_a. The independent descendant superfields of $(W_{\alpha\beta}, W^{\dot{\alpha}\dot{\beta}})$ are all annihilated by K_a. However, under S-supersymmetry, they transform nontrivially, as given in the appendix A.

The gauge group of conformal supergravity is denoted by $\mathcal{G}$. It is generated by *c*ovariant general coordinate transformations, δ_{cgct}, associated with a local superdiffeomorphism parameter ξ^A and *s*tandard superconformal transformations, $\delta_{\mathcal{H}}$, associated with the local superfield parameters: the dilatation σ, Lorentz $\Lambda^{ab} = -\Lambda^{ba}$, $SU(2)_R$ $\Lambda^{ij} = \Lambda^{ji}$, $U(1)_R$ Λ, and special conformal (bosonic and fermionic) transformations $\Lambda^A = (\eta^i_\alpha, \bar{\eta}^{\dot{\alpha}}_i, \Lambda^a_K)$. The covariant derivatives transform as

$$\delta_{\mathcal{G}}\nabla_A = [\mathcal{K}, \nabla_A] , \tag{2.2.61a}$$

with

$$\mathcal{K} = \xi^C\nabla_C + \frac{1}{2}\Lambda^{ab}M_{ab} + \Lambda^{ij}J_{ij} + \sigma\mathbb{D} + \Lambda Y + \Lambda^A K_A . \tag{2.2.61b}$$

A covariant (or tensor) superfield U transforms as

$$\delta_{\mathcal{G}}U = (\delta_{cgct} + \delta_{\mathcal{H}})U = \mathcal{K}U . \tag{2.2.62}$$

The superfield U is said to be *superconformal primary* of dimension Δ and $\mathrm{U}(1)_R$ charge q_{R} if $K_A U = 0$ (it suffices to require that $S^\alpha_i U = \bar{S}^i_{\dot{\alpha}} U = 0$), $\mathbb{D}U = \Delta U$, and $YU = q_{\mathrm{R}}U$.

Let us now explain how to systematically obtain various component fields of the standard Weyl multiplet from the superspace geometry described above. The vielbein ($e_m{}^a$) and gravitini ($\psi_m{}^\alpha_i$, $\bar{\psi}_m{}^i_{\dot{\alpha}}$) appear as the $\theta = 0$ projections of the coefficients of $\mathrm{d}x^m$ in the supervielbein E^A one-form,

$$e^a = \mathrm{d}x^m e_m{}^a = E^a\| , \quad \psi^\alpha_i = \mathrm{d}x^m\psi_m{}^\alpha_i = 2E^\alpha_i\| , \quad \bar{\psi}^i_{\dot{\alpha}} = \mathrm{d}x^m\bar{\psi}_m{}^i_{\dot{\alpha}} = 2E^i_{\dot{\alpha}}\| . \tag{2.2.63}$$

Here we have defined the double bar projection of a superform as $\Omega\| \equiv \Omega\|_{\theta=\mathrm{d}\theta=0}$. On the other hand, a single bar next to a superfield denotes the usual bar projection $X| \equiv X|_{\theta=0}$. The remaining component one-forms are defined as

$$A := \Phi\| , \quad \phi^{kl} := \Phi^{kl}\| , \quad b := B\| , \quad \omega^{cd} := \Omega^{cd}\| , \tag{2.2.64}$$

$$\phi^k_\gamma := 2\mathfrak{F}^k_\gamma\| , \quad \bar{\phi}^{\dot{\gamma}}_k := 2\mathfrak{F}^{\dot{\gamma}}_k\| , \quad \mathfrak{f}_c := \mathfrak{F}_c\| . \tag{2.2.65}$$

The covariant matter fields W_{ab}, D, and $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha}i})$ (we denote these component fields with the same symbols as the super-Weyl superfield and its descendants) arise as some of the components of the multiplet described by the super-Weyl tensor. In particular, it holds that

$$W_{ab}(x) := W_{ab}(z)| , \qquad D = \frac{1}{12}\nabla^{\alpha\beta}W_{\alpha\beta}| = \frac{1}{12}\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\alpha}\dot{\beta}}| , \tag{2.2.66a}$$

$$\Sigma^{\alpha i} = \frac{1}{3}\nabla^i_\beta W^{\alpha\beta}| \ , \quad \bar{\Sigma}_{\dot{\alpha} i} = -\frac{1}{3}\bar{\nabla}^{\dot{\beta}}_i \bar{W}_{\dot{\alpha}\dot{\beta}}| \ . \tag{2.2.66b}$$

The other components of the super Weyl tensor are given by $W_{\alpha\beta\gamma}{}^i := W_{\alpha\beta\gamma}{}^i|$, $\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}|$, $\Sigma_{\alpha\beta}{}^{ij} := \Sigma_{\alpha\beta}{}^{ij}|$, $W_{\alpha\beta\gamma\delta} := W_{\alpha\beta\gamma\delta}|$, and $\Sigma_{\alpha\beta\gamma}{}^k := \Sigma_{\alpha\beta\gamma}{}^k|$ along with their complex conjugates.[2] The local superconformal transformations of the gauge fields listed above can be straightforwardly derived by taking the $\theta = 0$ projection of the superspace transformations (2.2.24). The transformations of W_{ab}, D, and $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i})$ can be obtained by applying the transformation rule for covariant superfields, eq. (2.2.24), and the definition of the descendant fields in eq. (2.2.66).

By taking the double bar projection of the superspace covariant derivative one-form ∇, eq. (2.2.54), $\nabla = E^A\nabla_A$, the component vector covariant derivative ∇_a is defined to coincide with the projection of the superspace derivative $\nabla_a|$,

$$\begin{aligned} e_m{}^a\nabla_a =& \partial_m - \frac{1}{2}\psi_m{}^\alpha_i \nabla^i_\alpha| - \frac{1}{2}\bar{\psi}_m{}^i_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}_i| - \frac{1}{2}\omega_m{}^{ab}M_{ab} - b_m\mathbb{D} \\ & - A_m Y - \phi_m{}^{ij}J_{ij} - \frac{1}{2}\phi_m{}^{\alpha i}S_{\alpha i} - \frac{1}{2}\bar{\phi}_{m\dot{\alpha} i}\bar{S}^{\dot{\alpha} i} - \mathfrak{f}_m{}^a K_a \ . \end{aligned} \tag{2.2.67}$$

Here, the projected spinor covariant derivative $\nabla^i_\alpha|$ corresponds to the generator of Q-supersymmetry. It is defined such that if $\mathcal{U} = U|$, then $Q^i_\alpha\mathcal{U} := \nabla^i_\alpha|\mathcal{U} := (\nabla^i_\alpha U)|$. For the other generators, e.g., $M_{ab}\mathcal{U} = (M_{ab}U)|$, there is no ambiguity in identifying the bar projection; hence, local diffeomorphisms, Q-supersymmetry transformations, and so forth descend naturally from their corresponding rule in superspace. With these reduction rules, the algebra of component covariant derivatives acting on a covariant field is also completely determined by the geometry of conformal superspace. All the component torsions and curvatures are simply the $\theta = 0$ projections of the superspace ones.

The component supercovariant curvature tensors are given by

$$\begin{aligned} [\nabla_a, \nabla_b] = & -R(P)_{ab}{}^c\nabla_c - R(Q)_{ab}{}^\alpha_i\nabla^i_\alpha| - R(\bar{Q})_{ab}{}^i_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}_i| - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} - R(J)_{ab}{}^{ij}J_{ij} \\ & - R(\mathbb{D})_{ab}\mathbb{D} - R(Y)_{ab}Y - R(S)_{ab}{}^{\gamma k}S_{\gamma k} - R(\bar{S})_{ab\dot{\lambda} k}\bar{S}^{\dot{\lambda} k} - R(K)_{ab}{}^c K_c \ . \end{aligned} \tag{2.2.68}$$

We have introduced $R(P)_{ab}{}^c = \mathscr{T}_{ab}{}^c|$ and $R(Q)_{ab}{}^i_\alpha = \mathscr{T}_{ab}{}^i_\alpha|$. $R(M)_{ab}{}^{cd}$, $R(J)_{ab}{}^{ij}$, $R(\mathbb{D})_{ab}$, $R(S)_{ab}{}^{\gamma k}$, and $R(K)_{ab}{}^c$ coincide with the lowest components of the corresponding superspace curvature tensors.

5D $N = 1$ conformal superspace

The $N = 1$ conformal superspace is parametrised by local bosonic (x^m) and fermionic (θ_i) coordinates $z^M = (x^m, \theta^\mu_i)$, where $m = 0,1,2,3,4$, $\mu = 1,\cdots,4$, and $i = 1,2$. To gauge the superconformal algebra, one introduces covariant derivatives $\nabla_A = (\nabla_a, \nabla^i_\alpha)$ which have the form

$$\begin{aligned} \nabla_A = E_A - \omega_A{}^{\underline{b}}X_{\underline{b}} &= E_A - \frac{1}{2}\Omega_A{}^{ab}M_{ab} - \Phi_A{}^{ij}J_{ij} - B_A\mathbb{D} - \mathfrak{F}_A{}^B K_B \ , && (2.2.69a) \\ &= E_A - \frac{1}{2}\Omega_A{}^{ab}M_{ab} - \Phi_A{}^{ij}J_{ij} - B_A\mathbb{D} - \mathfrak{F}_A{}^{\alpha i}S_{\alpha i} - \mathfrak{F}_A{}^a K_a \ . && (2.2.69b) \end{aligned}$$

[2]In the case of $W_{\alpha\beta}$, $W_{ab}{}^i_\alpha$, $\Phi_{ab}{}^{ij}$, and $W_{\alpha\beta\gamma\delta}$, unless specified, it should be clear from the context if we refer to the superfields or their component projections.

Here $E_A = E_A{}^M \partial_M$ is the inverse super-vielbein, M_{ab} are the Lorentz generators, J_{ij} are generators of the $SU(2)_R$ R-symmetry group, $\mathbb{D}$ is the dilatation generator, and $K_A = (K_a, S_{\alpha i})$ are the special superconformal generators. The super-vielbein one-form is $E^A = \mathrm{d}z^M E_M{}^A$ with $E_M{}^A E_A{}^N = \delta_M^N$ and $E_A{}^M E_M{}^B = \delta_A^B$. Associated with each generator $X_{\underline{a}} = (M_{ab}, J_{ij}, \mathbb{D}, S_{\alpha i}, K_a)$ is the connection super one-form $\omega^{\underline{a}} = (\Omega^{ab}, \Phi^{ij}, B, \mathfrak{F}^{\alpha i}, \mathfrak{F}^a) = \mathrm{d}z^M \omega_M{}^{\underline{a}} = E^A \omega_A{}^{\underline{a}}$.

The algebra of covariant derivatives

$$
\begin{aligned}
[\nabla_A, \nabla_B\} = &-\mathscr{T}_{AB}{}^C \nabla_C - \frac{1}{2}\mathscr{R}(M)_{AB}{}^{cd} M_{cd} - \mathscr{R}(J)_{AB}{}^{kl} J_{kl} \\
&- \mathscr{R}(\mathbb{D})_{AB}\mathbb{D} - \mathscr{R}(S)_{AB}{}^{\gamma k} S_{\gamma k} - \mathscr{R}(K)_{AB}{}^c K_c \ , \qquad (2.2.70)
\end{aligned}
$$

is constrained to be expressed in terms of a single primary superfield, the super-Weyl tensor $W_{\alpha\beta}$, which has the following properties

$$
W_{\alpha\beta} = W_{\beta\alpha} \ , \quad K_A W_{\alpha\beta} = 0 \ , \quad \mathbb{D} W_{\alpha\beta} = W_{\alpha\beta} \ , \qquad (2.2.71)
$$

and satisfies the Bianchi identity

$$
\nabla^k_\gamma W_{\alpha\beta} = \nabla^k_{(\alpha} W_{\beta\gamma)} + \frac{2}{5}\varepsilon_{\gamma(\alpha}\nabla^{\delta k} W_{\beta)\delta} \ . \qquad (2.2.72)
$$

In (2.2.70), $\mathscr{T}_{AB}{}^C$ is the torsion, while $\mathscr{R}(M)_{AB}{}^{cd}$, $\mathscr{R}(J)_{AB}{}^{kl}$, $\mathscr{R}(\mathbb{D})_{AB}$, $\mathscr{R}(S)_{AB}{}^{\gamma k}$, and $\mathscr{R}(K)_{AB}{}^c$ are the curvatures associated with Lorentz, $SU(2)_R$, dilatation, S-supersymmetry, and special conformal boosts, respectively.

In this paper we make use of the "traceless" frame conventional constraints for the conformal superspace algebra employed in appendix C of [29] as well as in [3]. The full algebra of covariant derivatives (2.2.70) (including the explicit expressions for the torsion and curvature tensors) are given in appendix B of our paper.

Let us introduce the dimension-3/2 superfields

$$
W_{\alpha\beta\gamma}{}^k := \nabla^k_{(\alpha} W_{\beta\gamma)} \ , \quad X^i_\alpha := \frac{2}{5}\nabla^{\beta i} W_{\beta\alpha} \ , \qquad (2.2.73a)
$$

and the following dimension-2 descendant superfields

$$
W_{\alpha\beta\gamma\delta} := \nabla^k_{(\alpha} W_{\beta\gamma\delta)k} \ , \quad X_{\alpha\beta}{}^{ij} := \nabla^{(i}_{(\alpha} X^{j)}_{\beta)} \ , \quad Y := \mathrm{i}\nabla^{\gamma k} X_{\gamma k} \ . \qquad (2.2.73b)
$$

It can be checked that only the superfields (2.2.73) and their vector derivatives appear upon taking successive spinor derivatives of $W_{\alpha\beta}$. Specific relations that connect the various descendant fields which we will need later are given below:

$$
\begin{aligned}
\nabla^k_\gamma W_{\alpha\beta} &= W_{\alpha\beta\gamma}{}^k + \varepsilon_{\gamma(\alpha} X^k_{\beta)} \ , &(2.2.74a)\\
\nabla^i_\alpha X^j_\beta &= X_{\alpha\beta}{}^{ij} + \frac{\mathrm{i}}{8}\varepsilon^{ij}\varepsilon_{\alpha\beta} Y - \frac{3\mathrm{i}}{2}\varepsilon^{ij}(\Gamma^a)_\alpha{}^\rho \nabla_a W_{\beta\rho} - 2\mathrm{i}\varepsilon^{ij} W_\alpha{}^\rho W_{\beta\rho} \\
&\quad + \frac{\mathrm{i}}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta} W^{\gamma\delta} W_{\gamma\delta} - \frac{\mathrm{i}}{2}\varepsilon^{ij}(\Gamma^a)_\beta{}^\rho \nabla_a W_{\alpha\rho} \ , &(2.2.74b)\\
\nabla^i_\alpha W_{\beta\gamma\lambda}{}^j &= -\frac{1}{2}\varepsilon^{ij}\Big(W_{\alpha\beta\gamma\lambda} + 3\mathrm{i}(\Gamma_a)_{\alpha(\beta}\nabla^a W_{\gamma\lambda)} + 3\mathrm{i}\varepsilon_{\alpha(\beta}(\Gamma_a)_\gamma{}^\tau \nabla^a W_{\lambda)\tau}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3}{2}\varepsilon_{\alpha(\beta}X_{\gamma\lambda)}{}^{ij} \ , && (2.2.74c)\\
\nabla^i_\alpha W_{\beta\gamma\lambda\rho} &= -4\mathrm{i}(\Gamma_a)_{\alpha(\beta}\nabla^a W_{\gamma\lambda\rho)}{}^i - 6\mathrm{i}W_{\alpha(\beta\gamma}{}^i W_{\lambda\rho)} + 6\mathrm{i}W_{\alpha(\beta}W_{\gamma\lambda\rho)}{}^i \\
&\quad +6\mathrm{i}\varepsilon_{\alpha(\beta}\Big(W_{\gamma\lambda}X^i_{\rho)} - 2(\Gamma_a)_\gamma{}^\tau\nabla^a W_{\lambda\rho)\tau}{}^i - W_\gamma{}^\tau W_{\lambda\rho)\tau}{}^i\Big) \ , && (2.2.74d)\\
\nabla^i_\alpha X_{\beta\gamma}{}^{jk} &= \mathrm{i}\varepsilon^{i(j}\Big(-3W_{(\beta}{}^\lambda W_{\gamma)\alpha\lambda}{}^{k)} - \varepsilon_{\alpha(\beta}W^{\rho\tau}W_{\gamma)\rho\tau}{}^{k)} - W_{\alpha\lambda}W_{\beta\gamma}{}^{k)\lambda} - \frac{3}{2}W_{\beta\gamma}X^{k)}_\alpha \\
&\quad +\frac{1}{2}W_{\alpha(\beta}X^{k)}_{\gamma)} + \frac{3}{2}\varepsilon_{\alpha(\beta}W_{\gamma)\lambda}X^{k)\lambda} + 2(\Gamma^a)_\alpha{}^\rho\nabla_a W_{\beta\gamma\rho}{}^{k)} \\
&\quad +2(\Gamma^a)_{(\beta}{}^\rho\nabla_a W_{\gamma)\alpha\rho}{}^{k)} - (\Gamma^a)_{\alpha(\beta}\nabla_a X^{k)}_{\gamma)} + \varepsilon_{\alpha(\beta}(\Gamma^a)_{\gamma)\lambda}\nabla_a X^{k)\lambda}\Big) \ , && (2.2.74e)\\
\nabla^i_\alpha Y &= 8(\Gamma^a)_\alpha{}^\beta\nabla_a X^i_\beta + 8W_\alpha{}^\beta X^i_\beta \ . && (2.2.74f)
\end{aligned}
$$

As implied by (2.2.72), the dimension-2 superfields $X_{\alpha\beta}{}^{ij}$ and $W_{\alpha\beta\gamma\delta}$ obey the following Bianchi identities in the traceless frame:

$$
\begin{aligned}
\nabla_{(\alpha}{}^\gamma X_{\beta)\gamma}{}^{ij} &= -\frac{1}{2}X^{\gamma(i}W_{\alpha\beta\gamma}{}^{j)} \ , && (2.2.75a)\\
\nabla_{(\alpha}{}^\lambda W_{\beta\gamma\tau)\lambda} &= 3\mathrm{i}\nabla_{(\alpha}{}^\lambda\Big(W_{\beta\gamma}W_{\tau)\lambda}\Big) \ . && (2.2.75b)
\end{aligned}
$$

The independent descendant superfields of $W_{\alpha\beta}$, specifically $W_{\alpha\beta\gamma}{}^k, X^i_\alpha, W_{\alpha\beta\gamma\delta}, X_{\alpha\beta}{}^{ij}, Y$, are all annihilated by K_a. However, under S-supersymmetry, they transform nontrivially, as given in the appendix B. We also stress that in the paper, we will often use the following notation:

$$
\Phi_{ab}{}^{ij} := \mathscr{R}(J)_{ab}{}^{ij} = -\frac{3\mathrm{i}}{4}X_{ab}{}^{ij} \ . \qquad (2.2.76)
$$

The gauge group of conformal supergravity is denoted by $\mathcal{G}$. It is generated by *c*ovariant general coordinate transformations, δ_{cgct}, associated with a local superdiffeomorphism parameter ξ^A and *s*tandard superconformal transformations, $\delta_{\mathcal{H}}$, associated with the local superfield parameters: the dilatation σ, Lorentz $\Lambda^{ab} = -\Lambda^{ba}$, $SU(2)_R$ $\Lambda^{ij} = \Lambda^{ji}$, and special conformal (bosonic and fermionic) transformations $\Lambda^A = (\eta^{\alpha i}, \Lambda^a_K)$. The covariant derivatives transform as

$$
\delta_{\mathcal{G}}\nabla_A = [\mathcal{K}, \nabla_A] \ , \qquad (2.2.77a)
$$

with

$$
\mathcal{K} = \xi^C\nabla_C + \frac{1}{2}\Lambda^{ab}M_{ab} + \Lambda^{ij}J_{ij} + \sigma\mathbb{D} + \Lambda^A K_A \ . \qquad (2.2.77b)
$$

A covariant (or tensor) superfield U transforms as

$$
\delta_{\mathcal{G}}U = (\delta_{cgct} + \delta_{\mathcal{H}})U = \mathcal{K}U \ . \qquad (2.2.78)
$$

The superfield U is said to be *superconformal primary* of dimension Δ if $K_A U = 0$ (it suffices to require that $S_{\alpha i}U = 0$) and $\mathbb{D}U = \Delta U$.

Let us now explain how to systematically obtain various component fields of the standard Weyl multiplet from the superspace geometry described above. The vielbein ($e_m{}^a$) and gravitini ($\psi_m{}^i_\alpha$) appear as the coefficients of $\mathrm{d}x^m$ of the super-vielbein $E^A = (E^a, E^\alpha_i) = \mathrm{d}z^M E_M{}^A$,

$$
e_m{}^a(x) := E_m{}^a(z)| \ , \qquad \psi_m{}^i_\alpha(x) := 2E_m{}^i_\alpha(z)| \ , \qquad (2.2.79)
$$

where a single vertical line next to a superfield denotes the usual component projection to $\theta = 0$, i.e., $V(z)| := V(z)|_{\theta=0}$. This operation can be written in a coordinate-independent way using the so-called double-bar projection [143, 144]

$$e^a = \mathrm{d}x^m e_m{}^a = E^a\| \ , \qquad \psi^i_\alpha = \mathrm{d}x^m \psi_m{}^i_\alpha = 2E^i_\alpha\| \ , \tag{2.2.80}$$

where the double-bar denotes setting $\theta = \mathrm{d}\theta = 0$. The remaining fundamental and composite one-forms are also obtained by taking the projections of the corresponding superspace one-forms,

$$\phi^{ij} := \Phi^{ij}\| \ , \quad b := B\| \ , \quad \omega^{ab} := \Omega^{ab}\| \ , \quad \phi^{\alpha i} := 2\mathfrak{F}^{\alpha i}\| \ , \quad \mathfrak{f}^a := \mathfrak{F}^a\| \ . \tag{2.2.81}$$

The covariant matter fields are contained within the super-Weyl tensor $W_{\alpha\beta}$ and its independent descendants,

$$\begin{aligned} W_{\alpha\beta} &:= W_{\alpha\beta}| \ , & (2.2.82a)\\ \chi^i_\alpha &:= \frac{3\mathrm{i}}{32} X^i_\alpha| = \frac{3\mathrm{i}}{80} \nabla^{\beta i} W_{\beta\alpha}| \ , & (2.2.82b)\\ D &:= -\frac{3}{128} Y| = -\frac{3\mathrm{i}}{320} \nabla^k_\alpha \nabla_{\beta k} W^{\alpha\beta}| \ . & (2.2.82c) \end{aligned}$$

The other components of the super Weyl tensor are given by $W_{ab}{}^i_\alpha := W_{ab}{}^i_\alpha|$, $W_{\alpha\beta\gamma\delta} := W_{\alpha\beta\gamma\delta}|$ and $\Phi_{ab}{}^{ij} := \Phi_{ab}{}^{ij}| = -\frac{3\mathrm{i}}{4} X_{ab}{}^{ij}| = -\frac{3\mathrm{i}}{4}(\Sigma_{ab})^{\alpha\beta} X_{\alpha\beta}{}^{ij}|$.[3] These will turn out to be composite and expressed in terms of the component curvatures.

Taking the double-bar projection of $\nabla = E^A \nabla_A$, the component vector covariant derivative ∇_a is defined to coincide with the projection of the superspace derivative $\nabla_a|$,

$$e_m{}^a \nabla_a = \partial_m - \frac{1}{2}\psi_m{}^\alpha_i \nabla^i_\alpha| - \frac{1}{2}\omega_m{}^{ab} M_{ab} - b_m \mathbb{D} - \phi_m{}^{ij} J_{ij} - \frac{1}{2}\phi_m{}^{\alpha i} S_{\alpha i} - \mathfrak{f}_m{}^a K_a \ . \tag{2.2.83}$$

Here, the projected spinor covariant derivative $\nabla^i_\alpha|$ corresponds to the generator of Q-supersymmetry. It is defined such that if $\mathcal{U} = U|$, then $Q^i_\alpha \mathcal{U} := \nabla^i_\alpha | \mathcal{U} := (\nabla^i_\alpha U)|$. For the other generators, e.g., $M_{ab}\mathcal{U} = (M_{ab}U)|$, there is no ambiguity in identifying the bar projection; hence, local diffeomorphisms, Q-supersymmetry transformations, and so forth descend naturally from their corresponding rule in superspace.

The component supercovariant curvature tensors are given by

$$\begin{aligned} [\nabla_a, \nabla_b] = &-R(P)_{ab}{}^c \nabla_c - R(Q)_{ab}{}^\alpha_i \nabla^i_\alpha| - \frac{1}{2} R(M)_{ab}{}^{cd} M_{cd} - R(J)_{ab}{}^{ij} J_{ij} \\ &- R(\mathbb{D})_{ab} \mathbb{D} - R(S)_{ab}{}^{\gamma k} S_{\gamma k} - R(K)_{ab}{}^c K_c \ . \end{aligned} \tag{2.2.84}$$

We have introduced $R(P)_{ab}{}^c = \mathscr{T}_{ab}{}^c|$ and $R(Q)_{ab}{}^i_\alpha = \mathscr{T}_{ab}{}^i_\alpha|$. $R(M)_{ab}{}^{cd}$, $R(J)_{ab}{}^{ij}$, $R(\mathbb{D})_{ab}$, $R(S)_{ab}{}^{\gamma k}$, and $R(K)_{ab}{}^c$ coincide with the lowest components of the corresponding superspace curvature tensors given in appendix B.

[3] In the case of $W_{\alpha\beta}$, $W_{ab}{}^i_\alpha$, $\Phi_{ab}{}^{ij}$, and $W_{\alpha\beta\gamma\delta}$, unless specified, it should be clear from the context if we refer to the superfields or their component projections.

6D $N=(1,0)$ **conformal superspace**

The 6D $N=(1,0)$ conformal superspace is parametrised by local bosonic (x^m) and fermionic (θ^μ_i) coordinates $z^M=(x^m,\theta^\mu_i)$, where $m=0,1,2,3,4,5$, $\mu=1,2,3,4$ and $i=1,2$. By gauging the full 6D $N=(1,0)$ superconformal algebra in superspace, we introduce covariant derivatives $\nabla_A=(\nabla_a,\nabla^i_\alpha)$ which take the form

$$\begin{aligned}\nabla_A=E_A-\omega_A{}^{\underline{b}}X_{\underline{b}} &= E_A-\frac{1}{2}\Omega_A{}^{ab}M_{ab}-\Phi_A{}^{ij}J_{ij}-B_A\mathbb{D}-\mathfrak{F}_{AB}K^B\ , && (2.2.85)\\ &= E_A-\frac{1}{2}\Omega_A{}^{ab}M_{ab}-\Phi_A{}^{ij}J_{ij}-B_A\mathbb{D}-\mathfrak{F}_A{}^i_\alpha S^\alpha_i-\mathfrak{F}_A{}^aK_a\ . && (2.2.86)\end{aligned}$$

Here $E_A=E_A{}^M\partial_M$ is the inverse super-vielbein (which plays a role of a connection for local super-translations), M_{ab} are the Lorentz generators, J_{ij} are generators of the $SU(2)_R$ R-symmetry group, $\mathbb{D}$ is the dilatation generator and $K^A=(K^a,S^\alpha_i)$ are the special superconformal generators.[4] The super-vielbein one-form is given by $E^A=\mathrm{d}z^ME_M{}^A$ and satisfies $E_M{}^AE_A{}^N=\delta^N_M$, $E_A{}^ME_M{}^B=\delta^B_A$. Associated with each structure group generator $X_{\underline{a}}=(M_{ab},J_{ij},\mathbb{D},S^\alpha_i,K_a)$ there is a connection superfield one-form given by $\omega^{\underline{a}}=(\Omega^{ab},\Phi^{ij},B,\mathfrak{F}^i_\alpha,\mathfrak{F}^a)=\mathrm{d}z^M\omega_M{}^{\underline{a}}=E^A\omega_A{}^{\underline{a}}$.

To describe the standard 6D $N=(1,0)$ Weyl multiplet in conformal superspace, the algebra of covariant derivatives

$$\begin{aligned}[\nabla_A,\nabla_B\}=&-\mathscr{T}_{AB}{}^C\nabla_C-\frac{1}{2}\mathscr{R}(M)_{AB}{}^{cd}M_{cd}-\mathscr{R}(J)_{AB}{}^{kl}J_{kl}\\ &-\mathscr{R}(\mathbb{D})_{AB}\mathbb{D}-\mathscr{R}(S)_{AB}{}^k_\gamma S^\gamma_k-\mathscr{R}(K)_{AB}{}^cK_c\ ,\end{aligned}\tag{2.2.87}$$

is constrained to be completely determined in terms of the symmetric super-Weyl tensor superfield $W^{\alpha\beta}$, which is a superconformal primary with conformal dimension one

$$W^{\alpha\beta}=W^{\beta\alpha}\ ,\quad K^AW^{\alpha\beta}=0\ ,\quad \mathbb{D}W^{\alpha\beta}=W^{\alpha\beta}\ ,\tag{2.2.88}$$

obeying the Bianchi identities

$$\begin{aligned}\nabla^{(i}_\alpha\nabla^{j)}_\beta W^{\gamma\delta} &= -\delta^{(\gamma}_{[\alpha}\nabla^{(i}_{\beta]}\nabla^{j)}_\rho W^{\delta)\rho}\ , && (2.2.89\text{a})\\ \nabla^k_\alpha\nabla_{\gamma k}W^{\beta\gamma}-\frac{1}{4}\delta^\beta_\alpha\nabla^k_\gamma\nabla_{\delta k}W^{\gamma\delta} &= 8\mathrm{i}\nabla_{\alpha\gamma}W^{\gamma\beta}\ . && (2.2.89\text{b})\end{aligned}$$

The relation $W^{\alpha\beta}=1/6(\tilde\gamma^{abc})^{\alpha\beta}W_{abc}$ means that the super-Weyl tensor $W^{\alpha\beta}$ is equivalent to an anti-self-dual rank-3 tensor superfield W_{abc}. In (2.2.87) $\mathscr{T}_{AB}{}^C$ is the torsion curvature, and $\mathscr{R}(M)_{AB}{}^{cd}$, $\mathscr{R}(J)_{AB}{}^{kl}$, $\mathscr{R}(\mathbb{D})_{AB}$, $\mathscr{R}(S)_{AB}{}^k_\gamma$, and $\mathscr{R}(K)_{AB}{}^c$ are the curvatures associated with Lorentz, $SU(2)_R$, dilatation, S-supersymmetry, and special conformal boosts, respectively.

Their expressions in terms of the super-Weyl tensor $W^{\alpha\beta}$ and its descendant superfields of dimension 3/2

$$X^{\alpha i}:=-\frac{\mathrm{i}}{10}\nabla^i_\beta W^{\alpha\beta}\ ,\quad X^{k\alpha\beta}_\gamma:=-\frac{\mathrm{i}}{4}\nabla^k_\gamma W^{\alpha\beta}-\delta^{(\alpha}_\gamma X^{\beta)k}\ ,\tag{2.2.90}$$

[4]Note here the change in the $SU(2)_R$ index structure of the 6D S-supersymmetry generator, S^α_i, relative to the 5D case where we it was originally introduced as $S_{\alpha i}$. Though this difference might seem unnatural, and introduce minus signs in similar expressions in 5D and 6D, we decided to keep adhering to the notations used in [29, 69].

and of dimension 2

$$Y_\alpha{}^{\beta ij} := -\frac{5}{2}\Big(\nabla_\alpha^{(i}X^{\beta j)} - \frac{1}{4}\delta_\alpha^\beta \nabla_\gamma^{(i}X^{\gamma j)}\Big) = -\frac{5}{2}\nabla_\alpha^{(i}X^{\beta j)} \ , \tag{2.2.91a}$$

$$Y := \frac{1}{4}\nabla_\gamma^k X_k^\gamma \ , \tag{2.2.91b}$$

$$Y_{\alpha\beta}{}^{\gamma\delta} := \nabla_{(\alpha}^k X_{\beta)k}{}^{\gamma\delta} - \frac{1}{6}\delta_\beta^{(\gamma}\nabla_\rho^k X_{\alpha k}{}^{\delta)\rho} - \frac{1}{6}\delta_\alpha^{(\gamma}\nabla_\rho^k X_{\beta k}{}^{\delta)\rho} \ , \tag{2.2.91c}$$

are given in appendix C. Just like the $5D$ case, we consider the superspace and component structures for $6D$ corresponding to the "traceless" choice of conventional constraints, which was first considered in [32]. The full algebra of covariant derivatives (2.2.87) (including the explicit expressions for the torsion and curvature tensors) are given in appendix C.

The superfields $X^{\alpha i}$, $X_\gamma^{k\alpha\beta}$, $Y_\alpha{}^{\beta ij}$, Y, and $Y_{\alpha\beta}{}^{\gamma\delta}$ are the only independent descendants of $W^{\alpha\beta}$. All the other higher dimension descendants obtained by the action of spinor derivatives on $W^{\alpha\beta}$ are vector derivatives of these independent fields as a result of the non-trivial Bianchi identities (2.2.89). Eq. (C.1.16) gives the action of the S-generators on these independent descendants that prove to all be annihilated by K^a.

The conformal supergravity gauge group $\mathcal{G}$ is generated by *c*ovariant general coordinate transformations, δ_{cgct}, associated with a local superdiffeomorphism parameter ξ^A and *s*tandard superconformal transformations, $\delta_{\mathcal{H}}$, associated with the following local superfield parameters: the dilatation σ, Lorentz $\Lambda^{ab} = -\Lambda^{ba}$, $SU(2)_R$ $\Lambda^{ij} = \Lambda^{ji}$, , and special conformal transformations $\Lambda_A = (\eta_\alpha^i, \Lambda_a)$. The covariant derivatives transform as

$$\delta_{\mathcal{G}}\nabla_A \quad = \quad [\mathcal{K}, \nabla_A] \ , \tag{2.2.92}$$

where $\mathcal{K}$ denotes the first-order differential operator

$$\mathcal{K} = \xi^C\nabla_C + \frac{1}{2}\Lambda^{ab}M_{ab} + \Lambda^{ij}J_{ij} + \sigma\mathbb{D} + \Lambda_A K^A \ . \tag{2.2.93}$$

A covariant (or tensor) superfield U transforms as

$$\delta_{\mathcal{G}}U = (\delta_{cgct} + \delta_{\mathcal{H}})U = \mathcal{K}U \ . \tag{2.2.94}$$

The superfield U is said to be *superconformal primary* and of dimension Δ if $K_A U = 0$ and $\mathbb{D}U = \Delta U$.

Similar to the $5D$ case, we begin by identifying the various component fields of the $6D$ $N = (1,0)$ standard Weyl multiplet [89] within the geometry of conformal superspace. The vielbein ($e_m{}^a$) and gravitino ($\psi_{m}{}_i^\alpha$) are identified with the coefficients of dx^m of the super-vielbein $E^A = (E^a, E_i^\alpha) = dz^M E_M{}^A$,

$$e_m{}^a(x) := E_m{}^a(z)| \ , \qquad \psi_m{}^i_\alpha(x) := 2E_m{}^i_\alpha(z)| \ . \tag{2.2.95}$$

In a local coordinate independent way they are given by

$$e^a = dx^m e_m{}^a = E^a\| \ , \qquad \psi_i^\alpha = dx^m \psi_m{}_i^\alpha = 2E_i^\alpha\| \ . \tag{2.2.96}$$

Similar to the $5D$ case, look at (2.2.79) and (2.2.80), the single bar denotes setting $\theta = 0$ and the double-bar denotes setting $\theta = \mathrm{d}\theta = 0$. Analogously, the remaining fundamental and composite one-forms correspond to double-bar projections of superspace one-forms,

$$\phi^{kl} := \Phi^{kl}\| \,, \quad b := B\| \,, \quad \omega^{cd} := \Omega^{cd}\| \,, \quad \phi_\gamma^k := 2\mathfrak{F}_\gamma^k\| \,, \quad \mathfrak{f}_c := \mathfrak{F}_c\| \,. \tag{2.2.97}$$

The covariant matter fields are contained within the super-Weyl tensor W_{abc} and its independent descendants,

$$\begin{aligned} T^-_{abc} &:= -2W_{abc}| \,, && (2.2.98a)\\ \chi^{\alpha i} &:= \frac{15}{2}X^{\alpha i}| = -\frac{3\mathrm{i}}{4}\nabla^i_\beta W^{\alpha\beta}| \,, && (2.2.98b)\\ D &:= \frac{15}{2}Y| = -\frac{3\mathrm{i}}{16}\nabla^k_\alpha \nabla_{\beta k}W^{\alpha\beta}| \,. && (2.2.98c) \end{aligned}$$

The lowest components of the other nontrivial descendants of $W^{\alpha\beta}$, specifically $X^i_\alpha{}^{\beta\gamma}|$, $Y_\alpha{}^{\beta kl}|$ and $Y_{\alpha\beta}{}^{\gamma\delta}|$, prove to be directly related to component curvatures and hence are composite fields.

The component gauge connections can now be used to define the locally superconformal covariant derivative ∇_a, which coincide with the bar projection of the conformal superspace covariant derivative $\nabla_a|$

$$e_m{}^a\nabla_a = \partial_m - \frac{1}{2}\psi_m{}^\alpha_i\nabla^i_\alpha| - \frac{1}{2}\omega_m{}^{cd}M_{cd} - \phi_m{}^{ij}J_{ij} - b_m\mathbb{D} - \frac{1}{2}\phi_m{}^i_\alpha S^\alpha_i - \mathfrak{f}_{ma}K^a \,. \tag{2.2.99}$$

This satisfies the algebra

$$\begin{aligned} [\nabla_a, \nabla_b] \;=\; & -R(P)_{ab}{}^c\nabla_c - R(Q)_{ab}{}^\alpha_i Q^i_\alpha - \frac{1}{2}R(M)_{ab}{}^{cd}M_{cd} - R(J)_{ab}{}^{kl}J_{kl} \\ & -R(\mathbb{D})_{ab}\mathbb{D} - R(S)_{ab}{}^i_\gamma S^\gamma_i - R(K)_{ab}{}^cK_c \,. \end{aligned} \tag{2.2.100}$$

where we identified $R(P)_{ab}{}^c = \mathscr{T}_{ab}{}^c|$, $R(Q)_{ab}{}^i_\alpha = \mathscr{T}_{ab}{}^i_\alpha|$, while $R(M)_{ab}{}^{cd}$, $R(J)_{ab}{}^{ij}$, $R(\mathbb{D})_{ab}$, $R(S)_{ab}{}^k_\gamma$, and $R(K)_{ab}{}^c$ are coinciding with the lowest components of the corresponding superspace curvature tensors given in appendix C.

The following publication has been incorporated as Chapter 3.

1. Gregory Gold, Jessica Hutomo, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *Components of curvature-squared invariants of minimal supergravity in five dimensions,* (Accepted JHEP) [2311.00679] [1].

Contributor	Statement of Contribution	%
Gregory Gold	Writing of text	15
	Proof-reading	30
	Theoretical derivations	10
	Computational derivations	70
	Initial concept	10
Jessica Hutomo	Writing of text	20
	Proof-reading	10
	Theoretical derivations	20
	Initial concept	10
	Supervision, guidance	30
Saurish Khandelwal	Writing of text	45
	Proof-reading	30
	Theoretical derivations	65
	Computational derivations	30
	Initial concept	10
Gabriele Tartaglino-Mazzucchelli	Writing of text	20
	Proof-reading	25
	Theoretical derivations	5
	Initial concept	70
	Supervision, guidance	70

Table 2.13: Contributions of each author to the work

Chapter 3

Components of curvature-squared invariants of minimal supergravity in five dimensions

We present for the first time the component structure of the supersymmetric completions for all curvature-squared invariants of five-dimensional, off-shell (gauged) minimal supergravity, including all fermions. This is achieved by using an interplay between superspace and superconformal tensor calculus techniques, and by employing results from arXiv:1410.8682 and arXiv:2302.14295. Our analysis is based on using a standard Weyl multiplet of conformal supergravity coupled to a vector and a linear multiplet compensator to engineer off-shell Poincaré supergravity. We compute all the descendants of the composite linear multiplets that describe gauged supergravity together with the three independent four-derivative invariants. These are the building blocks of the locally superconformal invariant actions. A derivation of the primary equations of motion for minimal gauged off-shell supergravity deformed by an arbitrary combination of these three locally superconformal invariants, is then provided. Finally, all the covariant descendants in the multiplets of equations of motion are obtained by applying a series of Q-supersymmetry transformations, equivalent to successively applying superspace spinor derivatives to the primary equations of motion.

3.1 Introduction

Even though the first (two-derivative) supergravity was constructed (for $N = 1$ supersymmetry in four dimensions) almost five decades ago [21, 145] (see also [146–155] and the pedagogical reviews [34–37, 40]), higher-order locally supersymmetric invariants are still largely unknown. However, in an effective field theory approach, quantum corrections in string theory take the form of an infinite series of (supersymmetric) higher-derivative terms; see, e. g., [156–160] and references therein. Many open problems in string theory, for example its vacua structure, are unresolved due to the lack of information about the full quantum corrected supergravity effective action. More complexity arises due to the fact that the purely gravitational higher-curvature terms are related by supersymmetry to contributions depending on p-forms, which describe part of the string spectrum. These terms, which have not yet been fully understood, play an important role in studying, for example, the moduli space in compactified string theory and the low-energy description of string dualities. Even the first α' corrections, associated with curvature-squared terms, have not been completely understood to date. Our paper represents an important step forward in this direction.

A remarkable success of supergravity in the last 25 years has been its role in the development of holographic correspondences such as the AdS/CFT [45, 46, 48] (see also the classic reviews [47, 161]). The amount of evidence that planar (large-N, at first order in the $1/N$ expansion) field theory can be described by a dual theory of (two-derivative) gravity, and vice versa, is absolutely outstanding. These results have revolutionised our understanding of both quantum field and gravity theories and have shed new light in several fields of research such as black hole physics, condensed matter and integrable systems, and quantum information theory, just to mention a few. One aspect to stress is that most of these developments were based on the leading-order effective field theory description of string theory. It is of fundamental importance for holography to advance to higher orders so that we may assess the validity of the correspondence. In fact, precision tests beyond the leading order have become increasingly important in the last few years. The reason is that, on the field theory side, a series of breakthroughs based on integrability and localisation techniques has allowed several observables in superconformal field theories (SCFT) to be computed exactly. Higher-order (in $1/N$) corrections in quantum field theories translate into higher-curvature terms on the gravity side, calling for new supergravity higher-derivative analyses.

When *off-shell* techniques are available for supergravity,[1] systematic approaches exist to construct locally supersymmetric higher-derivative invariants. In $\mathrm{D} \leq 6$ space-time dimensions, off-shell techniques are now thoroughly developed and understood for up to eight real supercharges – see [34–40] for reviews. By using these approaches, exact, off-shell higher-derivative supergravity models have been constructed, see for example, the following list of references [2, 6, 28, 29, 31, 32, 51, 55–76]. These results have recently enabled new pioneering works towards obtaining precision, higher-order, tests in AdS/CFT from the gravity side – see for example [2, 77–86] and references therein. We underline, by extending results obtained in [29] and more recently in [6], that in [2] and in the current paper, the construction of a complete basis to study (minimal) gauged supergravity in five dimensions modified by the three possible curvature-squared terms has been completed. As such, we have control of all first α' correction of a universal sector of string compactifications to five dimensions preserving at least eight real supercharges. This then allows inspection from the gravity side of new first order tests of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence.

The playground of our paper is minimal five-dimensional ($5D$) supergravity. On-shell, this was introduced four decades ago in [162, 163], and the first off-shell description was given in [164] by the use of superspace techniques. The matter couplings in $5D$ minimal supergravity have since been extensively studied at the component level, both by using on-shell [165–168] and off-shell approaches [90, 169–177]. The superspace approach to general off-shell $5D$ $N = 1$ supergravity-matter systems was then developed in [29, 124–126, 178]. In our paper we will employ an interplay between

[1] Off-shell means that the symmetry algebra closes without using equations of motion; see, e.g., [34–37].

component techniques based on the superconformal tensor calculus and superspace – see [37–40] for reviews – the merging of which goes under the name of *conformal superspace*. Conformal superspace was originally introduced by D. Butter for $4D$ $N = 1$ supergravity in a seminal work [25] and then extended to other space-time dimensions $2 \leq \mathrm{D} \leq 6$ for various amount of supersymmetry in [26, 27, 29, 31, 32, 142] – see [39, 40] for recent reviews.

In superconformal approaches to supergravity, the key idea is to enlarge the supergravity gauge group to be described by local superconformal transformations and, potentially, internal symmetries. Local Poincaré supersymmetry is then recovered by making appropriate choices of gauge fixing conditions for the non-physical symmetries within the conformal algebra implemented through the use of compensating multiplets. The extra symmetries make some of the local supersymmetry of the gravitational sector (both the transformations and symmetry algebra) more manageable and allow for increased freedom to appropriately fix gauges and frame choices in the final steps of various analyses – see [37–40] for pedagogical reviews. To achieve minimal $5D$ supergravity off-shell from superconformal techniques one can couple the standard Weyl multiplet of $5D$ conformal supergravity to two off-shell conformal compensators: a vector multiplet and a linear multiplet [29, 90, 124–126, 169–177]. These will be the off-shell multiplets employed in our paper.

Within the framework of these superconformal approaches, the minimal five dimensional two-derivative (gauged) supergravity theory is obtained by the combination of the locally superconformal two-derivative theories for the vector and linear multiplets together with a BF coupling of the two superconformal compensators that leads to a supersymmetric cosmological constant. Using this setup, locally supersymmetric completions of the Weyl tensor squared and the scalar curvature squared were constructed for the first time, respectively, in [55] and [64] by using component field techniques.

The third independent, locally superconformal invariant, which includes a Ricci tensor-squared term, was constructed in superspace in [29] by using a $5D$ analog of the “Log multiplet” construction in $4D$ $N = 2$ supergravity of [61]. However, due to the computational complexity associated to the construction of this invariant, it took almost 10 years to obtain the results starting from the very simple building block in superspace. For example, one must compute up to eight local Q-supersymmetry transformations on the logarithm of the primary field W of the vector multiplet compensator to obtain the component action of the $5D$ Log invariant. In our paper we present for the first time the component structure of this invariant in a standard Weyl multiplet background. See [2] for the bosonic terms based on using a dilaton Weyl multiplet instead of the standard Weyl multiplet of conformal supergravity. Moreover, by employing the techniques mentioned above, and by a substantial use of the computer algebra program *C*adabra [87, 88], in our work we have also obtained all the fermionic contributions for all the three curvature-squared invariant for the first time. The reader who is only interested in the bosonic results of all the curvature squared actions in a standard Weyl background can look directly to section 3.5 of our paper, specifically: equations (3.5.2) and (3.5.7) for the Weyl squared invariant; equations (3.5.12) and (3.5.20) for the Log invariant; and equations (3.5.22) and (3.5.23) for the scalar curvature-squared invariant. Due to the size of the fermionic structures arising for the building blocks of the invariants, much of the fermionic contributions are given in the (more than 200 pages long) supplementary file accompanying our paper [1].

Having obtained all the building blocks for minimal gauged supergravity with the addition of the three curvature-squared invariants, we are in the position to obtain for the first time the multiplets of equations of motions (EOMs) for this five-parameter class of theories. Because these models are obtained within a superconformal tensor calculus approach, all the EOMs turn out to be organised in terms of superconformal multiplets described by three main primary (super)fields: a scalar primary supercurrent operator J which arises from the variation of the fields of the standard Weyl multiplet; a composite vector multiplet of equations of motion which arises from the variation of the linear multiplet compensator; and a composite linear multiplet of equations of motion which arises from the variation of the vector multiplet compensator. The primary (super)fields associated with these three multiplets of EOMs were presented in [6]. In this paper we will present a derivation of the primary EOMs based on an interplay of superspace and component techniques together with all their superconformal descendants. An advantage in obtaining the equations of motion by this approach is that covariance will be manifest in every result, which is nontrivial because the various component actions have Chern-Simons terms and naked gravitini terms. Similar to the other building blocks of the curvature-squared invariants, the equations of motion have remarkable length and complexity. For this reason, once more, we have decided to relegate most of the complete fermionic results of the EOM analysis in the supplementary file associated to this paper. It is our hope that these results will be later used to advance our understanding of the on-shell structure of these models.

Our paper is organised as follows. In section 3.2 we review the structure of the $5D$, $N = 1$ superconformal matter multiplets that will be used in this work. We do present results both in conformal superspace and in components in the so called traceless frame employed in [29]. It is useful to mention that our notation and conventions correspond to that of [29] (see also [3], where some typos from [29] were fixed). In section 3.3 we review the definition of the BF action principle which is the building block for all the invariants studied in this paper. We then review the construction of all the two-derivative invariants that define minimal gauged supergravity in five dimensions. In section 3.4, elaborating on the results of [6], we describe the component structure of all these composite primary multiplets (including the fermionic terms that can be found in the supplementary file) which are used to construct the three independent four-derivative invariants studied in our paper. By using all the building blocks of sections 3.2–3.4, in section 3.5 we present the bosonic part of the three curvature-squared invariants, including the "Log invariant" which is presented for the first time in subsection 3.5.2 in a standard Weyl basis.[2] Sections 3.6 and 3.7 are respectively devoted to firstly provide a derivation of the primary equations of motion which first appeared in [6], and secondly to analyse all the descendants of the equations of motion. We conclude our paper in section 3.8 where we comment on possible developments and applications of our results. We accompany our paper with two appendices and a 200-page supplementary file. This supplementary file gives all the results including fermions, and can be found in [1]. Appendix B collects results about conformal superspace in the traceless frame of [3, 29] and explains how to map our notations and conventions to the ones of some

[2]The Log invariant in the gauged dilaton Weyl basis has been obtained recently in [2].

other groups.

3.2 Superconformal multiplets

This section is devoted to a review of several superconformal matter multiplets in five dimensions. Along with the standard Weyl multiplet discussed in subsections 2.2.2 and 2.2.3, they are the building blocks for the various supersymmetric curvature-squared invariants discussed in this paper. We will first describe the off-shell Abelian vector multiplet and then move on to the off-shell linear multiplet. For each of the previously mentioned multiplets, we will present their formulations in both superspace and components for completeness. Our analysis follows the superspace and components notation and results of [3, 6, 29].[3]

3.2.1 The Abelian vector multiplet

Following the presentation and conventions of [3, 6], let us turn to the description of an off-shell Abelian vector multiplet.

The Abelian vector multiplet in superspace

To describe the Abelian vector multiplet [181, 182] in conformal superspace [29], we introduce a real primary superfield W of dimension 1,

$$(W)^* = W \ , \qquad K_A W = 0 \ , \qquad \mathbb{D} W = W \ . \tag{3.2.1a}$$

The superfield W is subject to the Bianchi identity

$$\nabla_\alpha^{(i}\nabla_\beta^{j)} W = \frac{1}{4}\varepsilon_{\alpha\beta}\nabla^{\gamma(i}\nabla_\gamma^{j)} W \ . \tag{3.2.1b}$$

Acting with spinor covariant derivatives on W gives the following descendants:

$$\lambda_\alpha^i := -\mathrm{i}\nabla_\alpha^i W \ , \qquad X^{ij} := \frac{\mathrm{i}}{4}\nabla^{\alpha(i}\nabla_\alpha^{j)} W = -\frac{1}{4}\nabla^{\alpha(i}\lambda_\alpha^{j)} \ . \tag{3.2.2a}$$

These superfields, along with

$$F_{\alpha\beta} := -\frac{\mathrm{i}}{4}\nabla_{(\alpha}^k\nabla_{\beta)k} W - W_{\alpha\beta} W = \frac{1}{4}\nabla_{(\alpha}^k\lambda_{\beta)k} - W_{\alpha\beta} W \ , \tag{3.2.2b}$$

satisfy the identities:

$$\nabla_\alpha^i\lambda_\beta^j = -2\varepsilon^{ij}\big(F_{\alpha\beta} + W_{\alpha\beta} W\big) - \varepsilon_{\alpha\beta} X^{ij} - \varepsilon^{ij}\nabla_{\alpha\beta} W \ , \tag{3.2.3a}$$

$$\begin{aligned}\nabla_\alpha^i F_{\beta\gamma} = & -\mathrm{i}\nabla_{\alpha(\beta}\lambda_{\gamma)}^i - \mathrm{i}\varepsilon_{\alpha(\beta}\nabla_{\gamma)}{}^\delta\lambda_\delta^i - \frac{3\mathrm{i}}{2} W_{\beta\gamma}\lambda_\alpha^i - W_{\alpha\beta\gamma}{}^i W \\ & + \frac{\mathrm{i}}{2} W_{\alpha(\beta}\lambda_{\gamma)}^i - \frac{3\mathrm{i}}{2}\varepsilon_{\alpha(\beta} W_{\gamma)}{}^\delta\lambda_\delta^i \ ,\end{aligned} \tag{3.2.3b}$$

[3] For various discussions on off-shell multiplets in five dimensions, see also other works based on superspace, e.g., [123–126, 164, 164, 179–181], and component approaches, e.g., [90, 165–177].

$$\nabla^i_\alpha X^{jk} = 2\mathrm{i}\varepsilon^{i(j}\Big(\nabla_\alpha{}^\beta\lambda^{k)}_\beta - \frac{1}{2}W_{\alpha\beta}\lambda^{\beta k)} + \frac{3\mathrm{i}}{4}X^{k)}_\alpha W\Big) \ . \tag{3.2.3c}$$

Due to (3.2.1b), a dimension-2 superfield of a vector multiplet in the traceless frame obeys the following Bianchi identity:

$$\nabla_{(\alpha}{}^\gamma F_{\beta)\gamma} = \frac{1}{2}\lambda^{\gamma k}W_{\alpha\beta\gamma k} \ . \tag{3.2.4}$$

It is useful to note that the S-supersymmetry generator acts on the descendants as

$$S^i_\alpha\lambda^j_\beta = -2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}W \ , \qquad S^i_\alpha F_{\beta\gamma} = 4\varepsilon_{\alpha(\beta}\lambda^i_{\gamma)} \ , \qquad S^i_\alpha X^{jk} = -2\varepsilon^{i(j}\lambda^{k)}_\alpha \ , \tag{3.2.5}$$

while all the superfields are annihilated by K_a.

It is worth noting that there exists a prepotential formulation for the Abelian vector multiplet which was developed in [29], see also [123, 125, 126, 179] for related works in other superspaces. The formulation of [29] is based on a real primary superfield V_{ij} of dimension -2, i.e., $\mathbb{D}V_{ij} = -2V_{ij}$. Here V_{ij} transforms as an isovector under $\mathrm{S}U(2)_R$ transformations and is the $5D$ analogue of Mezincescu's prepotential [183–185] for the $4D$ $N = 2$ Abelian vector multiplet. This then allows us to represent the field strength W as

$$W = -\frac{3\mathrm{i}}{40}\nabla_{ij}\Delta^{ijkl}V_{kl} \ , \tag{3.2.6}$$

where we have defined the operators

$$\begin{aligned}\Delta^{ijkl} &:= -\frac{1}{96}\varepsilon^{\alpha\beta\gamma\delta}\nabla^{(i}_\alpha\nabla^j_\beta\nabla^k_\gamma\nabla^{l)}_\delta = -\frac{1}{32}\nabla^{(ij}\nabla^{kl)} = \Delta^{(ijkl)} \ , && (3.2.7a)\\ \nabla^{ij} &:= \nabla^{\alpha(i}\nabla^{j)}_\alpha \ . && (3.2.7b)\end{aligned}$$

Let us also point out that V_{ij} in (3.2.6) is defined modulo gauge transformations of the form

$$\delta V_{kl} = \nabla^p_\alpha\Lambda^\alpha{}_{klp} \ , \qquad \Lambda^\alpha{}_{klp} = \Lambda^\alpha{}_{(klp)} \ , \tag{3.2.8}$$

with the gauge parameter $\Lambda^\alpha{}_{klp}$ being a dimension $-5/2$ primary superfield,

$$S^i_\alpha\Lambda^\beta{}_{jkl} = 0 \ , \qquad \mathbb{D}\Lambda^\beta{}_{jkl} = -\frac{5}{2}\Lambda^\beta{}_{jkl} \ . \tag{3.2.9}$$

The Abelian vector multiplet in components

The component structure of the vector multiplet follows directly from the superfield definitions (3.2.2). It contains a real scalar field $W := W|$, gaugini $\lambda^i_\alpha := \lambda^i_\alpha|$, a triplet of auxiliary fields $X^{ij} := X^{ij}|$, and a real Abelian gauge connection $v_m := V_m|$ or, equivalently, its real field strength $f_{mn} := F_{mn}| = 2\partial_{[m}v_{n]}$. The field strength f_{mn} may be expressed in terms of the covariant field strength $F_{ab} := F_{ab}|$ via the relation

$$F_{ab} = f_{ab} + \mathrm{i}(\Gamma_{[a})_\alpha{}^\beta\psi_{b]}{}^\alpha_k\lambda^k_\beta + \frac{\mathrm{i}}{2}\psi_{[a}{}^\gamma_k\psi_{b]}{}^k_\gamma W \ , \qquad f_{ab} := e_a{}^m e_b{}^n f_{mn} \ . \tag{3.2.10}$$

The dilatation weights of the vector multiplet fields are summarised in Table 3.1.

	W	λ_α^i	X^{ij}	F_{ab}	v_m
$\mathbb{D}$	1	3/2	2	2	0

Table 3.1: Dilatation weights of the Abelian vector multiplet.

The transformations of the component fields in a standard Weyl multiplet background can be obtained from the corresponding superfields. They read

$$\begin{aligned}
\delta W &= \mathrm{i}\xi_i\lambda^i + \lambda_{\mathbb{D}} W \,, &(3.2.11a)\\
\delta\lambda_\alpha^i &= -(\Sigma^{ab}\xi^i)_\alpha F_{ab} - (\Sigma^{ab}\xi^i)_\alpha W_{ab} W + \xi_{\alpha j} X^{ij} + (\Gamma^a\xi^i)_\alpha \nabla_a W \\
&\quad + \frac{1}{2}\lambda^{ab}(\Sigma_{ab}\lambda^i)_\alpha + \lambda^i{}_j\lambda_\alpha^j + \frac{3}{2}\lambda_{\mathbb{D}}\lambda_\alpha^i + 2\mathrm{i}\eta_\alpha{}^i W \,, &(3.2.11b)\\
\delta X^{ij} &= -2\mathrm{i}\xi^{(i}\Gamma^a\nabla_a\lambda^{j)} - \frac{\mathrm{i}}{2}\xi^{(i}\Sigma^{ab}\lambda^{j)} W_{ab} - 16\mathrm{i}(\xi^{(i}\chi^{j)})W \\
&\quad + 2\lambda^{(i}{}_k X^{j)k} - 2\eta^{(i}\lambda^{j)} + 2\lambda_{\mathbb{D}} X^{ij} \,, &(3.2.11c)\\
\delta v_m &= \mathrm{i}(\xi^i\psi_{mi})W - \mathrm{i}(\xi^i\Gamma_m\lambda_i) + \partial_m\lambda_V \,, &(3.2.11d)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a W &= \mathcal{D}_a W - \frac{\mathrm{i}}{2}\psi_{ai}\lambda^i \,, &(3.2.12a)\\
\nabla_a\lambda_\alpha^i &= \mathcal{D}_a\lambda_\alpha^i + \frac{1}{2}(\Sigma^{bc}\psi_a{}^i)_\alpha\Big(F_{bc} + W_{bc}W\Big) - \frac{1}{2}\psi_{a\alpha j}X^{ij} \\
&\quad - \frac{1}{2}(\Gamma^b\psi_a{}^i)_\alpha\nabla_b W - \mathrm{i}\phi_{a\alpha}{}^i W \,. &(3.2.12b)
\end{aligned}$$

For completeness, we include

$$\begin{aligned}
\nabla_a X^{jk} &= \mathcal{D}_a X^{jk} + \mathrm{i}\psi_{ai}^{\alpha}\varepsilon^{i(j}\Big(\nabla_\alpha{}^\beta\lambda_\beta^{k)} - \frac{1}{2}W_{\alpha\beta}\lambda^{\beta k)} + \frac{3\mathrm{i}}{4}X_\alpha^{k)}W\Big) + \phi_a^{\alpha(j}\lambda_\alpha^{k)} \,, &(3.2.12c)\\
\nabla_a F_{\beta\gamma} &= \mathcal{D}_a F_{\beta\gamma} + \frac{\mathrm{i}}{2}\psi_{ai}^{\alpha}\Bigg(\nabla_{\alpha(\beta}\lambda_{\gamma)}^i + \varepsilon_{\alpha(\beta}\nabla_{\gamma)}{}^\delta\lambda_\delta^i + \frac{3}{2}W_{\beta\gamma}\lambda_\alpha^i - \mathrm{i}W_{\alpha\beta\gamma}{}^i W \\
&\quad - \frac{1}{2}W_{\alpha(\beta}\lambda_{\gamma)}^i + \frac{3}{2}\varepsilon_{\alpha(\beta}W_{\gamma)}{}^\delta\lambda_\delta^i\Bigg) + 2\phi_a{}_{(\beta}^i\lambda_{\gamma)i} \,. &(3.2.12d)
\end{aligned}$$

Note that we have also included in (3.2.11d) the gauge field transformation parametrised by the local real parameter λ_V.

3.2.2 The linear multiplet

We now turn to the discussion of the off-shell linear multiplet coupled to conformal supergravity.

The linear multiplet in superspace

The linear multiplet [117, 118, 138–140, 186–191], or $\mathcal{O}(2)$ multiplet, can be described in terms of the dimension three primary superfield $G^{ij} = G^{ji}$ with the properties

$$\nabla_\alpha^{(i}G^{jk)} = 0 \,, \qquad (3.2.13a)$$

$$K_A G^{ij} = 0 , \qquad \mathbb{D}G^{ij} = 3G^{ij} , \tag{3.2.13b}$$

where G^{ij} is assumed to be real, $(G^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}G^{kl}$.

The following tower of descending identities is useful to elaborate on the component structure of the superfield G^{ij}:

$$\nabla^i_\alpha G^{jk} = 2\varepsilon^{i(j}\varphi^{k)}_\alpha , \tag{3.2.14a}$$

$$\nabla^i_\alpha \varphi^j_\beta = -\frac{\mathrm{i}}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}F + \frac{\mathrm{i}}{2}\varepsilon^{ij}\mathcal{H}_{\alpha\beta} + \mathrm{i}\nabla_{\alpha\beta}G^{ij} , \tag{3.2.14b}$$

$$\nabla^i_\alpha F = -2\nabla_\alpha{}^\beta\varphi^i_\beta - 3W_{\alpha\beta}\varphi^{\beta i} - \frac{3}{2}X_{\alpha j}G^{ij} , \tag{3.2.14c}$$

$$\nabla^i_\alpha \mathcal{H}_a = 4(\Sigma_{ab})_\alpha{}^\beta\nabla^b\varphi^i_\beta - \frac{3}{2}(\Gamma_a)_\alpha{}^\beta W_{\beta\gamma}\varphi^{\gamma i} - \frac{1}{2}(\Gamma_a)_\gamma{}^\beta W_{\beta\alpha}\varphi^{\gamma i} , \tag{3.2.14d}$$

with the independent descendant superfields being defined as

$$\varphi^i_\alpha := \frac{1}{3}\nabla_{\alpha j}G^{ij} , \tag{3.2.15a}$$

$$F := \frac{\mathrm{i}}{12}\nabla^{\gamma i}\nabla^j_\gamma G_{ij} = -\frac{\mathrm{i}}{4}\nabla^{\gamma k}\varphi_{\gamma k} , \tag{3.2.15b}$$

$$\mathcal{H}_{abcd} := \frac{\mathrm{i}}{12}\varepsilon_{abcde}(\Gamma^e)^{\alpha\beta}\nabla^i_\alpha\nabla^j_\beta G_{ij} \equiv \varepsilon_{abcde}\mathcal{H}^e . \tag{3.2.15c}$$

Here $\mathcal{H}^a$ obeys the differential condition

$$\nabla_a\mathcal{H}^a = 0 , \qquad \mathcal{H}^a := -\frac{1}{4!}\varepsilon^{abcde}\mathcal{H}_{bcde} . \tag{3.2.16}$$

The descendants (3.2.15) are all annihilated by K_a. Under the action of S-supersymmetry, they transform as follows:

$$S^i_\alpha\varphi^j_\beta = -6\varepsilon_{\alpha\beta}G^{ij} , \qquad S^i_\alpha F = 6\mathrm{i}\varphi^i_\alpha , \qquad S^i_\alpha\mathcal{H}_b = -8\mathrm{i}(\Gamma_b)_\alpha{}^\beta\varphi^i_\beta . \tag{3.2.17}$$

We refer the reader to [29] for a superform description of the linear multiplet.

There exists a prepotential formulation for the linear multiplet in superspace [29]. The constraints (3.2.13) may be solved in terms of an arbitrary primary real dimensionless scalar prepotential Ω,

$$S^i_\alpha\Omega = 0 , \qquad \mathbb{D}\Omega = 0 , \tag{3.2.18}$$

which leads to

$$G^{ij} = -\frac{3\mathrm{i}}{40}\Delta^{ijkl}\nabla_{kl}\Omega . \tag{3.2.19}$$

An important property of G^{ij} defined by (3.2.19) is that it is invariant under gauge transformations of Ω of the form

$$\delta\Omega = -\frac{\mathrm{i}}{2}(\Gamma^a)^{\alpha\beta}\nabla^i_\alpha\nabla^j_\beta B_{aij} , \tag{3.2.20}$$

where the gauge parameter is assumed to have the properties

$$B_a{}^{ij} = B_a{}^{ji} , \qquad S^i_\alpha B_a{}^{jk} = 0 , \qquad \mathbb{D}B_a{}^{ij} = -B_a{}^{ij} , \tag{3.2.21}$$

and is otherwise arbitrary.

It is useful to note that given a system of n Abelian vector multiplets W^I, with $I = 1,2,\dots n$, all satisfying (3.2.1), we can construct the following composite linear multiplet and its descendants [29]:

$$\begin{aligned} H^{ij} &= C_{JK}\Big\{ 2W^J X^{ijK} - \mathrm{i}\lambda^{\alpha J(i}\lambda_\alpha^{j)K}\Big\} , &(3.2.22a)\\ \varphi_\alpha^i &= C_{JK}\Big\{ \mathrm{i}X^{ijJ}\lambda_{\alpha j}^K - 2\mathrm{i}F_{\alpha\beta}^J\lambda^{\beta iK} - \frac{3}{2}X_\alpha^i W^J W^K - 2\mathrm{i}W^J\nabla_{\alpha\beta}\lambda^{\beta iK} \\ &\quad -\mathrm{i}(\nabla_{\alpha\beta}W^J)\lambda^{\beta iK} - 3\mathrm{i}W_{\alpha\beta}W^J\lambda^{\beta iK}\Big\} , &(3.2.22b)\\ F &= C_{JK}\Big\{ X^{ijJ}X_{ij}^K - F^{abJ}F_{ab}^K + 4W^J\Box W^K + 2(\nabla^a W^J)\nabla_a W^K \\ &\quad +2\mathrm{i}(\nabla_\alpha{}^\beta\lambda_\beta^{iJ})\lambda_i^{\alpha K} - 6W^{ab}F_{ab}^J W^K - \frac{39}{8}W^{ab}W_{ab}W^J W^K \\ &\quad +\frac{3}{8}YW^J W^K + 6X^{\alpha i}\lambda_{\alpha i}^J W^K - 3\mathrm{i}W_{\alpha\beta}\lambda^{\alpha iJ}\lambda_i^{\beta K}\Big\} , &(3.2.22c)\\ \mathcal{H}_a &= C_{JK}\Big\{ -\frac{1}{2}\varepsilon_{abcde}F^{bcJ}F^{deK} + 4\nabla^b\Big(W^J F_{ba}^K + \frac{3}{2}W_{ba}W^J W^K\Big) \\ &\quad +2\mathrm{i}(\Sigma_{ba})^{\alpha\beta}\nabla^b(\lambda_\alpha^{iJ}\lambda_{\beta i}^K)\Big\} , &(3.2.22d) \end{aligned}$$

where $\Box := \nabla^a\nabla_a$ and $C_{JK} = C_{(JK)}$ is a constant symmetric in J and K. Equation (3.2.22) is the superspace analogue of the composite linear multiplet constructed for the first time in [90]. Note that it is possible to create a composite linear multiplet from a single Abelian vector multiplet by setting $C_{JK} = 1$ when $J = K = 1$ and zero otherwise. This case will play an important role.

The linear multiplet in components

The components of the linear multiplet follows directly from the superfield description: an $\mathrm{SU}(2)_R$ triplet of Lorentz scalar fields $G^{ij} = G^{ij}|$; a spinor field $\varphi_{\alpha i} = \varphi_{\alpha i}|$; a scalar field $F = F|$; and a covariant closed anti-symmetric four-form field strength $\mathcal{H}_{abcd} := \mathcal{H}_{abcd}|$. The latter is equivalent to a conserved dual vector $\mathcal{H}^a := -1/4!\,\varepsilon^{abcde}\mathcal{H}_{bcde}$.[4] It holds that

$$\mathcal{H}^a = h^a + 2(\Sigma^{ab})_\alpha{}^\beta\psi_{bi}{}^\alpha\varphi_\beta^i - \frac{\mathrm{i}}{2}\varepsilon^{abcde}(\Sigma_{bc})_{\alpha\beta}\psi_{di}{}^\alpha\psi_{e}{}_j^\beta G^{ij} . \qquad (3.2.23)$$

The covariant conservation equation for $\mathcal{H}_a$ is

$$\nabla^a\mathcal{H}_a = 0 . \qquad (3.2.24)$$

The constraint implies that there exists a gauge three-form potential, b_{mnp}, and its exterior derivative, such that $h_{mnpq} := 4\partial_{[m}b_{npq]}$, with $b_{mnp} := \mathcal{B}_{mnp}|$.

The local superconformal transformations of the covariant fields can be derived using (3.2.14) and (3.2.17), which give

$$\delta G_{ij} = -2\xi_{(i}\varphi_{j)} - 2\lambda_{(i}{}^k G_{j)k} + 3\lambda_{\mathbb{D}}G_{ij} , \qquad (3.2.25a)$$

[4] The Levi-Civita tensor with world indices is defined as $\varepsilon^{mnpqr} := \varepsilon^{abcde}e_a{}^m e_b{}^n e_c{}^p e_d{}^q e_e{}^r$, such that ε_{abcde} and ε^{abcde} are normalised as $\varepsilon_{01234} = -\varepsilon^{01234} = 1$.

$$\begin{aligned}
\delta\varphi_{\alpha i} &= -\frac{\mathrm{i}}{2}\xi_{\alpha i}F - \frac{\mathrm{i}}{2}\mathcal{H}_a(\Gamma^a\xi_i)_\alpha - \mathrm{i}(\Gamma^a\xi^j)_\alpha\nabla_a G_{ij} + 6\eta_\alpha{}^j G_{ij} \\
&\quad + \frac{1}{2}\lambda^{ab}(\Sigma_{ab}\varphi_i)_\alpha - \lambda_i{}^j\varphi_{j\alpha} + \frac{7}{2}\lambda_{\mathbb{D}}\varphi_{\alpha i} , && (3.2.25b)\\
\delta F &= 2\xi^i\Gamma^a\nabla_a\varphi_i - \frac{3}{2}(\xi^i\Sigma^{ab}\varphi_i)W_{ab} + 16\mathrm{i}(\xi^i\chi^j)G_{ij} \\
&\quad + 6\mathrm{i}\,\eta^i\varphi_i + 4\lambda_{\mathbb{D}}F , && (3.2.25c)\\
\delta\mathcal{H}_a &= -4\xi^i\Sigma_{ab}\nabla^b\varphi_i - \frac{1}{2}(\xi^i\Gamma^b\varphi_i)W_{ab} + \frac{1}{2}\varepsilon_{abcde}W^{bc}(\xi^i\Sigma^{de}\varphi_i) \\
&\quad + \lambda_a{}^b\mathcal{H}_b + 4\lambda_{\mathbb{D}}\mathcal{H}_a - 8\mathrm{i}\eta^i\Gamma_a\varphi_i , && (3.2.25d)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a G_{ij} &= \mathcal{D}_a G_{ij} + \psi_{a(i}\varphi_{j)} , && (3.2.26a)\\
\nabla_a\varphi_{\alpha i} &= \mathcal{D}_a\varphi_{\alpha i} + \frac{\mathrm{i}}{4}\psi_{a\alpha i}F + \frac{\mathrm{i}}{4}(\Gamma^b\psi_{ai})_\alpha\mathcal{H}_b + \frac{\mathrm{i}}{2}(\Gamma^b\psi_a{}^j)_\alpha\nabla_b G_{ij} - 3\phi_{a\alpha}{}^j G_{ij} . && (3.2.26b)
\end{aligned}$$

For completeness, we also have

$$\begin{aligned}
\nabla_a F &= \mathcal{D}_a F - (\Gamma_b\psi_{ai})_\alpha\nabla^b\varphi^{i\alpha} + \frac{3}{2}\psi_{ai}{}^\beta W_{\alpha\beta}\varphi^{i\alpha} - 8\mathrm{i}G^{ij}\psi_{aj}{}^\alpha\chi_{i\alpha} + 3\mathrm{i}\phi_{ai}{}^\alpha\varphi^i_\alpha , && (3.2.26c)\\
\nabla_a\mathcal{H}_b &= \mathcal{D}_a\mathcal{H}_b - 2(\Sigma_{bc}\psi_{ai})_\alpha\nabla^c\varphi^{i\alpha} + \frac{3}{4}(\Gamma_b\psi_{ai})_\alpha W^{\alpha\beta}\varphi^i_\beta + \frac{1}{4}\psi_{ai}{}^\beta W_{\alpha\beta}(\Gamma_b\varphi^i)^\alpha \\
&\quad + 4\mathrm{i}(\Gamma_b\phi_{ai})_\alpha\varphi^{i\alpha} . && (3.2.26d)
\end{aligned}$$

The locally superconformal transformations of b_{mnp} are

$$\delta b_{mnp} = 2\,\varepsilon_{abcde}e_m{}^a e_n{}^b e_p{}^c(\xi_i\Sigma^{de}\varphi^i) - 12\mathrm{i}(\psi_{[m}{}^i\Sigma_{np]}\xi^j)G_{ij} + 3\partial_{[m}l_{np]} , \qquad (3.2.27)$$

where we have also included the gauge transformation $\delta_l b_{mnp} = 3\partial_{[m}l_{np]}$ leaving h_{mnpq} and $\mathcal{H}^a$ invariant. The dilatation weights of the $\mathcal{O}(2)$ multiplet are summarised in Table 3.2.

	G_{ij}	$\varphi_{\alpha i}$	F	$\mathcal{H}_a$	b_{mnp}
$\mathbb{D}$	3	7/2	4	4	0

Table 3.2: Dilatation weights of the off-shell $\mathcal{O}(2)$ multiplet.

3.3 Superconformal actions

In this section we review a main action principle that has played an important role in constructing various locally superconformal invariants including the two-derivative gauged supergravity action. The reader should refer to [29] for a more complete analysis.

3.3.1 BF action

Consider a full superspace integral

$$S[\mathcal{L}] = \int \mathrm{d}^{5|8}z\, E\,\mathcal{L} , \qquad \mathrm{d}^{5|8}z := \mathrm{d}^5x\,\mathrm{d}^8\theta , \qquad E := \mathrm{Ber}(E_M{}^A) , \qquad (3.3.1)$$

where the Lagrangian $\mathcal{L}$ is a conformal primary superfield of dimension $+1$, i.e, $\mathbb{D}\mathcal{L} = \mathcal{L}$. The above action is locally superconformal, that is, invariant under the supergravity gauge transformations (2.2.77).

One of the main building blocks for the construction of general supergravity-matter couplings [90, 172–177] and higher-derivative invariants in [29] is the so-called BF action principle. It is based on an appropriate product of a linear multiplet with an Abelian vector multiplet, which may be described by the following form in superspace

$$S_{\mathrm{BF}} = \int \mathrm{d}^{5|8}z\, E\, \Omega W = \int \mathrm{d}^{5|8}z\, E\, G^{ij} V_{ij}\ . \tag{3.3.2a}$$

As shown above, the BF action can be written in different ways, see [29] for even more variants. In the first form in (3.3.2a), it involves the field strength of the vector multiplet, W, and the prepotential of the linear multiplet, Ω. Equivalently, the BF action can be written in terms of the Mezincescu's prepotential V_{ij} and the field strength G^{ij} of the linear multiplet described by the right-hand side of (3.3.2a). In addition, the functionals $\int \mathrm{d}^{5|8}z\, E\, \Omega W$ and $\int \mathrm{d}^{5|8}z\, E\, G^{ij} V_{ij}$ are, respectively, invariant under the gauge transformations (3.2.20) and (3.2.8), thanks to the defining differential constraints satisfied by W and G^{ij}, (3.2.1b) and (3.2.13a).

In components, and in our notation, the BF action takes the form [29]

$$\begin{aligned} S_{\mathrm{B}F} &= -\int \mathrm{d}^5x\, e\Big(v_a \mathcal{H}^a + WF + X_{ij}G^{ij} + 2\lambda^{\alpha k}\varphi_{\alpha k} \\ &\quad - \psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \varphi^i_\beta W - \mathrm{i}\psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \lambda_{\beta j} G^{ij} + \mathrm{i}\psi_a{}^\alpha_i (\Sigma^{ab})_\alpha{}^\beta \psi_{b\beta j} W G^{ij}\Big) \qquad (3.3.2b) \\ &= -\int \mathrm{d}^5x\, e\Big(-\frac{1}{12}\varepsilon^{abcde} f_{ab} b_{cde} + WF + X_{ij}G^{ij} + 2\lambda^{\alpha k}\varphi_{\alpha k} \\ &\quad - \psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \varphi^i_\beta W - \mathrm{i}\psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \lambda_{\beta j} G^{ij} + \mathrm{i}\psi_a{}^\alpha_i (\Sigma^{ab})_\alpha{}^\beta \psi_{b\beta j} W G^{ij}\Big)\ , \qquad (3.3.2c) \end{aligned}$$

where it should be noted that sometimes we associate the same symbol for the covariant component fields and the corresponding superfields, when the interpretation is clear from the context.

3.3.2 Vector multiplet compensator

The two-derivative action for the vector multiplet compensator can be constructed via the BF action principle (3.3.2a), with the linear multiplet being a composite superfield. We denote by H^{ij}_{VM} the composite linear multiplet (3.2.22),

$$\begin{aligned} H^{ij}_{\mathrm{VM}} &= \mathrm{i}(\nabla^{\alpha(i}W)\nabla^{j)}_\alpha W + \frac{\mathrm{i}}{2} W \nabla^{\alpha(i}\nabla^{j)}_\alpha W \\ &= -\mathrm{i}\lambda^{\alpha i}\lambda^j_\alpha + 2WX^{ij}\ . \end{aligned} \tag{3.3.3}$$

One can then check that H^{ij}_{VM} is a primary superfield, $S^k_\alpha H^{ij}_{\mathrm{VM}} = 0$ and has dimension 3. Thanks to the Bianchi identity (3.2.1b), it also satisfies the analyticity constraint

$$\nabla^{(i}_\alpha H^{jk)}_{\mathrm{VM}} = 0\ . \tag{3.3.4}$$

The vector multiplet action may then be written as an integral over the full superspace,

$$S_{\rm VM} = \frac{1}{4}\int {\rm d}^{5|8}z\,E\,V_{ij}H^{ij}_{\rm VM} . \tag{3.3.5}$$

This invariant also admits another representation, which can be obtained by applying (3.3.2a) once more

$$S_{\rm VM} = \frac{1}{4}\int {\rm d}^{5|8}z\,E\,\mathbf{\Omega}_{\rm VM}W , \tag{3.3.6a}$$

where we have introduced the primary superfield [29]

$$\mathbf{\Omega}_{\rm VM} = \frac{\rm i}{4}\Big(W\nabla^{ij}V_{ij} - 2(\nabla^{\alpha i}V_{ij})\nabla_\alpha^j W - 2V_{ij}\nabla^{ij}W\Big) . \tag{3.3.6b}$$

It can be shown that, by making use of (3.2.6), (3.2.19), (3.3.6), and integration by parts, we obtain the following relation

$$\begin{aligned}\frac{1}{4}\int {\rm d}^{5|8}z\,E\,V_{ij}\,\delta H^{ij}_{\rm VM} &= \frac{1}{2}\int {\rm d}^{5|8}z\,E\,\delta W\,\mathbf{\Omega}_{\rm VM} &(3.3.7a)\\ &= \frac{1}{2}\Big(-\frac{3{\rm i}}{40}\Big)\int {\rm d}^{5|8}z\,E\,\mathbf{\Omega}_{\rm VM}\nabla_{ij}\Delta^{ijkl}\,\delta V_{kl} &(3.3.7b)\\ &= \frac{1}{2}\int {\rm d}^{5|8}z\,E\,\delta V_{ij}H^{ij}_{\rm VM} . &(3.3.7c)\end{aligned}$$

Making use of the representations (3.3.5) and (3.3.6), it is seen that the variation of $S_{\rm VM}$, induced by an arbitrary variation of the Mezincescu's prepotential V_{ij} reads

$$\delta S_{\rm VM} = \frac{3}{4}\int {\rm d}^{5|8}z\,E\,\delta V_{ij}H^{ij}_{\rm VM} . \tag{3.3.8}$$

The above variation vanishes when δV_{ij} is a gauge transformation (3.2.8). This implies that

$$\int {\rm d}^{5|8}z\,E\,\Lambda^\alpha{}_{ijk}\nabla_\alpha^{(k}H^{ij)}_{\rm VM} = 0 , \tag{3.3.9}$$

that is, $\nabla_\alpha^{(i}H^{jk)}_{\rm VM} = 0$. This result holds for any dynamical system involving an Abelian vector multiplet [29]. The variation with respect to the prepotential V_{ij} couples to a composite linear multiplet which depends on the specific form of the associated action principle – let us call this, in general, $\mathbf{H}^{ij}$ which satisfies by construction the constraints (3.2.13). The equation of motion (EOM) for a vector multiplet is then $\mathbf{H}^{ij} = 0$. In the case of eq. (3.3.5), the EOM for the vector multiplet compensator is $H^{ij}_{\rm VM} = 0$.

Let us now work out the component form of $S_{\rm VM}$. The bosonic part of such an action takes the form

$$S_{\rm VM} = -\frac{1}{4}\int {\rm d}^5x\,e\Big\{v^a\mathcal{H}_{a\rm VM} + WF_{\rm VM} + X_{ij}H^{ij}_{\rm VM}\Big\} . \tag{3.3.10}$$

This amounts to taking the bosonic sector[5] of the bar projection of eqs. (3.2.22) for a single Abelian vector multiplet:

$$H^{ij}_{\rm VM}|_{bosonic} = 2WX^{ij} , \tag{3.3.11a}$$

[5]Here and in what follows, whenever we only explicitly present the bosonic sectors, any supercovariant field strength (which contains fermionic terms) can also be replaced with its purely bosonic analogue, e.g., $\mathcal{H}^a$ with h_a, F_{ab} with f_{ab} and so on, as this will only modify the suppressed fermionic terms.

$$
\begin{aligned}
F_{\mathrm{VM}}|_{bosonic} &= X^{ij}X_{ij} - f^{ab}f_{ab} + 4W\nabla^a\nabla_a W + 2(\nabla^a W)\nabla_a W \\
&\quad -6W\,W^{ab}f_{ab} - \frac{39}{8}W^2W^{ab}W_{ab} - 16W^2 D\,, \qquad (3.3.11b) \\
\mathcal{H}_{a\mathrm{VM}}|_{bosonic} &= -\frac{1}{2}\varepsilon_{abcde}f^{bc}f^{de} + 4\nabla^b(W f_{ba} + \frac{3}{2}W^2 W_{ba})\,, \qquad (3.3.11c)
\end{aligned}
$$

and plugging (3.3.11) back into (3.3.10). This procedure results in

$$
\begin{aligned}
S_{\mathrm{VM}} &= \int \mathrm{d}^5x\,e\Big\{-\frac{1}{2}W(\nabla^a W)\nabla_a W - W^2\nabla^a\nabla_a W - \frac{3}{4}WX^{ij}X_{ij} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} \\
&\quad + \frac{3}{4}W f^{ab}f_{ab} + \frac{9}{4}W^2W^{ab}f_{ab} + \frac{39}{32}W^3W^{ab}W_{ab} + 4W^3D\Big\}\,. \qquad (3.3.12)
\end{aligned}
$$

The expression (3.3.12) can be written in terms of the degauged covariant derivative $\mathcal{D}_a$, as defined in (2.2.32). The term "degauging" refers to the process of reducing part of the local symmetry of the theory. To obtain the supergravity action from a superconformal invariant action, one must degauge the superconformal covariant derivative down to the Poincaré covariant derivative. Throughout the main body of this paper, whenever the degauging procedure is applied, we effectively replace the superconformal vector covariant derivative ∇_a using the following relation

$$\nabla_a = \mathcal{D}_a - \mathfrak{f}_a{}^b K_b\,, \qquad (3.3.13)$$

focusing only on the bosonic part. For now, the dilatation and R-symmetry components remain gauged but can be degauged separately if required. For results including fermionic terms, we refer to the supplementary file of our paper [1]. As such, we have that

$$\nabla^a\nabla_a W = \mathcal{D}^a\mathcal{D}_a W + \frac{1}{8}W\mathcal{R}\,. \qquad (3.3.14)$$

After performing integration by parts, one then arrives at

$$
\begin{aligned}
S_{\mathrm{VM}} &= \int \mathrm{d}^5x\,e\Big\{-\frac{1}{8}W^3\mathcal{R} + \frac{3}{2}W(\mathcal{D}^a W)\mathcal{D}_a W - \frac{3}{4}WX^{ij}X_{ij} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} \\
&\quad + \frac{3}{4}W f^{ab}f_{ab} + \frac{9}{4}W^2W^{ab}f_{ab} + \frac{39}{32}W^3W^{ab}W_{ab} + 4W^3D\Big\}\,. \qquad (3.3.15)
\end{aligned}
$$

Upon gauge fixing dilatation by imposing $W = 1$, the first term gives a scalar curvature term, $\mathcal{R}$. The gauge fixing $W = 1$ can be achieved by requiring $W \neq 0$ meaning that the vector multiplet is a conformal compensator.

3.3.3 Linear multiplet compensator

Within the BF action principle (3.3.2a), the dynamical part of the linear multiplet action is described by a vector multiplet built out of the linear multiplet. We denote by $\mathbf{W}$ the composite vector multiplet field strength:

$$\mathbf{W} := \frac{\mathrm{i}}{16}G\nabla^{\alpha i}\nabla_\alpha^j\Big(\frac{G_{ij}}{G^2}\Big) = \frac{1}{4}FG^{-1} - \frac{\mathrm{i}}{8}G_{ij}\varphi^{i\alpha}\varphi_\alpha^j G^{-3}\,, \qquad (3.3.16)$$

with

$$G := \sqrt{\frac{1}{2}G^{ij}G_{ij}} \tag{3.3.17}$$

being nowhere vanishing, $G \neq 0$. At the component level, the vector multiplet (3.3.16) was first derived by Zucker [192] as a 5D analogue of the improved 4D $N = 2$ tensor multiplet [191]. The field strength $\mathbf{W}$ obeys the constraints (3.2.1).

The component fields of the above composite vector multiplet in our notation have been worked out in [6], see [192] for the first analysis in components. They consist of the $\theta = 0$ projection of $\mathbf{W}$, along with the descendant components of $\mathbf{W}$: $\boldsymbol{\lambda}_\alpha^i = -\mathrm{i}\nabla_\alpha^i \mathbf{W}|$, $\mathbf{X}^{ij} = \frac{\mathrm{i}}{4}\nabla^{\alpha(i}\nabla_\alpha^{j)}\mathbf{W}|$ and $\mathbf{F}_{ab} = \frac{1}{4}(\Sigma_{ab})^{\alpha\beta}\nabla^k_{(\alpha}\boldsymbol{\lambda}_{\beta)k}| - W_{ab}\mathbf{W}|$. Their explicit expressions are

$$\begin{aligned}
\boldsymbol{\lambda}_\alpha^i &= G^{-1}\Big\{-\frac{\mathrm{i}}{2}\nabla_{\alpha\beta}\varphi^{\beta i} + \frac{3\mathrm{i}}{4}W_{\alpha\beta}\varphi^{\beta i} + 4G^{ij}\chi_{\alpha j}\Big\} \\
&\quad + G^{-3}\Big\{-\frac{\mathrm{i}}{8}FG^{ij}\varphi_{\alpha j} - \frac{\mathrm{i}}{8}G^{ij}\mathcal{H}_{\alpha\beta}\varphi_j^\beta + \frac{\mathrm{i}}{4}G_{jk}\varphi^{\beta k}\nabla_{\alpha\beta}G^{ij} + \frac{1}{4}\varphi^{\beta i}\varphi_\beta^j\varphi_{\alpha j}\Big\} \\
&\quad + G^{-5}\Big\{-\frac{3}{8}G^{ij}G_{kl}\varphi^{\beta k}\varphi_\beta^l\varphi_{\alpha j}\Big\},
\end{aligned} \tag{3.3.18a}$$

$$\begin{aligned}
\mathbf{X}^{ij} &= G^{-1}\Big\{\frac{1}{2}\Box G^{ij} + \frac{3}{64}W^{ab}W_{ab}G^{ij} + 2DG^{ij} + 8\chi^{\alpha(i}\varphi_\alpha^{j)}\Big\} \\
&\quad + G^{-3}\Big\{-\frac{1}{16}F^2G^{ij} - \frac{1}{16}\mathcal{H}^a\mathcal{H}_aG^{ij} + \frac{1}{4}\mathcal{H}^aG^{k(i}\nabla_aG_k{}^{j)} - \frac{1}{4}G_{kl}\big(\nabla^aG^{k(i}\big)\nabla_aG^{j)l} \\
&\qquad -4G^{ij}G_{kl}\chi^{\alpha k}\varphi_\alpha^l - \frac{\mathrm{i}}{8}F\varphi^{\alpha(i}\varphi_\alpha^{j)} + \frac{3\mathrm{i}}{8}W^{\alpha\beta}G^{ij}\varphi_\alpha^k\varphi_{\beta k} \\
&\qquad + \frac{\mathrm{i}}{16}(\Gamma^a)^{\alpha\beta}\Big(\mathcal{H}_a\varphi_\alpha^{(i}\varphi_\beta^{j)} + 8G^{k(i}\big(\nabla_a\varphi_\alpha^{j)}\big)\varphi_{\beta k} + 2\varphi_\alpha^{(i}\big(\nabla_aG^{j)k}\big)\varphi_{\beta k}\Big)\Big\} \\
&\quad + G^{-5}\Big\{\frac{3\mathrm{i}}{16}FG^{ij}G_{kl}\varphi^{\alpha k}\varphi_\alpha^l + \frac{3\mathrm{i}}{16}G^{k(i}G^{j)l}(\Gamma^a)_{\alpha\beta}\mathcal{H}_a\varphi_k^\alpha\varphi_l^\beta - \frac{3\mathrm{i}}{8}G^{mn}G^{k(i}\big(\nabla_{\alpha\beta}G_m{}^{j)}\big)\varphi_k^\alpha\varphi_n^\beta \\
&\qquad + \frac{3}{8}G^{k(i}\varphi_\alpha^{j)}\varphi^{\alpha l}\varphi_k^\beta\varphi_{\beta l} - \frac{3}{8}G^{kl}\varphi^{\alpha(i}\varphi_\alpha^{j)}\varphi_k^\beta\varphi_{\beta l}\Big\} \\
&\quad + G^{-7}\Big\{\frac{15}{32}G^{ij}G_{kl}G_{mn}\varphi^{\alpha k}\varphi_\alpha^l\varphi^{\beta m}\varphi_\beta^n\Big\},
\end{aligned} \tag{3.3.18b}$$

$$\begin{aligned}
\mathbf{F}_{ab} &= G^{-1}\Big\{\frac{1}{2}\nabla_{[a}\mathcal{H}_{b]} + \frac{1}{2}G_{ij}\Phi_{ab}{}^{ij} + \frac{\mathrm{i}}{4}W_{ab\alpha}{}^i\varphi_i^\alpha\Big\} \\
&\quad + G^{-3}\Big\{\frac{1}{4}G_{ij}\mathcal{H}_{[a}\nabla_{b]}G^{ij} - \frac{1}{4}G_{ij}\big(\nabla_{[a}G^{ik}\big)\nabla_{b]}G_k{}^j + \frac{\mathrm{i}}{2}G_{ij}(\Gamma_{[a})^{\alpha\beta}\big(\nabla_{b]}\varphi_\alpha^i\big)\varphi_\beta^j \\
&\qquad - \frac{\mathrm{i}}{8}(\Gamma_{[a})^{\alpha\beta}\big(\nabla_{b]}G_{ij}\big)\varphi_\alpha^i\varphi_\beta^j\Big\}
\end{aligned}$$

$$+G^{-5}\left\{-\frac{3\mathrm{i}}{8}G^k{}_iG^l{}_j(\Gamma_{[a})^{\alpha\beta}\left(\nabla_{b]}G_{kl}\right)\varphi^i_\alpha\varphi^j_\beta\right\}. \tag{3.3.18c}$$

The action for the linear multiplet compensator may be expressed as

$$S_{\rm L}=\int \mathrm{d}^{5|8}z\,E\,\Omega\mathbf{W}\,. \tag{3.3.19}$$

Varying the prepotential Ω leads to

$$\delta S_{\rm L}=\int \mathrm{d}^{5|8}z\,E\,\delta\Omega\,\mathbf{W}\,. \tag{3.3.20}$$

The previous form holds for the first-order variation of a matter system which includes a linear multiplet with respect to its prepotential Ω. In particular, the variation must vanish if $\delta\Omega$ is given by (3.2.20). This holds provided $\mathbf{W}$ satisfies the Bianchi identity (3.2.1b). In general, any dynamical system involving a linear multiplet possesses a composite vector multiplet $\mathbf{W}$. The EOM for the linear multiplet is $\mathbf{W}=0$, and for the linear multiplet action of eq. (3.3.19), this is given by $\mathbf{W}$ defined in (3.3.16).

In components, the linear multiplet action, $S_{\rm L}$, is obtained by making use of the component BF action (3.3.2c), with the fields of the vector multiplet now being composite. That is, we substitute the expressions (3.3.18) into (3.3.2c). After degauging (3.3.13), the bosonic part of $S_{\rm L}$ is given by [29]

$$\begin{aligned} S_{\rm L} &= \int \mathrm{d}^5x\,e\left\{-\frac{3}{8}G\mathcal{R}-4DG-\frac{1}{8G}F^2-\frac{3}{32}W^{ab}W_{ab}G+\frac{1}{4}G^{-1}(\mathcal{D}_aG^{ij})\mathcal{D}^aG_{ij}\right.\\ &\left.-\frac{1}{8}G^{-1}\mathcal{H}^a\mathcal{H}_a+\frac{1}{24}\varepsilon^{abcde}b_{cde}\left(\frac{1}{2}G^{-3}(\mathcal{D}_aG_{ik})(\mathcal{D}_bG_j{}^k)G^{ij}+G^{-1}\Phi_{ab}{}^{ij}G_{ij}\right)\right\}. \end{aligned} \tag{3.3.21}$$

3.3.4 Gauged supergravity action

An off-shell formulation for $5D$ minimal supergravity can be constructed by coupling the standard Weyl multiplet to two off-shell compensators: vector and linear multiplets [29, 90, 169–177]. The complete (gauged) supergravity action, S_{gSG}, is given by the following two-derivative action [29]:

$$S_{gSG} = S_{VM}+S_{\rm L}+\kappa S_{BF}=\int \mathrm{d}^{5|8}z\,E\left\{\frac{1}{4}V_{ij}H^{ij}_{\rm VM}+\Omega\mathbf{W}+\kappa V_{ij}G^{ij}\right\} \tag{3.3.22a}$$

$$= \int \mathrm{d}^{5|8}z\,E\left\{\frac{1}{4}V_{ij}H^{ij}_{\rm VM}+\Omega\mathbf{W}+\kappa\Omega W\right\}. \tag{3.3.22b}$$

The BF action κS_{BF} describes a supersymmetric cosmological term. The $\kappa=0$ case corresponds to Poincaré supergravity, while $\kappa\neq 0$ leads to gauged or anti-de Sitter supergravity. In components, it reads

$$\kappa S_{BF}=-\kappa\int \mathrm{d}^5x\,e\left\{WF+X^{ij}G_{ij}+v_a\mathcal{H}^a\right\}+\text{fermions}\,. \tag{3.3.23}$$

Upon gauge fixing dilatation and superconformal symmetries (dilatation, S and K) and integrating out the various auxiliary fields, one obtains the on-shell Poincare supergravity action of [162, 163]. The contributions from the scalar curvature terms in (3.3.15) and (3.3.21) combine to give the normalised Einstein-Hilbert term $-\frac{1}{2}\mathcal{R}$ plus a cosmological constant, see, e.g., [29] for details.

3.4 Curvature-squared multiplets in components

In [29], three independent curvature-squared invariants [29, 55, 58, 63, 64] were defined in superspace in the standard Weyl multiplet background. The full expressions of all the composite primary superfields generating these invariants were presented recently in [6]. In this section we describe the component structure of all these composite primary multiplets (including the fermionic terms which can be found in the supplementary file [1]), thus elaborating the results of [6].

3.4.1 Weyl-squared

In superspace, a supersymmetric completion of a Weyl-squared term can be generated using a composite linear multiplet $H^{ij}_{\mathrm{W}eyl}$ built out of the super Weyl tensor and its descendants [29]. The appropriate composite linear multiplet is described by the primary superfield

$$H^{ij}_{\mathrm{Weyl}} := -\frac{\mathrm{i}}{2}W^{\alpha\beta\gamma i}W_{\alpha\beta\gamma}{}^{j} + \frac{3\mathrm{i}}{2}W^{\alpha\beta}X_{\alpha\beta}{}^{ij} - \frac{3\mathrm{i}}{4}X^{\alpha i}X^{j}_{\alpha} . \tag{3.4.1}$$

Here $H^{ij}_{\mathrm{W}eyl}$ satisfies the constraints (3.2.13), and corresponds to the composite linear multiplet which was first constructed in components by Hanaki, Ohashi, and Tachikawa in [55].

With the aid of relations (2.2.74), along with the definition of the Weyl multiplet components given in (2.2.82), the components of $H^{ij}_{\mathrm{W}eyl}$ are straightforward to compute. They include the $\theta = 0$ projection of $H^{ij}_{\mathrm{W}eyl}$, together with the descendant components:

$$\varphi^{\alpha i}_{\mathrm{W}eyl} = \frac{1}{3}\nabla^{\alpha}_{j}H^{ij}_{\mathrm{W}eyl}| , \tag{3.4.2a}$$

$$F_{\mathrm{W}eyl} = \frac{\mathrm{i}}{12}\nabla^{\alpha}_{i}\nabla_{\alpha j}H^{ij}_{\mathrm{Weyl}}| , \tag{3.4.2b}$$

$$\mathcal{H}^{a}_{\mathrm{W}eyl} = \frac{\mathrm{i}}{12}(\Gamma^{a})^{\alpha\beta}\nabla_{\alpha i}\nabla_{\beta j}H^{ij}_{\mathrm{Weyl}}| . \tag{3.4.2c}$$

The bosonic part of the descendants (3.4.2) was given in [55], while its complete structure including fermions is given for the first time in this paper and can be found in the supplementary file [1]. Below we have omitted the fermionic contributions in the expressions for $F_{\mathrm{W}eyl}$ and $\mathcal{H}^{a}_{\mathrm{W}eyl}$.

$$\begin{aligned}
H^{ij}_{\mathrm{Weyl}} &= -\frac{\mathrm{i}}{2}W^{\alpha\beta\gamma i}W_{\alpha\beta\gamma}{}^{j} - W^{ab}\Phi_{ab}{}^{ij} + \frac{256\mathrm{i}}{3}\chi^{\alpha i}\chi^{j}_{\alpha} , &\quad (3.4.3a)\\
\varphi^{i}_{\alpha\,\mathrm{W}eyl} &= C_{\alpha\beta\gamma\lambda}W^{\beta\gamma\lambda i} + \frac{3}{4}W_{\alpha\beta}W_{\gamma\lambda}W^{\beta\gamma\lambda i} - \frac{3}{4}W_{\beta\gamma\lambda}{}^{i}\nabla_{\alpha}{}^{\beta}W^{\gamma\lambda}\\
&\quad + \frac{3}{4}W_{\alpha\beta\lambda}{}^{i}\nabla^{\beta\gamma}W_{\gamma}{}^{\lambda} - \Phi^{\beta\gamma ij}W_{\alpha\beta\gamma j} - \frac{32\mathrm{i}}{3}\Phi_{\alpha\beta}{}^{ij}\chi^{\beta}_{j} - \frac{3}{2}W^{\gamma\lambda}\nabla_{\alpha}{}^{\beta}W_{\beta\gamma\lambda}{}^{i}\\
&\quad - 8\mathrm{i}W^{\beta\gamma}W_{\beta\gamma}\chi^{i}_{\alpha} - \frac{3}{2}W_{\beta\lambda}\nabla^{\beta\gamma}W_{\alpha\gamma}{}^{\lambda i} - 8\mathrm{i}W_{\gamma\lambda}\nabla_{\alpha}{}^{\gamma}\chi^{\lambda i} + 8\mathrm{i}W_{\alpha\beta}\nabla^{\beta\rho}\chi^{i}_{\rho}\\
&\quad - \frac{128\mathrm{i}}{3}D\chi^{i}_{\alpha} - 12\mathrm{i}\chi^{\lambda i}\nabla_{\alpha}{}^{\gamma}W_{\gamma\lambda} + 4\mathrm{i}\chi^{i}_{\beta}\nabla^{\beta\rho}W_{\alpha\rho} , &\quad (3.4.3b)\\
F_{\mathrm{W}eyl} &= \frac{128}{3}D^{2} + 8W^{ab}W_{ab}D - \frac{2}{3}\Phi^{abij}\Phi_{abij} + \frac{1}{4}C^{abcd}C_{abcd} + \frac{15}{8}C^{abcd}W_{ab}W_{cd}\\
&\quad + \frac{81}{32}W^{ab}W_{bc}W^{cd}W_{da} - 3W_{cd}\nabla_{b}\nabla^{c}W^{bd} - \frac{33}{128}W^{ab}W_{ab}W^{cd}W_{cd}
\end{aligned}$$

$$+\frac{3}{2}\nabla^a W^{ce}\nabla_c W_{ae} - \frac{3}{2}\nabla^a W^{ce}\nabla_a W_{ce} + \frac{9}{16}\varepsilon^{abcde}W_{ab}W_{cd}\nabla^f W_{ef} \ , \tag{3.4.3c}$$

$$\mathcal{H}^a_{\text{Weyl}} = \frac{1}{8}\varepsilon^{abcde}C_{bc}{}^{e_1e_2}C_{dee_1e_2} - \frac{1}{6}\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij} + \nabla_b\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} \ , \tag{3.4.3d}$$

where

$$\begin{aligned}\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} &= 8W^{ab}D + \frac{3}{2}C^{abcd}W_{cd} + \frac{3}{16}W^{ab}W^{cd}W_{cd} - \frac{9}{4}W^{ac}W_{cd}W^{db} \\ &\quad -\frac{9}{4}\varepsilon^{abcde}W_{ce_1}\nabla_d W_e{}^{e_1} - \frac{3}{2}\varepsilon^{abcde}W_{ce_1}\nabla^{e_1}W_{de} \ . \end{aligned} \tag{3.4.3e}$$

Here the Weyl tensor C_{abcd} is defined through

$$\begin{aligned} C_{abcd} := \mathscr{R}(M)_{abcd} &= -\frac{\mathrm{i}}{4}W_{abcd} - \frac{1}{2}W_{ab}W_{cd} - \frac{1}{8}\eta_{a[c}\eta_{d]b}W^{ef}W_{ef} \\ &\quad +\frac{1}{2}W_{a[c}W_{d]b} - W_{e[a}\eta_{b][c}W^e{}_{d]} \ . \end{aligned} \tag{3.4.4}$$

We also note the relation

$$\nabla_a\nabla_b W_{cd} = \mathcal{D}_a\mathcal{D}_b W_{cd} - 2\mathfrak{f}_{ab}W_{cd} + 4\mathfrak{f}_{a[c}W_{d]b} - 4\mathfrak{f}_{af}\eta_{b[c}W_{d]}{}^f + \cdots \ , \tag{3.4.5}$$

with

$$\mathfrak{f}_{ab} = -\frac{1}{6}\mathcal{R}_{ab} + \frac{1}{48}\eta_{ab}\mathcal{R} + \dots \ , \qquad \mathfrak{f}_a{}^a = -\frac{1}{16}\mathcal{R} + \dots \ , \tag{3.4.6}$$

and the ellipsis corresponding to omitted fermionic contributions. Having obtained the full component expressions, the first non-trivial check we have done is to show that the divergence of $\mathcal{H}^a_{\text{Weyl}}$ vanishes up to fermions, that is $\nabla_a\mathcal{H}^a_{\text{Weyl}} = 0$. One can show that this holds with the use of the Bianchi identities on C_{abcd} and $\Phi_{ab}{}^{ij}$, along with the fact that $\nabla_a\nabla_b\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} = 0$ up to fermions as $\tilde{\mathcal{H}}^{ab}_{\text{Weyl}}$ is a K-invariant, $SU(2)$-singlet, antisymmetric tensor. In addition, we have checked explicitly that all the component fields described in (3.4.3) are K-primary.

3.4.2 Log

As an extension of the higher-derivative chiral invariant in $4D$ $\mathcal{N}=2$ supergravity [61], Ref. [29] proposed a composite linear superfield which describes a supersymmetric Ricci tensor-squared term. This primary superfield, $H^{ij}_{\log}$, makes use of the standard Weyl multiplet coupled to the off-shell vector multiplet compensator. It is given by[6]

$$H^{ij}_{\log} = -\frac{3\mathrm{i}}{40}\Delta^{ijkl}\nabla_{kl}\log W = \frac{3\mathrm{i}}{1280}\nabla^{(ij}\nabla^{kl)}\nabla_{kl}\log W \ . \tag{3.4.7}$$

Due to the complexity in computing the action of up to eight spinor derivatives on the "log multiplet", the component structure of $H^{ij}_{\log}$ has just been studied recently. With the aid of the *Cadabra* software [87, 88], the full expression of $H^{ij}_{\log}$ in its expanded form in terms of the descendants of W and $W_{\alpha\beta}$ appeared for the first time in [6]. Its lowest component, $H^{ij}_{\log}|$, is given by

$$H^{ij}_{\log} = \frac{1}{2}W_{ab}\Phi^{abij} - \frac{272\mathrm{i}}{3}\chi^{\alpha i}\chi^j_\alpha$$

[6]Note the overall minus sign difference between the definition of the $\log W$ invariant in this paper and the one of [29].

$$
\begin{aligned}
&+W^{-1}\Big\{-6X^{ij}D+\frac{1}{2}F_{ab}\Phi^{abij}-\frac{1}{2}\Box X^{ij}-\frac{9}{64}X^{ij}W^{ab}W_{ab}+\frac{1}{4}W^{ab}W_{ab}{}^{\alpha(i}\lambda_\alpha^{j)}\\
&\qquad+8\mathrm{i}(\Gamma^a)^{\alpha\beta}\chi_\alpha^{(i}\nabla_a\lambda_\beta^{j)}-8\mathrm{i}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^{(i}\nabla_a\chi_\beta^{j)}\Big\}\\
&+W^{-2}\Big\{\frac{1}{2}X^{ij}\Box W+\frac{1}{2}\big(\nabla^aW\big)\nabla_aX^{ij}+\frac{1}{4}F^{ab}W_{ab}{}^{\alpha(i}\lambda_\alpha^{j)}-\frac{\mathrm{i}}{2}\lambda^{\alpha(j}\Box\lambda_\alpha^{i)}\\
&\qquad-\frac{\mathrm{i}}{4}\big(\nabla^a\lambda^{\alpha i}\big)\nabla_a\lambda_\alpha^j-\frac{3\mathrm{i}}{16}\varepsilon^{abcde}(\Sigma_{ab})^{\alpha\beta}W_{de}\lambda_\alpha^{(i}\nabla_c\lambda_\beta^{j)}-\frac{3\mathrm{i}}{8}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^i\lambda_\beta^j\nabla^cW_{ac}\\
&\qquad-2\mathrm{i}D\lambda^{\alpha i}\lambda_\alpha^j-4\mathrm{i}(\Sigma_{ab})^{\alpha\beta}F^{ab}\chi_\alpha^{(i}\lambda_\beta^{j)}+\frac{3\mathrm{i}}{128}W^{ab}W_{ab}\lambda^{\alpha i}\lambda_\alpha^j\\
&\qquad+\frac{9\mathrm{i}}{256}\varepsilon^{abcde}(\Gamma_a)^{\alpha\beta}W_{bc}W_{de}\lambda_\alpha^i\lambda_\beta^j+4\mathrm{i}X^{ij}\chi^{\alpha k}\lambda_{\alpha k}\Big\}\\
&+W^{-3}\Big\{\frac{1}{8}X^{ij}F^{ab}F_{ab}-\frac{1}{8}X^{ij}X^{kl}X_{kl}-\frac{1}{4}X^{ij}\big(\nabla^aW\big)\nabla_aW\\
&\qquad-\frac{\mathrm{i}}{8}\varepsilon^{abcde}(\Sigma_{ab})^{\alpha\beta}F_{de}\lambda_\alpha^{(i}\nabla_c\lambda_\beta^{j)}-\frac{\mathrm{i}}{4}(\Gamma^a)^{\alpha\beta}F_{ab}\lambda_\alpha^{(i}\nabla^b\lambda_\beta^{j)}\\
&\qquad-\frac{\mathrm{i}}{4}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^i\lambda_\beta^j\nabla^cF_{ac}+\frac{\mathrm{i}}{4}(\Gamma^a)^{\alpha\beta}X^{ij}\lambda_\alpha^k\nabla_a\lambda_{\beta k}+\frac{\mathrm{i}}{4}(\Gamma^a)^{\alpha\beta}X^{k(i}\lambda_\alpha^{j)}\nabla_a\lambda_{\beta k}\\
&\qquad-\frac{\mathrm{i}}{4}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^{(i}\big(\nabla_aX^{j)l}\big)\lambda_{\beta l}+\frac{\mathrm{i}}{4}\lambda^{\alpha i}\lambda_\alpha^j\Box W+\frac{3\mathrm{i}}{4}\big(\nabla^aW\big)\lambda^{\alpha(i}\nabla_a\lambda_\alpha^{j)}\\
&\qquad-\frac{\mathrm{i}}{2}(\Sigma_{ab})^{\alpha\beta}\big(\nabla^aW\big)\lambda_\alpha^{(i}\nabla^b\lambda_\beta^{j)}-\frac{3\mathrm{i}}{16}(\Gamma^a)^{\alpha\beta}W_{ab}\lambda_\alpha^i\lambda_\beta^j\nabla^bW+\frac{3\mathrm{i}}{32}W_{ab}F^{ab}\lambda^{\alpha i}\lambda_\alpha^j\\
&\qquad+\frac{9\mathrm{i}}{64}\varepsilon^{abcde}(\Gamma_a)^{\alpha\beta}W_{bc}F_{de}\lambda_\alpha^i\lambda_\beta^j-\frac{3\mathrm{i}}{32}(\Sigma_{ab})^{\alpha\beta}X^{ij}W^{ab}\lambda_\alpha^k\lambda_{\beta k}\\
&\qquad-4\chi^{\alpha k}\lambda_\alpha^{(i}\lambda^{\beta j)}\lambda_{\beta k}-4\chi^{\alpha k}\lambda^{\beta i}\lambda_\beta^j\lambda_{\alpha k}\Big\}\\
&+W^{-4}\Big\{-\frac{3\mathrm{i}}{16}\lambda^{\alpha i}\lambda_\alpha^j\big(\nabla^aW\big)\nabla_aW-\frac{3\mathrm{i}}{8}(\Gamma^a)^{\alpha\beta}X^{k(i}\lambda_\alpha^{j)}\lambda_{\beta k}\nabla_aW\\
&\qquad+\frac{3\mathrm{i}}{8}(\Gamma^a)^{\alpha\beta}F_{ab}\lambda_\alpha^i\lambda_\beta^j\nabla^bW+\frac{3\mathrm{i}}{32}F^{ab}F_{ab}\lambda^{\alpha i}\lambda_\alpha^j+\frac{3\mathrm{i}}{64}\varepsilon^{abcde}(\Gamma_a)^{\alpha\beta}F_{bc}F_{de}\lambda_\alpha^i\lambda_\beta^j\\
&\qquad-\frac{3\mathrm{i}}{16}(\Sigma_{ab})^{\alpha\beta}X^{ij}F^{ab}\lambda_\alpha^k\lambda_{\beta k}-\frac{3\mathrm{i}}{16}X^{ij}X^{kl}\lambda_k^\alpha\lambda_{\alpha l}-\frac{3\mathrm{i}}{32}X^{kl}X_{kl}\lambda^{\alpha i}\lambda_\alpha^j\\
&\qquad-\frac{15}{64}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^i\lambda_\beta^j\lambda^{\gamma k}\nabla_a\lambda_{\gamma k}-\frac{9}{32}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^{(i}\lambda^{\rho j)}\lambda_\beta^k\nabla_a\lambda_{\rho k}\\
&\qquad-\frac{15}{32}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^{(i}\lambda^{\rho j)}\lambda_\rho^k\nabla_a\lambda_{\beta k}-\frac{15}{64}(\Gamma^a)^{\alpha\beta}\lambda^{\rho i}\lambda_\rho^j\lambda_\alpha^k\nabla_a\lambda_{\beta k}\\
&\qquad+\frac{9}{32}(\Sigma_{ab})^{\alpha\beta}W^{ab}\lambda_\alpha^{(i}\lambda^{\rho j)}\lambda_\beta^k\lambda_{\rho k}+\frac{9}{64}(\Sigma_{ab})^{\alpha\beta}W^{ab}\lambda^{\rho i}\lambda_\rho^j\lambda_\alpha^k\lambda_{\beta k}\Big\}\\
&+W^{-5}\Big\{\frac{3}{8}(\Gamma^a)^{\alpha\beta}\lambda_\alpha^{(i}\lambda^{\rho j)}\lambda_\beta^k\lambda_{\rho k}\nabla_aW+\frac{3}{8}(\Sigma_{ab})^{\alpha\beta}F^{ab}\lambda^{\rho i}\lambda_\rho^j\lambda_\alpha^k\lambda_{\beta k}\\
&\qquad+\frac{3}{8}X^{kl}\lambda^{\alpha i}\lambda_\alpha^j\lambda_k^\beta\lambda_{\beta l}\Big\}
\end{aligned}
$$

$$+W^{-6}\Big\{\frac{3\mathrm{i}}{32}\lambda^{\alpha i}\lambda_\alpha^j\lambda^{\beta k}\lambda_\beta^l\lambda_k^\gamma\lambda_{\gamma l}+\frac{3\mathrm{i}}{16}\lambda^{\alpha i}\lambda_\alpha^k\lambda^{\beta j}\lambda_\beta^l\lambda_k^\gamma\lambda_{\gamma l}\Big\}\,. \tag{3.4.8a}$$

The full expressions of the descendant component fields of $H_{\log}^{ij}$, including the fermionic terms, are very involved and can be found in the supplementary file [1]. In this subsection, we will only present the bosonic sectors. The component $F_{\log}=\frac{\mathrm{i}}{12}\nabla_i^\alpha\nabla_{\alpha j}H_{\log}^{ij}|$ takes the form

$$\begin{aligned}
F_{\log} \;=\;& -\frac{1}{2}W^{ab}W^{cd}C_{abcd}+\frac{8}{3}D^2+4\Box D+\frac{368}{3}W^{ab}W_{ab}D-\frac{1}{6}\Phi^{abij}\Phi_{abij}\\
&+2W_{cd}\nabla^c\nabla_aW^{ad}-\frac{1}{2}W_{cd}\nabla_a\nabla^cW^{ad}+\frac{39}{16}W_{cd}\Box W^{cd}-\frac{1005}{2048}W^{ab}W_{ab}W^{cd}W_{cd}\\
&-\frac{9}{4}\nabla^aW_{ab}\nabla_cW^{bc}-\frac{3}{4}\nabla^aW^{bc}\nabla_bW_{ac}+\frac{33}{16}\nabla^aW^{bc}\nabla_aW_{bc}+\frac{9}{16}\varepsilon^{abcde}W_{bc}W_{de}\nabla^fW_{af}\\
&+W^{-1}\Big\{-F_{cd}\nabla^c\nabla_aW^{ad}-\frac{1}{2}F_{cd}\nabla_a\nabla^cW^{ad}+\frac{3}{2}F_{cd}\Box W^{cd}+2W_{cd}\nabla^c\nabla_aF^{ad}\\
&\qquad+\frac{1}{2}W_{cd}\Box F^{cd}-4D\Box W-\Box\Box W-4\nabla^aW\nabla_aD-\frac{1}{4}W^{ab}F^{cd}C_{abcd}\\
&\qquad-\frac{9}{2}W_a{}^cW_{cb}\nabla^a\nabla^bW+\frac{9}{2}W_{cd}\nabla_aW\nabla^cW^{ad}-\frac{3}{16}W_{cd}\nabla_aW\nabla^aW^{cd}\\
&\qquad+\frac{9}{2}W_{cd}\nabla^cW\nabla_aW^{ad}-\frac{3}{32}W^{cd}W_{cd}\Box W-\frac{3}{2}\nabla^aF_{ab}\nabla_cW^{bc}\\
&\qquad+\frac{3}{2}\nabla^aF^{bc}\nabla_aW_{bc}+\frac{9}{16}\varepsilon^{abcde}\Big(2W_{cd}F_{ef}\nabla^fW_{ab}+W_{bc}W_{de}\nabla^fF_{af}\Big)\Big\}\\
&+W^{-2}\Big\{-F^{ab}F_{ab}D-\frac{1}{2}F_{cd}\Box F^{cd}+\frac{1}{4}F^{ab}F^{cd}C_{abcd}-\frac{75}{128}W^{ab}W_{ab}F^{cd}F_{cd}\\
&\qquad+\frac{9}{4}W^{ab}W_{bc}F^{cd}F_{da}-\frac{3}{2}F_{cd}\nabla_aW\nabla^aW^{cd}-\frac{3}{2}W_{cd}\nabla_aW\nabla^aF^{cd}\\
&\qquad-\frac{3}{2}W^{cd}F_{cd}\Box W+2D\nabla^aW\nabla_aW+2\nabla^aW\nabla_a\Box W+\Box W\Box W\\
&\qquad+\frac{1}{2}\nabla^a\nabla^bW\nabla_a\nabla_bW+3X^{ij}X_{ij}D+\frac{1}{2}X^{ij}\Box X_{ij}+\frac{1}{4}\nabla^aX^{ij}\nabla_aX_{ij}\\
&\qquad+\frac{9}{4}W_a{}^cW_{cb}\nabla^aW\nabla^bW+\frac{3}{64}W^{cd}W_{cd}\nabla^aW\nabla_aW-\frac{1}{4}\nabla^aF^{bc}\nabla_aF_{bc}\\
&\qquad-\frac{1}{2}X^{ij}F^{ab}\Phi_{abij}+\frac{9}{128}X^{ij}X_{ij}W^{ab}W_{ab}+\frac{3}{8}\varepsilon^{abcde}W_{cd}F_{ef}\nabla^fF_{ab}\\
&\qquad-\frac{3}{8}\varepsilon^{abcde}F_{bc}F_{de}\nabla^fW_{af}\Big\}\\
&+W^{-3}\Big\{-\frac{3}{4}W^{ab}F^{cd}F_{ab}F_{cd}+\frac{3}{2}W^{ab}F^{cd}F_{ac}F_{bd}-F_a{}^cF_{cb}\nabla^a\nabla^bW\\
&\qquad+F_{cd}\nabla^aW\nabla_aF^{cd}+F_{cd}\nabla^cW\nabla_aF^{ad}+\frac{1}{4}F^{cd}F_{cd}\Box W\\
&\qquad+\frac{3}{2}W^{cd}F_{cd}\nabla^aW\nabla_aW-\frac{3}{2}\nabla^aW\nabla_aW\Box W-\nabla^aW\nabla^bW\nabla_a\nabla_bW\\
&\qquad-\frac{3}{4}X^{ij}X_{ij}\Box W-\frac{3}{2}X^{ij}\nabla^aW\nabla_aX_{ij}-\frac{1}{8}\varepsilon^{abcde}F_{bc}F_{de}\nabla^fF_{af}\Big\}
\end{aligned}$$

$$+W^{-4}\Big\{-\frac{3}{32}F^{ab}F_{ab}F^{cd}F_{cd}+\frac{3}{8}F^{ab}F_{bc}F^{cd}F_{da}+\frac{3}{2}F_a{}^cF_{cb}\nabla^aW\nabla^bW$$
$$-\frac{3}{8}F^{cd}F_{cd}\nabla^aW\nabla_aW-\frac{3}{16}X^{ij}X_{ij}F^{ab}F_{ab}+\frac{3}{8}(\nabla^aW\nabla_aW)^2$$
$$+\frac{9}{8}X^{ij}X_{ij}\nabla^aW\nabla_aW+\frac{3}{32}X^{ij}X_{ij}X^{kl}X_{kl}\Big\}$$
$$+\,\text{fermions}\,. \tag{3.4.8b}$$

The initial expression for the top component, $\mathcal{H}^a_{\log}=\frac{\mathrm{i}}{12}(\Gamma^a)^{\alpha\beta}\nabla_{\alpha i}\nabla_{\beta j}H^{ij}_{\log}|$, obtained from the computer algebra program *Cadabra*, is quite extensive and contains 261 terms in the bosonic part alone. For completeness, this expression is given in the supplementary file. However, there are ample opportunities for simplification by utilising the algebra of commutators (B.2.4), symmetry properties of the Weyl tensor and the Bianchi identities (2.2.75), (3.2.4). Additionally, many terms in the expression can be further combined as they exhibit similar structures and are linearly dependent, see supplementary file [1].

The ultimate goal is to express $\mathcal{H}^a_{\log}$ in terms of total derivatives as much as possible, that is, $\mathcal{H}^a_{\log}=\nabla_b\tilde{\mathcal{H}}^{ab}_{\log}+\dots$. The reason is that, once coupled to the vector potential v_a in the component BF action (3.3.2b), this term will be proportional to $F_{ab}\tilde{\mathcal{H}}^{ab}_{\log}$ upon integration by parts. This task, however, proves to be challenging, as determining the most suitable combinations of terms to combine into total derivatives is not obvious beforehand. Despite these difficulties, the expression provided by *Cadabra* remains approachable, and progress has been made by manually combining terms to reach a final expression of the form

$$\begin{aligned}\mathcal{H}^a_{\log} &= \frac{1}{12}\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij}-\frac{3}{8}\nabla_bC^{abcd}W_{cd}\\ &\quad -\frac{3}{8}W^{-1}\nabla_bC^{abcd}F_{cd}+\nabla_b\tilde{\mathcal{H}}^{ab}_{\log}+\text{fermions}\,,\end{aligned} \tag{3.4.8c}$$

where $\tilde{\mathcal{H}}^{ab}_{\log}$ is antisymmetric and is given by

$$\begin{aligned}\tilde{\mathcal{H}}^{ab}_{\log} &= -4W^{ab}D+\frac{9}{4}\varepsilon^{abcde}W_{cd}\nabla^fW_{ef}+\frac{3}{4}\varepsilon^{abcde}W_{cf}\nabla^fW_{de}-\frac{3}{32}W^{ab}W^{cd}W_{cd}\\
&\quad -\frac{9}{2}W^{ac}W_{cd}W^{bd}-\frac{3}{2}C^{abcd}W_{cd}+\frac{3}{2}\Box W^{ab}\\
&\quad +W^{-1}\Big\{4F^{ab}D+\frac{3}{4}\varepsilon^{abcde}F_{cf}\nabla^fW_{de}+\frac{3}{2}\varepsilon^{abcde}F_{de}\nabla^fW_{cf}+\frac{3}{2}\varepsilon^{abcde}W_{de}\nabla^fF_{cf}\\
&\qquad +\frac{3}{4}\varepsilon^{abcde}W_{ef}\nabla^fF_{cd}+\frac{9}{8}\varepsilon^{cdef[b}\nabla^{a]}WW_{cd}W_{ef}+3\nabla^cW^{[b}{}_c\nabla^{a]}W\\
&\qquad +6\nabla^{[a}W_c{}^{b]}\nabla^cW+3W^{ab}\Box W+\Box F^{ab}+\frac{75}{32}W^{cd}W_{cd}F^{ab}\\
&\qquad +9W^{c[a}W_{cd}F^{b]d}+\Phi^{ab}{}_{ij}X^{ij}-C^{abcd}F_{cd}\Big\}\\
&\quad +W^{-2}\Big\{\frac{1}{4}\varepsilon^{abcde}F_{cf}\nabla^fF_{de}+\frac{1}{2}\varepsilon^{abcde}F_{cd}\nabla^fF_{ef}-\frac{3}{8}\varepsilon^{abcde}W_{cd}F_{ef}\nabla^fW\\
&\qquad +\frac{3}{8}\varepsilon^{abcde}W_{cf}F_{de}\nabla^fW+\frac{3}{8}\varepsilon^{cdef[a}W_{cd}F_{ef}\nabla^{b]}W-\frac{1}{2}F^{ab}\Box W\end{aligned}$$

$$
\begin{aligned}
&+F^{c[a}\nabla^{b]}\nabla_c W+\nabla^{[a}W\nabla_c F^{b]c}-\frac{3}{2}W^{ab}\nabla^c W\nabla_c W+3W^{c[a}F_{cd}F^{b]d}\\
&+\frac{3}{4}W^{ab}F^{cd}F_{cd}+\frac{3}{2}F^{c[a}W_{cd}F^{b]d}-\frac{1}{2}\nabla^c W\nabla_c F^{ab}+\frac{3}{2}F^{ab}W^{cd}F_{cd}\Big\}\\
+W^{-3}\Big\{&\frac{1}{4}\varepsilon^{cdef[a}F_{cd}F_{ef}\nabla^{b]}W+\frac{1}{2}F^{ab}\nabla^c W\nabla_c W\\
&+\frac{1}{4}F^{ab}F^{cd}F_{cd}+F^{ac}F_{cd}F^{db}+\frac{1}{4}X^{ij}X_{ij}F^{ab}\Big\}\ . \qquad (3.4.8d)
\end{aligned}
$$

We also note the following useful relations (up to fermions)

$$
\begin{aligned}
\nabla_a\nabla_b W &= \mathcal{D}_a\mathcal{D}_b W-2\mathfrak{f}_{ab}W+\dots\ , && (3.4.9a)\\
\nabla_a\nabla_b X^{ij} &= \mathcal{D}_a\mathcal{D}_b X^{ij}-4\mathfrak{f}_{ab}X^{ij}+\dots\ , && (3.4.9b)\\
\nabla_a\nabla_b F_{cd} &= \mathcal{D}_a\mathcal{D}_b F_{cd}-4\mathfrak{f}_{ab}F_{cd}+4\mathfrak{f}_{a[c}F_{d]b}-4\mathfrak{f}_{af}\eta_{b[c}F_{d]}{}^f+\dots\ , && (3.4.9c)\\
\nabla_a\Box W &= \mathcal{D}_a\mathcal{D}_b\mathcal{D}^b W-2\mathcal{D}_a(\mathfrak{f}_b{}^b W)+2\mathfrak{f}_{ab}\mathcal{D}^b W+\dots\ , && (3.4.9d)\\
\Box\Box W &= \mathcal{D}_a\mathcal{D}^a\mathcal{D}_b\mathcal{D}^b W-2\mathcal{D}_a\mathcal{D}^a(\mathfrak{f}_b{}^b W)+4\mathfrak{f}_{ab}\mathcal{D}^a\mathcal{D}^b W\\
&\quad+2\mathcal{D}^a\mathfrak{f}_{ab}\mathcal{D}^b W-6\mathfrak{f}_a{}^a\mathcal{D}_b\mathcal{D}^b W-4\mathfrak{f}_{ab}\mathfrak{f}^{ab}W+12\mathfrak{f}_a{}^a\mathfrak{f}_b{}^b W+\dots\ , && (3.4.9e)\\
\nabla_a\nabla_b D &= \mathcal{D}_a\mathcal{D}_b D-4\mathfrak{f}_{ab}D+\dots\ . && (3.4.9f)
\end{aligned}
$$

It is useful to express the component field $\mathcal{H}^a_{\log}$ in terms of the degauged covariant derivative $\mathcal{D}_a$. Degauging all the covariant derivative ∇_a in (3.4.8c) but not those implicit in $\tilde{\mathcal{H}}^{ab}_{\log}$ results in the following expression:

$$
\begin{aligned}
\mathcal{H}^a_{\log} &= \frac{1}{12}\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij}+\mathcal{D}_b\tilde{\mathcal{H}}^{ab}_{\log}-\mathfrak{f}_{bc}K^c\tilde{\mathcal{H}}^{ab}_{\log}\\
&\quad-\frac{3}{8}\mathcal{D}_b C^{abcd}W_{cd}-\frac{3}{8}W^{-1}\mathcal{D}_b C^{abcd}F_{cd}\ . \qquad (3.4.10a)
\end{aligned}
$$

This can equivalently be written as

$$
\mathcal{H}^a_{\log} = \frac{1}{12}\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij}+\mathcal{D}_b(\tilde{\mathcal{H}}^{ab}_{\log}+\hat{\mathcal{H}}^{ab}_{\log})\ , \qquad (3.4.10b)
$$

where $\hat{\mathcal{H}}^{ab}_{\log}$ is antisymmetric and is explicitly defined as

$$
\hat{\mathcal{H}}^{ab}_{\log} = \frac{5}{16}\mathcal{R}W^{ab}+\mathcal{R}^{c[a}W^{b]}{}_c+\frac{5}{16}\mathcal{R}F^{ab}W^{-1}+\mathcal{R}^{c[a}F^{b]}{}_c W^{-1}. \qquad (3.4.10c)
$$

In order to derive (3.4.10b), we have made use of the following Riemann tensor properties

$$
\begin{aligned}
C^{abcd} &= \mathcal{R}^{abcd}+4\eta^{c[a}\mathfrak{f}^{b]d}-4\eta^{d[a}\mathfrak{f}^{b]c}\ , && (3.4.11a)\\
\mathcal{D}_{[a}\mathcal{R}_{bc]de} &= 0\ , && (3.4.11b)\\
\mathcal{D}_d\mathcal{R}^{abcd} &= -2\mathcal{D}^{[a}\mathcal{R}^{b]c}\ , && (3.4.11c)\\
\mathcal{D}_a\mathcal{R}^{ab} &= \frac{1}{2}\mathcal{D}^b\mathcal{R}\ . && (3.4.11d)
\end{aligned}
$$

Having obtained the full component expressions, the first non-trivial check we have done is to show that the divergence of $\mathcal{H}^a_{log}$ vanishes up to fermions, that is $\nabla_a \mathcal{H}^a_{log} = 0$. Unlike the Weyl case, $\tilde{\mathcal{H}}^{ab}_{\log}$ is not K-invariant. Therefore,

$$\begin{aligned} \nabla_a \nabla_b \tilde{\mathcal{H}}^{ab}_{\log} &= -\frac{1}{2} R(K)_{ab}{}^c K_c \tilde{\mathcal{H}}^{ab}_{\log} \\ &= \frac{3}{8} \nabla^b C_{abcd} \left(\nabla^a W^{cd} + W^{-1} \nabla^a F^{cd} - W^{-2} F^{cd} \nabla^a W \right) . \end{aligned} \tag{3.4.12}$$

The above identity, along with the following relation

$$\nabla_a \nabla_b C^{abcd} = 0 , \tag{3.4.13}$$

$$\nabla_{[a} \Phi_{bc]}{}^{ij} = 0 , \tag{3.4.14}$$

implies $\nabla_a \mathcal{H}^a_{log} = 0$. In addition, we have checked explicitly the K-invariance of all the composite descendant components of H^{ij}_{log} including fermions.

3.4.3 Scalar curvature squared

Following the component field construction of [64], given a composite vector multiplet (3.3.16) and its corresponding descendants (3.3.18), one can then use (3.2.22) to construct a composite primary superfield based on a single composite vector multiplet [29]

$$\begin{aligned} H^{ij}_{R^2} := H^{ij}_{\rm VM}[\mathbf{W}] &= \mathrm{i}(\nabla^{\alpha(i} \mathbf{W}) \nabla^{j)}_\alpha \mathbf{W} + \frac{\mathrm{i}}{2} \mathbf{W} \nabla^{\alpha(i} \nabla^{j)}_\alpha \mathbf{W} \\ &= -\mathrm{i} \boldsymbol{\lambda}^{\alpha i} \boldsymbol{\lambda}^j_\alpha + 2 \mathbf{W} \mathbf{X}^{ij} . \end{aligned} \tag{3.4.15}$$

Its component structure can be compactly described in terms of (3.3.18). They read

$$H^{ij}_{R^2} = 2\mathbf{W}\mathbf{X}^{ij} - \mathrm{i}\boldsymbol{\lambda}^{\alpha(i}\boldsymbol{\lambda}^{j)}_\alpha , \tag{3.4.16a}$$

$$\begin{aligned} \varphi^i_{\alpha R^2} &= \mathrm{i}\mathbf{X}^{ij}\boldsymbol{\lambda}_{\alpha j} - 2\mathrm{i}\mathbf{F}_{\alpha\beta}\boldsymbol{\lambda}^{\beta i} + 16\mathrm{i}\mathbf{W}^2\chi^i_\alpha - 2\mathrm{i}\mathbf{W}\nabla_{\alpha\beta}\boldsymbol{\lambda}^{\beta i} \\ &\quad - \mathrm{i}\boldsymbol{\lambda}^{\beta i}\nabla_{\alpha\beta}\mathbf{W} - 3\mathrm{i}W_{\alpha\beta}\mathbf{W}\boldsymbol{\lambda}^{\beta i} , \end{aligned} \tag{3.4.16b}$$

$$\begin{aligned} F_{R^2} &= \mathbf{X}^{ij}\mathbf{X}_{ij} - \mathbf{F}^{ab}\mathbf{F}_{ab} + 4\mathbf{W}\nabla^a\nabla_a\mathbf{W} + 2(\nabla^a\mathbf{W})(\nabla_a\mathbf{W}) - 2\mathrm{i}(\nabla_{\alpha\beta}\boldsymbol{\lambda}^{\beta i})\boldsymbol{\lambda}^\alpha_i \\ &\quad - 6W^{ab}\mathbf{F}_{ab}\mathbf{W} - \frac{39}{8}W^{ab}W_{ab}\mathbf{W}^2 - 16D\mathbf{W}^2 + 64\mathrm{i}\chi^\alpha_i\boldsymbol{\lambda}^i_\alpha\mathbf{W} - 3\mathrm{i}W_{\alpha\beta}\boldsymbol{\lambda}^{\alpha i}\boldsymbol{\lambda}^\beta_i , \end{aligned} \tag{3.4.16c}$$

$$\mathcal{H}^a_{R^2} = -\frac{1}{2}\varepsilon^{abcde}\mathbf{F}_{bc}\mathbf{F}_{de} - \nabla_b\left(4\mathbf{W}\mathbf{F}^{ab} + 6W^{ab}\mathbf{W}^2 + 2\mathrm{i}(\Sigma^{ab})^{\alpha\beta}\boldsymbol{\lambda}^i_\alpha\boldsymbol{\lambda}_{\beta i}\right) . \tag{3.4.16d}$$

In addition to eqs. (3.2.26), the following identities are useful in which fermions[7] are suppressed at two or more covariant vector derivatives:

$$\nabla_a \nabla_b G^{ij} = \mathcal{D}_a \mathcal{D}_b G^{ij} - 6 G^{ij} \mathfrak{f}_{ab} , \tag{3.4.17a}$$

$$\nabla_a \nabla_b F = \mathcal{D}_a \mathcal{D}_b F - 8 F \mathfrak{f}_{ab} , \tag{3.4.17b}$$

$$\nabla_a \nabla_b \mathcal{H}_c = \mathcal{D}_a \mathcal{D}_b \mathcal{H}_c - 8\mathcal{H}_c \mathfrak{f}_{ab} - 2\mathcal{H}_b \mathfrak{f}_{ac} + 2\eta_{bc}\mathcal{H}_d \mathfrak{f}_a{}^d , \tag{3.4.17c}$$

[7] See the supplementary file [1] for the results including fermions.

$$\nabla_a \Box G^{ij} = \mathcal{D}_a \mathcal{D}_b \mathcal{D}^b G^{ij} - 6\mathcal{D}_a(\mathfrak{f}_b{}^b G^{ij}) - 6\mathfrak{f}_{ab}\mathcal{D}^b G^{ij} . \tag{3.4.17d}$$

The composite multiplet (3.4.16) will be used in section 3.5.3 to describe the supersymmetric completion of the scalar curvature-squared invariant.

Having obtained the full component expressions, the first non-trivial check we have done is to show that the divergence of $\mathcal{H}^a_{R^2}$ vanishes up to fermions, that is, $\nabla_a \mathcal{H}^a_{R^2} = 0$. Another consistency check is the K-invariance for the component fields (3.4.16). Both of these can be shown in a straighforward way because the composite vector multiplet fields are all primary.

3.5 Curvature-squared actions in a standard Weyl background

Within the superconformal approach, supersymmetric completions of the Weyl tensor squared and the scalar curvature squared were constructed for the first time, respectively in [55] and [64] by using component field techniques. The third independent invariant necessary to obtain all the curvature-squared models in five dimensions includes the Ricci tensor-squared term. In the standard Weyl multiplet background, this invariant was defined in superspace in [29].

In section 3.4, we have presented the bosonic component structure (and their fermionic counterparts in the supplementary file [1]) of all the composite primary linear multiplets generating the three curvature-squared terms for $5D$ minimal supergravity based on the standard Weyl multiplet: Weyl, Ricci tensor, and scalar curvature squared. In this section we will use these multiplets and the BF action principle to reproduce the results of [55] and [64]. In particular, by exploiting the computer algebra program *Cadabra* [87, 88], the full bosonic part of the new "Log invariant" will be presented for the first time in subsection 3.5.2.[8]

3.5.1 Weyl-squared

The off-shell Weyl-squared action in the standard Weyl multiplet background was first constructed in [55]. In our notation, such an invariant can be constructed using the BF action principle

$$S_{\text{Weyl}} = \int \mathrm{d}^{5|8}z\, E\, V_{ij} H^{ij}_{\text{Weyl}} \tag{3.5.1a}$$
$$= \int \mathrm{d}^5 x\, e\, \mathcal{L}_{\text{Weyl}} , \tag{3.5.1b}$$

where

$$\mathcal{L}_{\text{Weyl}} = -\Big(v_a \mathcal{H}^a_{\text{Weyl}} + W F_{\text{Weyl}} + X_{ij} H^{ij}_{\text{Weyl}} - 2\lambda^k_\alpha \varphi^\alpha_{k\,\text{Weyl}} - \psi^\alpha_{ai}(\Gamma^a)_\alpha{}^\beta \varphi^i_{\beta\,\text{Weyl}} W - \mathrm{i}\psi^\alpha_{ai}(\Gamma^a)_\alpha{}^\beta \lambda_{\beta j} H^{ij}_{\text{Weyl}} + \mathrm{i}\psi^\alpha_{ai}(\Sigma^{ab})_\alpha{}^\beta \psi_{\beta b j} W H^{ij}_{\text{Weyl}} \Big) . \tag{3.5.1c}$$

Note that the fields of the linear multiplet are composed in terms of the super Weyl tensor and its descendants. Upon plugging the composite expressions (3.4.3) into (3.5.1c) and disregarding total

[8] The Log invariant in the gauged dilaton Weyl basis has been obtained recently in [2].

derivatives, the bosonic part of the invariant reads

$$\begin{aligned}
\mathcal{L}_{\text{Weyl}} &= -\frac{1}{8}\varepsilon^{abcde}v_a C_{bc}{}^{fg}C_{defg} + \frac{1}{6}\varepsilon^{abcde}v_a \Phi_{bc}{}^{ij}\Phi_{deij} - 4F_{ab}W^{ab}D - \frac{3}{4}F_{ab}C^{abcd}W_{cd} \\
&\quad -\frac{3}{32}F_{ab}W^{ab}W^{cd}W_{cd} + \frac{9}{8}F_{ab}W^{ac}W_{cd}W^{db} + \frac{9}{8}\varepsilon^{abcde}F_{ab}W_{cf}\nabla_d W_e{}^f \\
&\quad +\frac{3}{4}\varepsilon^{abcde}F_{ab}W_{cf}\nabla^f W_{de} - \frac{9}{16}W\varepsilon^{abcde}W_{ab}W_{cd}\nabla^f W_{ef} - \frac{128}{3}WD^2 - 8WW^{ab}W_{ab}D \\
&\quad +\frac{2}{3}W\Phi^{abij}\Phi_{abij} - \frac{1}{4}WC^{abcd}C_{abcd} - \frac{3}{8}WC^{abcd}W_{ab}W_{cd} - \frac{81}{32}WW^{ab}W_{bc}W^{cd}W_{da} \\
&\quad +\frac{33}{128}WW^{ab}W_{ab}W^{cd}W_{cd} + 3WW_{cd}\nabla^c\nabla_b W^{bd} - \frac{3}{2}W\nabla^a W^{ce}\nabla_c W_{ae} \\
&\quad +\frac{3}{2}W\nabla^a W^{ce}\nabla_a W_{ce} + X_{ij}W^{ab}\Phi_{ab}{}^{ij} \, . \qquad (3.5.2)
\end{aligned}$$

In obtaining (3.5.2), we have performed integration by parts on v_a dependent terms. It is useful to look at these terms more closely. Substituting (3.4.3e) and turning off the fermions, we get

$$\int \mathrm{d}^5x\, e\, v_a \mathcal{H}^a_{\text{Weyl}} = \int \mathrm{d}^5x\, e\, v_a \left\{ \frac{1}{8}\varepsilon^{abcde}C_{bc}{}^{fg}C_{defg} - \frac{1}{6}\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij} + \nabla_b \tilde{\mathcal{H}}^{ab}_{\text{Weyl}} \right\} , \qquad (3.5.3)$$

with

$$\begin{aligned}
\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} &= 8W^{ab}D + \frac{3}{2}C^{abcd}W_{cd} + \frac{3}{16}W^{ab}W^{cd}W_{cd} - \frac{9}{4}W^{ac}W_{cd}W^{db} \\
&\quad -\frac{9}{4}\varepsilon^{abcde}W_{cf}\nabla_d W_e{}^f - \frac{3}{2}\varepsilon^{abcde}W_{cf}\nabla^f W_{de} \, . \qquad (3.5.4)
\end{aligned}$$

Writing the superconformally covariant vector derivative ∇_a in terms of the degauged covariant derivative $\mathcal{D}_a$ gives

$$\nabla_b \tilde{\mathcal{H}}^{ab}_{\text{Weyl}} = \mathcal{D}_b \tilde{\mathcal{H}}^{ab}_{\text{Weyl}} - \mathfrak{f}_b{}^c K_c \tilde{\mathcal{H}}^{ab}_{\text{Weyl}} + \text{gravitini terms} \, . \qquad (3.5.5)$$

One can explicitly check that $K_c\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} = 0$ and $\nabla_b\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} = \mathcal{D}_b\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} + \mathrm{f}ermions$. Thus,

$$\int \mathrm{d}^5x\, e\, v_a \mathcal{H}^a_{\text{Weyl}} = \int \mathrm{d}^5x\, e \left\{ \frac{1}{8}\varepsilon^{abcde}v_a C_{bc}{}^{fg}C_{defg} - \frac{1}{6}\varepsilon^{abcde}v_a\Phi_{bc}{}^{ij}\Phi_{deij} + \frac{1}{2}F_{ab}\tilde{\mathcal{H}}^{ab}_{\text{Weyl}} \right\} . \qquad (3.5.6)$$

Finally, in order to extract the K-connections, which encode the Ricci tensor contributions, we degauge the action (3.5.6). Making use of (2.2.36) and (3.4.9), and setting $W = 1$, the bosonic part of the Weyl-squared Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{\text{Weyl}} &= -\frac{1}{8}\varepsilon^{abcde}v_a \mathcal{R}_{bc}{}^{fg}\mathcal{R}_{defg} + \frac{1}{6}\varepsilon^{abcde}v_a\Phi_{bc}{}^{ij}\Phi_{deij} \\
&\quad -4F_{ab}W^{ab}D - \frac{3}{4}F_{ab}\mathcal{R}^{abcd}W_{cd} - \frac{1}{8}\mathcal{R}F_{ab}W^{ab} - F_{ab}\mathcal{R}^{[a}{}_c W^{b]c} - \frac{3}{32}F_{ab}W^{ab}W^{cd}W_{cd} \\
&\quad +\frac{9}{8}F_{ab}W^{ac}W_{cd}W^{db} + \frac{9}{8}\varepsilon^{abcde}F_{ab}W_{cf}\mathcal{D}_d W_e{}^f + \frac{3}{4}\varepsilon^{abcde}F_{ab}W_{cf}\mathcal{D}^f W_{de} - \frac{128}{3}D^2 \\
&\quad -8W^{ab}W_{ab}D + \frac{2}{3}\Phi^{abij}\Phi_{abij} - \frac{1}{4}\mathcal{R}^{abcd}\mathcal{R}_{abcd} + \frac{1}{3}\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{1}{24}\mathcal{R}^2 - \frac{3}{8}\mathcal{R}^{abcd}W_{ab}W_{cd} \\
&\quad -\frac{81}{32}W^{ab}W_{bc}W^{cd}W_{da} + 3W_{cd}\mathcal{D}^c\mathcal{D}_b W^{bd} - \frac{3}{2}\mathcal{R}_{ab}W^{bd}W^a{}_d + \frac{3}{16}\mathcal{R}W^{ab}W_{ab} \\
&\quad +\frac{33}{128}W^{ab}W_{ab}W^{cd}W_{cd} - \frac{3}{2}\mathcal{D}^a W^{ce}\mathcal{D}_c W_{ae} + \frac{3}{2}\mathcal{D}^a W^{ce}\mathcal{D}_a W_{ce}
\end{aligned}$$

$$-\frac{9}{16}\varepsilon^{abcde}W_{ab}W_{cd}\mathcal{D}^f W_{ef}+X_{ij}W^{ab}\Phi_{ab}{}^{ij}\ . \tag{3.5.7}$$

Note that we have also used

$$C_{abcd} = \mathcal{R}_{abcd}+\frac{1}{6}\eta_{c[a}\eta_{b]d}\mathcal{R}-\frac{2}{3}\eta_{c[a}\mathcal{R}_{b]d}+\frac{2}{3}\eta_{d[a}\mathcal{R}_{b]c}\ . \tag{3.5.8}$$

The action (3.5.7), up to a change of notations, coincides with the result of [55].

3.5.2 Log

Making use of the BF action principle, eqs. (3.3.2a) and (3.3.2b), the primary superfield $H^{ij}_{\log}$ can be used to define the following locally superconformal invariant in a standard Weyl multiplet background

$$S_{\log} = \int \mathrm{d}^{5|8}z\,E\,V_{ij}H^{ij}_{\log} \tag{3.5.9a}$$

$$= -\int \mathrm{d}^5x\,e\Big(v_a\mathcal{H}^a_{\log}+WF_{\log}+X_{ij}H^{ij}_{\log}+2\lambda^k\varphi_{k\log} -\psi_{ai}\Gamma^a\varphi^i_{\log}W-\mathrm{i}\psi_{ai}\Gamma^a\lambda_jH^{ij}_{\log}+\mathrm{i}\psi_{ai}\Sigma^{ab}\psi_{bj}WH^{ij}_{\log}\Big)\ . \tag{3.5.9b}$$

As can be seen from the expressions of the descendants of $H^{ij}_{\log}$ given in (3.4.8) and the supplementary file [1] for fermionic counterpart, the component form of (3.5.9b) is fairly involved. Let us now focus on the bosonic sector of (3.5.9b):

$$\mathcal{L}_{\log}=-\Big(v_a\mathcal{H}^a_{\log}+WF_{\log}+X_{ij}H^{ij}_{\log}\Big). \tag{3.5.10}$$

The relevant term in the Lagrangian can then be integrated by parts:

$$\begin{aligned}\mathcal{L}_{\log} &= -\frac{1}{12}v_a\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij}-v_a\mathcal{D}_b(\hat{H}^{ab}_{\log}+\tilde{H}^{ab}_{\log})-WF_{\log}-X_{ij}H^{ij}_{\log}\\ &= -\frac{1}{12}v_a\varepsilon^{abcde}\Phi_{bc}{}^{ij}\Phi_{deij}-\frac{1}{2}F_{ab}(\hat{H}^{ab}_{\log}+\tilde{H}^{ab}_{\log})-WF_{\log}-X_{ij}H^{ij}_{\log}.\end{aligned} \tag{3.5.11}$$

Inserting the composite expressions (3.4.8) and (3.4.10b) into the above action yields

$$\mathcal{L}_{\log} = \mathcal{L}^{cov}_{\log}+\mathcal{L}'_{\log}\ , \tag{3.5.12a}$$

where we have defined

$$\begin{aligned}\mathcal{L}^{cov}_{\log} = W\Big\{&\frac{3}{4}C_{abcd}W^{ab}W^{cd}-\frac{8}{3}D^2-4\Box D+\frac{23}{8}W^{ab}W_{ab}D+\frac{1}{6}\Phi^{abij}\Phi_{abij}\\ &-\frac{3}{2}W_{cd}\nabla^c\nabla_aW^{ad}-\frac{39}{16}W_{cd}\Box W^{cd}+\frac{1005}{2048}W^{ab}W^{cd}W_{ab}W_{cd}\\ &-\frac{9}{4}\nabla^cW_{cd}\nabla_aW^{ad}+\frac{3}{4}\nabla^cW^{ad}\nabla_aW_{cd}-\frac{33}{16}\nabla^aW^{cd}\nabla_aW_{cd}\\ &-\frac{9}{16}\varepsilon^{abcde}W_{ab}W_{cd}\nabla^fW_{ef}\Big\}\\ +\Big\{&-\frac{1}{12}\varepsilon^{abcde}v_a\Phi_{bc}{}^{ij}\Phi_{deij}-\frac{1}{2}X^{ij}W_{ab}\Phi^{ab}{}_{ij}+\frac{5}{4}C^{abcd}F_{ab}W_{cd}+2W^{ab}F_{ab}D\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{64}W^{ab}W^{cd}W_{ab}F_{cd}+\frac{9}{4}W^{ab}F^{cd}W_{ac}W_{bd}-\frac{9}{8}\varepsilon^{abcde}W_{ab}F_{cd}\nabla^{f}W_{ef}\\
&-\frac{3}{8}\varepsilon^{abcde}W_{af}F_{bc}\nabla^{f}W_{de}-\frac{9}{4}F_{cd}\Box W^{cd}+\frac{3}{2}F_{cd}\nabla^{c}\nabla_{a}W^{ad}\\
&-2W_{cd}\nabla^{c}\nabla_{a}F^{ad}-\frac{1}{2}W_{cd}\Box F^{cd}+4D\Box W+4\nabla^{a}W\nabla_{a}D\\
&+\frac{9}{2}W^{ac}W_{cd}\nabla_{a}\nabla^{d}W-\frac{9}{2}W_{cd}\nabla_{a}W\nabla^{c}W^{ad}+\frac{3}{16}W_{cd}\nabla^{a}W\nabla_{a}W^{cd}\\
&-\frac{9}{2}W_{cd}\nabla^{c}W\nabla_{a}W^{ad}+\frac{3}{32}W^{cd}W_{cd}\Box W-\frac{3}{2}\nabla^{c}F_{cd}\nabla_{a}W^{ad}\\
&-\frac{3}{2}\nabla^{a}F^{cd}\nabla_{a}W_{cd}-\frac{9}{16}\varepsilon^{abcde}\nabla^{f}\left(W_{ab}W_{cd}F_{ef}\right)+\Box\Box W\Bigg\}\\
+W^{-1}\Bigg\{&-\frac{1}{2}X^{ij}F_{ab}\Phi^{ab}{}_{ij}+\frac{1}{4}C^{abcd}F_{ab}F_{cd}+3X^{ij}X_{ij}D-F^{ab}F_{ab}D\\
&+\frac{9}{128}X^{ij}X_{ij}W^{ab}W_{ab}-\frac{75}{128}W^{ab}W_{ab}F^{cd}F_{cd}+\frac{9}{4}W^{ab}W_{ac}F^{cd}F_{bd}\\
&-\frac{3}{8}\varepsilon^{abcde}F_{ab}F_{cf}\nabla^{f}W_{de}-\frac{3}{8}\varepsilon^{abcde}F_{ab}F_{cd}\nabla^{f}W_{ef}-\frac{3}{4}\varepsilon^{abcde}W_{ab}F_{cd}\nabla^{f}F_{ef}\\
&-\frac{3}{8}\varepsilon^{abcde}W_{af}F_{bc}\nabla^{f}F_{de}-\frac{3}{8}\varepsilon^{abcde}W_{ab}F_{cf}\nabla^{f}F_{de}\\
&+\frac{9}{16}\varepsilon^{abcde}W_{ab}W_{cd}F_{ef}\nabla^{f}W-\frac{3}{2}F_{cd}\nabla_{a}W^{ac}\nabla^{d}W\\
&-3F_{ac}\nabla^{c}W^{a}{}_{d}\nabla^{d}W+\frac{3}{2}F_{cd}\nabla^{a}W\nabla_{a}W^{cd}+\frac{3}{2}W_{cd}\nabla^{a}W\nabla_{a}F^{cd}\\
&-2D\nabla^{a}W\nabla_{a}W-\frac{1}{4}\nabla^{a}X^{ij}\nabla_{a}X_{ij}+\frac{9}{4}W_{ac}W^{a}{}_{d}\nabla^{c}W\nabla^{d}W\\
&-\frac{3}{64}W^{cd}W_{cd}\nabla^{a}W\nabla_{a}W+\frac{1}{4}\nabla^{a}F^{cd}\nabla_{a}F_{cd}-2\nabla_{a}W\nabla^{a}\Box W\\
&-\Box W\Box W-\frac{1}{2}\nabla_{a}\nabla_{b}W\nabla^{a}\nabla^{b}W\Bigg\}\\
+W^{-2}\Bigg\{&-\frac{3}{8}W^{ab}F^{cd}F_{ab}F_{cd}+\frac{3}{4}W_{ab}F^{ac}F^{bd}F_{cd}-\frac{1}{8}\varepsilon^{abcde}F_{ab}F_{cf}\nabla^{f}F_{de}\\
&-\frac{1}{8}\varepsilon^{abcde}F_{ab}F_{cd}\nabla^{f}F_{ef}-\frac{3}{16}\varepsilon^{abcde}W_{af}F_{bc}F_{de}\nabla^{f}W\\
&+\frac{1}{2}F_{a}{}^{b}F_{bc}\nabla^{a}\nabla^{c}W-\frac{1}{2}F_{cd}\nabla^{c}W\nabla_{a}F^{ad}-\frac{3}{4}W^{cd}F_{cd}\nabla^{a}W\nabla_{a}W\\
&-\frac{3}{4}F_{cd}\nabla^{a}W\nabla_{a}F^{cd}+\frac{3}{2}\nabla^{a}W\nabla_{a}W\Box W+\nabla^{a}W\nabla^{c}W\nabla_{a}\nabla_{c}W\\
&+\frac{1}{4}X^{ij}X_{ij}\Box W+X^{ij}\nabla^{a}W\nabla_{a}X_{ij}\Bigg\}\\
+W^{-3}\Bigg\{&-\frac{1}{32}F^{ab}F^{cd}F_{ab}F_{cd}+\frac{1}{8}F^{ab}F^{cd}F_{ac}F_{bd}-\frac{1}{16}X^{ij}X_{ij}F^{ab}F_{ab}\\
&+\frac{1}{32}X^{ij}X_{ij}X^{kl}X_{kl}-\frac{1}{8}\varepsilon^{abcde}F_{ab}F_{cd}F_{ef}\nabla^{f}W\\
&+\frac{1}{8}F^{cd}F_{cd}\nabla^{a}W\nabla_{a}W+\frac{3}{2}F_{ac}F^{a}{}_{d}\nabla^{c}W\nabla^{d}W
\end{aligned}
$$

$$-\frac{7}{8}X^{ij}X_{ij}\nabla^aW\nabla_aW-\frac{3}{8}\nabla^aW\nabla_aW\nabla^cW\nabla_cW\Bigg\}\ , \tag{3.5.12b}$$

$$\mathcal{L}'_{\log} \ = \ -\frac{5}{32}\mathcal{R}W^{ab}F_{ab}+\frac{1}{2}\mathcal{R}_{ab}W^a{}_dF^{bd}-\frac{5}{32}\mathcal{R}F^{ab}F_{ab}W^{-1}+\frac{1}{2}\mathcal{R}_{ab}F^a{}_dF^{bd}W^{-1}\ . \tag{3.5.12c}$$

Here, $\mathcal{L}'_{\log}$ emerges from (3.4.10c) and can be expressed in a superconformal covariant form as ∇^aX_a up to total derivative terms, i.e.,

$$\mathcal{L}'_{\log}=\nabla^aX_a-\mathcal{D}^aX_a=-\mathfrak{f}^{ab}K_bX_a\ , \tag{3.5.13}$$

where, inside the integral, $\mathcal{D}^aX_a$ vanishes. Therefore, we are seeking an X_a such that $-\mathfrak{f}^{ab}K_bX_a$ coincides with $\mathcal{L}'_{\log}$, and consequently allows the action to be expressed as:

$$\mathcal{L}_{\log}=\mathcal{L}^{cov}_{\log}+\nabla^aX_a\ . \tag{3.5.14}$$

We proceed by writing the most general ansatz for X_a that generates the desired curvature terms:

$$\begin{aligned} X_a \ = \ & x_1W_{bc}\nabla_aF^{bc}+x_2W_{ac}\nabla_bF^{bc}+x_3F_{bc}\nabla_aW^{bc}+x_4F_{ac}\nabla_bW^{bc}+x_5F^{bc}\nabla_bW_{ac}\\ &+x_6W^{-1}\nabla_aWF_{bc}W^{bc}+x_7W^{-1}\nabla_bWF_{ac}W^{bc}+x_8W^{-1}\nabla_bWF^{bc}W_{ac}\\ &+x_9W^{-1}\nabla_a(F_{bc}F^{bc})+x_{10}W^{-1}\nabla_b(F_{ac}F^{bc})\ , \end{aligned} \tag{3.5.15}$$

and demand that

$$\begin{aligned} -\mathfrak{f}^{ab}K_bX_a \ &= \ -\frac{5}{32}\mathcal{R}W^{ab}F_{ab}+\frac{1}{2}\mathcal{R}_{ab}\eta^{ac}W_{cd}F^{bd}\\ &\quad -\frac{5}{32}\mathcal{R}F^{ab}F_{ab}W^{-1}+\frac{1}{2}\mathcal{R}_{ab}\eta^{ac}F^{bd}F_{cd}W^{-1}\ ,\\ K^a\mathcal{L}_{\log} \ &= \ 0\ . \end{aligned} \tag{3.5.16}$$

The second constraint ensures the K-invariance of the Log action. Although these constraints result in five linearly independent equations, they are insufficient to uniquely fix X_a. One interesting choice of solution that we opted for in this case is

$$\begin{aligned} X_a \ = \ & \frac{3}{16}\nabla_a(W_{bc}F^{bc})-\frac{3}{2}\nabla_b(F_{ac}W^{bc})+\frac{3}{16}W^{-1}(\nabla_aW)W_{bc}F^{bc}\\ &-\frac{3}{2}W^{-1}(\nabla_bW)F_{ac}W^{bc}+\frac{3}{16}W^{-1}\nabla_a(F_{bc}F^{bc})-\frac{3}{2}W^{-1}\nabla_b(F_{ac}F^{bc})\ , \end{aligned} \tag{3.5.17}$$

which then leads to the final expression

$$\begin{aligned} \mathcal{L}_{\log}=\mathcal{L}^{cov}_{\log}+\nabla^a\Bigg\{ & \frac{3}{16}\nabla_a(W_{bc}F^{bc})-\frac{3}{2}\nabla_b(F_{ac}W^{bc})+\frac{3}{16}W^{-1}(\nabla_aW)W_{bc}F^{bc}\\ &-\frac{3}{2}W^{-1}(\nabla_bW)F_{ac}W^{bc}+\frac{3}{16}W^{-1}\nabla_a(F_{bc}F^{bc})-\frac{3}{2}W^{-1}\nabla_b(F_{ac}F^{bc})\Bigg\}\ . \end{aligned} \tag{3.5.18}$$

Note that the resulting component action (3.5.18) includes, for example, a $(\Box\Box W)$ term, which upon using the degauge identity (3.4.9), gives rise to a Ricci tensor squared combination.

Unlike the Weyl-squared case, manually degauging the Log action (3.5.18) proves to be technically involved, primarily due to its considerable length. Consequently, we made use of *Cadabra* to carry out the degauging procedure and then applied various simplification techniques to refine the initial expression. These include integration by parts, symmetries of the Riemann tensor, and degauged Bianchi identities $\mathcal{D}_{[a}F_{bc]}=0$, $\mathcal{D}_{[a}\Phi_{bc]}^{ij}=0$ (up to fermions). Furthermore, the following identity is also useful for simplification at final stages

$$\varepsilon^{abcde}W_{ab}W_{cf}\mathcal{D}^f F_{de} = -\frac{1}{2}\varepsilon^{abcde}W_{ab}W_{cd}\mathcal{D}^f F_{ef} \ . \tag{3.5.19}$$

Setting $W=1$, the bosonic sector of the Log invariant reads

$$\begin{aligned}
\mathcal{L}_{\log} \;=\; & -\frac{1}{12}\varepsilon^{abcde}v_a\Phi_{bc}{}^{ij}\Phi_{deij}+\frac{1}{4}\mathcal{R}_{abcd}\big(F^{ab}F^{cd}+W^{ab}W^{cd}\big)+\frac{1}{4}\mathcal{R}_{ab}\big(4W^{ac}F^b{}_c-3W^{ac}W^b{}_c\big)\\
&-\frac{1}{256}\mathcal{R}\big(24F^{ab}F_{ab}+80W^{ab}F_{ab}+27W^{ab}W_{ab}+128D-8X^{ij}X_{ij}\big)\\
&-\frac{1}{6}\mathcal{R}_{ab}\mathcal{R}^{ab}+\frac{23}{384}\mathcal{R}^2-\frac{8}{3}D^2-\frac{1}{2}X^{ij}\big(W^{ab}\Phi_{abij}+F^{ab}\Phi_{abij}\big)\\
&+\frac{1}{128}X^{ij}X_{ij}\big(9W^{ab}W_{ab}-8F^{ab}F_{ab}+4X^{kl}X_{kl}\big)\\
&+\frac{1}{8}D\big(16W^{ab}F_{ab}-8F^{ab}F_{ab}+23W^{ab}W_{ab}+24X^{ij}X_{ij}\big)\\
&+W^{ab}F^{cd}\left(\frac{3}{64}W_{ab}W_{cd}+\frac{9}{4}W_{ac}W_{bd}-\frac{75}{128}W_{ab}F_{cd}+\frac{9}{4}W_{ac}F_{bd}-\frac{3}{8}F_{ab}F_{cd}+\frac{3}{4}F_{ac}F_{bd}\right)\\
&-\frac{1}{32}F^{ab}F^{cd}\big(F_{ab}F_{cd}-4F_{ac}F_{bd}\big)+\frac{1005}{2048}W^{ab}W^{cd}W_{ab}W_{cd}+\frac{1}{6}\Phi_{ab}{}^{ij}\Phi^{ab}{}_{ij}\\
&-\frac{3}{16}\varepsilon^{abcde}(\mathcal{D}^f W_{af})\big(4W_{bc}F_{de}+F_{bc}F_{de}+3W_{bc}W_{de}\big)\\
&-\frac{1}{16}\varepsilon^{abcde}(\mathcal{D}^f F_{af})\big(6W_{bc}F_{de}+F_{bc}F_{de}+3W_{bc}W_{de}\big)\\
&-\frac{3}{8}W^{cd}\mathcal{D}^a\mathcal{D}_aW_{cd}-\frac{3}{2}F^{ac}\mathcal{D}_c\mathcal{D}^dW_{ad}-\frac{1}{4}(\mathcal{D}^aX^{ij})\mathcal{D}_aX_{ij}+\frac{1}{4}(\mathcal{D}^aF^{bc})\mathcal{D}_aF_{bc} \ .
\end{aligned} \tag{3.5.20}$$

Note that the degauged action (3.5.20) contains both the supersymmetric Ricci tensor and Ricci scalar terms.

3.5.3 Scalar curvature squared

We now construct an action describing the supersymmetric completion of the Ricci scalar squared in the standard Weyl multiplet. It is obtained by plugging the composite linear multiplet (3.4.16) into the following BF action

$$S_{R^2} \;=\; \int \mathrm{d}^{5|8}z\,E\,V_{ij}H^{ij}_{R^2} \tag{3.5.21a}$$

$$\begin{aligned}
\;=\; & -\int \mathrm{d}^5x\,e\Big(v_a\mathcal{H}^a_{R^2}+WF_{R^2}+X_{ij}H^{ij}_{R^2}-2\lambda_i^\alpha\varphi^i_{\alpha R^2}\\
&-\psi_{ai}^\alpha(\Gamma^a)_\alpha{}^\beta\varphi^i_{\beta R^2}W-\mathrm{i}\psi_{ai}^\alpha(\Gamma^a)_\alpha{}^\beta\lambda_{\beta j}H^{ij}_{R^2}+\mathrm{i}\psi_{ai}^\alpha(\Sigma^{ab})_\alpha{}^\beta\psi_{b\beta j}WH^{ij}_{R^2}\Big) \ .
\end{aligned} \tag{3.5.21b}$$

The bosonic part of such an action reads

$$S_{R^2} \;=\; -\int \mathrm{d}^5x\,e\Big(-\frac{1}{2}\varepsilon_{abcde}v^a\mathbf{F}^{bc}\mathbf{F}^{de}-v^a\nabla^b\left(4\mathbf{W}\mathbf{F}_{ab}+6W_{ab}\mathbf{W}^2\right)$$

$$
\begin{aligned}
&+W\mathbf{X}^{ij}\mathbf{X}_{ij} - W\mathbf{F}^{ab}\mathbf{F}_{ab} + 4W\mathbf{W}\nabla^a\nabla_a\mathbf{W} + 2W(\nabla^a\mathbf{W})(\nabla_a\mathbf{W}) \\
&-6WW^{ab}\mathbf{F}_{ab}\mathbf{W} - \frac{39}{8}WW^{ab}W_{ab}\mathbf{W}^2 - 16WD\mathbf{W}^2 + 2X_{ij}\mathbf{W}\mathbf{X}^{ij}\Big) , \qquad (3.5.22a)
\end{aligned}
$$

which, upon integrating by parts, gives

$$
\begin{aligned}
S_{R^2} \;=\; \int \mathrm{d}^5x\, e \,\Big(&\frac{1}{2}\varepsilon_{abcde}v^a\mathbf{F}^{bc}\mathbf{F}^{de} + F^{ab}\left(2\mathbf{W}\mathbf{F}_{ab} + 3W_{ab}\mathbf{W}^2\right) \\
&-W\mathbf{X}^{ij}\mathbf{X}_{ij} + W\mathbf{F}^{ab}\mathbf{F}_{ab} - 4W\mathbf{W}\nabla^a\nabla_a\mathbf{W} - 2W(\nabla^a\mathbf{W})\nabla_a\mathbf{W} \\
&+6WW^{ab}\mathbf{F}_{ab}\mathbf{W} + \frac{39}{8}WW^{ab}W_{ab}\mathbf{W}^2 + 16DW\mathbf{W}^2 - 2X_{ij}\mathbf{W}\mathbf{X}^{ij}\Big) . \qquad (3.5.22b)
\end{aligned}
$$

Here we note that the composite expressions for the vector multiplet are given explicitly in (3.3.16) and (3.3.18). After degauging and omitting the fermionic terms, the composite vector multiplet now takes the form

$$
\mathbf{W} = \frac{1}{4}FG^{-1} , \qquad (3.5.23a)
$$

$$
\begin{aligned}
\mathbf{X}^{ij} = \;& G^{-1}\Big\{\frac{1}{2}\mathcal{D}^a\mathcal{D}_aG^{ij} + \frac{3}{16}\mathcal{R}G^{ij} + \frac{3}{64}W^{ab}W_{ab}G^{ij} + 2DG^{ij}\Big\} \\
&+G^{-3}\Big\{-\frac{1}{16}F^2G^{ij} - \frac{1}{16}\mathcal{H}^a\mathcal{H}_aG^{ij} + \frac{1}{4}\mathcal{H}^aG^{k(i}\mathcal{D}_aG_k{}^{j)} \\
&\qquad -\frac{1}{4}G_{kl}\left(\mathcal{D}^aG^{k(i}\right)\mathcal{D}_aG^{j)l}\Big\} , \qquad (3.5.23b)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{ab} = \;& G^{-1}\Big\{\frac{1}{2}\mathcal{D}_{[a}\mathcal{H}_{b]} - \frac{3\mathrm{i}}{8}G_{ij}X_{ab}{}^{ij}\Big\} \\
&+G^{-3}\Big\{\frac{1}{4}G_{ij}\mathcal{H}_{[a}\mathcal{D}_{b]}G^{ij} - \frac{1}{4}G_{ij}\left(\mathcal{D}_{[a}G^{ik}\right)\mathcal{D}_{b]}G_k{}^j\Big\} . \qquad (3.5.23c)
\end{aligned}
$$

Inserting the above expressions into (3.5.22b) and imposing the gauge fixing condition $W = 1$ leads to the supersymmetric extension of the Ricci scalar squared action [64].

Let us point out that while the component actions presented in this section do not include fermionic terms, we can, in principle, construct the complete actions for all the three invariants, using the composite expressions given in the supplementary file [1]. In the remainder of this paper, we will require some of these fermionic terms to derive multiplet of EOMs.

3.6 Derivation of superconformal equations of motion

The goal of this section is to obtain superconformal primary equations of motion in superspace that describe minimal $5D$ gauged supergravity based on a standard Weyl multiplet and deformed by an arbitrary combination of the three curvature-squared invariants described by the action

$$
S_{\mathrm{HD}} = S_{\mathrm{g}SG} + \alpha S_{\mathrm{W}eyl} + \beta S_{\log} + \gamma S_{R^2} , \qquad (3.6.1)
$$

where the three independent curvature-squared invariants are described by eqs. (3.5.1), (3.5.9), and (3.5.21b). The results in this section were first presented in [6] but with no derivation.

In principle, one can obtain these equations of motion by varying the superspace action (3.6.1) with respect to the superfield prepotentials of the standard Weyl multiplet ($\mathfrak{U}$), the vector multiplet compensator (V_{ij}), and the linear multiplet compensator (Ω). These variations lead to the supercurrent superfield $\mathcal{J}$, the linear multiplet of the EOM of V_{ij}, and the vector multiplet of the EOM of Ω, respectively. The result would be manifestly covariant, as is well known from the very well developed results for theories with four supercharges, see [35, 36]. Alternatively, we can reduce (3.6.1) to components and vary it with respect to the highest dimension independent component fields (D, X_{ij}, and F) of the corresponding multiplets. Because the prepotential formulation of the five-dimensional standard Weyl multiplet in superspace has not been developed to date, we will use mostly a component approach, in particular to obtain $\mathcal{J}$. We also stress that we are interested in obtaining the full equations of motions, including all the potential fermionic terms.

It is worth pointing out that in superspace the equations of motion derived by varying prepotentials would be manifestly covariant. Hence, one expects the same to be true once the superspace results are reduced to component fields. However, in the component approach of finding the EOMs, the component action computed from (3.6.1) includes thousands of terms when fermions are considered, and it is not manifestly covariant due to the presence of naked gravitini and Chern-Simons terms. Although the action lacks manifest covariance, it has recently been explicitly demonstrated in components that for any supergravity theory, there exist covariant equations of motion that are equivalent to the regular field equations [193, 194]. These covariant equations are obtained by covariantising the regular field equations, resulting in a multiplet of field equations [193, 194].

Adopting this approach, in this section we derive the covariant field equations for the Weyl multiplet, linear multiplet, and vector multiplet for the minimal $5D$ gauged supergravity deformed by an arbitrary combination of the three curvature-squared invariants described by the action $S_{\text{H}D}$. The above argument suggests that to obtain these equations of motion, it is sufficient to initially vary the gravitini-independent part (or by setting gravitini to zero), in a degauged action, to derive the regular field equations. Hence, we begin with

$$S_{\text{B}F} \;=\; -\int \mathrm{d}^5x\, e\Big(v_a\mathcal{H}^a + WF + X_{ij}G^{ij} + 2\lambda^{\alpha k}\varphi_{\alpha k}\Big)\,, \tag{3.6.2}$$

which, by suitably choosing primary composite linear or vector multiplets (e.g., eqs. (3.3.18), (3.4.3), and (3.4.8)), becomes the building block in finding EOMs for various two- and four-derivative invariants. Subsequently, we vary the degauged action with respect to the highest dimension auxiliary fields D, X_{ij}, and F to derive the regular field equations for Weyl, vector, and linear multiplets, respectively. To obtain covariant EOMs, we then appropriately replace the degauged derivative to the conformal covariant derivative and translate component fields into superfields. This uplifting process effectively eliminates the curvature terms present in the degauged EOMs. The resulting covariant EOMs then describe the primary fields, i.e., the top components of the multiplets of the equations of motion that arise from the variation of the full superfields. It is then straightforward to reinterpret them as the locally superconformal multiplet of field equations. Once this result is obtained, it is then possible to systematically produce the entire multiplet of equations of motion by the successive action of superspace spinor derivatives (equivalently Q-supersymmetry transformations) on the primary EOMs. Once again, it is important to emphasise that in our analysis, we can carefully disregard the influence of gravitini in both the degauging process and the subsequent transformation of regular EOMs into superconformally covariant EOMs. We then run several consistency checks to ensure the results are correct.

3.6.1 Two-derivative EOMs

The on-shell structure of the two-derivative gauged supergravity action, S_{gSG}, has been discussed in [29]. Recall that

$$\begin{aligned} S_{gSG} &= S_{VM} + S_{\rm L} + \kappa S_{\rm BF} = \int {\rm d}^{5|8}z\, E \left\{ \frac{1}{4} V_{ij} H^{ij}_{\rm VM} + \Omega \mathbf{W} + \kappa V_{ij} G^{ij} \right\} && (3.6.3a) \\ &= \int {\rm d}^{5|8}z\, E \left\{ \frac{1}{4} V_{ij} H^{ij}_{\rm VM} + \Omega \mathbf{W} + \kappa \Omega W \right\} . && (3.6.3b) \end{aligned}$$

Also, the BF action is

$$\begin{aligned} S_{\rm BF} = &- \int {\rm d}^5 x\, e \Big(v_a \mathcal{H}^a + WF + X_{ij} G^{ij} + 2\lambda^{\alpha k} \varphi_{\alpha k} \\ &- \psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \varphi^i_\beta W - {\rm i} \psi_a{}^\alpha_i (\Gamma^a)_\alpha{}^\beta \lambda_{\beta j} G^{ij} + {\rm i} \psi_a{}^\alpha_i (\Sigma^{ab})_\alpha{}^\beta \psi_{b\beta j} W G^{ij} \Big) . \qquad (3.6.4) \end{aligned}$$

Here we elaborate more on the derivation of the EOMs using the component form of S_{gSG}. While only the bosonic part of the component actions (3.3.15), (3.3.21), and (3.3.23) for the gauged supergravity (3.6.3a) are provided, it is possible to derive the full expressions including fermionic terms, by directly examining the composite fields (3.2.22) and (3.3.18) from which the action is derived.

Standard Weyl multiplet

The conformal supergravity equation of motion is obtained by varying the superspace form of S_{gSG} with respect to the standard Weyl multiplet prepotential superfield $\mathfrak{U}$ or equivalently the component form of S_{gSG} with respect to the auxiliary field D. To derive the equation of motion for the Weyl multiplet from the component action, the D-dependencies arise from the composite fields (3.2.22c) and (3.3.18b). Consequently, the equation of motion can be expressed as follows (note that no fermions are involved):

$$\begin{aligned} 0 = J_{EH} &= -\frac{1}{4} W \frac{\delta F_{\rm VM}}{\delta D} - \frac{\delta \mathbf{X}^{ij}}{\delta D} G_{ij} , \\ 0 = J_{EH} &= 4(W^3 - G) . && (3.6.5) \end{aligned}$$

Note that here and in what follows, we have neglected the gravitini-dependent terms throughout because of the argument given in the beginning of this section. It can be proven explicitly that their contribution either cancels out or adds up to make some of the fields in the equations of motion manifestly superconformally covariant.

Vector multiplet

For the two-derivative gauged supergravity action (3.3.22a), the EOM for the vector multiplet was given in [29]. In superspace, it was obtained by varying $S_{\text{g}SG}$ with respect to the Mezincescu's superfield prepotential V_{ij}. An arbitrary variation of V_{ij} leads to

$$\begin{aligned} \delta S_{\text{g}SG} &= \delta S_{\text{V}M} + \kappa\,\delta S_{\text{BF}} \\ &= \frac{1}{4}\int \mathrm{d}^{5|8}z\,E\left[\delta V_{ij}H^{ij}_{\text{V}M} + V_{ij}\delta H^{ij}_{\text{V}M}\right] + \kappa\int \mathrm{d}^{5|8}z\,E\,\delta V_{ij}G^{ij}\,. \end{aligned} \tag{3.6.6}$$

Making use of the relation (3.3.8), and various integration by parts of spinor covariant derivatives,[9] the total variation of $S_{\text{g}SG}$ induced by an arbitrary variation of the vector multiplet prepotential V_{ij} is given by

$$\delta S_{\text{g}SG} = \frac{3}{4}\int \mathrm{d}^{5|8}z\,E\,\delta V_{ij}H^{ij}_{\text{V}M} + \kappa\int \mathrm{d}^{5|8}z\,E\,\delta V_{ij}G^{ij}\,, \tag{3.6.7}$$

and, hence,

$$\frac{\delta S_{\text{g}SG}}{\delta V_{ij}} = 0 \qquad \Longrightarrow \qquad \frac{3}{4}H^{ij}_{\text{V}M} + \kappa G^{ij} = 0\,. \tag{3.6.8}$$

To obtain the EOM for the vector multiplet from the component action, we begin by taking the variation of the component gauged supergravity action with respect to the auxiliary field X^{ij}. Upon examining the composite fields (3.2.22) and (3.3.18) for their dependence on X^{ij}, the resulting EOM can be expressed as follows:

$$\begin{aligned} 0 &= -\frac{1}{4}W\frac{\delta F_{\text{V}M}}{\delta X^{ij}} - \frac{1}{4}H^{ij}_{\text{V}M} - \frac{1}{4}X_{kl}\frac{\delta H^{kl}_{\text{V}M}}{\delta X^{ij}} - \frac{1}{2}\lambda^{\alpha k}\frac{\delta \varphi_{\alpha k \text{V}M}}{\delta X^{ij}} - \kappa G^{ij}\,, \\ \Longrightarrow 0 &= -\frac{3}{4}H^{ij}_{\text{V}M} - \kappa G^{ij}\,, \end{aligned} \tag{3.6.9}$$

where we have neglected the gravitini-dependent terms in our analysis. This, as expected, corresponds to the lowest component of the superfield EOM (3.6.8). Note the relative sign difference between the superspace equation (3.6.8) and the component equation (3.6.9). This relative sign arises when considering the variation of the action with respect to either the superfield prepotential or its corresponding auxiliary field. A similar relative sign difference between the superspace and component EOMs will also exist for other EOMs.

Linear multiplet

Lastly, we consider the EOM for the linear multiplet compensator. In the superspace approach, this can be done by varying with respect to the superfield prepotential Ω; hence, at the two-derivative level, from (3.3.22b) we find that

$$\mathbf{W} + \kappa W = 0\,. \tag{3.6.10}$$

From the component action side, the linear multiplet EOM is obtained by varying the action with respect to auxiliary field F. As no composite fields in (3.2.22) and (3.3.18) have dependence on F, the

[9] See comments in [29], and also the recent reviews [39,40], for subtleties in performing (5D) conformal superspace integration by parts.

EOM is given by:

$$0 \;=-\mathbf{W}-\kappa W\;, \tag{3.6.11}$$

where we have neglected the gravitini dependent terms in our analysis for the EOM.

3.6.2 Four-derivative: Standard Weyl multiplet EOMs

We now proceed in examining the D-dependent terms in each curvature-squared invariant.

Weyl-squared

As can be seen from the explicit expressions (3.4.3), each of the descendant components of the composite H^{ij}_{Weyl} carry D-dependence. Suppressing the explicit gravitini terms, the D-dependent terms of the Weyl-squared Lagrangian (3.5.1c) are

$$\begin{aligned}
\mathcal{L}_{\text{Weyl},D} &= -\Big(v_a\mathcal{H}^a_{\text{Weyl}}+WF_{\text{Weyl}}-2\lambda_i^{\alpha}\varphi^i_{\alpha\,\text{Weyl}}\Big) && (3.6.12\text{a})\\
&= -8v_a\mathcal{D}_b(DW^{ab})-W\Big(\frac{128}{3}D^2+8W^{ab}W_{ab}D\Big)+2\lambda_i^{\alpha}\Big(-\frac{128}{3}\mathrm{i}D\chi^i_{\alpha}\Big) && (3.6.12\text{b})\\
&= -\frac{128}{3}WD^2-8WW^{ab}W_{ab}D-8v_a\mathcal{D}_b\left(DW^{ab}\right)-\frac{256}{3}\mathrm{i}\lambda_i^{\alpha}\chi^i_{\alpha}D\,. && (3.6.12\text{c})
\end{aligned}$$

Upon integrating by parts, an arbitrary variation of D leads to the following EOM

$$\frac{\partial}{\partial D}\mathcal{L}_{\text{Weyl},D}=-\frac{256}{3}WD-8WW^{ab}W_{ab}-4F^{ab}W_{ab}+\frac{256}{3}\mathrm{i}\lambda^{\alpha i}\chi_{\alpha i}=0\,. \tag{3.6.13}$$

We can uplift the above EOM to superspace [193, 194], by promoting every component field to superfield. This amounts to replacing $D\to-\frac{3}{128}Y$ and $\chi_{\alpha i}\to\frac{3}{32}\mathrm{i}X_{\alpha i}$ while all other fields should now be understood as superfields. The covariant superconformal primary EOM is then

$$0=J_{\text{Weyl}}=-\frac{3}{64}WY+\frac{3}{16}WW^{ab}W_{ab}+\frac{3}{32}F_{ab}W^{ab}-\frac{3}{16}\lambda_i^{\alpha}X^i_{\alpha}\,. \tag{3.6.14}$$

Note the extra factor of $-\frac{3}{128}$ in the above equation. This arises when we replace D with Y on the left-hand side of (3.6.13).

Log

Next, we examine the Log invariant and explicitly write out the D-dependent contributions. Suppressing the gravitini terms, upon substituting the composite fields (3.4.8) (of which their explicit fermionic dependence given in subsection 2.2 of the supplementary file [1]) into the BF Lagrangian (3.6.2), the relevant D-dependent terms are given by

$$\begin{aligned}
\mathcal{L}_{\log,D} &= -v_a\mathcal{H}^a_{\log}-WF_{\log}-X_{ij}H^{ij}_{\log}+2\lambda_i^{\alpha}\varphi^i_{\alpha\,\log}\,, && (3.6.15)\\
&= 2v_a\mathcal{D}_b\Big(2W^{ab}D-2W^{-1}F^{ab}D+\mathrm{i}W^{-2}(\Sigma^{ab})^{\alpha\beta}\lambda^i_{\alpha}\lambda_{\beta i}D\Big)\\
&\quad -W\bigg\{\frac{8}{3}D^2-\frac{23}{8}W^{ab}W_{ab}D-W^{-2}F^{ab}F_{ab}D+3W^{-2}X^{ij}X_{ij}D
\end{aligned}$$

$$
\begin{aligned}
&-4\mathrm{i}W^{-2}(\Gamma^a)^{\alpha\beta}\lambda^i_\alpha(\mathcal{D}_a\lambda_{\beta i})D-4W^{-1}D\mathcal{D}^a\mathcal{D}_aW+2W^{-2}D(\mathcal{D}^aW)\mathcal{D}_aW\\
&+2\mathrm{i}W^{-3}(\Sigma^{ab})^{\alpha\beta}F_{ab}\lambda^i_\alpha\lambda_{\beta i}D+4\mathrm{i}W^{-3}X^{ij}\lambda_i^\alpha\lambda_{\alpha j}D\\
&-\frac{3}{2}W^{-4}\lambda^{\alpha i}\lambda^j_\alpha\lambda_i^\beta\lambda_{\beta j}D-4W^{-1}(\mathcal{D}_aW)\mathcal{D}^aD+4\mathcal{D}^a\mathcal{D}_aD+\frac{1}{2}\mathcal{R}D\Big\}\\
&-X_{ij}\Big\{-6W^{-1}X^{ij}D-2\mathrm{i}W^{-2}\lambda^{\alpha i}\lambda^j_\alpha D\Big\}\\
&+2\lambda_j^\alpha\Big\{-\frac{8}{3}\mathrm{i}\chi^j_\alpha D+4\mathrm{i}(\Gamma_b)_{\alpha\beta}W^{-1}(\mathcal{D}^b\lambda^{\beta j})D-\mathrm{i}W^{-2}(\Sigma^{ab})_{\alpha\beta}F_{ab}\lambda^{\beta j}D\\
&+2\mathrm{i}W^{-2}X^{ij}\lambda_{\alpha i}D+2\mathrm{i}(\Gamma_b)_{\alpha\beta}\lambda^{\beta j}\mathcal{D}^b(W^{-1}D)-W^{-3}\lambda^{\beta j}\lambda^i_\alpha\lambda_{\beta i}D\Big\}\ .
\end{aligned}
\tag{3.6.16}
$$

Upon integrating by parts, an arbitrary variation of D leads to the following EOM:

$$
\begin{aligned}
0=\frac{\partial\mathcal{L}_{\log,D}}{\partial D}\ =\ &-\frac{16}{3}WD+\frac{23}{8}W^{ab}W_{ab}W-4\mathcal{D}_a\mathcal{D}^aW-\frac{1}{2}\mathcal{R}W+2F_{ab}W^{ab}-\frac{16\mathrm{i}}{3}\chi^{\alpha i}\lambda_{\alpha i}\\
&-F^{ab}F_{ab}W^{-1}+3X^{ij}X_{ij}W^{-1}-4\mathrm{i}(\Gamma^a)^{\alpha\beta}W^{-1}\lambda^i_\alpha\mathcal{D}_a\lambda_{\beta i}\\
&-2W^{-1}(\mathcal{D}^aW)\mathcal{D}_aW+\mathrm{i}(\Sigma^{ab})^{\alpha\beta}F_{ab}\lambda^i_\alpha\lambda_{\beta i}W^{-2}\\
&+2\mathrm{i}X^{ij}\lambda_i^\alpha\lambda_{\alpha j}W^{-2}+\frac{1}{2}\lambda^{\alpha i}\lambda_i^\beta\lambda^j_\alpha\lambda_{\beta j}W^{-3}\ .
\end{aligned}
\tag{3.6.17}
$$

We can uplift this EOM to superspace to get the covariant superconformal primary EOM

$$
\begin{aligned}
0=J_{\log}\ =\ &-\frac{3}{1024}WY-\frac{69}{1024}W^{ab}W_{ab}W+\frac{3}{32}\nabla_a\nabla^aW-\frac{3}{64}F_{ab}W^{ab}-\frac{3}{256}\lambda_j^\alpha X^j_\alpha\\
&+\frac{3}{128}F^{ab}F_{ab}W^{-1}-\frac{9}{128}X^{ij}X_{ij}W^{-1}+\frac{3\mathrm{i}}{32}(\Gamma^a)^{\alpha\beta}W^{-1}\lambda^i_\alpha\nabla_a\lambda_{\beta i}\\
&+\frac{3}{64}W^{-1}(\nabla^aW)\nabla_aW-\frac{3\mathrm{i}}{128}(\Sigma^{ab})^{\alpha\beta}F_{ab}\lambda^i_\alpha\lambda_{\beta i}W^{-2}\\
&-\frac{3\mathrm{i}}{64}X^{ij}\lambda_i^\alpha\lambda_{\alpha j}W^{-2}-\frac{3}{256}\lambda^{\alpha i}\lambda_i^\beta\lambda^j_\alpha\lambda_{\beta j}W^{-3}\ ,
\end{aligned}
\tag{3.6.18}
$$

where, in the uplifting process the curvature term has been incorporated into the degauged d'Alembertian operator making it superconformal, i.e., $\mathcal{D}^a\mathcal{D}_aW\to\nabla^a\nabla_aW$. All the fields should now be understood as superfields. Note the additional factor of $-\frac{3}{128}$ in the above equation as a result of replacing D with Y on the left-hand side of (3.6.17).

Scalar curvature squared

As shown in subsection 3.3.3, the scalar curvature-squared action can be expressed more compactly in terms of the composite vector multiplet fields (3.3.18). From (3.3.18) we also see that only $\mathbf{X}_{ij}$ has a D-dependent term, $\frac{\delta}{\delta D}\mathbf{X}^{ij}=2G^{-1}G^{ij}$. As such, to derive the EOM for D, the relevant terms in the Lagrangian are given by

$$
\begin{aligned}
\mathcal{L}_{R^2,D}\ &=\ -v_a\mathcal{H}^a_{R^2}-WF_{R^2}-X_{ij}H^{ij}_{R^2}+2\lambda_i^\alpha\varphi^i_{\alpha R^2}\\
&=\ -W\mathbf{X}^{ij}\mathbf{X}_{ij}+16WD\mathbf{W}^2-2X_{ij}\mathbf{W}\mathbf{X}^{ij}+2\mathrm{i}\lambda_i^\alpha\mathbf{X}^{ij}\lambda_{\alpha j}\ .
\end{aligned}
\tag{3.6.19}
$$

In the above we have neglected the gravitini-dependent terms due to a similar reasoning as in the previous subsections. An arbitrary variation with respect to D leads to the following EOM:

$$
0=\frac{\partial\mathcal{L}_{R^2,D}}{\partial D}\ =\ 16W\mathbf{W}^2-4G^{-1}\Big(WG^{ij}\mathbf{X}_{ij}+X_{ij}\mathbf{W}G^{ij}-\mathrm{i}\lambda_i^\alpha G^{ij}\lambda_{\alpha j}\Big)\ .
\tag{3.6.20}
$$

We can uplift this EOM to superspace. Once again recall the overall factor of $-\frac{3}{128}$ as we replace D with Y on the left-hand side of (3.6.20). The covariant superconformal primary EOM for Y then reads

$$0 = J_{R^2} \quad = \quad -\frac{3}{8} W \mathbf{W}^2 + \frac{3}{32} G^{-1} \Big(W G^{ij} \mathbf{X}_{ij} + G^{ij} X_{ij} \mathbf{W} - \mathrm{i} G^{ij} \lambda_i^{\alpha} \boldsymbol{\lambda}_{\alpha j} \Big) \, . \tag{3.6.21}$$

3.6.3 Four-derivative: Vector multiplet EOMs

In the Weyl-squared case, the composite primary superfield (3.4.1)

$$H^{ij}_{\mathrm{Weyl}} := -\frac{\mathrm{i}}{2} W^{\alpha\beta\gamma i} W_{\alpha\beta\gamma}{}^{j} + \frac{3\mathrm{i}}{2} W^{\alpha\beta} X_{\alpha\beta}{}^{ij} - \frac{3\mathrm{i}}{4} X^{\alpha i} X^{j}_{\alpha} \, , \tag{3.6.22}$$

and subsequently its descendants (3.4.3) are constructed solely out of the Weyl multiplet. Therefore,

$$\frac{\delta S_{\mathrm{W}eyl}}{\delta V_{ij}} = H^{ij}_{\mathrm{W}eyl} \, , \qquad \frac{\delta S_{\mathrm{W}eyl}}{\delta X_{ij}} = -H^{ij}_{\mathrm{W}eyl} \tag{3.6.23}$$

and then the component equation of motion is simply given by

$$-H^{ij}_{\mathrm{W}eyl} = 0 \, . \tag{3.6.24}$$

Similar arguments hold for the scalar curvature-squared invariant. Here the generating primary superfield $H^{ij}_{R^2}$ is given by (3.4.15)

$$H^{ij}_{R^2} \quad = \quad -\mathrm{i} \boldsymbol{\lambda}^{\alpha i} \boldsymbol{\lambda}^{j}_{\alpha} + 2 \mathbf{W} \mathbf{X}^{ij} \, , \tag{3.6.25}$$

and subsequently its descendants (3.3.18) are constructed solely out of the linear multiplet. This again implies that

$$\frac{\delta S_{R^2}}{\delta V_{ij}} = H^{ij}_{R^2} \, , \qquad \frac{\delta S_{R^2}}{\delta X_{ij}} = -H^{ij}_{R^2} \, , \tag{3.6.26}$$

and then the component equation of motion is simply

$$-H^{ij}_{R^2} = 0 \, . \tag{3.6.27}$$

As for the Log invariant, it requires more explanations on the variation of the primary superfield $H^{ij}_{\log}$ with respect to the V_{kl} prepotential. For this we recall that

$$W = -\frac{3\mathrm{i}}{40} \nabla_{ij} \Delta^{ijkl} V_{kl} \, , \quad \mathbb{D} V_{ij} = -2 V_{ij} \, , \quad S^{\gamma}_{k} V_{ij} = 0 \, . \tag{3.6.28}$$

while

$$H^{ij}_{\log} = -\frac{3\mathrm{i}}{40} \Delta^{ijkl} \nabla_{kl} \log W \, . \tag{3.6.29}$$

The BF action can then be written as

$$S_{\log} = \int \mathrm{d}^{5|8} z \, E \, V_{ij} H^{ij}_{\log} \, . \tag{3.6.30}$$

If we vary with respect to V_{kl} the above we obtain

$$\delta S_{\log} = \int \mathrm{d}^{5|8}z\, E \left\{ \delta V_{ij} H^{ij}_{\log} + V_{ij} \delta H^{ij}_{\log} \right\} , \tag{3.6.31}$$

and then

$$\delta S_{\log} = \int \mathrm{d}^{5|8}z\, E \left\{ \delta V_{ij} H^{ij}_{\log} + V_{ij} \Big[-\frac{3\mathrm{i}}{40} \Delta^{ijkl} \nabla_{kl} W^{-1} \delta W \Big] \right\} . \tag{3.6.32}$$

After integration by parts of the last term, it follows

$$\delta S_{\log} = \int \mathrm{d}^{5|8}z\, E \left\{ \delta V_{ij} H^{ij}_{\log} + \delta W \right\} = \int \mathrm{d}^{5|8}z\, E \left\{ \delta V_{ij} H^{ij}_{\log} \right\} . \tag{3.6.33}$$

Here we have used the fact that the full conformal superspace integral of a vector multiplet is (expected to be) zero as a total derivative. This is clearly true in flat superspace. Moreover, we know the same holds for a tensor multiplet in $6D$ $N = (1,0)$ and a vector multiplet in $4D$ $N = 2$ conformal superspace [31,32,195], and we expect the same in $5D$. We can prove it by using the A-action principle in $5D$ conformal superspace, a result that we plan to present elsewhere.

To obtain the EOM from the component action, we begin by taking the variation of the Log supergravity action with respect to the auxiliary field X^{ij}. As can be seen from the explicit expressions (3.4.8), each of the descendant components of the composite $H^{ij}_{\log}$ carry X^{ij} dependence. The resulting EOM is given by:

$$-H^{ij}_{\log} - v_a \frac{\delta \mathcal{H}^a_{\log}}{\delta X^{ij}} - W \frac{\delta F_{\log}}{\delta X^{ij}} - X_{kl} \frac{\delta H^{kl}_{\log}}{\delta X^{ij}} - 2\lambda^{\alpha k} \frac{\delta \varphi_{\alpha k \log}}{\delta X^{ij}} = 0 . \tag{3.6.34}$$

Next, we will demonstrate that the bosonic contribution arising from the X^{ij} dependencies found in $H^{ij}_{\log}$ and its descendants does indeed sum up to zero within the EOM, i.e.,

$$-v_a \frac{\delta \mathcal{H}^a_{\log}}{\delta X^{ij}} - W \frac{\delta F_{\log}}{\delta X^{ij}} - X_{kl} \frac{\delta H^{kl}_{\log}}{\delta X^{ij}} \tag{3.6.35}$$

(at least explicitly up to fermions) vanishes. We expect that this will hold even when we extend our analysis to include fermionic terms. This aligns with the expectations derived from the superspace argument that the full superspace integral of a vector multiplet is zero as a total derivative. To see it explicitly, note that

$$\begin{aligned}
X_{kl} \frac{\delta H^{kl}_{\log}}{\delta X^{ij}} &= -\frac{1}{16} W^{-1} \mathcal{R} X_{ij} - 6 W^{-1} X_{ij} D - \frac{1}{2} \mathcal{D}^a \mathcal{D}_a (W^{-1} X_{ij}) - \frac{9}{64} W^{-1} X_{ij} W^{ab} W_{ab} \\
&\quad + \frac{1}{2} X_{ij} W^{-2} \mathcal{D}^a \mathcal{D}_a (W) - \frac{1}{2} \mathcal{D}_a \big(W^{-2} X_{ij} (\mathcal{D}^a W) \big) + \frac{1}{8} X_{ij} W^{-3} F^{ab} F_{ab} \\
&\quad - \frac{3}{8} X_{ij} X^{kl} X_{kl} W^{-3} - \frac{1}{4} X_{ij} W^{-3} \big(\mathcal{D}^a W \big) \mathcal{D}_a W , \qquad (3.6.36a) \\
W \frac{\delta F_{\log}}{\delta X^{ij}} &= \frac{1}{16} W^{-1} \mathcal{R} X_{ij} + 6 W^{-1} X_{ij} D + \frac{1}{2} W^{-1} \mathcal{D}^a \mathcal{D}_a X_{ij} + \frac{1}{2} \mathcal{D}^a \mathcal{D}_a (W^{-1} X_{ij}) \\
&\quad - \frac{1}{2} \mathcal{D}^a (W^{-1} \mathcal{D}_a X_{ij}) - \frac{1}{2} W^{-1} F^{ab} \Phi_{abij} + \frac{9}{64} W^{-1} X_{ij} W^{ab} W_{ab} \\
&\quad - \frac{3}{2} X_{ij} W^{-2} \mathcal{D}^a \mathcal{D}_a (W) - \frac{3}{2} W^{-2} \mathcal{D}^a W \mathcal{D}_a X_{ij} + \frac{3}{2} \mathcal{D}_a (W^{-2} X_{ij} \mathcal{D}^a W)
\end{aligned}$$

$$-\frac{3}{8}W^{-3}X_{ij}F^{ab}F_{ab}+\frac{9}{4}W^{-3}X_{ij}\mathcal{D}^aW\mathcal{D}_aW+\frac{3}{8}W^{-3}X_{ij}X^{kl}X_{kl}\ , \tag{3.6.36b}$$

$$\begin{aligned} v_a\frac{\delta\mathcal{H}^a_{\log}}{\delta X^{ij}} &= v_a\mathcal{D}_b(W^{-1}\Phi^{ab}{}_{ij}+\frac{1}{2}W^{-3}X_{ij}F^{ab})\ , \\ &= \frac{1}{2}W^{-1}F_{ab}\Phi^{ab}{}_{ij}+\frac{1}{4}W^{-3}X_{ij}F_{ab}F^{ab}+\text{total derivative}\ . \end{aligned} \tag{3.6.36c}$$

Substituting the above expressions in (3.6.35) all terms cancel out implying that it holds

$$-v_a\frac{\delta\mathcal{H}^a_{\log}}{\delta X^{ij}}-W\frac{\delta F_{\log}}{\delta X^{ij}}-X_{kl}\frac{\delta H^{kl}_{\log}}{\delta X^{ij}}=0\ . \tag{3.6.37}$$

The vector multiplet EOM for the Log-invariant then reads

$$-H^{ij}_{\log}=0\ . \tag{3.6.38}$$

3.6.4 Four-derivative: Linear multiplet EOMs

Both the Weyl-squared and Log component actions do not have F-dependent terms, so we only have to vary the scalar curvature-squared action (3.5.21b). The relevant terms are given by

$$\mathcal{L}_{R^2,F} = -\Big(v_a\mathcal{H}^a_{R^2}+WF_{R^2}+X_{ij}H^{ij}_{R^2}-2\lambda_i^{\alpha}\varphi^i_{\alpha R^2}\Big)\ . \tag{3.6.39a}$$

After substituting $H^{ij}_{R^2}, \varphi^i_{\alpha R^2}, F_{R^2}$, and $\mathcal{H}^a_{R^2}$ according to (3.4.16) and performing integration by parts, we find that

$$\begin{aligned} \mathcal{L}_{R^2,F} = {} & F^{ab}\left(2\mathbf{W}\mathbf{F}_{ab}+3W_{ab}\mathbf{W}^2\right)-W\mathbf{X}^{ij}\mathbf{X}_{ij}-4W\mathbf{W}\mathcal{D}^a\mathcal{D}_a\mathbf{W}-\frac{1}{2}\mathcal{R}W\mathbf{W}^2 \\ & -2W(\mathcal{D}^a\mathbf{W})\mathcal{D}_a\mathbf{W}+6WW^{ab}\mathbf{F}_{ab}\mathbf{W}+\frac{39}{8}WW^{ab}W_{ab}\mathbf{W}^2+16DW\mathbf{W}^2 \\ & -2X_{ij}\mathbf{W}\mathbf{X}^{ij}+\mathrm{i}(\Sigma^{ab})^{\alpha\beta}F_{ab}\boldsymbol{\lambda}^i_{\alpha}\boldsymbol{\lambda}_{\beta i}-2\mathrm{i}W(\mathcal{D}_{\alpha}{}^{\beta}\boldsymbol{\lambda}^i_{\beta})\boldsymbol{\lambda}_i^{\alpha}+64W\chi^{\alpha i}\boldsymbol{\lambda}_{\alpha i}\mathbf{W} \\ & +3\mathrm{i}WW_{\alpha\beta}\boldsymbol{\lambda}^{\alpha i}\boldsymbol{\lambda}_i^{\beta}+\mathrm{i}X_{ij}\boldsymbol{\lambda}^{\alpha i}\boldsymbol{\lambda}^j_{\alpha} \\ & +2\lambda_i^{\alpha}\Big(\mathrm{i}\mathbf{X}^{ij}\boldsymbol{\lambda}_{\alpha j}-2\mathrm{i}\mathbf{F}_{\alpha\beta}\boldsymbol{\lambda}^{\beta i}+16\mathrm{i}\chi^i_{\alpha}\mathbf{W}^2-2\mathrm{i}\mathbf{W}\mathcal{D}_{\alpha\beta}\boldsymbol{\lambda}^{\beta i}-\mathrm{i}(\mathcal{D}_{\alpha\beta}\mathbf{W})\boldsymbol{\lambda}^{\beta i} \\ & \quad -3\mathrm{i}W_{\alpha\beta}\mathbf{W}\boldsymbol{\lambda}^{\beta i}\Big)\ . \end{aligned} \tag{3.6.39b}$$

Since the above is expressed in terms of the composite vector multiplet fields, we make use of (3.3.18) to work out the F-dependence. Specifically,

$$\frac{\delta}{\delta F}\mathbf{W} = \frac{1}{4}G^{-1}\ , \tag{3.6.40a}$$

$$\frac{\delta}{\delta F}\boldsymbol{\lambda}^i_{\alpha} = -\frac{\mathrm{i}}{8}G^{-3}G^{ij}\varphi_{\alpha j}\ , \tag{3.6.40b}$$

$$\frac{\delta}{\delta F}\mathbf{X}^{ij} = -\frac{1}{8}G^{-3}FG^{ij}-\frac{\mathrm{i}}{8}G^{-3}\varphi^{\alpha i}\varphi_{\alpha}{}^{j}+\frac{3\mathrm{i}}{16}G^{-5}G^{ij}G^{kl}\varphi^{\alpha}_k\varphi_{\alpha l}\ . \tag{3.6.40c}$$

Varying $\mathcal{L}_{R^2,F}$ with respect to F yields

$$0 = -G^{-1}\Bigg[\frac{1}{2}X^{ij}\mathbf{X}_{ij}-\frac{1}{2}F^{ab}\mathbf{F}_{ab}+W\mathcal{D}^a\mathcal{D}_a\mathbf{W}+\mathbf{W}\mathcal{D}^a\mathcal{D}_aW+\big(\mathcal{D}^aW\big)\mathcal{D}_a\mathbf{W}$$

$$
\begin{aligned}
&+\frac{1}{4}\mathcal{R}W\mathbf{W}-8DW\mathbf{W}-\frac{3}{2}W^{ab}\big(F_{ab}\mathbf{W}+\mathbf{F}_{ab}W\big)-\frac{39}{16}W^{ab}W_{ab}W\mathbf{W}\\
&+\frac{\mathrm{i}}{2}\lambda^{\alpha i}\mathcal{D}_\alpha{}^\beta\boldsymbol{\lambda}_{\beta i}+\frac{\mathrm{i}}{2}\boldsymbol{\lambda}^{\alpha i}\mathcal{D}_\alpha{}^\beta\lambda_{\beta i}-16\mathrm{i}\chi^{\alpha i}\big(W\boldsymbol{\lambda}_{\alpha i}+\mathbf{W}\lambda_{\alpha i}\big)-\frac{3\mathrm{i}}{2}W^{\alpha\beta}\lambda^i_\alpha\boldsymbol{\lambda}_{\beta i}\Big]\\
-G^{-3}\Big[&\frac{1}{4}G^{ij}\varphi_{\beta i}\big(\lambda_{\alpha j}\mathcal{D}^{\alpha\beta}\mathbf{W}+\boldsymbol{\lambda}_{\alpha j}\mathcal{D}^{\alpha\beta}W\big)+\frac{1}{2}G^{ij}\varphi_{\beta i}\big(W\mathcal{D}^{\alpha\beta}\boldsymbol{\lambda}_{\alpha j}+\mathbf{W}\mathcal{D}^{\alpha\beta}\lambda_{\alpha j}\big)\\
&-\frac{1}{2}G^{ij}\varphi_{\alpha i}\big(F^{\alpha\beta}\boldsymbol{\lambda}_{\beta j}+\mathbf{F}^{\alpha\beta}\lambda_{\beta j}\big)-\frac{1}{4}G^{ij}F\big(W\mathbf{X}_{ij}+\mathbf{W}X_{ij}\big)\\
&-\frac{3}{4}G^{ij}W^{\alpha\beta}\varphi_{\alpha i}\big(\mathbf{W}\lambda_{\beta j}+W\boldsymbol{\lambda}_{\beta j}\big)+\frac{\mathrm{i}}{4}FG^{ij}\lambda_i^\alpha\boldsymbol{\lambda}_{\alpha j}+8G_{ij}\chi^{\alpha i}\varphi^j_\alpha W\mathbf{W}\\
&+\frac{1}{4}G_{ij}\varphi^{\alpha i}\big(X^{jk}\boldsymbol{\lambda}_{\alpha k}+\mathbf{X}^{jk}\lambda_{\alpha k}\big)-\frac{\mathrm{i}}{4}\varphi^{\alpha i}\varphi^j_\alpha\big(X_{ij}\mathbf{W}+\mathbf{X}_{ij}W\big)-\frac{1}{4}\varphi^{\alpha i}\varphi^j_\alpha\lambda_i^\beta\boldsymbol{\lambda}_{\beta j}\Big]\\
-G^{-5}\Big[&\frac{3\mathrm{i}}{8}G^{ij}G^{kl}\varphi^\alpha_k\varphi_{\alpha l}\Big(X_{ij}\mathbf{W}+\mathbf{X}_{ij}W-\mathrm{i}\lambda_i^\beta\boldsymbol{\lambda}_{\beta j}\Big)\Big]\,.
\end{aligned}
\tag{3.6.41}
$$

After covariantising the previous result, one obtains the following superconformal primary EOM [6]:

$$0=-W_{R^2}\ , \tag{3.6.42}$$

with

$$
\begin{aligned}
W_{R^2}\ =\ G^{-1}\Big[&\frac{1}{2}X^{ij}\mathbf{X}_{ij}-\frac{1}{2}F^{ab}\mathbf{F}_{ab}+W\Box\mathbf{W}+\mathbf{W}\Box W+\big(\nabla^a W\big)\nabla_a\mathbf{W}\\
&+\frac{3}{16}YW\mathbf{W}-\frac{3}{2}W^{ab}\big(F_{ab}\mathbf{W}+\mathbf{F}_{ab}W\big)-\frac{39}{16}W^{ab}W_{ab}W\mathbf{W}\\
&+\frac{\mathrm{i}}{2}\lambda^{\alpha i}\nabla_\alpha{}^\beta\boldsymbol{\lambda}_{\beta i}+\frac{\mathrm{i}}{2}\boldsymbol{\lambda}^{\alpha i}\nabla_\alpha{}^\beta\lambda_{\beta i}+\frac{3}{2}X^{\alpha i}\big(W\boldsymbol{\lambda}_{\alpha i}+\mathbf{W}\lambda_{\alpha i}\big)-\frac{3\mathrm{i}}{2}W^{\alpha\beta}\lambda^i_\alpha\boldsymbol{\lambda}_{\beta i}\Big]\\
+G^{-3}\Big[&\frac{1}{4}G^{ij}\varphi_{\beta i}\big(\lambda_{\alpha j}\nabla^{\alpha\beta}\mathbf{W}+\boldsymbol{\lambda}_{\alpha j}\nabla^{\alpha\beta}W\big)+\frac{1}{2}G^{ij}\varphi_{\beta i}\big(W\nabla^{\alpha\beta}\boldsymbol{\lambda}_{\alpha j}+\mathbf{W}\nabla^{\alpha\beta}\lambda_{\alpha j}\big)\\
&-\frac{1}{2}G^{ij}\varphi_{\alpha i}\big(F^{\alpha\beta}\boldsymbol{\lambda}_{\beta j}+\mathbf{F}^{\alpha\beta}\lambda_{\beta j}\big)-\frac{1}{4}G^{ij}F\big(W\mathbf{X}_{ij}+\mathbf{W}X_{ij}\big)\\
&-\frac{3}{4}G^{ij}W^{\alpha\beta}\varphi_{\alpha i}\big(\mathbf{W}\lambda_{\beta j}+W\boldsymbol{\lambda}_{\beta j}\big)+\frac{\mathrm{i}}{4}FG^{ij}\lambda_i^\alpha\boldsymbol{\lambda}_{\alpha j}+\frac{3\mathrm{i}}{4}G_{ij}X^{\alpha i}\varphi^j_\alpha W\mathbf{W}\\
&+\frac{1}{4}G_{ij}\varphi^{\alpha i}\big(X^{jk}\boldsymbol{\lambda}_{\alpha k}+\mathbf{X}^{jk}\lambda_{\alpha k}\big)-\frac{\mathrm{i}}{4}\varphi^{\alpha i}\varphi^j_\alpha\big(X_{ij}\mathbf{W}+\mathbf{X}_{ij}W-\mathrm{i}\lambda_i^\beta\boldsymbol{\lambda}_{\beta j}\big)\Big]\\
+G^{-5}\Big[&\frac{3\mathrm{i}}{8}G^{ij}G^{kl}\varphi^\alpha_k\varphi_{\alpha l}\Big(X_{ij}\mathbf{W}+\mathbf{X}_{ij}W-\mathrm{i}\lambda_i^\beta\boldsymbol{\lambda}_{\beta j}\Big)\Big]\,.
\end{aligned}
\tag{3.6.43}
$$

3.6.5 Consistency checks

This subsection is devoted to the various consistency checks that have been performed on the vector, linear, and Weyl multiplet EOMs. As we will comment more in detail soon, and as expected, these EOMs are primary superfields. Furthermore, it has been checked that the vector multiplet EOMs satisfy the linear multiplet constraint (3.2.13), while the linear multiplet EOM adheres to the vector multiplet constraint (3.2.1). The Weyl multiplet EOM is found to satisfy the conformal supercurrent conservation equation [181], which we will elaborate further.

The vector multiplet EOM for the higher-derivative action $S_{\rm HD}$ (3.6.1) is

$$0=\frac{3}{4}H^{ij}_{\rm VM}+\kappa G^{ij}+\alpha H^{ij}_{\rm Weyl}+\beta H^{ij}_{\log}+\gamma H^{ij}_{R^2}\ . \tag{3.6.44}$$

Here the first two terms correspond to the EOM for the vector multiplet in the two-derivative supergravity theory S_{gSG} while the remaining three terms describe the contribution coming from the three independent curvature-squared invariants. It is clear that, as expected, the right-hand side of (3.6.44) is a primary superfield satisfying the linear multiplet constraint (3.2.13).

Next, the resulting linear multiplet EOM in the higher-derivative action $S_{\rm HD}$ (3.6.1) reads

$$0 = \mathbf{W} + \kappa W + \gamma W_{R^2} \ . \tag{3.6.45}$$

It is possible to check from the explicit form (3.6.43) that W_{R^2} is primary, $S^i_\alpha W_{R^2} = 0$. Here we have made use of the relations (3.2.5) and (3.2.17) as well as the analogous identities for the bold superfields

$$\begin{aligned}
S^i_\alpha \boldsymbol{\lambda}^j_\beta &= -2{\rm i}\varepsilon_{\alpha\beta}\varepsilon^{ij}\boldsymbol{W} \ , && (3.6.46a)\\
S^i_\alpha \boldsymbol{F}_{\beta\gamma} &= 4\varepsilon_{\alpha(\beta}\boldsymbol{\lambda}_{\gamma)}{}^i \ , && (3.6.46b)\\
S^i_\alpha \boldsymbol{X}^{jk} &= -2{\rm i}\varepsilon^{i(j}\boldsymbol{\lambda}^{k)}_\alpha \ , && (3.6.46c)\\
S^i_\alpha \nabla_\beta{}^\gamma\boldsymbol{\lambda}^j_\gamma &= 5{\rm i}\varepsilon_{\alpha\beta}\boldsymbol{X}^{ij} + 2{\rm i}\varepsilon^{ij}\boldsymbol{F}_{\alpha\beta} - {\rm i}\varepsilon^{ij}\nabla_{\alpha\beta}\boldsymbol{W} \ , && (3.6.46d)\\
S^i_\alpha \Box\boldsymbol{W} &= -\frac{9}{2}W_{\alpha\beta}\boldsymbol{\lambda}^{\beta i} + 2\nabla_{\alpha\beta}\boldsymbol{\lambda}^{\beta i} \ . && (3.6.46e)
\end{aligned}$$

Alternatively, it is useful to note that W_{R^2} can be expressed as

$$W_{R^2} = \frac{\rm i}{32}G\nabla_{ij}\mathcal{R}^{ij}_1 \ , \tag{3.6.47a}$$

where

$$\mathcal{R}^{ij}_1 = G^{-2}\left(\delta^i_k\delta^j_l - \frac{1}{2G^2}G^{ij}G_{kl}\right)H^{kl}_{\text{bilinear}} \ , \tag{3.6.47b}$$

and

$$H^{kl}_{\text{bilinear}} = 2W\mathbf{X}^{kl} + 2\mathbf{W}X^{kl} - 2\lambda^{\alpha k}\boldsymbol{\lambda}^l_\alpha \ . \tag{3.6.47c}$$

This remarkably simple form of (3.6.47) guarantees that the right-hand side of eq. (3.6.45), and in particular (3.6.43), is a primary superfield satisfying the vector multiplet constraints (3.2.1), as expected.

Finally, the Weyl multiplet EOM for the higher-derivative action is

$$0 \ = \ \mathcal{J} = J_{\rm EH} + \alpha J_{\rm Weyl} + \beta J_{\log} + \gamma J_{R^2} \ , \tag{3.6.48a}$$

with

$$\begin{aligned}
J_{\rm EH} &= \frac{3}{32}(G - W^3) \ , && (3.6.48b)\\
J_{\rm Weyl} &= -\frac{3}{64}WY + \frac{3}{16}WW^{ab}W_{ab} + \frac{3}{32}F_{ab}W^{ab} - \frac{3}{16}\lambda^\alpha_i X^i_\alpha \ , && (3.6.48c)\\
J_{\log} &= -\frac{3}{1024}WY - \frac{69}{1024}W^{ab}W_{ab}W + \frac{3}{32}\Box W - \frac{3}{64}F_{ab}W^{ab} - \frac{3}{256}\lambda^\alpha_j X^j_\alpha\\
&\quad + \frac{3}{128}F^{ab}F_{ab}W^{-1} - \frac{9}{128}X^{ij}X_{ij}W^{-1} + \frac{3{\rm i}}{32}(\Gamma^a)^{\alpha\beta}W^{-1}\lambda^i_\alpha\nabla_a\lambda_{\beta i}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{64}W^{-1}(\nabla^a W)\nabla_a W-\frac{3\mathrm{i}}{128}(\Sigma^{ab})^{\alpha\beta}F_{ab}\lambda_\alpha^i\lambda_{\beta i}W^{-2}-\frac{3\mathrm{i}}{64}X^{ij}\lambda_i^\alpha\lambda_{\alpha j}W^{-2}\\
&-\frac{3}{256}\lambda^{\alpha i}\lambda_i^\beta\lambda_\alpha^j\lambda_{\beta j}W^{-3}\ , \qquad (3.6.48\mathrm{d})
\end{aligned}
$$

$$
J_{R^2} = -\frac{3}{8}W\mathbf{W}^2+\frac{3}{32}G^{-1}\Big(WG^{ij}\mathbf{X}_{ij}+G^{ij}X_{ij}\mathbf{W}-\mathrm{i}G^{ij}\lambda_i^\alpha\boldsymbol{\lambda}_{\alpha j}\Big)\ . \qquad (3.6.48\mathrm{e})
$$

Here $J_{\mathrm{E}H}$ is the EOM from the two-derivative gauged supergravity action, $S_{\mathrm{g}SG}$, which does not have any contribution from the cosmological constant term κ.

Let us comment on the Weyl multiplet EOM (3.6.48). As previously explained in [6,29], the 5D Weyl multiplet may be described by a single unconstrained real prepotential $\mathfrak{U}$, in complete analogy with the case of 4D $N=2$ supergravity [196]. Given a system of matter superfields φ^i, a Noether coupling between $\mathfrak{U}$ and the matter supercurrent $\mathfrak{J}$ can be constructed

$$
S[\varphi^i] = \int \mathrm{d}^{5|8}z\,E\,\mathfrak{U}\mathfrak{J}=\int \mathrm{d}^5x\,e\Big(DJ+\cdots\Big)\ , \qquad (3.6.49)
$$

where $J=\mathfrak{J}|$. The Weyl multiplet EOM (3.6.48) is obtained by varying the supergravity action with respect to $\mathfrak{U}$

$$
\frac{\delta S[\varphi^i]}{\delta\mathfrak{U}}=\mathfrak{J}=0\ . \qquad (3.6.50)
$$

In five dimensions, the $N=1$ and $N=2$ supercurrent multiplets were introduced by Howe and Lindström [181], see also [164]. The conformal supercurrent, $\mathfrak{J}$, is a dimension-3 primary real scalar superfield which satisfies the conservation equation [197]

$$
\nabla^{ij}\mathfrak{J}=0\ , \qquad (3.6.51)
$$

provided the dynamical matter superfields obey their EOMs, $\delta S[\varphi^i]/\delta\varphi^i=0$. Hence, we shall prove that the expression $\mathfrak{J}$ in (3.6.48) satisfies the conservation constraint (3.6.51). It has been shown in [29] that this constraint holds for $J_{\mathrm{E}H}$. For each invariant, it has indeed been verified in [6] that the corresponding J is a primary superfield of dimension 3. Furthermore, a very non-trivial consistency check of (3.6.48b)–(3.6.48e) is to verify that $\nabla^{ij}J=0$, provided the vector and linear multiplets equations of motion of eqs. (3.6.44) and (3.6.45), respectively, are imposed. Using *Cadabra* an explicit calculation shows that, off-shell, it holds

$$
\begin{aligned}
\nabla^{ij}J_{\mathrm{W}eyl} &= \frac{3\mathrm{i}}{4}WH^{ij}_{\mathrm{W}eyl}\ , &(3.6.52\mathrm{a})\\
\nabla^{ij}J_{\log} &= \frac{3\mathrm{i}}{4}WH^{ij}_{\log}\ , &(3.6.52\mathrm{b})\\
\nabla^{ij}J_{R^2} &= \frac{3\mathrm{i}}{4}WH^{ij}_{R^2}-\frac{3\mathrm{i}}{4}G^{ij}W_{R^2}\ . &(3.6.52\mathrm{c})
\end{aligned}
$$

The right-hand sides of (3.6.52) are all proportional to the composite vector and linear multiplets appearing in (3.6.44) and (3.6.45). Consequently, the supercurrent conservation equation (3.6.51) is satisfied once the EOMs for the compensators are used.

3.7 EOM descendants

In section 3.6, we derived the manifestly covariant superconformal primary superfield equations of motions (EOMs) for the vector, linear, and Weyl multiplet. In this section, we will find the descendants of these EOMs, which are obtained by successive applications of superspace spinor derivatives or, equivalently, successive application of Q-supersymmetry transformations. Computing the descendants is of importance. It suffices to mention that the EOM resulting from the variaton of the vielbein in components is at the bottom of the multiplet of EOMs. One advantage of deriving all the EOMs as descendants of the primary ones is that covariance is manifest at every stage. This fact would be very non-trivial if one attempted to obtain the EOMs directly by varying the component actions, including all fermions.

The component structure of the vector multiplet compensator EOM is straightforward to obtain, as it naturally forms a (composite) linear multiplet. More specifically, in the two-derivative gauge supergravity S_{gSG}, the EOM (3.6.8) leads to a combination of G^{ij} and $H^{ij}_{\rm VM}$, whose descendants can be found using (3.2.15) and (3.2.22). In the four-derivative case, the EOMs in the Weyl-squared, Log, and scalar curvature-squared invariants are described by $H^{ij}_{\rm W\mathit{eyl}}$, $H^{ij}_{\log}$, and $H^{ij}_{R^2}$, respectively. Their corresponding descendants have already been presented in eqs. (3.4.3), (3.4.8), and (3.4.16).

The component structure of the linear multiplet compensator EOM is straightforward to obtain, as it naturally forms a (composite) vector multiplet. More specifically, in S_{gSG}, the EOM (3.6.10) leads to a combination of W and $\mathbf{W}$, whose descendants can be found using (3.2.2) and (3.3.18). In the four-derivative case, since both the Weyl-squared and Log invariants do not have any dependence on the linear multiplet (e.g., there are no F-dependent terms), the only contribution to the EOM comes from the scalar curvature-squared invariant which leads to the non-trivial structure W_{R^2}, eq. (3.6.43). We will present its new component structure in subsection 3.7.5.

To our knowledge, the component structure for the Weyl multiplet EOM, specifically the supercurrent, was not known before. Hence, it is the subject of the next subsection to derive the independent components of the supercurrent multiplet and use them to compute its descendants for all two and four-derivative invariants.

3.7.1 Component structure of the conformal supercurrent

The $5D$ $N=1$ conformal supercurrent is described by a real primary scalar superfield J of dimension 3, which obeys the conservation equation

$$\nabla^2_{ij} J = 0 \,, \tag{3.7.1a}$$

where we have defined[10]

$$\nabla^2_{ij} = \frac{1}{2}\nabla_{(i}\nabla_{j)} \,. \tag{3.7.1b}$$

[10]Note that in this section, our $\nabla^2_{ij} = \frac{1}{2}\nabla_{ij}$. We insert the superscript “2” to easily keep track the number of spinor derivatives acting on J.

To define the independent components of the conformal supercurrent J, one can systematically act with spinor derivatives on J. At dim-$\frac{7}{2}$, we introduce

$$J^1_{\alpha i} = \nabla_{\alpha i} J \ . \tag{3.7.2}$$

At dim-4, we start by applying two spinor derivatives on J. A generic two-spinor-derivative combination can be expanded as

$$\nabla_{\alpha i}\nabla_{\beta j} = \mathrm{i}\varepsilon_{ij}(\Gamma^a)_{\alpha\beta}\nabla_a - \frac{1}{2}\varepsilon_{ij}(\Sigma^{ab})_{\alpha\beta}\nabla^2_{ab} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^2_{ij} + \frac{1}{2}(\Gamma^a)_{\alpha\beta}\nabla^2_{aij} + l.d. \ , \tag{3.7.3}$$

where $l.d.$ contains terms related to lower-dimensional supercurrent descendants and Weyl multiplet. Furthermore, the irreducible ∇^2 operators are defined as

$$\nabla^2_{aij} = \frac{1}{2}\nabla_{(i}\Gamma_a\nabla_{j)} \ , \qquad \text{and} \qquad \nabla^2_{ab} = \frac{1}{2}\nabla^i\Sigma_{ab}\nabla_i \ . \tag{3.7.4}$$

It follows from the constraint (3.7.1) that the independent descendants at dim-4 are

$$J^2_{ab} = \nabla^2_{ab}J \ , \quad J^2_{aij} = \nabla^2_{aij}J \ . \tag{3.7.5}$$

Note that, given the superconformal primary supercurrent J, dim-$\frac{7}{2}$ and dim-4 descendants prove to be K-primary.

At dim-$\frac{9}{2}$, following the discussion of [198], there exist three irreducible ∇^3 operators, specifically

$$\nabla^3_{\alpha i} = \{\nabla^j_\alpha, \nabla^2_{ij}\} \ , \quad \nabla^3_{a\alpha i} = \{\nabla^j_\alpha, \nabla^2_{aij}\} \ , \quad \nabla^3_{\alpha ijk} = \{\nabla_{\alpha(i}, \nabla^2_{jk)}\} \ \cong \ 2\nabla_{\alpha(i}\nabla^2_{jk)} \ , \tag{3.7.6}$$

where we have omitted writing $l.d.$ and $\cong$ means that the equality holds only up to $l.d.$ terms. It is easy to verify that a generic object with three spinor derivatives can be expanded in terms of these three irreducible ∇^3 operators (3.7.6), since

$$\nabla\nabla\nabla \propto \nabla\nabla^2 = \frac{1}{2}\{\nabla,\nabla^2\} + \frac{1}{2}[\nabla,\nabla^2] \ \cong \ \frac{1}{2}\{\nabla,\nabla^2\} \ , \tag{3.7.7}$$

where the anticommutator can be expanded in (3.7.6) according to eq. (A.13) of [198]. However, it is important to note that eq. (A.13) there was given in flat space. When uplifted to curved space, one then picks up additional contributions to $l.d.$ terms. Furthermore, when acting on J, the following two ∇^3 operators do not lead to independent components as a result of the conservation constraint (3.7.1):

$$\nabla^3_{\alpha ijk}J \ \cong \ 2\nabla_{\alpha(i}\nabla^2_{jk)}J = 0 \ , \tag{3.7.8a}$$

$$\nabla^3_{\alpha i}J \ = \ \{\nabla^j_\alpha, \nabla^2_{ij}\}J = \nabla^2_{ij}\nabla^j_\alpha J = [\nabla^2_{ij}, \nabla^j_\alpha]J \propto \nabla_a J^1_{\alpha i} + l.d. \ . \tag{3.7.8b}$$

Therefore, the only independent component at dim-$\frac{9}{2}$ is $\nabla^3_{a\alpha i}J$. Now, using the fact that

$$(\Gamma^a)_\alpha{}^\beta\nabla^3_{a\beta i} = -\nabla^3_{\alpha i} \ , \tag{3.7.9}$$

this implies that the gamma trace part of $\nabla^3_{a\alpha i}J$ is a total derivative, i.e.,

$$(\Gamma^a)_\alpha{}^\beta\nabla^3_{a\beta i}J = -\nabla^3_{\alpha i}J \propto -\nabla_a J^1_{\alpha i} \ . \tag{3.7.10}$$

Hence, we can subtract the trace by modifying the above as follows

$$\nabla^3_{a\alpha i}J+\frac{1}{5}(\Gamma_a)_\alpha{}^\beta(\Gamma^b)_\beta{}^\gamma\nabla^3_{b\gamma i}J\,. \tag{3.7.11}$$

Lastly, we modify the descendant to make it K-primary. This amounts to adding a total derivative term to obtain the dim-$\frac{9}{2}$ component

$$J^3_{a\alpha i}=\left(\delta^b_a\delta^\gamma_\alpha+\frac{1}{5}(\Gamma_a)_\alpha{}^\beta(\Gamma^b)_\beta{}^\gamma\right)\left(\nabla^3_{b\gamma i}J-3\mathrm{i}\nabla_bJ^1_{\gamma i}\right)\,. \tag{3.7.12}$$

Next, we move on to dim-5 descendants. All the independent components at this level will come from

$$\begin{aligned}\nabla_{\beta j}\nabla^3_{a\alpha i}J &= 2\nabla_{\beta j}\nabla^k_\alpha\nabla^2_{aik}J+\nabla_{\beta j}[\nabla^2_{aik},\nabla^k_\alpha]J\\ &\cong (\Sigma^{cd})_{\beta\alpha}\nabla^2_{cd}\nabla^2_{aij}J-(\Gamma^c)_{\beta\alpha}\nabla^2_{cjk}\nabla^2_{ai}{}^kJ\,.\end{aligned} \tag{3.7.13}$$

Let us consider the Σ trace part:

$$\begin{aligned}\nabla_{(\beta(i}\nabla^3_{|a|\alpha)j)}J &= (\Sigma^{cd})_{\beta\alpha}\nabla^2_{cd}\nabla^2_{aij}J\\ &= -\nabla^k_{(\alpha}\nabla_{\beta)k}\nabla^2_{aij}J\\ &= -\frac{1}{2}\nabla^k_{(\alpha}\{\nabla_{\beta)k},\nabla^2_{aij}\}J-\frac{1}{2}\nabla^k_{(\alpha}[\nabla_{\beta)k},\nabla^2_{aij}]J\\ &\cong \frac{1}{3}\nabla_{(\alpha(i}\nabla^3_{|a|\beta)j)}J\,.\end{aligned} \tag{3.7.14}$$

The above equation implies that the Σ-trace part is a $l.d.$ term and hence, does not give any independent component. We are only left with the Γ-trace part, which is given by:

$$\begin{aligned}\nabla_{[\beta j}\nabla^3_{a\alpha]i}J &\cong -(\Gamma^c)_{\beta\alpha}\nabla^2_{cjk}\nabla^2_{ai}{}^kJ\\ &\cong \frac{2}{3}\nabla_{[\beta j}\nabla^3_{a\alpha]i}J+\frac{1}{6}\varepsilon_{ji}\nabla_{[\beta}{}^k\nabla^3_{a\alpha]k}J\,.\end{aligned} \tag{3.7.15}$$

The symmetric part in (ij) of the above equation implies that $\nabla_{[\beta(j}\nabla^3_{a\alpha]i)}J$ is again a $l.d.$ term, while the antisymmetric part is unfixed. Therefore, the only independent part left of $\nabla_{\beta j}\nabla^3_{a\alpha i}J$ is the following:

$$\begin{aligned}\nabla_{[\beta[j}\nabla^3_{a\alpha]i]}J &\cong -(\Gamma^c)_{\beta\alpha}\nabla^2_{c[j|k|}\nabla^2_{ai]}{}^kJ\\ &\cong \frac{1}{2}(\Gamma^c)_{\beta\alpha}\varepsilon_{ji}\nabla^2_{(c}{}^{kl}\nabla^2_{a)kl}J\,.\end{aligned} \tag{3.7.16}$$

The independent component is thus $\nabla^2_{(c}{}^{kl}\nabla^2_{a)kl}J$. Once again the η-trace part of this component is a total derivative and can be removed as follows

$$\nabla^2_{(a}{}^{kl}\nabla^2_{b)kl}J-\frac{1}{5}\eta_{ab}\nabla^{2ckl}\nabla^2_{ckl}J\,. \tag{3.7.17}$$

Lastly, we modify the descendant to make it K-primary. This amounts to adding a total derivative term to the above descendant to obtain the dim-5 component

$$J^4_{ab}=\left(\delta^c_{(a}\delta^d_{b)}-\frac{1}{5}\eta_{ab}\eta^{cd}\right)\left(\nabla^{2kl}_c\nabla^2_{dkl}J-6\nabla_c\nabla_dJ\right)\,. \tag{3.7.18}$$

In summary, we have derived the independent components of the conformal supercurrent multiplet J. They are defined as follows:

$$\begin{aligned}
&\text{dim-3} && J\ , && (3.7.19a)\\
&\text{dim-}\frac{7}{2} && J^1_{\alpha i} = \nabla_{\alpha i} J\ , && (3.7.19b)\\
&\text{dim-4} && J^2_{ab} = \nabla^2_{ab} J\ , \qquad J^2_{aij} = \nabla^2_{aij} J\ , && (3.7.19c)\\
&\text{dim-}\frac{9}{2} && J^3_{a\alpha i} = \left(\delta_a^b \delta_\alpha^\gamma + \frac{1}{5}(\Gamma_a)_\alpha{}^\beta (\Gamma^b)_\beta{}^\gamma\right)\left(\nabla^3_{b\gamma i} J - 3\mathrm{i}\nabla_b J^1_{\gamma i}\right)\ , && (3.7.19d)\\
&\text{dim-5} && J^4_{ab} = \left(\delta^c_{(a}\delta^d_{b)} - \frac{1}{5}\eta_{ab}\eta^{cd}\right)\left(\nabla^{2kl}_c \nabla^2_{dkl} J - 6\nabla_c \nabla_d J\right)\ . && (3.7.19e)
\end{aligned}$$

The component fields in the supercurrent multiplet will be defined by using the same names as the superfields, i.e., $J := J|$, $J^1_{\alpha i} := J^1_{\alpha i}|$, and so forth. As previously discussed by Howe and Lindström in [164, 181], associated to each component of the above current multiplet, we may then identify a corresponding field in the Weyl multiplet whose variation leads to the various Js in eq. (3.7.19). This is summarised in Table 3.3 below:

Current multiplet	J	$J^1_{\alpha i}$	J^2_{ab}	J^2_{aij}	$J^3_{a\alpha i}$	J^4_{ab}
Fields	D	$\chi^{\alpha i}$	W_{ab}	V_m^{ij}	$\hat{\psi}_m{}_i^\alpha$	$\hat{g}_{mn}$

Table 3.3: Current multiplet and the corresponding fields to which they couple via a Noether coupling. With $\hat{\psi}_m{}_i^\alpha$ and $\hat{g}_{mn}$ we denote the gamma traceless part of the gravitini and the traceless part of the metric, respectively.

3.7.2 Einstein–Hilbert

In this subsection we derive the EOM descendants of the Weyl multiplet in the two-derivative gauged supergravity action, $S_{\mathrm{g}SG}$. The descendants of the vector compensator EOM can be readily obtained from (3.2.15) and (3.2.22). To compute the descendants of the linear compensator EOM, the relevant equations are given in (3.2.2) and (3.3.18).

Recall the Weyl multiplet EOM for the action $S_{\mathrm{g}SG}$ given in subsection 3.6.1:

$$J_{\mathrm{E}H} = \frac{3}{32}\left(G - W^3\right)\ . \tag{3.7.20}$$

We present here the results for each component descendants of $J_{\mathrm{E}H}$ based on the structures found in eqs. (3.7.19):

$$\begin{aligned}
J^1_{\alpha i,\mathrm{E}H} &= \frac{3}{32}G_{ij}\varphi^j_\alpha G^{-1} - \frac{9}{32}\mathrm{i}\lambda_{i\alpha}\ , && (3.7.21a)\\
J^2_{ab,\mathrm{E}H} &= \frac{9}{16}\mathrm{i}F_{ab}W^2 + \frac{9}{16}\mathrm{i}W_{ab}W^3 - \frac{3}{32}(\Sigma_{ab})^{\alpha\beta}\varphi^i_\alpha\varphi_{i\beta}G^{-1} - \frac{9}{32}(\Sigma_{ab})^{\alpha\beta}\lambda^i_\alpha\lambda_{i\beta}W\ , && (3.7.21b)\\
J^2_{aij,\mathrm{E}H} &= \frac{3}{64}(\Gamma_a)_{\alpha\beta}\varphi^\alpha_i\varphi^\beta_j G^{-1} + \frac{3}{32}\mathrm{i}\mathcal{H}_a G_{ij}G^{-1} - \frac{3}{16}\mathrm{i}G^{-1}G^k{}_{(i}\nabla_a G_{j)k}\\
&\quad + \frac{3}{64}(\Gamma_a)^{\alpha\beta}G_{ik}G_{jl}G^{-3}\varphi^k_\alpha\varphi^l_\beta - \frac{9}{32}(\Gamma_a)_{\alpha\beta}W\lambda_i^\alpha\lambda_j^\beta\ , && (3.7.21c)
\end{aligned}$$

$$
\begin{aligned}
J^3_{a\alpha i,\mathrm{EH}} \;=\;& G^{-1}\Big\{\frac{9}{20}\mathrm{i}\Big(\delta_a^b\delta_\alpha^\beta-\frac{1}{2}(\Sigma_a{}^b)_\alpha{}^\beta\Big)\Big(\frac{1}{2}\nabla_b G_{ij}\varphi^j_\beta-\frac{3}{2}G_{ij}\nabla_b\varphi^j_\beta-\mathcal{H}_b\varphi_{i\beta}\Big)\\
&\quad-\frac{9\mathrm{i}}{320}\Big(\varepsilon_{abcde}(\Sigma^{de})_\alpha{}^\beta+3\eta_{ab}(\Gamma_c)_\alpha{}^\beta\Big)G_{ij}W^{bc}\varphi^j_\beta\Big\}\\
&+G^{-3}\Big\{\frac{27}{80}\mathrm{i}G_{ij}G_{kl}\Big(\delta_a^b\delta_\alpha^\beta-\frac{1}{2}(\Sigma_a{}^b)_\alpha{}^\beta\Big)\nabla_b G^{kl}\varphi^j_\beta\\
&\quad+\frac{3}{16}\Big(\delta_\alpha^\lambda(\Gamma_a)^{\beta\rho}-\frac{1}{5}\varepsilon^{\beta\rho}(\Gamma_a)_\alpha{}^\lambda\Big)G_{ij}\varphi^k_\lambda\varphi^j_\beta\varphi_{k\rho}\Big\}\\
&+\quad\Big\{\frac{9\mathrm{i}}{16}\Big(\delta_\alpha^\lambda(\Gamma_a)^{\beta\rho}-\frac{1}{5}\varepsilon^{\beta\rho}(\Gamma_a)_\alpha{}^\lambda\Big)\lambda_{j\lambda}\lambda_{i\beta}\lambda^j_\rho\Big\}\\
&+W\Big\{\frac{27}{40}\Big(\varepsilon_{abcde}(\Sigma^{de})_{\alpha\beta}+3\eta_{ab}(\Gamma_c)_{\alpha\beta}\Big)F^{bc}\lambda_i^\beta-\frac{27}{20}\Big(\delta_a^b\delta_\alpha^\beta-\frac{1}{2}(\Sigma_a{}^b)_\alpha{}^\beta\Big)\lambda_{i\beta}\nabla_b W\Big\}\\
&+W^2\Big\{\frac{189}{320}\Big(\varepsilon_{abcde}(\Sigma^{de})_{\alpha\beta}+3\eta_{ab}(\Gamma_c)_{\alpha\beta}\Big)W^{bc}\lambda_i^\beta\\
&\quad+\frac{27}{40}\Big(\delta_a^b\delta_\alpha^\beta-\frac{1}{2}(\Sigma_a{}^b)_\alpha{}^\beta\Big)\nabla_b\lambda_{i\beta}\Big\}\,,
\end{aligned}
\tag{3.7.21d}
$$

$$
\begin{aligned}
J^4_{ab,\mathrm{EH}} \;=\;& \Big(\delta^c_{(a}\delta^d_{b)}-\frac{1}{5}\eta_{ab}\eta^{cd}\Big)\times\Big[\\
&G\Big\{\frac{27}{8}W_c{}^eW_{de}\Big\}\\
&+G^{-1}\Big\{-\frac{9}{16}\mathcal{H}_c\mathcal{H}_d-\frac{9}{32}\nabla_c G_{ij}\nabla_d G^{ij}+\frac{27}{32}G_{ij}\nabla_c\nabla_d G^{ij}-\frac{9\mathrm{i}}{8}(\Gamma_c)^{\alpha\beta}\nabla_d\varphi^i_\alpha\varphi_{i\beta}\\
&\quad-\frac{9\mathrm{i}}{16}(\Sigma_c{}^e)^{\alpha\beta}W_{de}\varphi^i_\alpha\varphi_{i\beta}\Big\}\\
&+G^{-3}\Big\{-\frac{27}{64}G_{ij}G_{kl}\nabla_c G^{ij}\nabla_d G^{kl}+\frac{9}{32}\mathrm{i}\mathcal{H}_c G_{ij}(\Gamma_d)^{\alpha\beta}\varphi^i_\alpha\varphi^j_\beta\\
&\quad-\frac{9}{16}\mathrm{i}G_{ij}(\Gamma_c)^{\alpha\beta}\nabla_d G^i{}_k\varphi^j_\alpha\varphi^k_\beta\Big\}\\
&+\quad\Big\{\frac{27\mathrm{i}}{8}(\Sigma_{ce})^{\alpha\beta}F_d{}^e\lambda_{i\alpha}\lambda^i_\beta\Big\}\\
&+\;W\Big\{-\frac{27}{4}F_c{}^eF_{de}-\frac{27}{8}\nabla_c W\nabla_d W+\frac{27\mathrm{i}}{16}(\Sigma_c{}^e)^{\alpha\beta}W_{de}\lambda_{i\alpha}\lambda^i_\beta\\
&\quad+\frac{27\mathrm{i}}{8}(\Gamma_c)^{\alpha\beta}\lambda_{i\alpha}\nabla_d\lambda^i_\beta\Big\}\\
&+W^2\Big\{-\frac{27}{2}W_c{}^eF_{de}+\frac{27}{16}\nabla_c\nabla_d W\Big\}\\
&+W^3\Big\{-\frac{27}{4}W_c{}^eW_{de}\Big\}\Big]\,.
\end{aligned}
\tag{3.7.21e}
$$

Note that, we have proven each to be explicitly K-primary.

3.7.3 Weyl-squared

As previously discussed, the Weyl-squared invariant does not contribute to the linear multiplet EOM, while the descendants of the vector multiplet EOM have been described earlier in eq. (3.4.3).

Thus, in this subsection we are mainly focusing on deriving the descendants of the Weyl multiplet EOM.

In superspace, the Weyl multiplet EOM was presented in subsection 3.6.2. It defines a conformal supercurrent multiplet

$$J_{\mathrm{W}eyl} = -\frac{3}{64}WY + \frac{3}{16}WW^{ab}W_{ab} + \frac{3}{32}F_{ab}W^{ab} - \frac{3}{16}\lambda_i{}^{\alpha}X^i_{\alpha} \ . \tag{3.7.22}$$

We present here the results for one- and two-derivative descendants of $J_{\mathrm{W}eyl}$ based on the structures found in eqs. (3.7.19). Note that, we have explicitly proven each to be K-primary. In subsection 4.2 of the supplementary file [1], the reader can find the complete expressions of $J^3_{a\alpha i,\mathrm{W}eyl}$, along with the degauged, gauged fixed $(W=1)$, bosonic part of $J^4_{ab,\mathrm{W}eyl}$. It holds that

$$\begin{aligned}
J^1_{\alpha i,\mathrm{W}eyl} \ = \ & -\frac{3}{128}\mathrm{i}Y\lambda_{i\alpha} - \frac{3}{8}(\Gamma_a)_\alpha{}^\beta W\nabla^a X_{i\beta} + \frac{3}{32}\mathrm{i}W_{ab}W^{ab}\lambda_{i\alpha} + \frac{9}{128}WW_{ab}W^{ab}{}_{\alpha i} \\
& + \frac{9}{256}\varepsilon^{abcde}(\Gamma_e)_\alpha{}^\beta WW_{ab}W_{cd\beta i} - \frac{9}{32}(\Sigma^{ac})_\alpha{}^\beta WW_a{}^bW_{cb\beta i} + \frac{3}{128}F^{ab}W_{ab\alpha i} \\
& + \frac{3}{256}\varepsilon^{abcde}(\Gamma_e)_\alpha{}^\beta F_{ab}W_{cd\beta i} - \frac{3}{32}(\Sigma^{ac})_\alpha{}^\beta F_a{}^bW_{cb\beta i} + \frac{3}{32}(\Sigma_{ab})_{\alpha\beta}F^{ab}X_i^\beta \\
& + \frac{3}{16}\mathrm{i}(\Gamma^a)_{\alpha\beta}W_{ab}\nabla^b\lambda_i^\beta + \frac{1}{8}\mathrm{i}\Phi_{ab\,ij}(\Sigma^{ab})_\alpha{}^\beta\lambda^j_\beta + \frac{3}{32}\mathrm{i}\varepsilon^{abcde}(\Sigma_{ab})_{\alpha\beta}\lambda_i^\beta\nabla_eW_{cd} \\
& + \frac{3}{32}\mathrm{i}(\Gamma^a)_{\alpha\beta}\lambda_i^\beta\nabla^bW_{ab} - \frac{3}{16}X_{ij}X^j_\alpha + \frac{3}{16}(\Gamma^a)_{\alpha\beta}X_i^\beta\nabla_aW \ ,
\end{aligned} \tag{3.7.23a}$$

$$\begin{aligned}
J^2_{aij,\mathrm{W}eyl} \ = \ & -\frac{1}{16}\mathrm{i}\varepsilon_{abcde}F^{bc}\Phi^{de}{}_{ij} - \frac{1}{2}\mathrm{i}\nabla^b(W\Phi_{abij}) - \frac{3}{64}\mathrm{i}(\Gamma_a)_{\alpha\beta}W^{cd}W_{cd}{}^\alpha{}_{(i}\lambda^\beta_{j)} \\
& + \frac{9}{32}(\Gamma_a)_{\alpha\beta}WX^\alpha_{(i}X^\beta_{j)} - \frac{9}{32}\mathrm{i}(\Gamma^b)_{\alpha\beta}W_{ab}X^\alpha_{(i}\lambda^\beta_{j)} + \frac{3}{64}\varepsilon_{abcde}W^{bc}{}_{\alpha(i}W^{de\alpha}{}_{j)}W \\
& + \frac{3}{16}\mathrm{i}W_{[a}{}^cW_{b]c\alpha(i}(\Gamma^b)^{\alpha\rho}\lambda_{j)\rho} + \frac{3}{128}\mathrm{i}\varepsilon_{abcde}W^{bc}W^{de}{}_{\alpha(i}\lambda^\alpha_{j)} \\
& - \frac{3}{64}\mathrm{i}(\Sigma_{ab})_{\alpha\beta}\varepsilon^{bcdef}W_{cd}W_{ef}{}^\alpha{}_{(i}\lambda^\beta_{j)} - \frac{3}{32}\mathrm{i}\varepsilon_{abcde}(\Sigma^{de})^\alpha{}_\lambda W^{bf}W^c{}_{f\alpha(i}\lambda^\lambda_{j)} \\
& - \frac{3}{8}\mathrm{i}\nabla^b(X_{ij}W_{ab}) - \frac{3}{8}\mathrm{i}(\Sigma_{ab})_{\alpha\beta}\nabla^b(\lambda^\alpha_{(i}X^\beta_{j)}) - \frac{3}{8}\mathrm{i}\nabla^b(W_{ab\alpha(i}\lambda^\alpha_{j)}) \ ,
\end{aligned} \tag{3.7.23b}$$

$$\begin{aligned}
J^2_{ab,\mathrm{W}eyl} \ = \ & \frac{3}{128}\mathrm{i}F_{ab}Y + \frac{9}{128}\mathrm{i}WW_{ab}Y + \frac{1}{4}\mathrm{i}\Phi_{abij}X^{ij} - \frac{9}{16}\mathrm{i}WC_{abcd}W^{cd} - \frac{3}{16}\mathrm{i}C_{abcd}F^{cd} \\
& + \frac{81}{32}\mathrm{i}WW_{c[a}W_{b]d}W^{cd} + \frac{45}{128}\mathrm{i}WW_{ab}W^{cd}W_{cd} + \frac{9}{16}\mathrm{i}W_{c[a}F_{b]d}W^{cd} \\
& + \frac{9}{32}\mathrm{i}W_{c[a}W_{b]d}F^{cd} - \frac{3}{128}\mathrm{i}F_{ab}W^{cd}W_{cd} + \frac{3}{16}\mathrm{i}\varepsilon_{abcde}W^{e(f}\nabla_fF^{d)c} \\
& + \frac{9}{64}\mathrm{i}\varepsilon_{abcde}F^{cd}\nabla_fW^{fe} + \frac{3}{64}\mathrm{i}\varepsilon_{abcde}F^{cf}\nabla_fW^{de} + \frac{3}{32}\mathrm{i}\varepsilon_{cdef[a}F^{cd}\nabla^fW_{b]}{}^e \\
& - \frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}\nabla^fF^{cd}W_{b]}{}^e - \frac{15}{32}\mathrm{i}\varepsilon_{cdef[a}F_{b]}{}^c\nabla^fW^{de} - \frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}\nabla^fF_{b]}{}^cW^{de} \\
& - \frac{9}{64}\mathrm{i}\varepsilon_{abcde}WW^{cf}\nabla_fW^{de} - \frac{27}{64}\mathrm{i}\varepsilon_{abcde}WW^{de}\nabla_fW^{cf} + \frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}W\nabla^c(W^{de}W_{b]}{}^f) \\
& - \frac{9}{64}\mathrm{i}\varepsilon_{abcde}W^{cd}W^{ef}\nabla_fW - \frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}W_{b]}{}^cW^{de}\nabla^fW - \frac{3}{2}\mathrm{i}W\nabla^c\nabla_{[a}W_{b]c} \\
& - \frac{3}{4}\mathrm{i}W\nabla_{[a}\nabla^cW_{b]c} - \frac{3}{2}\mathrm{i}\nabla^cW\nabla_{[a}W_{b]c} - \frac{3}{4}\mathrm{i}\nabla_{[a}W\nabla^cW_{b]c} - \frac{3}{4}\mathrm{i}W_{c[a}\nabla^c\nabla_{b]}W \\
& - \frac{3}{4}\mathrm{i}\nabla_cW\nabla^cW_{ab} - \frac{3}{4}\mathrm{i}W\nabla_c\nabla^cW_{ab} + \frac{15}{32}W_{ab}{}^i{}_\alpha X^\alpha_iW + \frac{9}{32}\mathrm{i}W_{ab}\lambda^i_\alpha X^\alpha_i
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{16}W_{[a}{}^{c\alpha i}W_{b]c\alpha i}W+\frac{9}{32}\mathrm{i}W_{[a}{}^{c}W_{b]c}{}^{\alpha}{}_{i}\lambda^{i}_{\alpha}-\frac{3}{16}\mathrm{i}(\Sigma_{c[a})^{\alpha\beta}W^{cd}W_{b]d\alpha i}\lambda^{i}_{\beta}\\
&+\frac{3}{16}\mathrm{i}(\Sigma_{c[a})^{\alpha\beta}W_{b]d}W^{cd}{}_{\alpha i}\lambda^{i}_{\beta}+\frac{3}{16}\mathrm{i}(\Sigma^{cd})^{\alpha\beta}W_{c[a}W_{b]d\alpha i}\lambda^{i}_{\beta}+\frac{3}{64}\mathrm{i}(\Sigma_{ab})^{\alpha\beta}W^{cd}W_{cd\,\alpha i}\lambda^{i}_{\beta}\\
&+\frac{3}{64}\mathrm{i}(\Sigma_{cd})^{\alpha\beta}W^{cd}W_{ab\alpha i}\lambda^{i}_{\beta}+\frac{3}{64}\mathrm{i}(\Sigma_{cd})^{\alpha\beta}W_{ab}W^{cd}{}_{\alpha i}\lambda^{i}_{\beta}\\
&-\frac{3}{128}\mathrm{i}\varepsilon_{abcde}(\Gamma_{f})^{\alpha\beta}W^{cf}W^{de}{}_{\alpha i}\lambda^{i}_{\beta}+\frac{3}{64}\mathrm{i}\varepsilon_{cdef[a}(\Gamma^{c})^{\alpha\beta}W^{de}W_{b]}{}^{f}{}_{\alpha i}\lambda^{i}_{\beta}\\
&-\frac{3}{16}\mathrm{i}\varepsilon_{abcde}\lambda^{\alpha}_{i}(\Sigma^{cd})_{\alpha}{}^{\beta}\nabla^{e}X^{i}_{\beta}-\frac{3}{16}\mathrm{i}\varepsilon_{abcde}X^{\alpha}_{i}(\Sigma^{cd})_{\alpha}{}^{\beta}\nabla^{e}\lambda^{i}_{\beta}-\frac{3}{16}\mathrm{i}\lambda_{i\alpha}(\Gamma_{[a})^{\alpha\beta}\nabla_{b]}X^{i}_{\beta}\\
&-\frac{3}{16}\mathrm{i}X_{i\alpha}(\Gamma_{[a})^{\alpha\beta}\nabla_{b]}\lambda^{i}_{\beta}-\frac{3}{64}\mathrm{i}W_{ab}{}^{\alpha}{}_{i}(\Gamma_{c})_{\alpha}{}^{\beta}\nabla^{c}\lambda^{i}_{\beta}+\frac{3}{32}\mathrm{i}W_{c[a}{}^{\alpha i}(\Gamma^{c})_{|\alpha|}{}^{\beta}\nabla_{b]}\lambda_{i\beta}\\
&-\frac{3}{32}\mathrm{i}W_{c[a}{}^{\alpha i}(\Gamma_{b]})_{\alpha}{}^{\beta}\nabla^{c}\lambda_{i\beta}+\frac{3}{64}\mathrm{i}\varepsilon_{abcde}W^{cd\,\alpha}{}_{i}(\Sigma^{ef})_{\alpha}{}^{\beta}\nabla_{f}\lambda^{i}_{\beta}\\
&-\frac{3}{32}\mathrm{i}\varepsilon_{cdef[a}W_{b]}{}^{c\alpha}{}_{i}(\Sigma^{de})_{\alpha}{}^{\beta}\nabla^{f}\lambda^{i}_{\beta}-\frac{9}{128}\mathrm{i}\varepsilon_{abcde}W^{cd\,\alpha}{}_{i}\nabla^{e}\lambda^{i}_{\alpha}+\frac{39}{128}\mathrm{i}\varepsilon_{abcde}\lambda^{\alpha}_{i}\nabla^{e}W^{cd}{}_{\alpha}{}^{i}\\
&-\frac{3}{64}\mathrm{i}\varepsilon_{abcde}\lambda^{\alpha}_{i}(\Sigma^{ef})_{\alpha}{}^{\beta}\nabla_{f}W^{cd}{}_{\beta}{}^{i}+\frac{3}{32}\mathrm{i}\varepsilon_{cdef[a}\lambda^{\alpha}_{i}(\Sigma^{de})_{|\alpha|}{}^{\beta}\nabla^{f}W_{b]}{}^{c}{}_{\beta}{}^{i}\\
&-\frac{3}{32}\mathrm{i}\lambda^{\alpha}_{i}(\Gamma^{c})_{\alpha}{}^{\beta}\nabla_{[a}W_{b]c\beta}{}^{i}+\frac{3}{32}\mathrm{i}\lambda^{\alpha}_{i}(\Gamma_{[a})_{\alpha}{}^{\beta}\nabla^{c}W_{b]c\beta}{}^{i}\\
&-\frac{3}{64}\mathrm{i}\lambda^{\alpha}_{i}(\Gamma_{c})_{\alpha}{}^{\beta}\nabla^{c}W_{ab\beta}{}^{i}\,. \qquad (3.7.23c)
\end{aligned}
$$

3.7.4 Log

The Log invariant does not contribute to the linear multiplet EOM and the descendants of the vector multiplet EOM have been described in (3.4.8). Our main focus in this subsection is thus to derive the descendants of the Weyl multiplet EOM.

The Weyl multiplet EOM in superspace was derived in subsection 3.6.2. It takes the following form

$$
\begin{aligned}
J_{\log} \;=\; &-\frac{3}{1024}WY-\frac{69}{1024}W^{ab}W_{ab}W+\frac{3}{32}\Box W-\frac{3}{64}F_{ab}W^{ab}-\frac{3}{256}\lambda^{\alpha}_{j}X^{j}_{\alpha}\\
&+\frac{3}{128}F^{ab}F_{ab}W^{-1}-\frac{9}{128}X^{ij}X_{ij}W^{-1}+\frac{3\mathrm{i}}{32}(\Gamma^{a})^{\alpha\beta}W^{-1}\lambda^{i}_{\alpha}\nabla_{a}\lambda_{\beta i}\\
&+\frac{3}{64}W^{-1}(\nabla^{a}W)\nabla_{a}W-\frac{3\mathrm{i}}{128}(\Sigma^{ab})^{\alpha\beta}F_{ab}\lambda^{i}_{\alpha}\lambda_{\beta i}W^{-2}-\frac{3\mathrm{i}}{64}X^{ij}\lambda^{\alpha}_{i}\lambda_{\alpha j}W^{-2}\\
&-\frac{3\mathrm{i}}{32}(\Gamma_{b})^{\alpha\beta}\,\lambda^{j}_{\alpha}\lambda_{\beta j}W^{-2}\nabla^{b}W-\frac{3}{256}\lambda^{\alpha i}\lambda^{\beta}_{i}\lambda^{j}_{\alpha}\lambda_{\beta j}W^{-3}\,. \qquad (3.7.24)
\end{aligned}
$$

We present here the results for one and two-derivative descendants of $J_{\log}$ based on the structures found in (3.7.19). Note that, we have proven each to be explicitly K-primary. Due to their size, only the bosonic parts of $J^{2}_{aij,\log}$ and $J^{2}_{ab,\log}$ are provided below. In subsection 4.3 of the supplementary file [1], the reader can find the complete expressions of $J^{2}_{aij,\log}$, $J^{2}_{ab,\log}$, $J^{3}_{a\alpha i,\log}$, along with the degauged, gauged fixed ($W=1$), bosonic part of $J^{4}_{ab,\log}$. It holds:

$$
\begin{aligned}
J^{1}_{\alpha i,\log} \;=\; & W\left\{\frac{3}{256}\varepsilon_{abcde}(\Gamma^{a})_{\alpha\beta}W^{bc}W^{de\beta}{}_{i}+\frac{3}{32}(\Sigma^{ab})_{\alpha\beta}W_{a}{}^{c}W_{cb}{}^{\beta}{}_{i}-\frac{3}{128}W_{ab}W^{ab}{}_{\alpha i}\right.\\
&\left.+\frac{81}{512}(\Sigma_{ab})_{\alpha\beta}W^{ab}X^{\beta}_{i}+\frac{3}{128}(\Gamma^{a})_{\alpha\beta}\nabla_{a}X^{\beta}_{i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \Big\{ \frac{1}{16}\mathrm{i}(\Sigma_{ab})_{\alpha\beta}\Phi^{ab}{}_{ij}\lambda^{j\beta} - \frac{3}{2048}\mathrm{i}Y\lambda_{i\alpha} - \frac{231}{2048}\mathrm{i}W^{ab}W_{ab}\lambda_{i\alpha} + \frac{3}{32}\mathrm{i}\Box\lambda_{i\alpha} \\
& - \frac{27}{2048}\mathrm{i}\varepsilon_{abcde}(\Gamma^a)_{\alpha\beta}W^{bc}W^{de}\lambda_i^\beta + \frac{3}{256}(\Gamma^a)_{\alpha\beta}X_i^\beta\nabla_a W \\
& + \frac{3}{256}\mathrm{i}\varepsilon_{abcde}(\Sigma^{ab})_\alpha{}^\beta\lambda_{i\beta}\nabla^c W^{de} + \frac{9}{128}\mathrm{i}\varepsilon_{abcde}(\Sigma^{ab})_{\alpha\beta}W^{cd}\nabla^e\lambda_i^\beta \\
& + \frac{21}{128}\mathrm{i}(\Gamma^a)_{\alpha\beta}\lambda_i^\beta\nabla^b W_{ab} - \frac{3}{32}\mathrm{i}(\Gamma^a)_{\alpha\beta}W_{ab}\nabla^b\lambda_i^\beta \\
& - \frac{3}{256}\varepsilon_{abcde}(\Gamma^a)_\alpha{}^\beta F^{bc}W^{de}{}_{\beta i} - \frac{3}{128}F^{ab}W_{ab\alpha i} - \frac{3}{32}(\Sigma_{ab})_\alpha{}^\beta F^{ac}W_c{}^b{}_{\beta i} \\
& + \frac{15}{256}(\Sigma_{ab})_{\alpha\beta}F^{ab}X_i^\beta + \frac{51}{256}X_{ij}X_\alpha^j \Big\} \\
+W^{-1} & \Big\{ \frac{3}{64}\mathrm{i}\varepsilon^{abcde}(\Sigma_{ab})_{\alpha\beta}F_{cd}\nabla_e\lambda_i^\beta + \frac{27}{512}\mathrm{i}\varepsilon_{abcde}(\Gamma^a)_\alpha{}^\beta W^{bc}F^{de}\lambda_{i\beta} \\
& - \frac{27}{256}\mathrm{i}W^{ab}F_{ab}\lambda_{i\alpha} - \frac{3}{16}\mathrm{i}(\Gamma_a)_\alpha{}^\beta X_{ij}\nabla^a\lambda_\beta^j + \frac{9}{64}\mathrm{i}\lambda_i^\beta\lambda_{j\beta}X_\alpha^j \\
& + \frac{3}{16}\mathrm{i}(\Sigma_{ab})_{\alpha\beta}\nabla^a W\nabla^b\lambda_i^\beta + \frac{3}{64}\mathrm{i}\varepsilon_{abcde}(\Sigma^{ab})_\alpha{}^\beta\lambda_{i\beta}\nabla^c F^{de} \\
& + \frac{3}{32}\mathrm{i}(\Gamma^a)_{\alpha\beta}\lambda_i^\beta\nabla^b F_{ab} + \frac{27}{128}\mathrm{i}(\Gamma^a)_{\alpha\beta}W_{ab}\lambda_i^\beta\nabla^b W - \frac{3}{32}\mathrm{i}(\Gamma_a)_\alpha{}^\beta\lambda_\beta^j\nabla^a X_{ij} \\
& + \frac{3}{32}\mathrm{i}\lambda_{i\alpha}\Box W + \frac{3}{16}\mathrm{i}(\Sigma_{ab})_{\alpha\beta}\lambda_i^\beta\nabla^a\nabla^b W + \frac{9}{32}\mathrm{i}\lambda_{\alpha(i}\lambda_{j)}^\beta X_\beta^j \Big\} \\
+W^{-2} & \Big\{ \frac{9}{128}\mathrm{i}\lambda_{i\alpha}X_{jk}X^{jk} - \frac{3}{32}(\Gamma_a)^{\beta\rho}\lambda_{i\alpha}\lambda_{j\beta}\nabla^a\lambda_\rho^j - \frac{3}{64}\mathrm{i}\lambda_{i\alpha}\nabla_a W\nabla^a W \\
& + \frac{3}{64}(\Gamma_a)_{\alpha\beta}\lambda_j^\beta\lambda^{j\rho}\nabla^a\lambda_{i\rho} + \frac{9}{256}(\Sigma_{ab})^{\beta\rho}W^{ab}\lambda_{i\alpha}\lambda_{j\beta}\lambda_\rho^j \\
& - \frac{3}{64}(\Gamma_a)^{\beta\rho}\lambda_{j\alpha}\lambda_\beta^j\nabla^a\lambda_{i\rho} + \frac{9}{256}(\Sigma_{ab})^{\beta\rho}W^{ab}\lambda_{j\alpha}\lambda_{i\beta}\lambda_\rho^j \\
& - \frac{3}{256}\mathrm{i}\varepsilon_{abcde}(\Gamma^a)_{\alpha\beta}F^{bc}F^{de}\lambda_i^\beta + \frac{3}{64}\mathrm{i}(\Sigma_{ab})_\alpha{}^\beta X_{ij}F^{ab}\lambda_\beta^j \\
& + \frac{3}{128}\mathrm{i}\varepsilon_{abcde}(\Sigma^{ab})_\alpha{}^\beta F^{cd}\lambda_{i\beta}\nabla^e W - \frac{3}{64}\mathrm{i}(\Gamma^a)_{\alpha\beta}F_{ab}\lambda_i^\beta\nabla^b W \\
& + \frac{3}{32}(\Gamma_a)_\alpha{}^\beta\lambda_i^\rho\lambda_{j\rho}\nabla^a\lambda_\beta^j + \frac{3}{32}\mathrm{i}X_{ij}X^j{}_k\lambda_\alpha^k + \frac{3}{32}\mathrm{i}(\Gamma_a)_\alpha{}^\beta X_{ij}\lambda_\beta^j\nabla^a W \\
& + \frac{9}{256}(\Sigma_{ab})_{\alpha\beta}W^{ab}\lambda_i^\rho\lambda_j^\beta\lambda_\rho^j \Big\} \\
+W^{-3} & \Big\{ \frac{3}{64}(\Sigma_{ab})^{\beta\rho}F^{ab}\lambda_{i\alpha}\lambda_{j\beta}\lambda_\rho^j - \frac{3}{32}X_{jk}\lambda_{i\alpha}\lambda^{j\beta}\lambda_\beta^k + \frac{3}{64}(\Sigma_{ab})_{\alpha\beta}F^{ab}\lambda_i^\rho\lambda_j^\beta\lambda_\rho^j \\
& + \frac{3}{64}X_{ij}\lambda_{k\alpha}\lambda^{j\beta}\lambda_\beta^k + \frac{3}{64}(\Gamma_a)_{\alpha\beta}\lambda_i^\rho\lambda_j^\beta\lambda_\rho^j\nabla^a W \Big\} \\
+W^{-4} & \Big\{ \frac{9}{256}\mathrm{i}\lambda_{i\alpha}\lambda^{\beta j}\lambda_j^\rho\lambda_\beta^k\lambda_{k\rho} \Big\} \, ,
\end{aligned} \tag{3.7.25a}
$$

$$
\begin{aligned}
J^2_{aij,\log} = & -\frac{1}{8}\mathrm{i}W\nabla^b\Phi_{abij} - \frac{1}{8}\mathrm{i}\Phi_{abij}\nabla^b W + \frac{1}{32}\mathrm{i}\varepsilon_a{}^{bcde}\Phi_{deij}F_{bc} + \frac{3}{16}\mathrm{i}X_{ij}\nabla^b W_{ab} \\
& + \frac{3}{16}\mathrm{i}W_{ab}\nabla^b X_{ij} + \frac{3}{16}\mathrm{i}F_{ab}W^{-1}\nabla^b X_{ij} + \frac{3}{16}\mathrm{i}X_{ij}W^{-1}\nabla^b F_{ab} \\
& + \frac{3}{16}\mathrm{i}X_{(i}{}^k W^{-1}\nabla_a X_{j)k} - \frac{3}{16}\mathrm{i}X_{ij}F_{ab}W^{-2}\nabla^b W + \text{f}ermions \, ,
\end{aligned} \tag{3.7.25b}
$$

$$
\begin{aligned}
J^2_{ab,\log} \quad = \quad & W\left\{\frac{3}{16}\mathrm{i}C_{abcd}W^{cd}-\frac{9}{256}\mathrm{i}W_{ab}Y+\frac{117}{256}\mathrm{i}W_{ab}W^{cd}W_{cd}-\frac{3}{16}\mathrm{i}W^{cd}W_{c[a}W_{b]d}\right.\\
&+\frac{3}{16}\mathrm{i}W_{c[a}W_{b]d}W^{cd}-\frac{27}{64}\mathrm{i}\varepsilon_{abcde}W^{cd}\nabla_f W^{ef}-\frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}W_{b]}{}^c\nabla^d W^{ef}\\
&\left.-\frac{9}{64}\mathrm{i}\varepsilon_{abcde}W^{cf}\nabla_f W^{de}+\frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}W^{cd}\nabla^e W_{b]}{}^f-\frac{3}{16}\mathrm{i}\Box W_{ab}\right\}\\
+\quad & \left\{\frac{3}{16}\mathrm{i}C_{abcd}F^{cd}-\frac{3}{256}\mathrm{i}F_{ab}Y-\frac{1}{8}\mathrm{i}\Phi_{ab}{}^{ij}X_{ij}+\frac{3}{256}\mathrm{i}F_{ab}W_{cd}W^{cd}\right.\\
&-\frac{3}{32}\mathrm{i}W_{ab}F_{cd}W^{cd}-\frac{21}{16}\mathrm{i}W_{c[a}F_{b]d}W^{cd}-\frac{9}{16}\mathrm{i}W_{c[a}W_{b]d}F^{cd}\\
&+\frac{3}{32}\mathrm{i}W_{ab}W_{cd}F^{cd}+\frac{3}{16}\mathrm{i}W_{c[a}F_{b]d}W^{cd}-\frac{9}{64}\mathrm{i}\varepsilon_{abcde}W^{cd}W^{ef}\nabla_f W\\
&-\frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}W_{b]}{}^c W^{de}\nabla^f W-\frac{3}{16}\mathrm{i}\nabla_c W\nabla^c W_{ab}-\frac{3}{4}\mathrm{i}\nabla_{[a}W\nabla^c W_{b]c}\\
&+\frac{3}{4}\mathrm{i}\nabla^c W\nabla_{[a}W_{b]c}-\frac{9}{32}\mathrm{i}\varepsilon_{abcde}F^{cd}\nabla_f W^{ef}-\frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}F_{b]}{}^c\nabla^d W^{ef}\\
&-\frac{3}{32}\mathrm{i}\varepsilon_{abcde}W^{cf}\nabla_f F^{de}+\frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}W^{cd}\nabla^e F_{b]}{}^f+\frac{9}{8}\mathrm{i}W_{c[a}\nabla^c\nabla_{b]}W\\
&-\frac{3}{4}\mathrm{i}W_{c[a}\nabla_{b]}\nabla^c W-\frac{3}{16}\mathrm{i}\varepsilon_{abcde}W^{cd}\nabla_f F^{ef}+\frac{3}{8}\mathrm{i}\varepsilon_{cdef[a}W_{b]}{}^c\nabla^d F^{ef}\\
&\left.-\frac{3}{32}\mathrm{i}\varepsilon_{abcde}F^{cf}\nabla_f W^{de}+\frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}F^{cd}\nabla^e W_{b]}{}^f-\frac{3}{16}\mathrm{i}\Box F_{ab}-\frac{9}{16}\mathrm{i}W_{ab}\Box W\right\}\\
& +W^{-1}\left\{-\frac{9}{8}\mathrm{i}W_{c[a}F_{b]d}F^{cd}-\frac{9}{32}\mathrm{i}W_{ab}F_{cd}F^{cd}-\frac{3}{64}\mathrm{i}\varepsilon_{abcde}F^{cf}\nabla_f F^{de}\right.\\
&+\frac{3}{32}\mathrm{i}\varepsilon_{cdef[a}F^{cd}\nabla^e F_{b]}{}^f-\frac{9}{64}\mathrm{i}\varepsilon_{abcde}F^{cd}\nabla_f F^{ef}+\frac{9}{32}\mathrm{i}\varepsilon_{cdef[a}F_{b]}{}^c\nabla^d F^{ef}\\
&+\frac{9}{32}\mathrm{i}\varepsilon_{abcde}W^{cd}F^{ef}\nabla_f W-\frac{3}{8}\mathrm{i}F_{ab}\Box W+\frac{3}{8}\mathrm{i}F_{c[a}\nabla^c\nabla_{b]}W-\frac{3}{4}\mathrm{i}F_{c[a}\nabla_{b]}\nabla^c W\\
&+\frac{3}{16}\mathrm{i}\nabla_c W\nabla^c F_{ab}-\frac{3}{4}\mathrm{i}\nabla_{[a}W\nabla^c F_{b]c}+\frac{3}{4}\mathrm{i}\nabla^c W\nabla_{[a}F_{b]c}\\
&\left.+\frac{3}{8}\mathrm{i}\varepsilon_{abcde}\nabla^c W\nabla^d\nabla^e W-\frac{9}{8}\mathrm{i}W_{c[a}\nabla_{b]}W\nabla^c W\right\}\\
& +W^{-2}\left\{-\frac{3}{32}\mathrm{i}F_{ab}F_{cd}F^{cd}-\frac{3}{16}\mathrm{i}F_{c[a}F_{b]d}F^{cd}+\frac{3}{32}\mathrm{i}\varepsilon_{abcde}F^{cd}F^{ef}\nabla_f W\right.\\
&+\frac{3}{16}\mathrm{i}\varepsilon_{cdef[a}F_{b]}{}^c F^{de}\nabla^f W+\frac{3}{8}\mathrm{i}F_{c[a}\nabla_{b]}W\nabla^c W\\
&\left.+\frac{3}{16}\mathrm{i}F_{ab}\nabla_c W\nabla^c W\right\}+\text{f}ermions\ . \qquad (3.7.25c)
\end{aligned}
$$

3.7.5 Scalar curvature squared

In this subsection, we will first describe the descendants of the supercurrent multiplet associated to the scalar curvature squared J_{R^2}. Unlike the Weyl-squared and Log invariants, the EOM for the linear multiplet compensator presented in subsection 3.6.4 leads to new nontrivial descendants which we will compute in subsection 3.7.5. The EOM for the vector multiplet compensator gives rise to the composite linear multiplet superfield $H^{ij}_{R^2}$, whose descendants have been given earlier in eq. (3.4.16).

Standard Weyl multiplet

The Weyl multiplet EOM for the curvature-squared invariant was given in 3.6.2. It defines a conformal supercurrent multiplet

$$\begin{aligned}
J_{R^2} &= -\frac{3}{8}W\mathbf{W}^2+\frac{3}{32}G^{-1}\Big(WG^{ij}\mathbf{X}_{ij}+G^{ij}X_{ij}\mathbf{W}-\mathrm{i}G^{ij}\lambda_i^{\alpha}\boldsymbol{\lambda}_{\alpha j}\Big)\\
&= \frac{3}{32}G^{-1}\Big(G^{ij}W\mathbf{X}_{ij}+G^{ij}X_{ij}\mathbf{W}-\mathrm{i}G^{ij}\lambda_i^{\alpha}\boldsymbol{\lambda}_{\alpha j}-W\mathbf{W}F\Big)\\
&\quad+\frac{3\mathrm{i}}{64}G^{-3}W\mathbf{W}G_{ij}\varphi^{i\alpha}\varphi_\alpha^j\,. \qquad (3.7.26)
\end{aligned}$$

We present here the results for one and two-derivative descendants of J_{R^2} based on the structures found in eqs. (3.7.19). Note that, we have proven each to be explicitly K-primary. Due to their size, only the bosonic parts of J^2_{aij,R^2} and J^2_{ab,R^2} are provided below. In subsection 4.4 of the supplementary file [1], the reader can find the complete expressions of J^2_{aij,R^2}, J^2_{ab,R^2}, $J^3_{a\alpha i,R^2}$, along with the degauged, gauged fixed ($W=1$), bosonic part of J^4_{ab,R^2}. It holds that

$$\begin{aligned}
J^1_{\alpha i,R^2} &= G^{-1}\Big\{W\boldsymbol{W}\Big(-\frac{3}{16}(\Gamma_a)_{\alpha\beta}\nabla^a\varphi_i^\beta+\frac{9}{64}(\Sigma_{ab})_{\alpha\beta}W^{ab}\varphi_i^\beta-\frac{27}{64}G_{ij}X_\alpha^j\Big)\\
&\quad-\frac{3}{32}\mathrm{i}F\left(\boldsymbol{W}\lambda_{i\alpha}+W\boldsymbol{\lambda}_{i\alpha}\right)+\frac{3}{32}\mathrm{i}G_{jk}\Big(\mathbf{X}^{jk}\lambda_{i\alpha}+X^{jk}\boldsymbol{\lambda}_{i\alpha}\Big)\\
&\quad+\frac{3}{16}\left(W\mathbf{X}_{ij}+\boldsymbol{W}X_{ij}\right)\varphi_\alpha^j+\frac{3}{16}\mathrm{i}G_{ij}(\Gamma_a)_\alpha{}^\beta\Big(W\nabla^a\boldsymbol{\lambda}_\beta^j+\boldsymbol{W}\nabla^a\lambda_\beta^j\Big)\\
&\quad+\frac{9}{64}\mathrm{i}G_{ij}(\Sigma_{ab})_\alpha{}^\beta W^{ab}\Big(W\boldsymbol{\lambda}_\beta^j+\boldsymbol{W}\lambda_\beta^j\Big)-\frac{3}{32}\mathrm{i}\Big(\lambda_i^\beta\boldsymbol{\lambda}_{j\beta}+\boldsymbol{\lambda}_i^\beta\lambda_{j\beta}\Big)\varphi_\alpha^j\\
&\quad+\frac{3}{32}\mathrm{i}G_{ij}(\Sigma_{ab})_\alpha{}^\beta\Big(F^{ab}\boldsymbol{\lambda}_\beta^j+\mathbf{F}^{ab}\lambda_\beta^j\Big)-\frac{3}{32}\mathrm{i}G_{jk}\Big(X_i{}^j\boldsymbol{\lambda}_\alpha^k+\mathbf{X}_i{}^j\lambda_\alpha^k\Big)\\
&\quad+\frac{3}{32}\mathrm{i}G_{ij}(\Gamma_a)_\alpha{}^\beta\Big(\lambda_\beta^j\nabla^a\boldsymbol{W}+\boldsymbol{\lambda}_\beta^j\nabla^aW\Big)\Big\}\\
&\quad+G^{-3}\Big\{\frac{3}{64}FG_{ij}W\boldsymbol{W}\varphi_\alpha^j-\frac{3}{64}G_{jk}\left(\boldsymbol{W}\lambda_{i\alpha}+W\boldsymbol{\lambda}_{i\alpha}\right)\varphi^{j\beta}\varphi_\beta^k+\frac{3}{32}\mathrm{i}W\boldsymbol{W}\varphi_i^\beta\varphi_{j\alpha}\varphi_\beta^j\\
&\quad-\frac{3}{64}\mathcal{H}_aG_{ij}(\Gamma^a)_\alpha{}^\beta W\boldsymbol{W}\varphi_\beta^j-\frac{3}{32}G_{jk}(\Gamma_a)_\alpha{}^\beta W\boldsymbol{W}\nabla^aG_i{}^j\varphi_\beta^k\\
&\quad-\frac{3}{32}G_{ij}G_{kl}\Big(W\mathbf{X}^{kl}+\boldsymbol{W}X^{kl}\Big)\varphi_\alpha^j+\frac{3}{32}\mathrm{i}G_{ij}G_{kl}\lambda^{k\beta}\boldsymbol{\lambda}^l_\beta\varphi_\alpha^j\Big\}\\
&\quad+G^{-5}\Big\{-\frac{9}{64}\mathrm{i}G_{ij}G_{kl}W\boldsymbol{W}\varphi_\alpha^j\varphi^{k\beta}\varphi_\beta^l\Big\}\,, \qquad (3.7.27a)\\
J^2_{aij,R^2} &= G^{-1}\Big\{\frac{3}{16}\mathrm{i}\mathcal{H}_a\left(W\mathbf{X}_{ij}+\boldsymbol{W}X_{ij}\right)-\frac{3}{32}\mathrm{i}\varepsilon_a{}^{bcde}G_{ij}F_{bc}\mathbf{F}_{de}\\
&\quad-\frac{9}{8}\mathrm{i}G_{ij}\nabla^b\left(W\boldsymbol{W}W_{ab}\right)+\frac{3}{8}\mathrm{i}\left(W\mathbf{X}_{k(i}+\boldsymbol{W}X_{k(i}\right)\nabla_aG_{j)}{}^k\\
&\quad-\frac{3}{8}\mathrm{i}G_{ij}\nabla^b\left(W\mathbf{F}_{ab}+\boldsymbol{W}F_{ab}\right)-\frac{3}{8}\mathrm{i}G^k{}_{(i}\nabla_a\left(W\mathbf{X}_{j)k}+\boldsymbol{W}X_{j)k}\right)\Big\}\\
&\quad+G^{-3}\Big\{-\frac{3}{32}\mathrm{i}\mathcal{H}_aG_{ij}G_{kl}\Big(W\mathbf{X}^{kl}+\boldsymbol{W}X^{kl}\Big)\\
&\quad-\frac{3}{16}\mathrm{i}G_{lm}\Big(W\mathbf{X}^{lm}+\boldsymbol{W}X^{lm}\Big)G_{k(i}\nabla_aG_{j)}{}^k\Big\}+\text{fermions}\,, \qquad (3.7.27b)
\end{aligned}$$

$$
\begin{aligned}
J^2_{ab,R^2} \;=\;& G^{-1}\left\{ \frac{3}{16}\mathrm{i}F\left(\boldsymbol{W}F_{ab}+W\mathbf{F}_{ab}\right)+\frac{9}{16}\mathrm{i}FW\boldsymbol{W}W_{ab}+\frac{3}{8}\mathrm{i}W\boldsymbol{W}\nabla_{[a}\mathcal{H}_{b]}\right.\\
&\left.+\frac{3}{8}\mathrm{i}\Phi_{abij}G^{ij}W\boldsymbol{W}G^{-1}\right\}\\
&+G^{-3}\left\{ \frac{3}{16}\mathrm{i}G_{ij}W\boldsymbol{W}\mathcal{H}_{[a}\nabla_{b]}G^{ij}-\frac{3}{16}\mathrm{i}G_{ij}W\boldsymbol{W}\nabla_{[a}G^{ik}\nabla_{b]}G^{j}{}_{k}\right\}\\
&+\mathrm{f}ermions\;. \qquad (3.7.27c)
\end{aligned}
$$

Linear multiplet

Recall that the EOM for the auxiliary field F for the curvature-squared invariant was given in subsection 3.6.4. It leads to the expression W_{R^2}, eq. (3.6.43), which defines a composite vector multiplet. We present here, for the first time, the results for the bosonic parts of the component descendants of W_{R^2} based on the structures defined in eqs. (3.2.2). Note that, we have proven the bosons for each to be explicitly K-primary. Due to their size, the entire component results with fermions of $X^{ij}_{R^2}$ and F_{ab,R^2} as well as the fermionic descendant λ^i_{α,R^2} are given in subsection 4.4 of the corresponding supplementary file [1]. It holds:

$$
\begin{aligned}
X^{ij}_{R^2} \;=\;& G^{-1}\left\{\Box(W\mathbf{X}^{ij}+\mathbf{W}X^{ij})+\frac{3}{32}\left(W^{ab}W_{ab}-Y\right)\left(W\mathbf{X}^{ij}+\mathbf{W}X^{ij}\right)\right\}\\
&+G^{-3}\left\{\left(\frac{1}{2}\mathcal{H}_aG^{k(i}-G^{kl}\nabla_aG^{(i}{}_{l}\right)\nabla^a\left(\mathbf{X}^{j)}{}_{k}W+X^{j)}{}_{k}\mathbf{W}\right)\right.\\
&+\left(\frac{1}{2}\mathcal{H}_aG^{ij}-G^{k(i}\nabla_aG^{j)}{}_{k}\right)\left[\nabla_b(W\mathbf{F}^{ab})+\nabla_b(\mathbf{W}F^{ab})+3\nabla_b\left(W\mathbf{W}W^{ab}\right)\right.\\
&\left.+\frac{1}{4}\varepsilon^{abcde}F_{bc}\mathbf{F}_{de}\right]+\frac{3}{8}FG^{k(i}\left(X^{j)l}\mathbf{X}_{kl}+\mathbf{X}^{j)l}X_{kl}\right)\\
&-\frac{1}{2}FG^{ij}\left(W\Box\mathbf{W}+\mathbf{W}\Box W\right)+\frac{3}{4}FG^{ij}W_{ab}\left(W\mathbf{F}^{ab}+\mathbf{W}F^{ab}\right)\\
&+\frac{3}{32}\left(13W^{ab}W_{ab}-Y\right)FG^{ij}W\mathbf{W}-\frac{1}{8}\left(F^2+\mathcal{H}_a\mathcal{H}^a\right)\left(W\mathbf{X}^{ij}+\mathbf{W}X^{ij}\right)\\
&+\left[-\frac{1}{2}\nabla^aG^{ki}\nabla_aG^{jl}-\frac{1}{2}G^{kl}\Box G^{ij}\right.\\
&\left.+\frac{3}{64}G^{ij}G^{kl}(Y-W^{ab}W_{ab})\right]\left(W\mathbf{X}_{kl}+\mathbf{W}X_{kl}\right)\\
&-\frac{5}{8}FG^{kl}\left(X^{ij}\mathbf{X}_{kl}+\mathbf{X}^{ij}X_{kl}\right)+\frac{1}{2}\mathcal{H}_a\left(\mathbf{W}X^{k(i}+W\mathbf{X}^{k(i}\right)\nabla^aG^{j)}{}_{k}\\
&\left.+\frac{1}{4}FG^{ij}\mathbf{F}^{ab}F_{ab}-\frac{1}{2}FG^{ij}\nabla^aW\nabla_a\mathbf{W}+\frac{5}{4}FG^{kl}X^{(i}{}_{k}\mathbf{X}^{j)}{}_{l}\right\}\\
&+G^{-5}\left\{\left[\frac{3}{4}G_{kl}\nabla^aG^{ik}\nabla_aG^{jl}+\frac{3}{16}\left(\mathcal{H}_a\mathcal{H}^a+F^2\right)G^{ij}\right.\right.\\
&\left.\left.-\frac{3}{4}\mathcal{H}_aG^{k(i}\nabla^aG^{j)}{}_{k}\right]\left(W\mathbf{X}_{mn}G^{mn}+\mathbf{W}X_{mn}G^{mn}\right)\right\}\\
&+\mathrm{f}ermions\;, \qquad (3.7.28a)
\end{aligned}
$$

$$
\begin{aligned}
F_{ab,R^2} \;=\;& 6\nabla_{[b}\left(G^{-1}W_{a]}{}^{d}\left(W\nabla_d\mathbf{W}+\mathbf{W}\nabla_dW\right)\right)+2\nabla_{[b}\left(G^{-1}\nabla^e\left(W\mathbf{F}_{a]e}+\mathbf{W}F_{a]e}\right)\right)\\
&-6\nabla_{[a}\left(G^{-1}\mathbf{W}W\nabla^eW_{b]e}\right)-\frac{1}{4}\varepsilon_{abcde}\nabla_f\left(G^{-1}\left(F^{cd}\mathbf{F}^{ef}+\mathbf{F}^{cd}F^{ef}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\varepsilon_{cdef[a}\nabla^{f}\Big(G^{-1}\Big(\mathbf{F}_{b]}{}^{c}F^{de}+F_{b]}{}^{c}\mathbf{F}^{de}\Big)\Big)\\
&-\frac{1}{2}\nabla_{[a}\Big(G^{-3}G^{ij}\mathcal{H}_{b]}\left(W\mathbf{X}_{ij}+\mathbf{W}X_{ij}\right)\Big)\\
&+G^{-3}\nabla_{[a}\Big(G_{i}{}^{k}\left(\mathbf{X}^{ij}W+X^{ij}\mathbf{W}\right)\nabla_{b]}G_{jk}\Big)\\
&-\frac{G^{-3}}{2}\left(W\mathbf{X}^{ij}+\mathbf{W}X^{ij}\right)\nabla_{a}G_{i}{}^{k}\nabla_{b}G_{jk}\\
&+\frac{3}{4}G^{-5}G_{ij}G^{kl}\left(W\mathbf{X}^{ij}+\mathbf{W}X^{ij}\right)\nabla_{a}G_{k}{}^{m}\nabla_{b}G_{lm}\\
&+\text{fermions}\ . \qquad (3.7.28b)
\end{aligned}
$$

Having described in detail all the descendants of the EOMs for the Weyl multiplet and compensators, let us end this section by commenting on the physical relevance of our results; in particular, how these might be useful for understanding higher-order corrections to Einstein's field equations. Recall that, after gauge fixing, the Poincaré supergravity multiplet contains the metric g_{mn}, gravitini $\psi_{m\alpha}^{\ i}$, graviphoton A_m, along with the matter fields D, χ^i_α, and W_{ab}. As indicated in Table 3 and looking at the descendants of the Weyl multiplet EOM, the component current $J^4_{mn} = e_m{}^a e_n{}^b J^4_{ab}$ is associated to the variation of the traceless part of the metric, $\delta\hat{g}_{mn}$. Hence, to compute higher-order corrections to the traceless part of Einstein's equations, one essentially needs to consider J^4_{ab} from all the three four-derivative invariants: $J^4_{ab,\mathrm{W}eyl}$, $J^4_{ab,\log}$, and J^4_{ab,R^2}. The latter are given in the auxiliary file, specifically in subsections 4.2.5, 4.3.5 and 4.4.5, respectively. On the other hand, the trace part of Einstein's equations comes from taking a linear combination of the components $F_{\mathrm{W}eyl}$ (3.4.3) and $F_{\log}$ (3.4.8).

The component current $J^3_{a\alpha i}$ is associated to variation of the gamma-traceless part of the gravitini, $\delta\hat{\psi}_{m i}{}^{\alpha}$, while the variation with respect to the trace part is given by the fermionic descendant of the vector compensator EOM (3.6.44). The composite field $\mathcal{H}_a$ coming from (3.6.44) is associated to the variation of graviphoton of the Poincaré supergravity multiplet. Finally, the composite field strength F_{ab} coming from (3.6.45) (see also (3.7.28b)) is associated to the variation of the Hodge dual of the three-form potential b_{mnp}.

3.8 Conclusion

By using an interplay between superspace and superconformal tensor calculus techniques, in our paper we have presented for the first time a comprehensive description of the component structure of the supersymmetric completions for all curvature-squared invariants of five-dimensional, off-shell (gauged) minimal supergravity. Our results include both bosonic as well as fermionic contributions for the five-parameter family of theories described by the action

$$
S_{\mathrm{HD}} = S_{\mathrm{g}SG} + \alpha S_{\mathrm{W}eyl} + \beta S_{\log} + \gamma S_{R^2}\ , \qquad (3.8.1)
$$

where $S_{\mathrm{g}SG}$ describes the supersymmetric two-derivative (gauged) minimal Poincaré supergravity with a cosmological constant, while the three independent curvature-squared invariants are described by eqs. (3.5.1), (3.5.9), (3.5.21b), and further results in section 3.5. As a first application of our results, we have also analysed the structure of the multiplets of equations of motion for the theory defined by (3.8.1). It did take more than fifteen years from the first papers on curvature-squared supergravity in five dimensions [29, 55, 63, 64] to complete the detailed construction of these invariants. We expect that the results in our paper will find several developments and applications. We comment here on a few directions.

A key motivation of our analysis was the recent developments of [78–86] studying α' corrected gravity for precision tests of the AdS/CFT correspondence. In [81, 85, 86] the formulation of minimal gauged supergravity in $5D$ based on the standard Weyl multiplet was used, as in our paper. Only the Weyl-squared and scalar curvature-squared invariants were employed in these works, since the Log (Ricci squared) invariant had not been constructed by using component fields. As far as the first α' corrections were concerned, arguments were given to justify why only two invariants might suffice to compute (BPS) on-shell observables (as for example the entropy of five-dimensional black holes). However, it remains an open question to understand to which extent only a subset of curvature squared invariants suffice for a general analysis. For example, the on-shell analysis employs a set of field redefinitions up to order α', but it remains an open question whether higher-order corrections in the solutions of the auxiliary fields equations of motions would still admit on-shell simplifications found at the first order. In view of this, and other open questions, we expect that our results about the three curvature-squared invariants could play an important role in the analysis of various observables in quantum corrected minimal gauged supergravity.

In [82] three invariants were considered in a dilaton Weyl background to engineer Poincaré supergravity. However, this analysis did assume that the gauged two-derivative supergravity could be added to the three ungauged curvature-squared invariants of [58, 64] to obtain a five-parameter family of locally supersymmetric invariants. A description of the invariants in a gauged dilaton Weyl background was given in [2], see also [2] for an extended upcoming analysis. We have discussed off-shell formulations based on the standard Weyl multiplet and the dilaton Weyl multiplet. The dilaton Weyl multiplet mentioned above is the one first introduced in [90, 174] and extended in [199] to a version suitable for gauged supergravity. This is defined as an on-shell vector multiplet coupled to the standard Weyl multiplet – for this reason we will denote it as vector-dilaton Weyl multiplet. Recently, it was noticed in [3, 4] that there exist even more variant versions of Weyl multiplets. In fact, by considering an on-shell hypermultiplet in a standard Weyl multiplet background one can obtain another variant Weyl multiplet of off-shell conformal supergravity which was referred to as hyper-dilaton Weyl. Thanks to the results in our paper, we can straightforwardly obtain a basis of independent curvature-squared invariants in all the dilaton Weyl backgrounds. The vector-dilaton Weyl multiplet has been largely used to engineer general matter couplings in Poincaré supergravity [90, 174], including the recent curvature-squared analysis in [2, 2]. By using another new independent off-shell scalar curvature-squared invariant that we constructed in [6], it would be possible to engineer general off-shell supergravities based on the hyper-dilaton Weyl multiplet of [3, 4]. An interesting feature of this formulation, once coupled to a system of vector multiplets, is how it is straightforward to describe general gaugings off-shell. It is in fact a natural avenue of future investigations to extend the results in this paper to general off-shell matter-coupled supergravity, at least for arbitrary couplings with physical vector multiplets. These models would be important for applications involving α' corrected string compactifications to five dimensions where physical scalar multiples would describe coordinates of the internal geometry and their moduli.

Another natural question concerning curvature-squared invariants is how the results we have obtained here and in [2,2] connects with the known invariants in lower and higher dimensions. In [200] an off-shell dimensional reduction approach was developed. There it was noticed how the Weyl-squared invariant in $5D$, once reduced to $4D$, leads to a linear combination of the $4D$ Weyl squared and the $4D$ Log invariants. It would be interesting to extend this analysis by using all the independent $5D$ invariants that are now available from this year. Knowing the precise connection between $4D$ and $5D$ might help to answer questions in AdS/CFT realising the roles that half BPS (e.g., $4D$ F-term) and D-term invariants can generally have in both dimensions. The classification of all curvature-squared invariants for six-dimensional $N=(1,0)$ minimal Poincaré supergravity was obtained six years ago in [68, 69]. One could develop an off-shell $5D/6D$ map and identify how the invariants in different dimensions are related – see [58] for the $5D/6D$ Riemann squared based on a dilaton Weyl multiplet. One wonders if understanding off-shell relations between higher-derivative invariants in different dimensions might shed some light to better understand higher-derivative supergravities in dimensions higher than six where superconformal techniques are not (directly) applicable.

It is known that among all the curvature-squared terms one predominant role is played by the Gauss-Bonnet combination. This is a topological term in four dimensions. In D >4 supersymmetric Gauss-Bonnet invariants are expected to play a key role in the description of the first α' corrections to compactified string theory [201, 202]. A reason being that, though higher derivative, the Gauss-Bonnet combination,

$$\mathcal{L}_{\rm GB} \quad = \quad -\frac{1}{4}\mathcal{R}^{abcd}\mathcal{R}_{abcd}+\mathcal{R}^{ab}\mathcal{R}_{ab}-\frac{1}{4}\mathcal{R}^2 \ , \tag{3.8.2}$$

leads to the same physical degrees of freedom as standard massless gravity with a cosmological constant when studying fluctuations around maximally symmetric backgrounds. In five dimensions, one remarkable property of α' corrected minimal gauged supergravity is that, up to perturbatively integrating out auxiliary fields and making curvature dependent field redefinitions, the complicated Lagrangian in (3.8.1) simplifies to the following one [2]

$$e^{-1}\mathcal{L}_{2\partial+4\partial}=a_0\mathcal{R}+a_1\kappa^2+a_2f^{ab}f_{ab}+a_3\varepsilon^{abcde}v_af_{bc}f_{de}+\alpha\mathcal{L}_{\rm GB}^{\rm on-\mathit{shell}} \ . \tag{3.8.3}$$

Here, the various constants a_i are functions of the five independent constants in (3.8.1), and $\mathcal{L}_{\rm GB}^{\rm on-\mathit{shell}}$ is an appropriate linear combination of (3.8.2) and the following terms: $f^{ab}f_{ab}$, $C_{abcd}f^{ab}f^{cd}$, and $\varepsilon^{abcde}v_a\mathcal{R}_{bc}{}^{fg}\mathcal{R}_{defg}$; see [2] for more detail.[11]

[11] Note that the result obtained in [2] is based on the gauged dilaton Weyl multiplet. However, the on-shell action obtained by perturbatively integrating out auxiliary fields and performing field redefinitions would agree with the one obtained by starting from the standard Weyl multiplet invariants of our current paper. We also underline that the result in equation (3.8.3) in a standard Weyl multiplet and in gauged supergravity was first obtained in [86] – see also [81, 85] – by using only the Weyl-squared and the scalar curvature-squared invariants.

One might ask what is an off-shell extension of $\mathcal{L}_{\mathrm{GB}}|_{onshell}$. By looking only at the curvature-squared invariants that we constructed in this paper, it is straightforward to realise that combinations of the independent curvature-squared invariants with proper coefficients leads to off-shell supersymmetric completions of the Gauss-Bonnet combination. One can show that this can be obtained by using just two curvature-squared invariants: the Weyl squared and Log. It is simple to show that the combination

$$\mathcal{L}_{\mathrm{GB}}^{off-shell} = \mathcal{L}_{\mathrm{W}eyl} - 4\mathcal{L}_{\log} , \tag{3.8.4}$$

is such that the kinetic terms for the $\mathrm{SU}(2)_R$ connection $\phi_m{}^{ij}$ cancel between the Weyl squared (3.5.7) and the Log (3.5.20) invariants. These observations indicate that there is no massive graviton nor dynamical massive vector generated by (3.8.4), as expected from a Gauss-Bonnet invariant. The equation of motion for the field D are algebraic, as it is clear from the D dependent terms in the bosonic part of $\mathcal{L}_{\mathrm{GB}}^{off-shell}$,

$$\begin{aligned}\mathcal{L}_{\mathrm{GB},D}^{off-shell} &= -32D^2 + 2\mathcal{R}D - 12F_{ab}W^{ab}D + 4DF^{ab}F_{ab} \\ &\quad -\frac{39}{2}DW^{ab}W_{ab} - 12DX^{ij}X_{ij} .\end{aligned} \tag{3.8.5}$$

It is then possible to eliminate D by substituting its equation of motion in the action. One can easily check that the final result is $\mathcal{L}_{GB}$ plus terms that leads to a supersymmetric completions. Moreover, one can prove that in an ungauged vector-dilaton Weyl background, the combination above leads to the off-shell Gauss-Bonnet invariant first constructed in [63,64]. The simplifications arising with the $5D$ Gauss-Bonnet invariant make easier the study of solutions of gauged supergravity modified with four-derivative terms, which are known to be difficult to analyse in general. One might now construct new classes of solutions and prove if and how solutions of the two-derivative theory are modified.

So far we have commented mostly on four-derivative corrections. However, the building blocks of our paper allow to construct invariants with arbitrary high orders in α'. For example, now that we have obtained all the information about the three independent composite linear multiplets associated to the Weyl squared, Log, and scalar curvature-squared invariants, we might construct an arbitrary (appropriately constrained) function of these invariants as for the system of linear multiplets engineered in [74].[12] To be more precise, we label $G_I^{ij} = (G^{ij}, H_{\mathrm{VM}}^{ij}, H_{\mathrm{W}eyl}^{ij}, H_{\log}^{ij}, H_{R^2}^{ij})$ with $I = 0,1,2,3,4$ as the different linear multiplets used in our paper, the compensator, and the various composite ones. Then, in a covariant projective superspace approach,[13] one can formulate a half BPS invariant built out of the linear multiplets with the projective superspace Lagrangian given by

$$\mathcal{L}^{(2)} = G_I^{(2)}\mathcal{F}^I[G_J^{(2)}] , \qquad G_I^{(2)} = v_i v_j G_I^{ij} . \tag{3.8.6}$$

[12]The results of [74] extend to five dimension the four dimensional construction of [73]. See [185], and [29] in $5D$, for their superspace formulation and for other (infinite series of) higher-derivative terms.

[13]We refer the reader to [29, 185], and the recent review [39] and references therein, for detail about the covariant projective formalism and its application to higher derivative supergravity. More in general, projective superspace [117–119], together with the related harmonic superspace [116, 121], are powerful techniques to formulate general off-shell matter couplings in supersymmetric models with eight real supercharges in $\mathrm{D} \leq 6$ dimensions where infinite number of auxiliary/matter fields might be necessary. See [29, 39, 120, 123–128, 130–133] for further references in the curved (supergravity) case.

Here v_i are constant isotwistor (twistor-like) coordinates describing an auxiliary $\mathbb{C}P^1$ manifold which is used to engineer a $5D$ locally supersymmetric invariant associated to the projective Lagrangian $\mathcal{L}^{(2)}$, see [29]. The functions $\mathcal{F}^I[G_J^{(2)}]$ are only required to be homogeneous of degree zero in their argument:

$$G_K^{(2)} \frac{\partial \mathcal{F}^I[G_J^{(2)}]}{\partial G_K^{(2)}} = 0 \,. \tag{3.8.7}$$

One could also consider a single function $\mathbb{F}[G_I^{(2)}]$ homogenous of degree one in $G_I^{(2)}$, which is locally equivalent to the description above. The details are not important here, but what we want to stress is the existence of a function that can lead to arbitrarily high curvature corrections in $5D$ minimal supergravity. In four dimensions, it is known how for a large class of observables and applications to AdS/CFT an F-term constructed out of an holomorphic function with arguments given by the composite vector multiplets describing the $4D$ curvature squared invariants of $N = 2$ supergravity [51, 57, 61, 66, 75, 76] suffices to computing observables based on the on-shell action as for example the entropy of (supersymmetric) black holes – see [83, 84], and references therein, for inspiring recent results in the four dimensional case. Along this line, it would be interesting to investigate if and how the model described by (3.8.6) encodes enough information to study general higher-derivative correction in minimal supergravity in the AdS/CFT context.

Acknowledgements:

We are grateful to M. Ozkan and Y. Pang for discussions and collaboration related to this work. G.T.-M. thanks N. Bobev, and K. Hristov, for recent discussions. G.T.-M. is also grateful to D. Butter, S. M. Kuzenko, and J. Novak for the joint collaboration on [29] that led to several of the structures used in this paper. This work was supported by the Australian Research Council (ARC) Future Fellowship FT180100353, ARC Discovery Project DP240101409, and the Capacity Building Package of the University of Queensland. G.G. and S.K. are supported by the postgraduate scholarships at the University of Queensland. The work of J.H. is also supported by the European Union under the Marie Sklodowska-Curie grant agreement number 101107602.[14]

[14]Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

The following publication has been incorporated as Chapter 4.

1. Gregory Gold, Jessica Hutomo, **Saurish Khandelwal**, Mehmet Ozkan, Yi Pang, Gabriele Tartaglino-Mazzucchelli, *All Gauged Curvature Squared Supergravities in Five Dimensions*, ***Phys. Rev. Lett.*** **131** (2023) 25, 251603 [2309.07637] [2].

Contributor	Statement of Contribution	%
Gregory Gold	Writing of text	15
	Proof-reading	20
	Off-shell theoretical derivations	17.5
	Off-shell computational derivations	80
	On-shell analysis	2.5
	Initial concept	5
Jessica Hutomo	Writing of text	15
	Proof-reading	10
	Off-shell theoretical derivations	25
	On-shell analysis	2.5
	Initial concept	5
Saurish Khandelwal	Writing of text	15
	Proof-reading	25
	Off-shell theoretical derivations	47.5
	Off-shell computational derivations	20
	On-shell analysis	2.5
	Initial concept	5
Mehmet Ozkan	Writing of text	18
	Proof-reading	15
	On-shell analysis	45
	Initial concept	22.5
Yi Pang	Writing of text	18
	Proof-reading	15
	On-shell analysis	45
	Initial concept	22.5
Gabriele Tartaglino-Mazzucchelli	Writing of text	18
	Proof-reading	15
	Off-shell theoretical derivations	10
	On-shell analysis	2.5
	Initial concept	30

Table 3.4: Contributions of each author to the work

Chapter 4

All Gauged Curvature Squared Supergravities in Five Dimensions

We present a complete basis to study gauged curvature-squared supergravity in five dimensions. We replace the conventional ungauged Riemann-squared action with the Log-invariant discussed in chapter 3, offering a comprehensive framework for all gauged curvature-squared supergravities. Our findings address long-standing challenges and have implications for precision tests in the AdS/CFT correspondence.

4.1 Introduction

Twenty-six years after its discovery, the AdS/CFT correspondence has entered a new era in which precision tests beyond the leading order have become increasingly important, owing to developments in both field theory and gravity. On the one hand, integrability and localization techniques allow one to compute the observables in superconformal field theories (SCFT) exactly at finite couplings. On the other hand, the development of superconformal tensor calculus and superspace techniques – see the reviews [37–40] – in conjunction with the computational capabilities offered by computer algebra programs, has significantly advanced the construction of exact, off-shell higher-derivative supergravity models.

In this letter, we present all gauged curvature-squared supergravity invariants in five dimensions based on the off-shell dilaton Weyl multiplet. After going on-shell, our invariants describe the most general four-derivative corrections to the five-dimensional minimal gauged supergravity, which is a universal sector to all string compactifications preserving at least eight supercharges. The gauged aspect is necessary to accommodate a supersymmetric anti-de Sitter (AdS) solution, and thus is of broad interest in holography. In particular, due to recent advancements in AdS black hole microstate counting [203–210] using the dual CFT, a precise matching between the gravity and CFT results at the next to leading order clearly requires the knowledge of the complete curvature-squared supergravity actions. Previous works have made attempts to compute four-derivative corrections based on partial results in the literature, and certain assumptions were made. Using our full results, it can be shown that, in fact, some of the assumptions are invalid, thus finally furnishing the stage for new, next-to-leading order analyses on the gravity side of the AdS/CFT correspondence.

The construction of gauged curvature-squared invariants is notoriously hard, as opposed to their ungauged counterparts, which have been fully known for more than a decade [58, 63, 64]. The primary difficulty stems from the absence of a straightforward transition from ungauged to gauged theories. In fact, the complete basis of invariants must be constructed from completely different starting points. For instance, certain ungauged curvature-squared models are attainable through the application of superconformal tensor calculus, utilizing the dilaton Weyl multiplet. In contrast, their gauged counterparts necessitate the use of a modified version of the same multiplet [199], which has an entirely different field content and transformation rules. Furthermore, the deformation necessary for the construction of gauged supergravity models renders certain established higher-derivative supergravity building techniques impractical, thus further complicating the task. In fact, it takes an interplay between superconformal tensor calculus [199] and superspace techniques [29], together with a series of new, daunting computations finalized only in the results presented here and in chapter 3, to yield the complete set of gauged curvature-squared invariants.

This letter aims to explicitly show that the past challenges can be overcome by changing the basis of curvature-squared supergravities, which previously employed the Weyl tensor squared, Riemann tensor squared, and Ricci scalar squared as fundamental building blocks. We demonstrate that by replacing the Riemann-squared action with the Log-invariant, which is discussed in detail in chapter 3 where the leading term involves the Ricci tensor squared, it is possible to explicitly establish all gauged curvature-squared supergravities in five dimensions. The outcomes presented in our letter mark a significant advancement, paving the way to a complete study of physical results beyond the leading supergravity approximation in five dimensions. This development holds particular promise for precision tests of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. In this context, we derive the anomaly coefficients in the dual SCFT_4, which apparently depend on all curvature-squared couplings.

4.2 Construction of the invariants

We start by introducing the field content of the standard Weyl multiplet of conformal supergravity in five dimensions [90]. The notation and conventions used in this chapter follow those of [90]. These differ from the notation and conventions used in the previous chapter, which adhere to [29]. For clarity, Table B.1 provides a concise translation scheme between the different conventions employed in [90, 174, 211], all of which contributed to the development of superconformal tensor calculus in five dimensions. We denote the spacetime indices by $\mu, \nu, \cdots$, Lorentz indices by $a, b, \cdots$, SU(2) indices by $i, j, \cdots$, and spinor indices by $\alpha, \beta, \cdots$. The multiplet is described by a set of independent gauge fields: the vielbein $e_\mu{}^a$, the gravitino $\psi_\mu{}^i_\alpha$, the SU(2) gauge fields $V_\mu{}^{ij}$, and a dilatation gauge field b_μ. The other gauge fields associated with the remaining symmetries, including the spin connection $\omega_\mu{}^{ab}$, the S-supersymmetry connection $\phi_\mu{}^i_\alpha$, and the special conformal connection $f_\mu{}^a$, are composite fields, i.e., they are determined in terms of the other fields by imposing certain curvature constraints. The standard Weyl multiplet also contains a set of matter fields: a real antisymmetric tensor T_{ab}[1], a fermion χ^i_α, and a real scalar D. A more detailed discussion of the superconformal transformations of the various fields can be found, e.g., in [90, 199].

Below we will make use of a variant multiplet of conformal supergravity, known as the gauged vector-dilaton Weyl multiplet [199]. For this multiplet, the independent gauge fields remain the same as the standard Weyl multiplet, but the matter content is replaced with $\{\sigma, C_\mu, B_{\mu\nu}, L_{ij}, E_{\mu\nu\rho}, N, \psi^i, \varphi^i\}$. This is obtained by coupling the standard Weyl multiplet to on-shell vector and linear multiplets. The vector multiplet consists of a scalar field σ, the gaugino ψ^i_α, an abelian gauge vector C_μ with field strength $G_{\mu\nu} = 2\partial_{[\mu}C_{\nu]}$, and an SU(2) triplet of auxiliary fields $Y^{ij} = Y^{(ij)}$. The linear multiplet contains an SU(2) triplet of scalars $L^{ij} = L^{(ij)}$, a gauge three-form $E_{\mu\nu\rho}$, a scalar N, and an SU(2) doublet φ^i_α. The bosonic matter fields of the vector and the standard Weyl multiplet are then expressed as follows [199]

$$\begin{aligned}
Y^{ij} &= -\tfrac{g}{2}\sigma^{-1}L^{ij} + \text{f.}t.\ , \\
T_{ab} &= \tfrac{1}{8}\sigma^{-1}G_{ab} + \tfrac{1}{48}\sigma^{-2}\varepsilon_{abcde}H^{cde} + \text{f.}t.\ , \\
D &= \tfrac{1}{4}\sigma^{-1}\nabla^a\nabla_a\sigma + \tfrac{1}{8}\sigma^{-2}(\nabla^a\sigma)\nabla_a\sigma - \tfrac{1}{32}R \\
&\quad -\tfrac{1}{16}\sigma^{-2}G^{ab}G_{ab} - (\tfrac{26}{3}T^{ab} - 2\sigma^{-1}G^{ab})T_{ab} \\
&\quad +\tfrac{g}{4}\sigma^{-2}N + \tfrac{g^2}{16}\sigma^{-4}L^2 + \text{f.}t.\ ,
\end{aligned} \tag{4.2.1}$$

where "f.t." stands for omitted fermionic terms and $H_{abc} = e_a{}^\mu e_b{}^\nu e_c{}^\rho H_{\mu\nu\rho}$ denotes the three-form field strength $H_{\mu\nu\rho} := 3\partial_{[\mu}B_{\nu\rho]} + \frac{3}{2}C_{[\mu}G_{\nu\rho]} + \frac{1}{2}gE_{\mu\nu\rho}$. In the above, the covariant derivative is denoted by

$$\nabla_a = e_a{}^\mu\left(\partial_\mu - \omega_\mu{}^{bc}M_{bc} - b_\mu\mathbb{D} - V_\mu{}^{ij}U_{ij}\right)\ , \tag{4.2.2}$$

with M_{ab}, $\mathbb{D}$, and U_{ij} being the Lorentz, dilatation, and SU(2) generators, respectively. The dilatation connection b_μ is pure gauge and will be set to zero throughout. The mapping (4.2.1) allows us to easily convert every invariant involving a coupling to the standard Weyl multiplet to that written in terms of the gauged dilaton Weyl multiplet. The ungauged map and the models can simply be obtained by setting $g = 0$ in (4.2.1). In this case, the fields of the linear multiplet decouple from the map (4.2.1), and the multiplet reduces to the ungauged dilaton Weyl multiplet with $32+32$ off-shell degrees of freedom [90, 174].

In the superconformal tensor calculus, the so-called BF action principle plays a fundamental role in the construction of general supergravity-matter couplings, see [29, 90, 173–177] for the $5D$ case. It is based on an appropriate product of a linear multiplet with an Abelian vector multiplet:

$$e^{-1}\mathcal{L}_{\text{BF}} = A_aE^a + \rho N + \mathcal{Y}_{ij}L^{ij} + \text{f.}t.\ . \tag{4.2.3}$$

Here we use $\{\rho, A_\mu, \mathcal{Y}_{ij}, \lambda^i_\alpha\}$ to denote the field content in an arbitrary vector multiplet, and the bosonic part of the constrained vector E_a is related to the three-form gauge field E_{abc} via $E_a = -\frac{1}{12}\varepsilon_{abcde}\nabla^bE^{cde}$.

[1] In the previous chapter, we used W_{ab} instead of T_{ab} to denote the antisymmetric tensor, reflecting a difference in conventions. Specifically, the two are related by $W_{ab} = \frac{16}{3}T_{ab}$. For a complete comparison of the notation, see Table B.1.

In any construction that involves composite expressions for the fields of the linear multiplet in terms of the vector multiplet, the BF-action yields a vector-coupled action in the (gauged) dilaton Weyl background. Using the off-shell map given in [63], a vector multiplet can be identified with fields in the gauged dilaton Weyl multiplet as

$$\begin{aligned} Y^{ij} &\to \frac{1}{4}\mathrm{i}\sigma^{-1}\bar{\psi}^i\psi^j - \frac{g}{2}\sigma^{-1}L^{ij}\,, \qquad & \rho &\to \sigma\,, \\ A_\mu &\to C_\mu\,, & \lambda^i &\to \psi^i\,, \end{aligned} \tag{4.2.4}$$

which gives rise to off-shell models that are purely expressed in terms of the fields of the (gauged) dilaton Weyl multiplet. By appropriately choosing primary composite linear multiplets, eq. (4.2.3) becomes the building block for constructing various curvature-squared invariants.

In the superconformal approach, the off-shell formulation of minimal $5D$ supergravity can be achieved by coupling the standard Weyl multiplet to two off-shell conformal compensators: a vector multiplet and a linear multiplet. Within this setup, supersymmetric completions of the Weyl tensor squared and Ricci scalar squared were constructed in [55] and [64], respectively. The Weyl tensor-squared invariant is based on a composite linear multiplet comprised solely of standard Weyl multiplet fields. To construct the Ricci scalar-squared invariant, one starts by defining a composite vector multiplet in terms of a linear multiplet. This composite vector multiplet is then substituted into the vector multiplet action obtained using (4.2.3).

While the Weyl tensor-squared and Ricci scalar-squared actions based on the dilaton Weyl multiplet were presented as ungauged models [64], they can simply be gauged by using the maps (4.2.1) and (4.2.4). It is worthwhile to mention at this point that the on-shell results for these gauged actions differ from those presented by [82]. The reason is that Ref. [82] assumes that the map between the standard Weyl and the dilaton Weyl multiplet is not modified in the gauged case. However, as shown in [199] and presented as in eq. (4.2.1), these expressions are indeed deformed. For completeness, we present the bosonic sectors of the off-shell Weyl-squared and scalar curvature-squared invariants in the gauged dilaton Weyl background. These results can be obtained by starting from the invariants written in a standard Weyl multiplet background, then applying the map of fields, eq. (4.2.1), which defines the gauged dilaton Weyl multiplet. For the purpose of this letter, it is enough to present their bosonic sector in the dilaton Weyl background in the gauge

$$\sigma = 1\,, \qquad b_\mu = 0\,, \qquad \psi^i = 0\,. \tag{4.2.5}$$

The bosonic sector of the gauged Weyl-squared invariant in the gauged dilaton Weyl background, eq. (4.2.1), and in the gauge eq. (4.2.5), is given by:

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{Weyl}^2} \;=\;& -\frac{1}{8}\varepsilon^{abcde}C_aR_{bcfg}R_{de}{}^{fg} + \frac{1}{6}\varepsilon^{abcde}C_aV_{bc}{}^{ij}V_{deij} + \frac{2}{3}V^{abij}V_{abij} \\ & -\frac{1}{4}R_{abcd}R^{abcd} + \frac{1}{3}R_{ab}R^{ab} - \frac{1}{12}R^2 + \frac{1}{3}R_{abcd}(G^{ab}G^{cd} - 2H^{ab}H^{cd} - 3H^{ab}G^{cd}) \\ & -\frac{4}{3}R_{ab}H^{ac}G^b{}_c + \frac{16}{3}R^{ab}H^2_{ab} + \frac{1}{3}RH_{ab}G^{ab} - \frac{4}{3}RH^2 - 4(H^2)^2 - 8H^4 \\ & -\frac{16}{3}H^2H_{cd}G^{cd} - \frac{40}{3}H^2_{ad}H^a{}_cG^{cd} + \frac{8}{3}H^2G^2 + \frac{2}{3}H_{ab}H_{cd}G^{ab}G^{cd} + \frac{1}{12}(G^2)^2 \end{aligned}$$

$$
\begin{aligned}
&-\frac{16}{3}H^2_{ab}G^{2ab}-\frac{4}{3}H_{ab}H_{cd}G^{ac}G^{bd}-\frac{1}{3}H_{ab}G^{ab}G^2+2G^{2ab}G^c{}_bH_{ca}\\
&-\frac{1}{3}(\nabla^a G_{bc})\nabla_a G^{bc}+\frac{8}{3}(\nabla^a H_{bc})\nabla_a H^{bc}-\frac{1}{2}G^4+\frac{4}{3}\varepsilon^{abcde}H_{ab}H_{cd}\nabla^f G_{ef}\\
&-2\varepsilon^{abcde}H_{bf}(\nabla_a H_c{}^f)G_{de}-\frac{2}{3}\varepsilon^{abcde}H_{ab}(\nabla^f G_{cf})G_{de}-\frac{1}{24}\varepsilon^{abcde}(\nabla^f G_{af})G_{bc}G_{de}\\
&-g\left(-\frac{4}{3}NH_{ab}G^{ab}+\frac{4}{3}NG^2-8NH^2+\frac{4}{3}V_{ab}{}^{ij}L_{ij}H^{ab}-\frac{2}{3}V_{ab}{}^{ij}L_{ij}G^{ab}-\frac{2}{3}RN\right)\\
&+g^2\left(\frac{1}{3}L^2H^{ab}G_{ab}-\frac{1}{3}L^2G^2+2L^2H^2-\frac{8}{3}N^2+\frac{1}{6}RL^2\right)\\
&-\frac{4}{3}g^3NL^2-\frac{1}{6}g^4L^4\ , \qquad (4.2.6)
\end{aligned}
$$

where we have used the following notations: $H^{ab}=-\frac{1}{12}\varepsilon^{abcde}H_{cde}$, $H^2=H^{ab}H_{ab}$, $G^2=G^{ab}G_{ab}$, $H^2_{ab}:=H_a{}^cH_{bc}$, $G^2_{ab}:=G_a{}^cG_{bc}$, $G^4=G^{2ab}G^2_{ab}$, and $H^4=H^{2ab}H^2_{ab}$.

The bosonic sector of the gauged scalar curvature-squared invariant in the gauged dilaton Weyl background, eq. (4.2.1), and in the gauge eq. (4.2.5), is given by:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{R^2} \;=\;& \mathbf{Y}^{ij}\mathbf{Y}_{ij}-2\nabla^a(NL^{-1})\nabla_a(NL^{-1})\\
&-\frac{1}{8}\varepsilon_{abcde}C^a\mathbf{G}^{bc}\mathbf{G}^{de}+NL^{-1}G^{ab}\mathbf{G}_{ab}-N^2L^{-2}G^{ab}G_{ab}\\
&+4N^2L^{-2}H^{ab}G_{ab}-\frac{1}{4}\mathbf{G}^{ab}\mathbf{G}_{ab}-4NL^{-1}H_{ab}\mathbf{G}^{ab}\\
&+g^2\Big(\frac{1}{4}L^{ij}\nabla^a\nabla_aL_{ij}-\frac{1}{4}RL^2-H^2L^2+\frac{1}{8}G^2L^2-\frac{5}{2}N^2-\frac{1}{2}E^aE_a-\frac{1}{2}\nabla^aL\nabla_aL\Big)\\
&-4gN^3L^{-2}+\frac{1}{16}g^4L^4\ , \qquad (4.2.7a)
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{G}_{ab} \;=\;& 4\nabla_{[a}(L^{-1}E_{b]})+2L^{-1}L_{ij}(V_{ab}{}^{ij})-2L^{-3}L_{ij}\big(\nabla_{[a}L^{ik}\big)\nabla_{b]}L_k{}^j\ ,\\
\mathbf{Y}^{ij} \;=\;& \frac{1}{4}L^{-1}\left(4\nabla^a\nabla_aL^{ij}-2RL^{ij}-8H^2L^{ij}+G^2L^{ij}\right)\\
&+L^{-3}\left(-N^2L^{ij}-E^aE_aL^{ij}-2E^aL^{k(i}\nabla_aL_k{}^{j)}-L_{kl}\nabla^aL^{k(i}\nabla_aL^{j)l}\right)\ . \qquad (4.2.7b)
\end{aligned}
$$

The third invariant necessary to obtain all the curvature-squared models in five dimensions was constructed as the Riemann-squared invariant in the ungauged dilaton Weyl basis in [58]. However, this model does not refer to the standard Weyl multiplet; hence, the prescription to obtain gauged models cannot be applied. Furthermore, the construction methodology cannot be extended to the gauged dilaton Weyl multiplet.

Alternatively, a third independent, locally superconformal invariant containing the Ricci tensor-squared term can be constructed [6,29], which provides the correct basis to study gauged curvature-squared supergravity, as we shall discuss momentarily. In this case, the lowest component of the composite linear multiplet is given by the field L^{ij}_{Log}. This is obtained by making use of the standard Weyl multiplet and by acting with six Q-supersymmetry transformations on the field $\log\rho$, with ρ

being the lowest component of a compensating vector multiplet[2]. The rest of the composite "Log multiplet" is then obtained by acting with up to two more Q-supersymmetry transformations on L^{ij}_{Log}. Due to the complexity of computing up to eight supersymmetry transformations, the explicit form of the Log multiplet, including all fermionic terms, has been obtained only recently with the aid of the *Cadabra* software [87, 88]. These lengthy results were published in [1, 6] and discussed in chapter 3. Inserting the resulting composite multiplet into (4.2.3) yields the explicit form of a new "Log invariant", which is presented in chapter 3 in eq. (3.5.20) within the standard Weyl basis. Then, the Log invariant in the gauged dilaton Weyl background can be obtained by employing the map (4.2.1) and (4.2.4).

The gauged Log invariant, which includes a Ricci-squared term, reads

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{Log}} \;=\;& -\tfrac{1}{6}R_{ab}R^{ab}+\tfrac{1}{24}R^2+\tfrac{1}{6}R^{ab}G^2_{ab}+\tfrac{1}{3}RH_{ab}G^{ab}-\tfrac{4}{3}R_{ab}H^{ac}G^{b}{}_{c}-\tfrac{1}{3}RH^2\\
&-\tfrac{1}{12}\varepsilon^{abcde}C_aV_{bc}{}^{ij}V_{deij}+\tfrac{1}{6}V^{abij}V_{abij}-2(H^2)^2+\tfrac{16}{3}H^2_{ab}H^{ac}G^{b}{}_{c}-\tfrac{4}{3}H^2H_{ab}G^{ab}\\
&+\tfrac{2}{3}H_{ab}H_{cd}\big(G^{ab}G^{cd}-2G^{ac}G^{bd}\big)+\tfrac{2}{3}H^2G^2-\tfrac{4}{3}H^{2ab}G^2_{ab}-\tfrac{1}{3}H_{ab}G^{ab}G^2\\
&+G^2_{ab}H^{ac}G^{b}{}_{c}-\tfrac{1}{48}(G^2)^2-\tfrac{1}{24}G^4-\tfrac{1}{6}\nabla_cG^{ac}\nabla^bG_{ab}\\
&+2\nabla_aH_{bc}\nabla^{[a}H^{bc]}+\tfrac{1}{48}\varepsilon^{abcde}\nabla^fG_{ef}(4H_{ab}-G_{ab})(4H_{cd}-G_{cd})\\
&+\tfrac{g}{6}\Big(RN-4NH_{ab}G^{ab}-2NG^2+V_{ab}{}^{ij}L_{ij}(G^{ab}+4H^{ab})+12NH^2-6\nabla^a\nabla_aN\Big)\\
&-\tfrac{g^2}{24}\Big(2RL^2-L^2(G^2-4G^{ab}H_{ab}-24H^2)+4N^2+6\nabla^aL^{ij}\nabla_aL_{ij}\Big)\\
&+\tfrac{2}{3}NL^2g^3+\tfrac{5}{24}L^4g^4\,,
\end{aligned}\tag{4.2.9}$$

where $V_{ab}{}^{ij}=2\partial_{[a}V_{b]}{}^{ij}-2V_{[a}{}^{k(i}V_{b]k}{}^{j)}$. Furthermore, we have used the notations defined after equation (4.2.6).

Note that the supersymmetric Riemann-squared invariant can be obtained by taking the following linear combination of the ungauged Weyl-squared invariant presented in [58] and setting $g=0$ in the Log invariant (4.2.9)

$$\mathcal{L}_{\text{R}iem^2}=\mathcal{L}_{\text{W}eyl^2}+2\mathcal{L}_{\text{Log}}|_{g=0}\,. \tag{4.2.10}$$

The resulting action is identical to the one presented in [58] up to total derivatives.

4.3 Going on shell and dual CFT

Now let us study a certain linear combination of the Einstein-Hilbert and all three curvature-squared invariants

$$(16\pi G)\mathcal{L}_{2\partial+4\partial}=\mathcal{L}_{\text{E}H}+\lambda_1\mathcal{L}_{\text{W}eyl^2}+\lambda_2\mathcal{L}_{\text{Log}}+\lambda_3\mathcal{L}_{R^2}\,. \tag{4.3.1}$$

[2] In superspace the primary superfield $\mathcal{L}^{ij}_{\text{Log}}$ associated with the Log multiplet is

$$\mathcal{L}^{ij}_{\text{Log}}=\tfrac{3\mathrm{i}}{1280}\nabla^{(ij}\nabla^{kl)}\nabla_{kl}\log\mathcal{W}\,, \tag{4.2.8}$$

with $\mathcal{W}$ being the superfield describing the primary field strength of a compensating vector multiplet. In (3.4.7) $\nabla^{ij}=\nabla^{\alpha(i}\nabla^{j)}_{\alpha}$, and ∇^i_α is the conformal superspace spinor derivative. The primary component fields are then obtained by projection to their lowest components, $\sigma=\mathcal{W}|_{\theta=0}$ and $L^{ij}_{\text{Log}}=\mathcal{L}^{ij}_{\text{Log}}|_{\theta=0}$ – see [6, 29] for detail.

where G is Newton's constant and all the invariants are given in the gauged dilaton Weyl multiplet background. $\mathcal{L}_{\mathrm{W}eyl^2}$ and $\mathcal{L}_{R^2}$ respectively denote the Weyl tensor squared and Ricci scalar squared actions which are obtained by employing the maps (4.2.1) and (4.2.4) in the standard Weyl multiplet results of [64]. Their explicit form is not crucial here, but they are given in the Supplemental Material for the reader's convenience. The two-derivative Lagrangian $\mathcal{L}_{\mathrm{E}H}$ is obtained by using the linear multiplet action in the standard Weyl multiplet basis [63, 199] and the sequential use of the maps (4.2.1) and (4.2.4). Note that, in this section, we rescaled the Lagrangians such that the coefficient of their leading curvature-squared term is normalized to unity. To go on-shell, we fix the gauge according to (4.2.5) and break SU(2) down to U(1) by choosing

$$L_{ij} = \tfrac{1}{\sqrt{2}}\delta_{ij}L\,, \qquad\qquad V_a^{ij} = V_a^{\prime ij} + \tfrac{1}{2}\delta^{ij}V_a\,. \tag{4.3.2}$$

Consequently, the two-derivative Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{E}H} \;=\;& L(R - \tfrac{1}{2}G_{ab}G^{ab} + 4H_{ab}H^{ab} + 2V_a^{\prime ij}V_{ij}^{\prime a}) \\ & + L^{-1}\partial_a L\partial^a L - 2L^{-1}E_aE^a - 2\sqrt{2}\,E_aV^a - 2N^2L^{-1} \\ & - 4gC_aE^a - 2gNL - 4gN - \tfrac{1}{2}g^2L^3 + 2g^2L^2\,. \end{aligned} \tag{4.3.3}$$

From the total Lagrangian (4.3.1), several auxiliary fields can be solved from their field equations up to $\mathcal{O}(\lambda_i)$

$$\begin{aligned} N &= -\tfrac{1}{2}gL(2+L) + \mathcal{O}(\lambda_i)\,, \\ E_a &= \mathcal{O}(\lambda_i)\,, \quad V_a^{\prime ij} = \mathcal{O}(\lambda_i)\,. \end{aligned} \tag{4.3.4}$$

To arrive at the five-dimensional gauged minimal supergravity, we first dualize $B_{\mu\nu}$ to a new 1-form gauge field $\widetilde{C}_\mu$ following the procedure in [63, 199]. We then truncate the model consistently by imposing

$$L = 1 + \mathcal{O}(\lambda_i)\,, \quad \widetilde{C}_a = C_a + \mathcal{O}(\lambda_i)\,. \tag{4.3.5}$$

Following (4.3.5), the field equation of E_{abc} now implies

$$V_a = -\tfrac{3}{\sqrt{2}}gC_a + \mathcal{O}(\lambda_i)\,. \tag{4.3.6}$$

Plugging (4.3.4)–(4.3.6) back to the total Lagrangian (4.3.1), one obtains the on-shell theory up to first order in λ_i. It is important to note that in the procedure outlined above, the $\mathcal{O}(\lambda_i)$ terms arising from substituting (4.3.4)–(4.3.6) to the two-derivative action either vanish (proportional to the leading order equations of motion of auxiliary fields) or can be removed by field redefinitions [212]. To recover the standard convention of minimal supergravity, we rescale the graviphoton and the U(1) coupling according to $C_a \to \frac{1}{\sqrt{3}}C_a$, $g \to \sqrt{2}\,g$. To conclude, following [213], the resulting Lagrangian can be further simplified by redefining the metric and the U(1) gauge field. Eventually, the on-shell model is recast in the form below

$$(16\pi G)e^{-1}\mathcal{L}_{2\partial+4\partial} \;=\; c_0R + 12c_1g^2 - \tfrac{1}{4}c_2G_{ab}G^{ab}$$

$$+\tfrac{1}{12\sqrt{3}}c_3\varepsilon^{abcde}C_aG_{bc}G_{de}+\lambda_1\mathcal{L}_{\rm GB}|_{onshell}\ , \qquad (4.3.7a)$$

where the various coefficients are

$$\begin{aligned}
c_0 &= 1+(\tfrac{28}{3}\lambda_1-20\lambda_2-4\lambda_3)g^2\,,\\
c_1 &= 1+(\tfrac{50}{9}\lambda_1-\tfrac{28}{3}\lambda_2+\tfrac{52}{3}\lambda_3)g^2\ ,\\
c_2 &= 1+(\tfrac{64}{9}\lambda_1-\tfrac{92}{3}\lambda_2-\tfrac{76}{3}\lambda_3)g^2\ ,\\
c_3 &= 1-12(\lambda_1+3\lambda_2+3\lambda_3)g^2\ ,
\end{aligned} \qquad (4.3.7b)$$

and the on-shell Gauss-Bonnet invariant is given by

$$\begin{aligned}
\mathcal{L}_{\rm GB}|_{onshell} &= R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2+\tfrac{1}{8}G^4\\
&\quad -\tfrac{1}{2}W_{abcd}G^{ab}G^{cd}+\tfrac{1}{2\sqrt{3}}\varepsilon^{abcde}C_aR_{bc}{}^{fg}R_{defg}\ ,
\end{aligned} \qquad (4.3.7c)$$

where W_{abcd} is the Weyl tensor. This on-shell action is consistent with the generic result presented in [86] for the proper choice of parameters. Based on the on-shell model (4.3.7) we find that the AdS$_5$ radius receives corrections from the higher-derivative terms and is given by

$$\ell = g^{-1}(1+\tfrac{8}{9}g^2\lambda_1-\tfrac{16}{3}g^2\lambda_2-\tfrac{32}{3}g^2\lambda_3)\ . \qquad (4.3.8)$$

The effective Newton's constant from eq. (4.3.7) is then

$$G_{eff} = G+G(-\tfrac{28}{3}\lambda_1+20\lambda_2+4\lambda_3)g^2\ . \qquad (4.3.9)$$

The AdS$_5$ vacuum preserves maximal eight supercharges [214, 215] and the dual field theory should be a $D=4, N=1$ CFT. Utilizing (4.3.8) and (4.3.9) the a and c Weyl anomaly coefficients of the dual CFT can be obtained via the standard holographic renormalization procedure [216, 217]

$$\begin{aligned}
a &= \frac{\pi}{8g^3G}-\frac{9\pi(\lambda_2+\lambda_3)}{2gG}\ ,\\
c &= \frac{\pi}{8g^3G}+\frac{\pi(2\lambda_1-9\lambda_2-9\lambda_3)}{2gG}\ ,
\end{aligned} \qquad (4.3.10)$$

using which one finds the results above are consistent with R-symmetry anomaly whose coefficients are related to those of the Weyl anomaly via [134, 135]

$$5a-3c=\frac{\pi c_3}{4g^3G}\ , \quad a-c=-\frac{\pi\lambda_1}{gG}\ . \qquad (4.3.11)$$

4.4 Conclusions and outlook

In this letter, we provide the correct and complete basis to study curvature-squared gauged supergravity in five dimensions. Our new results show explicitly that in the ungauged case, only the Weyl-squared invariant is relevant in computing physical quantities as the rest can be redefined away, generalizing the non-renormalization theorems for $D=4, N=2$ [76]. Based on the new results, we successfully computed the anomaly coefficients governing dual four-dimensional SCFTs. As four-dimensional SCFTs are characterized by two anomaly coefficients, one would naturally anticipate the emergence of only two independent linear combinations among the three four-derivative couplings. Indeed, our analysis confirms this expectation, with λ_2 and λ_3 consistently appearing together in the anomaly coefficients as the combination $\lambda_2+\lambda_3$. However, when examining the on-shell action (4.3.7), it becomes evident that λ_2 and λ_3 do not share this combination. Indeed, it can be shown that the Log invariant and the Ricci scalar invariant both contribute to the thermodynamics of general non-BPS AdS black holes, and the λ_2, λ_3 dependence cannot be arranged into the combinaiton $\lambda_2+\lambda_3$ [218, 219] as in some (BPS) quantities [77, 81, 85, 86].

We can also generalize the results by coupling multiple vector multiplets which also enjoy an off-shell formulation. Given the simple form the $6D$ ungauged Gauss-Bonnet invariant [68, 69, 160] and the relation between dilaton Weyl multiplets in these two dimensions [58], it may be feasible to reformulate the $5D$ Gauss-Bonnet invariant into a more elegant expression that facilitates the construction of intriguing solutions. For instance, the non-existence of supersymmetric AdS_5 black ring solutions in the two-derivative theory [220] raises the intriguing question of whether this situation changes in the presence of higher-derivative interactions. Our new invariants enable the computation of corrections to the entropy of $\frac{1}{16}$-BPS black holes [214, 215, 221–223], thereby extending the precision test of black hole microstate counting to the next to leading order. It is also interesting to extend the recently proposed equivariant localization [224, 225] beyond the leading two-derivative cases.

Acknowledgements:
The work of G.G., J.H., S.K., and G.T.-M. was supported by the Australian Research Council (ARC) Future Fellowship FT180100353, and by the Capacity Building Package of the University of Queensland. G.G. and S.K. are supported by the postgraduate scholarships at the University of Queensland. G.T.-M. thank the MATRIX Institute in Creswick for hospitality and support during part of this project. M.O. is supported in part by TUBITAK grant 121F064. M.O. acknowledges the support by the Distinguished Young Scientist Award BAGEP of the Science Academy. M.O. also acknowledges the support by the Outstanding Young Scientist Award of the Turkish Academy of Sciences (TUBA-GEBIP). The work of Y.P. is supported in part by National Natural Science Foundation of China (NSFC) under grant No. 12175164. Y.P. also acknowledges support by the National Key Research and Development Program under grant No. 2022YFE0134300.

The following publication has been incorporated as Chapter 5.

1. Gregory Gold, **Saurish Khandelwal**, William Kitchin, Gabriele Tartaglino-Mazzucchelli, *Hyper-dilaton Weyl multiplet of* 4*D*, $\mathcal{N} = 2$ *conformal supergravity*, JHEP 09 (2022) 016 [2203.12203] [4].

Contributor	Statement of Contribution	%
Gregory Gold	Writing of text	10
	Proof-reading	30
	Theoretical derivations	20
	Initial concept	10
Saurish Khandelwal	Writing of text	20
	Proof-reading	30
	Theoretical derivations	20
	Initial concept	10
William Kitchen	Writing of text	35
	Proof-reading	20
	Theoretical derivations	50
	Initial concept	20
Gabriele Tartaglino-Mazzucchelli	Writing of text	35
	Proof-reading	20
	Theoretical derivations	10
	Initial concept	60
	Supervision and guidance	100

Table 4.1: Contributions of each author to the work

Chapter 5

Hyper-Dilaton Weyl Multiplet of $4D$, $N = 2$ Conformal Supergravity

We define a new dilaton Weyl multiplet of $N = 2$ conformal supergravity in four dimensions. This is constructed by reinterpreting the equations of motion of an on-shell hypermultiplet as constraints that render some of the fields of the standard Weyl multiplet composite. The independent bosonic components include four scalar fields and a triplet of gauge two-forms. The resulting, so-called, hyper-dilaton Weyl multiplet defines a $24+24$ off-shell representation of the local $N = 2$ superconformal algebra. By coupling the hyper-dilaton Weyl multiplet to an off-shell vector multiplet compensator, we obtain one of the two minimal $32+32$ off-shell multiplets of $N = 2$ Poincaré supergravity constructed by Müller in 1986. On-shell, this contains the minimal $N = 2$ Poincaré supergravity multiplet together with a hypermultiplet where one of its physical scalars plays the role of a dilaton, while its three other scalars are dualised to a triplet of real gauge two-forms. Interestingly, a BF-coupling induces a scalar potential for the dilaton without a standard gauging.

5.1 Introduction

Conformal supergravity has played an important role in several research avenues in the last five decades — we refer the reader to a few books for reviews and a more detailed list of references [35–38]. The main aim of our paper is to revise some of the ingredients of the superconformal tensor calculus for matter-coupled Poincaré supergravity focusing on the four-dimensional (4D), $N = 2$ case, and more generally on theories with eight real supercharges. For instance, we will show how to define a new off-shell $24+24$ Weyl multiplet of $N = 2$ conformal supergravity.

After the seminal papers on $4D$, $N = 1$ supergravity [22–24, 226], the superconformal tensor calculus for the $4D$, $N = 2$ case was first constructed in the 80s in [137–141] and also extended to the cases of $6D$, $N = (1,0)$ in [89]; 5D, $N = 1$ in [90, 172, 174–177]; and more recently to three space-time dimensions in [27, 28]. Similar to superspace approaches (see [35, 36] for introductory reviews and, e. g., [26, 41, 116, 121, 127, 128, 131–133, 227–230] and references therein, for the $4D$, $N = 2$ case) a main advantage of the superconformal tensor calculus is to provide an off-shell description of general supergravity-matter couplings. This allows one to formulate general supergravity-matter couplings where supersymmetry is engineered in a completely model independent way. The approach has been very successful in helping to decipher many of the intricate geometrical structures associated to (two-derivatives) sigma-models in supergravity-matter systems with eight real supercharges, see e. g., [38, 42, 231–234]. The off-shell nature of the formalism has been a central ingredient in its employment to the study of supersymmetric localisation and supersymmetric quantum field theories on curved space-times — see [235] for a recent extensive review. Moreover, off-shell supersymmetry has also been a crucial ingredient when using the superconformal tensor calculus to construct higher-derivative supergravity invariants [28, 29, 31, 32, 51, 55–57, 59–72]. These play an important role, e. g., in the study of black-hole entropy in next to leading order AdS/CFT — see the recent works [79–81] and references therein.

Within the superconformal tensor calculus, general supergravity-matter couplings are engineered by a few ingredients. First of all, one needs a conformal supergravity multiplet — named the *Weyl* multiplet — which forms an off-shell representation of the local superconformal algebra and contains the vielbein as one of its independent fields. This multiplet defines the geometry (soft algebra) associated with the gauging of the superconformal space-time symmetry. Next, one identifies off-shell matter multiplets with local superconformal transformation rules in a Weyl multiplet background. These two ingredients provide the kinematic data of a specific supergravity-matter system. Finally, one engineers locally superconformal invariant action principles constructed out of these multiplets to obtained well-defined supergravity theories.[1]

Assuming the matter multiplets contain enough "compensating" degrees of freedom, one can suitably gauge fix part of the superconformal group, specifically dilatations, special conformal transformations, S-supersymmetry, and R-symmetry, to obtain supergravity models where only the super-Poincaré symmetry survives and is gauged. For instance, pure $4D$, $N = 2$ Poincaré supergravity can arise by the coupling of the *standard Weyl multiplet* [137–141] to two compensating multiplets. There is significant freedom in doing so. Typically, one uses a vector multiplet and a hypermultiplet (e. g., a linear, non-linear, or hypermultiplet with or without a central charge) as compensators — see [38] for a recent review. Note that, in this endeavour, for forty years the first step has predominantly been the same (standard Weyl multiplet), while most of the freedom that has been used concerned the matter (compensators) side of this story. However, it is natural to ask whether it is possible to use alternative Weyl multiplets and if it is useful to do so. These are the types of questions that led to our paper.

Answers to these questions are already known. For instance, if one considers the superconformal tensor calculus for $6D$, $N = (1,0)$ supergravity, it has been known since 1986 [89] that there can be more than one Weyl multiplet. In $6D$, the existence of a variant *dilaton Weyl* multiplet engineered as a standard Weyl multiplet coupled to an on-shell tensor multiplet has been a key ingredient to obtain 6D,

[1]These tasks can be simplified by manifestly gauging the superconformal algebra in superspace through the so-called *conformal superspace*. Conformal superspace was first introduced for $4D$, $N = 1,2$ supergravity in [25, 26] (see also the seminal work [236]) and it was then developed for 3D, N-extended supergravity [27], 5D, $N = 1$ supergravity [29], and recently $6D$, $N = (1,0)$ supergravity [31, 32].

$N=(1,0)$ Poincaré supergravity by using superconformal techniques. Similar ideas were employed to construct a variant dilaton Weyl multiplet for $5D$, $N=1$ conformal supergravity as the standard Weyl multiplet coupled to an on-shell vector multiplet [90]. Among interesting applications of these variant Weyl multiplets, it is worth mentioning that both in $5D$ and $6D$ the use of the dilaton Weyl multiplet allowed the construction of the component actions for the supersymmetric extensions of all curvature squared combinations — Riemann-squared, Ricci-squared, and scalar-curvature-squared — in Poincaré supergravities, see [29, 56, 59, 60, 63–65, 68, 69].

For the $4D$, $N=2$ case the existence of a variant representation of the Weyl multiplet of conformal supergravity was argued in [91] and was explicitly constructed only recently in [92]. For reasons that will soon be clear, we will refer to this multiplet as the *vector-dilaton Weyl* multiplet. Its construction closely mimics the $5D$ case [90]. More specifically, in [92] the system described by a $4D$, $N=2$ on-shell vector multiplet in a standard Weyl multiplet background was interpreted as a new $24+24$ multiplet of conformal supergravity. Using the equations of motion for the vector multiplet, the existing covariant matter fields of the standard Weyl multiplet, i.e., the real antisymmetric tensor, $T_{ab}{}^{ij}$, the real scalar field, D, and the spinor field, χ^i, together with the $\mathrm{U}(1)_R$ symmetry connection, were traded for fields of the on-shell vector multiplet [92]. The complex scalar field, X, of the vector multiplet then becomes an independent physical field whose real part plays the role of a dilaton in a Poincaré supergravity constructed in this framework. This Poincaré supergravity was constructed by coupling the vector-dilaton Weyl multiplet to a $8+8$ linear multiplet compensator [72, 92]. Upon gauge fixing dilatation, special conformal transformations, S-supersymmetry, and $\mathrm{U}(1)_R \times SU(2)_R$ R-symmetry (up to a residual $\mathrm{U}(1)_R$), the resulting $32+32$ Poincaré supergravity multiplet comprises the following set of fields

$$\{e_m{}^a, \psi_{m}{}^{\alpha}_i, \bar{\psi}_{m}{}^{i}_{\dot{\alpha}}, a_m | a'_m, \lambda^i_\alpha, \bar{\lambda}^{\dot{\alpha}}_i, C, t_{mn} | v_m, t'_{mn}, M, b_a{}^{ij}\} \, . \tag{5.1.1}$$

Here, $\{e_m{}^a, \psi_{m}{}^{\alpha}_i, \bar{\psi}_{m}{}^{i}_{\dot{\alpha}}, a_m\}$ are the fields of the $N=2$ on-shell supergravity multiplet, respectively: the vielbein, the gravitino and its conjugate, and the real graviphoton gauge vector field. The fields $\{v_m, t'_{mn}, M, b_a{}^{ij}\}$, that are respectively a real vector, a real antisymmetric gauge two-form, a real scalar, and a triplet of real vectors, are all auxiliary fields. The remaining fields, $\{a'_m, \lambda^i_\alpha, \bar{\lambda}^{\dot{\alpha}}_i, C, t_{mn}\}$, are physical and describe an on-shell vector multiplet where the imaginary part of the complex scalar field X has been traded for a dual antisymmetric real gauge two-form, t_{mn}.

The previous $32+32$ off-shell Poincaré supergravity turned out to coincide with the one engineered in 1986 by Müller in [237] — we will refer to this as the *vector-Müller* supergravity. It is useful to compare Müller's supergravity to the well-known $40+40$ off-shell supergravity of [137, 238–240]. One initial feature is that, from the point of view of off-shell supersymmetry, Müller's multiplet is irreducible while the $40+40$ multiplet is not. In fact, the vector-Müller multiplet arises from a $24+24$ conformal supergravity multiplet coupled to a single $8+8$ off-shell compensator, while the $40+40$ multiplet requires two $8+8$ compensating multiplets. The on-shell theories are however different. The $40+40$ off-shell supergravity leads to a dynamical system containing only the irreducible on-shell $N=2$ supergravity while Müller's $32+32$ off-shell supergravity comes with an extra physical on-shell "dilaton" vector multiplet.

Interestingly, in 1986 Müller constructed another minimal $32+32$ off-shell $N=2$ Poincaré supergravity [241]. We will refer to this as the *hyper-dilaton Poincaré* supergravity. The multiplet of [241] comprises the following fields

$$\{e_m{}^a, \psi_{m}{}_i^\alpha, \bar{\psi}_m{}^i_{\dot\alpha}, a_m | C, t_{mn}{}^{ij}, \rho^i_\alpha, \bar{\rho}_i^{\dot\alpha} | X^{ij}, W_{ab}, b_a\} \ . \tag{5.1.2}$$

Besides the fields of minimal $N=2$ on-shell supergravity, $\{e_m{}^a, \psi_{m}{}_i^\alpha, \bar{\psi}_m{}^i_{\dot\alpha}, a_m\}$, the fields $\{X^{ij}, W_{ab}, b_a\}$ are respectively a real SU(2) triplet of Lorentz scalars, a real antisymmetric tensor, and a real vector. These three are auxiliary fields [241]. The remaining fields, $\{C, t_{mn}{}^{ij}, \rho^i_\alpha, \bar{\rho}_i^{\dot\alpha}\}$, are physical and describe an on-shell hypermultiplet where three of the four hypermultiplet's scalar fields have been traded for an SU(2) triplet of dual antisymmetric real gauge two-forms, $t_{mn}{}^{ij} = t_{mn}{}^{ji} = -t_{nm}{}^{ij}$. Precisely as for the vector-Müller supergravity, even though off-shell the hyper-dilaton Poincaré multiplet is irreducible with $32+32$ degrees of freedom, the on-shell theory contains an extra $4+4$ dilaton multiplet which, in this case, is described by a variant version of a hypermultiplet where C plays the role of a dilaton field. Considering the recent superconformal description of the vector-Müller supergravity [72, 92], it is natural to ask if and how one can engineer the hyper-dilaton Poincaré supergravity by using the superconformal tensor calculus. The main aim of our paper is to show how this can be done by using a $24+24$ variant representation of the $N=2$ conformal supergravity multiplet that we will refer to as the *hyper-dilaton Weyl* multiplet.

The definition of the hyper-dilaton Weyl multiplet is fairly simple. In fact, it closely mimics the description of vector-dilaton Weyl multiplets with the crucial difference that one starts with an on-shell hypermultiplet [186, 242] in a standard Weyl multiplet background [138–141, 232], rather than with an on-shell vector multiplet. The constraints that arise by requiring the algebra of local superconformal transformations to close on the fields of the hypermultiplet can then be interpreted as algebraic equations for some of the fields of the standard Weyl multiplet. More precisely, in the hyper-dilaton Weyl multiplet, the standard Weyl multiplet's matter fields $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot\alpha i})$ and D, together with the $SU(2)_R$ symmetry connection $\phi_m{}^{ij}$ become composite fields. On the other hand, the four bosonic $q^{i\underline{i}}$ and four fermionic $(\rho^{\underline{i}}_\alpha, \bar{\rho}_{\underline{i}}^{\dot\alpha})$ fields of the hypermultiplet, together with an emerging triplet of real gauge two-forms $b_{mn}{}^{\underline{ij}} = b_{mn}{}^{\underline{ji}} = -b_{nm}{}^{\underline{ij}}$, are independent and not subject to any equations of motion. By then coupling the $24+24$ hyper-dilaton Weyl multiplet to a single $8+8$ off-shell vector multiplet compensator, upon gauge fixing dilatation, special conformal transformations, S-supersymmetry, and the whole $\mathrm{U}(1)_R \times SU(2)_R$ R-symmetry, one readily obtains the $32+32$ hyper-dilaton Poincaré supergravity with the field content described in (5.1.2).

To the best of our knowledge, despite their simplicity, the reinterpretation that we advocate in this paper for the on-shell hypermultiplet as the hyper-dilaton Weyl multiplet and their connection to Müller's supergravity in a superconformal framework has never explicitly appeared before in the literature. The advantage of our novel superconformal formulation compared to the original work of Müller [241] is the potentially straightforward extension to more general matter couplings. As a simple example, in our paper we show this by extending Müller's Poincaré supergravity action by including a new invariant that leads to a non-trivial potential for the dilaton field. Intriguingly, such a scalar potential is generated without a standard gauging which, in an $N=2$ standard Weyl multiplet setting, is associated to integrating out an independent auxiliary field given by the $SU(2)_R$ gauge connection.

This paper is organised as follows. In section 5.2 we first review the definition of the standard Weyl multiplet of off-shell $N=2$ conformal supergravity and, by using our notation, present results we need for the rest of the paper. We then describe the structure of an on-shell hypermultiplet in a standard Weyl multiplet background and explain how such a system can be reinterpreted as a variant hyper-dilaton Weyl multiplet of off-shell $N=2$ conformal supergravity. Section 5.3 is devoted to first prove how the off-shell Poincaré supergravity theory constructed by Müller in [241] can be engineered as the hyper-dilaton Weyl multiplet coupled to an off-shell vector multiplet conformal compensator. We then extend the results of [241] by adding a new BF-coupling which induces a scalar potential for the dilaton without a standard R-symmetry gauging. Section 5.4 includes a final discussion and an outline of some future directions based on the results of our paper.

5.2 The hyper-dilaton Weyl multiplet

The aim of this section is to construct the $24+24$ hyper-dilaton Weyl multiplet of off-shell $N=2$ conformal supergravity. Subsections 2.2.2 and 2.2.3 reviews well-known results about the standard Weyl multiplet and serves to introduce the notation that we employ. Subsection 5.2.1 describes the on-shell hypermultiplet in a standard Weyl multiplet background and the resulting interpretation of this system as an independent multiplet of conformal supergravity.

5.2.1 On-shell hypermultiplet and hyper-dilaton Weyl multiplet

A single on-shell hypermultiplet comprises $4+4$ degrees of freedom described by a Lorentz scalar field $q^{i\underline{i}}$ and spinor fields $(\rho_\alpha^{\underline{i}}, \bar{\rho}_{\underline{i}}^{\dot{\alpha}})$ — see [138–141,186,242] together with [37,38,232] and references therein for superconformal approaches to systems of on-shell hypermultiplets. The index $\underline{i}=\underline{1},\underline{2}$ is an SU(2) flavour index and the fields satisfy the following reality conditions

$$(q^{i\underline{i}})^* = q_{i\underline{i}} \, , \qquad (\rho_\alpha^{\underline{i}})^* = \bar{\rho}_{\dot{\alpha}\underline{i}} \, , \tag{5.2.1}$$

together with the following dilatation and chiral weight identities

$$\begin{aligned} \mathbb{D}q^{i\underline{i}} = q^{i\underline{i}} \, , \quad \mathbb{D}\rho_\alpha^{\underline{i}} &= \frac{3}{2}\rho_\alpha^{\underline{i}} \, , \quad \mathbb{D}\bar{\rho}_{\dot{\alpha}\underline{i}} = \frac{3}{2}\bar{\rho}_{\dot{\alpha}\underline{i}} \, , && (5.2.2\text{a}) \\ Yq^{i\underline{i}} = 0 \, , \quad Y\rho_\alpha^{\underline{i}} &= \rho_\alpha^{\underline{i}} \, , \quad Y\bar{\rho}_{\dot{\alpha}\underline{i}} = -\bar{\rho}_{\dot{\alpha}\underline{i}} \, . && (5.2.2\text{b}) \end{aligned}$$

The multiplet, which has the field $q^{i\underline{i}}$ as its superconformal primary, is characterised by the following local superconformal transformations [37,38,138–141,232]

$$\begin{aligned} \delta q^{i\underline{i}} &= \frac{1}{2}\xi^i\rho^{\underline{i}} - \frac{1}{2}\bar{\xi}^i\bar{\rho}^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + \lambda_{\mathbb{D}} q^{i\underline{i}} \, , && (5.2.3\text{a}) \\ \delta\rho_\alpha^{\underline{i}} &= -4\mathrm{i}(\sigma^a\bar{\xi}_k)_\alpha \nabla_a q^{k\underline{i}} + \frac{1}{2}\lambda_{ab}(\sigma^{ab}\rho^{\underline{i}})_\alpha + \mathrm{i}\lambda_Y\rho_\alpha^{\underline{i}} + \frac{3}{2}\lambda_{\mathbb{D}}\rho_\alpha^{\underline{i}} + 8\eta_\alpha^k q_k{}^{\underline{i}} \, , && (5.2.3\text{b}) \end{aligned}$$

$$\delta\bar{\rho}^{\dot{\alpha}}_{\underline{i}} = 4\mathrm{i}(\tilde{\sigma}^a\xi^k)^{\dot{\alpha}}\nabla_a q_{k\underline{i}} + \frac{1}{2}\lambda_{ab}(\tilde{\sigma}^{ab}\bar{\rho}_{\underline{i}})^{\dot{\alpha}} - \mathrm{i}\lambda_Y\bar{\rho}^{\dot{\alpha}}_{\underline{i}} + \frac{3}{2}\lambda_{\mathbb{D}}\bar{\rho}^{\dot{\alpha}}_{\underline{i}} - 8\bar{\eta}^{\dot{\alpha}}_k q^k{}_{\underline{i}}\,, \tag{5.2.3c}$$

where

$$\nabla_a q^{i\underline{i}} = \mathcal{D}_a q^{i\underline{i}} - \frac{1}{4}\psi_a{}^i\rho^{\underline{i}} + \frac{1}{4}\bar{\psi}_a{}^i\bar{\rho}^{\underline{i}}\,. \tag{5.2.4}$$

In contrast with the standard Weyl multiplet described in the previous subsection, the algebra of the local transformations (5.2.3) closes only when equations of motion for the fields are imposed, see for example [38, 232] for a detailed analysis. In our notations, the covariant equations of motion of $q^{i\underline{i}}$ and $(\rho^{\underline{i}}_{\alpha}, \bar{\rho}^{\dot{\alpha}}_{\underline{i}})$ are:

$$\begin{aligned}
\left(\nabla_a\rho^{\underline{i}}\sigma^a\right)_{\dot{\alpha}} &= \frac{\mathrm{i}}{2}(\bar{\rho}^{\underline{i}}\tilde{\sigma}^{cd})_{\dot{\alpha}}W^-_{cd} + 6\mathrm{i}\bar{\Sigma}_{\dot{\alpha}k}q^{k\underline{i}}\,, &(5.2.5a)\\
\left(\nabla_a\bar{\rho}_{\underline{i}}\tilde{\sigma}^a\right)^{\alpha} &= -\frac{\mathrm{i}}{2}(\rho_{\underline{i}}\sigma^{cd})^{\alpha}W^+_{cd} + 6\mathrm{i}\Sigma^{\alpha k}q_{k\underline{i}}\,, &(5.2.5b)\\
\Box q^{i\underline{i}} &= -\frac{3}{2}Dq^{i\underline{i}}\,, \qquad \Box := \nabla^a\nabla_a\,. &(5.2.5c)
\end{aligned}$$

The expressions for $\nabla_a\rho^{\underline{i}}_{\alpha}$, $\nabla_a\bar{\rho}^{\dot{\alpha}}_{\underline{i}}$, and $\Box q^{i\underline{i}}$ in terms of the derivatives $\mathcal{D}_a$ are given by

$$\begin{aligned}
\nabla_a\rho^{\underline{i}}_{\alpha} &= \mathcal{D}_a\rho^{\underline{i}}_{\alpha} + 2\mathrm{i}(\sigma^b\bar{\psi}_{ak})_{\alpha}\left(\mathcal{D}_b q^{k\underline{i}} - \frac{1}{4}\psi_b{}^k\rho^{\underline{i}} + \frac{1}{4}\bar{\psi}_b{}^k\bar{\rho}^{\underline{i}}\right) + 4\phi_{a\alpha k}q^{k\underline{i}}\,, &(5.2.6a)\\
\nabla_a\bar{\rho}^{\dot{\alpha}}_{\underline{i}} &= \mathcal{D}_a\bar{\rho}^{\dot{\alpha}}_{\underline{i}} - 2\mathrm{i}(\tilde{\sigma}^b\psi_a{}^k)^{\dot{\alpha}}\left(\mathcal{D}_b q_{k\underline{i}} - \frac{1}{4}\psi_{bk}\rho_{\underline{i}} + \frac{1}{4}\bar{\psi}_{bk}\bar{\rho}_{\underline{i}}\right) - 4\bar{\phi}_a{}^{\dot{\alpha}k}q_{k\underline{i}}\,, &(5.2.6b)\\
\Box q^{i\underline{i}} &= \mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} - 2\mathfrak{f}_a{}^a q^{i\underline{i}} - \frac{1}{4}\rho^{\underline{i}}\mathcal{D}_a\psi^{ai} + \frac{1}{4}\bar{\rho}^{\underline{i}}\mathcal{D}_a\bar{\psi}^{ai} - \frac{1}{2}\psi^{ai}\mathcal{D}_a\rho^{\underline{i}} + \frac{1}{2}\bar{\psi}^{ai}\mathcal{D}_a\bar{\rho}^{\underline{i}}\\
&\quad - \frac{\mathrm{i}}{4}\phi_a{}^i\sigma^a\bar{\rho}^{\underline{i}} + \frac{\mathrm{i}}{4}\bar{\phi}^i_a\tilde{\sigma}^a\rho^{\underline{i}} + \mathrm{i}(\psi_a{}^{(i}\sigma^b\bar{\psi}^{k)a})\mathcal{D}_b q_k{}^{\underline{i}} + \frac{3\mathrm{i}}{4}(\psi_a{}^i\sigma^a\bar{\Sigma}_l)q^{l\underline{i}} + \frac{3\mathrm{i}}{4}(\bar{\psi}^i_a\tilde{\sigma}^a\Sigma_l)q^{l\underline{i}}\\
&\quad - \frac{\mathrm{i}}{16}(\bar{\psi}_a{}^i\tilde{\sigma}^a\sigma^{cd}\rho^{\underline{i}})W^+_{cd} - \frac{\mathrm{i}}{16}(\psi_a{}^i\sigma^a\tilde{\sigma}^{cd}\bar{\rho}^{\underline{i}})W^-_{cd} - (\psi_a{}^i\phi^a{}_k)q^{k\underline{i}} + (\bar{\psi}_a{}^i\bar{\phi}^a{}_k)q^{k\underline{i}}\\
&\quad - \frac{\mathrm{i}}{4}(\psi_a{}^{(i}\sigma^b\bar{\psi}^{k)a})(\psi_{bk}\rho^{\underline{i}}) + \frac{\mathrm{i}}{4}(\psi_a{}^{(i}\sigma^c\bar{\psi}^{k)a})(\bar{\psi}_{ck}\bar{\rho}^{\underline{i}})\,. &(5.2.6c)
\end{aligned}$$

It is important to stress that equations (5.2.5) are typically read as equations of motion for the hypermultiplet fields, see e. g., [37, 38, 138–141, 232]. They certainly are dynamical equations for $q^{i\underline{i}}$ and $(\rho^{\underline{i}}_{\alpha}, \bar{\rho}^{\dot{\alpha}}_{\underline{i}})$ in a flat background (with no central charges as in our case) where all conformal supergravity fields are set to zero [186, 242]. For this reason, the multiplet is typically referred to as the on-shell hypermultiplet. However, such an interpretation is not necessary in a curved background described by the standard Weyl multiplet. In fact, the equations (5.2.5) can be interpreted as algebraic equations for the standard Weyl multiplet that determine the fields $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha}i})$ and D in terms of $q^{i\underline{i}}$ and $(\rho^{\underline{i}}_{\alpha}, \bar{\rho}^{\dot{\alpha}}_{\underline{i}})$ together with the other independent fields of the standard Weyl multiplet. If we assume that $q^{i\underline{i}}$ is an invertible matrix, which is equivalent to imposing

$$q^2 := q^{i\underline{i}}q_{i\underline{i}} = \varepsilon_{ij}\varepsilon_{\underline{i}\underline{j}}q^{i\underline{i}}q^{j\underline{j}} = 2\det q^{i\underline{i}} \neq 0\,, \tag{5.2.7}$$

then the following relations hold

$$\Sigma^{\alpha i} = 2q^{-2}q^{i\underline{i}}\left[-\frac{\mathrm{i}}{2}(\mathcal{D}_a\bar{\rho}_{\underline{i}}\tilde{\sigma}^a)^{\alpha} + (\psi_a{}^j\sigma^b\tilde{\sigma}^a)^{\alpha}\left(\mathcal{D}_b q_{j\underline{i}} - \frac{1}{4}\psi_{bj}\rho_{\underline{i}} + \frac{1}{4}\bar{\psi}_{bj}\bar{\rho}_{\underline{i}}\right)\right.$$

$$
+\frac{2}{3}\big(\Psi_{ab\,j}\sigma^{ab}\big)^{\alpha}q^{j}{}_{\underline{i}}+\frac{1}{4}(\rho_{\underline{i}}\sigma^{cd})^{\alpha}W^{+}_{cd}+\frac{\mathrm{i}}{6}(\bar{\psi}_{a\,j}\tilde{\sigma}^{a}\sigma^{cd})^{\alpha}q^{j}{}_{\underline{i}}W^{+}_{cd}\Bigg] , \qquad (5.2.8a)
$$

$$
\begin{aligned}
\bar{\Sigma}_{\dot\alpha i} &= 2q^{-2}q_{i\underline{i}}\Bigg[-\frac{\mathrm{i}}{2}(\mathcal{D}_a\rho^{\underline{i}}\sigma^{a})_{\dot\alpha}-(\bar{\psi}_{a\,j}\tilde{\sigma}^{b}\sigma^{a})_{\dot\alpha}\left(\mathcal{D}_b q^{j\underline{i}}-\frac{1}{4}\psi_b{}^{j}\rho^{\underline{i}}+\frac{1}{4}\bar{\psi}_b{}^{j}\bar{\rho}^{\underline{i}}\right)\\
&\quad -\frac{2}{3}\big(\overline{\Psi}_{ab}{}^{j}\tilde{\sigma}^{ab}\big)_{\dot\alpha}q_j{}^{\underline{i}}-\frac{1}{4}(\bar{\rho}^{\underline{i}}\tilde{\sigma}^{cd})_{\dot\alpha}W^{-}_{cd}+\frac{\mathrm{i}}{6}(\psi_a{}^{j}\sigma^{a}\tilde{\sigma}^{cd})_{\dot\alpha}q_j{}^{\underline{i}}W^{-}_{cd}\Bigg] , \qquad (5.2.8b)\\
D &= q^{-2}q_{i\underline{i}}\Bigg[\mathcal{D}^{a}\mathcal{D}_a q^{i\underline{i}}+\frac{1}{6}Rq^{i\underline{i}}-\frac{\mathrm{i}}{8}(\bar{\psi}_a{}^{i}\tilde{\sigma}^{a}\sigma^{cd}\rho^{\underline{i}})W^{+}_{cd}-\frac{\mathrm{i}}{2}\phi_a{}^{i}\sigma^{a}\bar{\rho}^{\underline{i}}-\frac{1}{2}\rho^{\underline{i}}\mathcal{D}_a\psi^{ai}\\
&\quad -\psi^{ai}\mathcal{D}_a\rho^{\underline{i}}+2(\psi_a{}^{i}\phi^{aj})q_j{}^{\underline{i}}+\frac{3\mathrm{i}}{2}(\psi_a{}^{i}\sigma^{a}\bar{\Sigma}_j)q^{j\underline{i}}+\frac{\mathrm{i}}{2}(\psi_{aj}\sigma^{a}\bar{\Sigma}^{j})q^{i\underline{i}}\\
&\quad +\frac{1}{6}\varepsilon^{mnpq}(\bar{\psi}^{j}_{m}\tilde{\sigma}_n\mathcal{D}_p\psi_{qj})q^{i\underline{i}}+\frac{1}{3}W^{ab+}(\bar{\psi}^{j}_{a}\bar{\psi}_{bj})q^{i\underline{i}}+\mathrm{i}(\psi_a{}^{(i}\sigma^{b}\bar{\psi}^{j)a})\mathcal{D}_b q_j{}^{\underline{i}}\\
&\quad -\frac{\mathrm{i}}{2}(\psi_a{}^{(i}\sigma^{b}\bar{\psi}^{j)a})(\psi_{bj}\rho^{\underline{i}})\Bigg]+\text{c.c.}\ . \qquad (5.2.8c)
\end{aligned}
$$

In the expression for D, eq. (5.2.8c), remember that $(\Sigma^{i},\bar{\Sigma}_i)$ and $(\phi^{i},\bar{\phi}_i)$, together with the spin connection $\omega_m{}^{cd}$, are composite fields. Note that so far we have only used one of the four equations that are equivalent to (5.2.5c) to solve for D in eq. (5.2.8c). It is simple to show that the remaining independent three equations are equivalent to the following

$$
\nabla^{a}(q^{i(\underline{i}}\nabla_a q_i{}^{\underline{j})})=0\ . \qquad (5.2.9)
$$

As we are going to explain in detail below, this equation is solved by turning the $\mathrm{SU}(2)_R$ connection $\phi_m{}^{kl}$ into a composite field.

As a next step in the construction of the hyper-dilaton Weyl multiplet, we note that, accompanied to an on-shell hypermultiplet there is always a triplet of composite linear multiplets [138–140, 186]. An $N=2$ off-shell linear multiplet [117, 118, 139, 187–191] comprises the following covariant fields: an $\mathrm{SU}(2)_R$ triplet of Lorentz scalar fields G^{ij} subject to the reality condition $(G^{ij})^{*}=G_{ij}$; spinor fields $(\chi_{\alpha i},\bar{\chi}^{\dot\alpha i})$; a complex scalar field $(F,\bar{F})$; and a covariant real closed anti-symmetric three-form H_{abc}, which is equivalent to a conserved dual vector $\tilde{H}^{a}:=\frac{1}{6}\varepsilon^{abcd}H_{bcd}$. Their local superconformal transformations in a standard Weyl multiplet background are given by [139]

$$
\begin{aligned}
\delta G_{ij} &= 2\xi_{(i}\chi_{j)}+2\bar{\xi}_{(i}\bar{\chi}_{j)}-2\lambda_{(i}{}^{k}G_{j)k}+2\lambda_{\mathbb{D}}G_{ij}\ , \qquad (5.2.10a)\\
\delta\chi_{\alpha i} &= -\xi_{\alpha i}F-4\mathrm{i}\tilde{H}_a(\sigma^{a}\bar{\xi}_i)_{\alpha}+\mathrm{i}(\sigma^{a}\bar{\xi}^{j})_{\alpha}\nabla_a G_{ij}+4\eta^{j}_{\alpha}G_{ji}\\
&\quad +\frac{1}{2}\lambda^{ab}(\sigma_{ab}\chi_i)_{\alpha}-\lambda_i{}^{j}\chi_{j\alpha}+\frac{5}{2}\lambda_{\mathbb{D}}\chi_{\alpha i}+\mathrm{i}\lambda_Y\chi_{\alpha i}\ , \qquad (5.2.10b)\\
\delta\bar{\chi}^{\dot\alpha i} &= -\bar{\xi}^{\dot\alpha i}\bar{F}+4\mathrm{i}\tilde{H}_a(\tilde{\sigma}^{a}\xi^{i})^{\dot\alpha}+\mathrm{i}(\tilde{\sigma}^{a}\xi_j)^{\dot\alpha}\nabla_a G^{ij}+4\bar{\eta}^{\dot\alpha}_{j}G^{ji}\\
&\quad +\frac{1}{2}\lambda^{ab}(\tilde{\sigma}_{ab}\bar{\chi}^{i})^{\dot\alpha}+\lambda^{i}{}_{j}\bar{\chi}^{j\dot\alpha}+\frac{5}{2}\lambda_{\mathbb{D}}\bar{\chi}^{\dot\alpha i}-\mathrm{i}\lambda_Y\bar{\chi}^{\dot\alpha i}\ , \qquad (5.2.10c)\\
\delta F &= -2\mathrm{i}\bar{\xi}^{i}\tilde{\sigma}^{a}\nabla_a\chi_i+(\bar{\xi}^{i}\tilde{\sigma}^{cd}\bar{\chi}_i)W^{-}_{cd}-6(\bar{\xi}^{i}\bar{\Sigma}^{j})G_{ij}+4\eta^{i}\chi_i\\
&\quad +3\lambda_{\mathbb{D}}F+2\mathrm{i}\lambda_Y F\ , \qquad (5.2.10d)\\
\delta\bar{F} &= -2\mathrm{i}\xi_i\sigma^{a}\nabla_a\bar{\chi}^{i}-(\xi_i\sigma^{cd}\chi^{i})W^{+}_{cd}+6(\xi^{i}\Sigma^{j})G_{ij}+4\bar{\eta}_i\bar{\chi}^{i}
\end{aligned}
$$

$$
\begin{aligned}
&+3\lambda_{\mathbb{D}}\bar{F}-2\mathrm{i}\lambda_Y\bar{F}\ , && (5.2.10e)\\
\delta\tilde{H}_a &= \frac{1}{2}\xi_i\sigma_{ab}\nabla^b\chi^i-\frac{\mathrm{i}}{16}(\xi_i\sigma_a\tilde{\sigma}^{cd}\bar{\chi}^i)W^-_{cd}-\frac{3\mathrm{i}}{8}(\xi_i\sigma_a\bar{\Sigma}_j)G^{ij}\\
&\quad -\frac{1}{2}\bar{\xi}^i\tilde{\sigma}_{ab}\nabla^b\bar{\chi}_i-\frac{\mathrm{i}}{16}(\bar{\xi}^i\tilde{\sigma}_a\sigma^{cd}\chi_i)W^+_{cd}-\frac{3\mathrm{i}}{8}(\bar{\xi}^i\tilde{\sigma}_a\Sigma^j)G_{ij}\\
&\quad +\lambda_a{}^b\tilde{H}_b+3\lambda_{\mathbb{D}}\tilde{H}_a-\frac{3\mathrm{i}}{4}\eta^i\sigma_a\bar{\chi}_i+\frac{3\mathrm{i}}{4}\bar{\eta}_i\tilde{\sigma}_a\chi^i\ , && (5.2.10f)
\end{aligned}
$$

where

$$
\begin{aligned}
\nabla_a G_{ij} &= \mathcal{D}_a G_{ij}-\psi_{a(i}\chi_{j)}-\bar{\psi}_{a(i}\bar{\chi}_{j)}\ , && (5.2.11a)\\
\nabla_a\chi_{\alpha i} &= \mathcal{D}_a\chi_{\alpha i}+\frac{1}{2}\psi_{a\alpha i}F+2\mathrm{i}(\sigma^b\bar{\psi}_{ai})_\alpha\tilde{H}_b-\frac{\mathrm{i}}{2}(\sigma^b\bar{\psi}_a{}^j)_\alpha\nabla_b G_{ij}-2\phi_{a\alpha}^{\ j}G_{ij}\ , && (5.2.11b)\\
\nabla_a\bar{\chi}^{\dot{\alpha}i} &= \mathcal{D}_a\bar{\chi}^{\dot{\alpha}i}+\frac{1}{2}\bar{\psi}_a{}^{\dot{\alpha}i}\bar{F}-2\mathrm{i}(\tilde{\sigma}^b\psi_a{}^i)_\alpha\tilde{H}_b-\frac{\mathrm{i}}{2}(\tilde{\sigma}^b\psi_{aj})^{\dot{\alpha}}\nabla_b G^{ij}-2\bar{\phi}_{a\,j}^{\ \dot{\alpha}}G^{ij}\ . && (5.2.11c)
\end{aligned}
$$

The covariant conservation equation for $\tilde{H}_a$ is

$$
\nabla^a\tilde{H}_a = \frac{3}{8}\Sigma^i\chi_i+\frac{3}{8}\bar{\Sigma}_i\bar{\chi}^i\ . \qquad (5.2.12)
$$

The constraint implies the existence of a gauge two-form potential, $b_{mn}=-b_{nm}$, and its exterior derivative $h_{mnp}:=3\partial_{[m}b_{np]}$. The solution of (5.2.12) is

$$
\tilde{H}_a=\frac{1}{6}\varepsilon_a{}^{bcd}\Big(h_{bcd}-\frac{3\mathrm{i}}{4}\psi_{bi}\sigma_{cd}\chi^i-\frac{3\mathrm{i}}{4}\bar{\psi}_b{}^i\tilde{\sigma}_{cd}\bar{\chi}_i-\frac{3}{4}(\psi_b{}^i\sigma_c\bar{\psi}_d{}^j)G_{ij}\Big)\ , \qquad (5.2.13)
$$

where $h_{abc}=e_a{}^m e_b{}^n e_c{}^p h_{mnp}$. The locally superconformal transformations of b_{mn} are

$$
\delta b_{mn} = \frac{\mathrm{i}}{2}\xi_i\sigma_{mn}\chi^i+\frac{\mathrm{i}}{2}\bar{\xi}^i\tilde{\sigma}_{mn}\bar{\chi}_i+\frac{1}{2}\Big(\bar{\psi}_{[m}{}^i\sigma_{n]}\xi^j-\psi_{[m}{}^i\sigma_{n]}\bar{\xi}^j\Big)G_{ij}+2\partial_{[m}l_{n]}\ , \qquad (5.2.14)
$$

where we have also included the vector gauge transformation $\delta_l b_{mn}=2\partial_{[m}l_{n]}$ that leaves h_{mnp} and $\tilde{H}^a$ invariant. For convenience, we have summarised the dilatation and chiral weights of the fields of the linear multiplet in Table 5.1.

	G_{ij}	$\chi_{\alpha i}$	$\bar{\chi}^{\dot{\alpha}i}$	F	$\bar{F}$	$\tilde{H}^a$	b_{mn}
$\mathbb{D}$	2	5/2	5/2	3	3	3	0
Y	0	1	-1	2	-2	0	0

Table 5.1: Summary of the dilatation and chiral weights in the off-shell linear multiplet.

Now that we have reviewed the structure of a locally superconformal linear multiplet, a straightforward analysis shows that, assuming $q^{i\underline{i}}$ and $(\rho_\alpha^{\underline{i}},\bar{\rho}^{\dot{\alpha}}_{\underline{i}})$ describe an on-shell hypermultiplet in a standard Weyl multiplet background with transformation rules (5.2.3), the following composite fields define a triplet of linear multiplets [141]

$$
\begin{aligned}
G_{ij}{}^{\underline{ij}} &= q_{(i}{}^{\underline{i}}q_{j)}{}^{\underline{j}}=q_i{}^{(\underline{i}}q_j{}^{\underline{j})}\ , && (G_{ij}{}^{\underline{ij}})^*=G^{ij}{}_{\underline{ij}}\ , && (5.2.15a)\\
\chi_{\alpha i}{}^{\underline{ij}} &= \frac{1}{2}q_i{}^{(\underline{i}}\rho_\alpha^{\underline{j})}\ ,\quad \bar{\chi}^{\dot{\alpha}i}{}_{\underline{ij}}=-\frac{1}{2}q^i{}_{(\underline{i}}\bar{\rho}^{\dot{\alpha}}_{\underline{j})}\ , && (\chi_{\alpha i}{}^{\underline{ij}})^*=\bar{\chi}^i_{\dot{\alpha}\underline{ij}}\ , && (5.2.15b)\\
F^{\underline{ij}} &= \frac{1}{8}\rho^{(\underline{i}}\rho^{\underline{j})}\ ,\quad \bar{F}_{\underline{ij}}=\frac{1}{8}\bar{\rho}_{(\underline{i}}\bar{\rho}_{\underline{j})}\ , && (F^{\underline{ij}})^*=\bar{F}_{\underline{ij}}\ , && (5.2.15c)
\end{aligned}
$$

$$\tilde{H}^{a\underline{ij}} = -\frac{1}{4}q^{i(\underline{i}}\nabla^a q_i{}^{\underline{j})} + \frac{\mathrm{i}}{32}\rho^{(\underline{i}}\sigma^a\bar{\rho}^{\underline{j})} , \qquad (\tilde{H}^{a\underline{ij}})^* = \tilde{H}^a{}_{\underline{ij}} . \tag{5.2.15d}$$

These fields all transform according to (5.2.10) and each of the previous fields is symmetric in $\underline{i}$ and $\underline{j}$. Within the previous composite fields, the field $\tilde{H}^{a\underline{ij}}$ is particularly interesting. In fact, equation (5.2.15d) together with (5.2.13) represent the solution to the constraint (5.2.9) and can be used to express the $SU(2)_R$ connection $\phi_m{}^{ij}$ as a composite field. By introducing the derivative

$$\mathbf{D}_a = e_a{}^m\left(\partial_m - \frac{1}{2}\omega_m{}^{cd}M_{cd} - \mathrm{i}A_m Y - b_m\mathbb{D}\right) = \mathcal{D}_a + e_a{}^m\phi_m{}^{ij}J_{ij} , \tag{5.2.16}$$

and by using (5.2.4), eq. (5.2.15d) can be rearranged for the SU(2)$_R$ gauge connection as follows

$$\phi_a{}^{ij} = 4q^{-4}q^{(i}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\left[q^{k\underline{i}}\mathbf{D}_a q_k{}^{\underline{j}} - \frac{1}{4}q^{k\underline{i}}(\psi_{ak}\rho^{\underline{j}}) + \frac{1}{4}q^{k\underline{i}}(\bar{\psi}_{ak}\bar{\rho}^{\underline{j}}) - \frac{\mathrm{i}}{8}\rho^{\underline{i}}\sigma_a\bar{\rho}^{\underline{j}} + 4\tilde{H}_a{}^{\underline{ij}}\right] , \tag{5.2.17}$$

where $\tilde{H}_a{}^{\underline{ij}}$ is given by (5.2.13).

This concludes the definition of the hyper-dilaton Weyl multiplet. The final result of our analysis is that we have identified a new representation of the off-shell local $4D$, $N = 2$ superconformal algebra in terms of the following independent fields: $e_m{}^a$, b_m, A_m, W_{ab}, $q^{i\underline{i}}$, $b_{mn}{}^{\underline{ij}}$, $(\psi_{mi}, \bar{\psi}_m{}^i)$, and $(\rho^{\underline{i}}, \bar{\rho}_{\underline{i}})$. The multiplet has precisely the same number of off-shell degrees of freedom as the standard Weyl multiplet, $24+24$. Table 5.2 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields. Note that with the ingredients provided so far, it is a straightforward exercise

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_{ma}$	$\phi_m{}^{ij}$	A_m	ψ_{mi}	$\phi_m{}^i$	W_{ab}	$\rho^{\underline{i}}$	$q^{i\underline{i}}$	$b_{mn}{}^{\underline{ij}}$
$16B$	0	$4B$	0	0	$4B$	$32F$	0	$6B$	$8F$	$4B$	$18B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Y	Q	S				$\lambda_m{}^{\underline{ij}}$-sym
$-4B$	$-6B$	$-1B$	$-4B$	$-3B$	$-1B$	$-8F$	$-8F$				$-9B$
Result: $24+24$ degrees of freedom											

Table 5.2: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components of these fields – composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that the parameter $\lambda_m{}^{\underline{ij}}$ describes the vector symmetry associated with the gauge two-forms $b_{mn}{}^{\underline{ij}}$ with field strength three-forms $h_{mnp}{}^{\underline{ij}}$ and $\tilde{H}^{a\underline{ij}}$. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. Row five gives the resulting number of degrees of freedom.

to obtain the locally superconformal transformations of the fundamental fields of the hyper-dilaton Weyl multiplet written only in terms of fundamental fields. These are given by (2.2.24a)–(2.2.24c), (2.2.24e)–(2.2.24g), (5.2.14), and (5.2.3a)–(5.2.3c) after using the appropriate identities for all the composite fields $\omega_m{}^{cd}$, $\mathfrak{f}_{ma}$, $\phi_m{}^{ij}$, $(\phi_{mi}, \bar{\phi}_m{}^i)$, $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha}i})$, and D respectively given by eqs. (2.2.15), (5.2.17), (2.2.16), and (5.2.8).

It is important to underline that the local gauge transformations of the hyper-dilaton Weyl multiplet form an algebra that closes off-shell on a local extension of SU$(2,2|2)$. In fact, by construction the resulting algebra is identical to the one of the standard Weyl multiplet transformations (2.2.2) (see [137] and [26, 41] for detail on the local algebra), with the only important subtlety being that the structure functions will have more composite fields.

5.3 Gauge fixing and Müller's Poincaré supergravity

As explained in the introduction, one of the motivations of our analysis was to show that the $32+32$ off-shell multiplet of $4D$, $N=2$ Poincaré supergravity constructed by Müller in [241] could be derived by superconformal techniques starting from the hyper-dilaton Weyl multiplet. In this section we explain how this goes. We first focus on the structure of the multiplet and then explain how to construct the Poincaré supergravity action derived in [241]. At the end of this section we also extend the results of [241] by adding a new BF-coupling which induces a scalar potential for the dilaton without a standard R-symmetry gauging.

5.3.1 Hyper-Dilaton Poincaré supergravity multiplet

To recover a multiplet of Poincaré supergravity, compensating multiplets must be coupled to the off-shell conformal supergravity multiplet to fix some of the local superconformal symmetries — see [37, 38] for reviews. Below we will describe how to recover the multiplet described in [241] which we denote as the hyper-dilaton Poincaré multiplet. The construction is straightforward. We simply need to couple the hyper-dilaton Weyl multiplet to a single off-shell vector multiplet compensator and then appropriately gauge fix to eliminate all symmetries except local supersymmetry, Lorentz, and the vector gauge symmetry of the gauge two-forms $b_{mn}{}^{\underline{ij}}$.

It is straightforward to define an off-shell $4D$, $N=2$ Abelian vector multiplet in a hyper-dilaton Weyl multiplet background. As a first step consider an Abelian vector multiplet [242, 243] in a standard Weyl multiplet background [42, 137, 141, 231]. This is described by a complex scalar field ϕ and its conjugate $\bar\phi=(\phi)^*$, gaugini $(\lambda_\alpha^i,\bar\lambda_i^{\dot\alpha})$ such that $(\lambda_\alpha^i)^*=\bar\lambda_{\dot\alpha i}$, a triplet of auxiliary fields $X^{ij}=X^{ji}$ satisfying the reality condition $(X^{ij})^*=X_{ij}$, and a real Abelian gauge connection v_m or, equivalently, its covariant real field strength F_{ab} given by

$$F_{ab} \;=\; e_a{}^m e_b{}^n f_{mn}-\frac{\mathrm{i}}{2}\psi_{[ak}\sigma_{b]}\bar\lambda^k+\frac{\mathrm{i}}{2}\bar\psi_{[a}{}^k\tilde\sigma_{b]}\lambda_k-\frac{1}{2}(\psi_{ak}\psi_b{}^k)\bar\phi+\frac{1}{2}(\bar\psi_a{}^k\bar\psi_{bk})\phi\ , \tag{5.3.1}$$

where $f_{mn}=2\partial_{[m}v_{n]}$. By construction F_{ab} satisfies the Bianchi identity

$$\nabla_{[a}F_{bc]}=-\frac{\mathrm{i}}{2}R(Q)_{[ab\,j}\sigma_{c]}\bar\lambda^j+\frac{\mathrm{i}}{2}R(\bar Q)_{[ab}{}^j\tilde\sigma_{c]}\lambda_j\ , \tag{5.3.2}$$

that is solved by (5.3.1). The non-trivial dilatation and chiral weights of the vector multiplet fields are summarised in Table 5.3.

	ϕ	$\bar\phi$	λ_α^i	$\bar\lambda_i^{\dot\alpha}$	X^{ij}	F_{ab}	v_m
$\mathbb{D}$	1	1	3/2	3/2	2	2	0
Y	-2	2	-1	1	0	0	0

Table 5.3: Summary of the dilatation and chiral weights in the off-shell Abelian vector multiplet.

The transformation rules of the vector multiplet fields in a standard Weyl multiplet background are

$$\delta\phi \;=\; \xi_i\lambda^i+\lambda_{\mathbb{D}}\phi-2\mathrm{i}\lambda_Y\phi\ , \tag{5.3.3a}$$

$$\begin{aligned}
\delta\bar{\phi} &= \bar{\xi}^i\bar{\lambda}_i + \lambda_{\mathbb{D}}\bar{\phi} + 2\mathrm{i}\lambda_Y\bar{\phi}\ , && (5.3.3\text{b})\\
\delta\lambda_\alpha^i &= 2(\sigma^{ab}\xi^i)_\alpha F_{ab} + (\sigma^{ab}\xi^i)_\alpha W_{ab}^+\bar{\phi} - \frac{1}{2}\xi_{\alpha j}X^{ij} + 2\mathrm{i}(\sigma^a\bar{\xi}^i)_\alpha\nabla_a\phi \\
&\quad + \frac{1}{2}\lambda^{ab}(\sigma_{ab}\lambda^i)_\alpha + \lambda^i{}_j\lambda_\alpha^j + \frac{3}{2}\lambda_{\mathbb{D}}\lambda_\alpha^i - \mathrm{i}\lambda_Y\lambda_\alpha^i + 4\eta_\alpha^i\phi\ , && (5.3.3\text{c})\\
\delta\bar{\lambda}_i^{\dot\alpha} &= -2(\tilde\sigma^{ab}\bar{\xi}_i)^{\dot\alpha}F_{ab} - (\tilde\sigma^{ab}\bar{\xi}_i)^{\dot\alpha}W_{ab}^-\phi - \frac{1}{2}\bar{\xi}^{\dot\alpha j}X_{ij} + 2\mathrm{i}(\tilde\sigma^a\xi_i)^{\dot\alpha}\nabla_a\bar{\phi} \\
&\quad + \frac{1}{2}\lambda^{ab}(\tilde\sigma_{ab}\bar{\lambda}_i)^{\dot\alpha} - \lambda_i{}^j\bar{\lambda}_j^{\dot\alpha} + \frac{3}{2}\lambda_{\mathbb{D}}\bar{\lambda}_i^{\dot\alpha} + \mathrm{i}\lambda_Y\bar{\lambda}_i^{\dot\alpha} + 4\bar{\eta}_i^{\dot\alpha}\bar{\phi}\ , && (5.3.3\text{d})\\
\delta X^{ij} &= -4\mathrm{i}\xi^{(i}\sigma^a\nabla_a\bar{\lambda}^{j)} - 4\mathrm{i}\bar{\xi}^{(i}\tilde\sigma^a\nabla_a\lambda^{j)} + 2\lambda^{(i}{}_k X^{j)k} + 2\lambda_{\mathbb{D}}X^{ij}\ , && (5.3.3\text{e})\\
\delta F_{ab} &= \left[-\mathrm{i}\xi_k\sigma_{[a}\nabla_{b]}\bar{\lambda}^k + 2\big(\xi_k R(Q)_{ab}{}^k\big)\bar{\phi} - \frac{1}{2}(\xi_k\lambda^k)W_{ab}^- + 2\eta^k\sigma_{ab}\lambda_k + \text{c.c.}\right] \\
&\quad + 2\lambda_{\mathbb{D}}F_{ab} - 2\lambda_{[a}{}^c F_{b]c}\ , && (5.3.3\text{f})\\
\delta v_m &= (\xi_k\psi_m{}^k)\bar{\phi} - (\bar{\xi}^k\bar{\psi}_{mk})\phi + \partial_m\lambda_V\ , && (5.3.3\text{g})
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a\phi &= \mathcal{D}_a\phi - \frac{1}{2}\psi_{ai}\lambda^i\ , && (5.3.4\text{a})\\
\nabla_a\bar{\phi} &= \mathcal{D}_a\bar{\phi} - \frac{1}{2}\bar{\psi}_a{}^i\bar{\lambda}_i\ , && (5.3.4\text{b})\\
\nabla_a\lambda_\alpha^i &= \mathcal{D}_a\lambda_\alpha^i - (\sigma^{cd}\psi_a{}^i)_\alpha\Big(F_{cd}^+ + \frac{1}{2}W_{cd}^+\bar{\phi}\Big) + \frac{1}{4}\psi_{a\alpha j}X^{ij} \\
&\quad - \mathrm{i}(\sigma^b\bar{\psi}_a{}^i)_\alpha\nabla_b\phi - 2\phi_a{}_\alpha^i\phi\ , && (5.3.4\text{c})\\
\nabla_a\bar{\lambda}_i^{\dot\alpha} &= \mathcal{D}_a\bar{\lambda}_i^{\dot\alpha} + (\tilde\sigma^{cd}\bar{\psi}_{ai})^{\dot\alpha}\Big(F_{cd}^- + \frac{1}{2}W_{cd}^-\phi\Big) + \frac{1}{4}\bar{\psi}_a{}^{\dot\alpha j}X_{ij} \\
&\quad - \mathrm{i}(\tilde\sigma^b\psi_{ai})^{\dot\alpha}\nabla_b\bar{\phi} - 2\bar{\phi}_a{}_i^{\dot\alpha}\bar{\phi}\ , && (5.3.4\text{d})
\end{aligned}$$

and we have also included in (5.3.3g) the gauge field transformation parametrised by the local real parameter λ_V. The transformations of the vector multiplet in a hyper-dilaton Weyl multiplet background are precisely the same with the only subtlety that one has to interpret several standard Weyl multiplet fields as composite of $q^{i\underline{i}}$, $(\rho_\alpha^{\underline{i}}, \bar{\rho}_{\underline{i}}^{\dot\alpha})$, and $b_{mn}{}^{\underline{ij}}$.

The compensating vector multiplet contains $8+8$ off-shell degrees of freedom. Once added to the hyper-dilaton Weyl multiplet we obtain the right number of off-shell degrees of freedom, $32+32$, of the hyper-dilaton Poincaré multiplet [241] but in a manifestly superconformal setting. We can then obtain the structure of the Poincaré multiplet, including its local transformation rules, after gauge fixing.

The first set of gauge fixing conditions are

$$\phi = 1\ , \quad \bar{\phi} = 1\ , \tag{5.3.5a}$$

$$b_m = 0\ . \tag{5.3.5b}$$

The condition (5.3.5a) fixes dilatation and $\mathrm{U}(1)_R$ symmetries, while (5.3.5b) fixes special conformal K^a symmetry. Next we impose

$$\lambda_\alpha^i = 0\ , \quad \bar{\lambda}_i^{\dot\alpha} = 0\ , \tag{5.3.5c}$$

which gauge fixes S-supersymmetry. A characterising feature of the hyper-dilaton Weyl multiplet is that it contains an SU(2)$_R$ compensator, the $q^{i\underline{i}}$ fields. As a last gauge fixing condition we then impose

$$q^{i\underline{i}} = -\varepsilon^{i\underline{i}}\mathrm{e}^{-U} \iff q^{i}{}_{\underline{i}} = \delta^{i}_{\underline{i}}\mathrm{e}^{-U} \iff q_{i}{}^{\underline{i}} = -\delta^{\underline{i}}_{i}\mathrm{e}^{-U} \iff q_{i\underline{i}} = \varepsilon_{i\underline{i}}\mathrm{e}^{-U} \ , \tag{5.3.5d}$$

which breaks SU(2)$_R$. After imposing the previous gauge fixing conditions, the remaining fundamental fields in the multiplet match those of the hyper-dilaton Poincaré supergravity multiplet [241] as summarised in table 5.4. The fundamental fields are the vielbein $e_m{}^a$, the gravitini $(\psi_m{}_i^\alpha, \bar\psi_m{}^i_{\dot\alpha})$, a real

$e_m{}^a$	$\omega_m{}^{cd}$	A_m	$(\psi_m{}_i^\alpha, \bar\psi_m{}^i_{\dot\alpha})$	W_{ab}	$(\rho^{\underline{i}}_\alpha, \bar\rho_{\underline{i}}^{\dot\alpha})$	U	$b_{mn}{}^{\underline{ij}}$	X^{ij}	v_m
$16B$	0	$4B$	$32F$	$6B$	$8F$	$1B$	$18B$	$3B$	$4B$
P_a	M_{ab}		Q				$(\lambda_m{}^{\underline{ij}})$		(λ_V)
$-4B$	$-6B$		$-8F$				$-9B$		$-1B$
Result: $32+32$ degrees of freedom									

Table 5.4: Hyper-Dilaton Poincaré multiplet. Row 1 gives all fields in the multiplet. Row two gives the number of independent components of these fields. Row three gives the surviving gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. Row five gives the resulting degrees of freedom. The parameter $\lambda_m{}^{\underline{ij}}$ describes the vector symmetry associated with the triplet of gauge two-form $b_{mn}{}^{\underline{ij}}$. The gauge parameter λ_V describes the scalar symmetry of v_m.

vector field A_m, a real antisymmetric tensor W_{ab}, a real scalar field that plays the role of a dilaton U, a real triplet of scalar fields X^{ij}, a triplet of gauge two forms $b_{mn}{}^{\underline{ij}}$, a gauge field v_m that plays the role of the graviphoton, and spinor fields $(\rho^{\underline{i}}_\alpha, \bar\rho_{\underline{i}}^{\dot\alpha})$. The residual gauge transformations of the multiplet are described by covariant general coordinate transformations (ξ^a), local Lorentz transformations (λ_{ab}), local supersymmetry ($\xi_i^\alpha, \bar\xi^i_{\dot\alpha}$), and Abelian scalar ($\lambda_V$) and vector ($\lambda_m{}^{\underline{ij}}$) gauge transformations. Note that we have kept the distinction of SU(2)$_R$ and SU(2) flavour indices. However, thanks to the second gauge condition in (5.3.5d), after gauge fixing the two indices can be identified.

The transformation rules of the resulting Poincaré supergravity multiplet [241] are those that preserve the previous gauge conditions (5.3.5). To preserve the gauge condition (5.3.5a) we need to impose $\lambda_{\mathbb{D}} \equiv 0$ and $\lambda_Y \equiv 0$. Since Q-supersymmetry do not preserve the gauge, it is necessary to accompany these transformations with appropriate S-supersymmetry, special conformal, and SU(2)$_R$ compensating transformations. To preserve the gauge condition (5.3.5c), by examining the transformations (5.3.3c) and (5.3.3d), it is straightforward to show that any Q-supersymmetry transformation has to be accompanied by a compensating S-supersymmetry transformation with parameter

$$\eta^i_\alpha(\xi) = -\frac{1}{2}(\sigma^{cd}\xi^i)_\alpha\Big(F^+_{cd}+\frac{1}{2}W^+_{cd}\Big)+\frac{1}{8}\xi_{\alpha j}X^{ji}+(\sigma^a\bar\xi^i)_\alpha A_a \ , \tag{5.3.6a}$$

$$\bar\eta_i^{\dot\alpha}(\xi) = \frac{1}{2}(\tilde\sigma^{cd}\bar\xi_i)^{\dot\alpha}\Big(F^-_{cd}+\frac{1}{2}W^-_{cd}\Big)+\frac{1}{8}\bar\xi^{\dot\alpha j}X_{ji}-(\tilde\sigma^a\xi_i)^{\dot\alpha}A_a \ . \tag{5.3.6b}$$

A similar analysis shows that to preserve the gauge condition $b_m = 0$ one needs to enforce nontrivial compensating special conformal K-transformations with a parameter $\lambda^a(\xi)$. However, since all the other supergravity fields are conformal (not necessarily superconformal) primaries, not transforming under special conformal boosts, in practice we will never have to worry about inserting the compensating $\lambda^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations.

The last gauge fixing condition which is not preserved is (5.3.5d). It is straightforward to check that we can consistently have $\delta q^{(i\underline{i})} = 0$ by implementing in (5.2.3a) a compensating SU(2)$_R$ transformation with the following parameter

$$\lambda^{ij}(\xi) = -\mathrm{e}^U \left[\xi^{(i}\rho^{j)} - \bar{\xi}^{(i}\bar{\rho}^{j)}\right] , \tag{5.3.7}$$

where $\rho^i = \delta^i_{\underline{i}}\rho^{\underline{i}}$ and $\bar{\rho}_i = \delta_i^{\underline{i}}\bar{\rho}_{\underline{i}}$.

At this stage, one has all the ingredients to obtain the transformation rules of any matter multiplet in the gauge fixed, Poincaré supergravity frame, by appropriately implementing the previous compensating gauge parameters in the superconformal transformation rules. The resulting local transformation rules of the hyper-dilaton Poincaré multiplet form an algebra that closes off-shell on a local extension of the $N = 2$ super-Poincaré algebra with no residual R-symmetry. The structure of the algebra coincides, up to notation, with results in [241]. A detailed presentation of the hyper-dilaton Poincaré multiplet and its coupling to matter in the superconformal framework of this section will be given elsewhere as it is not necessary for the rest of our paper. It is worth underlining that, as explained by Müller in [241], the resulting $32+32$ multiplet describes an irreducible off-shell representation. This differs from the case of the standard $40+40$ multiplet of off-shell $N = 2$ Poincaré supergravity [137, 238–240] which, for example, can arise by coupling the standard Weyl multiplet to two compensators given by an off-shell vector and a hypermultiplet (the simplest of which is probably an off-shell linear multiplet).

5.3.2 Hyper-Dilaton Poincaré supergravity action

With the hyper-dilaton Poincaré's multiplet recovered using a superconformal approach, we can now describe how to obtain the Poincaré supergravity action first constructed in [241]. Once more, the construction is straightforward. In fact, the action derives from the kinetic action of the vector multiplet compensator in a hyper-dilaton Weyl multiplet background after imposing the gauge fixing conditions (5.3.5). We will describe this construction by focusing only on the bosonic fields.

As a starting point we consider the bosonic sector of the chiral density formula for a system of vector multiplets possessing scalar fields ϕ^I with prepotential $\mathcal{F}(\phi^I)$ in a standard Weyl multiplet background [41, 42, 141, 231]. This supersymmetric invariant has the following bosonic Lagrangian

$$\begin{aligned} e^{-1}\,\mathcal{L}|_{bosonic} \;=\; & \mathcal{F}_I \Box \bar{\phi}^I + 3\mathcal{F}_I\bar{\phi}^I D + \frac{1}{32}\mathcal{F}_{IJ}X^{Iij}X^J_{ij} - \mathcal{F}_{IJ}f^{+Iab}f^{+J}_{ab} - \frac{1}{2}\mathcal{F}W^{-ab}W^-_{ab} \\ & -\mathcal{F}_I W^{-ab}f^{-I}_{ab} - \mathcal{F}_{IJ}\bar{\phi}^I W^{+ab}f^{+J}_{ab} - \frac{1}{4}\mathcal{F}_{IJ}\bar{\phi}^I\bar{\phi}^J W^{+ab}W^+_{ab} + \text{c.c.}\,, \end{aligned} \tag{5.3.8}$$

where $\mathcal{F}_I = \frac{\partial \mathcal{F}(\phi)}{\partial \phi^I}$ and $\mathcal{F}_{IJ} = \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^I \partial \phi^J}$. We refer the reader to equation (3.30) of [41] for a derivation of the previous Lagrangian. In our case, we have only one vector multiplet and the only possible function $\mathcal{F}$ we can choose which leads to a locally superconformal invariant is given by

$$\mathcal{F}(\phi) = -\frac{1}{4}\phi^2 \,. \tag{5.3.9}$$

Here the overall factor is chosen for later convenience. Once we insert (5.3.9) into (5.3.8) and take into consideration that we are working with a hyper-dilaton Weyl multiplet rather than a standard Weyl multiplet (meaning that (5.2.8c) has to be used), we obtain the following

$$e^{-1}\mathcal{L}|_{bosonic} = -\frac{1}{4}|\phi|^2R - \frac{|\phi|^2}{q^2}q_{\underline{ii}}\mathcal{D}^a\mathcal{D}_aq^{\underline{ii}} - \frac{1}{2}\phi\mathcal{D}^a\mathcal{D}_a\bar{\phi} - \frac{1}{64}X^{ij}X_{ij} + \frac{1}{2}f^{+ab}f^+_{ab}$$
$$+\frac{1}{2}\phi W^{-ab}f^-_{ab} + \frac{1}{2}\bar{\phi}W^{+ab}f^+_{ab} + \frac{1}{8}\phi^2W^{-ab}W^-_{ab} + \frac{1}{8}\bar{\phi}^2W^{+ab}W^+_{ab} + \text{c.c.}\,. \quad (5.3.10)$$

Note that in the previous Lagrangian there is a dependence upon the triplet of gauge two-forms $b_{mn}{}^{\underline{ij}}$ which is still hidden in the $\mathrm{SU}(2)_R$ connection inside the $\mathcal{D}_a$ derivatives, see eq. (5.2.17).

The final step to obtain the bosonic sector of the Poincaré supergravity of [241] is to impose the gauge fixing conditions (5.3.5). Upon implementing these conditions, the resulting Poincaré supergravity Lagrangian turns out to be

$$e^{-1}\mathcal{L}|_{bosonic} = -\frac{1}{2}R + \frac{1}{2}f^{ab}f_{ab} + W^{ab}f_{ab} + \frac{1}{4}W^{ab}W_{ab} - 2(\partial_mU)\partial^mU$$
$$+16\,\mathrm{e}^{4U}\tilde{h}^a{}_{kl}\tilde{h}_a{}^{kl} - \frac{1}{32}X^{ij}X_{ij} + 4A^aA_a\,. \quad (5.3.11)$$

Note that here $b_{mn}{}^{kl} = \delta^k_{\underline{k}}\delta^l_{\underline{l}}b_{mn}{}^{\underline{kl}}$ and $\tilde{h}_a{}^{kl} = \delta^k_{\underline{k}}\delta^l_{\underline{l}}\tilde{h}_a{}^{\underline{kl}}$ since we have stopped distinguishing between underlined and non-underlined SU(2) indices after gauge fixing.

The structure of our and Müller's Lagrangian in [241] coincide up to change of notation. It is a straightforward exercise to derive the fermionic extension of the previous Lagrangian. This result will be presented elsewhere together with a discussion of more general supergravity-matter couplings based on the hyper-dilaton Weyl multiplet and the associated hyper-dilaton Poincaré supergravity.

To conclude let us analyse the on-shell structure of (5.3.11). It is clear that W_{ab}, X^{ij}, and A_a are auxiliary fields that can be algebraically integrated out by using the equations of motion

$$W_{ab} = -2f_{ab}\,, \quad X^{ij} = 0\,, \quad A_a = 0\,. \quad (5.3.12)$$

Once the previous equations are used in (5.3.11), one obtains the on-shell Lagrangian

$$e^{-1}\mathcal{L}|_{bosonic} = -\frac{1}{2}R - \frac{1}{2}f^{mn}f_{mn} - 2(\partial_mU)\partial^mU + 16\,\mathrm{e}^{4U}\tilde{h}^m{}_{kl}\tilde{h}_m{}^{kl}\,. \quad (5.3.13)$$

The first two terms describe the standard kinetic terms for minimal on-shell $N = 2$ Poincaré supergravity with a dynamical graviton and graviphoton. The last two terms describe a dilaton and a triplet of dynamical gauge two-forms which are not part of the minimal on-shell $N = 2$ Poincaré supergravity multiplet. In fact, these fields describe the bosonic sector of an on-shell hypermultiplet where three of the scalars have been dualised into real gauge two-forms [241]. The same holds by including the fermionic sector.

5.3.3 *BF*-coupling and dilaton potential

To conclude this section we consider an extension of the original result of Müller from [241] and show how to construct by using superconformal techniques a new off-shell supersymmetric invariant that, e. g., leads to a non-trivial scalar potential for the dilaton.

Given a vector multiplet and a linear multiplet, we consider the local supersymmetric extension of a BF-action in a standard Weyl multiplet background [138]. We refer the reader to [41] for a derivation of the locally superconformal invariant, including fermionic terms, in the notation used in our paper. The bosonic part of such an invariant is

$$
\begin{aligned}
e^{-1}\mathcal{L}_{BF}|_{bosonic} &= F\phi + \bar{F}\bar{\phi} + \frac{1}{4}G_{ij}X^{ij} - 2\varepsilon^{mnpq}b_{mn}f_{pq}\ , &\text{(5.3.14a)}\\
&= F\phi + \bar{F}\bar{\phi} + \frac{1}{4}G_{ij}X^{ij} - 8\tilde{h}^{m}v_{m}\ . &\text{(5.3.14b)}
\end{aligned}
$$

By construction, the supersymmetric BF-action is also well defined as an invariant in a hyper-dilaton Weyl background. We can readily construct an invariant of this form by considering the off-shell vector multiplet compensator used in this section and an off-shell linear multiplet given by

$$
\begin{aligned}
&G_{\xi\, ij} := \xi_{\underline{ij}}G_{ij}{}^{\underline{ij}}\ ,\quad \chi_{\xi\,\alpha i} := \xi_{\underline{ij}}\chi_{\alpha i}{}^{\underline{ij}}\ ,\quad \bar{\chi}_{\xi}{}^{\dot{\alpha} i} = \xi^{\underline{ij}}\bar{\chi}^{\dot{\alpha} i}{}_{\underline{ij}}\ , &\text{(5.3.15a)}\\
&F_{\xi} := \xi_{\underline{ij}}F^{\underline{ij}}\ ,\quad \bar{F}_{\xi} = \xi^{\underline{ij}}\bar{F}_{\underline{ij}}\ ,\quad b_{\xi\, mn} := \xi_{\underline{ij}}b_{mn}{}^{\underline{ij}}\ ,\quad \tilde{H}_{\xi}{}^{a} = \xi_{\underline{ij}}\tilde{H}^{a\underline{ij}}\ . &\text{(5.3.15b)}
\end{aligned}
$$

Here $G_{ij}{}^{\underline{ij}}$, $\chi_{\alpha i}{}^{\underline{ij}}$, $\bar{\chi}^{\dot{\alpha} i}{}_{\underline{ij}}$, $F^{\underline{ij}}$, $\bar{F}_{\underline{ij}}$, $b_{mn}{}^{\underline{ij}}$, and $\tilde{H}^{a\underline{ij}}$ are fields of the composite triplet of linear multiplets (5.2.15) constructed in terms of fundamental fields of the hyper-dilaton Weyl multiplet, while

$$
\xi_{\underline{ij}} = \xi_{\underline{ji}}\ ,\quad (\xi_{\underline{ij}})^{*} = \xi^{\underline{ij}}\ , \qquad \text{(5.3.16)}
$$

is a real triplet of (structure group invariant) constants. The bosonic part of the resulting Lagrangian is

$$
e^{-1}\mathcal{L}_{\xi}|_{bosonic} = \xi_{\underline{ij}}\Big(\frac{1}{4}q_{i}{}^{\underline{i}}q_{j}{}^{\underline{j}}X^{ij} - 2\varepsilon^{mnpq}b_{mn}{}^{\underline{ij}}f_{pq}\Big) = \xi_{\underline{ij}}\Big(\frac{1}{4}q_{i}{}^{\underline{i}}q_{j}{}^{\underline{j}}X^{ij} - 8\tilde{h}^{m\underline{ij}}v_{m}\Big)\ . \qquad \text{(5.3.17)}
$$

After imposing the gauge fixing conditions (5.3.5), and adding the previous term into (5.3.11), we obtain the following Lagrangian

$$
\begin{aligned}
e^{-1}\,\mathcal{L}|_{bosonic} = {}& -\frac{1}{2}R + \frac{1}{2}f^{ab}f_{ab} + W^{ab}f_{ab} + \frac{1}{4}W^{ab}W_{ab} - 2(\partial_{m}U)\partial^{m}U + 4A^{a}A_{a}\\
&+16\,\mathrm{e}^{4U}\tilde{h}^{a}{}_{kl}\tilde{h}_{a}{}^{kl} - 2\xi_{ij}\,\varepsilon^{mnpq}b_{mn}{}^{ij}f_{pq} - \frac{1}{32}X^{ij}X_{ij} + \frac{1}{4}\xi_{ij}\,\mathrm{e}^{-2U}X^{ij}\ , \qquad \text{(5.3.18)}
\end{aligned}
$$

where, after gauge fixing, we have used $\xi_{ij} = \delta_{i}^{\underline{i}}\delta_{j}^{\underline{j}}\xi_{\underline{ij}}$ and $b_{mn}{}^{ij} = \delta^{i}_{\underline{i}}\delta^{j}_{\underline{j}}b_{mn}{}^{\underline{ij}}$. As for the undeformed Lagrangian (5.3.11), W_{ab}, X^{ij}, and A_{a} are auxiliary fields that can be algebraically integrated out. With the ξ-deformation turned on, the equations of motion obtained from (5.3.18) are

$$
W_{ab} = -2f_{ab}\ ,\quad X^{ij} = -4\,\xi^{ij}\mathrm{e}^{-2U}\ ,\quad A_{a} = 0\ . \qquad \text{(5.3.19)}
$$

Once these equations are used in (5.3.18), we obtain the on-shell Lagrangian

$$
\begin{aligned}
e^{-1}\,\mathcal{L}|_{bosonic} = {}& -\frac{1}{2}R - \frac{1}{2}f^{mn}f_{mn} - 2(\partial_{m}U)\partial^{m}U + 16\,\mathrm{e}^{4U}\tilde{h}^{m}{}_{kl}\tilde{h}_{m}{}^{kl}\\
&+\xi^{2}\,\mathrm{e}^{-4U} + 2\xi_{ij}\,\varepsilon^{mnpq}b_{mn}{}^{ij}f_{pq}\ , \qquad \text{(5.3.20)}
\end{aligned}
$$

where

$$
\xi^{2} := \frac{1}{2}\xi^{ij}\xi_{ij} \geq 0\ . \qquad \text{(5.3.21)}
$$

The first line coincides with the on-shell hyper-dilaton Poincaré supergravity (5.3.13) containing the standard minimal on-shell $N = 2$ Poincaré supergravity coupled to a dilaton and a triplet of dynamical real gauge two-forms. Interestingly, the ξ-deformation induces a scalar potential for the dilaton together with a BF-coupling between the graviphoton and one of the three gauge two-forms of the hyper-dilaton Poincaré multiplet (the component $b_{\xi\, mn} = \xi_{\underline{ij}}\, b_{mn}{}^{\underline{ij}}$ parallel to the ξ_{ij} direction). We now conclude by commenting the results obtained in this subsection.

By considering a flat limit with $e_m{}^a \to \delta_m^a$, $b_{mn}{}^{\underline{ij}} \to 0$, $U \to 0$, and $q_i{}^{\underline{i}} \to \delta_i^{\underline{i}}$, and by keeping dynamical the vector multiplet with auxiliary field X^{ij}, the Lagrangian (5.3.17) turns into

$$\mathcal{L}_\xi^{flat}|_{bosonic} = \frac{1}{4}\xi_{ij}X^{ij} \ . \tag{5.3.22}$$

This is a standard (electric) Fayet–Iliopoulos (FI) term for an $N = 2$ vector multiplet [242, 244]. The invariant (5.3.17) can be considered as a curved extension of a FI term in a hyper-dilaton Weyl multiplet background. If one were to choose a different gauge fixing to Poincaré supergravity where $q^i{}_{\underline{i}} = \delta^i_{\underline{i}}$ (a condition that would lead to the same model but in a string frame), the dependence upon the dilaton would disappear from the term linear in X^{ij}. This straightforwardly shows that if one restricts to a sector with constant dilaton, the ξ-deformation leads to a negative cosmological constant $\Lambda = -\xi^2 \le 0$. Hence, the deformed model (5.3.20) admits an AdS_4 vacuum with constant negative curvature proportional to ξ^2.

Another interesting aspect to comment about is how the $SU(2)_R$ symmetry plays a sharply different role for the FI terms in off-shell $N = 2$ supergravity based on the standard Weyl multiplet compared to the hyper-dilaton Weyl multiplet case and the ξ-deformed Müller supergravity described above. When working with the standard Weyl multiplet (and even vector-dilaton Weyl multiplets), there is a close interplay between the $SU(2)_R$ symmetry, the gauging of isometries of scalar field manifolds, and the emergence of non-trivial scalar potentials. Within the superconformal tensor calculus, this was already noticed in early investigations of systems of Abelian vector multiplets [141, 231], and then extended to general hypermultiplet sigma-models [233, 234].[2] By working with the standard Weyl multiplet, the $SU(2)_R$ connection is an auxiliary field. In the presence of FI terms, its equations of motion identify the R-symmetries of the theory with the symmetries gauged by the $N = 2$ vector multiplets. This leads to non-trivial scalar potentials together with charges and masses for the gravitini — see [38, 250] for reviews. The simplest case is the one of the standard Weyl multiplet coupled to a single vector multiplet and a single hypermultiplet compensator. In this case, the scalar potential is a simple negative cosmological constant and an FI term identifies on-shell the R-symmetry connection with the graviphoton. In contrast, by using the hyper-dilaton Weyl multiplet, the $SU(2)_R$ connection is a composite field. After gauge fixing, on-shell the R-symmetry is completely broken, its connection is identified with the field strength of a dynamical gauge two-form and the ξ-deformation introduces a dynamical BF-coupling. As a result, one has an alternative procedure to obtain non-trivial scalar

[2] See also [245–249] for discussions concerning gauging and scalar potentials by using alternative on-shell supergravity approaches.

potentials compared to a setting based on gaugings in an $N = 2$ standard Weyl multiplet.[3] It will be worth exploring this mechanism for more general off-shell matter systems coupled to a hyper-dilaton Weyl multiplet, potentially including more physical hypermultiplets.

5.4 Conclusion and future directions

In our paper we have defined a new $24+24$ so-called hyper-dilaton Weyl multiplet of $N = 2$ conformal supergravity in four dimensions. The construction is based on reinterpreting the equations of motion for an on-shell hypermultiplet as constraints that render some of the fields of the standard Weyl multiplet composite. By coupling the hyper-dilaton Weyl multiplet to an off-shell vector multiplet compensator, we have obtained a minimal $32+32$ off-shell multiplet of $N = 2$ Poincaré supergravity that was constructed by Müller in [241] and then, by using superconformal techniques, we have shown how to reproduce the supergravity action of [241]. This contains the minimal on-shell $N = 2$ Poincaré supergravity coupled to a hypermultiplet where one of its physical scalars plays the role of a dilaton while its three other scalars are dualised to a triplet of real gauge two-forms. We have then described how a superconformal BF-coupling induces a scalar potential for the dilaton without a standard gauging. There are several future directions that our work is opening up. In the following we are going to mention a few.

As mentioned in the introduction, vector-dilaton Weyl multiplets were used in the past to study off-shell supergravity in five and six space-time dimensions, see [89, 90] for descriptions in terms of component fields and also [29, 31, 69, 126] for analyses in superspace. We are currently working towards extending our construction for hyper-dilaton Weyl multiplets in other $\mathrm{D} \leq 6$ space-time dimensions. Note also that much of the results obtained in our paper were obtained by using the conformal superspace approach to $N = 2$ conformal supergravity described in [26, 41]. We will present superspace analyses together with more detailed derivations of our results, and extensions to $\mathrm{D} \leq 6$ dimensions in the near future.

One of the main motivations of our work was to explore alternative, yet simple, off-shell engineering of non-trivial scalar potentials in $4D$, $N = 2$ supergravity. The results in subsection 5.3.3 are a first step in this direction. While we have only presented in this paper an off-shell Poincaré supergravity based on the hyper-dilaton Weyl multiplet coupled to a single off-shell vector multiplet compensator, a straightforward generalisation, part of a current work in progress, is to look at generic systems of (Abelian) vector multiplets. It is well known that for these systems, non-trivial scalar potentials in $4D$, $N = 2$ supergravity are associated to Fayet-Iliopoulos (FI) terms. These couplings are known to take two forms, either electric or magnetic FI terms. The electric and magnetic nomenclature arise from the role that extensions of electro-magnetic duality of Maxwell theory play in $4D$, $N = 2$ supersymmetry. In the case of global supersymmetry, electric and magnetic FI terms are well understood both on-shell and off-shell, see [93, 95, 252–256]. They play an important role in the description of spontaneous

[3]If one considers higher-derivative interactions, it is possible to construct very general scalar potentials without gauged R-symmetry by using a standard Weyl multiplet and new types of $N = 2$ FI terms, see [251].

full and partial breaking of supersymmetry. They are also key ingredients in supergravity descriptions of compactified string theories with fluxes and various patterns of supersymmetry breaking, see e. g., [257] and references therein. In supergravity, the off-shell description of $4D$, $N = 2$ magnetic FI terms (and magnetic gaugings) has not been developed in full generality yet, though they are expected to play an important role in engineering scalar potentials in supergravity models possessing vacua with both positive and negative cosmological constant – see for instance the recent discussion of magnetic $4D$, $N = 1$ FI terms [112]. The curved superspace constraints for off-shell magnetic FI terms were introduced in [258, 259] and in depth supergravity analyses in components (though not fully off-shell) were presented earlier in [260, 261]. By using a hyper-dilaton Weyl multiplet it is straightforward to engineer generic electric and magnetic FI-type terms by means of composite linear multiplets. We have already described how supergravity extensions of electric ξ-deformations can be obtained by using the BF-coupling (5.3.14) in terms of the composite linear multiplet (5.3.15). Off-shell magnetic FI-type deformations in a hyper-dilaton Weyl multiplet background can easily be engineered in terms of the same composite linear multiplet. This would, for example, appear as an imaginary deformation of the X^{ij}-auxiliary real field of a vector multiplet. Such deformations would be parametrised by the composite field $G_{\zeta ij} = \zeta_{\underline{ij}} q_i{}^{\underline{i}} q_j{}^{\underline{j}}$ with $\zeta_{\underline{ij}} = \zeta_{\underline{ji}}$, $(\zeta_{\underline{ij}})^* = \zeta^{\underline{ij}}$ constants that generalise the magnetic FI terms of global supersymmetry. Given a system of $n+1$ vector multiplets with scalar fields ϕ^I (with $I = 0, 1, \cdots, n$) coupled to the off-shell hyper-dilaton Weyl multiplet, it is then straightforward to introduce $3(n+1)$ off-shell deformations each associated to either a $\xi_I^{\underline{ij}}$ electric deformation or a $\zeta^I_{\underline{ij}}$ magnetic deformation. These induce non-trivial scalar potentials and vacuum structures. We plan to report in the near future on work in progress based on this direction and to extend these analyses also by including more physical hypermultiplets.

Up until now, dilaton Weyl multiplets for $4D$ $N = 2$ conformal supergravity have been constructed by coupling the standard Weyl multiplet to either an on-shell vector multiplet [92] or an on-shell hypermultiplet (the latter in our current paper). It is quite clear that other variant dilaton Weyl multiplets might exist. A natural possibility is to couple the standard Weyl multiplet to either an on-shell linear (tensor) multiplet or an on-shell vector-tensor multiplet – see, e.g., [41, 190, 262–267] for references on the vector-tensor multiplet including its coupling to conformal supergravity. It would be interesting to make these constructions explicitly and explore the peculiarities of these possible other dilaton Weyl multiplets in the study of off-shell $4D$, $N = 2$ Poincaré supergravity.

Another natural direction for future research is the construction of higher-derivative actions based on the hyper-dilaton Weyl and the hyper-dilaton Poincaré multiplets. Higher-derivative supergravity naturally arise in the low-energy description of string theory but, despite its importance, it is still poorly understood. Vector-dilaton Weyl multiplets have been successfully used to construct several off-shell higher-derivative supergravities in $4 \leq \mathrm{D} \leq 6$ dimensions, see [29, 56, 59, 60, 63–65, 68, 69, 72]. It is natural to look at this problem starting from a hyper-dilaton Weyl multiplet coupled to systems of vector multiplets with electric and magnetic FI-type terms. We expect to be able to overcome some of the other Weyl multiplet's restrictions to engineer off-shell gauged supergravity.

Acknowledgements:
We are grateful to I. Antoniadis, D. Butter, J.-P. Derendinger, J. Hutomo, H. Jiang, S. Kuzenko, A. Van Proeyen, and J. Woods for discussions related to this work. This work is supported by the Australian Research Council (ARC) Future Fellowship FT180100353, and by the Capacity Building Package of the University of Queensland. G.G. and S.K. are supported by the postgraduate scholarships at the University of Queensland.

The following publication has been incorporated as Chapter 6.

1. Jessica Hutomo, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, Jesse Woods, *Hyper-dilaton Weyl multiplets of* 5*D and* 6*D minimal conformal supergravity*, *Phys. Rev. D* 107 (2023) 4, 046009 [2209.05748] [3].

Contributor	Statement of Contribution	%
Jessica Hutomo	Writing of text	40
	Proof-reading	30
	Minimal 5*D* HDWM construction	70
	Minimal 6*D* HDWM construction	10
	Initial concept	20
	Supervision, guidance	15
Saurish Khandelwal	Writing of text	40
	Proof-reading	30
	Minimal 5*D* HDWM construction	10
	Minimal 6*D* HDWM construction	70
	Initial concept	20
	Supervision, guidance	10
Gabriele Tartaglino-Mazzucchelli	Writing of text	20
	Proof-reading	30
	Minimal 5*D* HDWM construction	10
	Minimal 6*D* HDWM construction	10
	Initial concept	60
	Supervision, guidance	75
Jesse Woods	Proof-reading	20
	Minimal 5*D* HDWM construction	10
	Minimal 6*D* HDWM construction	10

Table 5.5: Contributions of each author to the work

Chapter 6

Hyper-Dilaton Weyl Multiplets of 5D and 6D Minimal Conformal Supergravity

By extending the recent analysis of arXiv:2203.12203 for $N=2$ conformal supergravity in four dimensions, which is discussed in detail in chapter 5, we define new hyper-dilaton Weyl multiplets for five-dimensional $N=1$, and six-dimensional $N=(1,0)$ conformal supergravities. These are constructed by coupling the five- and six-dimensional standard Weyl multiplets to on-shell hypermultiplets and reinterpreting the systems as new multiplets of conformal supergravity. In the five-dimensional case, we also construct a new hyper-dilaton Poincaré supergravity by coupling to an off-shell vector multiplet compensator. As in four dimensions, a BF-coupling induces a non-trivial scalar potential for the five-dimensional dilaton that admits AdS_5 vacua.

6.1 Introduction

A key ingredient to efficiently engineer off-shell supergravity-matter couplings is the fact that Poincaré gravity can be formulated as conformal gravity coupled to a compensating scalar field [268, 269]. This approach plays an equally important role both in the superconformal tensor calculus and in superspace supergravity formalisms — see [37, 38] and [35, 36] for reviews and references. In the supersymmetric case, conformal gravity is turned into an off-shell representation of the local superconformal algebra containing the vielbein as one of its independent fields. Such multiplet is referred to as the *Weyl* multiplet of conformal supergravity. The scalar compensator is also lifted to an off-shell locally superconformal multiplet. Depending on the amount of supersymmetry, due to the existence of several possible choices of compensating multiplets, it is possible to obtain several different off-shell Poincaré supergravity theories. Moreover, the fact that the Weyl multiplets themselves are in general not unique adds to the richness of the off-shell representations.

The first instance where variant representations of the Weyl multiplets were presented is six-dimensional (6D) minimal $N=(1,0)$ supergravity [89] — see also [31, 32, 59, 60, 65, 69, 129, 130] for further references on 6D conformal supergravity. In this case, it was noted that the so-called *standard Weyl* multiplet could be turned into a *dilaton Weyl* multiplet by reinterpreting the system described by an on-shell tensor multiplet in a standard Weyl multiplet background as a new conformal supergravity multiplet. Such a *tensor-dilaton* Weyl multiplet plays an important role since, once coupled to a second off-shell conformal compensating multiplet, is the one used to construct the simplest versions of two-derivatives Poincaré supergravity theories. Extending the idea of [89], dilaton Weyl multiplets have been discovered also for five-dimensional (5D) $N = 1$ [90] and, more recently, for four-dimensional (4D) $N = 2$ conformal supergravities in [92] and [4].

In 5*D* $N = 1$ supergravity the known dilaton Weyl multiplet is constructed by coupling an on-shell vector multiplet to the standard Weyl multiplet [90]. See [29, 126, 172, 174–177] for more discussions of 5*D* conformal supergravity and its matter couplings. The 4*D* $N = 2$ analogue of this type of *vector-dilaton* Weyl multiplet was constructed in [92]. It is natural to expect that different on-shell multiplets containing a scalar field playing the role of a dilaton could be used to engineer other multiplets of conformal supergravity. In fact, as discussed in chapter 5 for the 4*D* $\mathcal{N} = 2$ case and based on our work in [4], a so-called *hyper-dilaton* Weyl multiplet was constructed using an on-shell hypermultiplet. The scope of this chapter is to present the extension of the analysis of chapter 5 to 5*D* $N = 1$ and 6*D* $N = (1,0)$ supergravities.

Besides a mathematically oriented interest in classifying variant Weyl multiplets, it is worth to explore new options to define off-shell Poincaré supergravities with an eye on their broad range of applications. For example, in our opinion, for theories with eight supercharges it remains an open problem to have a simple, though general, off-shell approach for gauged supergravities with no physical matter hypermultiplets. In the presence of physical charged hypermultiplets (with no central charge) one will need to use representations containing an infinite number of auxiliary/matter fields [29, 116–133], but with only physical vector multiplets it might be beneficial to use simpler approaches, if possible. Exploring options to address this gap was one of the primary motivations behind the construction of the 4*D* $\mathcal{N} = 2$ *hyper-dilaton* Weyl multiplet [4], as discussed in chapter 5.

Interesting applications of off-shell approaches to (gauged) supergravity includes the construction of locally supersymmetric higher-derivative invariants [28, 29, 31, 32, 51, 55–57, 59–72]. This topic has recently attracted a renewed attention due to advances in the study of black-hole entropy and next to leading order AdS/CFT calculations — see the very recent works [79–81, 84–86] and references therein. Vector-dilaton Weyl multiplets have been a main ingredient to construct off-shell higher-derivative supergravities in five and six dimensions, see [29, 56, 59, 60, 63–65, 68, 69, 72]. We hope new hyper-dilaton Weyl multiplets might play an interesting role to extend some of these results to gauged supergravity. The construction of alternative Weyl multiplets could also play an interesting role to develop alternative approaches to localisation of quantum field theories on curved space-times — see [235] for a recent extensive review. In this context, off-shell supersymmetry has been a central ingredient for localisation and new Weyl multiplets could offer alternative starting points.

This paper is organised as follows. In section 6.2 we use the conformal superspace approach to 5*D* $N = 1$ supergravity [29] to review the locally superconformal multiplets used in our analysis. Specifically, we introduce the 5*D* standard Weyl multiplet and its geometric superspace construction, the on-shell hypermultiplet, the linear (or $\mathcal{O}(2)$) multiplet, and the Abelian vector multiplet. Section 6.3 describes the construction of the new 32+32 5D $N = 1$ hyper-dilaton Weyl multiplet. In section 6.4, we couple the hyper-dilaton Weyl multiplet to a vector multiplet compensator to recover a new 40+40 Poincaré supergravity. This can be thought of as a 5D analogue of the 4D off-shell $N = 2$ supergravity introduced by Müller in 1986 [241] and redefined by using superconformal techniques in [4]. A distinctive feature of the dilaton Poincaré supergravity is the fact that the off-shell multiplet is irreducible, while the two-derivative supergravity action leads to the minimal on-shell Poincaré supergravity multiplet coupled to an extra physical matter multiplet containing the dilaton. Interestingly, as in the 4D analysis of chapter 5, in subsection 6.4.2 we show how it is possible to engineer a non-trivial scalar potential for the dilaton and obtain an AdS$_5$ vacua in a framework different than standard gauged supergravity. In sections 6.5, and 6.6 we closely repeat the 5D analysis of sections 6.2, and 6.3 in the case of 6D $N = (1,0)$ conformal supergravity. The paper also contains two technical appendices, B and C, where we collect conformal superspace identities from [29] and [31,32] used in our paper.

6.2 Superconformal multiplets in 5D $N = 1$ superspace

This section reviews the salient details of several superconformal matter multiplets pertinent to this work. Along with the standard Weyl multiplet discussed in subsections 2.2.2 and 2.2.3, they serve as the building blocks for the invariants of 5D conformal supergravity discussed in this paper. Here we make use of the formulation and results of [29]. We also refer the reader to the following list of papers for other work on flat and curved superspace and off-shell multiplets in five dimensions [123–126, 164, 179–181].

6.2.1 The on-shell hypermultiplet

The on-shell realisation for the hypermultiplet contains $4+4$ degrees of freedom, exactly as in the four-dimensional case [186, 242]. In conformal superspace, it is described by a Lorentz scalar superfield $q^{i\underline{i}}$ subject to the constraint

$$\nabla_\alpha^{(i} q^{j)\underline{j}} = 0 \ , \tag{6.2.1}$$

which is equivalent to

$$\nabla_\alpha^i q^{k\underline{k}} = -\frac{1}{2}\varepsilon^{ik}\rho_\alpha^{\underline{k}} \ , \qquad \rho_\alpha^{\underline{k}} := \nabla_\alpha^j q_j{}^{\underline{k}} \ . \tag{6.2.2}$$

Here, the index $\underline{i} = \underline{1}, \underline{2}$ denotes an SU(2) flavour index. The superfield $q^{i\underline{i}}$ is a Lorentz scalar and superconformal primary,

$$M_{ab} q^{i\underline{i}} = 0 \ , \qquad K_A q^{i\underline{i}} = 0 \ , \qquad J^{jk} q^{i\underline{i}} = \varepsilon^{i(j} q^{k)\underline{i}} \ . \tag{6.2.3}$$

Eqs. (6.2.1), (6.2.3), and the relation (B.2.1k) tell us that $\mathbb{D} q^{i\underline{i}} = \frac{3}{2} q^{i\underline{i}}$.

The independent descendants of $q^{i\underline{i}}$ are obtained by acting on it with spinor derivatives. We obtain several implications of the (anti-)commutation relations (B.2.2), (B.2.3), along with the constraints (6.2.2), and (6.2.3):

$$\nabla^i_\alpha \nabla^j_\beta q_j{}^{\underline{i}} = \nabla^i_\alpha \rho^{\underline{i}}_\beta = -4\mathrm{i}\nabla_{\alpha\beta} q^{i\underline{i}} , \tag{6.2.4a}$$

$$\nabla^{\alpha i} \rho^{\underline{i}}_\alpha = 0 , \tag{6.2.4b}$$

with $\nabla_{\alpha\beta} := (\Gamma^a)_{\alpha\beta}\nabla_a$. Next, we shall consider

$$\{\nabla^k_\alpha, \nabla_{\beta k}\}\rho^{\beta\underline{i}} = 4\mathrm{i}\nabla_{\alpha\beta}\rho^{\beta\underline{i}} - 2\mathrm{i}W_\alpha{}^\beta \rho^{\underline{i}}_\beta + 18\mathrm{i}X^k_\alpha q_k{}^{\underline{i}} , \tag{6.2.5}$$

where we have made use of (B.2.2) and the S-supersymmetry transformation

$$S_{\alpha i}\rho^{\underline{i}}_\beta = 12\varepsilon_{\alpha\beta}\, q_i{}^{\underline{i}} \qquad \Longrightarrow \qquad K_a\, \rho^{\underline{i}}_\beta = 0 . \tag{6.2.6}$$

On the other hand, by virtue of (6.2.4), we also have that

$$\{\nabla^k_\alpha, \nabla_{\beta k}\}\rho^{\beta\underline{i}} = 4\mathrm{i}\nabla^\beta_k \nabla_{\alpha\beta} q^{k\underline{i}} = 4\mathrm{i}\,[\nabla^\beta_k, \nabla_{\alpha\beta}]q^{k\underline{i}} - 4\mathrm{i}\nabla_{\alpha\beta}\rho^{\beta\underline{i}} . \tag{6.2.7}$$

Applying the commutation relation (B.2.3), we can then equate (6.2.5) and (6.2.7) to obtain

$$(\nabla_a \rho^{\underline{i}}\Gamma^a)^\alpha = -\frac{3}{4}(\rho^{\underline{i}}\Sigma^{bc})^\alpha W_{bc} - \frac{3}{2}X^{\alpha k} q_k{}^{\underline{i}} . \tag{6.2.8}$$

We can then hit both sides of (6.2.8) with ∇^i_α and make use of (6.2.4), (B.2.3), and the identity (2.2.73). This results in the equation

$$\Box q^{i\underline{i}} = \frac{3\mathrm{i}}{16}X^i\rho^{\underline{i}} + \frac{3}{64}\big(Y - W^{ab}W_{ab}\big)\, q^{i\underline{i}} , \qquad \Box := \nabla^a\nabla_a . \tag{6.2.9}$$

Both (6.2.8) and (6.2.9) describe on-shell conditions for the hypermultiplet's fields when $W_{ab} = 0$. However, as we will discuss in more detail later, in a non-trivial curved background these equations can be reinterpreted as conditions linking $q^{i\underline{i}}$ and $\rho^{\underline{i}}_\alpha$ with fields of the standard Weyl multiplet.

By restricting to $\xi^a \equiv 0$, the local superconformal $\delta = \delta_Q + \delta_{\mathcal{H}}$ transformations of the covariant superfields $q^{i\underline{i}}$ and $\rho^{\underline{i}}_\alpha$ can be derived using the relations (6.2.2), (6.2.4), and (6.2.6). This leads to

$$\delta q^{i\underline{i}} = \frac{1}{2}\xi^i\rho^{\underline{i}} + \Lambda^i{}_k q^{k\underline{i}} + \frac{3}{2}\sigma q^{i\underline{i}} , \tag{6.2.10a}$$

$$\delta\rho^{\underline{i}}_\alpha = -4\mathrm{i}(\Gamma^a\xi_i)_\alpha \nabla_a q^{i\underline{i}} + \frac{1}{2}\Lambda_{ab}(\Sigma^{ab}\rho^{\underline{i}})_\alpha + 2\sigma\rho^{\underline{i}}_\alpha - 12\eta^i_\alpha q_i{}^{\underline{i}} . \tag{6.2.10b}$$

As we will describe later, these will lead to the analogue transformations of the component fields in the hypermultiplet.

6.2.2 The $\mathcal{O}(2)$ multiplet

The linear multiplet [117, 118, 138–140, 186–191], or $\mathcal{O}(2)$ multiplet, can be described in 5D $N = 1$ conformal superspace [29] in terms of the superfield $G^{ij} = G^{ji}$, with $(G^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}G^{kl}$ and satisfies the defining constraint

$$\nabla^{(i}_\alpha G^{jk)} = 0 . \tag{6.2.11}$$

Here G^{ij} is a superconformal primary dimension-3 superfield,

$$K_A G^{ij} = 0 \ , \qquad \mathbb{D} G^{ij} = 3 G^{ij} \ . \tag{6.2.12}$$

To elaborate on the component structure of the superfield G^{ij}, we list the following useful identities:

$$\nabla_\alpha^i G^{jk} = 2\varepsilon^{i(j}\varphi_\alpha^{k)} \ , \tag{6.2.13a}$$
$$\nabla_\alpha^i \varphi_\beta^j = -\frac{\mathrm{i}}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}F + \frac{\mathrm{i}}{2}\varepsilon^{ij}\mathcal{H}_{\alpha\beta} + \mathrm{i}\nabla_{\alpha\beta}G^{ij} \ , \tag{6.2.13b}$$
$$\nabla_\alpha^i F = -2\nabla_\alpha{}^\beta\varphi_\beta^i - 6W_{\alpha\beta}\varphi^{\beta i} - 9X_{\alpha j}G^{ij} \ , \tag{6.2.13c}$$
$$\nabla_\alpha^i \mathcal{H}_a = 4(\Sigma_{ab})_\alpha{}^\beta\nabla^b\varphi_\beta^i - \frac{3}{2}(\Gamma_a)_\alpha{}^\beta W_{\beta\gamma}\varphi^{\gamma i} - \frac{1}{2}(\Gamma_a)_\gamma{}^\beta W_{\beta\alpha}\varphi^{\gamma i} \ , \tag{6.2.13d}$$

where we have defined the independent descendant superfields

$$\varphi_\alpha^i := \frac{1}{3}\nabla_{\alpha j}G^{ij} \ , \tag{6.2.14a}$$
$$F := \frac{\mathrm{i}}{12}\nabla^{\gamma i}\nabla_\gamma^j G_{ij} = -\frac{\mathrm{i}}{4}\nabla^{\gamma k}\varphi_{\gamma k} \ , \tag{6.2.14b}$$
$$\mathcal{H}_{abcd} := \frac{\mathrm{i}}{12}\varepsilon_{abcde}(\Gamma^e)^{\alpha\beta}\nabla_\alpha^i\nabla_\beta^j G_{ij} \equiv \varepsilon_{abcde}\mathcal{H}^e \ . \tag{6.2.14c}$$

It can be checked that $\mathcal{H}^a$ obeys the differential condition

$$\nabla_a \mathcal{H}^a = 0 \ , \qquad \mathcal{H}^a := -\frac{1}{4!}\varepsilon^{abcde}\mathcal{H}_{bcde} \ . \tag{6.2.15}$$

The descendants (6.2.14) prove to be annihilated by K_a and to satisfy

$$S_\alpha^i \varphi_\beta^j = -6\varepsilon_{\alpha\beta}G^{ij} \ , \tag{6.2.16a}$$
$$S_\alpha^i F = 6\mathrm{i}\varphi_\alpha^i \ , \tag{6.2.16b}$$
$$S_\alpha^i \mathcal{H}_b = -8\mathrm{i}(\Gamma_b)_\alpha{}^\beta\varphi_\beta^i \ . \tag{6.2.16c}$$

We refer the reader to [29] for a superform description of the $\mathcal{O}(2)$ multiplet.

6.2.3 The Abelian vector multiplet

In conformal superspace [29], an Abelian vector multiplet is described by a superfield W, which is superconformal primary of dimension 1, $K_A W = 0$ and $\mathbb{D}W = W$. Moreover, it is real, $(W)^* = W$, and obeys the Bianchi identity

$$\nabla_\alpha^{(i}\nabla_\beta^{j)}W = \frac{1}{4}\varepsilon_{\alpha\beta}\nabla^{\gamma(i}\nabla_\gamma^{j)}W \ . \tag{6.2.17}$$

It is useful to introduce the following descendant superfields constructed from spinor derivatives of W:

$$\lambda_\alpha^i := -\mathrm{i}\nabla_\alpha^i W \ , \qquad X^{ij} := \frac{\mathrm{i}}{4}\nabla^{\alpha(i}\nabla_\alpha^{j)}W = -\frac{1}{4}\nabla^{\alpha(i}\lambda_\alpha^{j)} \ . \tag{6.2.18a}$$

These superfields, along with

$$\mathcal{F}_{\alpha\beta} := -\frac{\mathrm{i}}{4}\nabla^k_{(\alpha}\nabla_{\beta)k}W - W_{\alpha\beta}W = \frac{1}{4}\nabla^k_{(\alpha}\lambda_{\beta)k} - W_{\alpha\beta}W\ , \tag{6.2.18b}$$

satisfy the following identities:

$$\nabla^i_\alpha\lambda^j_\beta = -2\varepsilon^{ij}\big(\mathcal{F}_{\alpha\beta} + W_{\alpha\beta}W\big) - \varepsilon_{\alpha\beta}X^{ij} - \varepsilon^{ij}\nabla_{\alpha\beta}W\ , \tag{6.2.19a}$$

$$\begin{aligned}\nabla^i_\alpha\mathcal{F}_{\beta\gamma} = {}& -\mathrm{i}\nabla_{\alpha(\beta}\lambda^i_{\gamma)} - \mathrm{i}\varepsilon_{\alpha(\beta}\nabla_{\gamma)}{}^\delta\lambda^i_\delta - \frac{3\mathrm{i}}{2}W_{\beta\gamma}\lambda^i_\alpha - W_{\alpha\beta\gamma}{}^iW \\ & + \frac{\mathrm{i}}{2}W_{\alpha(\beta}\lambda^i_{\gamma)} - \frac{3\mathrm{i}}{2}\varepsilon_{\alpha(\beta}W_{\gamma)}{}^\delta\lambda^i_\delta\ ,\end{aligned} \tag{6.2.19b}$$

$$\nabla^i_\alpha X^{jk} = 2\mathrm{i}\varepsilon^{i(j}\Big(\nabla_\alpha{}^\beta\lambda^{k)}_\beta - \frac{1}{2}W_{\alpha\beta}\lambda^{\beta k)} + \frac{3\mathrm{i}}{4}X^{k)}_\alpha W\Big)\ . \tag{6.2.19c}$$

We also note the relation $\mathcal{F}_{\alpha\beta} = \frac{1}{2}(\Sigma^{ab})_{\alpha\beta}\mathcal{F}_{ab}$. The S-supersymmetry generator acts on these descendants as

$$S^i_\alpha\lambda^j_\beta = -2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}W\ , \qquad S^i_\alpha\mathcal{F}_{\beta\gamma} = 4\varepsilon_{\alpha(\beta}\lambda^i_{\gamma)}\ , \qquad S^i_\alpha X^{jk} = -2\varepsilon^{i(j}\lambda^{k)}_\alpha\ , \tag{6.2.20}$$

while all the fields are annihilated by the K_a generators.

For the construction of Poincaré supergravity models later in subsection 6.4.2, it is important to note that given a system of n Abelian vector multiplets W^I, with $I = 1,2,\dots n$, we can construct the following composite $\mathcal{O}(2)$ multiplet and its descendant superfields [29]:

$$G^{ij}_I = C_{IJK}\Big(2W^JX^{ijK} - \mathrm{i}\lambda^{\alpha J(i}\lambda^{j)K}_\alpha\Big)\ , \tag{6.2.21a}$$

$$\begin{aligned}\varphi^i_{\alpha I} = {}& C_{IJK}\Big(\mathrm{i}X^{ijJ}\lambda^K_{\alpha j} - 2\mathrm{i}\mathcal{F}^J_{\alpha\beta}\lambda^{\beta iK} - \frac{3}{2}X^i_\alpha W^JW^K - 2\mathrm{i}W^J\nabla_{\alpha\beta}\lambda^{\beta iK} \\ & -\mathrm{i}\nabla_{\alpha\beta}W^J\lambda^{\beta iK} - 3\mathrm{i}W_{\alpha\beta}W^J\lambda^{\beta iK}\Big)\ ,\end{aligned} \tag{6.2.21b}$$

$$\begin{aligned}F_I = {}& C_{IJK}\Big(X^{ijJ}X^K_{ij} - \mathcal{F}^{abJ}\mathcal{F}^K_{ab} + 4W^J\nabla^a\nabla_aW^K + 2(\nabla^aW^J)\nabla_aW^K \\ & +2\mathrm{i}(\nabla_\alpha{}^\beta\lambda^{iJ}_\beta)\lambda^{\alpha K}_i - 6W^{ab}\mathcal{F}^J_{ab}W^K - \frac{39}{8}W^{ab}W_{ab}W^JW^K + \frac{3}{8}YW^JW^K \\ & +6X^{\alpha i}\lambda^J_{\alpha i}W^K - 3\mathrm{i}W_{\alpha\beta}\lambda^{\alpha iJ}\lambda^{\beta K}_i\Big)\ ,\end{aligned} \tag{6.2.21c}$$

$$\begin{aligned}\mathcal{H}_{aI} = {}& C_{IJK}\Big(-\frac{1}{2}\varepsilon_{abcde}\mathcal{F}^{bcJ}\mathcal{F}^{deK} + 4\nabla^b\big(W^J\mathcal{F}^K_{ba} + \frac{3}{2}W_{ba}W^JW^K\big) \\ & +2\mathrm{i}(\Sigma_{ba})^{\alpha\beta}\nabla^b(\lambda^{iJ}_\alpha\lambda^K_{\beta i})\Big)\ ,\end{aligned} \tag{6.2.21d}$$

where $C_{IJK} = C_{(IJK)}$ is a completely symmetric constant. These are the superspace analogue of the composite linear multiplets constructed in [90].

6.3 The hyper-dilaton Weyl multiplet in $5D$

The aim of this section is to construct a new $32+32$ hyper-dilaton Weyl multiplet of off-shell $N = 1$ conformal supergravity in five dimensions. The analysis closely follows the $4D$ $N = 2$ case of [4].

In constructing such a hyper-dilaton Weyl multiplet, our starting point is the component structure of the on-shell hypermultiplet. This can be readily extracted from the previous superspace realisation (see subsection 6.2.1) via the bar projection. As was shown before, taking successive spinor derivatives of $q^{i\underline{i}}$ leads to $\rho_\alpha^{\underline{i}}$ or the vector derivatives of $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$. Hence, the independent components of the on-shell hypermultiplet are simply the Lorentz scalar field $q^{i\underline{i}}|$ which is superconformal primary, and the spinor field $\rho_\alpha^{\underline{i}}|$.

In what follows, we will associate the same symbol for the covariant component fields and the corresponding superfields, when the interpretation is clear from the context. The superfields $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ are all annihilated by K^a; hence, all their bar projections are K-primary fields. The local superconformal transformations of the component fields follow directly from the projections of (6.2.10), which give

$$\delta q^{i\underline{i}} = \frac{1}{2}\xi^i\rho^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + \frac{3}{2}\lambda_{\mathbb{D}} q^{i\underline{i}} , \tag{6.3.1a}$$

$$\delta\rho_\alpha^{\underline{i}} = -4\mathrm{i}(\Gamma^a\xi_i)_\alpha\nabla_a q^{i\underline{i}} + \frac{1}{2}\lambda_{ab}(\Sigma^{ab}\rho^{\underline{i}})_\alpha + 2\lambda_{\mathbb{D}}\rho_\alpha^{\underline{i}} - 12\eta_\alpha{}^i q_i{}^{\underline{i}} , \tag{6.3.1b}$$

where

$$\nabla_a q^{i\underline{i}} = \mathcal{D}_a q^{i\underline{i}} - \frac{1}{4}\psi_a{}^i\rho^{\underline{i}} . \tag{6.3.2}$$

The algebra of the local transformations (6.3.1) closes only when the following equations of motion for the fields $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ are imposed:

$$(\nabla_a\rho^{\underline{i}}\Gamma^a)^\alpha = -\frac{3}{4}(\rho^{\underline{i}}\Sigma^{bc})^\alpha w_{bc} + 16\mathrm{i}\,\chi^{\alpha k}q_k{}^{\underline{i}} , \tag{6.3.3a}$$

$$\Box q^{i\underline{i}} = 2\chi^i\rho^{\underline{i}} - 2D\,q^{i\underline{i}} - \frac{3}{64}w^{ab}w_{ab}q^{i\underline{i}} , \qquad \Box := \nabla^a\nabla_a . \tag{6.3.3b}$$

The above equations are obtained by bar-projecting (6.2.8), (6.2.9), and using the definitions (2.2.73). The expressions for $\nabla_a\rho^{\alpha\underline{i}}$ and $\Box q^{i\underline{i}}$ in terms of the derivatives $\mathcal{D}_a$ are given by

$$\nabla_a\rho^{\alpha\underline{i}} = \mathcal{D}_a\rho^{\alpha\underline{i}} + 2\mathrm{i}(\psi_{ak}\Gamma^b)^\alpha\left(\mathcal{D}_b q^{k\underline{i}} - \frac{1}{4}\psi_b{}^k\rho^{\underline{i}}\right) + 6\phi_a{}^{\alpha k}q_k{}^{\underline{i}} , \tag{6.3.4a}$$

$$\begin{aligned}\Box q^{i\underline{i}} = {} & \mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} - 3\mathfrak{f}_a{}^a q^{i\underline{i}} - \frac{1}{4}\rho^{\underline{i}}\mathcal{D}_a\psi^{ai} - \frac{1}{2}\psi^{ai}\mathcal{D}_a\rho^{\underline{i}} \\ & + \frac{\mathrm{i}}{4}\phi_a{}^i\Gamma^a\rho^{\underline{i}} + \frac{\mathrm{i}}{2}(\psi_a{}^{(i}\Gamma^b\psi^{k)a})\mathcal{D}_b q_k{}^{\underline{i}} + 4\mathrm{i}(\psi_a{}^i\Gamma^a\chi_k)q^{k\underline{i}} \\ & - \frac{1}{8}w^{cd}(\psi_c{}^i\Gamma_d\rho^{\underline{i}}) + \frac{3}{2}(\psi^{ai}\phi_{ak})q^{k\underline{i}} - \frac{\mathrm{i}}{8}(\psi_a{}^{(i}\Gamma^b\psi^{k)a})(\psi_{bk}\rho^{\underline{i}}) .\end{aligned} \tag{6.3.4b}$$

Equations (6.3.3) can then be interpreted as algebraic equations for the standard Weyl multiplet that determine the auxiliary fields $\chi^{\alpha i}$ and D in terms of $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$, together with the other independent fields of the standard Weyl multiplet. Assuming that $q^{i\underline{i}}$ is an invertible matrix,

$$q^2 := q^{i\underline{i}}q_{i\underline{i}} = \varepsilon_{ij}\varepsilon_{\underline{i}\underline{j}}q^{i\underline{i}}q^{j\underline{j}} = 2\det q^{i\underline{i}} \neq 0 , \tag{6.3.5}$$

then the following relations hold

$$\chi^{\alpha i} = \frac{\mathrm{i}}{8}q^{-2}q^{i\underline{i}}\Big[-(\nabla_a\rho_{\underline{i}}\Gamma^a)^\alpha - \frac{3}{4}w_{cd}(\rho_{\underline{i}}\Sigma^{cd})^\alpha\Big] , \tag{6.3.6a}$$

$$D = -\frac{1}{2}q^{-2}q_{i\underline{i}}\Box q^{i\underline{i}} + \frac{\mathrm{i}}{16}q^{-2}\Big[-(\nabla_a\rho_{\underline{i}}\Gamma^a)^\alpha - \frac{3}{4}w_{cd}(\rho_{\underline{i}}\Sigma^{cd})^\alpha\Big]\rho^{\underline{i}}_\alpha - \frac{3}{128}w_{ab}w^{ab} . \quad (6.3.6b)$$

Note that so far, we have only used one of the four equations that are equivalent to (6.3.3b) to solve for D in eq. (6.3.6b). It is simple to show that the remaining independent three equations are equivalent to the following

$$\nabla^a(q^{i(\underline{i}}\nabla_a q_i{}^{\underline{j})}) = 0 . \quad (6.3.7)$$

As in the 4D $N=2$ case [4], this equation is solved by turning the SU(2)$_R$ connection $\phi_m{}^{kl}$ into a composite field. Let us describe how.

Following the analysis of the $N=2$ hyper-dilaton Weyl multiplet in 4D [4], here we also note that, associated to an on-shell hypermultiplet, there is always a triplet of composite linear multiplets [138–140,186]. The covariant component fields of the 5D $N=1$ off-shell linear (or $\mathcal{O}(2)$) multiplet are defined in terms of the bar projections of (6.2.14): an SU$(2)_R$ triplet of Lorentz scalar fields $G^{ij} = G^{ij}|$; a spinor field $\varphi_{\alpha i} = \varphi_{\alpha i}|$; a scalar field $F = F|$; and a covariant closed anti-symmetric four-form field strength $H_{abcd} := \mathcal{H}_{abcd}|$. The latter is equivalent to a conserved dual vector $H^a := -1/4!\,\varepsilon^{abcde}H_{bcde}$.[1] At the component level it holds that

$$H^a = h^a + 2\psi_{bi}\Sigma^{ab}\varphi^i + \frac{\mathrm{i}}{2}\varepsilon^{abcde}\psi_{bi}\Sigma_{cd}\psi_{ej}G^{ij} . \quad (6.3.8)$$

The covariant conservation equation for H_a is

$$\nabla^a H_a = 0 . \quad (6.3.9)$$

The constraint implies the existence of a gauge three-form potential, b_{mnp}, and its exterior derivative $h_{mnpq} := 4\partial_{[m}b_{npq]}$, with $b_{mnp} := \mathcal{B}_{mnp}|$.

In the standard Weyl multiplet background, the local superconformal transformations of the covariant fields can be derived using the relations (6.2.13) and (6.2.16), which lead to

$$\begin{aligned}
\delta G_{ij} &= -2\xi_{(i}\varphi_{j)} - 2\lambda_{(i}{}^k G_{j)k} + 3\lambda_{\mathbb{D}}G_{ij} , && (6.3.10a)\\
\delta\varphi_{\alpha i} &= -\frac{\mathrm{i}}{2}\xi_{\alpha i}F - \frac{\mathrm{i}}{2}H_a(\Gamma^a\xi_i)_\alpha - \mathrm{i}(\Gamma^a\xi^j)_\alpha\nabla_a G_{ij} + 6\eta_\alpha{}^j G_{ij} \\
&\quad + \frac{1}{2}\lambda^{ab}(\Sigma_{ab}\varphi_i)_\alpha - \lambda_i{}^j\varphi_{j\alpha} + \frac{7}{2}\lambda_{\mathbb{D}}\varphi_{\alpha i} , && (6.3.10b)\\
\delta F &= 2\xi^i\Gamma^a\nabla_a\varphi_i - \frac{3}{2}(\xi^i\Sigma^{ab}\varphi_i)w_{ab} + 16\mathrm{i}(\xi^i\chi^j)G_{ij} \\
&\quad + 6\mathrm{i}\,\eta^i\varphi_i + 4\lambda_{\mathbb{D}}F , && (6.3.10c)\\
\delta H_a &= -4\xi^i\Sigma_{ab}\nabla^b\varphi_i + \frac{3}{2}(\xi^i\Gamma^b\varphi_i)w_{ab} + \frac{1}{2}\varepsilon_{abcde}w^{bc}(\xi^i\Sigma^{de}\varphi_i) \\
&\quad + \lambda_a{}^b H_b + 4\lambda_{\mathbb{D}}H_a - 8\mathrm{i}\eta^i\Gamma_a\varphi_i , && (6.3.10d)
\end{aligned}$$

where

$$\nabla_a G_{ij} = \mathcal{D}_a G_{ij} + \frac{3}{4}\psi_{a(i}\phi_{j)} , \quad (6.3.11a)$$

[1] The Levi-Civita tensor with world indices is defined as $\varepsilon^{mnpqr} := \varepsilon^{abcde}e_a{}^m e_b{}^n e_c{}^p e_d{}^q e_e{}^r$, such that ε_{abcde} and ε^{abcde} are normalised as $\varepsilon_{01234} = -\varepsilon^{01234} = 1$.

$$\nabla_a\varphi_{\alpha i} = \mathcal{D}_a\varphi_{\alpha i} - \frac{\mathrm{i}}{4}\psi_{a\alpha i}F - \frac{\mathrm{i}}{4}(\Gamma^b\psi_{ai})_\alpha H_b + \frac{\mathrm{i}}{2}(\Gamma^b\psi_a{}^j)_\alpha\nabla_b G_{ij} - 3\phi_{a\alpha}{}^j G_{ij} \,. \quad (6.3.11b)$$

The locally superconformal transformations of b_{mnp} are

$$\delta b_{mnp} = 2\,\varepsilon_{abcde}e_m{}^a e_n{}^b e_p{}^c(\xi_i\Sigma^{de}\varphi^i) - 12\mathrm{i}(\psi_{[m}{}^i\Sigma_{np]}\xi^j)G_{ij} + 3\partial_{[m}l_{np]} \,, \quad (6.3.12)$$

where we have also included the gauge transformation $\delta_l b_{mnp} = 3\partial_{[m}l_{np]}$ leaving h_{mnpq} and H^a invariant. For convenience, we have summarised the dilatation weights of the fields of the $\mathcal{O}(2)$ multiplet in Table 6.1.

	G_{ij}	$\varphi_{\alpha i}$	F	H_a	b_{mnp}
$\mathbb{D}$	3	7/2	4	4	0

Table 6.1: Dilatation weights of the off-shell $\mathcal{O}(2)$ multiplet.

Given that $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ describe an on-shell hypermultiplet in a standard Weyl multiplet background with transformation rules (6.3.1), it can be verified that the following composite fields define a triplet of $\mathcal{O}(2)$ multiplets

$$\begin{aligned} G_{ij}{}^{\underline{ij}} &= q_{(i}{}^{\underline{i}}q_{j)}{}^{\underline{j}} = q_i{}^{(\underline{i}}q_j{}^{\underline{j})} \,, && (6.3.13a)\\ \varphi_{\alpha i}{}^{\underline{ij}} &= -\frac{1}{2}q_i{}^{(\underline{i}}\rho_\alpha^{\underline{j})} \,, && (6.3.13b)\\ F^{\underline{ij}} &= \frac{\mathrm{i}}{8}\rho^{(\underline{i}}\rho^{\underline{j})} \,, && (6.3.13c)\\ H^{a\underline{ij}} &= 2q^{i(\underline{i}}\nabla^a q_i{}^{\underline{j})} - \frac{\mathrm{i}}{8}\rho^{(\underline{i}}\Gamma^a\rho^{\underline{j})} \,. && (6.3.13d) \end{aligned}$$

These fields all transform according to (6.3.10) and each of the previous fields is symmetric in $\underline{i}$ and $\underline{j}$. The field $H^{a\underline{ij}}$ can be used to express the $SU(2)_R$ connection $\phi_m{}^{ij}$ as a composite field. To see this, we introduce a new covariant derivative

$$\mathbf{D}_a = e_a{}^m\left(\partial_m - \frac{1}{2}\omega_m{}^{cd}M_{cd} - b_m\mathbb{D}\right) = \mathcal{D}_a + e_a{}^m\phi_m{}^{ij}J_{ij} \,, \quad (6.3.14)$$

which then allows us to rearrange eq. (6.3.13d) for the SU(2)$_R$ gauge connection:

$$\phi_a{}^{ij} = 4q^{-4}q^{(i}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\left[q^{k\underline{i}}\mathbf{D}_a q_k{}^{\underline{j}} - \frac{1}{4}q^{k\underline{i}}(\psi_{ak}\rho^{\underline{j}}) - \frac{\mathrm{i}}{16}\rho^{\underline{i}}\Gamma_a\rho^{\underline{j}} - \frac{1}{2}H_a{}^{\underline{ij}}\right] \,. \quad (6.3.15)$$

Our analysis demonstrates that the hyper-dilaton Weyl multiplet defines a new representation of the off-shell local 5D $N=1$ superconformal algebra. The multiplet comprises of the following independent fields: $e_m{}^a$, b_m, w_{ab}, $q^{i\underline{i}}$, $b_{mnp}{}^{\underline{ij}}$, ψ_{mi}, and $\rho^{\underline{i}}$. It also possesses the same number of off-shell degrees of freedom as the standard Weyl multiplet, $32+32$. Table 6.2 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields.

Note that with the ingredients provided so far, it is a straightforward exercise to obtain the locally superconformal transformations of the fields of the hyper-dilaton Weyl multiplet written only in terms of the independent fields and they are given as follows:

$$\delta e_m{}^a = \mathrm{i}(\xi_i\Gamma^a\psi_m{}^i) - \lambda_{\mathbb{D}}e_m{}^a + \lambda^a{}_b e_m{}^b \,, \quad (6.3.16a)$$

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\phi_m{}^{ij}$	ψ_{mi}	$\phi_m{}^i$	w_{ab}	$\rho^{\underline{i}}$	$q^{i\underline{i}}$	$b_{mnp}{}^{\underline{ij}}$
$25B$	0	$5B$	0	0	$40F$	0	$10B$	$8F$	$4B$	$30B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Q	S				$\lambda_{mn}{}^{\underline{ij}}$-sym
$-5B$	$-10B$	$-1B$	$-5B$	$-3B$	$-8F$	$-8F$				$-18B$
Result: $32+32$ degrees of freedom										

Table 6.2: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the component fields. Row two gives the number of independent components of these fields — composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that $\lambda_{mn}{}^{\underline{ij}} = -\lambda_{nm}{}^{\underline{ij}}$ corresponds to the symmetry associated with the gauge three-forms $b_{mnp}{}^{\underline{ij}}$ with field strength four-forms $h_{mnpq}{}^{\underline{ij}}$ and $H^{a\underline{ij}}$. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom.

$$\begin{aligned}
\delta\psi_{m\alpha}^{\,i} &= 2\mathbf{D}_m\xi^i_\alpha + 8q^{-4}q^{(i}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\left[q^{k\underline{i}}\mathbf{D}_m q_k{}^{\underline{j}} - \frac{1}{4}q^{k\underline{i}}(\psi_{mk}\rho^{\underline{j}}) - \frac{\mathrm{i}}{16}\rho^{\underline{i}}\Gamma_m\rho^{\underline{j}} - \frac{1}{2}H_m{}^{\underline{ij}}\right]\xi_{\alpha j} \\
&\quad -\frac{1}{4}w_{cd}\Big((\Gamma_m\Sigma^{cd})_\alpha{}^\beta - 3(\Sigma^{cd}\Gamma_m)_\alpha{}^\beta\Big)\xi^i_\beta + 2\mathrm{i}\,(\Gamma_m\eta^i)_\alpha \\
&\quad +\frac{1}{2}\lambda^{ab}(\Sigma_{ab}\psi_m{}^i)_\alpha + \lambda^i{}_j\,\psi^{\,j}_{m\alpha} - \frac{1}{2}\lambda_{\mathbb{D}}\,\psi^{\,i}_{m\alpha}\,, &(6.3.16b)\\
\delta b_m &= \partial_m\lambda_{\mathbb{D}} + \frac{1}{3}\,q^{-2}q^{i\underline{i}}(\xi_i\Gamma_m)^\alpha[(\nabla_a\rho_{\underline{i}}\Gamma^a\varepsilon)_\alpha + \frac{3}{4}w_{cd}(\rho_{\underline{i}}\Sigma^{cd}\varepsilon)_\alpha] \\
&\quad -\xi_i\phi_m{}^i - \psi_m{}^i\eta_i - 2\lambda_m\,, &(6.3.16c)\\
\delta w_{ab} &= 2\mathrm{i}\,\xi_i R(Q)_{ab}{}^i + \frac{4}{3}q^{-2}q^{i\underline{i}}(\xi_i\Sigma_{ab})^\alpha[(\nabla_c\rho_{\underline{i}}\Gamma^c\varepsilon)_\alpha + \frac{3}{4}w_{cd}(\rho_{\underline{i}}\Sigma^{cd}\varepsilon)_\alpha] \\
&\quad -2\lambda_{[a}{}^c w_{b]c} + \lambda_{\mathbb{D}}w_{ab}\,, &(6.3.16d)\\
\delta q^{i\underline{i}} &= \frac{1}{2}\xi^i\rho^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + \frac{3}{2}\lambda_{\mathbb{D}}q^{i\underline{i}}\,, &(6.3.16e)\\
\delta\rho^{\underline{i}}_\alpha &= -4\mathrm{i}(\Gamma^a\xi_i)_\alpha\nabla_a q^{i\underline{i}} + \frac{1}{2}\lambda_{ab}(\Sigma^{ab}\rho^{\underline{i}})_\alpha + 2\lambda_{\mathbb{D}}\rho^{\underline{i}}_\alpha - 12\eta_\alpha{}^i q_i{}^{\underline{i}}\,, &(6.3.16f)\\
\delta b_{mnp} &= 2\,\varepsilon_{abcde}e_m{}^a e_n{}^b e_p{}^c(\xi_i\Sigma^{de}\varphi^i) - 12\mathrm{i}(\psi_{[m}{}^i\Sigma_{np]}\xi^j)G_{ij} + 3\partial_{[m}l_{np]}\,, &(6.3.16g)
\end{aligned}$$

It would be useful to have (6.3.4) expressions in terms of the derivative $\mathbf{D}_a$ instead of $\mathcal{D}_a$, which has an implicit dependence on this new composite field $\phi_a{}^{ij}$. It holds that

$$\begin{aligned}
\nabla_a q^{i\underline{i}} &= \frac{1}{2}\mathbf{D}_a q^{i\underline{i}} - \frac{1}{8}\psi^i_a\rho^{\underline{i}} - q^{-2}q^i{}_{\underline{j}}q^{k\underline{i}}\mathbf{D}_a q_k{}^{\underline{j}} + \frac{1}{4}q^{-2}q^i{}_{\underline{j}}q^{k\underline{i}}(\psi_{ak}\rho^{\underline{j}}) \\
&\quad +\frac{\mathrm{i}}{8}q^{-2}q^i{}_{\underline{j}}(\rho^{(\underline{i}}\Gamma_a\rho^{\underline{j})}) + q^{-2}q^i{}_{\underline{j}}H_a{}^{\underline{ij}} &(6.3.17a)\\
\nabla_a\rho^{\alpha\underline{i}} &= \mathbf{D}_a\rho^{\alpha\underline{i}} + 2\mathrm{i}(\psi_{ak}\Gamma^b)^\alpha\nabla_b q^{k\underline{i}} + 6\phi_a{}^{\alpha k}q_k{}^{\underline{i}}\,, &(6.3.17b)\\
\Box q^{i\underline{i}} &= \mathbf{D}^a\nabla_a q^{i\underline{i}} - 3\mathfrak{f}_a{}^a q^{i\underline{i}} \\
&\quad +(\nabla_a q_k{}^{\underline{i}})\Big[2q^{-2}q^{(i}{}_{\underline{j}}\mathbf{D}^a q^{k)\underline{j}} - 2q^{-4}q^{(i}{}_{\underline{i}}q^{k)}{}_{\underline{j}}H^{a\underline{ij}} - \frac{1}{2}q^{-2}q^{(i}{}_{\underline{j}}(\psi^{ak)}\rho^{\underline{j}}) \\
&\qquad -\frac{\mathrm{i}}{4}q^{-4}q^{(i}{}_{\underline{i}}q^{k)}{}_{\underline{j}}\rho^{\underline{i}}\Gamma^a\rho^{\underline{j}}\Big] \\
&\quad -\frac{1}{8}w^{cd}(\psi_c{}^i\Gamma_d\rho^{\underline{i}}) + 4\mathrm{i}(\psi_a{}^i\Gamma^a\chi_k)q^{k\underline{i}} - \frac{1}{4}\psi^{ai}\nabla_a\rho^{\underline{i}} + \frac{\mathrm{i}}{4}\phi_a{}^i\Gamma^a\rho^{\underline{i}}\,. &(6.3.17c)
\end{aligned}$$

where the composite connection $\phi_m{}^i$ and $\mathfrak{f}_a{}^b$ are now given in terms of $\mathbf{D}_a$ by:

$$\mathrm{i}\,\phi_m{}^i = \frac{2}{3}(\Gamma^{[p}\delta_m{}^{q]} + \frac{1}{4}\Gamma_m\Sigma^{pq})\Big[\mathbf{D}_{[p}\psi_{q]}{}^i + \frac{1}{8}w_{cd}\big(3\Sigma^{cd}\Gamma_{[p}\psi_{q]}{}^i - \Gamma_{[p}\Sigma^{cd}\psi_{q]}{}^i\big)$$

$$+4q^{-4}q^{(i}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\Big\{q^{k\underline{i}}\mathbf{D}_{[p}q_k{}^{\underline{j}}-\frac{1}{4}q^{k\underline{i}}(\psi_{[pk}\rho^{\underline{j}})-\frac{\mathrm{i}}{16}\rho^{\underline{i}}\Gamma_{[p}\rho^{\underline{j}}-\frac{1}{2}H_{[p}{}^{\underline{ij}}\Big\}\psi_{q]j}\Big] \,, \tag{6.3.18a}$$

$$\begin{aligned}\mathfrak{f}_a{}^b \;=\;& -\frac{1}{6}\mathcal{R}(\omega)_{ac}{}^{bc}+\frac{1}{48}\delta_a{}^b\mathcal{R}(\omega)_{cd}{}^{cd}-\frac{\mathrm{i}}{6}\psi_{cj}\Gamma^{[b}R(Q)_a{}^{c]j}-\frac{\mathrm{i}}{12}\psi_{cj}\Gamma_aR(Q)^{bcj}\\ &+\frac{1}{12}q^{-2}q^{i\underline{i}}(\psi_{ai}\Gamma^b)^\alpha\Big[(\nabla_a\rho_{\underline{i}}\Gamma^a\varepsilon)_\alpha+\frac{3}{4}w_{cd}(\rho_{\underline{i}}\Sigma^{cd}\varepsilon)_\alpha\Big]\\ &+\frac{1}{3}\psi_{[aj}\Sigma^{bd}\phi_{d]}{}^j-\frac{1}{24}\delta_a{}^b(\psi_{cj}\Sigma^{cd}\phi_d{}^j)-\frac{\mathrm{i}}{12}\psi_{aj}\psi_c{}^jw^{bc}+\frac{\mathrm{i}}{24}(\psi_{aj}\Gamma_e\psi_d{}^j)\tilde{w}^{bde}\\ &+\frac{\mathrm{i}}{192}\delta_a{}^b\Big[2(\psi_{cj}\psi_d{}^j)w^{cd}-(\psi_{cj}\Gamma_e\psi_d{}^j)\tilde{w}^{cde}\Big]\,.\end{aligned} \tag{6.3.18b}$$

Note that the expression of $\mathfrak{f}_a{}^b$ has explicit as well as implicit dependence on the composite connection $\phi_a{}^i$ via $\nabla_a\rho_{\underline{i}}$, which can now be substituted from (6.3.18a). For later use, it is convenient to have the bosonic expression of $\Box q^{i\underline{i}}$, which, by also using that $\mathbf{D}^aH_a{}^{\underline{ij}}=0$ up to fermions, is given by:

$$\begin{aligned}\Box q^{i\underline{i}} \;=\;& \frac{1}{2}\mathbf{D}^a\mathbf{D}_aq^{i\underline{i}}+\frac{3}{4}q^{-2}q^{i\underline{i}}(\mathbf{D}^aq^{k\underline{k}})\mathbf{D}_aq_{k\underline{k}}-q^{-2}q^i{}_{\underline{j}}q^{k\underline{i}}\mathbf{D}^a\mathbf{D}_aq_k{}^{\underline{j}}+\frac{1}{2}q^{-2}q^k{}_{\underline{j}}(\mathbf{D}^aq^{i\underline{j}})\mathbf{D}_aq_k{}^{\underline{i}}\\ &+q^{-4}q_{l\underline{l}}q^i{}_{\underline{j}}q^{k\underline{i}}(\mathbf{D}^aq^{l\underline{l}})\mathbf{D}_aq_k{}^{\underline{j}}-\frac{1}{2}q^{-4}q^{i\underline{i}}H^{a\underline{jk}}H_{a\underline{jk}}+\frac{3}{16}\mathcal{R}q^{i\underline{i}}+\text{fermions}\,.\end{aligned} \tag{6.3.19}$$

6.4 Recovering Poincaré supergravity in 5D

The goal of this section is to explicitly show that our hyper-dilaton Weyl multiplet constructed in the previous section can be used to derive a $40+40$ off-shell multiplet of $5D$ $N=1$ Poincaré supergravity, by making use of superconformal approaches. We first elaborate on the structure of the multiplet and then explain how to construct a Poincaré supergravity action, pointing out some peculiarities which do not hold in the $4D$ $N=2$ supergravity case [4]. As an extension of the results of [4], we describe a new type of BF-coupling which induces a scalar potential for the dilaton without a standard R-symmetry gauging that admits AdS_5 vacua.

6.4.1 Hyper-dilaton multiplet of Poincaré supergravity

To recover a multiplet of Poincaré supergravity, compensating multiplets must be coupled to an off-shell conformal supergravity multiplet to gauge fix some of the local superconformal symmetries. As will be discussed below, from the symmetry point of view, it suffices to use the components of the new hyper-dilaton Weyl multiplet alone to appropriately gauge fix and eliminate all symmetries except local supersymmetry, Lorentz, and the gauge symmetry of the gauge three-forms $b_{mnp}{}^{\underline{ij}}$. This peculiar feature is different from the construction of the $4D$ $N=2$ hyper-dilaton Poincaré multiplet [4]. In the latter case, we are required to gauge fix the scalar field of the compensating vector multiplet in order to fix the extra $\mathrm{U}(1)_R$ symmetry. In the $5D$ case, the only purpose to couple to a compensator is simply to obtain the Einstein-Hilbert kinetic term in a Poincaré supergravity action. The simplest choice is to couple our hyper-dilaton Weyl multiplet to a single off-shell, Abelian vector multiplet compensator. The scalar field in the compensator is assumed to be nowhere vanishing.

Let us first define an off-shell $5D$ $N=1$ Abelian vector multiplet in a standard Weyl multiplet background. Its component structure follows directly from the superfield definitions (6.2.18). The multiplet contains a real scalar field $\phi := W|$, gaugini $\lambda_\alpha^i := \lambda_\alpha^i|$, a triplet of auxiliary fields $X^{ij} := X^{ij}|$, and a real Abelian gauge connection $v_m := \mathcal{V}_m|$ or, equivalently, its real field strength $f_{mn} := \mathcal{F}_{mn}| = 2\partial_{[m}v_{n]}$. The field strength f_{mn} may be expressed in terms of the bar-projected, covariant field strength $F_{ab} := \mathcal{F}_{ab}|$ via the relation

$$F_{ab} = f_{ab} + \mathrm{i}(\Gamma_{[a})_\alpha{}^\beta \psi_{b]k}^\alpha \lambda_\beta^k + \frac{\mathrm{i}}{2}\psi_{[a k}^\gamma \psi_{b]\gamma}^k \phi \ , \qquad f_{ab} := e_a{}^m e_b{}^n f_{mn} \ . \tag{6.4.1}$$

The dilatation weights of the vector multiplet fields are summarised in Table 6.3.

	ϕ	λ_α^i	X^{ij}	F_{ab}	v_m
$\mathbb{D}$	1	3/2	2	2	0

Table 6.3: Dilatation weights of the Abelian vector multiplet.

The transformation rules of the vector multiplet fields in a standard Weyl multiplet background can be obtained from the corresponding superfields. They read

$$\begin{aligned}
\delta\phi &= \mathrm{i}\xi_i\lambda^i + \lambda_{\mathbb{D}}\phi \ , && (6.4.2a)\\
\delta\lambda_\alpha^i &= -(\Sigma^{ab}\xi^i)_\alpha F_{ab} - (\Sigma^{ab}\xi^i)_\alpha w_{ab}\phi + \xi_{\alpha j}X^{ij} + (\Gamma^a\xi^i)_\alpha \nabla_a\phi \\
&\quad + \frac{1}{2}\lambda^{ab}(\Sigma_{ab}\lambda^i)_\alpha + \lambda^i{}_j\lambda_\alpha^j + \frac{3}{2}\lambda_{\mathbb{D}}\lambda_\alpha^i + 2\mathrm{i}\eta_\alpha{}^i\phi \ , && (6.4.2b)\\
\delta X^{ij} &= -2\mathrm{i}\xi^{(i}\Gamma^a\nabla_a\lambda^{j)} + \frac{3\mathrm{i}}{2}\xi^{(i}\Sigma^{ab}\lambda^{j)}w_{ab} - 16\mathrm{i}(\xi^{(i}\chi^{j)})\phi \\
&\quad + 2\lambda^{(i}{}_k X^{j)k} - 2\eta^{(i}\lambda^{j)} + 2\lambda_{\mathbb{D}}X^{ij} \ , && (6.4.2c)\\
\delta v_m &= \mathrm{i}(\xi^i\psi_{mi})\phi - \mathrm{i}(\xi^i\Gamma_m\lambda_i) + \partial_m\lambda_V \ , && (6.4.2d)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a\phi &= \mathcal{D}_a\phi - \frac{\mathrm{i}}{2}\psi_{ai}\lambda^i \ , && (6.4.3a)\\
\nabla_a\lambda_\alpha^i &= \mathcal{D}_a\lambda_\alpha^i + \frac{1}{2}(\Sigma^{bc}\psi_a{}^i)_\alpha\Big(F_{bc} + w_{bc}\phi\Big) - \frac{1}{2}\psi_{a\alpha j}X^{ij} \\
&\quad - \frac{1}{2}(\Gamma^b\psi_a{}^i)_\alpha\nabla_b\phi - \mathrm{i}\phi_{a\alpha}{}^i\phi \ . && (6.4.3b)
\end{aligned}$$

Note that we have also included in (6.4.2d) the gauge field transformation parametrised by the local real parameter λ_V. The transformations of the vector multiplet in a hyper-dilaton Weyl multiplet background are precisely the same as above. The only subtlety is that one has to interpret several standard Weyl multiplet fields as composites of $q^{i\underline{i}}$, $\rho_\alpha^{\underline{i}}$ and $b_{mnp}{}^{\underline{i}\underline{j}}$. It should be emphasised that, being a compensator, one may require that the lowest component of the vector multiplet is non-zero and positive, $\phi > 0$.

Gauge fixing in a string frame

We now describe the structure of the supergravity multiplet by imposing several gauge fixing constraints. In a string frame, we choose the gauge fixing condition

$$q^{i\underline{i}} = -\varepsilon^{i\underline{i}} \iff q^i{}_{\underline{i}} = \delta_{\underline{i}}^i \iff q_i{}^{\underline{i}} = -\delta_i^{\underline{i}} \iff q_{i\underline{i}} = \varepsilon_{i\underline{i}} \ . \tag{6.4.4a}$$

This condition fixes dilatation and $\mathrm{SU}(2)_R$ symmetries. By imposing

$$b_m = 0 \,, \tag{6.4.4b}$$

the special conformal K^a symmetry is now fixed. In order to fix S-supersymmetry, we impose the constraint

$$\rho_{\alpha}^{\underline{i}} = 0 \,. \tag{6.4.4c}$$

The compensating vector multiplet contains $8+8$ off-shell degrees of freedom. Once added to the remaining fundamental fields in the hyper-dilaton Weyl multiplet, we obtain $40+40$ off-shell degrees of freedom of a Poincaré supergravity multiplet, as shown in Table 6.4. The fundamental fields are

${e_m}^a$	${\omega_m}^{ab}$	$\psi_{m}{}_{\underline{i}}^{\alpha}$	w_{ab}	${b_{mnp}}^{\underline{ij}}$	ϕ	$\lambda_{\alpha}^{\underline{i}}$	X^{ij}	v_m
$25B$	0	$40F$	$10B$	$30B$	$1B$	$8F$	$3B$	$5B$
P_a	M_{ab}	Q		$({\lambda_{mn}}^{\underline{ij}})$				(λ_V)
$-5B$	$-10B$	$-8F$		$-18B$				$-1B$
Result: $40+40$ degrees of freedom								

Table 6.4: A Poincaré supergravity multiplet. Row one gives all fields in the multiplet. Row two gives the number of independent components of these fields. Row three gives the surviving gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. The parameter ${\lambda_{mn}}^{\underline{ij}}$ describes the symmetry associated with the triplet of gauge three-form ${b_{mnp}}^{\underline{ij}}$. The gauge parameter λ_V describes the scalar symmetry of v_m.

the vielbein ${e_m}^a$, the gravitino $\psi_{m}{}_{\underline{i}}^{\alpha}$, a real antisymmetric tensor w_{ab}, a real scalar field that plays the role of a dilaton ϕ, a real triplet of scalar fields X^{ij}, a triplet of gauge three forms ${b_{mnp}}^{\underline{ij}}$, a gauge field v_m that plays the role of the graviphoton, and a spinor field $\lambda_{\alpha}^{\underline{i}}$. Note that we kept the distinction of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)$ flavour indices. However, the gauge condition (6.4.4a) implies that the two indices can be identified, after gauge fixing.

The transformation rules of the resulting Poincaré supergravity multiplet are those that preserve the previous set of gauge conditions (6.4.4). Since we fix $\rho_{\alpha}^{\underline{i}} = 0$, to preserve (6.4.4a), we require $\lambda_{\mathbb{D}} = \lambda^{ij} = 0$. To preserve (6.4.4c), it can be shown that any Q-supersymmetry transformation must be accompanied by a compensating S-supersymmetry transformation with the following parameter

$$\eta_{\alpha}{}^{i}(\xi) \;=\; -\frac{\mathrm{i}}{3}(\Gamma^a \xi_i)_{\alpha}\, \phi_a{}^{ij} \,, \tag{6.4.5}$$

A similar analysis shows that to preserve the condition $b_m = 0$ one needs to enforce nontrivial compensating special conformal K-transformations with a parameter $\lambda_K^a(\xi)$. However, since all the other supergravity fields are conformal (not necessarily superconformal) primaries, not transforming under special conformal boosts, in practice we will never have to worry about inserting the compensating $\lambda_K^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations.

Gauge fixing in the Einstein frame

It is possible to choose a different gauge fixing to Poincaré supergravity, where we also impose constraints on some of the fields of the compensating vector multiplet. This gauge fixing choice, which is analogous to that in the $4D$ $N = 2$ case [4], corresponds to the Einstein frame and leads to a different Poincaré supergravity multiplet.

We now adopt the gauge where

$$\phi = 1 \, , \tag{6.4.6a}$$

$$b_m = 0 \, . \tag{6.4.6b}$$

Condition (6.4.6a) fixes dilatation symmetry, while (6.4.6b) fixes special conformal K^a symmetry. In order to fix S-supersymmetry, we impose

$$\lambda_\alpha^{\underline{i}} = 0 \, . \tag{6.4.6c}$$

A characterising feature of the hyper-dilaton Weyl multiplet is that it contains an $\mathrm{SU}(2)_R$ compensator, the $q^{i\underline{i}}$ fields. By imposing

$$q^{i\underline{i}} = -\varepsilon^{i\underline{i}} \mathrm{e}^{-U} \iff q^i{}_{\underline{i}} = \delta^i_{\underline{i}} \mathrm{e}^{-U} \iff q_i{}^{\underline{i}} = -\delta_i^{\underline{i}} \mathrm{e}^{-U} \iff q_{i\underline{i}} = \varepsilon_{i\underline{i}} \mathrm{e}^{-U} \, , \tag{6.4.6d}$$

we break the $\mathrm{SU}(2)_R$ symmetry. The resulting Poincaré supergravity multiplet is shown in Table 6.5. The fundamental fields are the vielbein $e_m{}^a$, the gravitini $\psi_{m i}^{\alpha}$, a real antisymmetric tensor w_{ab}, a real

$e_m{}^a$	$\omega_m{}^{ab}$	$\psi_{m i}^{\alpha}$	w_{ab}	$\rho_\alpha^{\underline{i}}$	U	$b_{mnp}{}^{\underline{ij}}$	X^{ij}	v_m
$25B$	0	$40F$	$10B$	$8F$	$1B$	$30B$	$3B$	$5B$
P_a	M_{ab}	Q				$(\lambda_{mn}{}^{\underline{ij}})$		(λ_V)
$-5B$	$-10B$	$-8F$				$-18B$		$-1B$
Result: $40+40$ degrees of freedom								

Table 6.5: A variant Poincaré supergravity multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components. Row three gives the surviving gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. The parameter $\lambda_{mn}{}^{\underline{ij}}$ corresponds to the symmetry associated with the triplet of gauge three-form $b_{mnp}{}^{\underline{ij}}$. The gauge parameter λ_V describes the scalar symmetry of v_m.

scalar field that plays the role of a dilaton U, a real triplet of scalar fields X^{ij}, a triplet of gauge three forms $b_{mnp}{}^{\underline{ij}}$, a gauge field v_m that plays the role of the graviphoton, and a spinor field $\rho_\alpha^{\underline{i}}$.

The transformation rules of the resulting Poincaré supergravity multiplet are those preserving (6.4.6). To preserve (6.4.6a), we require $\lambda_{\mathbb{D}} \equiv 0$. Since Q-supersymmetry does not preserve the gauge fixing conditions, it is necessary to accompany these transformations with appropriate S-supersymmetry, special conformal, and $\mathrm{SU}(2)_R$ compensating transformations. To preserve (6.4.6c), it can be shown that any Q-supersymmetry transformation must be accompanied by a compensating S-supersymmetry transformation with the following parameter

$$\eta_\alpha{}^i(\xi) \;=\; -\frac{\mathrm{i}}{2}(\Sigma^{ab}\xi^i)_\alpha\Big(F_{ab}+w_{ab}\Big)+\frac{\mathrm{i}}{2}\xi_{\alpha j}X^{ij} \, . \tag{6.4.7}$$

A similar analysis shows that to preserve the condition $b_m = 0$, one needs to enforce nontrivial compensating special conformal K-transformations with a parameter $\lambda_K^a(\xi)$. However, since all the other supergravity fields are conformal (not necessarily superconformal) primaries, not transforming under special conformal boosts, in practice we will never have to worry about inserting the compensating $\lambda_K^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations. Finally, we can easily check that the requirement $\delta q^{(\underline{i}\underline{i})} = 0$ is satisfied by implementing in (6.3.1a) a compensating SU(2)$_R$ transformation with the parameter

$$\lambda^{ij}(\xi) = -\frac{1}{2}\mathrm{e}^{U}\xi^{(i}\rho^{j)} , \tag{6.4.8}$$

where $\rho^i = \delta^i_{\underline{i}}\rho^{\underline{i}}$.

6.4.2 Hyper-dilaton Poincaré supergravity action and dilaton potential

We turn to deriving a Poincaré supergravity action by considering the two-derivative action of the vector multiplet compensator [29] in a hyper-dilaton Weyl multiplet background and then imposing appropriate gauge fixing conditions leading to the two frames described above. As shown in [29], the component form of such a vector multiplet action may be derived from the bosonic part of the *BF* Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{BF}|_{bosonic} &= -\frac{1}{4}\Big(F\phi + G_{ij}X^{ij} - \frac{1}{12}\varepsilon^{abcde}f_{ab}b_{cde}\Big) \\ &= -\frac{1}{4}\Big(F\phi + G_{ij}X^{ij} + v^a h_a\Big) , \end{aligned} \tag{6.4.9}$$

with the fields of the $\mathcal{O}(2)$ multiplet being composite. More precisely, this amounts to taking the bosonic sector of the bar projection of eqs. (6.2.21):

$$G^{ij}|_{bosonic} = 2\phi X^{ij} , \tag{6.4.10a}$$

$$\begin{aligned} F|_{bosonic} &= X^{ij}X_{ij} - f^{ab}f_{ab} + 4\phi\nabla^a\nabla_a\phi + 2(\nabla^a\phi)\nabla_a\phi \\ &\quad -6\phi\, w^{ab}f_{ab} - \frac{39}{8}\phi^2 w^{ab}w_{ab} - 16\phi^2 D , \end{aligned} \tag{6.4.10b}$$

$$h_a|_{bosonic} = -\frac{1}{2}\varepsilon_{abcde}f^{bc}f^{de} + 4\nabla^b(\phi f_{ba} + \frac{3}{2}\phi^2 w_{ba}) , \tag{6.4.10c}$$

and plugging (6.4.10) back into (6.4.9). This procedure results in

$$\begin{aligned} e^{-1}\mathcal{L}_{BF}|_{bosonic} &= -\frac{1}{2}\phi(\nabla^a\phi)\nabla_a\phi - \phi^2\nabla^a\nabla_a\phi - \frac{3}{4}\phi X^{ij}X_{ij} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} \\ &\quad +\frac{3}{4}\phi f^{ab}f_{ab} + \frac{9}{4}\phi^2 w^{ab}f_{ab} + \frac{39}{32}\phi^3 w^{ab}w_{ab} + 4\phi^3 D . \end{aligned} \tag{6.4.11}$$

The expression (6.4.11) can be written in terms of the degauged covariant derivative $\mathcal{D}_a$, where we note the following relation

$$\nabla^a\nabla_a\phi = \mathcal{D}^a\mathcal{D}_a\phi + \frac{1}{8}\phi\,\mathcal{R} . \tag{6.4.12}$$

After performing integration by parts, one then arrives at the following bosonic Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{BF}|_{bosonic} &= -\frac{1}{8}\phi^3\mathcal{R} + \frac{3}{2}\phi(\mathcal{D}^a\phi)\mathcal{D}_a\phi - \frac{3}{4}\phi X^{ij}X_{ij} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} \\ &\quad +\frac{3}{4}\phi f^{ab}f_{ab} + \frac{9}{4}\phi^2 w^{ab}f_{ab} + \frac{39}{32}\phi^3 w^{ab}w_{ab} + 4\phi^3 D . \end{aligned} \tag{6.4.13}$$

The action (6.4.13) is given in the standard Weyl multiplet background. When working with a hyper-dilaton Weyl multiplet, we need to take into account that the auxiliary field D is composite (and that (6.3.6b) has to be used). The algebraic expression for D takes the following form

$$D = -\frac{3}{32}\mathcal{R} - \frac{3}{128}w^{ab}w_{ab} - \frac{1}{2q^2}q_{i\underline{i}}\mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} + \text{fermionic terms} . \tag{6.4.14}$$

Upon substituting this, one obtains

$$\begin{aligned} e^{-1}\mathcal{L}|_{bosonic} &= -\frac{1}{2}\phi^3\mathcal{R} - \frac{2}{q^2}\phi^3 q_{i\underline{i}}\mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} + \frac{3}{2}\phi(\mathcal{D}^a\phi)\mathcal{D}_a\phi - \frac{3}{4}\phi X^{ij}X_{ij} \\ &\quad + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc} f^{de} + \frac{3}{4}\phi f^{ab}f_{ab} + \frac{9}{4}\phi^2 w^{ab}f_{ab} + \frac{9}{8}\phi^3 w^{ab}w_{ab} . \end{aligned} \tag{6.4.15}$$

In eq. (6.4.15), there is still a dependence upon the triplet of gauge three-forms $b_{mnp}{}^{\underline{ij}}$, which is hidden in the SU(2)$_R$ connection inside the $\mathcal{D}_a$ derivatives. It is straightforward to obtain the analogue expressions in terms of $\mathbf{D}_a$.

As a direct generalisation, coupling to n vector multiplets leads to the following action

$$\begin{aligned} e^{-1}\mathcal{L}|_{bosonic} &= C_{IJK}\Big(-\frac{1}{2}\phi^I\phi^J\phi^K\mathcal{R} - \frac{2}{q^2}\phi^I\phi^J\phi^K q_{i\underline{i}}\mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} + \frac{3}{2}\phi^I(\mathcal{D}^a\phi^J)\mathcal{D}_a\phi^K \\ &\quad -\frac{3}{4}\phi^I X^{ijJ}X^K_{ij} + \frac{1}{8}\varepsilon_{abcde}v^{aI}f^{bcJ}f^{deK} + \frac{3}{4}\phi^I f^{abJ}f^K_{ab} \\ &\quad +\frac{9}{4}\phi^I\phi^J w^{ab}f^K_{ab} + \frac{9}{8}\phi^I\phi^J\phi^K w^{ab}w_{ab}\Big) . \end{aligned} \tag{6.4.16}$$

The final step to obtain the bosonic sector of the Poincaré supergravity action is to impose the set of gauge fixing conditions on (6.4.15) or appropriate generalisations when physical matter multiplets are included. Here we give the gauge-fixed supergravity action in both the string and Einstein frames. The difference between the Poincaré supergravity multiplets in these two frames comes from the specific gauge-fixing choices used in each. If the gauge-fixing is undone and the theory is viewed within a conformal supergravity framework, one can see that the two frames are linked by supergravity gauge transformations, including superconformal ones. This means the two frames are related by a supersymmetric version of a Weyl transformation.

String frame

Upon implementing the constraints (6.4.4), the resulting BF action turns out to be

$$\begin{aligned} e^{-1}\mathcal{L}|_{bosonic} &= -\frac{1}{2}\phi^3\mathcal{R} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} + \frac{3}{4}\phi f^{ab}f_{ab} + \frac{9}{4}\phi^2 w^{ab}f_{ab} + \frac{9}{8}\phi^3 w^{ab}w_{ab} \\ &\quad +\frac{3}{2}\phi(\partial^m\phi)\partial_m\phi + \frac{1}{4}h^a{}_{ij}h_a{}^{ij} - \frac{3}{4}\phi X^{ij}X_{ij} . \end{aligned} \tag{6.4.17}$$

Here $h_a{}^{ij} = \delta^i_{\underline{i}}\delta^j_{\underline{j}}h_a{}^{\underline{ij}}$ since we have stopped distinguishing between underlined and non-underlined SU(2) indices after gauge fixing. We also stress that $\phi > 0$ as the compensator is nowhere vanishing.

We can further analyse the on-shell structure of (6.4.17). It is clear that w_{ab} and X^{ij} are auxiliary fields that can be algebraically integrated out by using the equations of motion

$$f_{ab} + \phi\, w_{ab} = 0 , \qquad X^{ij} = 0 . \tag{6.4.18}$$

The on-shell Lagrangian then reads

$$e^{-1}\mathcal{L}|_{bosonic} = -\frac{1}{2}\phi^3\mathcal{R}+\frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de}-\frac{3}{8}\phi f^{ab}f_{ab} +\frac{3}{2}\phi(\partial^m\phi)\partial_m\phi+\frac{1}{4}h^a{}_{ij}h_a{}^{ij} \ . \tag{6.4.19}$$

The first three terms are kinetic terms for minimal on-shell $N=1$ Poincaré supergravity with a dynamical graviton and graviphoton, described in a string frame. The last two terms describe a dilaton and a triplet of dynamical gauge three-forms which are not part of the minimal on-shell $N=1$ Poincaré supergravity multiplet.

By construction, the supersymmetric BF-action (3.3.10) is also well defined as an invariant in a hyper-dilaton Weyl background. We can readily construct an invariant of this form by considering the off-shell vector multiplet compensator used in this section and an off-shell linear multiplet given by

$$e^{-1}\mathcal{L}_\xi|_{bosonic} = -\frac{1}{4}\Big(\phi\,F_\xi + G_{\xi\,ij}X^{ij}+v_a h_\xi{}^a\Big) \ , \tag{6.4.20a}$$

where we have defined

$$G_{\xi\,ij} := \xi_{\underline{ij}}\,G_{ij}{}^{\underline{ij}} \ , \quad \varphi_{\xi\,\alpha i} := \xi_{\underline{ij}}\,\varphi_{\alpha i}{}^{\underline{ij}} \ , \\ F_\xi := \xi_{\underline{ij}}F^{\underline{ij}} \ , \quad b_{\xi\,mnp} := \xi_{\underline{ij}}\,b_{mnp}{}^{\underline{ij}} \ , \quad h_\xi{}^a = \xi_{\underline{ij}}\,h^{a\underline{ij}} \ . \tag{6.4.20b}$$

Here $G_{ij}{}^{\underline{ij}}$, $\varphi_{\alpha i}{}^{\underline{ij}}$, $F^{\underline{ij}}$, $b_{mnp}{}^{\underline{ij}}$, and $h^{a\underline{ij}}$ are fields of the composite triplet of linear multiplets (6.3.13) constructed in terms of fundamental fields of the hyper-dilaton Weyl multiplet, while $\xi_{\underline{ij}} = \xi_{\underline{ji}}$ is a real triplet of (structure group invariant) constants. The bosonic part of the resulting Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L}_\xi|_{bosonic} &= -\frac{1}{4}\xi_{\underline{ij}}\Big(q_i{}^{\underline{i}}q_j{}^{\underline{j}}X^{ij}-\frac{1}{12}\varepsilon^{mnpqr}b_{mnp}{}^{\underline{ij}}f_{qr}\Big) \\ &= -\frac{1}{4}\xi_{\underline{ij}}\Big(q_i{}^{\underline{i}}q_j{}^{\underline{j}}X^{ij}+h^{m\underline{ij}}v_m\Big) \ . \end{aligned} \tag{6.4.21}$$

Upon imposing the gauge fixing (6.4.4) and adding (6.4.21) into (6.4.17), we get

$$\begin{aligned} e^{-1}\mathcal{L}|_{bosonic} = & -\frac{1}{2}\phi^3\mathcal{R}+\frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de}+\frac{3}{4}\phi f^{ab}f_{ab}+\frac{9}{4}\phi^2 w^{ab}f_{ab}+\frac{9}{8}\phi^3 w^{ab}w_{ab} \\ & +\frac{3}{2}\phi(\partial^m\phi)\partial_m\phi+\frac{1}{4}h^a{}_{ij}h_a{}^{ij}-\frac{3}{4}\phi X^{ij}X_{ij} \\ & +\frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}{}^{ij}f_{qr}-\frac{1}{4}\xi_{ij}X^{ij} \ , \end{aligned} \tag{6.4.22}$$

where, after gauge fixing, we have used $\xi_{ij} = \delta_{\underline{i}}^i\delta_{\underline{j}}^j\xi_{\underline{ij}}$ and $b_{mnp}{}^{ij} = \delta_{\underline{i}}^i\delta_{\underline{j}}^j b_{mnp}{}^{\underline{ij}}$. We can then integrate w_{ab} and X^{ij} out as they are auxiliary fields. With the ξ-deformation turned on, the equations of motion obtained from (6.4.22) become

$$f_{ab}+\phi\,w_{ab} = 0 \ , \qquad \phi X^{ij}+\frac{1}{6}\xi^{ij} = 0 \ . \tag{6.4.23}$$

This leads to the on-shell Lagrangian

$$e^{-1}\mathcal{L}|_{bosonic} = -\frac{1}{2}\phi^3\mathcal{R}+\frac{1}{8}\varepsilon_{mnpqr}v^m f^{np}f^{qr}-\frac{3}{8}\phi f^{mn}f_{mn}+\frac{3}{2}\phi(\partial^m\phi)\partial_m\phi+\frac{1}{4}h^m{}_{kl}h_m{}^{kl}$$

$$+\frac{1}{24\phi}\xi^2+\frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}{}^{ij}f_{qr}\ , \tag{6.4.24}$$

where

$$\xi^2:=\frac{1}{2}\xi^{ij}\xi_{ij}\geq 0\ . \tag{6.4.25}$$

As a result, we have obtained a non-trivial, negatively defined, potential for the dilaton. The previous Lagrangian admits a constant dilaton, AdS_5 vacua.

Einstein frame

If we instead adopt the gauge fixing conditions (6.4.6) to (6.4.15), we obtain the following Poincaré supergravity action

$$\begin{aligned} e^{-1}\,\mathcal{L}|_{bosonic} &= -\frac{1}{2}\mathcal{R}+\frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de}+\frac{3}{4}f^{ab}f_{ab}+\frac{9}{4}w^{ab}f_{ab}+\frac{9}{8}w^{ab}w_{ab} \\ &\quad -2(\partial^m U)\partial_m U+\frac{1}{4}\mathrm{e}^{4U}h^a{}_{ij}h_a{}^{ij}-\frac{3}{4}X^{ij}X_{ij}\ . \end{aligned} \tag{6.4.26}$$

We can further analyse the on-shell structure of (6.4.26). It is clear that w_{ab} and X^{ij} are auxiliary fields that can be algebraically integrated out by using the equations of motion

$$w_{ab}=-f_{ab}\ ,\quad X^{ij}=0\ . \tag{6.4.27}$$

The on-shell Lagrangian then reads

$$\begin{aligned} e^{-1}\,\mathcal{L}|_{bosonic} &= -\frac{1}{2}\mathcal{R}+\frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de}-\frac{3}{8}f^{ab}f_{ab} \\ &\quad -2(\partial^m U)\partial_m U+\frac{1}{4}\mathrm{e}^{4U}h^a{}_{ij}h_a{}^{ij}\ . \end{aligned} \tag{6.4.28}$$

The first three terms describe the standard kinetic terms for minimal on-shell $N=1$ Poincaré supergravity with a dynamical graviton and graviphoton. The last two terms describe a dilaton and a triplet of dynamical gauge three-forms which are not part of the minimal on-shell $N=1$ Poincaré supergravity multiplet. This is a standard feature of dilaton multiplets, where on-shell a physical dilaton multiplet adds to the degrees of freedom of the multiplet. In the case of hyper-dilaton Poincaré supergravity, the extra multiplet is a hypermultiplet where three of the scalars have been dualised to a triplet of gauge three forms in complete analogy to the $4D$ $N=2$ case of [4, 241].

Let us now add the second supersymmetric invariant (6.4.21) to the action (6.4.15). Upon imposing the gauge fixing conditions (6.4.6), we arrive at

$$\begin{aligned} e^{-1}\,\mathcal{L}|_{bosonic} &= -\frac{1}{2}\mathcal{R}+\frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de}+\frac{3}{4}f^{ab}f_{ab}+\frac{9}{4}w^{ab}f_{ab}+\frac{9}{8}w^{ab}w_{ab} \\ &\quad -2(\partial^m U)\partial_m U+\frac{1}{4}\mathrm{e}^{4U}h^a{}_{ij}h_a{}^{ij}-\frac{3}{4}X^{ij}X_{ij} \\ &\quad +\frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}{}^{ij}f_{qr}-\frac{1}{4}\xi_{ij}\mathrm{e}^{-2U}X^{ij}\ . \end{aligned} \tag{6.4.29}$$

We can then integrate w_{ab} and X^{ij} out as they are auxiliary fields. With the ξ-deformation turned on, the equations of motion obtained from (6.4.29) become

$$w_{ab}=-f_{ab}\ ,\quad X^{ij}=-\frac{1}{6}\xi^{ij}\mathrm{e}^{-2U}\ . \tag{6.4.30}$$

This leads to the on-shell Lagrangian

$$
\begin{aligned}
e^{-1}\,\mathcal{L}|_{bosonic} \;&=\; -\frac{1}{2}\mathcal{R}+\frac{1}{8}\varepsilon_{mnpqr}v^m f^{np} f^{qr}-\frac{3}{8}f^{mn}f_{mn}-2(\partial^m U)\partial_m U+\frac{1}{4}\mathrm{e}^{4U}h^m{}_{kl}h_m{}^{kl} \\
&\quad +\frac{1}{24}\mathrm{e}^{-4U}\xi^2+\frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}{}^{ij}f_{qr}\,. \qquad (6.4.31)
\end{aligned}
$$

6.5 Superconformal multiplets in 6D $N=(1,0)$ superspace

The following section reviews the relevant details of various superconformal matter multiplets required in this work. Along with the standard Weyl multiplet discussed in subsections 2.2.2 and 2.2.3, they serve as the building blocks for the invariants of 6D conformal supergravity discussed in this paper. Here we make use of the conformal superspace formulation in the traceless frame [32] and results from [69]. We also refer the reader to the following list of papers for other work on flat superspace and multiplets in six dimensions [270–276] while see also [129, 130, 277–283] for alternative curved superspace approaches to describe supergravity multiplets in six dimensions.

6.5.1 The on-shell hypermultiplet

Analogously to the 5D case, our starting point is the on-shell realisation for the 6D $N=(1,0)$ hypermultiplet with $4+4$ degrees of freedom. In conformal superspace, it is described by a Lorentz scalar superfield $q^{i\underline{i}}$ subject to the constraint

$$
\nabla_\alpha^{(i}q^{j)\underline{j}}=0\,. \qquad (6.5.1)
$$

Here, the index $\underline{i}=\underline{1},\underline{2}$ denotes an SU(2) flavour index. The superfield $q^{i\underline{i}}$ is a Lorentz scalar superconformal primary,

$$
M_{ab}q^{i\underline{i}}=0\,,\qquad S^\alpha_j q^{i\underline{i}}=K_a q^{i\underline{i}}=0\,,\qquad J^{kl}q^{i\underline{i}}=\varepsilon^{i(k}q^{l)\underline{i}}\,. \qquad (6.5.2)
$$

Eqs. (6.5.1), (6.5.2), and the relation (C.1.11) tell us that $\mathbb{D}q^{i\underline{i}}=2q^{i\underline{i}}$. The only independent descent superfield of $q^{i\underline{i}}$ is a dimension 5/2 spinor superfield

$$
\rho_\alpha^{\underline{k}}:=\nabla_\alpha^j q_j{}^{\underline{k}}\,. \qquad (6.5.3)
$$

Equation (6.5.1) can now equivalently be written in terms of this spinor superfield

$$
\nabla_\alpha^i q^{k\underline{k}}=-\frac{1}{2}\varepsilon^{ik}\rho_\alpha^{\underline{k}}\,. \qquad (6.5.4)
$$

Applying spinor derivatives on these independent fields leads to several implications of the (anti-)commutation relations (C.1.13), (C.1.14), along with the constraints (6.5.2), and (6.5.4):

$$
\nabla_\alpha^i\nabla_\beta^j q_j{}^{\underline{i}} \;=\; \nabla_\alpha^i\rho_\beta^{\underline{i}}=-4\mathrm{i}\nabla_{\alpha\beta}q^{i\underline{i}}\,, \qquad (6.5.5)
$$

with $\nabla_{\alpha\beta}:=(\gamma^a)_{\alpha\beta}\nabla_a$. Next, we shall consider

$$
\varepsilon^{\gamma\delta\alpha\beta}\{\nabla_\alpha^k,\nabla_{\beta k}\}\rho_\delta^{\underline{i}} \;=\; 8\mathrm{i}\tilde{\nabla}^{\gamma\delta}\rho_\delta^{\underline{i}}-24\mathrm{i}W^{\gamma\delta}\rho_\delta^{\underline{i}}+288X^{\gamma k}q_k{}^{\underline{i}}\,, \qquad (6.5.6)
$$

here $\tilde{\nabla}^{\alpha\beta} := (\tilde{\gamma}^a)^{\alpha\beta}\nabla_a$ and we have made use of (C.1.13) and the S-supersymmetry transformation

$$S_i^\gamma \rho_\beta^{\underline{i}} = 16\delta_\beta^\gamma q_i{}^{\underline{i}} \qquad \Longrightarrow \qquad K_a \rho_\beta^{\underline{i}} = 0 \, . \tag{6.5.7}$$

On the other hand, by virtue of (6.5.5), we also have that

$$\begin{aligned} \varepsilon^{\gamma\delta\alpha\beta}\{\nabla_\alpha^k, \nabla_{\beta k}\}\rho_\delta^{\underline{i}} &= 16\mathrm{i}(\tilde{\gamma}^\alpha)^{\alpha\gamma}\nabla_\alpha^i\nabla_a q_i{}^{\underline{i}} \\ &= 16\mathrm{i}(\tilde{\gamma}^\alpha)^{\alpha\gamma}[\nabla_\alpha^i, \nabla_a]q_i{}^{\underline{i}} + 16\mathrm{i}(\tilde{\gamma}^\alpha)^{\alpha\gamma}\nabla_a\rho_\alpha{}^{\underline{i}} \, . \end{aligned} \tag{6.5.8}$$

Applying the commutation relation (C.1.14), we can then equate (6.5.6) and (6.5.8) to obtain

$$(\tilde{\gamma}^a\nabla_a\rho^{\underline{i}})^\alpha = -W^{\alpha\beta}\rho_\beta^{\underline{i}} - 4\mathrm{i}X^{\alpha k}q_k{}^{\underline{i}} \, . \tag{6.5.9}$$

We can then hit both sides of (6.5.9) with ∇_α^i and make use of (6.5.5), (C.1.14), and the identity (C.1.17). This results in the equation

$$\Box q^{i\underline{i}} = \frac{1}{2}X^i\rho^{\underline{i}} - \frac{1}{2}Y q^{i\underline{i}} \, , \qquad \Box := \nabla^a\nabla_a \, . \tag{6.5.10}$$

The local superconformal $\delta = \delta_Q + \delta_{\mathcal{H}}$ transformations (except translation, i.e. $\xi^a = 0$) of the covariant superfields $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ can be derived using (2.2.93) and the relations (6.5.4), (6.5.5), and (6.5.7). This leads to

$$\delta q^{i\underline{i}} = \frac{1}{2}\xi^i\rho^{\underline{i}} + \Lambda^i{}_k q^{k\underline{i}} + 2\sigma q^{i\underline{i}} \, , \tag{6.5.11a}$$

$$\delta\rho_\alpha^{\underline{i}} = -4\mathrm{i}(\xi_i\gamma^a)_\alpha\nabla_a q^{i\underline{i}} - \frac{1}{4}\Lambda_{ab}(\gamma^{ab}\rho^{\underline{i}})_\alpha + \frac{5}{2}\sigma\rho_\alpha^{\underline{i}} + 16\eta_\alpha^i q_i{}^{\underline{i}} \, . \tag{6.5.11b}$$

6.5.2 The $\mathcal{O}(2)$ multiplet

The 6D linear multiplet, or $\mathcal{O}(2)$ multiplet can be described in terms of an $\mathrm{SU}(2)_R$ triplet of Lorentz scalar superfields L^{ij}, with $(L^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}L^{kl}$ and satisfies the defining constraint

$$\nabla_\alpha^{(i}L^{jk)} = 0 \, . \tag{6.5.12}$$

Here L^{ij} is a superconformal primary dimension-4 superfield,

$$S_k^\gamma L^{ij} = K^a L^{ij} = 0 \, , \qquad \mathbb{D}L^{ij} = 4L^{ij} \, , \qquad J^{ij}L^{kl} = \varepsilon^{k(i}L^{j)l} + \varepsilon^{l(i}L^{j)k} \, . \tag{6.5.13}$$

The tower of component fields of the superfield L^{ij} is given by the following set of useful identities:

$$\nabla_\alpha^i L^{jk} = -2\varepsilon^{i(j}\varphi_\alpha^{k)} \, , \tag{6.5.14a}$$

$$\nabla_\alpha^i\varphi_\beta^j = -\frac{\mathrm{i}}{2}\varepsilon^{ij}H_{\alpha\beta} - \mathrm{i}\nabla_{\alpha\beta}L^{ij} \, , \tag{6.5.14b}$$

$$\nabla_\gamma^k H_{\alpha\beta} = -8\nabla_{\gamma[\alpha}\varphi_{\beta]}^k - 2\nabla_{\alpha\beta}\varphi_\gamma^k + 2\varepsilon_{\alpha\beta\gamma\delta}W^{\delta\rho}\varphi_\rho^k \, , \tag{6.5.14c}$$

where we have defined the independent descendant superfields

$$\varphi_\alpha^i := -\frac{1}{3}\nabla_{\alpha j}L^{ij} \, , \tag{6.5.15a}$$

$$H_a := -\frac{\mathrm{i}}{4}(\tilde{\gamma}_a)^{\alpha\beta}\nabla_\alpha^k \varphi_{\beta k} \;, \qquad H_{\alpha\beta} := (\gamma^a)_{\alpha\beta} H_a \;. \tag{6.5.15b}$$

We will be using these results later when analysing the component structure of the multiplet. Further, it can be checked that H^a obeys the differential condition

$$\nabla_a H^a = 0 \;, \qquad H^a := \frac{1}{5!}\varepsilon^{abcdef} H_{bcdef} \;. \tag{6.5.16}$$

The descendants (6.5.15) prove to be annihilated by K_a and to satisfy

$$S_i^\alpha \varphi_\beta^j = 8\delta_\beta^\alpha L^{ij} \;, \tag{6.5.17a}$$

$$S_k^\gamma H_{\alpha\beta} = -40\mathrm{i}\delta^\gamma_{[\alpha}\varphi_{\beta]k} \;. \tag{6.5.17b}$$

We refer the reader to [69] for a superform description of the $\mathcal{O}(2)$ multiplet.

6.6 The hyper-dilaton Weyl multiplet in $6D$

The aim of this section is to construct the $40+40$ hyper-dilaton Weyl multiplet of off-shell $N=(1,0)$ conformal supergravity in six dimensions. The construction mimics the $5D$ $N=1$ case elaborated earlier in our paper and the $4D$ $N=2$ case of [4].

We begin with the component structure of the on-shell hypermultiplet. This can be readily extracted from the previous superspace realisation via the bar projection. The independent components of the on-shell hypermultiplet are simply: the Lorentz scalar field $q^{i\underline{i}}|$ which is a superconformal primary, and the spinor field $\rho_\alpha^{\underline{i}}|$, which account for $4+4$ on-shell degrees of freedom. All other descendants are derivatives of these two fields.

In what follows, we will associate the same symbol for the covariant component fields and the corresponding superfields, when the interpretation is clear from the context. The local superconformal transformations of the component fields follow directly from the projections of (6.5.11), which give

$$\delta q^{i\underline{i}} = \frac{1}{2}\xi^i\rho^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + 2\lambda_{\mathbb{D}} q^{i\underline{i}} \;, \tag{6.6.1a}$$

$$\delta\rho_\alpha^{\underline{i}} = -4\mathrm{i}(\xi_i\gamma^a)_\alpha \nabla_a q^{i\underline{i}} - \frac{1}{4}\lambda_{ab}(\gamma^{ab}\rho^{\underline{i}})_\alpha + \frac{5}{2}\lambda_{\mathbb{D}}\rho_\alpha^{\underline{i}} + 16\eta_\alpha^i q_i{}^{\underline{i}} \;, \tag{6.6.1b}$$

where

$$\nabla_a q^{i\underline{i}} = \mathcal{D}_a q^{i\underline{i}} - \frac{1}{4}\psi_a{}^i\rho^{\underline{i}} \;. \tag{6.6.2}$$

Unlike the standard Weyl multiplet, the algebra of the local transformations (6.6.1) closes only when the equations of motion for the fields $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ are imposed. These equations of motion can be obtained by taking the bar projection of (6.5.9) and (6.5.10). Specifically, in the traceless frame, the lowest component of the descendants $\nabla_a\nabla_\alpha^i q_i{}^{\underline{i}}|$ and $\Box q^{i\underline{i}}|$ leads to the following two constraints:

$$(\nabla_a\rho^{\underline{i}}\tilde{\gamma}^a)^\alpha = -\frac{1}{12}(\rho^{\underline{i}}\tilde{\gamma}^{abc})^\alpha T^-_{abc} + \frac{8\mathrm{i}}{15}\chi^{\alpha k} q_k{}^{\underline{i}} \;, \tag{6.6.3a}$$

$$\Box q^{i\underline{i}} = \frac{1}{15}\chi^i\rho^{\underline{i}} - \frac{1}{15}Dq^{i\underline{i}} \;, \qquad \Box := \nabla^a\nabla_a \;. \tag{6.6.3b}$$

The expressions for $\nabla_a \rho_\alpha^{\underline{i}}$ and $\Box q^{i\underline{i}}$ in terms of the derivatives $\mathcal{D}_a$ are given by

$$\begin{aligned}
\nabla_a \rho_\alpha^{\underline{i}} &= \mathcal{D}_a \rho_\alpha^{\underline{i}} + 2\mathrm{i}(\psi_{ak}\gamma^c)_\alpha \left(\mathcal{D}_c q^{k\underline{i}} - \frac{1}{4}\psi_c{}^k \rho^{\underline{i}}\right) - 8\phi_{a\alpha}^{\ k} q_k{}^{\underline{i}} \,, &(6.6.4a)\\
\Box q^{i\underline{i}} &= \mathcal{D}^a \mathcal{D}_a q^{i\underline{i}} - 4\mathfrak{f}_a{}^a q^{i\underline{i}} - \frac{1}{4}\mathcal{D}_a(\psi^{ai}\rho^{\underline{i}}) - \frac{\mathrm{i}}{4}\phi_a{}^i \tilde{\gamma}^a \rho^{\underline{i}} \\
&\quad + \frac{1}{4}\psi_a{}^{\alpha i}\Big[-\mathcal{D}^a \rho_\alpha^{\underline{i}} - 2\mathrm{i}(\psi^a{}_j \gamma^c)_\alpha \mathcal{D}_c q^{j\underline{i}} + \frac{1}{24}(\rho^{\underline{i}}\tilde{\gamma}^{bcd}\gamma^a)_\alpha T^-_{bcd} \\
&\qquad - \frac{8\mathrm{i}}{15}(\chi^k \gamma^a)_\alpha q_k^{\underline{i}} + \frac{\mathrm{i}}{2}(\psi^a{}_j \gamma^c)_\alpha \psi_c{}^j \rho^{\underline{i}} + 8\phi^{a}{}_{\alpha}^{j} q_j{}^{\underline{i}} \Big] \,. &(6.6.4b)
\end{aligned}$$

Equations (6.6.3) can then be interpreted as algebraic equations for the standard Weyl multiplet that determine the auxiliary fields $\chi^{\alpha i}$ and D in terms of $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$, together with the other independent fields of the standard Weyl multiplet. It can be noted that in the traceless frame equations (6.6.4a) and (6.6.4b) do not depend on the fields $\chi^{\alpha i}$ and D respectively making it trivial to find these auxiliary fields. If we assume that $q^{i\underline{i}}$ is an invertible matrix, which is equivalent to imposing

$$q^2 := q^{i\underline{i}} q_{i\underline{i}} = \varepsilon_{ij}\varepsilon_{\underline{i}\underline{j}} q^{i\underline{i}} q^{j\underline{j}} = 2\det q^{i\underline{i}} \neq 0 \,, \tag{6.6.5}$$

then the following relations hold

$$\begin{aligned}
\chi_i^\alpha &= \frac{15\mathrm{i}}{4} q^{-2} q_{i\underline{i}} \left[(\nabla_a \rho^{\underline{i}} \tilde{\gamma})^\alpha + \frac{1}{12}(\rho^{\underline{i}} \tilde{\gamma}^{abc})^\alpha T^-_{abc} \right] , &(6.6.6a)\\
D &= -15 q^{-2} q^{i\underline{i}} \Box q_{i\underline{i}} + \frac{15\mathrm{i}}{8} q^{-2} \left[\nabla_a \rho^{\underline{j}} \tilde{\gamma}^a \rho_{\underline{j}} + \frac{1}{12}(\rho^{\underline{j}} \tilde{\gamma}^{abc} \rho_{\underline{j}}) T^-_{abc} \right] . &(6.6.6b)
\end{aligned}$$

Once more, we stress that the right-hand side of equation (6.6.6a) does not have any dependence on field χ, thus making it a composite field. Similarly, the right-hand side of equation (6.6.6b) does not depend on D, however it has an implicit dependence on χ through the special conformal connection $\mathfrak{f}_{ab}$, eq. (2.2.40c) and see (6.6.4b), that is hidden in the expression of $\Box q^{i\underline{i}}$. It is straightforward to pull out the explicit dependence upon χ and then use (6.6.6a). As a result, both χ and D are composite.

As a next step in the construction of the hyper-dilaton Weyl multiplet, we note that associated to an on-shell hypermultiplet one can construct a triplet of linear multiplets, exactly as in the $4D$ $N = 2$ and $5D$ $N = 1$ cases. The component fields of the $N = 1$ off-shell linear (or $\mathcal{O}(2)$) multiplet are defined in terms of the bar projections of (6.5.15): an $\mathrm{SU}(2)_R$ triplet of Lorentz scalar fields $L^{ij} = L^{ij}|$; a spinor field $\varphi_\alpha^i = \varphi_\alpha^i|$; and a closed anti-symmetric five-form field strength $h_{mnpqr} := H_{mnpqr}|$, which is equivalent to a conserved dual vector $h^a := \frac{1}{5!}\varepsilon^{amnpqr} h_{mnpqr}$. Defining $H^a = H^a|$, at the component level it holds that

$$H_a = h_a - \psi_{bi}\gamma^{ab}\varphi^i - \frac{\mathrm{i}}{2}\psi_{bi}\gamma^{abc}\psi_{cj}L^{ij} \quad . \tag{6.6.7}$$

The covariant conservation equation of H_a is

$$\nabla^a H_a \;=\; 0\,. \tag{6.6.8}$$

The constraint implies the existence of a gauge four-form potential, b_{mnpq}, and its exterior derivative $h_{mnpqr} := 5\partial_{[m} b_{npqr]}$. The local superconformal transformations of the linear multiplet in a standard

Weyl multiplet background are given by

$$\begin{aligned}
\delta L^{ij} &= 2\xi^{\alpha(i}\varphi_\alpha^{j)} + 2\lambda^{(i}{}_k L^{j)k} + 4\lambda_{\mathbb{D}} L^{ij} \ , && (6.6.9a)\\
\delta\varphi_\alpha^i &= \frac{\mathrm{i}}{2}\xi^{\beta i} H_{\beta\alpha} - \mathrm{i}\xi_j^\beta \nabla_{\beta\alpha} L^{ij} + \frac{1}{4}\lambda^{ab}(\varphi^i\tilde{\gamma}_{ab})_\alpha - \lambda^{ij}\varphi_{\alpha j} + \frac{9}{2}\lambda_{\mathbb{D}}\varphi_\alpha^i + 8\eta_\alpha^j L_j{}^i \ , && (6.6.9b)\\
\delta H_a &= 2(\xi_i\gamma_{ab})^\beta\nabla^b\varphi_\beta^i + \frac{1}{12}(\xi^i\gamma_a\tilde{\gamma}^{bcd}\varphi_i)T^-_{bcd} + \lambda_a{}^d H_d + 5\lambda_{\mathbb{D}} H_a \\
&\quad -10\mathrm{i}\eta^i\tilde{\gamma}_a\varphi_i \ , && (6.6.9c)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_a L^{ij} &= \mathcal{D}_a L^{ij} - \psi_a{}^{(i}\varphi^{j)} \ , && (6.6.10a)\\
\nabla_a\varphi^i &= \mathcal{D}_a\varphi^i - \frac{\mathrm{i}}{4}\psi_a{}^i\gamma_b H^b - \frac{\mathrm{i}}{2}\psi_{aj}\tilde{\nabla\!\!\!/} L^{ij} + 4\phi_{aj}L^{ij} \ . && (6.6.10b)
\end{aligned}$$

The locally superconformal transformations of b_{mnpq} are

$$\delta b_{mnpq} = -\varepsilon_{mnpqef}(\xi_i\gamma^{ef}\varphi^i) + 8\mathrm{i}(\psi_{[mi}\gamma_{npq]}\xi_j)L^{ij} + 4\partial_{[m}l_{npq]} \ , \qquad (6.6.11)$$

where we have also included the gauge transformation $\delta_l b_{mnpq} = 4\partial_{[m}l_{npq]}$ leaving h_{mnpqr} and H^a invariant. For convenience, we have summarised the dilatation weights of the fields of the $\mathcal{O}(2)$ multiplet in Table 6.6.

	L_{ij}	$\varphi_{\alpha i}$	H_a	b_{mnpq}
$\mathbb{D}$	4	9/2	5	0

Table 6.6: Summary of the dilatation weights in the off-shell $\mathcal{O}(2)$ multiplet.

Now that we have reviewed the structure of a locally superconformal $\mathcal{O}(2)$ multiplet, we return to constructing a triplet of linear multiplets from an on-shell hypermultiplet. Given that $q^{i\underline{i}}$ and $\rho_\alpha^{\underline{i}}$ describe an on-shell hypermultiplet in a standard Weyl multiplet background with transformation rules (6.6.1), it can be shown that the following composite fields define a triplet of $\mathcal{O}(2)$ multiplets

$$\begin{aligned}
L_{ij}{}^{\underline{ij}} &= q_{(i}{}^{\underline{i}}q_{j)}{}^{\underline{j}} = q_i{}^{(\underline{i}}q_j{}^{\underline{j})} \ , && (6.6.12a)\\
\varphi_{\alpha i}{}^{\underline{ij}} &= \frac{1}{2}q_i{}^{(\underline{i}}\rho_\alpha{}^{\underline{j})} \ , && (6.6.12b)\\
H^{a\underline{ij}} &= 2q^{i(\underline{i}}\nabla^a q_i{}^{\underline{j})} - \frac{\mathrm{i}}{8}\rho^{(\underline{i}}\tilde{\gamma}^a\rho^{\underline{j})} \ . && (6.6.12c)
\end{aligned}$$

These fields all transform according to (6.6.9) and each of the previous fields is symmetric in $\underline{i}$ and $\underline{j}$. The field $H^{a\underline{ij}}$, in particular, is interesting as it can be used to express the $SU(2)_R$ connection $\phi_m{}^{ij}$ as a composite field. To see this, we introduce a new covariant derivative

$$\mathbf{D}_a = e_a{}^m\left(\partial_m - \frac{1}{2}\omega_m{}^{cd}M_{cd} - b_m\mathbb{D}\right) = \mathcal{D}_a + e_a{}^m\phi_m{}^{ij}J_{ij} \ , \qquad (6.6.13)$$

which then allows us to rearrange eq. (6.6.12c) for the SU(2)$_R$ gauge connection:

$$\phi_a{}^{ij} = 4q^{-4}q^{(i}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\left[q^{k\underline{i}}\mathbf{D}_a q_k{}^{\underline{j}} - \frac{1}{4}q^{k\underline{i}}(\psi_{ak}\rho^{\underline{j}}) - \frac{\mathrm{i}}{16}\rho^{\underline{i}}\tilde{\gamma}_a\rho^{\underline{j}} - \frac{1}{2}H_a{}^{\underline{ij}}\right] \ . \qquad (6.6.14)$$

This concludes the definition of the hyper-dilaton Weyl multiplet. Our analysis demonstrates that the hyper-dilaton Weyl multiplet defines a new representation of the off-shell local $6D$, $N=(1,0)$ superconformal algebra. The multiplet comprises the following independent fields: $e_m{}^a$, b_m, T^-_{abc}, $q^{i\underline{i}}$, $b_{mnpq}{}^{\underline{ij}}$, ψ_{mi}, and $\rho^{\underline{i}}$. It also possesses the same number of off-shell degrees of freedom as the standard Weyl multiplet, $40+40$. Table 6.7 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields. Note that with the ingredients provided so far, it is a straightforward

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_m{}^a$	$\phi_m{}^{ij}$	ψ_{mi}	$\phi_m{}^i$	T^-_{abc}	$\rho^{\underline{i}}$	$q^{i\underline{i}}$	$b_{mnpq}{}^{\underline{ij}}$
36	0	$6B$	0	0	$48F$	0	$10B$	$8F$	$4B$	$45B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Q	S				$\lambda_{mnp}{}^{\underline{ij}}$-sym
$-6B$	$-15B$	$-1B$	$-6B$	$-3B$	$-8F$	$-8F$				$-30B$
Result: $40+40$ degrees of freedom										

Table 6.7: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components of these fields — composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that the parameter $\lambda_{mnp}{}^{\underline{ij}}$ describes the symmetry associated with the gauge four-forms $b_{mnpq}{}^{\underline{ij}}$ with field strength five-forms $h_{mnpqr}{}^{\underline{ij}}$ and $E^{a\underline{ij}}$. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom.

exercise to obtain the locally superconformal transformations of the fields of the hyper-dilaton Weyl multiplet written only in terms of the independent fields and they are given as follows:

$$
\begin{aligned}
\delta e_m{}^a &= -\mathrm{i}\,\xi_i\gamma^a\psi_m{}^i + \lambda^a{}_b e_m{}^b - \lambda_{\mathbb{D}} e_m{}^a\ , && (6.6.15a)\\
\delta\psi_{mi}^{\alpha} &= 2\mathcal{D}_m\xi_i^{\alpha} - \frac{1}{12}T^-_{abc}(\xi_i\gamma_m\tilde{\gamma}^{abc})^{\alpha} + \frac{1}{4}\lambda^{ab}(\psi_{mi}\gamma_{ab})^{\alpha} - \lambda_i{}^j\,\psi_{mj}^{\alpha} \\
&\quad -\frac{1}{2}\lambda_{\mathbb{D}}\,\psi_{mi}^{\alpha} - 2\mathrm{i}(\eta_i\tilde{\gamma}_m)^{\alpha}\ , && (6.6.15b)\\
\delta b_m &= \frac{1}{4}q^{-2}q_{i\underline{i}}(\xi^i\gamma_m)_{\alpha}\left[\frac{1}{12}(\rho^{\underline{i}}\tilde{\gamma}^{abc})^{\alpha}T^-_{abc} - (\tilde{\gamma}^a\nabla_a\rho^{\underline{i}})^{\alpha}\right] \\
&\quad +\xi_i\phi_m{}^i + \partial_m\lambda_{\mathbb{D}} - \psi_m{}^i\eta_i - 2\lambda_m\ , && (6.6.15c)\\
\delta T^-_{abc} &= -\frac{\mathrm{i}}{8}\xi^k\gamma^{ef}\gamma_{abc}R(Q)_{efk} \\
&\quad -\frac{1}{2}q^{-2}q_{i\underline{i}}(\xi^i\gamma_{abc})_{\alpha}\left[\frac{1}{12}(\rho^{\underline{i}}\tilde{\gamma}^{abc})^{\alpha}T^-_{abc} - (\tilde{\gamma}^a\nabla_a\rho^{\underline{i}})^{\alpha}\right] \\
&\quad -3\lambda^e{}_{[a}T^-_{bc]e} + \lambda_{\mathbb{D}}T^-_{abc}\ , && (6.6.15d)\\
\delta q^{i\underline{i}} &= \frac{1}{2}\xi^i\rho^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + 2\lambda_{\mathbb{D}}q^{i\underline{i}}\ , && (6.6.15e)\\
\delta\rho^{\underline{i}}_{\alpha} &= -4\mathrm{i}(\xi_i\gamma^c)_{\alpha}\nabla_c q^{i\underline{i}} + \frac{1}{4}\lambda_{ab}(\rho^{\underline{i}}\tilde{\gamma}^{ab})_{\alpha} + \frac{5}{2}\lambda_{\mathbb{D}}\rho^{\underline{i}}_{\alpha} + 16\eta^k_{\alpha}q_k{}^{\underline{i}}\ , && (6.6.15f)\\
\delta b_{mnpq} &= -\varepsilon_{mnpqef}\xi_i\gamma^{ef}\varphi^i + 8\mathrm{i}(\psi_{[mi}\gamma_{npq]}\xi_j)L^{ij} + 4\partial_{[m}l_{npq]}\ . && (6.6.15g)
\end{aligned}
$$

For completeness, here we present the expressions relevant to the transformation in terms of this new covariant derivative $\mathbf{D}_a$ instead of $\mathcal{D}_a$, which has an implicit dependence on the composite $SU(2)_R$ connection $\phi_a{}^{ij}$.

$$
\begin{aligned}
\nabla_a q^{i\underline{i}} &= \frac{1}{2}\mathbf{D}_a q^{i\underline{i}} - \frac{1}{8}\psi_a{}^i\rho^{\underline{i}} - q^{-2}q^i{}_{\underline{k}}q^{k\underline{i}}\mathbf{D}_a q_k{}^{\underline{k}} + \frac{1}{4}q^{-2}q^i{}_{\underline{k}}q^{k\underline{i}}(\psi_{ak}\rho^{\underline{k}}) \\
&\quad +\frac{\mathrm{i}}{8}q^{-2}q^i{}_{\underline{k}}(\rho^{(\underline{k}}\tilde{\gamma}_a\rho^{\underline{i})}) + q^{-2}q^i{}_{\underline{k}}H_a{}^{k\underline{i}}\ , && (6.6.16a)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_a\rho^{\underline{i}}_{\alpha} &= \mathbf{D}_a\rho^{\underline{i}}_{\alpha} + 2\mathrm{i}(\psi_{ak}\gamma^c)_\alpha\nabla_c q^{k\underline{i}} - 8\phi_{a\alpha}^{\ k}q_k{}^{\underline{i}}\ , &(6.6.16\text{b})\\
\Box q^{i\underline{i}} &= \mathbf{D}_a\nabla^a q^{i\underline{i}} - 4\mathfrak{f}_a{}^a q^{i\underline{i}}\\
&\quad +4q^{-4}q^{(i}{}_{\underline{k}}q^{j)}{}_{\underline{l}}(\nabla^a q_j{}^{\underline{i}})\left[q^{k\underline{k}}\mathbf{D}_a q_k{}^{\underline{l}} - \frac{1}{2}H_a{}^{\underline{kl}} - \frac{1}{4}q^{k\underline{k}}(\psi_{ak}\rho^{\underline{l}})\frac{\mathrm{i}}{16}\rho^{\underline{k}}\tilde{\gamma}_a\rho^{\underline{l}}\right]\\
&\quad -\frac{1}{96}(\psi_a{}^i\gamma^a\tilde{\gamma}^{bcd}\rho^{\underline{i}})T^-_{bcd} + \frac{2\mathrm{i}}{15}(\psi_a{}^i\gamma^a\chi^k)q^{\underline{i}}_k - \frac{1}{4}\psi_a{}^{\alpha i}\nabla_a\rho^{\underline{i}}_\alpha - \frac{\mathrm{i}}{4}\phi_a{}^i\tilde{\gamma}^a\rho^{\underline{i}}\ , &(6.6.16\text{c})
\end{aligned}
$$

where the composite connection $\phi_m{}^i$ and $\mathfrak{f}_a{}^b$ are now given in terms of $\mathbf{D}_a$ by:

$$
\begin{aligned}
\phi_m{}^k &= \frac{\mathrm{i}}{16}\Big(\gamma^{bc}\gamma_m - \frac{3}{5}\gamma_m\tilde{\gamma}^{bc}\Big)\Big[2\mathbf{D}_{[b}\psi_{c]}{}^k + \frac{1}{12}T^-_{def}\tilde{\gamma}^{def}\gamma_{[b}\psi_{c]}{}^k\\
&\quad +8q^{-4}q^{(k}{}_{\underline{i}}q^{j)}{}_{\underline{j}}\Big\{q^{k\underline{i}}\mathbf{D}_{[b}q_k{}^{\underline{j}} - \frac{1}{2}H_{[b}{}^{\underline{ij}} - \frac{1}{4}q^{k\underline{i}}(\psi_{[bk}\rho^{\underline{j}}) - \frac{\mathrm{i}}{16}(\rho^{\underline{i}}\tilde{\gamma}_{[b}\rho^{\underline{j}})\Big\}\psi_{c]j}\Big]\ , &(6.6.17\text{a})\\
\mathfrak{f}_a{}^b &= -\frac{1}{8}\mathcal{R}_a{}^b(\omega) + \frac{1}{80}\delta^b_a\mathcal{R}(\omega) + \frac{\mathrm{i}}{16}\psi_{cj}\gamma_a R(Q)^{bcj} + \frac{\mathrm{i}}{8}\psi_{cj}\gamma^{[b}R(Q)_a{}^{c]j}\\
&\quad -\frac{1}{16}q^{-2}q^j{}_{\underline{i}}(\psi_{aj}\gamma^b)_\alpha\left[\frac{1}{12}(\rho^{\underline{i}}\tilde{\gamma}^{cde})^\alpha T^-_{cde} - (\tilde{\gamma}^c\nabla_c\rho^{\underline{i}})^\alpha\right] + \frac{1}{8}\psi_{[aj}\gamma^{bc}\phi_{c]}{}^j\\
&\quad -\frac{1}{80}\delta^b_a\psi_{cj}\gamma^{cd}\phi_d{}^j + \frac{\mathrm{i}}{16}\psi_a{}^j\gamma_c\psi_{dj}T^{-bcd} - \frac{\mathrm{i}}{160}\delta^b_a\psi_c{}^j\gamma_d\psi_{ej}T^{-cde}\ . &(6.6.17\text{b})
\end{aligned}
$$

Note that the expression of $\mathfrak{f}_a{}^b$ has an explicit as well as implicit dependence on the composite connection $\phi_a{}^i$ via $\nabla_a\rho^{\underline{i}}$, which can now be substituted from (6.6.17a). It is also convenient to provide the bosonic part of $\Box q^{i\underline{i}}$. By using that $\mathbf{D}^a H_a{}^{\underline{ij}} = 0$ up to fermions, it holds:

$$
\begin{aligned}
\Box q^{i\underline{i}} &= \frac{1}{2}\mathbf{D}^a\mathbf{D}_a q^{i\underline{i}} + \frac{3}{4}q^{-2}q^{i\underline{i}}(\mathbf{D}^a q^{k\underline{k}})\mathbf{D}_a q_{k\underline{k}} - q^{-2}q^i{}_{\underline{j}}q^{k\underline{i}}\mathbf{D}^a\mathbf{D}_a q_k{}^{\underline{j}} + \frac{1}{2}q^{-2}q^k{}_{\underline{j}}\mathbf{D}^a q^{i\underline{j}}\mathbf{D}_a q_k{}^{\underline{i}}\\
&\quad +q^{-4}q_{l\underline{l}}q^i{}_{\underline{j}}q^{k\underline{i}}(\mathbf{D}^a q^{l\underline{l}})\mathbf{D}_a q_k{}^{\underline{j}} - \frac{1}{2}q^{-4}q^{i\underline{i}}H^{a\underline{jk}}H_{a\underline{jk}} + \frac{1}{5}\mathcal{R}q^{i\underline{i}} + \text{fermions}\ . &(6.6.18)
\end{aligned}
$$

In analogy to the $5D$ $N=1$ hyper-dilaton Weyl multiplet, we will end this subsection by underlining the following two remarks about the $6D$ $N=(1,0)$ hyper-dilaton Weyl multiplet:

1. From a symmetry point of view, the hyper-dilaton Weyl multiplet contains all the fields that are required to gauge fix the extra symmetries of the superconformal group, i.e., it contains a triplet of scalar field $q^{i\underline{i}}$, which can be used to gauge fix dilatation and $SU(2)_R$ symmetry; the spinor field $\rho^{\underline{i}}_\alpha$ and the dilatation connection b_m can be used to fix S-supersymmetry and special conformal symmetry, respectively. An example of such a gauge choice is as follows:

$$q^{i\underline{i}} = -\varepsilon^{i\underline{i}}\ , \qquad (6.6.19\text{a})$$

$$\rho^{\underline{i}}_\alpha = 0\ , \qquad (6.6.19\text{b})$$

$$b_m = 0\ . \qquad (6.6.19\text{c})$$

   This indicates that in the gauge fixed version we would obtain an off-shell irreducible multiplet of Poincaré supergravity.

2. The second point would be to obtain a supersymmetric completion of the Einstein-Hilbert term by using an appropriate compensating multiplet. In $4D$ $N=2$ and $5D$ $N=1$ this was achieved

by using an off-shell vector multiplet compensator. In $6D$ the $N=(1,0)$ vector multiplet has no scalar fields [270, 271] that can be used for this purpose. The natural choice would be a tensor multiplet. However, known versions of an off-shell tensor multiplet involve infinite number of auxiliary fields [129, 130]. One might wonder whether it suffices to use an off-shell linear multiplet. The action for an improved linear multiplet in a standard Weyl multiplet background take the form of the following BF Lagrangian [69, 89]

$$\begin{aligned} e^{-1}\mathcal{L}|_{bosonic} &= -\frac{2}{15}\big(3\mathcal{R}+D\big)L-\frac{1}{8L}H^aH_a-\frac{1}{2L}H^a\phi_a^{ij}L_{ij}-\mathcal{D}^a\mathcal{D}_aL \\ &\quad +\frac{1}{4L}(\mathcal{D}^aL^{ij})\mathcal{D}_aL_{ij}-\frac{1}{8L^3}\tilde{b}^{mn}L_i{}^j(\partial_mL^{ki})\partial_nL_{jk}\ . \end{aligned} \tag{6.6.20}$$

When working with a hyper-dilaton Weyl multiplet, we need to take into account that the auxiliary field D and the $SU(2)_R$ connection are composite fields (and that (6.6.6b) and (6.6.14) have to be used). The combination $(3\mathcal{R}+D)$ turns out to not depend on the scalar curvature $\mathcal{R}$,

$$\begin{aligned} 3\mathcal{R}+D &= -15q^{-2}\Big\{q_{i\underline{i}}\mathbf{D}_a\mathbf{D}^aq^{i\underline{i}}+(\mathbf{D}^aq^{i\underline{i}})\mathbf{D}_aq_{i\underline{i}}-q^{-2}q^{i\underline{i}}q^{j\underline{j}}(\mathbf{D}_aq_{i\underline{i}})\mathbf{D}^aq_{j\underline{j}} \\ &\quad -\frac{1}{2}q^{-2}H_{ai\underline{j}}H^{ai\underline{j}}\Big\}\ . \end{aligned} \tag{6.6.21}$$

Clearly, by plugging this into (6.6.20), the result is independent of $\mathcal{R}$ and fails to be a good starting point to engineer a supersymmetric extension of the Einstein-Hilbert term to obtain a two-derivative Poincaré' supergravity Lagrangian.

It is worth mentioning that coupling the hyper-Dilaton Weyl multiplet to any number of linear multiplet will encounter the same problem. This is not too surprising since the linear multiplet is on-shell equivalent to the on-shell hypermultiplet up to trading a scalar field with a gauge four-form. We expect that the same would be true by using other variant off-shell hypermultiplets, such as the off-shell charged hypermultiplet, coupled to conformal supergravity [129, 130]. We will come back in the future to engineer $6D$ $N=(1,0)$ off-shell Poincaré supergravity theories, and their matter couplings, by using our new hyper-dilaton Weyl multiplet in a superconformal setting.

Acknowledgements:

We are grateful to Gregory Gold and William Kitchin for discussions and collaborations related to this work. This work is supported by the Australian Research Council (ARC) Future Fellowship FT180100353, and by the Capacity Building Package of the University of Queensland. S.K. is supported by the postgraduate scholarships at the University of Queensland. J.W., during part of this work, thanks the support of The Australian Mathematical Sciences Institute (AMSI) Vacation Research Scholarship 2021/22 for providing financial support, and the European Organization for Nuclear Research (CERN) for recently providing financial support, hospitality, and further research experience during June-August 2022. J.H., S.K., and G.T.-M. thank the MATRIX Institute in Creswick for hospitality and support during part of this project. J.H. thanks the Simons Fondation for support through a MATRIX travel grant.

The following manuscript has been incorporated as Chapter 7.

1. Gregory Gold, **Saurish Khandelwal**, Gabriele Tartaglino-Mazzucchelli, *On 4D,* $\mathcal{N} = 2$ *deformed vector multiplets and partial supersymmetry breaking in off-shell supergravity* (upcoming) [5].

Contributor	Statement of Contribution	%
Gregory Gold	Writing of text	33
	Proof-reading	33
	Theoretical derivations	40
	Computational derivations	40
	Initial concept	25
Saurish Khandelwal	Writing of text	33
	Proof-reading	33
	Theoretical derivations	40
	Computational derivations	40
	Initial concept	25
Gabriele Tartaglino-Mazzucchelli	Writing of text	34
	Proof-reading	34
	Theoretical derivations	20
	Computational derivations	20
	Initial concept	50
	Supervision and guidance	100

Table 6.8: Contributions of each author to the work

Chapter 7

On 4D, $N = 2$ deformed vector multiplets and partial supersymmetry breaking in off-shell supergravity

Electric and magnetic Fayet-Ilioupulous (FI) terms are used to engineer partial breaking of $N = 2$ *global supersymmetry for systems of vector multiplets. The magnetic FI term induces a deformation of the off-shell field transformations associated with an imaginary constant shift of the triplet of auxiliary fields of the vector multiplet. In this paper, we elaborate on the deformation of off-shell vector multiplets in supergravity, both in components and superspace. In a superconformal framework, the deformations are associated with (composite) linear multiplets. We engineer an off-shell model that exhibits partial local supersymmetry breaking with a zero cosmological constant. This is based on the hyper-dilaton Weyl multiplet introduced in arXiv:2203.12203, coupled to the SU(1,1)/U(1) special-Kähler sigma model in a symplectic frame admitting a holomorphic prepotential, with one compensating and one physical vector multiplet, the latter magnetically deformed.*

7.1 Introduction

Partial supersymmetry (SUSY) breaking is a fascinating subject that has been studied for several decades. It has witnessed no-go theorems [17, 96] and their eventual disproof [93, 98–100, 284], and to date, still remains a subject with various interesting open questions. In the case of four space-time dimensions (4D), which is the subject of this work, partial breaking of global $N = 2 \to N = 1$ supersymmetry requires the deformation of supersymmetry transformations [93–95, 252, 253, 255, 256, 285]. This can be engineered with off-shell supersymmetry, meaning that the algebra of supersymmetry transformations closes without the aid of any equations of motion. In particular, possibly the simplest model exhibiting spontaneous global supersymmetry breaking is the one introduced by Antoniadis-Partouche-Taylor (APT) in 1995 [93] where a single $N = 2$ vector multiplet deformed by both electric and a magnetic Fayet-Iliopoulos (FI) terms suffices to engineer the $N = 2 \to N = 1$ supersymmetry breaking. A motivation for our work is to look for APT-type models in supergravity engineered with manifest off-shell local supersymmetry.

An alternative way to construct partial supersymmetry breaking models employs non-linear realization techniques, including nilpotent Goldstone multiplet analyses, where only one supersymmetry is manifestly preserved and linearly realized [95, 254–256, 259, 286–292]. The non-linear realization of partial supersymmetry breaking also naturally takes place in theories with supersymmetric extended objects (like membranes), which lead to supersymmetric Dirac-Born-Infeld (DBI) type actions [254, 289, 293, 294]. Moreover, understanding the mechanisms of partial supersymmetry breaking is also motivated by phenomenology as it would be welcome to have feasible mechanisms to control the breaking of extended supersymmetries in some high-energy scale while allowing a single $N = 1$ supersymmetry at low energy, see, e.g., [295] and references therein.

Returning to the APT-type model, it is worth mentioning that electric and magnetic FI terms are related to each other under the electric-magnetic duality of the associated vector multiplets [93, 252, 253]. However, the two possess different features in the context of supersymmetry. The electric FI term is a supersymmetric deformation of an $N = 2$ Lagrangian that on shell can induce a vev for the triplet of real auxiliary fields of a vector multiplet. The magnetic FI term is a deformation of the supersymmetry algebra that results from a constant imaginary shift of the same vector multiplet, resulting in an inherited deformation of the supersymmetric constraint off shell. The possibility of turning on both types of FI terms in an off-shell setting and tuning them appropriately allows for the simple engineering of general matter systems with global spontaneous supersymmetry breaking. To the best of our knowledge, one of the remaining open questions on the subject is to explicitly engineer fully off-shell models that lead to local partial supersymmetry breaking in the supergravity context. In our paper, we revisit this question and propose a solution to this problem.

Local partial supersymmetry breaking in supergravity is a subject that has obtained substantial attention with a non-geodesic history. A limited set of references on the subject can be found here [38, 96–98, 100–102, 107, 109, 110, 257, 257, 296–298]. In line with the early result of [17], in 1984 Cecotti-Girardello-Porrati did prove a no-go theorem for local partial supersymmetry breaking in supergravity [96]. These were based on a clever analysis of the general aspects of local supersymmetry algebras with different field content and the employment of superconformal techniques [37, 38, 42, 140, 141, 231]. The next year, it was realised by the same authors that local partial supersymmetry breaking with a zero cosmological constant can be realised if one lifts the technical assumption of the existence of a holomorphic prepotential for the special-Kähler geometry of the vector multiplets [98]. The resulting model has one physical vector multiplet parametrising a $SU(1,1)/U(1)$ special-Kähler sigma model together with a hypermultiplet parametrising a $SO(4,1)/SO(4)$ quaternion-Kähler manifold and a set of (electric) gaugings. This set-up has seen several generalisations, see, e.g., [38, 101, 102, 107, 109, 110, 257, 297, 298], however two common features are: (i) these local partial supersymmetry breaking models include physical vector multiplets and at least one physical hypermultiplet; (ii) due to the presence of the hypermultiplet, which in the component superconformal tensor calculus with a finite number of auxiliary fields is on shell [38],[1] the resulting models have local supersymmetry that only closes on shell. We will see in our work that the second restriction can be lifted for an off-shell model of supergravity that has partial supersymmetry breaking and, in fact, is closely related to the original set-up of [98]. However, for the model in our paper, we will employ an off-shell magnetically deformed vector multiplet, together with a compensating vector multiplet in a hyper-dilaton Weyl multiplet background (which we will comment about shortly) with no other matter multiplets. The structure resembles the APT model with both electric and magnetic FI terms, and the resulting construction is fully off shell, due to the modification of one of the building blocks in the superconformal tensor calculus approach that leads to a new spectrum of fields.

Other than the examples mentioned above, it is worth reminding the reader that conformal supergravity has played an important role in several research avenues in the last five decades — we refer the reader to a few books and reviews for a more detailed discussion and list of references [35–40]. Similar to superspace approaches (see [35, 36, 39, 40] for introductory reviews and, e.g., [4, 26, 41, 116–119, 121, 127, 128, 131–133, 227–230, 299, 300] and references therein, for the 4D, $N = 2$ case) a main advantage of the superconformal tensor calculus is to provide an off-shell description of potentially general supergravity-matter couplings. This allows one to formulate models where local supersymmetry is engineered in a completely model-independent way. The approach has been very successful in helping to decipher many of the intricate geometrical structures associated to (two-derivative) sigma models in supergravity-matter systems with eight real supercharges, see, e.g., [38, 42, 231–234]. The off-shell nature of the formalism has been a central ingredient in its employment in the study of supersymmetric localisation and supersymmetric quantum field theories on curved space-times — see [235] for a recent extensive review. Moreover, off-shell supersymmetry has also been a crucial ingredient when using superconformal tensor calculus to construct higher-derivative supergravity invariants [1, 2, 6, 7, 28, 29, 31, 32, 51, 55–57, 59–72, 185, 301, 302]. These play an important role, e.g., in the study of black-hole entropy and other applications in next to leading order AdS/CFT — see the recent works [2, 79–82, 84, 86, 219, 303–305] and references therein.

Within the superconformal tensor calculus, general supergravity-matter couplings are engineered by a few ingredients. First, one needs a multiplet of conformal supergravity — named the *Weyl* multiplet — which forms an off-shell representation of the local superconformal algebra and contains the vielbein as one of its independent fields. This multiplet defines the geometry (soft algebra) associated with the gauging of the superconformal space-time symmetry. Next, one identifies off-shell matter multiplets with local superconformal transformation rules in a Weyl multiplet background. These two ingredients provide the kinematic data of a specific supergravity-matter system. Finally, one engineers locally superconformal invariant action principles constructed out of these multiplets to obtain well-defined

[1]One way to overcome this difficulty is to employ multiplets with gauged central charges, see for example [141], but, to the best of our knowledge, it remains an open question whether most general supergravity-matter couplings can be engineered this way, see also the discussion in [38].

supergravity theories.[2]

Assuming the matter multiplets contain enough "compensating" degrees of freedom, one can suitably gauge fix part of the superconformal group, specifically dilatations, special conformal transformations, S-supersymmetry, and R-symmetry, to obtain supergravity models where only the super-Poincaré symmetry survives and is gauged. For instance, pure 4D, $N=2$ Poincaré supergravity can arise by the coupling of the *standard Weyl multiplet* [137–141] to two compensating multiplets. There is significant freedom in doing so. Typically, one uses a vector multiplet and a hypermultiplet (the hyper can take several forms, e.g., a linear, non-linear, or hypermultiplet with or without a central charge) as compensators — see [38, 39] for recent reviews. Note that, in this approach, historically the first step has predominantly been the same (standard Weyl multiplet), while most of the freedom that has been used concerned the matter (compensators) side of this story. However, it is known that variant Weyl multiplets exist and can be used to engineer theories of Poincaré supergravity. These go by the name of dilaton Weyl multiplets.

The first example of a dilaton Weyl multiplet was introduced for 6D, $N=(1,0)$ supergravity in 1986 [89], and similar ideas were then employed to construct a variant dilaton Weyl multiplet for 5D, $N=1$ conformal supergravity [90, 199]. For the 4D, $N=2$ case, the existence of a variant representation of the Weyl multiplet of conformal supergravity was argued in [91] and was explicitly constructed only recently in [92] by coupling an on-shell vector multiplet to a standard Weyl multiplet — for this reason, we sometimes refer to this as the *vector-dilaton Weyl* multiplet. Two years ago, we did show the existence of a so-called *hyper-dilaton Weyl* multiplet for 4D, $N=2$ conformal supergravity engineered by coupling an on-shell hypermultiplet to the standard Weyl multiplet and by reinterpreting the resulting system as a variant off-shell Weyl multiplet [4]. This is the Weyl multiplet that we will use to engineer an off-shell model for local partial supersymmetry breaking. A similar analysis was then performed to define hyper-dilaton Weyl multiplets also in five and six dimensions [3]. It is also worth mentioning that new dilaton Weyl multiplets were recently engineered for maximal conformal supergravity in four and five space-time dimensions [306, 307].

Considering the role played by the hyper-dilaton Weyl multiplet in our paper, let us now review some of its key features. The off-shell standard Weyl multiplet of 4D, $N=2$ conformal supergravity comprises $24+24$ independent fields. Besides the vielbein, gravitini, $\mathrm{U}(1)_R \times SU(2)_R$, and dilatation symmetry connections, the multiplet comprises a set of covariant matter (auxiliary) fields: a real antisymmetric tensor, W_{ab}, the real scalar field, D, and the spinor fields that we denote by $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i})$. The presence of the matter fields is key to obtaining a set of local superconformal field transformations that close off shell. To define the hyper-dilaton Weyl multiplet one starts with an on-shell hypermultiplet [186, 242] in a standard Weyl multiplet background [138–141, 232]. The constraints that arise by requiring the algebra of local superconformal transformations to close on the fields of the hypermultiplet can then be interpreted as algebraic equations for some of the fields of the standard Weyl multiplet.

[2]These tasks can be simplified by manifestly gauging the superconformal algebra in superspace through so-called *conformal superspace*. Conformal superspace was first introduced for 4D, $N=1,2$ supergravity in [25, 26] (see also the seminal work [236]) and it was then developed for 3D, N-extended supergravity [27], 5D, $N=1$ supergravity [29], 6D, $N=(1,0)$ supergravity [31, 32], and recently 4D $N=3$ supergravity [33] — see [39, 40] for recent reviews.

More precisely, the standard Weyl multiplet's matter fields $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i})$ and D, together with the $SU(2)_R$ symmetry connection $\phi_m{}^{ij}$ become composite fields. On the other hand, the four bosonic $q^{i\underline{i}}$ and four fermionic $(\rho_{\alpha}^{\underline{i}}, \bar{\rho}_{\underline{i}}^{\dot{\alpha}})$ fields of the hypermultiplet, together with an emerging triplet of real gauge two-forms $b_{mn}{}^{\underline{ij}} = b_{mn}{}^{\underline{ji}} = -b_{nm}{}^{\underline{ij}}$, are independent and not subject to any equations of motion. In turn, the new set of independent fields describes another $24+24$ representation of the local superconformal algebra that closes off shell. An interesting feature of the hyper-dilaton Weyl multiplet is that not only dilatation but also $SU(2)_R$ becomes pure gauge, while a triplet of one-form symmetry takes place.

To construct a multiplet of $N=2$ Poincaré supergravity, where only local Lorentz symmetry and local Q-supersymmetry are unbroken, it then suffices to couple the hyper-dilaton Weyl multiplet to a single compensating vector multiplet. The result is an off-shell $32+32$ hyper-dilaton Poincaré supergravity multiplet, which was originally constructed by Müller in [241] with a different approach. Even though the off-shell field content is minimal, the on-shell theory is non-minimal and comprises the $N=2$ Poincaré supergravity multiplet with a vielbein, gravitini, and a graviphoton together with an on-shell hypermultiplet where three of the real scalar field are dualised to a triplet of gauge two-forms and one of the scalars plays the role of a dilaton. This is precisely the Poincaré supergravity that we will use to engineer an off-shell model that comprises an extra physical vector multiplet (similar to the one of the APT model) where local partial supersymmetry breaking easily takes place.

An interesting feature of Müller's supergravity, and our implementation in terms of the hyper-dilaton Weyl multiplet, is the alternative way with which it is possible to generate scalar field potentials with a mechanism different than that of gauging the R-symmetry. It is well known that in the standard engineering of general supergravity-matter couplings in 4D, extended ($N=2$) supergravity, scalar potentials are associated with moment maps of the embedding of the scalar fields sigma model gauged isometries in the $SU(2)_R$ group. In a superconformal setting based on the $N=2$ standard Weyl multiplet, this emerges by integrating out the $SU(2)_R$ gauge connection, which in a two-derivative theory is an auxiliary field — see [38] for review. In the hyper-dilaton Weyl and Müller multiplets, $SU(2)_R$ can be fixed without the aid of a compensating multiplet as, in fact, its gauge field is a composite field that turns into the Hodge dual of the field strength of a triplet of gauge two-forms $b_{mn}{}^{\underline{ij}}$. The result is a coupling of the supergravity multiplet to new two-form physical fields and not a mechanism that makes fields (as, for example, the gravitini) charged under the physical gauge group. This was explained in [4], and it generalises to generic couplings with vector multiplets. The result is an alternative, yet simple, off-shell engineering of non-trivial scalar potentials in 4D, $N=2$ supergravity.

In our work, we will focus on off-shell Poincaré supergravity based on the hyper-dilaton Weyl multiplet coupled to a system of off-shell Abelian vector multiplets, one of which is a conformal compensator. We will not add other matter fields in the system, in particular, no hypermultiplets other than the on-shell one which defines the hyper-dilaton Weyl multiplet. It is well known that, by using the standard Weyl multiplet, for pure systems of physical vector multiplets, non-trivial scalar potentials in 4D, $N=2$ supergravity are engineered through gauging by local FI terms. As in the rigid case, FI terms are either electric or magnetic. To the best of our knowledge, in supergravity, the off-shell description of 4D, $N=2$ magnetic FI terms (and magnetic gaugings) has not been developed in full generality yet, though they are expected to play an important role in engineering scalar potentials in supergravity models possessing vacua with both positive and negative cosmological constants – see, for instance the recent discussion of magnetic 4D, $N=1$ FI terms [112]. Part of our work is to extend this analysis to the 4D, $N=2$ case and elaborate on off-shell magnetically deformed vector multiplets.

The curved superspace constraints for off-shell magnetic FI-type terms were introduced in [258, 259] and in depth supergravity analyses in components (though not fully off shell) were presented earlier in [73, 260, 261]. By using a hyper-dilaton Weyl multiplet it is straightforward to engineer generic electric and magnetic FI-type terms by means of composite linear multiplets. The result is similar to the global case, where the two types of deformations induce a real or imaginary shift of the vector multiplets. The supergravity extensions of electric FI terms, which we will parameterize with ξ, can be obtained by using the BF-coupling between a vector and a linear multiplet. In a hyper-dilaton Weyl background, one can construct composite linear multiplets by using a quadratic combination of the fields of the on-shell hypermultiplet. In the case of a ξ-deformation, the bottom component of such a composite linear multiplet is given by $G_{\xi\,ij} = \xi_{\underline{i}\underline{j}} q_i{}^{\underline{i}} q_j{}^{\underline{j}}$. We will see that off-shell magnetic FI-type deformations in a hyper-dilaton Weyl multiplet background can easily be engineered in terms of the same type of composite linear multiplet. This would, for example, appear as an imaginary deformation of the X^{ij}-auxiliary real field of a vector multiplet. These deformations are parametrised by the composite field $G_{\zeta\,ij} = \zeta_{\underline{i}\underline{j}} q_i{}^{\underline{i}} q_j{}^{\underline{j}}$ with $\zeta_{\underline{i}\underline{j}} = \zeta_{\underline{j}\underline{i}}$, $(\zeta_{\underline{i}\underline{j}})^* = \zeta^{\underline{i}\underline{j}}$ constants that generalise the magnetic FI terms of global supersymmetry. Given a system of $n+1$ vector multiplets with scalar fields ϕ^I (with $I = 0, 1, \cdots, n$) coupled to the off-shell hyper-dilaton Weyl multiplet, it is then straightforward to introduce $3(n+1)$ off-shell deformations each associated to either a $\xi_I^{\underline{i}\underline{j}}$ electric deformation or a $\zeta^I_{\underline{i}\underline{j}}$ magnetic deformation. These in general induce non-trivial scalar potentials and vacuum structures.

Remarkably, due to the fact that each of the ξ and ζ deformations can take three SU(2) directions independently, there is enough freedom to obtain local partial supersymmetry breaking. We will prove this by considering a very simple model given by one physical and one compensating vector multiplets, the first magnetically deformed, the second having an electric deformation turned on. By taking a special-Kähler holomorphic prepotential of the form $\mathcal{F} = c\phi\boldsymbol{\phi}$ (the reader can look at [37–40] for reviews on structures of general Lagrangians for off-shell vector multiplets), with c a nonzero real constant while ϕ being the complex scalar field of the compensating vector multiplet and $\boldsymbol{\phi}$ the same for the physical vector multiplets, and by choosing the determinant of the matrix

$$\mathbf{M}_{\underline{i}\underline{j}} = -\frac{2}{c}\xi_{\underline{i}\underline{j}} + \mathrm{i}\zeta_{\underline{i}\underline{j}}\,, \tag{7.1.1}$$

to be zero, $\det\mathbf{M} = 0$, we find local partial supersymmetry breaking in a Minkowski vacua. A zero determinant condition of a matrix given by a linear combination of an electric and a magnetic FI term, is precisely the one for partial breaking in the global APT model. In fact, in the local model that we consider, the mechanism is very similar, since all shift symmetry terms in the supersymmetry variation of the fermions, together with the fermionic mass matrices, are all parametrized by the matrix $\mathbf{M}_{\underline{i}\underline{j}}$ given above. A difference, however, is the fact that here, one FI term belongs to the physical multiplet and the other to the compensator. Still, the simplicity of the construction is inspiring, also because at all steps supersymmetry is off shell, allowing one to potentially add other couplings to the model, including higher-derivative ones, in a fairly straightforward way.

Before moving to the technical part of our paper, we would like to comment on the form of the special-Kähler potential, $\mathcal{F} = c\phi\boldsymbol{\phi}$, that we have chosen for the example that we discuss in detail in this paper. This is a natural choice. In fact, as mentioned, e.g., in footnote four of [100], this holomorphic prepotential is precisely the one of the $SU(1,1)/U(1)$ special-Kähler sigma model which was employed in the seminal work on local partial supersymmetry breaking [98] but in a symplectic frame, obtained after an electric-magnetic duality, where a holomorphic potential actually exists. From the point of view of the special-Kähler geometry, our exemplary model is inspired by the one of [98] after a duality transformation, where, however, the hypermultiplet sector in our case becomes part of the conformal supergravity multiplet with three scalar fields turned into gauge two-forms. The emergence of a magnetically deformed vector multiplet is then expected. However, the absence of gauging in our setup, as well as the new spectrum of the on-shell theory, are intriguing features of working with hyper-dilaton supergravity.

This paper is organised as follows. In Section 7.2, we introduce the relevant superconformal multiplets in superspace and components. This includes the standard Weyl multiplet, the abelian vector multiplet, the linear multiplet (often referred to tensor multiplet), and the hyper-dilaton Weyl multiplet (constructed by an on-shell hypermultiplet). The reader familiar with these results can skip this review section, but note that this section does review a wealth of results in our notations. In Section 7.3, we introduce the deformed abelian vector multiplet with which we induce the previously mentioned magnetic deformation. In Section 7.4, we give the component actions used in this paper being the Abelian deformed vector multiplet action and the standard FI term by a linear multiplet action. In this section, the linear multiplet is considered as a general one (not necessarily composite), whereas in the successive sections of the paper, all deformations will be parametrised by linear multiplets that are composite of a hypermultiplet. The general off-shell action for deformed $N = 2$ conformal supergravity in a hyper-dilaton Weyl multiplet background is then given in Section 7.5 followed by the covariant equations of motion for this general model's auxiliary fields. In Section 7.6, we then proceed with a specific choice being the SU(1,1)/U(1) special-Kähler sigma model and give the corresponding off-shell action. This is followed by a process of gauge fixing and integrating out auxiliary fields resulting in an on-shell supergravity model with partial supersymmetry breaking. Finally, in Section 7.7, we collect concluding comments and an outlook for our paper. We also present a few technical appendices in Appendices A. This first includes our notations and conventions and details on 4D, $N = 2$ conformal superspace. For the reader's convenience, we then give the S-supersymmetry and local superconformal transformations of various multiplet fields seen throughout this paper. Lastly, we accompany our paper with a supplementary file where we give the fermionic counterparts to various component actions, the bosonic parts of which are given in Sections 7.5 and 7.6 of this paper. This supplementary file can be found in [5].

7.2 Superconformal multiplets in 4D, $N = 2$

This section is devoted to a review of several superconformal matter multiplets in 4D, $N = 2$ theory. Along with the standard Weyl multiplet discussed in subsections 2.2.2 and 2.2.3, they serve as the building blocks for the invariants of 4D, $N = 2$ conformal supergravity discussed in this paper.

7.2.1 The abelian vector multiplet

One-form geometry of the abelian vector multiplet and its descendents

The field strength two-form F of an Abelian vector multiplet is given in terms of its one-form potential $V = \mathrm{d}z^M V_M = E^A V_A$ by $F = \mathrm{d}V = \frac{1}{2}E^B \wedge E^A F_{AB}$, or equivalently,

$$F_{AB} = 2\nabla_{[A}V_{B\}} - T_{AB}{}^C V_C \ . \tag{7.2.1}$$

Due to the existence of the one-form potential the field strength must satisfy the Bianchi identity

$$\mathrm{d}F = 0 \implies \nabla_{[A}F_{BC\}} - T_{[AB}{}^D F_{|D|C\}} = 0 \ . \tag{7.2.2}$$

At mass dimension-1 we impose the constraints

$$F_{\alpha}^{i}{}_{\beta}^{j} = -2\varepsilon^{ij}\varepsilon_{\alpha\beta}\overline{W} \ , \quad F^{\dot\alpha}_{i}{}^{\dot\beta}_{j} = 2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}W \ , \quad F_{\alpha}^{i}{}^{\dot\beta}_{j} = 0 \ , \tag{7.2.3}$$

where W is a primary superfield with dimension 1 and $\mathrm{U}(1)_R$ weight -2,

$$K_A W = 0 \ , \quad \mathbb{D}W = W \ , \quad YW = -2W \ . \tag{7.2.4}$$

Then the Bianchi identities may be solved giving

$$F_a{}^j_\beta = \frac{\mathrm{i}}{2}(\sigma_a)_\beta{}^{\dot\gamma}\bar\nabla^j_{\dot\gamma}\overline{W} \ , \qquad F_a{}^{\dot\beta}_j = -\frac{\mathrm{i}}{2}(\sigma_a)_\gamma{}^{\dot\beta}\nabla^\gamma_j W \ , \tag{7.2.5a}$$

$$F_{ab} = -\frac{1}{8}(\sigma_{ab})_{\alpha\beta}(\nabla^{\alpha\beta}W + 4W^{\alpha\beta}\overline{W}) + \frac{1}{8}(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}(\bar\nabla^{\dot\alpha\dot\beta}\overline{W} + 4\overline{W}^{\dot\alpha\dot\beta}W) \ . \tag{7.2.5b}$$

The Bianchi identities also require W to be a reduced chiral superfield,

$$\bar\nabla^i_{\dot\alpha}W = 0 \ , \qquad \nabla^{ij}W = \bar\nabla^{ij}\overline{W} \ . \tag{7.2.6}$$

Note that we have introduced the notation

$$\nabla^{ij} := \nabla^{\gamma(i}\nabla^{j)}_\gamma \ , \quad \bar\nabla^{ij} := \bar\nabla^{(i}_{\dot\gamma}\bar\nabla^{\dot\gamma j)} \ , \qquad \nabla^{\alpha\beta} := \nabla^{(\alpha k}\nabla^{\beta)}_k \ , \quad \bar\nabla^{\dot\alpha\dot\beta} := \bar\nabla^{(\dot\alpha}_k\bar\nabla^{\dot\beta)k} \ . \tag{7.2.7}$$

Acting with spinor covariant derivatives on W gives the following independent descendants:

$$\lambda_\alpha^i := \nabla^i_\alpha W \ , \quad \bar\lambda^{\dot\alpha}_i := \bar\nabla^{\dot\alpha}_i\overline{W} \ , \quad X^{ij} := \nabla^{ij}W = \bar\nabla^{ij}\overline{W} \ , \tag{7.2.8a}$$

$$F_{ab} := -\frac{1}{8}(\sigma_{ab})_{\alpha\beta}(\nabla^{\alpha\beta}W + 4W^{\alpha\beta}\overline{W}) + \frac{1}{8}(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}(\bar\nabla^{\dot\alpha\dot\beta}\overline{W} + 4\overline{W}^{\dot\alpha\dot\beta}W) \ , \tag{7.2.8b}$$

$$F_{\alpha\beta} := \frac{1}{2}(\sigma^{ab})_{\alpha\beta}F_{ab} = -\frac{1}{8}(\nabla_{\alpha\beta}W + 4W_{\alpha\beta}\overline{W}) \ , \tag{7.2.8c}$$

$$\bar{F}^{\dot\alpha\dot\beta} := -\frac{1}{2}(\tilde\sigma^{ab})^{\dot\alpha\dot\beta}F_{ab} = -\frac{1}{8}(\bar\nabla^{\dot\alpha\dot\beta}\overline{W} + 4\overline{W}^{\dot\alpha\dot\beta}W) . \tag{7.2.8d}$$

These superfields satisfy the following tower relations that are particularly useful in analysing the structure of invariants:

$$\nabla^i_\alpha\lambda^j_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}X^{ij} + 4\varepsilon^{ij}F_{\alpha\beta} + 2\varepsilon^{ij}W_{\alpha\beta}\overline{W} , \tag{7.2.9a}$$

$$\bar\nabla^{\dot\alpha}_i\lambda^j_\beta = -2\mathrm{i}\delta^j_i\nabla_\beta{}^{\dot\alpha}W , \qquad \nabla^i_\alpha\bar\lambda^{\dot\beta}_j = -2\mathrm{i}\delta^i_j\nabla_\alpha{}^{\dot\beta}\overline{W} , \tag{7.2.9b}$$

$$\bar\nabla^{\dot\alpha}_i\bar\lambda^{\dot\beta}_j = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}X_{ij} + 4\varepsilon_{ij}\bar F^{\dot\alpha\dot\beta} + 2\varepsilon_{ij}\overline{W}^{\dot\alpha\dot\beta}W , \tag{7.2.9c}$$

$$\nabla^i_\alpha X^{jk} = -4\mathrm{i}\varepsilon^{i(j}\nabla_\alpha{}^{\dot\alpha}\bar\lambda^{k)}_{\dot\alpha} , \quad \bar\nabla^{\dot\alpha}_i X^{jk} = 4\mathrm{i}\delta^{(j}_i\nabla_\alpha{}^{\dot\alpha}\lambda^{k)\alpha} , \tag{7.2.9d}$$

$$\nabla^i_\gamma F_{\alpha\beta} = \varepsilon_{\gamma(\alpha}\Sigma^i_{\beta)}\overline{W} - \frac{1}{2}W_{\alpha\beta}{}^i_\gamma\overline{W} + \frac{1}{2}\mathrm{i}\varepsilon_{\gamma(\alpha}\nabla_{\beta)}{}^{\dot\alpha}\bar\lambda^i_{\dot\alpha} , \tag{7.2.9e}$$

$$\bar\nabla^{\dot\gamma}_k F_{\alpha\beta} = \frac{\mathrm{i}}{2}\nabla_{(\alpha}{}^{\dot\gamma}\lambda_{\beta)k} - \frac{1}{2}W_{\alpha\beta}\bar\lambda^{\dot\gamma}_k , \tag{7.2.9f}$$

$$\nabla^i_\gamma\bar F^{\dot\alpha\dot\beta} = \frac{\mathrm{i}}{2}\nabla_\gamma{}^{(\dot\alpha}\bar\lambda^{\dot\beta)k} - \frac{1}{2}\overline{W}^{\dot\alpha\dot\beta}\lambda^k_\gamma , \tag{7.2.9g}$$

$$\bar\nabla^{\dot\gamma}_k\bar F^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\gamma(\dot\alpha}\bar\Sigma^{\dot\beta)}_k W - \frac{1}{2}\overline{W}^{\dot\alpha\dot\beta}{}^{\dot\gamma}_k W + \frac{1}{2}\mathrm{i}\varepsilon^{\dot\gamma(\dot\alpha}\nabla_\alpha{}^{\dot\beta)}\lambda^\alpha_k . \tag{7.2.9h}$$

These descendant superfields transform under S-supersymmetry as given in the Appendix A of our work.

The abelian vector multiplet in components

We define the component fields of the abelian vector multiplet as follows

$$\phi := W| , \quad \lambda^i_\alpha := \lambda^i_\alpha| = \nabla^i_\alpha W| , \quad X^{ij} := X^{ij}| = \nabla^{ij}W| , \quad F_{ab} := F_{ab}| . \tag{7.2.10}$$

See [42, 137, 141, 231, 242, 243] for seminal works on the $N = 2$ vector multiplet. The reality of X^{ij} follows from the Bianchi identity. The remaining component field being the gauge connection v_m is given by the lowest component of the corresponding superspace connection, $v_m = V_m|$. It is worth underlining that the following definition for F_{ab} from (7.2.8b) can be directly projected to components

$$F_{ab} \;=\; -\frac{1}{8}(\sigma_{ab})_{\alpha\beta}(\nabla^{\alpha\beta}W + 4W^{\alpha\beta}\overline{W})| + \frac{1}{8}(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}(\bar\nabla^{\dot\alpha\dot\beta}\overline{W} + 4\overline{W}^{\dot\alpha\dot\beta}W)| . \tag{7.2.11}$$

The component two-form field strength is constructed from a projection of the superspace two-form,

$$f_{mn} = F_{mn}| = 2\partial_{[m}V_{n]}| = 2\partial_{[m}v_{n]} . \tag{7.2.12}$$

Making use of the identity

$$F_{mn} = E_m{}^A E_n{}^B F_{AB}(-)^{ab}, \tag{7.2.13}$$

and projecting to its lowest component, we may solve for $F_{ab}|$ to give

$$F_{ab} := F_{ab}| = e_a{}^m e_b{}^n f_{mn} - \frac{\mathrm{i}}{2}(\sigma_{[a})_\alpha{}^{\dot\alpha}\psi_{b]}{}^\alpha_k\bar\lambda^k_{\dot\alpha} + \frac{\mathrm{i}}{2}(\tilde\sigma_{[a})_{\dot\alpha}{}^\alpha\lambda^k_\alpha\bar\psi_{b]}{}^{\dot\alpha}_k$$

$$-\frac{1}{2}\psi_a{}^\gamma_k\psi_b{}^k_\gamma\bar{\phi}+\frac{1}{2}\bar{\psi}_a{}^{\dot\gamma}_k\bar{\psi}_b{}^k_{\dot\gamma}\phi\ . \tag{7.2.14}$$

In the superconformal tensor calculus' language, F_{ab} is referred to as the supercovariant field strength (as it transforms covariantly under any local superconformal transformations) whereas $f_{mn}=2\partial_{[m}v_{n]}$ is the conventional field strength. By construction F_{ab} satisfies the Bianchi identity

$$\nabla_{[a}F_{bc]}=-\frac{\mathrm{i}}{2}R(Q)_{[ab\,j}\sigma_{c]}\bar{\lambda}^j+\frac{\mathrm{i}}{2}R(\bar{Q})_{[ab}{}^j\tilde{\sigma}_{c]}\lambda_j\ . \tag{7.2.15}$$

The local superconformal transformations of the fundamental fields of the vector multiplet fields in a standard Weyl multiplet background are given in Appendix 5.3.3.

7.2.2 The linear multiplet

Two-form geometry of the linear multiplet

The field strength three-form H is given in terms of its two-form gauge potential $B=\frac{1}{2}E^B\wedge E^A B_{AB}$ by

$$H=\mathrm{d}B=\frac{1}{3!}E^C\wedge E^B\wedge E^A H_{ABC}\ ,\qquad H_{ABC}=3\nabla_{[A}B_{BC\}}-3T_{[AB}{}^DB_{|D|C\}}\ . \tag{7.2.16}$$

The field strength remains invariant under gauge transformations $\delta B=\mathrm{d}V$ with V a one-form gauge parameter. The existence of the gauge potential requires that the Bianchi identity

$$\mathrm{d}H=0\implies\nabla_{[A}H_{BCD\}}-\frac{3}{2}T_{[AB}{}^EH_{|E|CD\}}=0\ , \tag{7.2.17}$$

be satisfied. As with the gauge one-form, we must impose constraints to reduce the multiplet. At mass dimension-$\frac{3}{2}$ they consist of

$$H_{\alpha\beta\gamma}^{i\,j\,k}=H_{i\ j\ k}^{\dot\alpha\dot\beta\dot\gamma}=H_{\alpha\beta k}^{i\,j\,\dot\gamma}=H_{i\ j\,\gamma}^{\dot\alpha\dot\beta k}=0\ . \tag{7.2.18}$$

The Bianchi identities for H can then be solved. The solution is

$$H_{a\alpha\beta}^{\ i\ j}=0\ ,\quad H_{ai\ j}^{\ \dot\alpha\dot\beta}=0\ ,\quad H_{a\alpha j}^{\ i\ \dot\alpha}=\frac{1}{2}(\sigma_a)_\alpha{}^{\dot\alpha}\mathcal{G}^i{}_j\ , \tag{7.2.19a}$$

$$H_{ab}{}^i_\alpha=\frac{\mathrm{i}}{6}(\sigma_{ab})_\alpha{}^\beta\nabla^k_\beta\mathcal{G}^i{}_k\ ,\quad H_{ab}{}^{\dot\alpha}_i=\frac{\mathrm{i}}{6}(\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\bar{\nabla}^{\dot\beta}_k\mathcal{G}^k{}_i\ , \tag{7.2.19b}$$

$$H_{abc}=\frac{\mathrm{i}}{96}\varepsilon_{abcd}(\sigma^d)^\alpha{}_{\dot\beta}[\nabla^i_\alpha,\bar{\nabla}^{\dot\beta}_j]\mathcal{G}^j{}_i=\varepsilon_{abcd}H^d\ , \tag{7.2.19c}$$

where $\mathcal{G}^{ij}$ is a real symmetric conformally primary dimension-2 superfield, i.e.,

$$K_A\mathcal{G}^{ij}=0\ ,\quad \mathbb{D}\mathcal{G}^{ij}=2\mathcal{G}^{ij}\ ,\quad Y\mathcal{G}^{ij}=0\ ,\quad (\mathcal{G}^{ij})^*=\mathcal{G}_{ij}=\varepsilon_{ik}\varepsilon_{jl}\mathcal{G}^{kl}\ . \tag{7.2.20}$$

The superfield $\mathcal{G}^{ij}$ also obeys the constraint

$$\bar{\nabla}^{(i}_{\dot\alpha}\mathcal{G}^{jk)}=\nabla^{(i}_\alpha\mathcal{G}^{jk)}=0\ , \tag{7.2.21}$$

which defines the $N=2$ linear multiplet. By acting with spinor covariant derivatives on $\mathcal{G}^{ij}$ gives the following descendants:

$$\chi_{\alpha i} := \frac{1}{3}\nabla^j_\alpha \mathcal{G}_{ij}\ , \qquad \bar{\chi}^{\dot{\alpha} i} := \frac{1}{3}\bar{\nabla}^{\dot{\alpha}}_j \mathcal{G}^{ij}\ , \tag{7.2.22a}$$

$$F := \frac{1}{12}\nabla^{ij}\mathcal{G}_{ij}\ , \qquad \bar{F} := \frac{1}{12}\bar{\nabla}^{ij}\mathcal{G}_{ij}\ , \tag{7.2.22b}$$

$$H_{abc} := \frac{\mathrm{i}}{96}\varepsilon_{abcd}(\sigma^d)^\alpha{}_{\dot{\beta}}[\nabla^i_\alpha, \bar{\nabla}^{\dot{\beta}}_j]\mathcal{G}^j{}_i = \varepsilon_{abcd}H^d\ , \tag{7.2.22c}$$

$$H_a = \frac{\mathrm{i}}{96}(\sigma_a)^\alpha{}_{\dot{\beta}}[\nabla^i_\alpha, \bar{\nabla}^{\dot{\beta}}_j]\mathcal{G}^j{}_i = \frac{1}{6}\varepsilon_{abcd}H^{bcd}\ . \tag{7.2.22d}$$

These superfields satisfy the following tower relations and are particularly useful in analysing the structure of invariants:

$$\nabla^i_\alpha \mathcal{G}^{jk} = -2\varepsilon^{i(j}\chi^{k)}_\alpha\ , \quad \bar{\nabla}^{\dot{\alpha}}_i \mathcal{G}_{jk} = -2\varepsilon_{i(j}\bar{\chi}^{\dot{\alpha}}_{k)}\ , \tag{7.2.23a}$$

$$\nabla^i_\alpha \chi_{\beta j} = \delta^i_j \varepsilon_{\alpha\beta}F\ , \quad \bar{\nabla}^{\dot{\alpha}}_i \chi_{\beta j} = -4\mathrm{i}\varepsilon_{ij}H_a(\sigma^a)_\beta{}^{\dot{\alpha}} - \mathrm{i}\nabla_\beta{}^{\dot{\alpha}}\mathcal{G}_{ij}\ , \tag{7.2.23b}$$

$$\bar{\nabla}^{\dot{\alpha}}_i \bar{\chi}^{\dot{\beta} j} = \delta^j_i \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{F}\ , \quad \nabla_{i\beta}\bar{\chi}^{\dot{\alpha}}_j = -4\mathrm{i}\varepsilon_{ij}H_a(\sigma^a)_\beta{}^{\dot{\alpha}} - \mathrm{i}\nabla_\beta{}^{\dot{\alpha}}\mathcal{G}_{ij}\ , \tag{7.2.23c}$$

$$\nabla^i_\alpha F = 0\ , \quad \bar{\nabla}^{\dot{\alpha}}_i F = 2\mathrm{i}\nabla_\alpha{}^{\dot{\alpha}}\chi^\alpha_i - 2\overline{W}^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\chi}^{\dot{\gamma}}_i - 6\bar{\Sigma}^{\dot{\alpha} j}G_{ij}\ , \tag{7.2.23d}$$

$$\bar{\nabla}^{\dot{\alpha}}_i \bar{F} = 0\ , \quad \nabla^i_\alpha \bar{F} = 2\mathrm{i}\nabla_\alpha{}^{\dot{\alpha}}\bar{\chi}^i_{\dot{\alpha}} + 2W_{\alpha\gamma}\chi^{\gamma i} - 6\Sigma^j_\alpha G^i{}_j\ , \tag{7.2.23e}$$

$$\nabla^i_\alpha H_a = \frac{1}{2}(\sigma_{ab})_\alpha{}^\beta\nabla^b\chi^i_\beta - \frac{\mathrm{i}}{8}(\sigma_a)_{\alpha\dot{\beta}}\left[\overline{W}^{\dot{\beta}\dot{\gamma}}\bar{\chi}^i_{\dot{\gamma}} + 3\bar{\Sigma}^{\dot{\beta}}_l G^{li}\right]\ , \tag{7.2.23f}$$

$$\bar{\nabla}_{\dot{\alpha} i}H_a = -\frac{1}{2}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\nabla^b\bar{\chi}^{\dot{\beta}}_i + \frac{\mathrm{i}}{8}(\tilde{\sigma}_a)_{\dot{\alpha}\beta}\left[W^{\beta\gamma}\chi_{\gamma i} + 3\Sigma^{\beta l}G_{li}\right]\ . \tag{7.2.23g}$$

These descendant superfields transform under S-supersymmetry as given in the Appendix A of our work.

It is possible to construct a superfield which automatically obeys the above constraints (7.2.21) by imposing constraints on the two-form B_{AB} itself. It holds that

$$B^{i\ j}_{\alpha\beta} = -2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{\Psi}\ , \quad B^{\dot{\alpha}\dot{\beta}}_{i\ j} = 2\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}\Psi\ , \quad B^{i\ \dot{\beta}}_{\alpha j} = 0\ , \tag{7.2.24}$$

where Ψ is a chiral superfield, $\bar{\nabla}^{\dot{\alpha}}_i\Psi = 0$, of dimension 1 and $\mathrm{U}(1)_R$ weight -2, but otherwise arbitrary

$$K_A\Psi = 0\ , \quad \mathbb{D}\Psi = \Psi\ , \quad Y\Psi = -2\Psi\ . \tag{7.2.25}$$

Constraints of this kind are quite natural since the gauge transformation $\delta B = \mathrm{d}\tilde{V} = \tilde{F}$ amounts to

$$\delta\Psi = \tilde{W}\ , \tag{7.2.26}$$

with $\tilde{W}$ a generic vector multiplet chiral field strength satisfying the Bianchi identities of eq. (7.2.6). We can then proceed to solve (7.2.16) for the full two-form B:

$$B_a{}^i_{\dot{\alpha}} = \frac{\mathrm{i}}{2}(\sigma_a)_\alpha{}^{\dot{\alpha}}\bar{\nabla}^i_{\dot{\alpha}}\bar{\Psi}\ , \ B_{ai}^{\ \dot{\alpha}} = -\frac{\mathrm{i}}{2}(\sigma_a)_\alpha{}^{\dot{\alpha}}\nabla^\alpha_i\Psi\ , \tag{7.2.27a}$$

$$B_{ab} = -\frac{1}{8}(\sigma_{ab})^{\alpha\beta}(\nabla_{\alpha\beta}\Psi + 4W_{\alpha\beta}\bar{\Psi}) + \frac{1}{8}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}(\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\Psi} + 4\overline{W}^{\dot{\alpha}\dot{\beta}}\Psi)\ . \tag{7.2.27b}$$

Inserting this solution into the definition of H leads to an expression for the linear multiplet in terms of an unconstrained chiral prepotential

$$\mathcal{G}^{ij} = -\frac{\mathrm{i}}{2}\big(\nabla^{ij}\Psi - \bar{\nabla}^{ij}\bar{\Psi}\big) = \mathrm{I}m[\nabla^{ij}\Psi] \ , \qquad \nabla^{ij}\Psi - \bar{\nabla}^{ij}\bar{\Psi} = 2\mathrm{i}\,\mathcal{G}^{ij} \ . \tag{7.2.28}$$

One can check that $\mathcal{G}^{ij}$ indeed obeys (7.2.20) and (7.2.21) and is invariant under (7.2.26).

Linear multiplet in components

The components of the linear multiplet are defined as follows — see [117, 118, 138–140, 186–191, 262] for seminal works. The linear multiplet is described by a real primary superfield $\mathcal{G}^{ij}$ of dimension 2 (7.2.20) satisfying the constraint (7.2.21). The corresponding 3-form field strength H in superspace is given by (7.2.18) and (7.2.19). Within the superfield $\mathcal{G}^{ij}$ are the matter components of the linear multiplet: a real isotriplet field G^{ij}, a fermion $\chi_{\alpha i}$, and a complex scalar F:

$$G^{ij} := \mathcal{G}^{ij}| \ , \tag{7.2.29a}$$

$$\chi_{\alpha i} := \frac{1}{3}\nabla^j_\alpha \mathcal{G}_{ij}| \ , \qquad \bar{\chi}^{\dot{\alpha} i} := \frac{1}{3}\bar{\nabla}^{\dot{\alpha}}_j \mathcal{G}^{ij}| \ , \tag{7.2.29b}$$

$$F := \frac{1}{12}\nabla^{ij}\mathcal{G}_{ij}| \ , \qquad \bar{F} := \frac{1}{12}\bar{\nabla}^{ij}\mathcal{G}_{ij}| \ , \tag{7.2.29c}$$

$$H_{abc} := \frac{\mathrm{i}}{96}\varepsilon_{abcd}(\sigma^d)^\alpha{}_{\dot{\beta}}[\nabla^i_\alpha, \bar{\nabla}^{\dot{\beta}}_j]\mathcal{G}^j{}_i| = \varepsilon_{abcd}H^d \ , \tag{7.2.29d}$$

$$H_a = \frac{\mathrm{i}}{96}(\sigma_a)^\alpha{}_{\dot{\beta}}[\nabla^i_\alpha, \bar{\nabla}^{\dot{\beta}}_j]\mathcal{G}^j{}_i| = \frac{1}{6}\varepsilon_{abcd}H^{bcd} \ . \tag{7.2.29e}$$

The remaining component field, the two-form, is given by $b_{mn} := B_{mn}|$. Owing to the superspace identity

$$\nabla^i_\alpha \nabla^j_\beta \mathcal{G}^{kl} = -\frac{1}{6}\varepsilon_{\alpha\beta}\varepsilon^{i(k}\varepsilon^{l)j}\nabla^{pq}\mathcal{G}_{pq} \ , \tag{7.2.30}$$

there are no other independent component fields. The local superconformal transformations of the fundamental fields of the linear multiplet in a standard Weyl multiplet background are given in 5.2.10.

The covariant conservation equation for H_a is

$$\nabla^a H_a \;=\; \frac{3}{8}\Sigma^i\chi_i + \frac{3}{8}\bar{\Sigma}_i\bar{\chi}^i \ . \tag{7.2.31}$$

The constraint locally implies the existence of a gauge two-form potential, $b_{mn} = -b_{nm}$, and its exterior derivative $h_{mnp} := 3\partial_{[m}b_{np]}$. The solution of (7.2.31) is

$$H_a = \frac{1}{6}\varepsilon_a{}^{bcd}\Big(h_{bcd} - \frac{3\mathrm{i}}{4}\psi_{bi}\sigma_{cd}\chi^i - \frac{3\mathrm{i}}{4}\bar{\psi}_b{}^i\tilde{\sigma}_{cd}\bar{\chi}_i - \frac{3}{4}(\psi_b{}^i\sigma_c\bar{\psi}_d{}^j)G_{ij}\Big) \ , \tag{7.2.32}$$

where $h_{abc} = e_a{}^m e_b{}^n e_c{}^p h_{mnp}$. The local superconformal transformations of b_{mn} are

$$\delta b_{mn} \;=\; \frac{\mathrm{i}}{2}\xi_i\sigma_{mn}\chi^i + \frac{\mathrm{i}}{2}\bar{\xi}^i\tilde{\sigma}_{mn}\bar{\chi}_i + \frac{1}{2}\Big(\bar{\psi}_{[m}{}^i\sigma_{n]}\xi^j - \psi_{[m}{}^i\sigma_{n]}\bar{\xi}^j\Big)G_{ij} + 2\partial_{[m}l_{n]} \ , \tag{7.2.33}$$

where we have also included the vector gauge transformation $\delta_l b_{mn} = 2\partial_{[m}l_{n]}$ that leaves h_{mnp} and H^a invariant. In constructing the superspace three-form $H_{mnp} = 3\partial_{[m}B_{np]}$ we can make use of the superspace identity

$$H_{mnp} = E_m{}^A E_n{}^B E_p{}^C H_{ABC}(-)^{ab+ac+bc} \ . \tag{7.2.34}$$

Projecting this equation to the lowest component, and defining

$$h_{mnp} := H_{mnp}| = 3\partial_{[m}b_{np]} \ , \quad h_{abc} := e_a{}^m e_b{}^n e_c{}^p h_{mnp} \ , \tag{7.2.35}$$

it is easy to show that

$$\begin{aligned} h_{mnp} &= \varepsilon_{mnpq}H^a|e_a{}^q + \frac{3\mathrm{i}}{4}(\sigma_{[mn})_\alpha{}^\beta \psi_{p]}{}_j^\alpha \chi_\beta^j + \frac{3\mathrm{i}}{4}(si\tilde{g}ma_{[mn})^{\dot\alpha}{}_{\dot\beta}\bar\psi_{p]}{}^j_{\dot\alpha}\bar\chi_j^{\dot\beta} \\ &\quad -\frac{3}{4}(\sigma_{[m})_{\alpha\dot\alpha}\psi_n{}^{\alpha i}\bar\psi_{p]}{}^{\dot\alpha j}G_{ij} \ , \end{aligned} \tag{7.2.36}$$

or, equivalently,

$$\begin{aligned} H^a &:= H^a| = \frac{1}{6}\varepsilon^{abcd}H_{bcd}| \\ &= \frac{1}{6}\varepsilon^{abcd}\Big(h_{bcd} - \frac{3\mathrm{i}}{4}(\sigma_{cd})_\alpha{}^\beta \psi_b{}_k^\alpha \chi_\beta^k - \frac{3\mathrm{i}}{4}(\tilde\sigma_{cd})^{\dot\alpha}{}_{\dot\beta}\bar\psi_b{}^k_{\dot\alpha}\bar\chi_k^{\dot\beta} + \frac{3}{4}(\sigma_b)_\alpha{}^{\dot\beta}\psi_c{}_k^\alpha \bar\psi_d{}^l_{\dot\beta}G^k{}_l\Big) \ . \end{aligned} \tag{7.2.37}$$

In the paper, we will denote the Hodge dual of a three-form component field h_{abc} with

$$\tilde h^a = \frac{1}{6}\varepsilon^{abcd}h_{bcd} \ . \tag{7.2.38}$$

We also keep using the same notation for the superfield H^a and the covariant component field $H^a|$, but we hope the reader will understand what we refer to depending on the context.

We have emphasized that the construction of the two-form multiplet is completely geometrical, but it is worth noting that, as discussed in Subsection 7.2.2, the two-form multiplet can be encoded in a chiral superfield Ψ. By making use of the gauge transformations (7.2.26), one can choose for the components of B_{AB} that $B_{\alpha\beta}| = \bar B^{\dot\alpha\dot\beta}| = 0$ and $B_{a\beta}| = \bar B^{a\dot\beta}| = 0$ while $B_{ab}|$ remains unconstrained by imposing the component constraints[3]

$$\Psi| = 0 \ , \quad \nabla_\alpha^i\Psi| = 0 \ , \quad \nabla^{ij}\Psi| = -\bar\nabla^{ij}\bar\Psi| \ . \tag{7.2.39}$$

One may easily construct b_{mn} using $b_{mn} = e_m{}^a e_n{}^b B_{ab}|$. As usual, the supersymmetry transformation laws of the component fields may be derived by using the constraints.

7.2.3 On-shell hypermultiplet and hyper-dilaton Weyl multiplet

In this subsection, we review the construction of the 4D, $N=2$ hyper-dilaton Weyl multiplet of [4]. This plays a central work in our work.

A single on-shell hypermultiplet is comprised of $4+4$ degrees of freedom described by a Lorentz scalar field $q^{i\underline{i}}$ and spinor fields $(\rho_\alpha^{\underline{i}}, \bar\rho_{\underline{i}}^{\dot\alpha})$ — see [138–141, 186, 242] together with [37, 38, 232] and references therein for superconformal approaches to systems of on-shell hypermultiplets. The index $\underline{i} = \underline{1}, \underline{2}$ is a SU(2) flavour index, and the fields satisfy the following reality conditions

$$(q^{i\underline{i}})^* = q_{i\underline{i}} \ , \qquad (\rho_\alpha^{\underline{i}})^* = \bar\rho_{\dot\alpha\underline{i}} \ . \tag{7.2.40}$$

[3]The third constraint is not actually necessary to eliminate the other components of the two-form, but it does substantially simplify the component evaluation later.

They also satisfy the following dilatation and chiral weight identities

$$
\begin{aligned}
\mathbb{D}q^{i\underline{i}} = q^{i\underline{i}} \,, \quad \mathbb{D}\rho_\alpha^{\underline{i}} &= \frac{3}{2}\rho_\alpha^{\underline{i}} \,, \quad \mathbb{D}\bar{\rho}_{\dot{\alpha}\underline{i}} = \frac{3}{2}\bar{\rho}_{\dot{\alpha}\underline{i}} \,, && (7.2.41a)\\
Y q^{i\underline{i}} = 0 \,, \quad Y\rho_\alpha^{\underline{i}} &= \rho_\alpha^{\underline{i}} \,, \quad Y\bar{\rho}_{\dot{\alpha}\underline{i}} = -\bar{\rho}_{\dot{\alpha}\underline{i}} \,. && (7.2.41b)
\end{aligned}
$$

The multiplet, which has the field $q^{i\underline{i}}$ as its superconformal primary, is characterised by the following local superconformal transformations [37,38,138–141,232]

$$
\begin{aligned}
\delta q^{i\underline{i}} &= \frac{1}{2}\xi^i\rho^{\underline{i}} - \frac{1}{2}\bar{\xi}^i\bar{\rho}^{\underline{i}} + \lambda^i{}_k q^{k\underline{i}} + \lambda_{\mathbb{D}} q^{i\underline{i}} \,, && (7.2.42a)\\
\delta\rho_\alpha^{\underline{i}} &= -4\mathrm{i}(\sigma^a\bar{\xi}_k)_\alpha \nabla_a q^{k\underline{i}} + \frac{1}{2}\lambda_{ab}(\sigma^{ab}\rho^{\underline{i}})_\alpha + \mathrm{i}\lambda_Y\rho_\alpha^{\underline{i}} + \frac{3}{2}\lambda_{\mathbb{D}}\rho_\alpha^{\underline{i}} + 8\eta_\alpha^k q_k{}^{\underline{i}} \,, && (7.2.42b)\\
\delta\bar{\rho}_{\underline{i}}^{\dot{\alpha}} &= 4\mathrm{i}(\tilde{\sigma}^a\xi^k)^{\dot{\alpha}} \nabla_a q_{k\underline{i}} + \frac{1}{2}\lambda_{ab}(\tilde{\sigma}^{ab}\bar{\rho}_{\underline{i}})^{\dot{\alpha}} - \mathrm{i}\lambda_Y\bar{\rho}_{\underline{i}}^{\dot{\alpha}} + \frac{3}{2}\lambda_{\mathbb{D}}\bar{\rho}_{\underline{i}}^{\dot{\alpha}} - 8\bar{\eta}_k^{\dot{\alpha}} q^k{}_{\underline{i}} \,, && (7.2.42c)
\end{aligned}
$$

where

$$
\nabla_a q^{i\underline{i}} = \mathcal{D}_a q^{i\underline{i}} - \frac{1}{4}\psi_a{}^i\rho^{\underline{i}} + \frac{1}{4}\bar{\psi}_a{}^i\bar{\rho}^{\underline{i}} \,. \tag{7.2.43}
$$

In conformal superspace, the multiplet is described by a dimension one primary superfield $Q^{i\underline{i}}$, neutral under $\mathrm{U}(1)_\mathrm{R}$, and satisfying the following analyticity constraint

$$
\nabla_\alpha^{(i}Q^{j)\underline{j}} = \bar{\nabla}_{\dot{\alpha}}^{(i}Q^{j)\underline{j}} \,. \tag{7.2.44}
$$

The transformation rules in (7.2.42) simply derive from the equation above together with self-consistency of the conformal superspace algebra of covariant derivatives and the following definition of the component fields

$$
q^{i\underline{i}} := Q^{i\underline{i}}| \,, \quad \rho_\alpha^{\underline{i}} := \nabla^i_\alpha Q_i{}^{\underline{i}}| \,, \quad \bar{\rho}_{\dot{\alpha}}^{\underline{i}} = \bar{\nabla}^i_{\dot{\alpha}} Q_i{}^{\underline{i}}| \,. \tag{7.2.45}
$$

In contrast with the standard Weyl multiplet described in a previous subsection, the algebra of the local superconformal transformations (7.2.42) closes only when equations of motion for the fields are imposed, see for example [38,232] for a detailed analysis. In our notations, the covariant equations of motion of $q^{i\underline{i}}$ and $(\rho_\alpha^{\underline{i}}, \bar{\rho}_{\underline{i}}^{\dot{\alpha}})$ are:

$$
\begin{aligned}
\left(\nabla_a\rho^{\underline{i}}\sigma^a\right)_{\dot{\alpha}} &= \frac{\mathrm{i}}{2}(\bar{\rho}^{\underline{i}}\tilde{\sigma}^{cd})_{\dot{\alpha}} W^-_{cd} + 6\mathrm{i}\bar{\Sigma}_{\dot{\alpha}k} q^{k\underline{i}} \,, && (7.2.46a)\\
\left(\nabla_a\bar{\rho}_{\underline{i}}\tilde{\sigma}^a\right)^\alpha &= -\frac{\mathrm{i}}{2}(\rho_{\underline{i}}\sigma^{cd})^\alpha W^+_{cd} + 6\mathrm{i}\Sigma^{\alpha k} q_{k\underline{i}} \,, && (7.2.46b)\\
\Box q^{i\underline{i}} &= -\frac{3}{2}D q^{i\underline{i}} \,, \qquad \Box := \nabla^a\nabla_a \,. && (7.2.46c)
\end{aligned}
$$

The expressions for $\nabla_a\rho_\alpha^{\underline{i}}$, $\nabla_a\bar{\rho}_{\underline{i}}^{\dot{\alpha}}$, and $\Box q^{i\underline{i}}$ in terms of the derivatives $\mathcal{D}_a$ are given by

$$
\begin{aligned}
\nabla_a\rho_\alpha^{\underline{i}} &= \mathcal{D}_a\rho_\alpha^{\underline{i}} + 2\mathrm{i}(\sigma^b\bar{\psi}_{ak})_\alpha \left(\mathcal{D}_b q^{k\underline{i}} - \frac{1}{4}\psi_b{}^k\rho^{\underline{i}} + \frac{1}{4}\bar{\psi}_b{}^k\bar{\rho}^{\underline{i}}\right) + 4\phi_{a\,\alpha k} q^{k\underline{i}} \,, && (7.2.47a)\\
\nabla_a\bar{\rho}_{\underline{i}}^{\dot{\alpha}} &= \mathcal{D}_a\bar{\rho}_{\underline{i}}^{\dot{\alpha}} - 2\mathrm{i}(\tilde{\sigma}^b\psi_a{}^k)^{\dot{\alpha}} \left(\mathcal{D}_b q_{k\underline{i}} - \frac{1}{4}\psi_{bk}\rho_{\underline{i}} + \frac{1}{4}\bar{\psi}_{bk}\bar{\rho}_{\underline{i}}\right) - 4\bar{\phi}_a{}^{\dot{\alpha}k} q_{k\underline{i}} \,, && (7.2.47b)
\end{aligned}
$$

$$
\begin{aligned}
\Box q^{i\underline{i}} \;=\;& \mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} - 2\mathfrak{f}_a{}^a q^{i\underline{i}} - \frac{1}{4}\rho^{\underline{i}}\mathcal{D}_a\psi^{ai} + \frac{1}{4}\bar\rho^{\underline{i}}\mathcal{D}_a\bar\psi^{ai} - \frac{1}{2}\psi^{ai}\mathcal{D}_a\rho^{\underline{i}} + \frac{1}{2}\bar\psi^{ai}\mathcal{D}_a\bar\rho^{\underline{i}} \\
& - \frac{\mathrm{i}}{4}\phi_a{}^i\sigma^a\bar\rho^{\underline{i}} + \frac{\mathrm{i}}{4}\bar\phi_a^i\tilde\sigma^a\rho^{\underline{i}} + \mathrm{i}(\psi_a{}^{(i}\sigma^b\bar\psi^{k)a})\mathcal{D}_b q_k{}^{\underline{i}} + \frac{3\mathrm{i}}{4}(\psi_a{}^i\sigma^a\bar\Sigma_l)q^{l\underline{i}} + \frac{3\mathrm{i}}{4}(\bar\psi_a^i\tilde\sigma^a\Sigma_l)q^{l\underline{i}} \\
& - \frac{\mathrm{i}}{16}(\bar\psi_a{}^i\tilde\sigma^a\sigma^{cd}\rho^{\underline{i}})W^+_{cd} - \frac{\mathrm{i}}{16}(\psi_a{}^i\sigma^a\tilde\sigma^{cd}\bar\rho^{\underline{i}})W^-_{cd} - (\psi_a{}^i\phi^a{}_k)q^{k\underline{i}} + (\bar\psi_a{}^i\bar\phi^a{}_k)q^{k\underline{i}} \\
& - \frac{\mathrm{i}}{4}(\psi_a{}^{(i}\sigma^b\bar\psi^{k)a})(\psi_{bk}\rho^{\underline{i}}) + \frac{\mathrm{i}}{4}(\psi_a{}^{(i}\sigma^c\bar\psi^{k)a})(\bar\psi_{ck}\bar\rho^{\underline{i}})\;. \qquad (7.2.47c)
\end{aligned}
$$

It is important to stress that eqs. (7.2.46) are typically read as equations of motion for the hypermultiplet fields, see for example, [37,38,138–141,232]. They certainly are dynamical equations for $q^{i\underline{i}}$ and $(\rho_\alpha^{\underline{i}},\bar\rho_{\underline{i}}^{\dot\alpha})$ in a flat background (with no central charges as in our case) where all conformal supergravity fields are set to zero [186,242]. For this reason, the multiplet is typically referred to as the on-shell hypermultiplet. However, such an interpretation is not necessary in a curved background described by the standard Weyl multiplet. In fact, the eqs. (7.2.46) can be interpreted as algebraic equations for the standard Weyl multiplet that determine the fields $(\Sigma^{\alpha i},\bar\Sigma_{\dot\alpha i})$ and D in terms of $q^{i\underline{i}}$ and $(\rho_\alpha^{\underline{i}},\bar\rho_{\underline{i}}^{\dot\alpha})$ together with the other independent fields of the standard Weyl multiplet. If we assume that $q^{i\underline{i}}$ is an invertible matrix, which is equivalent to imposing

$$
q^2 := q^{i\underline{i}}q_{i\underline{i}} = \varepsilon_{ij}\varepsilon_{\underline{i}\underline{j}}q^{i\underline{i}}q^{j\underline{j}} = 2\det q^{i\underline{i}} \neq 0\;, \qquad (7.2.48)
$$

then the following relations hold

$$
\begin{aligned}
\Sigma^{\alpha i} \;=\;& 2q^{-2}q^{i\underline{i}}\Bigg[-\frac{\mathrm{i}}{2}(\mathcal{D}_a\bar\rho_{\underline{i}}\tilde\sigma^a)^\alpha + (\psi_a{}^j\sigma^b\tilde\sigma^a)^\alpha\left(\mathcal{D}_b q_{j\underline{i}} - \frac{1}{4}\psi_{bj}\rho_{\underline{i}} + \frac{1}{4}\bar\psi_{bj}\bar\rho_{\underline{i}}\right) \\
& + \frac{2}{3}\left(\Psi_{abj}\sigma^{ab}\right)^\alpha q^j{}_{\underline{i}} + \frac{1}{4}(\rho_{\underline{i}}\sigma^{cd})^\alpha W^+_{cd} + \frac{\mathrm{i}}{6}(\bar\psi_{aj}\widetilde{sigma}^a\sigma^{cd})^\alpha q^j{}_{\underline{i}}W^+_{cd}\Bigg]\;, \qquad (7.2.49a)
\end{aligned}
$$

$$
\begin{aligned}
\bar\Sigma_{\dot\alpha i} \;=\;& 2q^{-2}q_{i\underline{i}}\Bigg[-\frac{\mathrm{i}}{2}(\mathcal{D}_a\rho^{\underline{i}}\sigma^a)_{\dot\alpha} - (\bar\psi_{aj}\tilde\sigma^b\sigma^a)_{\dot\alpha}\left(\mathcal{D}_b q^{j\underline{i}} - \frac{1}{4}\psi_b{}^j\rho^{\underline{i}} + \frac{1}{4}\bar\psi_b{}^j\bar\rho^{\underline{i}}\right) \\
& - \frac{2}{3}\left(\bar\Psi_{ab}{}^j\widetilde{sigma}^{ab}\right)_{\dot\alpha} q_j{}^{\underline{i}} - \frac{1}{4}(\bar\rho^{\underline{i}}\widetilde{sigma}^{cd})_{\dot\alpha}W^-_{cd} + \frac{\mathrm{i}}{6}(\psi_a{}^j\sigma^a\widetilde{sigma}^{cd})_{\dot\alpha}q_j{}^{\underline{i}}W^-_{cd}\Bigg] \qquad (7.2.49b)
\end{aligned}
$$

$$
\begin{aligned}
D \;=\;& q^{-2}q_{i\underline{i}}\Bigg[\mathcal{D}^a\mathcal{D}_a q^{i\underline{i}} + \frac{1}{6}Rq^{i\underline{i}} - \frac{\mathrm{i}}{8}(\bar\psi_a{}^i\tilde\sigma^a\sigma^{cd}\rho^{\underline{i}})W^+_{cd} - \frac{\mathrm{i}}{2}\phi_a{}^i\sigma^a\bar\rho^{\underline{i}} - \frac{1}{2}\rho^{\underline{i}}\mathcal{D}_a\psi^{ai} \\
& - \psi^{ai}\mathcal{D}_a\rho^{\underline{i}} + 2(\psi_a{}^i\phi^{aj})q_j{}^{\underline{i}} + \frac{3\mathrm{i}}{2}(\psi_a{}^i\sigma^a\bar\Sigma_j)q^{j\underline{i}} + \frac{\mathrm{i}}{2}(\psi_{aj}\sigma^a\bar\Sigma^j)q^{i\underline{i}} \\
& + \frac{1}{6}\varepsilon^{mnpq}(\bar\psi_m^j\tilde\sigma_n\mathcal{D}_p\psi_{qj})q^{i\underline{i}} + \frac{1}{3}W^{ab+}(\bar\psi_a^j\bar\psi_{bj})q^{i\underline{i}} + \mathrm{i}(\psi_a{}^{(i}\sigma^b\bar\psi^{j)a})\mathcal{D}_b q_j{}^{\underline{i}} \\
& - \frac{\mathrm{i}}{2}(\psi_a{}^{(i}\sigma^b\bar\psi^{j)a})(\psi_{bj}\rho^{\underline{i}})\Bigg] + \text{c.c.}\;. \qquad (7.2.49c)
\end{aligned}
$$

In the expression for D, eq. (7.2.49c), remember that $(\Sigma^i,\bar\Sigma_i)$ and $(\phi^i,\bar\phi_i)$, together with the spin connection $\omega_m{}^{cd}$, are composite fields. Note that so far we have only used one of the four equations that are equivalent to (7.2.46c) to solve for D in eq. (7.2.49c). It is simple to show that the remaining independent three equations are equivalent to the following

$$
\nabla^a(q^{i(\underline{i}}\nabla_a q_i{}^{\underline{j})}) = 0\;. \qquad (7.2.50)
$$

As we are going to explain in detail below, this equation is solved by turning the $SU(2)_R$ connection $\phi_m{}^{kl}$ into a composite field.

As a next step in the construction of the hyper-dilaton Weyl multiplet, we note that, accompanied to an on-shell hypermultiplet there is always a triplet of composite linear multiplets [138–140, 186]. The following composite fields define a triplet of linear multiplets [141]

$$
\begin{aligned}
G_{ij}{}^{\underline{ij}} &= q_{(i}{}^{\underline{i}} q_{j)}{}^{\underline{j}} = q_i{}^{(\underline{i}} q_j{}^{\underline{j})} , && (G_{ij}{}^{\underline{ij}})^* = G^{ij}{}_{\underline{ij}} , && (7.2.51a)\\
\chi_{\alpha i}{}^{\underline{ij}} &= \frac{1}{2} q_i{}^{(\underline{i}} \rho_\alpha^{\underline{j})} , \quad \bar\chi^{\dot\alpha i}{}_{\underline{ij}} = -\frac{1}{2} q^i{}_{(\underline{i}} \bar\rho^{\dot\alpha}_{\underline{j})} , && (\chi_{\alpha i}{}^{\underline{ij}})^* = \bar\chi^i_{\dot\alpha \underline{ij}} , && (7.2.51b)\\
F^{\underline{ij}} &= \frac{1}{8} \rho^{(\underline{i}} \rho^{\underline{j})} , \quad \bar F_{\underline{ij}} = \frac{1}{8} \bar\rho_{(\underline{i}} \bar\rho_{\underline{j})} , && (F^{\underline{ij}})^* = \bar F_{\underline{ij}} , && (7.2.51c)\\
H^{a\underline{ij}} &= -\frac{1}{4} q^{i(\underline{i}} \nabla^a q_i{}^{\underline{j})} + \frac{\mathrm{i}}{32} \rho^{(\underline{i}} \sigma^a \bar\rho^{\underline{j})} , && (H^{a\underline{ij}})^* = H^a{}_{\underline{ij}} . && (7.2.51d)
\end{aligned}
$$

These fields all transform according to linear multiplet transformation (5.2.10) and each of the previous fields is symmetric in $\underline{i}$ and $\underline{j}$. Within the previous composite fields, the field $H^{a\underline{ij}}$ is particularly interesting. In fact, eq. (7.2.51d) together with (7.2.32) represent the solution to the constraint (7.2.50) and can be used to express the $SU(2)_R$ connection $\phi_m{}^{ij}$ as a composite field. By introducing the derivative

$$
\mathbf{D}_a = e_a{}^m \left(\partial_m - \frac{1}{2} \omega_m{}^{cd} M_{cd} - b_m \mathbb{D} \right) = \mathcal{D}_a + e_a{}^m \phi_m{}^{ij} J_{ij} + \mathrm{i} A_m Y , \qquad (7.2.52)
$$

and by using eq. (7.2.43), eq. (7.2.51d) can be rearranged for the $SU(2)_R$ gauge connection as follows

$$
\phi_a{}^{ij} = 4 q^{-4} q^{(i}{}_{\underline{i}} q^{j)}{}_{\underline{j}} \left[q^{k\underline{i}} \mathbf{D}_a q_k{}^{\underline{j}} - \frac{1}{4} q^{k\underline{i}} (\psi_{ak} \rho^{\underline{j}}) + \frac{1}{4} q^{k\underline{i}} (\bar\psi_{ak} \bar\rho^{\underline{j}}) - \frac{\mathrm{i}}{8} \rho^{\underline{i}} \sigma_a \bar\rho^{\underline{j}} + 4 H_a{}^{\underline{ij}} \right] , \qquad (7.2.53)
$$

with

$$
H_a{}^{\underline{ij}} = \tilde h_a{}^{\underline{ij}} + \frac{1}{6} \varepsilon_{abcd} \Big(-\frac{3\mathrm{i}}{4} (\sigma^{cd})_\alpha{}^\beta \psi^{b\alpha}{}_k \chi_\beta^{k\underline{ij}} - \frac{3\mathrm{i}}{4} (\tilde\sigma^{cd})^{\dot\alpha}{}_{\dot\beta} \bar\psi^{bk}{}_{\dot\alpha} \bar\chi_k^{\dot\beta \underline{ij}} + \frac{3}{4} (\sigma^b)_\alpha{}^{\dot\beta} \psi^{c\alpha}{}_k \bar\psi^{dl}{}_{\dot\beta} G^k{}_l{}^{\underline{ij}} \Big) , \qquad (7.2.54)
$$

which plugged back into (7.2.53) along with (7.2.51) gives

$$
\begin{aligned}
\phi^{aij} = {} & 2 q^{-2} q^{(i}{}_{\underline{i}} \mathbf{D}^a q^{j)\underline{i}} - \frac{1}{2} \psi^{a\alpha(i} q^{-2} q^{j)}{}_{\underline{i}} \rho^{\underline{i}}_\alpha - \frac{1}{2} \bar\psi^{a(i}{}_{\dot\alpha} q^{-2} q^{j)\underline{i}} \bar\rho^{\dot\alpha}_{\underline{i}} + 16 q^{-4} q^i{}_{\underline{i}} q^j{}_{\underline{j}} \tilde h^{a\underline{ij}} \\
& - \frac{1}{2} \mathrm{i} (\sigma^a)^\alpha{}_{\dot\alpha} q^{-4} q^{(i}{}_{\underline{i}} q^{j)\underline{j}} \rho^{\underline{i}}_\alpha \bar\rho^{\dot\alpha}_{\underline{j}} - \frac{1}{2} \varepsilon^{abcd} (\sigma_b)^\alpha{}_{\dot\alpha} \psi_{c\alpha}{}^{(i} \bar\psi_d{}^{j)\dot\alpha} \\
& + q^{-2} (\sigma^{ab})^{\beta\alpha} \psi_{b\beta}^i q^j{}_{\underline{i}} \rho^{\underline{i}}_\alpha + q^{-2} (\tilde\sigma^{ab})_{\dot\beta\dot\alpha} \bar\psi_b{}^{\dot\beta(i} q^{j)\underline{i}} \bar\rho_{\underline{i}}{}^{\dot\alpha} . \qquad (7.2.55)
\end{aligned}
$$

The existence of the linear multiplets (7.2.51) is crucial in the analysis of deformations of vector multiplets that we will perform in our paper. Note that in terms of superfields, the composite linear multiplet is defined by

$$
\mathcal{G}_{ij}{}^{\underline{ij}} = Q_{(i}{}^{\underline{i}} Q_{j)}{}^{\underline{j}} = Q_i{}^{(\underline{i}} Q_j{}^{\underline{j})} , \qquad \nabla_\alpha^{(i} \mathcal{G}^{jk)}{}_{\underline{ij}} = \bar\nabla_{\dot\alpha}^{(i} \mathcal{G}^{jk)}{}_{\underline{ij}} = 0 , \qquad (7.2.56)
$$

where all component fields in (7.2.51) arise from the equations (7.2.29) together with (7.2.44) and (7.2.45).

This concludes the definition of the hyper-dilaton Weyl multiplet. The result of the analysis is a representation of the off-shell local 4D, $N = 2$ superconformal algebra in terms of the following independent fields: $e_m{}^a$, b_m, A_m, W_{ab}, $q^{i\underline{i}}$, $b_{mn}{}^{\underline{ij}}$, $(\psi_{mi}, \bar{\psi}_m{}^i)$, and $(\rho^{\underline{i}}, \bar{\rho}_{\underline{i}})$. The multiplet has precisely the same number of off-shell degrees of freedom as the standard Weyl multiplet, $24+24$. Table 7.1 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields. Note that with the ingredients provided so far, it is a straightforward exercise to obtain the locally

$e_m{}^a$	$\omega_m{}^{ab}$	b_m	$\mathfrak{f}_{ma}$	$\phi_m{}^{ij}$	A_m	ψ_{mi}	$\phi_m{}^i$	W_{ab}	$\rho^{\underline{i}}$	$q^{i\underline{i}}$	$b_{mn}{}^{\underline{ij}}$
$16B$	0	$4B$	0	0	$4B$	$32F$	0	$6B$	$8F$	$4B$	$18B$
P_a	M_{ab}	$\mathbb{D}$	K_a	J^{ij}	Y	Q	S				$\lambda_m{}^{\underline{ij}}$-sym
$-4B$	$-6B$	$-1B$	$-4B$	$-3B$	$-1B$	$-8F$	$-8F$				$-9B$
Result: $24+24$ degrees of freedom											

Table 7.1: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components of these fields – composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that the parameter $\lambda_m{}^{\underline{ij}}$ describes the vector symmetry associated with the gauge two-forms $b_{mn}{}^{\underline{ij}}$ with field strength three-forms $h_{mnp}{}^{\underline{ij}}$ and $H^{a\underline{ij}}$. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. Row five gives the resulting number of degrees of freedom.

superconformal transformations of the fundamental fields of the hyper-dilaton Weyl multiplet written only in terms of fundamental fields. These are given by (2.2.24a)–(2.2.24c), (2.2.24e)–(2.2.24g), (7.2.33), and (7.2.42a)–(7.2.42c) after using the appropriate identities for all the composite fields $\omega_m{}^{cd}$, $\mathfrak{f}_{ma}$, $\phi_m{}^{ij}$, $(\phi_{mi}, \bar{\phi}_m{}^i)$, $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha} i})$, and D respectively given by eqs. (2.2.15), (7.2.53), (2.2.16), and (7.2.49).

It is important to underline that the local gauge transformations of the hyper-dilaton Weyl multiplet form an algebra that closes off-shell on a local extension of $\mathrm{SU}(2,2|2)$, the 4D $N = 2$ superconformal group. In fact, by construction the resulting algebra is identical to the one of the standard Weyl multiplet transformations (2.2.2) (see [137] and [26, 41] for detail on the local algebra), with the only important subtlety being that the structure functions will have more composite fields. We also stress that the existence of the triplet of composite linear multiplets in eqs. (7.2.51) is a key ingredient to engineer FI-type terms in supergravity-matter couplings based on the hyper-dilaton Weyl multiplet.

Transformation check

As a side, here we comment more on the consistency of the composite linear multiplet constructed out of the on-shell hypermultiplet.

It is necessary to use the on-shell condition on the hypermultiplet to prove that the composite multiplet with its lowest component being $G_{ij}{}^{\underline{ij}}$ is a linear multiplet. It is straightforward to show that its descendant fields $\chi_{\alpha i}{}^{\underline{ij}}$, $\bar{\chi}^{\dot{\alpha} i}{}_{\underline{ij}}$, $F^{\underline{ij}}$, and $\bar{F}_{\underline{ij}}$ transform correctly as linear multiplet fields using only the hypermultiplet supersymmetry transformation rules. No equations of motion are needed. However, for the composite three-form field, one does need to use an equation of motion.

The local supersymmetry transformation of the composite three-form of the linear multiplet in flat space[4] is given by

$$\begin{aligned}\delta H^{a\underline{ij}} &= -\frac{1}{4}\delta q^{i(\underline{i}}\nabla^a q_i{}^{\underline{j})} - \frac{1}{4}q^{i(\underline{i}}\delta\nabla^a q_i{}^{\underline{j})} + \frac{\mathrm{i}}{32}\delta\rho^{(\underline{i}}\sigma^a\bar{\rho}^{\underline{j})} + \frac{\mathrm{i}}{32}\rho^{(\underline{i}}\sigma^a\delta\bar{\rho}^{\underline{j})} \\ &= \frac{1}{8}\xi^{\alpha i}q_i{}^{(\underline{i}}\nabla^a\rho_{\alpha}^{\underline{j})} + \frac{1}{4}(\sigma^{ab})_{\alpha\gamma}\xi^{\gamma i}\rho^{\alpha(\underline{i}}\nabla_b q_i{}^{\underline{j})} + \text{c.c.}\ .\end{aligned} \tag{7.2.57}$$

We use the on-shell hypermultiplet equation of motion to bring the first term into the desired form. In flat space, it is as follows

$$\left(\nabla_a\rho^{\underline{i}}\sigma^a\right)_{\dot{\alpha}} = 0 \qquad \Longrightarrow \qquad \nabla^a\rho_{\alpha}^{\underline{i}} = 2(\sigma^{ab})_{\alpha\beta}\nabla_b\rho^{\beta\underline{i}}\,.$$

Inserting this back into the transformation rule we have

$$\begin{aligned}\delta H^{a\underline{ij}} &= \frac{1}{4}(\sigma^{ab})_{\alpha\beta}\xi^{\alpha i}(\nabla_b\rho^{\beta(\underline{i}}q_i{}^{\underline{j})} + \rho^{\beta(\underline{i}}\nabla_b q_i{}^{\underline{j})}) + \text{c.c.} \\ &= \frac{1}{2}\xi_i\sigma_{ab}\nabla^b\chi^{i\underline{ij}} + \text{c.c.}\ ,\end{aligned} \tag{7.2.58}$$

as required. Besides working as a consistency check, the previous calculation indicates that we must interpret the on-shell hypermultiplet as matter fields of the hyper-dilaton Weyl multiplet when using the composite linear multiplet to construct supersymmetric invariants and deformations. This will be used in following sections.

7.3 The deformed abelian vector multiplet

In this section, we first revisit how the electric-magnetic duality is implemented in superspace for 4D, $N=2$ vector multiplets and how deformed off-shell abelian vector multiplets arise from this duality in the presence of an (electric) Fayet-Iliopoulos term.

7.3.1 EM duality in $N=2$ superspace

The purpose of this subsection is to review and motivate the magnetic deformations of vector multiplets, which we will study in more detail in the next subsection. We refer the reader to [252, 253] for a more extensive discussion of the duality in flat superspace.

We start from a gauge invariant $N=2$ superfield strength W which is chiral

$$\overline{D}^i_{\dot{\alpha}}W = 0\ , \tag{7.3.1}$$

and satisfies the additional constraint

$$D^{ij}W - \bar{D}^{ij}\overline{W} = 0\ , \qquad D^{ij} := D^{\alpha i}D^j_{\alpha}\ , \quad \bar{D}_{ij} = \bar{D}_{\dot{\alpha}i}\bar{D}^{\dot{\alpha}}_j\ , \tag{7.3.2}$$

[4] For simplicity we restrict to a flat geometry where the derivatives should be $\nabla_a \to \partial_a$, but it is straightforward to extend this analysis to a Weyl multiplet background.

where

$$D^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} + \mathrm{i}(\sigma^b)_{\alpha\dot\beta}\bar\theta^{\dot\beta i}\partial_b\ , \qquad \bar D^{\dot\alpha}_i = \frac{\partial}{\partial \bar\theta^i_{\dot\alpha}} + \mathrm{i}(\tilde\sigma^b)^{\dot\alpha\beta}\theta_{\beta i}\partial_b\ , \tag{7.3.3}$$

are the flat $N=2$ superspace spinor covariant derivatives. The constraints on W can be solved through the Mezincescu prepotential [183]

$$W = \frac{1}{4}\bar\Delta D^{ij}V_{ij}\ , \tag{7.3.4}$$

where V_{ij} satisfies $V_{ij}=V_{ji}$, $(V_{ij})^* = V^{ij}$, while

$$\bar\Delta := \frac{1}{48}\bar D^{ij}\bar D_{ij} = -\frac{1}{48}\bar D^{\dot\alpha\dot\beta}\bar D_{\dot\alpha\dot\beta}\ , \quad \bar D_{\dot\alpha\dot\beta} = \bar D_{\dot\alpha k}\bar D^k_{\dot\beta}\ , \tag{7.3.5}$$

is the $N=2$ chiral projecting operator such that

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\mathcal{L} = \int \mathrm{d}^4x\mathrm{d}^4\theta\,\bar\Delta\mathcal{L} = \int \mathrm{d}^4x\mathrm{d}^4\bar\theta\,\Delta\mathcal{L}\ . \tag{7.3.6}$$

The dynamics of a free abelian vector multiplet are described by the superspace Lagrangian

$$S^0_v = -\mathrm{I}m\left[\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4\theta\,\tau W^2\right]\ , \qquad \tau := \frac{i}{g^2} + \vartheta\ . \tag{7.3.7}$$

In the case of a self-interacting theory, the previous model can be lifted to

$$S_v = -\mathrm{I}m\Big[\int \mathrm{d}^4x\mathrm{d}^4\theta F(W)\Big]\ , \tag{7.3.8}$$

where $F(W)$ is an arbitrary function of W which is the special-Kähler geometry holomorphic prepotential. An $N=2$ (electric) FI term is defined by

$$S_{\mathrm{F}I} = \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,\xi^{ij}V_{ij}\ , \tag{7.3.9}$$

with ξ^{ij} being a triplet of real constants. The theory described by $S_e = S_v + S_{\mathrm{F}I}$ is referred to as electrically deformed [93, 252, 253].

The magnetic dual of S_e is described in terms of the Lagrangian

$$S_m = -\mathrm{I}m\Big[\int \mathrm{d}^4x\mathrm{d}^4\theta\,\hat F(\mathbf{W})\Big]\ , \tag{7.3.10a}$$

where, however, the superfield $\mathbf{W}$ satisfies a modified reduced chiral constraint as follows

$$D^{ij}\mathbf{W} - \bar D^{ij}\overline{\mathbf{W}} = 2\mathrm{i}\zeta^{ij}\ , \tag{7.3.10b}$$

with ζ^{ij} being a real triplet of constants [252, 253]. Here, we have used a hat to denote the function of the deformed vector multiplet as it can be proven to be related by a duality transformation to $F(W)$ of the electric S_e action. The duality between S_e and S_m can be implemented through the action

$$S_{\mathrm{d}uality} = -\mathrm{I}m\left[\int \mathrm{d}^4x\mathrm{d}^4\theta\,\hat F(\Upsilon)\right] - \frac{\mathrm{i}}{8}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta\,U_{ij}\left[D^{ij}\Upsilon - \bar D^{ij}\bar\Upsilon - 2\mathrm{i}\zeta^{ij}\right]\ , \tag{7.3.11}$$

where U_{ij} is an unconstrained real $(U_{ij})^* = U^{ij}$ superfield and Υ is an arbitrary (and long) chiral superfield. Υ can be represented by using the chiral projecting operator and an arbitrary complex prepotential superfield as

$$\Upsilon = \bar{\Delta}\Psi \, . \tag{7.3.12}$$

When integrating out U_{ij} and renaming $\Upsilon = \mathbf{W}$ in the previous action, one obtains the $N = 2$ vector multiplet action deformed by a magnetic FI term in eq. (7.3.10). After integration by parts and using (7.3.6) one can rewrite (7.3.11) as

$$S_{\mathrm{d}uality} \;=\; -\mathrm{I}m\left[\int \mathrm{d}^4x\mathrm{d}^4\theta\left(\hat{F}(\Upsilon) - \Upsilon W_U\right)\right] - \frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\,\zeta^{ij}U_{ij}\,, \tag{7.3.13a}$$

with

$$W_U = \frac{1}{4}\bar{\Delta}D^{ij}U_{ij}\,. \tag{7.3.14}$$

The variation of the previous action with respect to Υ, after using (7.3.12) and integrating by parts, is

$$\delta S_{\mathrm{d}uality} = \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\,\delta\Psi\Big(\frac{\partial\hat{F}(\Upsilon)}{\partial\Upsilon} - W_U\Big)\,, \tag{7.3.15}$$

implying that on shell it holds

$$\frac{\partial\hat{F}(\Upsilon)}{\partial\Upsilon} = W_U\,, \quad W_U = \frac{1}{4}\bar{\Delta}D^{ij}U_{ij}\,, \tag{7.3.16}$$

which turns (7.3.13a) into

$$S_e = -\mathrm{I}m\left[\int \mathrm{d}^4x\mathrm{d}^4\theta\,F(W_U)\right] + \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\,\widetilde{\xi}^{ij}U_{ij}\,, \qquad \widetilde{\xi}^{ij} = -\frac{1}{4}\zeta^{ij}\,. \tag{7.3.17}$$

This is equivalent to (7.3.8) plus a standard FI term if we define

$$F(W_U) := \hat{F}[\Upsilon(W_U)] - \Upsilon(W_U)W_U\,, \qquad \frac{\partial F(W_U)}{\partial W_U} = -\Psi\,. \tag{7.3.18}$$

This is a usual Legendre transform of the special-Kähler holomorphic prepotential F and its dual $\hat{F}$, and $\Upsilon(W_U)$ is an implicit solution (which we assume to exist) of $\frac{\partial\hat{F}(\Upsilon)}{\partial\Upsilon} = W_U$ satisfying

$$\frac{\partial\Upsilon(W_U)}{\partial W_U} = \left[\frac{\partial W_U}{\partial\Upsilon}\right]^{-1} = [\tau(\Upsilon)]^{-1} \equiv -\widetilde{\tau}(W_U)\,, \tag{7.3.19}$$

where

$$\tau(\Upsilon) = \frac{\partial^2\hat{F}(\Upsilon)}{(\partial\Upsilon)^2}\,, \qquad \widetilde{\tau}(W_U) = \frac{\partial^2\hat{F}(W_U)}{(\partial W_U)^2}\,. \tag{7.3.20}$$

These are standard results for the EM duality of a vector multiplet. They show that electric and magnetic FI terms are interchanged with the duality. The same arguments are well known to generalise to several (abelian) vector multiplets. An important comment is that for flat supersymmetry one can consider a vector multiplet that has both electric and magnetic deformations while still having preserved off-shell supersymmetry [93, 252, 253]

$$S = -\mathrm{I}m\left[\int \mathrm{d}^4x\mathrm{d}^4\theta\, F(\mathbf{W})\right] + \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\,\xi^{ij}V_{ij}\,, \tag{7.3.21a}$$

$$D^{ij}\mathbf{W} - \bar{D}^{ij}\overline{\mathbf{W}} = 2\mathrm{i}\zeta^{ij}\,, \quad \mathbf{W} = \frac{1}{4}\bar{\Delta}D^{ij}V_{ij} + \frac{\mathrm{i}}{2}\theta_{ij}\zeta^{ij}\,, \tag{7.3.21b}$$

where $\theta_{ij} := \theta_i^\alpha\theta_{\alpha j}$. The presence of both an electric and a magnetic deformation is the key to obtaining partial supersymmetry breaking in flat superspace with a single physical vector multiplet [93]. We stress that this is a feature of the globally supersymmetric case.

Note that the previous derivation can straightforwardly be lifted to conformal supergravity defined in conformal superspace. The key ingredient is to realise that the electric and magnetic deformations will turn into linear multiplets. This is potentially straightforward, though there will be various subtleties related to the choices of a conformal supergravity background and compensators which we will discuss in the coming sections. For instance, given a set of vector multiplets in a hyper-dilaton Weyl background, an electric FI-type deformation will be associated to the following full conformal superspace invariant

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\,E\,\mathcal{G}_{\xi}{}_I^{ij}V^I_{ij}\,, \qquad \mathcal{G}_{\xi}{}_I^{ij} := \xi_I^{\underline{ij}}Q^i{}_{\underline{i}}Q^j{}_{\underline{j}}\,, \tag{7.3.22}$$

where $Q^{i\underline{i}}$ is the on-shell hypermultiplet in conformal superspace, while $\xi_I^{\underline{ij}} = \xi_I^{\underline{ji}}$ is a triplet of real constant.

Let us now proceed by introducing the modification of the magnetic deformation of an abelian vector multiplet in a conformal supergravity background.

7.3.2 Deformed abelian vector multiplet in conformal superspace

Consider the following deformation of abelian vector multiplets [258, 259]

$$\nabla^{ij}\mathbf{W} - \bar{\nabla}^{ij}\overline{\mathbf{W}} = 2\mathrm{i}\mathcal{G}^{ij}\,, \tag{7.3.23}$$

where $\mathbf{W}$ is only required to be covariantly chiral. The constraint now holds in conformal supergravity where the ∇_A derivatives are the conformal superspace ones that we introduced before. This multiplet can be thought of as a deformation of a standard vector multiplet described by W by means of the shift

$$\mathbf{W} = W + \Psi\,, \tag{7.3.24}$$

where Ψ is the prepotential of $\mathcal{G}^{ij}$ and is a chiral superfield, $\bar{\nabla}_i^{\dot{\alpha}}\Psi = 0$, of dimension 1 and $\mathrm{U}(1)_R$ weight -2, but otherwise arbitrary — see eqs. (7.2.24)–(7.2.28).

It is straightforward to deduce that

$$\nabla^{ij}\mathbf{W} - \bar{\nabla}^{ij}\bar{\mathbf{W}} = \nabla^{ij}\Psi - \bar{\nabla}^{ij}\bar{\Psi} = 2\mathrm{i}\mathcal{G}^{ij}\,, \tag{7.3.25}$$

where $\mathcal{G}^{ij}$ is the linear multiplet superfield. Acting with spinor covariant derivatives on $\mathbf{W}$ gives the following independent descendants:

$$\boldsymbol{\lambda}^i_\alpha = \nabla^i_\alpha \mathbf{W} , \quad \bar{\boldsymbol{\lambda}}^{\dot\alpha}_i = \bar\nabla^{\dot\alpha}_i \bar{\mathbf{W}} , \tag{7.3.26a}$$

$$\mathbf{X}^{ij} = \frac{1}{2}(\nabla^{ij}\mathbf{W} + \bar\nabla^{ij}\bar{\mathbf{W}}) , \quad \mathcal{G}^{ij} = -\frac{\mathrm{i}}{2}(\nabla^{ij}\mathbf{W} - \bar\nabla^{ij}\overline{\mathbf{W}}) , \tag{7.3.26b}$$

$$\mathbf{F}_{ab} = -\frac{1}{8}(\sigma_{ab})^{\alpha\beta}(\nabla_{\alpha\beta}\mathbf{W} + 4W_{\alpha\beta}\bar{\mathbf{W}}) + \frac{1}{8}(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}(\bar\nabla^{\dot\alpha\dot\beta}\bar{\mathbf{W}} + 4\overline{W}^{\dot\alpha\dot\beta}\mathbf{W}) , \tag{7.3.26c}$$

$$\mathbf{F}_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}\mathbf{F}_{ab} = -\frac{1}{8}(\nabla_{\alpha\beta}\mathbf{W} + 4W_{\alpha\beta}\bar{\mathbf{W}}), \tag{7.3.26d}$$

$$\bar{\mathbf{F}}^{\dot\alpha\dot\beta} = -\frac{1}{2}(\tilde\sigma^{ab})^{\dot\alpha\dot\beta}\mathbf{F}_{ab} = -\frac{1}{8}(\bar\nabla^{\dot\alpha\dot\beta}\bar{\mathbf{W}} + 4\overline{W}^{\dot\alpha\dot\beta}\mathbf{W}) , \tag{7.3.26e}$$

$$\chi_{\alpha i} = \frac{1}{3}\nabla^j_\alpha \mathcal{G}_{ij} , \qquad \bar\chi^{\dot\alpha i} = \frac{1}{3}\bar\nabla^{\dot\alpha}_j \mathcal{G}^{ij} , \tag{7.3.26f}$$

$$F = \frac{1}{12}\nabla^{ij}\mathcal{G}_{ij} , \qquad \bar F = \frac{1}{12}\bar\nabla^{ij}\mathcal{G}_{ij} , \tag{7.3.26g}$$

$$H_{abc} = \frac{\mathrm{i}}{96}\varepsilon_{abcd}(\sigma^d)^\alpha{}_{\dot\beta}[\nabla^i_\alpha, \bar\nabla^{\dot\beta}_j]\mathcal{G}^j{}_i = \varepsilon_{abcd}H^d , \tag{7.3.26h}$$

$$H_a = \frac{\mathrm{i}}{96}(\sigma_a)^\alpha{}_{\dot\beta}[\nabla^i_\alpha, \bar\nabla^{\dot\beta}_j]\mathcal{G}^j{}_i = \frac{1}{6}\varepsilon_{abcd}H^{bcd} . \tag{7.3.26i}$$

These superfields satisfy the following tower relations that are particularly useful in analyzing the structure of invariants:

$$\nabla^i_\alpha \boldsymbol{\lambda}^j_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\mathbf{X}^{ij} + \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\mathcal{G}^{ij} + 4\varepsilon^{ij}\mathbf{F}_{\alpha\beta} + 2\varepsilon^{ij}W_{\alpha\beta}\bar{\mathbf{W}} , \tag{7.3.27a}$$

$$\bar\nabla^{\dot\alpha}_i \boldsymbol{\lambda}^j_\beta = -2\mathrm{i}\delta^j_i \nabla_\beta{}^{\dot\alpha}\mathbf{W} , \tag{7.3.27b}$$

$$\nabla^i_\alpha \bar{\boldsymbol{\lambda}}^{\dot\beta}_j = -2\mathrm{i}\delta^i_j \nabla_\alpha{}^{\dot\beta}\bar{\mathbf{W}} , \tag{7.3.27c}$$

$$\bar\nabla^{\dot\alpha}_i \bar{\boldsymbol{\lambda}}^{\dot\beta}_j = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\mathbf{X}^{ij} - \frac{\mathrm{i}}{2}\varepsilon^{\dot\alpha\dot\beta}\mathcal{G}^{ij} + 4\varepsilon_{ij}\bar{\mathbf{F}}^{\dot\alpha\dot\beta} + 2\varepsilon_{ij}\overline{W}^{\dot\alpha\dot\beta}\mathbf{W} , \tag{7.3.27d}$$

$$\nabla^i_\alpha \mathbf{X}^{jk} = -2\mathrm{i}\varepsilon^{i(j}\chi^{k)}_\alpha - 4\mathrm{i}\varepsilon^{i(j}\nabla_\alpha{}^{\dot\alpha}\bar{\boldsymbol{\lambda}}^{k)}_{\dot\alpha} , \tag{7.3.27e}$$

$$\bar\nabla^{\dot\alpha}_i \mathbf{X}_{jk} = 2\mathrm{i}\varepsilon_{i(j}\bar\chi^{\dot\alpha}_{k)} - 4\mathrm{i}\varepsilon_{i(j}\nabla_\alpha{}^{\dot\alpha}\boldsymbol{\lambda}^\alpha_{k)} , \tag{7.3.27f}$$

$$\nabla^i_\gamma \mathbf{F}_{\alpha\beta} = \varepsilon_{\gamma(\alpha}\Sigma^i_{\beta)}\bar{\mathbf{W}} - \frac{1}{2}W_{\alpha\beta}{}^i_\gamma\bar{\mathbf{W}} + \frac{1}{2}\mathrm{i}\varepsilon_{\gamma(\alpha}\chi^i_{\beta)} + \frac{1}{2}\mathrm{i}\varepsilon_{\gamma(\alpha}\nabla_{\beta)}{}^{\dot\alpha}\bar{\boldsymbol{\lambda}}^i_{\dot\alpha} , \tag{7.3.27g}$$

$$\bar\nabla^{\dot\gamma}_k \mathbf{F}_{\alpha\beta} = \frac{\mathrm{i}}{2}\nabla_{(\alpha}{}^{\dot\gamma}\boldsymbol{\lambda}_{\beta)k} - \frac{1}{2}W_{\alpha\beta}\bar{\boldsymbol{\lambda}}^{\dot\gamma}_k , \tag{7.3.27h}$$

$$\nabla^i_\gamma \bar{\mathbf{F}}^{\dot\alpha\dot\beta} = \frac{\mathrm{i}}{2}\nabla_\gamma{}^{(\dot\alpha}\bar{\boldsymbol{\lambda}}^{\dot\beta)k} - \frac{1}{2}\overline{W}^{\dot\alpha\dot\beta}\boldsymbol{\lambda}^k_\gamma , \tag{7.3.27i}$$

$$\bar\nabla^{\dot\gamma}_k \bar{\mathbf{F}}^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\gamma(\dot\alpha}\bar\Sigma^{\dot\beta)}_k\mathbf{W} - \frac{1}{2}\overline{W}^{\dot\alpha\dot\beta\dot\gamma}{}_k\mathbf{W} - \frac{1}{2}\mathrm{i}\varepsilon^{\dot\gamma(\dot\alpha}\bar\chi^{\dot\beta)}_k + \frac{1}{2}\mathrm{i}\varepsilon^{\dot\gamma(\dot\alpha}\nabla_\alpha{}^{\dot\beta)}\boldsymbol{\lambda}^\alpha_k . \tag{7.3.27j}$$

The tower of S-supersymmetry transformations is identical to the cases of a standard vector multiplet and a linear multiplet, up to appropriately renaming some descendant with bold symbols. As a result, the local superconformal transformation of the fundamental component fields of the deformed vector multiplet fields in a standard Weyl multiplet background are

$$\delta\boldsymbol{\phi} = \xi_i\boldsymbol{\lambda}^i + \lambda_{\mathbb{D}}\boldsymbol{\phi} - 2\mathrm{i}\lambda_Y\boldsymbol{\phi} , \tag{7.3.28a}$$

$$\delta\overline{\boldsymbol{\phi}} = \bar\xi^i\bar{\boldsymbol{\lambda}}_i + \lambda_{\mathbb{D}}\overline{\boldsymbol{\phi}} + 2\mathrm{i}\lambda_Y\overline{\boldsymbol{\phi}} , \tag{7.3.28b}$$

$$\begin{aligned}
\delta\boldsymbol{\lambda}_\alpha^i &= 2(\sigma^{ab}\xi^i)_\alpha\mathbf{F}_{ab}+(\sigma^{ab}\xi^i)_\alpha W_{ab}^+\overline{\boldsymbol{\phi}}-\frac{1}{2}\xi_{\alpha j}\mathbf{X}^{ij}-\frac{\mathrm{i}}{2}\xi_{\alpha j}G^{ij}+2\mathrm{i}(\sigma^a\bar{\xi}^i)_\alpha\nabla_a\boldsymbol{\phi}\\
&\quad+\frac{1}{2}\lambda^{ab}(\sigma_{ab}\boldsymbol{\lambda}^i)_\alpha+\lambda^i{}_j\boldsymbol{\lambda}_\alpha^j+\frac{3}{2}\lambda_{\mathbb{D}}\boldsymbol{\lambda}_\alpha^i-\mathrm{i}\lambda_Y\boldsymbol{\lambda}_\alpha^i+4\eta_\alpha^i\boldsymbol{\phi}\ , && (7.3.28c)\\
\delta\bar{\boldsymbol{\lambda}}_i^{\dot\alpha} &= -2(\tilde\sigma^{ab}\bar\xi_i)^{\dot\alpha}\mathbf{F}_{ab}-(\tilde\sigma^{ab}\bar\xi_i)^{\dot\alpha}W_{ab}^-\boldsymbol{\phi}-\frac{1}{2}\bar\xi^{\dot\alpha j}\mathbf{X}_{ij}+\frac{\mathrm{i}}{2}\bar\xi^{\dot\alpha j}G_{ij}+2\mathrm{i}(\tilde\sigma^a\xi_i)^{\dot\alpha}\nabla_a\overline{\boldsymbol{\phi}}\\
&\quad+\frac{1}{2}\lambda^{ab}(\tilde\sigma_{ab}\bar{\boldsymbol{\lambda}}_i)^{\dot\alpha}-\lambda_i{}^j\bar{\boldsymbol{\lambda}}_j^{\dot\alpha}+\frac{3}{2}\lambda_{\mathbb{D}}\bar{\boldsymbol{\lambda}}_i^{\dot\alpha}+\mathrm{i}\lambda_Y\bar{\boldsymbol{\lambda}}_i^{\dot\alpha}+4\bar\eta_i^{\dot\alpha}\overline{\boldsymbol{\phi}}\ , && (7.3.28d)\\
\delta\boldsymbol{X}^{ij} &= 2\mathrm{i}\xi^{\alpha(i}\chi_\alpha^{j)}+4\mathrm{i}\xi^{\alpha(i}\nabla_\alpha{}^{\dot\alpha}\bar{\boldsymbol{\lambda}}_{\dot\alpha}^{j)}-2\mathrm{i}\bar\xi_{\dot\alpha}^{(i}\bar\chi^{\dot\alpha j)}+4\mathrm{i}\bar\xi_{\dot\alpha}^{(i}\nabla_\alpha{}^{\dot\alpha}\boldsymbol{\lambda}^{j)\alpha}\\
&\quad+2\lambda^{(i}{}_k\boldsymbol{X}^{j)k}+2\lambda_{\mathbb{D}}\boldsymbol{X}^{ij}\ , && (7.3.28e)\\
\delta\boldsymbol{F}^{ab} &= \Big[-\mathrm{i}\xi_k\sigma_{[a}\nabla_{b]}\bar{\boldsymbol{\lambda}}^k+\Big(\xi_k\sigma_{ab}\Sigma^k-\frac{1}{2}\xi_k^\alpha(\sigma_{ab})^{\beta\gamma}W_{\alpha\beta\gamma}{}^k\Big)\bar{\boldsymbol{\phi}}-\frac{1}{2}(\xi_k\boldsymbol{\lambda}^k)W_{ab}^-\\
&\quad+\frac{\mathrm{i}}{2}\xi_k\sigma_{ab}\chi^k+2\eta^k\sigma_{ab}\boldsymbol{\lambda}_k+\mathrm{c.c.}\Big]+2\lambda_{\mathbb{D}}\boldsymbol{F}_{ab}-2\lambda_{[a}{}^c\boldsymbol{F}_{b]c}\ , && (7.3.28f)
\end{aligned}$$

while all the transformations of the descendants of $\mathbf{W}$ and $\bar{\mathbf{W}}$ associated to $\mathcal{G}^{ij}$ are exactly the same transformations as the linear multiplet component fields G^{ij}, $\chi_{\alpha i}$, $\bar\chi^{\dot\alpha i}$, F, and $\bar F$ given in (5.2.10).

Note that, in a hyper-dilaton Weyl background, thanks to the existence of the composite triplet of linear multiplets, see, e.g., eq. (7.2.56), it is natural to consider the deformations of a set of vector multiplets associated to the following deformed constraints

$$\nabla^{ij}\mathbf{W}^I-\bar\nabla^{ij}\overline{\mathbf{W}}^I=2\mathrm{i}\mathcal{G}_\zeta^{I\,ij}\ ,\qquad \mathcal{G}_\zeta^{I\,ij}:=\zeta^I_{\underline{ij}}Q^{i\underline{i}}Q^{j\underline{j}}\ , \tag{7.3.29}$$

where $Q^{i\underline{i}}$ is the on-shell hypermultiplet in conformal superspace, while $\zeta^I_{\underline{ij}}=\zeta^I_{\underline{ji}}$ is a triplet of real constants which plays a similar role to the global magnetic FI terms. This will be one of the ingredients that we use in the coming sections.

We proceed next with the definition of several locally superconformal action principles both in superspace and components.

7.4 Superconformal actions

In this section, we review the local superconformal action principles that we use to engineer the supergravity-matter systems studied in the rest of the paper. This includes the abelian vector multiplet action and the linear multiplet action with magnetic and electric deformations, respectively.

7.4.1 Chiral action principle

We introduce here the chiral action involving an integral over the chiral subspace

$$S=S_c+\mathrm{c.c.}\ ,\quad S_c=\int\mathrm{d}^8z\,\mathcal{E}\,\mathcal{L}_c\ ,\quad \mathrm{d}^8z:=\mathrm{d}^4x\,\mathrm{d}^4\theta\ , \tag{7.4.1}$$

where $\mathcal{L}_c$ is covariantly chiral, $\bar\nabla_i^{\dot\alpha}\mathcal{L}_c=0$, and $\mathcal{E}$ is a suitably chosen chiral measure [26, 308–310]. The Lagrangian $\mathcal{L}_c$ must be a conformally primary Lorentz and $SU(2)_R$ chiral scalar with conformal dimension two and $\mathrm{U}(1)_R$ weight -4:

$$\mathbb{D}\mathcal{L}_c = 2\mathcal{L}_c \ , \quad Y\mathcal{L}_c = -4\mathcal{L}_c \ , \quad J^{ij}\mathcal{L}_c = M_{ab}\mathcal{L}_c = K_a\mathcal{L}_c = S^i_\alpha\mathcal{L}_c = \bar{S}^{\dot{\alpha}}_i\mathcal{L}_c = 0 \ . \tag{7.4.2}$$

Any action involving an integral over the full superspace may be converted to one over the chiral subspace by the rule [26]

$$\int \mathrm{d}^{12}z\, E\, \mathcal{L} = \int \mathrm{d}^8 z\, \mathcal{E}\, \bar{\nabla}^4 \mathcal{L} \ , \quad \mathrm{d}^{12}z := \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}, \quad \bar{\nabla}^4 := \frac{1}{48}\bar{\nabla}^{ij}\bar{\nabla}_{ij} \ . \tag{7.4.3}$$

The chiral action in components [26], and in our notation that follow the ones of [41], takes the form of the following density formula

$$\begin{aligned} S_c = \int \mathrm{d}^4x\, e \Bigg(& \frac{1}{48}\nabla^{ij}\nabla_{ij} - \frac{\mathrm{i}}{12}\bar{\psi}_d{}^l_{\dot{\delta}}(\tilde{\sigma}^d)^{\dot{\delta}\alpha}\nabla^q_\alpha\nabla_{lq} + \frac{\mathrm{i}}{2}\bar{\psi}_d{}^l_{\dot{\delta}}(\sigma^d)_{\alpha\dot{\alpha}}\overline{W}^{\dot{\alpha}\dot{\delta}}\nabla^\alpha_l + \overline{W}^{\dot{\alpha}\dot{\beta}}\overline{W}_{\dot{\alpha}\dot{\beta}} \\ & + \frac{1}{4}\bar{\psi}_c{}^k_{\dot{\gamma}}\bar{\psi}_d{}^l_{\dot{\delta}}\Big((\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}}\nabla_{kl} - \frac{1}{2}\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon_{kl}(\sigma^{cd})_{\beta\gamma}\nabla^{\beta\gamma} - 4\varepsilon^{\dot{\gamma}\dot{\delta}}\varepsilon_{kl}(\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}}\overline{W}^{\dot{\alpha}\dot{\beta}} \Big) \\ & - \frac{1}{4}\varepsilon^{abcd}(\tilde{\sigma}_a)^{\dot{\beta}\alpha}\bar{\psi}_b{}^j_{\dot{\beta}}\bar{\psi}_c{}^k_{\dot{\gamma}}\bar{\psi}_d{}^{\dot{\gamma}}_k\nabla_{\alpha j} - \frac{\mathrm{i}}{4}\varepsilon^{abcd}\bar{\psi}_a{}^i_{\dot{\alpha}}\bar{\psi}_b{}^{\dot{\alpha}}_i\bar{\psi}_c{}^j_{\dot{\beta}}\bar{\psi}_d{}^{\dot{\beta}}_j \Bigg)\mathcal{L}_c| \ . \end{aligned} \tag{7.4.4}$$

Efficient ways to obtain this result make use of either a normal coordinate expansion in superspace, see [310], or alternatively by using the superform approach to constructing supersymmetric invariants, see [311, 312]. We stress that the component action (7.4.4) is the primary building block for the superconformal invariant actions considered throughout this work wherein a consistent choice for $\mathcal{L}_c$ in conformal superspace satisfying the above properties determines its structure.

7.4.2 Deformed abelian vector multiplet action

Let us now consider the general, superconformal chiral action $\mathcal{L}_c = \mathcal{F}(\mathbf{W}^I)$ of n deformed abelian vector multiplets $\mathbf{W}^I$. Note that $\mathcal{F}(\mathbf{W}^I)$ must be homogeneous of degree two in $\mathbf{W}^I$,

$$\mathbf{W}^I\mathcal{F}_I \equiv \mathbf{W}^I\frac{\partial}{\partial\mathbf{W}^I}\mathcal{F} = 2\mathcal{F} \ , \tag{7.4.5}$$

with

$$\nabla^{ij}\mathbf{W}^I - \bar{\nabla}^{ij}\overline{\mathbf{W}}^I = 2\mathrm{i}\mathcal{G}^{I,ij} \ . \tag{7.4.6}$$

Recall that the above follows from the fact that the deformed abelian vector multiplet can be defined by the shift

$$\mathbf{W}^I = W^I + \Psi^I \ , \tag{7.4.7}$$

where Ψ^I are the prepotential for the linear multiplets $\mathcal{G}^{I,ij}$

$$\mathcal{G}^{I,ij} = -\frac{\mathrm{i}}{2}\big(\nabla^{ij}\Psi^I - \bar{\nabla}^{ij}\overline{\Psi}^I\big) \ . \tag{7.4.8}$$

Here we do not specify whether the linear multiplets are composite (as for the hyper-dilaton Weyl case that we will study later on) or fundamental. The model is manifestly invariant under the gauge transformation

$$\hat{\delta}\mathbf{W}^I = 0 \ , \tag{7.4.9}$$

which is apparent from

$$\hat{\delta}\Psi^I = \hat{W}^I \,, \qquad \hat{\delta}W^I = -\hat{W}^I \,, \tag{7.4.10}$$

for a vector multiplet field strength $\hat{W}^I$ satisfying

$$\bar{\nabla}_i^{\dot\alpha}\hat{W}^I = 0 \,, \quad \nabla^{ij}\hat{W}^I = \bar{\nabla}^{ij}\overline{\hat{W}}^I \,. \tag{7.4.11}$$

By the component action principle of eq. (7.4.4), this action in components was generated computationally by the computer algebra software, *Cadabra* [87, 88, 313]. Note that a specific code repository [7] has been developed in parallel to this work that automatically generates 4D, $N=2$ actions in components. It has been directly applied here and to all further results in this paper. After further cleaning up by hand, we obtain the following result in components

$$\begin{aligned}
S_c = \int d^4x\,e\Bigg[&\mathcal{F}_I\Box\bar{\boldsymbol{\phi}}^I - 2\mathcal{F}_I\overline{W}_{\dot\alpha\dot\gamma}\bar{\mathbf{F}}^{I\dot\alpha\dot\gamma} - \mathcal{F}\overline{W}_{\dot\alpha\dot\beta}\overline{W}^{\dot\alpha\dot\beta} + 3\mathcal{F}_I D\bar{\phi}^I + \frac{3}{2}\mathcal{F}_I\bar{\Sigma}^k_{\dot\alpha}\bar{\boldsymbol{\lambda}}^{\dot\alpha}_k \\
&-\frac{\mathrm{i}}{2}\mathcal{F}_{IJ}\boldsymbol{\lambda}^{I\alpha j}\nabla_{\alpha\dot\alpha}\bar{\boldsymbol{\lambda}}^{J\dot\alpha}_j + \frac{1}{32}\mathcal{F}_{IJ}\mathbf{M}^{Iij}\mathbf{M}^J_{ij} - 2\mathcal{F}_{IJ}\mathbf{F}^{I\alpha\beta}\mathbf{F}^J_{\alpha\beta} \\
&-2\mathcal{F}_{IJ}\bar{\boldsymbol{\phi}}^I W^{\alpha\beta}\mathbf{F}^J_{\alpha\beta} - \frac{1}{2}\mathcal{F}_{IJ}\bar{\boldsymbol{\phi}}^I\bar{\boldsymbol{\phi}}^J W^{\alpha\beta}W_{\alpha\beta} + \frac{1}{16}\mathcal{F}_{IJK}\left(\boldsymbol{\lambda}^{Ii}\boldsymbol{\lambda}^{Jj}\right)\mathbf{M}^K_{ij} \\
&+\frac{1}{2}\mathcal{F}_{IJK}\boldsymbol{\lambda}^{I\alpha k}\boldsymbol{\lambda}^{J\beta}_k\mathbf{F}^K_{\alpha\beta} + \frac{1}{4}\mathcal{F}_{IJK}\bar{\boldsymbol{\phi}}^I\boldsymbol{\lambda}^{J\alpha k}\boldsymbol{\lambda}^{K\beta}_k W_{\alpha\beta} \\
&+\frac{1}{48}\mathcal{F}_{IJKL}\boldsymbol{\lambda}^{\alpha i,L}\boldsymbol{\lambda}^{j,K}_\alpha\boldsymbol{\lambda}^{\beta,J}_i\boldsymbol{\lambda}^K_{\beta j} - \frac{1}{2}\mathcal{F}_I\left(\bar{\psi}^j_m\tilde{\sigma}^m\sigma^b\right)_{\dot\alpha}\nabla_b\bar{\boldsymbol{\lambda}}^{I\dot\alpha}_j \\
&-\frac{\mathrm{i}}{8}\mathcal{F}_{IJ}\left(\bar{\psi}^i_m\tilde{\sigma}^m\boldsymbol{\lambda}^{Ij}\right)\mathbf{M}^J_{ij} - \mathrm{i}\mathcal{F}_{IJ}\left(\bar{\psi}^k_m\tilde{\sigma}^m\right)^\alpha\boldsymbol{\lambda}^{I\beta}_k\mathbf{F}^J_{\alpha\beta} \\
&-\frac{\mathrm{i}}{2}\mathcal{F}_{IJ}\bar{\boldsymbol{\phi}}^I\left(\bar{\psi}^k_m\tilde{\sigma}^m\right)^\alpha\boldsymbol{\lambda}^{J\beta}_k W_{\alpha\beta} - \frac{\mathrm{i}}{12}\mathcal{F}_{IJK}\left(\bar{\psi}^i_m\tilde{\sigma}^m\boldsymbol{\lambda}^{Ij}\right)\left(\boldsymbol{\lambda}^J_i\boldsymbol{\lambda}^K_j\right) \\
&+\frac{\mathrm{i}}{2}\mathcal{F}_I\bar{\psi}^j_{b\dot\gamma}\overline{W}_{\dot\beta}{}^{\dot\gamma}(\tilde{\sigma}^b)^{\dot\beta\alpha}\boldsymbol{\lambda}^I_{\alpha j} - \frac{1}{4}\mathcal{F}_I\left(\bar{\psi}^i_m\tilde{\sigma}^{mn}\bar{\psi}^j_n\right)\mathbf{M}^I_{ij} \\
&-\frac{1}{4}\mathcal{F}_{IJ}\left(\bar{\psi}^i_m\tilde{\sigma}^{mn}\bar{\psi}^j_n\right)\left(\boldsymbol{\lambda}^I_i\boldsymbol{\lambda}^J_j\right) + \mathcal{F}_I\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\sigma^{mn}\right)^{\alpha\beta}\mathbf{F}^I_{\alpha\beta} \\
&+\frac{1}{2}\mathcal{F}_I\bar{\boldsymbol{\phi}}^I\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\sigma^{mn}\right)^{\alpha\beta}W_{\alpha\beta} + \frac{1}{8}\mathcal{F}_{IJ}\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\boldsymbol{\lambda}^{Ik}\sigma^{mn}\boldsymbol{\lambda}^J_k\right) \\
&-\mathcal{F}\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\tilde{\sigma}^{mn}\right)^{\dot\alpha\dot\beta}\overline{W}_{\dot\alpha\dot\beta} + \frac{1}{4}\mathcal{F}_I\varepsilon^{mnpq}\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\bar{\psi}^i_p\tilde{\sigma}_q\boldsymbol{\lambda}^I_i\right) \\
&-\frac{\mathrm{i}}{4}\mathcal{F}\varepsilon^{mnpq}\left(\bar{\psi}_m\bar{\psi}_n\right)\left(\bar{\psi}_p\bar{\psi}_q\right) \\
&+\frac{\mathrm{i}}{2}\mathcal{F}_I F^I + \frac{\mathrm{i}}{2}\mathcal{F}_{IJ}\left(\boldsymbol{\lambda}^{j,I}\chi^J_j\right) + \frac{1}{2}\mathcal{F}_I\left(\bar{\psi}^j_m\tilde{\sigma}^m\chi^I_j\right)\Bigg] \,,
\end{aligned} \tag{7.4.12}$$

where we have introduced the following complex triplet of scalar fields

$$\mathbf{M}^I_{ij} = \mathbf{X}^I_{ij} + \mathrm{i}G^I_{ij} \,, \qquad \overline{\mathbf{M}}^I_{ij} = \mathbf{X}^I_{ij} - \mathrm{i}G^I_{ij} \,. \tag{7.4.13}$$

Note that the effect of the deformation is simply a shift by an imaginary unit times a linear multiplet and that the final line in (7.4.12) is comprised of terms that originated from the deformation. Otherwise, this action is equivalent to the chiral component action of the abelian vector multiplets without deformations, $\mathcal{L}_c = \mathcal{F}(W^I)$, as found in previous literature, see, e.g., [41], up to making various fields appropriately bold through the shift by a linear multiplet. It is useful to use the following properties in components in the action (7.4.12)

$$\nabla_b \bar{\boldsymbol{\lambda}}^{I\dot{\alpha}}_j = \mathcal{D}_b \bar{\boldsymbol{\lambda}}^{I\dot{\alpha}}_j - 2\bar{\phi}_b{}^{\dot{\alpha}}_j \bar{\boldsymbol{\phi}}^I + \frac{1}{4}\bar{\psi}_a{}^{\dot{\alpha}k}\overline{\mathbf{M}}^I_{jk} + \bar{\psi}_{a\dot{\gamma}j}(2\mathbf{F}^{I\dot{\gamma}\dot{\alpha}} + W^{\dot{\gamma}\dot{\alpha}}\boldsymbol{\phi}^I)$$
$$- \mathrm{i}\psi_{a\gamma j}\left(\mathcal{D}^{\dot{\alpha}\gamma}\bar{\boldsymbol{\phi}}^I + \frac{1}{2}(\sigma^c)^{\dot{\alpha}\gamma}(\bar{\psi}_{ck}\bar{\boldsymbol{\lambda}}^{Ik})\right) , \tag{7.4.14a}$$
$$\Box\bar{\boldsymbol{\phi}}^I = \mathcal{D}^a\mathcal{D}_a\bar{\boldsymbol{\phi}}^I + \frac{1}{2}\mathcal{D}_b(\bar{\psi}^b_k\bar{\boldsymbol{\lambda}}^{Ik}) + \frac{\mathrm{i}}{2}\left(\phi^j_m\sigma^m\bar{\boldsymbol{\lambda}}^I_j\right) + \frac{\mathrm{i}}{4}\left(\psi_{mj}\sigma^m\right)_{\dot{\alpha}}\overline{W}^{\dot{\alpha}\dot{\beta}}\bar{\boldsymbol{\lambda}}^{Ij}_{\dot{\beta}}$$
$$- \frac{3\mathrm{i}}{4}\left(\psi_{mj}\sigma^m\bar{\Sigma}^j\right)\bar{\boldsymbol{\phi}}^I - \frac{1}{2}\bar{\psi}^j_a\nabla^a\bar{\boldsymbol{\lambda}}^I_j - 2\mathfrak{f}_a{}^a\bar{\boldsymbol{\phi}}^I . \tag{7.4.14b}$$

As a final note, we underline that after covariant vector derivatives are degauged as seen above, due to the component gauge fixing conditions (7.2.39), the bold, “deformed” fields of this vector multiplet may be thought of as equal to their non-bold, “non-deformed” counterparts in components with one exception being

$$\mathbf{F}^I_{ab}| = F^I_{ab}| + B^I_{ab}| , \tag{7.4.15}$$

though one should keep in mind that the triplet of scalar auxiliary fields receives an imaginary shift.

Before we proceed, it is useful to make a comment on the effect that the deformation has on the theory’s scalar potential. A relevant term is given by $\frac{1}{32}\mathcal{F}_{IJ}\mathbf{M}^{Iij}\mathbf{M}^J_{ij} + \mathrm{c.}c.$ from (7.4.12). The quadratic term in X^I_{ij} is the one that, if the auxiliary fields acquire a vev (typically through an electric gauging by a standard FI term), could ubiquitously lead to a contribution to the vacuum energy. Now, the effect of the “magnetic” deformation is similar, but leads to different signs due to the imaginary unit. the difference is that the contribution is already in the Lagrangian without having to integrate out any auxiliary field. Depending on the structure of the model and its holomorphic prepotential, potentially the deformation can lead to Minkowski, anti de Sitter (AdS) or even de Sitter (dS) vacua.

7.4.3 Standard *BF* action/electric FI term

Let us now consider the supersymmetric *BF* action [41, 185]

$$S_{\text{standard FI}} = -2\mathrm{i}\int \mathrm{d}^8 z\, \mathcal{E}\, \Psi_I W^I + \text{c.c.} = \int \mathrm{d}^{12} z E\, G^{ij}_I V^I_{ij} , \tag{7.4.16}$$

where

$$G^{ij}_I := -\frac{\mathrm{i}}{2}\left(\nabla^{ij}\Psi_I - \bar{\nabla}^{ij}\bar{\Psi}_I\right) , \qquad W^I := \frac{1}{4}\bar{\Delta}\nabla^{ij}V^I_{ij} . \tag{7.4.17}$$

This is defined as a locally superconformal completion of a *BF* term and emerges as an appropriate product of a linear and a (undeformed) vector multiplet. Here we do not specify whether the linear multiplets, nor vector multiplets are composite. In all cases the previous action proves to be locally superconformal invariant.

The component action for $S_{\text{standard FI}}$ can be obtained by using eq. (7.4.4) in the first definition in (7.4.16). The result is

$$S_{\text{standard FI}} = \int d^4xe\Bigg[F_I\phi^I + \chi^\alpha_i{}_I\lambda^{iI}_\alpha + \frac{1}{8}G^{ij}_I X^I_{ij} - \varepsilon^{mnpq}b_{mnI}f^I_{pq}$$

$$-\frac{\mathrm{i}}{2}\bar{\psi}_d{}^l_{\dot{\delta}}(\tilde{\sigma}^d)^{\dot{\delta}\alpha}\left[2\chi_{\alpha lI}\phi^I+G_{lqI}\lambda_\alpha^{qI}\right]+\bar{\psi}_c{}^k_{\dot{\gamma}}\bar{\psi}_d{}^l_{\dot{\delta}}(\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}}G_{klI}\phi^I\Bigg]+\text{c.c.}\,, \tag{7.4.18a}$$

or equivalently

$$S_{\text{standard FI}}=\int d^4xe\Bigg[F_I\phi^I+\chi_i^\alpha{}_I\lambda_\alpha^{iI}+\frac{1}{8}G_I^{ij}X_{ij}^I+\frac{2}{3}\varepsilon^{mnpq}h_{mnpI}v_q^I$$
$$-\frac{\mathrm{i}}{2}\bar{\psi}^l_{d\dot{\delta}}(\tilde{\sigma}^d)^{\dot{\delta}\alpha}\left[2\chi_{\alpha lI}\phi^I+G_{lqI}\lambda_\alpha^{qI}\right]+\bar{\psi}^k_{c\dot{\gamma}}\bar{\psi}^l_{d\dot{\delta}}(\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}}G_{klI}\phi^I\Bigg]+\text{c.c.}\,. \tag{7.4.18b}$$

Notably, and consistently, the previous action is invariant under the defining shift symmetry of the linear multiplet prepotentials by vector multiplets,

$$\tilde{\delta}\Psi_I=\tilde{W}_I\,. \tag{7.4.19}$$

This is manifestly an invariance, assuming that $\tilde{\delta}W^I=0$ and

$$\bar{\nabla}_i^{\dot{\alpha}}\tilde{W}_I=0\,,\qquad \nabla^{ij}\tilde{W}_I=\bar{\nabla}^{ij}\overline{\tilde{W}}_I\,. \tag{7.4.20}$$

Here we have used the symbol $\tilde{W}_I$ to distinguish the closed, super two-form field strength vector multiplet gauge parameter from the physical vector multiplet W^I. The invariance can be trivially seen when looking at the second equation in (7.4.16) and by noticing that G_I^{ij} is identically zero if Ψ_I is replaced with the vector multiplet, $\tilde{W}_I$, in (7.4.17). Moreover, the action (7.4.16) is also invariant under the following gauge transformations of the vector multiplets prepotentials

$$\delta_\Lambda V_{ij}^I=\nabla_k^\alpha\Lambda_\alpha^{Iijk}+\bar{\nabla}_{\dot{\alpha}k}\bar{\Lambda}^{I\dot{\alpha}ijk}\,,\qquad \Lambda_\alpha^{Iijk}=\Lambda_\alpha^{I(ijk)}\,,\quad \bar{\Lambda}^{I\dot{\alpha}}_{ijk}=(\Lambda^{I\alpha ijk})^*\,, \tag{7.4.21}$$

for a set of complex gauge parameter superfields Λ^{Iijk}_α being arbitrary up to the algebraic and reality conditions stated above. This transformation leaves the field strengths W^I in (7.4.17) invariant and, after superspace integration by parts and using $\nabla_\alpha^{(i}G_I^{jk)}=0$, $\bar{\nabla}_{\dot{\alpha}}^{(i}G_I^{jk)}=0$, one can directly show the invariance of the second form of (7.4.16). The invariances under $\tilde{\delta}$ and δ_Λ manifest themselves in the component action (7.4.18) by the fact that the only term transforming would be $\varepsilon^{mnpq}b_{mnI}f_{pq}^I$, equivalent to $\varepsilon^{mnpq}h_{mnpI}v_q^I$, which transform as total derivatives under the $\tilde{\delta}$ and δ_Λ variations.

We did stress that in the global case, it is possible to simultaneously turn on an electric and a magnetic FI term preserving (and deforming) supersymmetry off shell [93, 252, 253]. This is a fundamental ingredient in engineering global partial supersymmetry breaking by the use of vector multiplets. It is natural to ask whether the same is possible in the local off-shell superconformal setting that we have described above. In contrast to the global case, due to the gauge symmetries of the linear multiplets involved in the two types of deformations, in the local case, it does not seem possible to have the two FI-type deformations turned on at the same time. Let us comment more on this.

Suppose that we consider magnetically deformed vector multiplets

$$\mathbf{W}^I=W^I+\Psi^I\,,\quad \nabla^{ij}\mathbf{W}^I-\bar{\nabla}^{ij}\overline{\mathbf{W}}^I=2\mathrm{i}\mathcal{G}^{Iij}\,, \tag{7.4.22}$$

which possess the gauge transformation

$$\hat{\delta}\mathbf{W}^I = 0\,, \qquad \hat{\delta}\Psi^I = \hat{W}^I\,, \quad \hat{\delta}W^I = -\hat{W}^I\,, \tag{7.4.23}$$

for a vector multiplet field strength $\hat{W}^I$ satisfying

$$\bar{\nabla}^{\dot{\alpha}}_i \hat{W}^I = 0\,, \quad \nabla^{ij}\hat{W}^I = \bar{\nabla}^{ij}\overline{\hat{W}}^I\,. \tag{7.4.24}$$

We do use $\hat{\delta}$ to distinguish from $\tilde{\delta}$ and also to distinguish $\hat{W}^I$ from $\tilde{W}^I$ and W^I. Assuming Ψ^I and Ψ_I are unrelated, the possible candidates for an electric FI-type deformation would be

$$-2\mathrm{i}\int \mathrm{d}^8z\,\mathcal{E}\,\Psi_I W^I + \text{c.c.} = \int \mathrm{d}^{12}z\,E\,G_I^{ij}V^I_{ij}\,, \qquad W^I := \frac{1}{4}\bar{\Delta}\nabla^{ij}V^I_{ij}\,, \tag{7.4.25}$$

which, is invariant under the $\tilde{\delta}\Psi_I = \tilde{W}_I$ transformation, but, with $\hat{\delta}\Psi_I = 0$, it is not invariant under $\hat{\delta}$ transformations, and

$$-2\mathrm{i}\int \mathrm{d}^8z\,\mathcal{E}\,\Psi_I\mathbf{W}^I + \text{c.c.}\,, \tag{7.4.26}$$

which, is invariant under the $\hat{\delta}$ transformation but not $\tilde{\delta}$. Note that the previous no-go argument holds also in cases where the electric and magnetic deformations are defined in terms of the same Ψ building block. This is for instance the case of the hyper-dilaton Weyl composite linear multiplet $\mathcal{G}_{ij}{}^{\underline{ij}}$ that leads to $\mathcal{G}_{\xi I}{}^{ij} = \xi_I^{\underline{ij}}\mathcal{G}^{ij}{}_{\underline{ij}}$ and $\mathcal{G}^I_{\zeta}{}^{ij} = \zeta^I_{\underline{ij}}\mathcal{G}^{ij\underline{ij}}$. In this case, assuming the existence of a potential $\Psi^{\underline{ij}}$ for $\mathcal{G}^{ij\underline{ij}}$, the $\tilde{\delta}$ and $\hat{\delta}$ transformations would coincide with a single one generated by $\underline{\delta}\Psi^{\underline{ij}} = \underline{W}^{\underline{ij}}$ for a triplet of vector multiplets with field strengths $\underline{W}^{\underline{ij}}$. The reader can check that by choosing $\Psi_{\xi I} := \xi_I^{\underline{ij}}\Psi_{\underline{ij}}$ and $\Psi^I_{\zeta} := \zeta^I_{\underline{ij}}\Psi^{\underline{ij}}$, together with $\mathbf{W}^I = W^I + \Psi^I_{\zeta}$, both (7.4.25) and (7.4.26) are in general not invariant under $\underline{\delta}$ transformations. Given the discussion above, in this paper, we will consider the existence of off-shell "electric" and "magnetic" deformations as mutually exclusive. Despite this difference compared to the off-shell global case, we will see that, also due to the presence of the compensating vector multiplet in supergravity, there is still enough freedom to obtain an off-shell model exhibiting local partial supersymmetry breaking.

7.5 Deformed $N = 2$ supergravity in a hyper-dilaton Weyl background

The action for a deformed abelian vector multiplet in a hyper-dilaton Weyl multiplet background can be derived by substituting the expressions (7.2.49) and (7.2.53) into (7.4.12). Because the special conformal $\mathfrak{f}_{mc}$ and S-supersymmetry $(\phi_{m\alpha}{}^i, \bar{\phi}_{mi}{}^{\dot{\alpha}})$ connections depend on D and $(\Sigma^{\alpha i}, \bar{\Sigma}_{\dot{\alpha}i})$ and because the $SU(2)_R$ connection $\phi_m{}^{ij}$ is composite in the hyper-dilaton Weyl background, we degauge the derivative ∇_a to $\mathbf{D}_a$ as defined in eq. (7.2.52). Also, note that the linear multiplet fields are composite of hypermultiplet fields, eq. (7.2.51) and $G_{\zeta}{}^{ijI} = \zeta^I_{\underline{ij}}G^{ij\underline{ij}}$, $\chi^I_{\alpha i} = \zeta^I_{\underline{ij}}\chi_{\alpha i}{}^{\underline{ij}}$, $\bar{\chi}^{\dot{\alpha}iI} = \zeta^I_{\underline{ij}}\bar{\chi}^{\dot{\alpha}i\underline{ij}}$, and $F^I = \zeta^I_{\underline{ij}}F^{\underline{ij}}$. With this in mind, the bosonic part of the action follows:[5]

$$\mathcal{L}_{c,\text{bosons}} = \frac{1}{2}\mathcal{F}_I\overline{\mathbf{W}}^I R + \frac{1}{2}\mathrm{i}\mathcal{F}_I F^I_{\zeta} + \frac{1}{4}\mathcal{F}W_{ab}W^{ab} - \frac{1}{8}\mathrm{i}\mathcal{F}\varepsilon_{abcd}W^{ab}W^{cd} - \frac{1}{2}\mathcal{F}_I W_{ab}\mathbf{F}^{abI}$$

[5]In the discussion in this section, we should use $\boldsymbol{\phi}^I$ rather than $\mathbf{W}^I$ since we are projecting to components several supersymmetric invariants. However, since we will obtain covariant superfields equations of motions, we continue to use $\mathbf{W}^I$ with the hope that it will be clear from the context whether we denote the superfield or its lowest component field.

$$
\begin{aligned}
&+\frac{1}{4}\mathrm{i}\mathcal{F}_I\varepsilon_{abcd}W^{ab}\mathbf{F}^{cdI}-\frac{1}{2}\mathcal{F}_{IJ}\mathbf{F}_{ab}{}^I\mathbf{F}^{abJ}-\frac{1}{4}\mathrm{i}\mathcal{F}_{IJ}\varepsilon_{abcd}\mathbf{F}^{abI}\mathbf{F}^{cdJ}-\frac{1}{32}\mathcal{F}_{IJ}G_{\zeta}{}^I_{ij}G_{\zeta}^{ijJ}\\
&+\frac{1}{32}\mathcal{F}_{IJ}\mathbf{X}_{ij}{}^I\mathbf{X}^{ijJ}+\mathcal{F}_I\mathbf{D}_a\mathbf{D}^a\overline{\mathbf{W}}^I-4\mathcal{F}_I\overline{\mathbf{W}}^I A_aA^a-\frac{1}{4}\mathcal{F}_IW_{ab}W^{ab}\mathbf{W}^I\\
&+\frac{1}{8}\mathrm{i}\mathcal{F}_I\varepsilon_{abcd}W^{ab}W^{cd}\mathbf{W}^I-\frac{1}{2}\mathcal{F}_{IJ}W_{ab}\overline{\mathbf{W}}^I\mathbf{F}^{abJ}-\frac{1}{4}\mathrm{i}\mathcal{F}_{IJ}\varepsilon_{abcd}W^{ab}\overline{\mathbf{W}}^I\mathbf{F}^{cdJ}\\
&+\frac{1}{16}\mathrm{i}\mathcal{F}_{JI}\mathbf{X}_{ij}{}^JG_{\zeta}^{ijI}-\frac{1}{8}\mathcal{F}_{IJ}W_{ab}W^{ab}\overline{\mathbf{W}}^I\overline{\mathbf{W}}^J-\frac{1}{16}\mathrm{i}\mathcal{F}_{IJ}\varepsilon_{abcd}W^{ab}W^{cd}\overline{\mathbf{W}}^I\overline{\mathbf{W}}^J\\
&-4\mathrm{i}\mathcal{F}_IA_a\mathbf{D}^a\overline{\mathbf{W}}^I-2\mathrm{i}\mathcal{F}_I\overline{\mathbf{W}}^I\mathbf{D}_aA^a-64\mathcal{F}_I\overline{\mathbf{W}}^Iq^{-4}H_{a\underline{i}\underline{j}}H^{a\underline{i}\underline{j}}\\
&+\mathcal{F}_I\overline{\mathbf{W}}^Iq^{-2}\left(2q_{i\underline{i}}\mathbf{D}_a\mathbf{D}^aq^{i\underline{i}}+\mathbf{D}_aq^{i\underline{i}}\mathbf{D}^aq_{i\underline{i}}-2q^{-2}q_{i\underline{i}}q_{j\underline{j}}\mathbf{D}_aq^{i\underline{j}}\mathbf{D}^aq^{j\underline{i}}\right)\ .
\end{aligned}
\tag{7.5.1}
$$

Note that implicit fermions exist here and can be seen by converting $H^{a\underline{i}\underline{j}}$, F^{ab}, and $\mathbf{F}^{ab}$ to $\tilde{h}^{a\underline{i}\underline{j}}$, f^{ab}, and $\mathbf{f}^{ab}$, respectively, by eqs. (7.2.37), (7.2.38), and (7.2.14). This notably includes the coupling of $\tilde{h}^{a\underline{i}\underline{j}}$ to two gravitini. Otherwise, the fermionic counterpart of the action is given in Section II of the supplementary file. The component action for $S_{\text{standard FI}}$ in the hyper-dilaton Weyl background will remain the same as in the standard Weyl multiplet background (7.4.18), as it does not depend on the composite fields D, $(\Sigma^{\alpha i},\bar{\Sigma}_{\dot{\alpha}i})$, and $\phi_m{}^{ij}$. Keep also in mind that some of the fields in the previous bosonic Lagrangian are composite and include fermions. For example, F_{ζ}^I in a hyper-dilaton Weyl background is purely quadratic in fermions while the $\mathbf{D}_a$ derivative is defined in terms of $\omega_m{}^{cd}$ which contains the torsion quadratic in gravitini.

7.5.1 Equations of motion

The goal of this section is to obtain superconformal primary equations of motion that describe gauged $N = 2$ deformed supergravity based on a hyper-dilaton Weyl multiplet and defined by the action

$$
\mathcal{S} = \mathcal{S}_c + \text{c.c.} + \mathcal{S}_{\text{standard FI}}\ . \tag{7.5.2}
$$

In all the expressions in this section, we will formally allow for arbitrary "electric" ($\mathcal{G}_{\xi}{}^{ij}_{I}$) and "magnetic" ($\mathcal{G}_{\zeta}{}^{I}_{ij}$) deformations but, as discussed before, the reader should keep in mind that we consider them to be mutually exclusive. Given a fixed value of the index I, we allow for either of the two to be turned on, but not both at the same time.

We obtain the equations of motion by the variation of the action (7.5.2) in components with respect to the auxiliary fields, i.e., the highest dimension independent fields, of each multiplet. The resulting equations of motion then describe the primary fields, i.e., the bottom components, of the multiplets of the equations of motion that arise from the variation of the full superfields. It is then straightforward to reinterpret them as the primary superfields of the equations of motion. See [1,6] for a recent analysis in a five-dimensional setting.

In components, the EOM for the vector multiplet is obtained by varying the action with respect to the auxiliary field $\mathbf{X}^{ijI}$. The $\mathbf{X}^{ijI}$-dependent terms in the action are

$$
\mathcal{L}_X = \frac{1}{16}\left(\mathcal{F}_{IJK}\boldsymbol{\lambda}_i^{\alpha I}\boldsymbol{\lambda}_{\alpha j}^{J}-\overline{\mathcal{F}}_{IJK}\bar{\boldsymbol{\lambda}}_i^{\dot{\alpha}I}\bar{\boldsymbol{\lambda}}_{\dot{\alpha}j}^{J}\right)\mathbf{X}^{ijK}+\frac{1}{32}N_{IJ}\mathbf{X}_{ij}^I\mathbf{X}^{ijJ}
$$

$$+\frac{1}{16}\mathrm{i}\left(\mathcal{F}_{IJ}-\bar{\mathcal{F}}_{IJ}\right)\mathbf{X}^I_{ij}G_\zeta^{ijJ}+\frac{1}{4}G_{\xi}{}^{ij}_{I}\mathbf{X}^I_{ij}\ , \tag{7.5.3}$$

where G_ζ^{ijI} and $G_{\xi}{}^{ij}_{I}$ are the magnetic and electric deformations, respectively. Thus the equations of motion follow (one for each selection of I):

$$0=\left(\mathcal{F}_{IJK}\boldsymbol{\lambda}_i^{\alpha J}\boldsymbol{\lambda}^K_{\alpha j}+\bar{\mathcal{F}}_{IJK}\bar{\boldsymbol{\lambda}}^J_{\dot\alpha i}\bar{\boldsymbol{\lambda}}^{\dot\alpha K}_j\right)+N_{IJ}\mathbf{X}^J_{ij}+\mathrm{i}\left(\mathcal{F}_{IJ}-\bar{\mathcal{F}}_{IJ}\right)G_{\zeta}{}^J_{ij}+4G_{\xi\, ijI}\ , \tag{7.5.4}$$

where we have defined the special Kähler metric

$$N_{IJ}=\mathcal{F}_{IJ}+\bar{\mathcal{F}}_{IJ}\ . \tag{7.5.5}$$

Next, we find the Euler-Lagrange equations of motion for the auxiliary fields $W_{\alpha\beta}$ and $\overline{W}^{\dot\alpha\dot\beta}$. It is worth pointing out that in superspace the equations of motion derived by varying prepotentials are manifestly covariant. Hence, one expects the same to be true once the superspace results are reduced to component fields. However, in the component approach of finding the EOMs, the component action computed from eq. (7.4.4) includes hundreds of terms when fermions are considered, and it is not manifestly covariant due to the presence of naked gravitini. Although the action lacks manifest covariance, it has recently been explicitly demonstrated in components that for any supergravity theory, there exist covariant equations of motion that are equivalent to the regular field equations [193, 194]. These covariant equations are obtained by covariantising the regular field equations, resulting in a multiplet of field equations [193, 194].

To find the covariant equations of motion for $W_{\alpha\beta}$, $\bar{W}^{\dot\alpha\dot\beta}$, we can directly use the above action as degauging is already completed. Collecting all terms up to all orders in fermions with $W_{\alpha\beta}$ and $\bar{W}^{\dot\alpha\dot\beta}$, we have

$$\begin{aligned}\mathcal{L}_{W_{\alpha\lambda}} \;=\;& -2\mathcal{F}_{IJ}W_{\alpha\lambda}\overline{\mathbf{W}}^I\mathbf{F}^{\alpha\lambda\,J}-\frac{1}{2}\mathcal{F}_{IJ}W_{\alpha\lambda}W^{\alpha\lambda}\overline{\mathbf{W}}^I\overline{\mathbf{W}}^J+\frac{1}{4}\mathcal{F}_{IJK}\varepsilon_{ij}W_{\alpha\lambda}\overline{\mathbf{W}}^I\boldsymbol{\lambda}^{i\alpha J}\boldsymbol{\lambda}^{j\lambda K}\\ &+\mathcal{F}_{IJ}W_{\alpha\lambda}\overline{\mathbf{W}}^I q_{i\underline{i}}\boldsymbol{\lambda}^{i\alpha J}\rho^{\underline{i}\lambda}q^{-2}+\frac{1}{4}\mathcal{F}_I\varepsilon_{\underline{ij}}W_{\alpha\lambda}\overline{\mathbf{W}}^I\rho^{\underline{i}\alpha}\rho^{\underline{j}\lambda}q^{-2}\\ &+\bar{\mathcal{F}}_I W_{\alpha\lambda}\boldsymbol{\lambda}^{\alpha iI}q_{i\underline{i}}\rho^{\lambda\underline{i}}q^{-2}-2\bar{\mathcal{F}}_I W_{\alpha\lambda}\mathbf{F}^{\alpha\lambda I}-\bar{\mathcal{F}}W_{\alpha\lambda}W^{\alpha\lambda}+\text{c.c.}\ .\end{aligned}$$

The Euler-Lagrange equations of motion for the auxiliary fields $W_{\alpha\beta}$, $\bar{W}^{\dot\alpha\dot\beta}$ follow

$$\begin{aligned}N_{IJ}W^{\alpha\lambda}\overline{\mathbf{W}}^I\overline{\mathbf{W}}^J \;=\;& -2N_{IJ}\overline{\mathbf{W}}^I\mathbf{F}^{\alpha\lambda\,J}+\frac{1}{4}\mathcal{F}_{IJK}\varepsilon_{ij}\overline{\mathbf{W}}^I\boldsymbol{\lambda}^{iJ(\alpha}\boldsymbol{\lambda}^{\lambda)jK}\\ &+(\bar{\mathcal{F}}_{IJ}-\mathcal{F}_{IJ})\overline{\mathbf{W}}^I q_{i\underline{i}}\boldsymbol{\lambda}^{iJ(\alpha}\rho^{\lambda)\underline{i}}q^{-2}+\frac{1}{4}\mathcal{F}_I\varepsilon_{\underline{ij}}\overline{\mathbf{W}}^I\rho^{\underline{i}(\alpha}\rho^{\lambda)\underline{j}}q^{-2}\ ,\end{aligned} \tag{7.5.6}$$

together with its complex conjugate. Note that the EOM for $W_{\alpha\beta}$ and $\bar{W}^{\dot\alpha\dot\beta}$ are manifestly covariant and do not depend on the gravitini as expected.

Next, to prove that the equation of motion for A_m is covariant, we need to perform integration by parts, which makes it essential to degauge the covariant derivative $\mathbf{D}_a$ with respect to M_{ab} in the above action and insert the composite expression for the spin connection $\omega_m{}^{ab}$ in terms of $\omega(e)_m{}^{ab}$ together with bilinear terms in gravitini, eq. (2.2.15). Once these steps are carried out, the terms involving $A_a=e_a{}^mA_m$ in the Lagrangian take the following form:

$$\mathcal{L}_{A_a}=-\mathrm{i}\mathcal{F}_{IJ}\varepsilon^{\alpha\beta}\varepsilon_{ij}\psi^{ai}{}_{\alpha}\overline{\mathbf{W}}^I\boldsymbol{\lambda}^{j\,J}_{\beta}A_a-\mathcal{F}_{IJ}\varepsilon^{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{ij}(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\mathbf{W}}^I\boldsymbol{\lambda}^{i\,J}_{\beta}q^{j\underline{i}}\bar{\rho}_{\underline{i}}{}^{\dot\beta}A^aq^{-2}$$

$$
\begin{aligned}
&+\frac{1}{2}\mathcal{F}_{IJ}\varepsilon^{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\sigma_a)_\alpha{}^{\dot\alpha}\boldsymbol{\lambda}^i_\beta{}^I\overline{\boldsymbol{\lambda}}^{\dot\beta}_i{}^J A^a+\mathrm{i}\mathcal{F}_I\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{ij}\bar\psi^a{}_i^{\dot\alpha}\overline{\boldsymbol{\lambda}}^{\dot\beta}_j{}^I A_a \\
&+\mathcal{F}_I\varepsilon^{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\underline{i}\underline{j}}(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\boldsymbol{\lambda}}^{\dot\beta}_i{}^I q^{i\underline{i}}\rho^{\underline{j}}{}_\beta A^a q^{-2}+\frac{1}{2}\mathcal{F}_I\varepsilon^{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\mathbf{W}}^I\rho^{\underline{i}}{}_\beta\bar\rho_{\underline{i}}{}^{\dot\beta}A^a q^{-2} \\
&-4\mathrm{i}\mathcal{F}_I A^a e_a{}^m\partial_m\overline{\mathbf{W}}^I-4\mathcal{F}_I\overline{\mathbf{W}}^I A^a A_a-2\mathrm{i}\mathcal{F}_I\overline{\mathbf{W}}^I e_a{}^m\partial_m A^a+\text{c.c.}\,. \qquad (7.5.7)
\end{aligned}
$$

The Euler-Lagrange equation of the auxiliary fields A^a give:

$$
\begin{aligned}
8NA_a \;=\;& -\mathrm{i}\mathcal{F}_{IJ}\psi_a{}^i_{\alpha}\overline{\mathbf{W}}^I\boldsymbol{\lambda}_i^{\alpha J}-\mathcal{F}_{IJ}(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\mathbf{W}}^I\boldsymbol{\lambda}^{i\alpha J}q_i{}^{\underline{i}}\bar\rho_{\underline{i}\dot\alpha}q^{-2}+\frac{1}{2}\mathcal{F}_{IJ}(\sigma_a)_\alpha{}^{\dot\alpha}\boldsymbol{\lambda}^{i\alpha I}\overline{\boldsymbol{\lambda}}^J_{i\dot\alpha} \\
&+\mathrm{i}\mathcal{F}_I\bar\psi_{ai}^{\dot\alpha}\overline{\boldsymbol{\lambda}}^i_{\dot\alpha}{}^I+\mathcal{F}_I(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\boldsymbol{\lambda}}^I_{i\dot\alpha}q^{i\underline{i}}\rho_{\underline{i}}^\alpha q^{-2} \\
&+\frac{1}{2}\mathcal{F}_I(\sigma_a)_\alpha{}^{\dot\alpha}\overline{\mathbf{W}}^I\rho^{\underline{i}\alpha}\bar\rho_{\underline{i}\dot\alpha}q^{-2}-2\mathrm{i}\mathcal{F}_{IJ}\mathbf{W}^J\mathcal{D}'_a\overline{\mathbf{W}}^I+2\mathrm{i}\mathcal{F}_{IJ}\overline{\mathbf{W}}^I\mathcal{D}'_a\mathbf{W}^J+\text{c.c.}\,, \qquad (7.5.8)
\end{aligned}
$$

where we have defined

$$
N=N_{IJ}\mathbf{W}^I\overline{\mathbf{W}}^J\,, \qquad (7.5.9)
$$

and the covariant derivative $\mathcal{D}'_a$ contains only the Lorentz connection (without gravitini torsion) and the inverse vielbein, i.e.,

$$
\mathcal{D}'_a=e_a{}^m\left(\partial_m-\frac{1}{2}\omega_m{}^{cd}(e)M_{cd}\right)\,. \qquad (7.5.10)
$$

Finally, we uplift the derivative to the superconformal covariant derivative. This will absorb leftover gravitini terms to get the covariant equation of motion for A^a

$$
\begin{aligned}
&N_{IJ}(\sigma_a)_\alpha{}^{\dot\alpha}\left(\overline{\mathbf{W}}^I\boldsymbol{\lambda}_i^{\alpha J}\bar\rho_{\underline{i}\dot\alpha}+\mathbf{W}^I\overline{\boldsymbol{\lambda}}^J_{i\dot\alpha}\rho_{\underline{i}}^\alpha\right)q^{i\underline{i}}q^{-2}+\frac{1}{2}N_{IJ}(\sigma_a)_\alpha{}^{\dot\alpha}\boldsymbol{\lambda}^{i\alpha I}\overline{\boldsymbol{\lambda}}^J_{i\dot\alpha} \\
&+\frac{1}{2}N(\sigma_a)_\alpha{}^{\dot\alpha}\rho^{\underline{i}\alpha}\bar\rho_{\underline{i}\dot\alpha}q^{-2}-2\mathrm{i}N_{IJ}\mathbf{W}^I\nabla^a\overline{\mathbf{W}}^J+2\mathrm{i}N_{IJ}\overline{\mathbf{W}}^I\nabla_a\mathbf{W}^J=0\,. \qquad (7.5.11)
\end{aligned}
$$

With these covariant equations of motion computed, we are in a position to integrate out the auxiliary fields prerequisite to going on shell. This is explored in the following section in the context of a SU(1,1)/U(1) model leading to partial supersymmetry breaking. We leave for future work a general analysis of the on-shell action for the model described by (7.5.2) in a hyper-dilaton Weyl setup.

7.6 Off-shell model with on-shell partial-susy breaking

In this section, we move to present a new off-shell model for partial supersymmetry breaking engineered by using off-shell deformed vector multiplets in a hyper-dilaton Weyl background. Before presenting the details of the construction, it is worth stressing some of the key features of local supersymmetry breaking that guide our analysis.

For simplicity, we seek for a model of local partial supersymmetry breaking on a Minkowski vacuum. Hence, once auxiliary fields are integrated out, we want an on-shell theory possessing a Minkowski solution and no (effective) cosmological constant. Partial supersymmetry breaking emerges once the on-shell transformations of the fermions, which we collectively denote here as f, all possess a shift symmetric term schematically of the form

$$
\delta_\xi f=\mathbf{M}\xi+\cdots\,. \qquad (7.6.1)
$$

Here $\xi = (\xi_i^\alpha, \bar{\xi}^i_{\dot\alpha})$ refers to the supersymmetry transformation parameters and $\mathbf{M}$ is a (field dependent) matrix. If $\mathbf{M}$ is degenerate, $\det \mathbf{M} = 0$, but non-trivial, then part of local supersymmetry is spontaneously broken. As a consequence of this fact, the fermionic mass terms are all parametrised in terms of $\mathbf{M}$, hence with part of the spectrum remaining massless. For instance, in the case of local $N = 2 \to N = 1$ susy breaking, one gravitino acquires a mass-like term while one remains massless.

As reviewed in the introduction, finding models based on vector multiplets, potentially coupled to appropriate hypermultiplets, that possess local partial supersymmetry breaking is a non-trivial task — see, for example, [38, 96–98, 100–102, 107, 109, 110, 257, 257, 296–298]. Three of the features of our new construction given in this section for supergravity with partial supersymmetry breaking are: (i) our model is manifestly off shell, which, to the best of our knowledge, is a first explicit example; (ii) the spectrum of the on-shell theory, which includes a triplet of gauged two-forms, differs from previous examples described in the literature; (iii) though based on electric and magnetic deformations of vector multiplets, our model is not based on a standard gauging procedure, and in fact the fermions, e.g. the gravitini, are not charged under any of the U(1) symmetries of the vector multiplets (a generic feature of working with an hyper-dilaton multiplet). Let us now move to the description of our construction to see these properties unfolding.

7.6.1 $\mathrm{S}U(1,1)/U(1)$ model

We consider two vector multiplets, and the holomorphic prepotential

$$\mathcal{F} = c\phi\boldsymbol{\phi} \, , \tag{7.6.2}$$

where ϕ is a compensator, $\boldsymbol{\phi}$ is a deformed physical vector multiplet, and c is a real, nonzero constant which we decide to leave as a free normalisation parameter. This model is directly inspired by the well-known SU(1,1)/U(1) special Kähler sigma model, which, in a different set up, is known to lead to partial supersymmetry breaking. In particular, as described in the introduction, (7.6.2) arises from the SU(1,1)/U(1) special-Kähler sigma model after performing a duality transformation in the geometry used in [98, 100] ending up into a symplectic frame where a holomorphic prepotential exists and is given by (7.6.2).

As shown in [98, 100], within the context of $N = 2$ supergravity in the standard Weyl multiplet background, the minimal matter content required for partial supersymmetry breaking includes a physical vector multiplet and a hypermultiplet. In this scenario, the vector multiplet parametrizes the SU(1,1)/U(1) special Kähler manifold, while the physical hypermultiplet parametrizes the SO(4,1)/SO(4) quaternionic manifold. This model was further generalized in [102] by coupling the standard Weyl multiplet to $n+1$ vector multiplets and m hypermultiplets in a set-up that inherently has supersymmetry closing (partially) on shell.

Considering the minimal field content mentioned above, it is natural to argue that in a hyper-dilaton Weyl multiplet background, only a single physical vector multiplet plus a compensator, which would parametrize the special Kähler manifold SU(1,1)/U(1), might be sufficient to achieve partial supersymmetry breaking. In fact, the hypermultiplet is already a part of the hyper-dilaton Weyl multiplet itself, though in an off-shell setting, and in a new type of matter content. We will see in this section that this intuition is correct.

Note that instead of using numbered indices for the vector multiplets, we employ a bold symbol to denote the physical "1" multiplet, while the unbold symbol represents the compensating "0" multiplet, which, after taking derivatives of the holomorphic prepotential and using eqs. (7.5.5) and (7.5.9), implies

$$
\begin{aligned}
&\mathcal{F}_0 = c\boldsymbol{\phi}, \;\; \mathcal{F}_1 = c\phi, \;\; \mathcal{F}_{00} = 0, \;\; \mathcal{F}_{10} = \mathcal{F}_{01} = c, \;\; \mathcal{F}_{11} = 0 \;, \\
&N_{00} = 0, \;\; N_{10} = N_{01} = 2c, \;\; N_{11} = 0, \;\; N = 2c(\phi\bar{\boldsymbol{\phi}} + \boldsymbol{\phi}\bar{\phi}) \;.
\end{aligned}
\tag{7.6.3}
$$

We also choose the electric and magnetic deformations to be $G_{\xi}{}_I^{ij} = (\xi_{\underline{i}\underline{j}}q^{i\underline{i}}q^{j\underline{j}}, 0)$ and $G_{\zeta}{}^I_{ij} = (0, \zeta_{\underline{i}\underline{j}}q_i{}^{\underline{i}}q_j{}^{\underline{j}})$, respectively. This means that the compensator is "electrically deformed", which generically induces a negative contribution to the vacuum energy, while the physical vector multiplet is "magnetically" deformed, which generically induces a positive contribution to the vacuum energy. By tuning appropriately $\xi_{\underline{i}\underline{j}}$ and $\zeta_{\underline{i}\underline{j}}$ we will find zero vacuum energy and partial supersymmetry breaking.

For simplicity, we consider the case where $c \neq 0$ is real. The off-shell component action for the SU(1,1)/U(1) model can be obtained by first substituting eq. (7.6.2) and its derivatives into the general, deformed off-shell action of eqs. (7.5.1) and the results given in the supplementary file (for bosons and fermions, respectively) along with their complex conjugates, and then adding the standard FI action of eq. (7.4.18). The bosonic part of the action takes the following form:

$$
\begin{aligned}
\mathcal{L}_{\text{bosons}} = \;& \frac{N}{2}R - 2cF_{ab}\mathbf{F}^{ab} + \frac{1}{8}cX_{ij}\mathbf{X}^{ij} - 4NA_aA^a \\
& - c(\overline{\boldsymbol{\phi}}F^{ab} + \overline{\phi}\mathbf{F}^{ab})W_{ab} - \frac{1}{2}\mathrm{i}c\varepsilon_{abcd}(\overline{\boldsymbol{\phi}}F^{ab} + \overline{\phi}\mathbf{F}^{ab})W^{cd} \\
& - c(\boldsymbol{\phi}F^{ab} + \phi\mathbf{F}^{ab})W_{ab} + \frac{1}{2}\mathrm{i}c\varepsilon_{abcd}(\boldsymbol{\phi}F^{ab} + \phi\mathbf{F}^{ab})W^{cd} \\
& + c\boldsymbol{\phi}\mathbf{D}_a\mathbf{D}^a\overline{\phi} + c\overline{\boldsymbol{\phi}}\mathbf{D}_a\mathbf{D}^a\phi + c\phi\mathbf{D}_a\mathbf{D}^a\overline{\boldsymbol{\phi}} + c\overline{\phi}\mathbf{D}_a\mathbf{D}^a\boldsymbol{\phi} \\
& - \frac{1}{2}cW_{ab}W^{ab}(\overline{\phi}\overline{\boldsymbol{\phi}} + \phi\boldsymbol{\phi}) - \frac{1}{4}\mathrm{i}c\varepsilon_{abcd}W^{ab}W^{cd}(\overline{\phi}\overline{\boldsymbol{\phi}} - \phi\boldsymbol{\phi}) \\
& - 4\mathrm{i}c(\boldsymbol{\phi} - \overline{\boldsymbol{\phi}})A_a\mathbf{D}^a\overline{\phi} - 4\mathrm{i}c(\phi - \overline{\phi})A_a\mathbf{D}^a\overline{\boldsymbol{\phi}} \\
& - 64Nq^{-4}H_{a\underline{i}\underline{j}}H^{a\underline{i}\underline{j}} + Nq^{-2}\mathbf{D}_aq_{i\underline{i}}\mathbf{D}^aq^{i\underline{i}} + 2Nq^{-2}q_{i\underline{i}}\mathbf{D}_a\mathbf{D}^aq^{i\underline{i}} \\
& - 2Nq^{-4}q_{i\underline{i}}q_{j\underline{j}}\mathbf{D}_aq^{i\underline{j}}\mathbf{D}^aq^{j\underline{i}} \\
& + \frac{1}{4}\xi_{\underline{i}\underline{j}}q_i{}^{\underline{i}}q_j{}^{\underline{j}}X^{ij} + \frac{4}{3}\xi_{\underline{i}\underline{j}}\varepsilon^{mnpq}h_{mnp}{}^{\underline{i}\underline{j}}v_q \;.
\end{aligned}
\tag{7.6.4}
$$

Note that, once again, implicit fermions exist from the conversion of $H^{a\underline{i}\underline{j}}$, F^{ab}, and $\mathbf{F}^{ab}$ to $\tilde{h}^{a\underline{i}\underline{j}}$, f^{ab}, and $\mathbf{f}^{ab}$, respectively, by eqs. (7.2.37), (7.2.38), and (7.2.14). Otherwise, the fermionic counterpart to this action can be found in Section III of the supplementary file. The Euler-Lagrange equations of motion for the vector multiplet auxiliary fields X^{ij} and $\mathbf{X}^{ij}$ can be obtained by substituting (7.6.2) into the general equation of motion (7.5.4) and are given by the following two equations

$$
X_{ij} = 0 \;, \tag{7.6.5a}
$$

$$\mathbf{X}_{ij} = -\frac{2}{c}\xi_{\underline{i}\underline{j}} q_i{}^{\underline{i}} q_j{}^{\underline{j}} \,. \tag{7.6.5b}$$

The second equation leads to the following expression for the shifted auxiliary field of eq. (7.4.13)

$$\mathbf{M}_{ij} := \left(-\frac{2}{c}\xi_{\underline{i}\underline{j}} + \mathrm{i}\zeta_{\underline{i}\underline{j}}\right) q_i{}^{\underline{i}} q_j{}^{\underline{j}} \,. \tag{7.6.5c}$$

The Euler-Lagrange equations of motion for the remaining auxiliary fields $W_{\alpha\beta}$, $\bar{W}^{\dot{\alpha}\dot{\beta}}$, and A_a can once again be obtained by substituting (7.6.2) into the general equations of motion for (7.5.6), its complex conjugate, and (7.5.8), respectively. They are all given by the following:

$$\begin{aligned}
4c\bar{\phi}\bar{\boldsymbol{\phi}}W^{\alpha\beta} &= -\frac{3}{2}c(\phi\bar{\boldsymbol{\phi}}+\bar{\phi}\boldsymbol{\phi})q^{-2}\rho_{\underline{j}}^{(\alpha}\rho^{\beta)\underline{j}} - 4c(\bar{\phi}\mathbf{F}^{\alpha\beta}+\bar{\boldsymbol{\phi}}F^{\alpha\beta})\,, && (7.6.6a)\\
4c\phi\boldsymbol{\phi}\overline{W}^{\dot{\alpha}\dot{\beta}} &= -\frac{3}{2}c(\phi\bar{\boldsymbol{\phi}}+\bar{\phi}\boldsymbol{\phi})q^{-2}\bar{\rho}^{\underline{j}(\dot{\alpha}}\bar{\rho}_{\underline{j}}^{\dot{\beta})} - 4c(\phi\bar{\mathbf{F}}^{\dot{\alpha}\dot{\beta}}+\boldsymbol{\phi}\bar{F}^{\dot{\alpha}\dot{\beta}})\,, && (7.6.6b)\\
8NA_a &= 4\mathrm{i}c\left(\bar{\phi}\mathcal{D}'_a\boldsymbol{\phi}+\bar{\boldsymbol{\phi}}\mathcal{D}'_a\phi-\phi\mathcal{D}'_a\bar{\boldsymbol{\phi}}-\boldsymbol{\phi}\mathcal{D}'_a\bar{\phi}\right)\\
&\quad -2\mathrm{i}c\psi_a{}^i_\alpha(\bar{\phi}\boldsymbol{\lambda}_i^\alpha+\bar{\boldsymbol{\phi}}\lambda_i^\alpha)+2\mathrm{i}c\bar{\psi}_{ai}^{\dot{\alpha}}(\phi\overline{\boldsymbol{\lambda}}^i{}_{\dot{\alpha}}+\boldsymbol{\phi}\overline{\lambda}^i{}_{\dot{\alpha}})\\
&\quad +c(\sigma_a)_\alpha{}^{\dot{\alpha}}(\lambda^{i\alpha}\overline{\boldsymbol{\lambda}}_{i\dot{\alpha}}+\boldsymbol{\lambda}^{i\alpha}\overline{\lambda}_{i\dot{\alpha}})+c(\sigma_a)_\alpha{}^{\dot{\alpha}}(\phi\bar{\boldsymbol{\phi}}+\bar{\phi}\boldsymbol{\phi})\rho^{\underline{i}\alpha}\overline{\rho}_{\underline{i}\dot{\alpha}}q^{-2}\\
&\quad +2c(\sigma_a)_\alpha{}^{\dot{\alpha}}(\bar{\phi}\boldsymbol{\lambda}^{i\alpha}+\bar{\boldsymbol{\phi}}\lambda^{i\alpha})q_{i\underline{i}}\overline{\rho}^{\underline{i}}_{\dot{\alpha}}q^{-2}\\
&\quad +2c(\sigma_a)_\alpha{}^{\dot{\alpha}}(\phi\overline{\boldsymbol{\lambda}}_{i\dot{\alpha}}+\boldsymbol{\phi}\overline{\lambda}_{i\dot{\alpha}})q^{i\underline{i}}\rho_{\underline{i}}^\alpha q^{-2}\,. && (7.6.6c)
\end{aligned}$$

Note that from the action in eq. (7.6.4) we see that the scalar potential

$$S_{potential} = \int d^4x\,e\left[\frac{1}{16}cX^{ij}(\mathbf{M}_{ij}+\overline{\mathbf{M}}_{ij})+\frac{1}{4}\xi_{\underline{i}\underline{j}}q_i{}^{\underline{i}}q_j{}^{\underline{j}}X^{ij}\right] = 0\,, \tag{7.6.7}$$

is zero on the vacuum where $X^{ij}=0$. Hence, we have zero cosmological constant. By analysing in the following subsection the supersymmetry variation of the fermions, we will see that the condition $\det\mathbf{M}=0$, together with the assumption that the rank of the matrix $\mathbf{M}$ is one, will ensure local partial supersymmetry breaking in Minkowski space-time, for any scalar fields configurations.

7.6.2 Gauge fixing and fermion shifts

In this section, we give the explicit expressions for the gauge fixing that lead to Poincaré supergravity. In particular, we will gauge fix all superconformal structure group transformations except local Q-supersymmetry and Lorentz. For the dilatations, we aim to have a standard kinetic term for gravity. Hence, we collect the terms with the scalar curvature and obtain the following gauge condition for dilatation

$$\mathbb{D}\text{-gauge:} \qquad (\phi\bar{\boldsymbol{\phi}}+\bar{\phi}\boldsymbol{\phi}) = -\tfrac{1}{2c}\,. \tag{7.6.8a}$$

The S-gauge can be obtained by simply taking the Q-supersymmetry transformation of the $\mathbb{D}$-gauge

$$S\text{-gauge:} \qquad \lambda^i_\alpha\bar{\boldsymbol{\phi}}+\bar{\phi}\boldsymbol{\lambda}^i{}_\alpha = 0\,. \tag{7.6.8b}$$

With this choice, the $\mathbb{D}$-gauge is invariant under Q-supersymmetry. Next, the consistent gauge choice for the $\mathrm{U}(1)_R$ symmetry [38] is

$$\text{U(1)-gauge:} \qquad \phi = \bar{\phi} = y\,, \tag{7.6.8c}$$

which clearly imposes the compensating vector multiplet field to be real. A characterising feature of the hyper-dilaton Weyl multiplet is that it contains an $SU(2)_R$ compensator being the $q_{i\underline{i}}$ fields. We then impose

$$\text{SU(2)-gauge}: \qquad q_{i\underline{i}} = \varepsilon_{i\underline{i}} e^{-U}\,, \tag{7.6.8d}$$

which gauge fixes $SU(2)_R$. Lastly, we take the standard choice of gauge fixing condition

$$K\text{-gauge}: \qquad b_m = 0\,, \tag{7.6.8e}$$

to fix special conformal symmetry.

The transformation rules of the resulting Poincaré supergravity multiplet [241] are those that preserve the previous gauge conditions of eqs. (7.6.8). To preserve the gauge condition (7.6.8a) we need to impose $\lambda_{\mathbb{D}} \equiv 0$. Because Q-supersymmetry does not preserve the gauge, it is necessary to accompany these transformations with appropriate S-supersymmetry, $\mathrm{U}(1)_R$, special conformal, and $SU(2)_R$ compensating transformations. To preserve (7.6.8b), by examining the transformations of eqs. (5.3.3) and (7.3.28), it is straightforward to show that any Q-supersymmetry transformation has to be accompanied by a compensating S-supersymmetry transformation with the following parameter

$$\begin{aligned}\eta^i_\alpha = \quad & \frac{c}{2}\Big(2(\sigma^{ab}\xi^i)_\alpha(F_{ab}\bar{\boldsymbol{\phi}} + \bar{\phi}\mathbf{F}_{ab}) + 2(\sigma^{ab}\xi^i)_\alpha W^+_{ab}(\bar{\phi}\bar{\boldsymbol{\phi}}) - \frac{1}{2}\xi_{\alpha j}(X^{ij}\bar{\boldsymbol{\phi}} + \bar{\phi}\mathbf{M}^{ij}) \\ & +2\mathrm{i}(\sigma^a\bar{\xi}^i)_\alpha(\nabla_a\phi\bar{\boldsymbol{\phi}} + \bar{\phi}\nabla_a\boldsymbol{\phi}) + \lambda^i_\alpha(\bar{\xi}^j\bar{\boldsymbol{\lambda}}_j) + (\bar{\xi}^j\bar{\lambda}_j)\boldsymbol{\lambda}^i_\alpha\Big)\,.\end{aligned} \tag{7.6.9}$$

To preserve the gauge condition (7.6.8c), by examining the transformations of eqs. (5.3.3) and (7.3.28), it is straightforward to show that any Q-supersymmetry transformation has to be accompanied by a compensating $\mathrm{U}(1)_R$-symmetry transformation with parameter

$$\lambda_Y = -\frac{\mathrm{i}}{4y}(\xi_i\lambda^i - \bar{\xi}^i\bar{\lambda}_i)\,. \tag{7.6.10}$$

A similar analysis shows that to preserve the gauge condition (7.6.8e) one needs to enforce non-trivial compensating special conformal K-transformations with a parameter $\lambda^a(\xi)$. However, because all the other supergravity fields are conformal primaries (though not necessarily superconformal primaries) that do not transform under special conformal boosts, in practice, we will never have to worry about inserting the compensating $\lambda^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations. The last gauge fixing condition that is not preserved is (7.6.8d). It is straightforward to check that we can consistently have $\delta q^{(i\underline{i})} = 0$ by implementing a compensating $SU(2)_R$ transformation with the following parameter

$$\lambda^{ij}(\xi) = -\frac{\mathrm{e}^U}{2}\left[\xi^{(i}\rho^{j)} - \bar{\xi}^{(i}\bar{\rho}^{j)}\right]\,, \tag{7.6.11}$$

where $\rho^i = \delta^i_{\underline{i}}\rho^{\underline{i}}$ and $\bar\rho_i = \delta_i^{\underline{i}}\bar\rho_{\underline{i}}$.

The local super-Poincaré transformations of the fermionic fields after gauge fixing are given by:

$$\begin{aligned}
\delta\lambda^i_\alpha &= 2(\sigma^{ab}\xi^i)_\alpha F_{ab} + (\sigma^{ab}\xi^i)_\alpha W^+_{ab}\bar\phi - \frac{1}{2}\xi_{\alpha j}X^{ij} + \frac{1}{2}\lambda^{ab}(\sigma_{ab}\lambda^i)_\alpha \\
&\quad +2\mathrm{i}(\sigma^a\bar\xi^i)_\alpha(\mathcal{D}'_a\phi + 2\mathrm{i}A_a\phi - \frac{1}{2}\psi_{a i}{}^\beta\lambda^i_\beta) + \frac{\mathrm{e}^U}{2}(\xi^{(i}\rho^{j)} - \bar\xi^{(i}\bar\rho^{j)})\lambda_{\alpha j} - \frac{1}{4y}(\xi_k\lambda^k - \bar\xi^k\bar\lambda_k)\lambda^i_\alpha \\
&\quad +2c\left[2(\sigma^{ab}\xi^i)_\alpha(F_{ab}\bar{\boldsymbol\phi} + \bar\phi\mathbf{F}_{ab}) + 2(\sigma^{ab}\xi^i)_\alpha W^+_{ab}(\bar\phi\bar{\boldsymbol\phi}) - \frac{1}{2}\xi_{\alpha j}(X^{ij}\bar{\boldsymbol\phi} + \bar\phi\mathbf{M}^{ij})\right]\phi \\
&\quad +2c\left[2\mathrm{i}(\sigma^a\bar\xi^i)_\alpha\left(\big(\mathcal{D}'_a\phi + 2\mathrm{i}A_a\phi - \frac{1}{2}\psi_{a i}{}^\beta\lambda^i_\beta\big)\bar{\boldsymbol\phi} + \bar\phi\big(\mathcal{D}'_a\boldsymbol\phi + 2\mathrm{i}A_a\boldsymbol\phi - \frac{1}{2}\psi_{a i}{}^\beta\boldsymbol\lambda^i_\beta\big)\right)\right]\phi \\
&\quad +2c\left[\lambda^i_\alpha(\bar\xi^j\bar{\boldsymbol\lambda}_j) + (\bar\xi^j\bar\lambda_j)\boldsymbol\lambda^i_\alpha\right]\phi\ , \qquad (7.6.12a)\\
\delta\psi_{m i}{}^\alpha &= \left(2\partial_m\xi^\alpha_i + \omega_m{}^{ab}(\xi_i\sigma_{ab})^\alpha + 2\phi_{mi}{}^j\xi^\alpha_j + 2\mathrm{i}A_m\xi^\alpha_i + b_m\xi^\alpha_i\right) - \frac{\mathrm{i}}{2}(\bar\xi_i\tilde\sigma_m\sigma^{cd})^\alpha W^+_{cd} \\
&\quad -\frac{1}{2}\lambda^{ab}(\psi_{mi}\sigma_{ab})^\alpha - \frac{\mathrm{e}^U}{2}(\xi_{(i}\rho_{j)} - \bar\xi_{(i}\bar\rho_{j)})\,\psi_m{}^{\alpha j} - \frac{1}{4y}(\xi_i\lambda^i - \bar\xi^i\bar\lambda_i)\,\psi_{m i}{}^\alpha \\
&\quad -\mathrm{i}c(\sigma_m)^\alpha{}_{\dot\alpha}\Big[-2(\tilde\sigma^{ab}\bar\xi_i)^{\dot\alpha}(\phi\mathbf{F}_{ab} + F_{ab}\boldsymbol\phi + W^-_{ab}\phi\boldsymbol\phi) - \frac{1}{2}\bar\xi^{\dot\alpha j}(X_{ij}\boldsymbol\phi + \phi\overline{\mathbf{M}}_{ij}) \\
&\quad +2\mathrm{i}(\tilde\sigma^a\xi_i)^{\dot\alpha}\big((\phi(\mathcal{D}'_a\bar{\boldsymbol\phi} - 2\mathrm{i}A_a\bar{\boldsymbol\phi} - \frac{1}{2}\bar\psi_a{}^i{}_{\dot\beta}\bar{\boldsymbol\lambda}^{\dot\beta}_i) + (\mathcal{D}'_a\bar\phi - 2\mathrm{i}A_a\bar\phi - \frac{1}{2}\bar\psi_a{}^i{}_{\dot\beta}\bar\lambda^{\dot\beta}_i)\boldsymbol\phi\big) \\
&\quad +(\xi_i\lambda^i)\bar{\boldsymbol\lambda}^{\dot\alpha}_i + \bar\lambda^{\dot\alpha}_i(\xi_i\boldsymbol\lambda^i)\Big]\ , \qquad (7.6.12b)\\
\delta\rho^{\underline{i}}_\alpha &= -4\mathrm{i}(\sigma^a\bar\xi_k)_\alpha\nabla_a q^{k\underline{i}} + \frac{1}{2}\lambda_{ab}(\sigma^{ab}\rho^{\underline{i}})_\alpha + \frac{1}{4y}(\xi_i\lambda^i - \bar\xi^i\bar\lambda_i)\rho^{\underline{i}}_\alpha \\
&\quad +4c\Big[2(\sigma^{ab}\xi^i)_\alpha(F_{ab}\bar{\boldsymbol\phi} + \bar\phi\mathbf{F}_{ab}) + 2(\sigma^{ab}\xi^i)_\alpha W^+_{ab}(\bar\phi\bar{\boldsymbol\phi}) - \frac{1}{2}\xi_{\alpha j}(X^{ij}\bar{\boldsymbol\phi} + \bar\phi\mathbf{M}^{ij}) \\
&\quad +2\mathrm{i}(\sigma^a\bar\xi^i)_\alpha\big((\mathcal{D}'_a\phi + 2\mathrm{i}A_a\phi - \frac{1}{2}\psi_{a i}{}^\beta\lambda^i_\beta)\bar{\boldsymbol\phi} + \bar\phi(\mathcal{D}'_a\boldsymbol\phi + 2\mathrm{i}A_a\boldsymbol\phi - \frac{1}{2}\psi_{a i}{}^\beta\boldsymbol\lambda^i_\beta)\big) \\
&\quad +\lambda^i_\alpha(\bar\xi^j\bar{\boldsymbol\lambda}_j) + (\bar\xi^j\bar\lambda_j)\boldsymbol\lambda^i_\alpha\Big]q_i{}^{\underline{i}}\ , \qquad (7.6.12c)
\end{aligned}$$

together with their complex conjugates. Note that we have only given the transformation of λ^i_α since the S-supersymmetry gauge condition (7.6.8b) implies that the other gaugino is not independent. In principle, we should substitute all the auxiliary fields equations of motion and gauge conditions, but we are mainly interested in the shift terms (where we ignore higher fermionic terms and tensorial structures, as, for example, F_{ab}, W_{ab}, etc.) as these are the ones to investigate the supersymmetry breaking pattern in the model. The resulting equations are

$$\begin{aligned}
\delta\lambda^i_\alpha &= -cy^2\mathbf{M}^{ij}\xi_{\alpha j} + \cdots\ , \qquad (7.6.13a)\\
\delta\psi_{m i}{}^\alpha &= \frac{cy\mathrm{i}}{2}(\sigma_m)^\alpha{}_{\dot\alpha}\overline{\mathbf{M}}_{ij}\bar\xi^{\dot\alpha j} + \cdots\ , \qquad (7.6.13b)\\
\delta\rho^{\underline{i}}_\alpha &= -2cyq^{i\underline{i}}\mathbf{M}_{ij}\xi^j_\alpha + \cdots\ . \qquad (7.6.13c)
\end{aligned}$$

We see immediately that all these terms are proportional to a single complex matrix, $\mathbf{M}_{ij}$. To achieve partial supersymmetry breaking and zero vacuum energy, we impose a vanishing determinant of this matrix, $\mathbf{M}_{ij}$. This implies the following expression is zero

$$\mathbf{M}_{ij}\mathbf{M}^{ij} = \frac{1}{c^2}\xi_{\underline{ij}}\xi^{\underline{ij}} - \frac{1}{4}\zeta_{\underline{ij}}\zeta^{\underline{ij}} - \frac{1}{c}\mathrm{i}\xi_{\underline{ij}}\zeta^{\underline{ij}} = 0\ , \qquad (7.6.14)$$

thereby giving the two independent conditions on the magnetic and electric deformation parameters

$$\frac{4}{c^2}\xi_{\underline{ij}}\xi^{\underline{ij}} - \zeta_{\underline{ij}}\zeta^{\underline{ij}} = 0 \, , \tag{7.6.15a}$$

$$\xi_{\underline{ij}}\zeta^{\underline{ij}} = 0 \, . \tag{7.6.15b}$$

It follows that for generic solutions of the previous equations, $\det \mathbf{M} = 0$ while the rank of the matrix $\mathbf{M}$ is one. This is irrespective of the values of the scalars in the model. In this case, we have one supersymmetry preserved and one broken on a Minkowski vacuum, as only one of the two supersymmetry transformations has a local shift term.

7.6.3 On-shell theory and fermionic mass matrix

The on-shell component action for the $SU(1,1)/U(1)$ model can be derived by substituting the equations of motion for all auxiliary fields as given in eqs. (7.6.5) and (7.6.6) into the off-shell action of eq. (7.6.4). This is followed by the imposition of gauge-fixing conditions as seen in eqs. (7.6.8) of the previous subsection. We give here some details of this process.

After imposing the $\mathrm{U(1)_R}$ gauge condition, the $\mathbb{D}$-gauge simplifies to the following:

$$y(\bar{\boldsymbol{\phi}} + \boldsymbol{\phi}) = -\frac{1}{2c} \, . \tag{7.6.16}$$

This implies that the real part of $\boldsymbol{\phi}$ is nonzero being a unique characteristic of this sigma model in the symplectic frame chosen. We should also then interpret this condition as requiring y to be a function of the real part of the physical vector multiplet field while its imaginary part remains independent. That is,

$$y := y(\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}) = -\frac{1}{2c\left(\boldsymbol{\phi} + \bar{\boldsymbol{\phi}}\right)} = -\frac{1}{4c\,\mathrm{Re}\boldsymbol{\phi}} \, . \tag{7.6.17}$$

One should therefore interpret y as in eq. (7.6.17) in all equations that follow. Next, one algebraically solve for $W_{\alpha\beta}$ and $\bar{W}_{\dot{\alpha}\dot{\beta}}$ in terms of the other independent fields by using (7.6.6a) and (7.6.6b). After imposing the $\mathrm{U(1)_R}$ gauge condition, we will have the inverse vector multiplet field and its conjugate, $\boldsymbol{\phi}^{-1}$ and $\bar{\boldsymbol{\phi}}^{-1}$, in our action as a direct result of integrating out these auxiliary fields. Remember that eq. (7.6.16) requires the real part of $\boldsymbol{\phi}$ to be nonzero, so their presence is not problematic. These condition both on the compensator and physical fields are a feature of the SU(1,1)/U(1) target space sigma model in the symplectic frame that we have chosen which admits the holomorphic prepotential (7.6.2). It is also straightforward to obtain the algebraic expression for A_a in terms of the independent component fields by solving (7.6.6c) after imposing all the gauge fixings. Finally, also note that the S-gauge (7.6.8b) can be applied in the following way

$$\boldsymbol{\lambda}^i_{\boldsymbol{\alpha}} = -y^{-1}\bar{\boldsymbol{\phi}}\lambda^i_\alpha \, , \tag{7.6.18}$$

thereby removing $\boldsymbol{\lambda}^i_\alpha$ from the final result. At the end of all these substitutions, the bosonic part of the on-shell action is

$$\mathcal{L}_{\text{bosons}} = -\frac{1}{2}R + \frac{1}{2}cy^{-1}\bar{\boldsymbol{\phi}} f_{ab}f^{ab} + \frac{1}{4}\mathrm{i}cy^{-1}\varepsilon_{abcd}\bar{\boldsymbol{\phi}} f^{ab}f^{cd} + \frac{1}{2}cy\bar{\boldsymbol{\phi}}^{-1}\mathbf{f}_{ab}\mathbf{f}^{ab}$$

$$
\begin{aligned}
&+\frac{1}{4}\mathrm{i}cy\varepsilon_{abcd}\bar{\boldsymbol{\phi}}^{-1}\mathbf{f}^{ab}\mathbf{f}^{cd}+\frac{1}{2}cy^{-1}\boldsymbol{\phi}f_{ab}f^{ab}-\frac{1}{4}\mathrm{i}cy^{-1}\varepsilon_{abcd}\boldsymbol{\phi}f^{ab}f^{cd} \\
&+\frac{1}{2}cy\boldsymbol{\phi}^{-1}\mathbf{f}_{ab}\mathbf{f}^{ab}-\frac{1}{4}\mathrm{i}cy\varepsilon_{abcd}\boldsymbol{\phi}^{-1}\mathbf{f}^{ab}\mathbf{f}^{cd}+cy\mathcal{D}'_a\mathcal{D}'^a\left(\boldsymbol{\phi}+\bar{\boldsymbol{\phi}}\right) \\
&+c^2y^2\mathcal{D}'_a\bar{\boldsymbol{\phi}}\mathcal{D}'^a\bar{\boldsymbol{\phi}}-2c^2y^2\mathcal{D}'_a\boldsymbol{\phi}\mathcal{D}'^a\bar{\boldsymbol{\phi}}+c^2y^2\mathcal{D}'_a\boldsymbol{\phi}\mathcal{D}'^a\boldsymbol{\phi} \\
&+16e^{4U}\tilde{h}^{aij}\tilde{h}_{aij}-2\mathcal{D}'_aU\mathcal{D}'^aU+2\mathcal{D}'_a\mathcal{D}'^aU+\frac{4}{3}\varepsilon^{mnpq}\xi_{ij}h_{mnp}{}^{ij}v_q\,.
\end{aligned}
\tag{7.6.19}
$$

Note that the Lagrangian has no scalar potential, as in the analogue model described in [98, 100]. The rest of the Lagrangian, including all fermionic terms, is given in Section IV of the supplementary file. Here we only present the fermionic mass terms, meaning quadratic terms in the fermions that do not have any derivative coupling or coupling to fields other than scalars. These terms take the form

$$
\begin{aligned}
\mathcal{L}_{\text{fermions mass terms}}=&-\frac{1}{16}cy\overline{\mathbf{M}}_{ij}e^{2U}\rho^{i\alpha}\rho^j{}_\alpha-\frac{1}{16}cy\mathbf{M}^{ij}e^{2U}\bar{\rho}_{i\dot\alpha}\bar{\rho}_j{}^{\dot\alpha} \\
&+\frac{1}{4}ce^U\overline{\mathbf{M}}_{ij}\lambda^{j\alpha}\rho^i{}_\alpha+\frac{1}{4}ce^U\mathbf{M}^{ij}\bar{\lambda}_{j\dot\alpha}\bar{\rho}_i{}^{\dot\alpha} \\
&-\frac{1}{4}\mathrm{i}c\mathbf{M}_{ij}(\sigma_a)^\alpha{}_{\dot\alpha}\psi^{a}{}_\alpha^{j}\bar{\lambda}^{i\dot\alpha}-\frac{1}{4}\mathrm{i}c\overline{\mathbf{M}}_{ij}(\sigma_a)^\alpha{}_{\dot\alpha}\bar{\psi}^{aj\dot\alpha}\lambda^i{}_\alpha \\
&+\frac{1}{4}\mathrm{i}cye^U\mathbf{M}_{ij}(\sigma_a)^\alpha{}_{\dot\alpha}\psi^{a}{}_\alpha^{j}\bar{\rho}^{i\dot\alpha}+\frac{1}{4}\mathrm{i}cye^U\overline{\mathbf{M}}_{ij}(\sigma_a)^\alpha{}_{\dot\alpha}\bar{\psi}^{aj\dot\alpha}\rho^i{}_\alpha \\
&-\frac{1}{2}cy\mathbf{M}_{ij}(\sigma_{ab})^{\alpha\beta}\psi^{ai}{}_\alpha\psi^{bj}{}_\beta-\frac{1}{2}cy\overline{\mathbf{M}}^{ij}(\tilde{\sigma}_{ab})_{\dot\alpha\dot\beta}\bar{\psi}^{a\dot\alpha}_{i}\bar{\psi}^{b\dot\beta}_{j}\,.
\end{aligned}
\tag{7.6.20}
$$

They are all proportional to either $\mathbf{M}_{ij}$ or its complex conjugate $\overline{\mathbf{M}}^{ij}$. This is expected from the analysis in subsection 7.6.2, where we did show that in the on-shell theory, for any scalar field configurations, half of the local supersymmetry is spontaneously broken while half is preserved. Imposing the zero determinant (but rank one) condition that we discussed in the previous subsections implies that half of the fermions remain massless, in agreement with local partial supersymmetry breaking in a Minkowski vacuum.

We repeat that the complete action up to all orders in fermions is given in Section IV of the supplementary file accompanying our paper. This notably includes the coupling between two gravitini and the three-form fields $h_{abc}{}^{\underline{ij}}$ that appear through their Hodge duals, $\tilde{h}_a{}^{\underline{ij}}$. Remember that, roughly, in the hyper-dilaton Weyl multiplet $\tilde{h}_a{}^{\underline{ij}}$ takes the place of the $\mathrm{S}U(2)_R$ connection $\phi_a{}^{ij}$ which within the supergravity-matter systems engineered in terms of the standard Weyl multiplet (where matter fields have ubiquitous couplings with $\phi_a{}^{ij}$) becomes a linear combination of the vector multiplet gauge connections together with terms arising from the hypermultiplet moment map of the quaternion-Kähler geometry. In the hyper-dilaton Weyl case, there is no gauging and only the coupling with $\tilde{h}_a{}^{\underline{ij}}$ appears in place of FI-type terms. It would be interesting in the future to explore in more detail such property of this off-shell engineering of 4D $N=2$ supergravity-matter systems.

7.7 Conclusion and outlook

In this paper, we have elaborated on the deformation of off-shell vector multiplets in supergravity, both in components and superspace. In a superconformal framework, the deformations are associated with (composite) linear multiplets. Analogue to the globally supersymmetric case where an interplay of electric and magnetic deformations can lead to (partial) breaking of $N = 2$ global supersymmetry for systems of vector multiplets [93, 242, 244, 252, 253], the aim of our work was to explore the off-shell engineering of local partial supersymmetry breaking. To construct new off-shell models, we made use of superconformal tensor calculus techniques where the multiplet of conformal supergravity was chosen to be the hyper-dilaton Weyl multiplet introduced in 2022 in [4], while general off-shell vector multiplets were deformed with what proves to be local analogous to global electric and magnetic FI terms. The hyper-dilaton Weyl multiplet was chosen since it naturally contains a triplet of composite linear multiples, and one can easily engineer non-parallel deformations, a prerequisite to obtaining partial supersymmetry breaking, in a fashion very similar to the global case of [93,252,253]. As a proof of concept, in this work, we did show that by considering the SU(1,1)/U(1) special-Kähler sigma model, originally employed in [98, 100], however working in a symplectic frame which admits a holomorphic prepotential given by (7.6.2), and with both electric and magnetic deformations appropriately turned on, we obtain local partial supersymmetry breaking.

It is inspiring that an off-shell model with partial supersymmetry breaking can be engineered by the hyper-dilaton Weyl multiplet and off-shell deformations, as there is potential in extending this simple example to more complicate supergravity-matter couplings. Our set-up is related to the off-shell work of Müller from 1986 [241] that has not been appreciated so far. We expect our results can be extended in various directions.

First of all, it would be interesting to extend the analysis of Section 7.6 to general special-Kähler target spaces leading to scalar potentials and also with more physical vector multiplets. In principle, one could revisit all the analyses performed in the past, see, e.g., [100–102, 107, 109, 110, 257, 257], by employing the off-shell setting offered by vector multiplets in a hyper-dilaton Weyl background. This might provide a new description of sectors of compactified string theories with fluxes and their various patterns of supersymmetry breaking, see, e.g., [257] and references therein.

Several features of our analysis resemble the global partial supersymmetry breaking APT model. It is natural to expect that one can obtain this theory as the global limit of our construction, as it was done with a different setting in [101] and more recently revisited in [110] for the case of a single physical vector multiplet. We aim to look at the global limit of our construction in the future.

It would also be intriguing to understand if and how the dilaton and triplet of gauge two-forms sector of our models, and Müller's Poincaré supergravity, is mapped to the quaternion-Kähler hypermultiplet target spaces that characterise the work in [100–102, 110]. For example, at least at the level of the on-shell Lagrangians, the $SO(4,1)/SO(4)$ hypermultiplet sector could arise by taking our model and dualising the triplet of physical gauge two-forms into scalars which could then organise with the dilaton to parametrise the conventional target space with two isometries. For general models based on the hyper-dilaton Weyl multiplet, it also remains unclear how various fermions would become charged in accordance with the standard gauging in $N = 2$ supergravity, see, e.g., [38, 245–250]. It is certainly an interesting option to further analyse, in $N = 2$ supergravity, this and the surprising and intriguing mechanism that leads to scalar potentials without gauging the $SU(2)_R$ symmetry in the hyper-dilaton and Müller frameworks. Moreover, perhaps similar, so far missed, structures might be found also in $N > 2$ extended supergravities.

As repeatedly mentioned, the fact that we have an off-shell construction allows us to straightforwardly extend our model without changing the local transformations of the multiplets. For example, it would be possible to add higher-derivative actions to these models based on the hyper-dilaton Weyl and the hyper-dilaton Poincaré multiplets. Higher-derivative supergravity naturally arise in the low-energy description of string theory but, despite its importance, is still poorly understood. Other types of dilaton Weyl multiplets have been key to the construction of several off-shell higher-derivative supergravities in $4 \leq \mathrm{D} \leq 6$ dimensions, see, e.g., [29, 56, 59, 60, 63–65, 68, 69, 72]. One can look at this problem starting from a hyper-dilaton Weyl multiplet coupled to systems of vector multiplets with electric and magnetic FI-type terms. Among higher-derivative couplings, it would also be interesting to study the interplay between the FI-type terms used in our paper and the new $N = 2$ FI term introduced in [251].

To conclude, it would be interesting to engineer other off-shell constructions for local partial supersymmetry breaking. In fact, though we do value the simplicity of the deformations in our hyper-dilaton Weyl set-up (which works well for constructions with vector multiplets but no arbitrary sector for physical charged hypermultiplets), an approach based on the standard Weyl multiplet and different types of off-shell matter multiplets would be welcome. In our paper, we have avoided the option of admitting the (composite) linear multiplets to be charged under central charge transformations, see for example [141]. It would be interesting to understand in detail if and how off-shell magnetic deformations could be implemented when charged under the action of a gauged central charge in a standard Weyl multiplet background. It is also worth mentioning that the most general $N = 2$ supergravity-matter couplings are expected to be engineered off shell by coupling the standard Weyl multiplet to matter multiplets defined by harmonic or projective superfields [26, 41, 116–119, 121, 127, 128, 131–133, 229, 230, 299, 300], which can include an infinite number of auxiliary fields. Though in this setup the electric gauging has been studied, to the best of our knowledge, the off-shell magnetic one has not. If one wanted to construct some composite multiplet similar to the linear ones but quadratic in a hypermultiplet with an infinite number of auxiliary fields, then an off-shell "magnetic" deformation would be associated with an extended vector multiplet field strength that should also have an infinite amount of component fields. This might lead to some new extended vector multiplet, and possibly new implementations of a central charge in the supergravity algebra. If understood, all these off-shell extensions might lead to new mechanisms of partial supersymmetry breaking, possibly even engineered with only (extended) vector multiplets and no physical hypers, in line with some of the on-shell results obtained in [314, 315]. We hope to come back to these various questions in the future.

Acknowledgements:
We thank W. Kitchin for collaboration at the early stage of this project and J. Hutomo for collaboration on related projects. G. T.-M. is also grateful to I. Antoniadis, J.-P. Derendinger, L. Girardello, F. Farakos, H. Jiang, S. Kuzenko, and A. Van Proeyen for several discussions and collaborations on topics related to this work. We are also grateful to F. Farakos and S. Kuzenko for the careful reading and the useful feedback on our manuscript. This work has been supported by the Australian Research Council (ARC)

Future Fellowship FT180100353, ARC Discovery Project DP240101409, and the Capacity Building Package of the University of Queensland. G. G. and S. K. have been supported by postgraduate scholarships at the University of Queensland. We acknowledge the kind hospitality and financial support at the MATRIX Program “New Deformations of Quantum Field and Gravity Theories,” that took place in Creswick (Australia) between 22 January and 2 February 2024, and the meeting “Integrability in low-supersymmetry theories,” held in Trani between 22 July and 2 Aug 2024 and funded by the COST Action CA22113 by INFN and by Salento University, where part of this work was performed.

Chapter 8

Conclusion and Outlook

All the results presented across chapters 3, 4, 5, 6 and 7 of this thesis include a dedicated conclusion and outlook section with the exception of chapter 6. Chapter 6 is an extension of chapter 5, and thus, the conclusions drawn in chapter 5 largely apply to chapter 6 as well. The purpose of this final chapter is to provide a broad perspective on future directions and the potential impact of this thesis in the field of supergravity and related areas.

In this thesis, we have addressed several gaps present in the literature on supergravity theory by utilizing off-shell superconformal methods, superconformal tensor calculus and conformal superspaces. We successfully classified all three curvature-squared invariant terms of the minimal five-dimensional off-shell gauged supergravity, resolving a problem that had persisted for two decades. This was achieved by developing a computer algebra program to algorithmically implement the off-shell techniques used for constructing supergravity theories. We explained how to obtain the superconformal primary equations of motion by varying the action with respect to either the superfield prepotential or its corresponding auxiliary field of the standard Weyl multiplet, the vector multiplet compensator, and the linear multiplet compensator. The component structure for the Weyl multiplet equations of motion, specifically the supercurrent, was not known before. Therefore, we derived the independent components of the supercurrent multiplet and used them to compute its descendants for all two and four-derivative invariants.

In analyzing the on-shell supergravity action deformed by curvature-squared invariant terms, we observed that an alternative representation of the conformal supergravity multiplet significantly simplifies calculations. This observation led us to define the hyper-dilaton Weyl multiplet for supergravity theories with 8 supercharges, they include $4D$ $N=2$, $5D$ $N=1$, and $6D$ $N=(1,0)$. These newly defined hyper-dilaton Weyl multiplets, reduce the dependence on the standard Weyl multiplet for constructing supergravity theories. By coupling this multiplet to an off-shell vector multiplet compensator and employing superconformal techniques, we demonstrated how to reproduce the supergravity action of [241] for $4D$ $N=2$ theory and subsequently generalized it for $5D$ $N=1$ theory.

Furthermore, the hyper-dilaton Weyl multiplet of $N=2$ conformal supergravity in four dimensions enabled us to construct, for the first time, an off-shell model of partial supersymmetry breaking in

Minkowski spacetime. This was achieved by coupling the hyper-dilaton Weyl multiplet to magnetically and electrically deformed off-shell vector and tensor multiplets. We presented a general off-shell gauged $N = 2$ supergravity deformed by an arbitrary number of deformed vector and tensor multiplets, and derived primary equations of motion for this general theory. We then focused on a detailed analysis of a specific $SU(1,1)/U(1)$ model, engineered fully off-shell, which exhibited on-shell partial supersymmetry breaking in Minkowski spacetime.

Outlined below are some potential future directions that stem directly from our work, with further elaboration in the subsequent discussion.

- Extending the results presented here on gauged curvature-squared invariant terms to general matter-coupled $5D$ supergravity.
- Exploring alternative dilaton-Weyl multiplets for various supergravity theories.
- Classifying all four-derivative invariant terms, including curvature-squared invariant terms, and other terms that are independent of these curvature-squared terms, such as $F\Box F$, $(F^2)^2 = (F_{ab}F^{ab})^2$, and $F^4 = F^{ab}F_{bc}F^{cd}F_{da}$, in both standard Weyl and various dilaton-Weyl multiplet backgrounds.
- Investigating higher-order corrections beyond the leading four-derivative terms, such as eight-derivative corrections.
- Understanding how these invariant terms affect the physics by calculating other physical quantities, such as the mass and entropy of black holes.
- Determining missing invariant terms (if any) by analyzing the off-shell reduction of the log invariant from 5 to 4 dimensions.
- Exploring models of partial supersymmetry breaking in spacetimes with positive, zero, or negative cosmological constants.

We now elaborate on the impact of the results presented in this thesis and potential future directions. Recent advances in gauge field theory, driven by integrability, supersymmetric localization, and other computational techniques, have initiated a new era in the study of higher-derivative supergravity theories within the context of gauge/gravity correspondence. While progress on the gravity side has lagged behind, this work on gauged curvature-squared supergravity in five dimensions partly addresses this gap. Although the structure of the on-shell gauged supergravity deformed by the three curvature-squared invariant terms agrees with those in [82] and [86], our results provide a complete and extended dependence on all constants associated with the five-parameter family of gauged curvature-squared supergravities (4.3.7). As far as the first α' corrections were concerned, arguments were given to justify why only two invariants might suffice to compute (BPS) on-shell observables (as for example the entropy of five-dimensional black holes). However, it remains an open question to understand to which extent only a subset of curvature squared invariants suffice for a general analysis. In the future, we plan to explore how all five parameters in the theory play a non-trivial role in the calculation of non-BPS observables.

The focus here on the gauged sector is pivotal for obtaining (asymptotically) AdS solutions, which play a central role in holography, both with and without supersymmetry. As an example, we computed the a and c Weyl anomaly coefficients for the dual CFT through holographic renormalization. Previous studies have addressed these coefficients using only two curvature-squared invariant terms, however, here we take into account all three invariants. While our results align with previous studies, incorporating the third invariant provides additional confidence in the findings.

This work has already gained significant attention [218, 301, 303, 304, 306, 316–318] highlighting its importance and impact in the supergravity literature. Although we have constructed the curvature-squared invariant terms for 5D gauged supergravity, they are without matter coupling. The off-shell construction of arbitrary supergravity-matter couplings, even up to curvature-squared terms, remains an open problem. In the future, we plan to extend these results on gauged curvature-squared invariant terms to general matter-coupled 5D supergravity.

Equipped with our computer algebra program for supergravity calculations, we are well-positioned to extend the current work and tackle several open problems in supergravity, as listed in the potential future directions above. We aim to explore higher-derivative corrections, starting by classifying all independent four-derivative invariant terms, including terms such as $F\Box F$, $(F^2)^2$, and F^4, which are independent of curvature-squared terms. A key question is whether these additional invariant terms can be constructed and, if so, how they will impact the on-shell theory and contribute to physical quantities.

Apart from AdS solutions, there exist other backgrounds of interest for minimal gauged supergravity, such as the Gödel spacetime. While Gödel spacetime backgrounds are well understood for 4D minimal gauged supergravity [76], a parallel 5D analysis could shed light on which higher derivative invariants in 5D correspond to those in 4D. This is possible due to a precise off-shell map between 4D and 5D Lagrangians [76]. By including only the curvature-squared correction terms to the known two-derivative action, we can understand the leading higher derivative corrections. If these leading corrections prove useful for the 4D/5D connection and for predicting other backgrounds such as black holes and black strings, this would provide further motivation to go beyond the leading four-derivative corrections to supergravity theory.

Furthermore, the newly defined hyper-dilaton Weyl multiplet of $N = 2$ conformal supergravity in four dimensions enabled us to obtain an off-shell partial supersymmetry breaking model in Minkowski spacetime. There are several interesting extensions of this work. We aim to systematically generalize these models to study partial supersymmetry breaking in spacetimes with positive, zero, or negative cosmological constants. Another direction, which we addressed in chapter 6, is to define the hyper-dilaton Weyl multiplet for other supergravity theories. We plan to explore other dilaton-Weyl multiplets as well. A natural possibility is to couple the standard Weyl multiplet to an on-shell vector-tensor multiplet—see [41, 190, 262–267] for references on the vector-tensor multiplet and its coupling to conformal supergravity. Furthermore, we intend to analyze the on-shell model corresponding to the general gauged supergravity deformed by an arbitrary number of deformed vectors and tensor multiplets. Finally, since higher-order invariant terms are crucial for understanding the effective field theory description of string theory, we aim to investigate these terms beyond four-derivatives by using various possible off-shell approaches.

The results presented in this thesis significantly advance our conceptual understanding of supergravity theories, while the computer algebra program available on GitHub [8] enhances our computational capabilities. These developments will be instrumental in carrying out the outlined projects on supergravity and holographic dualities, with valuable applications in quantum gravity, black holes, and cosmology.

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Appendix A

$4D$ $N=2$ Conformal superspace

In this appendix, we collect results about $4D$ $N=2$ conformal superspace focusing on the ingredients relevant to our discussion in Chapter 2, 5 and 7. The conventions we use here differ in numerous ways from those used originally in [26]. For the most part, we adhere to the notation and conventions of [36] and [127, 128].

A.1 Notations and conventions

Our notations and conventions follow mostly those in [36]. We briefly summarize them here.

We use two-component notation where dotted and undotted spinor indices are raised and lowered by ε tensors

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta , \quad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} , \tag{A.1.1}$$

obeying

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} , \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma , \quad \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}} , \quad \varepsilon^{12} = 1 .$$

Similarly $SU(2)$ indices are raised and lowered by ε_{ij} and ε^{ij} having the same properties as $\varepsilon_{\alpha\beta}$. Spinor indices are contracted as

$$\psi\chi := \psi^\alpha\chi_\alpha , \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} . \tag{A.1.2}$$

For spinors which are also isospinors, we define

$$\psi\chi = \psi_i^\alpha\chi_\alpha^i , \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}^i\bar{\chi}_i^{\dot{\alpha}} . \tag{A.1.3}$$

The metric is $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. The sigma matrices are defined as

$$(\sigma^a)_{\alpha\dot{\alpha}} = (1,\vec{\sigma}) , \quad (\tilde{\sigma}^a)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^a)_{\beta\dot{\beta}} = (1,-\vec{\sigma}) , \tag{A.1.4}$$

and have the properties

$$(\sigma_a)_{\alpha\dot{\beta}}(\tilde{\sigma}_b)^{\dot{\beta}\beta} = -\eta_{ab}\delta_\alpha^\beta - 2(\sigma_{ab})_\alpha{}^\beta , \tag{A.1.5}$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\beta}(\sigma_b)_{\beta\dot{\beta}} = -\eta_{ab}\delta^{\dot{\alpha}}_{\dot{\beta}} - 2(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \ , \tag{A.1.6}$$

together with the following useful identities

$$\begin{aligned}
(\sigma^a)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}} &= -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \ , \\
(\sigma_{ab})_{\alpha\beta}(\sigma^{ab})_{\gamma\delta} &= -2\varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta} \ , \\
(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}^{ab})_{\dot{\gamma}\dot{\delta}} &= -2\varepsilon_{\dot{\gamma}(\dot{\alpha}}\varepsilon_{\dot{\beta})\dot{\delta}} \ , \\
(\sigma_{ab})_{\alpha\beta}(\tilde{\sigma}^{ab})_{\dot{\gamma}\dot{\delta}} &= 0 \ , \\
\mathrm{tr}(\sigma_{ab}\sigma_{cd}) = (\sigma_{ab})_\alpha{}^\beta(\sigma_{cd})_\beta{}^\alpha &= -\eta_{a[c}\eta_{d]b} - \frac{\mathrm{i}}{2}\varepsilon_{abcd} \ , \\
\mathrm{tr}(\tilde{\sigma}_{ab}\tilde{\sigma}_{cd}) = (\tilde{\sigma}_{ab})_{\dot{\alpha}}{}^{\dot{\beta}}(\tilde{\sigma}_{cd})_{\dot{\beta}}{}^{\dot{\alpha}} &= -\eta_{a[c}\eta_{d]b} + \frac{\mathrm{i}}{2}\varepsilon_{abcd} \ , \\
(\sigma^a)_\alpha{}^{\dot{\alpha}}(\sigma_{ab})_{\beta\gamma} &= \varepsilon_{\alpha(\beta}(\sigma_b)_{\gamma)}{}^{\dot{\alpha}} \ , \\
(\sigma^a)_\alpha{}^{\dot{\alpha}}(\tilde{\sigma}_{ab})^{\dot{\beta}\dot{\gamma}} &= \varepsilon^{\dot{\alpha}(\dot{\beta}}(\sigma_b)_\alpha{}^{\dot{\gamma})} \ , \\
(\sigma_{[a})_{\alpha\dot{\beta}}(\sigma_{bc]})_{\gamma\delta} &= \frac{\mathrm{i}}{3}\varepsilon_{abcd}\varepsilon_{\alpha(\gamma}(\sigma^d)_{\delta)\dot{\beta}} \ , \\
\varepsilon^{abcd}\varepsilon_{a'b'c'd'} &= -4!\delta^a_{[a'}\delta^b_{b'}\delta^c_{c'}\delta^d_{d']} \ , \\
(\sigma_{ab}\sigma_c)_\alpha{}^{\dot{\alpha}} = (\sigma_{ab})_\alpha{}^\beta(\sigma_c)_\beta{}^{\dot{\alpha}} &= -\eta_{c[a}(\sigma_{b]})_\alpha{}^{\dot{\alpha}} - \frac{\mathrm{i}}{2}\varepsilon_{abcd}(\sigma^d)_\alpha{}^{\dot{\alpha}} \ , \\
(\tilde{\sigma}_{ab}\sigma_c)^{\dot{\alpha}}{}_\alpha = (\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}(\sigma_c)_\alpha{}^{\dot{\beta}} &= -\eta_{c[a}(\sigma_{b]})_\alpha{}^{\dot{\alpha}} + \frac{\mathrm{i}}{2}\varepsilon_{abcd}(\sigma^d)_\alpha{}^{\dot{\alpha}} \ , \\
\varepsilon_{abcd}(\sigma^{cd})_{\alpha\beta} = -2i(\sigma_{ab})_{\alpha\beta} \ , &\qquad \varepsilon_{abcd}(\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} = 2i(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \ .
\end{aligned}$$

The antisymmetric tensor is

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1 \ , \tag{A.1.8}$$

and (anti-)symmetrization includes a normalization factor, for example

$$V_{[ab]} = \frac{1}{2!}(V_{ab} - V_{ba}) \ , \quad \psi_{(\alpha\beta)} = \frac{1}{2!}(\psi_{\alpha\beta} + \psi_{\beta\alpha}) \ . \tag{A.1.9}$$

For superform indices, we introduce graded antisymmetrization, e.g.,

$$V_{[AB\}} = \frac{1}{2!}(V_{AB} - (-)^{ab}V_{BA}) \ . \tag{A.1.10}$$

When an index is not included, we separate it with vertical bars, e.g.,

$$\begin{aligned}
T_{[AB}{}^D F_{|D|C\}} = \frac{1}{3!}\Big(& T_{AB}{}^D F_{DC} - (-)^{ab}T_{BA}{}^D F_{DC} + (-)^{ca+cb}T_{CA}{}^D F_{DB} \\
& - (-)^{cb}T_{AC}{}^D F_{DB} + (-)^{ab+ac}T_{BC}{}^D F_{DA} - (-)^{ab+ac+cb}T_{CB}{}^D F_{DA}\Big) \ .
\end{aligned} \tag{A.1.11}$$

A vector V_a can be rewritten with spinor indices as

$$V_{\alpha\dot{\beta}} = (\sigma^a)_{\alpha\dot{\beta}}V_a \ , \quad V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\beta}\alpha}V_{\alpha\dot{\beta}} \ . \tag{A.1.12}$$

A real antisymmetric tensor, $F_{ab} = -F_{ba}$ is converted to spinor indices as

$$F_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}F_{ab} \ , \quad \bar{F}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}F_{ab} \ , \quad F_{ab} = (\sigma_{ab})^{\alpha\beta}F_{\alpha\beta} - (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\bar{F}^{\dot{\alpha}\dot{\beta}} \ . \tag{A.1.13}$$

A.2 4D $N = 2$ Conformal superspace identities

The Lorentz generators obey

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} \ , \quad [M_{ab}, \nabla_c] = 2\eta_{c[a}\nabla_{b]} \ ,$$
$$[M_{ab}, \nabla_\alpha^i] = (\sigma_{ab})_\alpha{}^\beta \nabla_\beta^i \ , \quad [M_{ab}, \bar{\nabla}_i^{\dot\alpha}] = (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta} \bar{\nabla}_i^{\dot\beta} \ . \tag{A.2.1}$$

The $SU(2)_R$, $U(1)_R$ and dilatation generators obey

$$[J_{ij}, J_{kl}] = -\varepsilon_{k(i}J_{j)l} - \varepsilon_{l(i}J_{j)k} \ , \quad [J_{ij}, \nabla_\alpha^k] = -\delta^k_{(i}\nabla_{\alpha j)} \ , \quad [J_{ij}, \bar{\nabla}_k^{\dot\alpha}] = -\varepsilon_{k(i}\bar{\nabla}_{j)}^{\dot\alpha} \ ,$$
$$[Y, \nabla_\alpha^i] = \nabla_\alpha^i \ , \quad [Y, \bar{\nabla}_i^{\dot\alpha}] = -\bar{\nabla}_i^{\dot\alpha} \ ,$$
$$[\mathbb{D}, \nabla_a] = \nabla_a \ , \quad [\mathbb{D}, \nabla_\alpha^i] = \frac{1}{2}\nabla_\alpha^i \ , \quad [\mathbb{D}, \bar{\nabla}_i^{\dot\alpha}] = \frac{1}{2}\bar{\nabla}_i^{\dot\alpha} \ . \tag{A.2.2}$$

The special superconformal generators K^A transform in the obvious way under Lorentz and $SU(2)_R$ rotations,

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]} \ , \quad [M_{ab}, S_i^\gamma] = -(\sigma_{ab})_\beta{}^\gamma S_i^\beta \ , \quad [M_{ab}, \bar{S}^i_{\dot\gamma}] = -(\tilde{\sigma}_{ab})^{\dot\beta}{}_{\dot\gamma}\bar{S}^i_{\dot\beta} \ ,$$
$$[J_{ij}, S_k^\gamma] = -\varepsilon_{k(i}S_{j)}^\gamma \ , \quad [J_{ij}, \bar{S}^k_{\dot\gamma}] = -\delta^k_{(i}\bar{S}_{\dot\gamma j)} \ , \tag{A.2.3}$$

while their transformation under $U(1)_R$ and dilatations is opposite that of ∇_A:

$$[Y, S_i^\alpha] = -S_i^\alpha \ , \quad [Y, \bar{S}^i_{\dot\alpha}] = \bar{S}^i_{\dot\alpha} \ ,$$
$$[\mathbb{D}, K_a] = -K_a \ , \quad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha \ , \quad [\mathbb{D}, \bar{S}^i_{\dot\alpha}] = -\frac{1}{2}\bar{S}^i_{\dot\alpha} \ . \tag{A.2.4}$$

Among themselves, the generators K^A obey the algebra

$$\{S_i^\alpha, \bar{S}^j_{\dot\alpha}\} = 2\mathrm{i}\delta_i^j(\sigma^a)^\alpha{}_{\dot\alpha}K_a \ . \tag{A.2.5}$$

Finally, the algebra of K^A with ∇_B is given by

$$[K^a, \nabla_b] = 2\delta_b^a\mathbb{D} + 2M^a{}_b \ ,$$
$$\{S_i^\alpha, \nabla_\beta^j\} = 2\delta_i^j\delta_\beta^\alpha\mathbb{D} - 4\delta_i^j M^\alpha{}_\beta - \delta_i^j\delta_\beta^\alpha Y + 4\delta_\beta^\alpha J_i{}^j \ ,$$
$$\{\bar{S}^i_{\dot\alpha}, \bar{\nabla}_j^{\dot\beta}\} = 2\delta_j^i\delta_{\dot\alpha}^{\dot\beta}\mathbb{D} + 4\delta_j^i\overline{M}_{\dot\alpha}{}^{\dot\beta} + \delta_j^i\delta_{\dot\alpha}^{\dot\beta}Y - 4\delta_{\dot\alpha}^{\dot\beta}J^i{}_j \ ,$$
$$[K^a, \nabla_\beta^j] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot\beta}\bar{S}^j_{\dot\beta} \ , \quad [K^a, \bar{\nabla}_j^{\dot\beta}] = -\mathrm{i}(\sigma^a)^{\dot\beta}{}_\beta S_j^\beta \ ,$$
$$[S_i^\alpha, \nabla_b] = \mathrm{i}(\sigma_b)^\alpha{}_{\dot\beta}\bar{\nabla}_i^{\dot\beta} \ , \quad [\bar{S}^i_{\dot\alpha}, \nabla_b] = \mathrm{i}(\sigma_b)_{\dot\alpha}{}^\beta\nabla_\beta^i \ , \tag{A.2.6}$$

where all other (anti-)commutations vanish.

The covariant derivatives obey (anti-)commutation relations of the form

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C\nabla_C + \frac{1}{2}R_{AB}{}^{cd}M_{cd} + R_{AB}{}^{kl}J_{kl}$$
$$+ \mathrm{i}R_{AB}(Y)Y + R_{AB}(\mathbb{D})\mathbb{D} + R_{AB}{}^C K_C \ , \tag{A.2.7}$$

where $T_{AB}{}^{C}$ is the torsion, and $R_{AB}{}^{cd}$, $R_{AB}{}^{kl}$, $R_{AB}(Y)$, $R_{AB}(\mathbb{D})$ and $R_{AB}{}^{C}$ are the curvatures. Some of the components of the torsion and curvature must be constrained. Following [26], the spinor derivative torsions and curvatures are chosen to obey

$$\{\nabla^i_\alpha, \nabla^j_\beta\} = -2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{\mathcal{W}} , \quad \{\bar\nabla_i^{\dot\alpha}, \bar\nabla_j^{\dot\beta}\} = 2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\mathcal{W} , \quad \{\nabla^i_\alpha, \bar\nabla_j^{\dot\beta}\} = -2\mathrm{i}\delta^i_j\nabla_\alpha{}^{\dot\beta} , \tag{A.2.8}$$

where $\mathcal{W}$ is some operator valued in the superconformal algebra. In [26], it was shown how to constrain $\mathcal{W}$ entirely in terms of a superfield $W_{\alpha\beta}$ so that the component structure reproduces $N=2$ conformal supergravity. In our notation, the constraints lead to

$$\{\nabla^i_\alpha, \nabla^j_\beta\} = 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\overline{W}_{\dot\gamma\dot\delta}\overline{M}^{\dot\gamma\dot\delta} + \frac{1}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar\nabla_{\dot\gamma k}\overline{W}^{\dot\gamma\dot\delta}\bar{S}^k_{\dot\delta} - \frac{1}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}\nabla_{\gamma\dot\delta}\overline{W}^{\dot\delta}{}_{\dot\gamma}K^{\gamma\dot\gamma} , \tag{A.2.9a}$$

$$\{\bar\nabla_i^{\dot\alpha}, \bar\nabla_j^{\dot\beta}\} = -2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}W^{\gamma\delta}M_{\gamma\delta} + \frac{1}{2}\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\nabla^{\gamma k}W_{\gamma\delta}S^\delta_k - \frac{1}{2}\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\nabla^{\gamma\dot\gamma}W_\gamma{}^\delta K_{\delta\dot\gamma} , \tag{A.2.9b}$$

$$\{\nabla^i_\alpha, \bar\nabla_j^{\dot\beta}\} = -2\mathrm{i}\delta^i_j\nabla_\alpha{}^{\dot\beta} , \tag{A.2.9c}$$

$$\begin{aligned}[\nabla_{\alpha\dot\alpha}, \nabla^i_\beta] = &-\mathrm{i}\varepsilon_{\alpha\beta}\overline{W}_{\dot\alpha\dot\beta}\bar\nabla^{\dot\beta i} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta i}\overline{W}_{\dot\alpha\dot\beta}\mathbb{D} - \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta i}\overline{W}_{\dot\alpha\dot\beta}Y + \mathrm{i}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta}_j\overline{W}_{\dot\alpha\dot\beta}J^{ij} \\ &- \mathrm{i}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\beta}\overline{W}_{\dot\gamma\dot\alpha}\overline{M}^{\dot\beta\dot\gamma} - \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\alpha}\bar\nabla^{\dot\beta}_k\overline{W}_{\dot\beta\dot\gamma}\bar{S}^{\dot\gamma k} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\gamma\dot\beta}\overline{W}_{\dot\alpha\dot\beta}S^i_\gamma \\ &+ \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\alpha}\nabla^\gamma{}_{\dot\gamma}\overline{W}^{\dot\gamma\dot\beta}K_{\gamma\dot\beta} ,\end{aligned} \tag{A.2.9d}$$

$$\begin{aligned}[\nabla_{\alpha\dot\alpha}, \bar\nabla^{\dot\beta}_i] = &\,\mathrm{i}\delta^{\dot\beta}_{\dot\alpha}W_{\alpha\beta}\nabla^\beta_i + \frac{\mathrm{i}}{2}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W_{\alpha\beta}\mathbb{D} - \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W_{\alpha\beta}Y + \mathrm{i}\delta^{\dot\beta}_{\dot\alpha}\nabla^{\beta j}W_{\alpha\beta}J_{ij} \\ &+ \mathrm{i}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W^\gamma{}_\alpha M_{\beta\gamma} + \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla_{\alpha i}\nabla^{\beta j}W_\beta{}^\gamma S_{\gamma j} - \frac{1}{2}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta{}_{\dot\gamma}W_{\alpha\beta}\bar{S}^{\dot\gamma}_i \\ &+ \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla_{\alpha i}\nabla^\gamma{}_{\dot\gamma}W_{\beta\gamma}K^{\beta\dot\gamma} .\end{aligned} \tag{A.2.9e}$$

The complex superfield $W_{\alpha\beta} = W_{\beta\alpha}$ and its complex conjugate $\overline{W}_{\dot\alpha\dot\beta} := \overline{W_{\alpha\beta}}$ are superconformally primary, $K_A W_{\alpha\beta} = 0$, and obey the additional constraints

$$\bar\nabla^{\dot\alpha}_i W_{\beta\gamma} = 0 , \qquad \nabla_{\alpha\beta}W^{\alpha\beta} = \bar\nabla^{\dot\alpha\dot\beta}\overline{W}_{\dot\alpha\dot\beta} , \tag{A.2.10}$$

where we introduce the notation

$$\nabla_{\alpha\beta} := \nabla^k_{(\alpha}\nabla_{\beta)k} , \quad \bar\nabla^{\dot\alpha\dot\beta} := \bar\nabla^{(\dot\alpha}_k\bar\nabla^{\dot\beta)k} . \tag{A.2.11}$$

Despite the appearance of the S-supersymmetry and special conformal K_a generators, the algebra of covariant derivatives (A.2.9) is significantly simpler to work with than the corresponding algebras of $SU(2)$ [127,319] or U(2) superspace [128,227,228].

The $S-$supersymmetry transformation of the descendant superfields of the super-Weyl tensor is given as follows:

$$S^k_\delta W_{\alpha\beta\gamma}{}^i = 8\varepsilon^{ki}\varepsilon_{\delta(\alpha}W_{\beta\gamma)} , \quad \bar{S}^{\dot\delta}_k W_{\alpha\beta\gamma}{}^i = 0 , \tag{A.2.12a}$$

$$\bar{S}^{\dot\delta}_k \overline{W}^{\dot\alpha\dot\beta\dot\gamma}{}_i = 8\varepsilon_{ki}\varepsilon^{\dot\delta(\dot\alpha}\overline{W}^{\dot\beta\dot\gamma)} , \quad S^k_\delta\overline{W}^{\dot\alpha\dot\beta\dot\gamma}{}_i = 0 , \tag{A.2.12b}$$

$$S^k_\alpha\Sigma^i_\beta = -\frac{4}{3}\varepsilon^{ki}W_{\alpha\beta} , \quad \bar{S}^{\dot\alpha}_k\Sigma^i_\beta = 0 , \tag{A.2.12c}$$

$$\bar{S}_k^{\dot{\alpha}}\bar{\Sigma}_i^{\dot{\beta}} = \frac{4}{3}\varepsilon_{ki}\overline{W}^{\dot{\alpha}\dot{\beta}} \ , \quad S_\alpha^k\bar{\Sigma}_i^{\dot{\beta}} = 0 \ , \tag{A.2.12d}$$

$$S_\lambda^k W_{\alpha\beta\gamma\delta} = 24\varepsilon_{\lambda(\alpha}W_{\beta\gamma\delta)}{}^k \ , \quad \bar{S}_k^{\dot{\lambda}}W_{\alpha\beta\gamma\delta} = 0 \ , \tag{A.2.12e}$$

$$\bar{S}_k^{\dot{\lambda}}\overline{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 24\varepsilon^{\dot{\lambda}(\dot{\alpha}}\overline{W}^{\dot{\beta}\dot{\gamma}\dot{\delta})}{}_k \ , \quad S_\lambda^k\overline{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0 \ , \tag{A.2.12f}$$

$$S_\lambda^k\Sigma_{\alpha\beta}{}^{ij} = \frac{28}{3}\varepsilon^{k(i}\varepsilon_{\lambda(\alpha}\Sigma_{\beta)}^{j)} + \frac{4}{3}\varepsilon^{k(i}W_{\alpha\beta\lambda}{}^{j)} \ , \quad \bar{S}_k^{\dot{\lambda}}\Sigma_{\alpha\beta}{}^{ij} = 0 \ , \tag{A.2.12g}$$

$$\bar{S}_k^{\dot{\lambda}}\bar{\Sigma}^{\dot{\alpha}\dot{\beta}}{}_{ij} = \frac{28}{3}\varepsilon_{k(i}\varepsilon^{\dot{\lambda}(\dot{\alpha}}\bar{\Sigma}_{j)}^{\dot{\beta})} - \frac{4}{3}\varepsilon_{k(i}\overline{W}^{\dot{\alpha}\dot{\beta}\dot{\lambda}}{}_{j)} \ , \quad S_\lambda^k\bar{\Sigma}^{\dot{\alpha}\dot{\beta}}{}_{ij} = 0 \ , \tag{A.2.12h}$$

$$S_\lambda^k\Sigma_{\alpha\beta} = \frac{32}{3}\varepsilon_{\lambda(\alpha}\Sigma_{\beta)}^k - \frac{4}{3}W_{\alpha\beta\lambda}{}^k \ , \quad \bar{S}_k^{\dot{\lambda}}\Sigma_{\alpha\beta} = 0 \ , \tag{A.2.12i}$$

$$\bar{S}_k^{\dot{\lambda}}\bar{\Sigma}^{\dot{\alpha}\dot{\beta}} = \frac{32}{3}\varepsilon^{\dot{\lambda}(\dot{\alpha}}\bar{\Sigma}_k^{\dot{\beta})} + \frac{4}{3}\overline{W}^{\dot{\alpha}\dot{\beta}\dot{\lambda}}{}_k \ , \quad S_\lambda^k\bar{\Sigma}^{\dot{\alpha}\dot{\beta}} = 0 \ , \tag{A.2.12j}$$

$$S_\lambda^k D = 0 \ , \quad \bar{S}_k^{\dot{\lambda}}D = 0 \ , \tag{A.2.12k}$$

$$S_\lambda^l\Sigma_{\alpha\beta\gamma}{}^k = \frac{2}{3}\varepsilon^{lk}W_{\alpha\beta\gamma\lambda} + 3\varepsilon^{lk}\varepsilon_{\lambda(\alpha}\Sigma_{\beta\gamma)} - 12\varepsilon_{\lambda(\alpha}\Sigma_{\beta\gamma)}{}^{lk} \ , \quad \bar{S}_l^{\dot{\lambda}}\Sigma_{\alpha\beta\gamma}{}^k = 0 \ , \tag{A.2.12l}$$

$$\bar{S}_l^{\dot{\lambda}}\bar{\Sigma}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_k = -\frac{2}{3}\varepsilon_{lk}\overline{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\lambda}} + 3\varepsilon_{lk}\varepsilon^{\dot{\lambda}(\dot{\alpha}}\bar{\Sigma}^{\dot{\beta}\dot{\gamma})} - 12\varepsilon^{\dot{\lambda}(\dot{\alpha}}\bar{\Sigma}^{\dot{\beta}\dot{\gamma})}{}_{lk} \ , \quad S_\lambda^l\bar{\Sigma}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_k = 0 \ . \tag{A.2.12m}$$

The $S-$supersymmetry transformation of the descendants of the abelian vector multiplet is given as follows:

$$S_i^\alpha\lambda_\beta^j = 4\delta_\beta^\alpha\delta_i^jW, \quad \bar{S}_{\dot{\alpha}}^i\lambda_\alpha^j = 0, \quad S_i^\alpha\bar{\lambda}_i^{\dot{\alpha}} = 0, \quad \bar{S}_{\dot{\alpha}}^i\bar{\lambda}_j^{\dot{\beta}} = 4\delta_{\dot{\alpha}}^{\dot{\beta}}\delta_j^i\overline{W}, \tag{A.2.13}$$

$$S_i^\gamma F_{\alpha\beta} = \frac{1}{2}\delta_{(\alpha}^\gamma\lambda_{\beta)i}, \quad \bar{S}_{\dot{\alpha}}^iF_{\alpha\beta} = 0 \ , \quad S_i^\gamma\bar{F}^{\dot{\alpha}\dot{\beta}} = 0, \quad \bar{S}_{\dot{\gamma}}^i\bar{F}^{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\delta_{\dot{\gamma}}^{(\dot{\alpha}}\lambda^{\dot{\beta})i} \ , \tag{A.2.14}$$

$$S_i^\alpha X^{kl} = 0, \quad \bar{S}_{\dot{\alpha}}^iX^{kl} = 0 \ . \tag{A.2.15}$$

The $S-$supersymmetry transformation of the descendants of the tensor multiplet is given as follows:

$$S_i^\alpha\chi_{\beta j} = 4\delta_\beta^\alpha G_{ij} \ , \quad \bar{S}_{\dot{\alpha}}^i\bar{\chi}^{\dot{\beta}j} = 4\delta_{\dot{\alpha}}^{\dot{\beta}}G^{ij} \ , \quad S_i^\alpha\bar{\chi}^{\dot{\beta}j} = 0 \ , \quad \bar{S}_{\dot{\alpha}}^i\chi_{\beta j} = 0 \ , \tag{A.2.16}$$

$$S_i^\alpha F = -4\chi_i^\alpha \ , \quad \bar{S}_{\dot{\alpha}}^i\bar{F} = -4\bar{\chi}_{\dot{\alpha}}^i \ , \quad S_i^\alpha\bar{F} = 0 \ , \quad \bar{S}_{\dot{\alpha}}^iF = 0 \ , \tag{A.2.17}$$

$$S_i^\alpha\hat{H}_a = \frac{3\mathrm{i}}{4}(\sigma_a)^\alpha{}_{\dot{\beta}}\bar{\chi}_i^{\dot{\beta}} \ , \quad \bar{S}_{\dot{\alpha}}^i\hat{H}_a = -\frac{3\mathrm{i}}{4}(\bar{\sigma}_a)_{\dot{\alpha}}{}^\beta\chi_\beta^i \ . \tag{A.2.18}$$

Appendix B

$5D\ N = 1$ conformal superspace

In this appendix we collect results about $5D\ N = 1$ conformal superspace in the traceless frame of [29] focusing on the ingredients relevant to our discussion in Chapter 2, 3, 4, and 6. We adhere to the notations and conventions of [29] with the exception of using the notation of [90] in chapter 4. The reader should also look at [3], where some typos from [29] were fixed.

B.1 Different $5D\ N = 1$ notation and conventions

Table B.1 provides a brief translation scheme between our conventions and other groups [90, 174, 211] that worked on the superconformal tensor calculus in five dimensions.

It is important to note that the definitions of supersymmetry are different between the various groups. Here the differences amount to not only to normalisations, but also to additional field-dependent S and K transformations in the definition of δ_Q. That is, given a transformation $\delta_Q + \delta_S + \delta_K$ in our conventions with respective parameters ξ^i_α, η^i_α, and Λ^a_K, we will find a transformation $\delta'_Q + \delta'_S + \delta'_K$ with new parameters ε^i, η'^i, and Λ'^a_K given in Table B.2.

We also emphasise that each group uses the same vector derivative D_a, corresponding to our ∇_a, modulo differing overall normalisations of the superconformal generators. The additional gravitino-dependent terms in the S-supersymmetry and special conformal connections in Table B.1 cancel against additional terms found within δ_Q, so that the vector derivative is unchanged.

For completeness, we also give in Table B.3 the relation between our conventions for the vector multiplet and the other groups. The conventions for the linear multiplet can be found in Table B.4.

Our conventions	de Wit and Katmadas	Bergshoeff et al.	Fujita et al.
η^{ab}	η^{ab}	η^{ab}	$-\eta^{ab}$
Γ^a	$-\mathrm{i}\gamma^a$	$\mathrm{i}\gamma^a$	γ^a
Σ^{ab}	$\frac{1}{2}\gamma^{ab}$	$\frac{1}{2}\gamma^{ab}$	$-\frac{1}{2}\gamma^{ab}$
ε^{abcde}	$-\mathrm{i}\varepsilon^{abcde}$	$-\varepsilon^{abcde}$	ε^{abcde}
${\psi_m}^i$	${\psi_\mu}^i$	${\psi_\mu}^i$	$2\,{\psi_\mu}^i$
${\mathcal{V}_m}^i{}_j$	$-\frac{1}{2}V_{\mu\,j}{}^i$	$-{V_\mu}^i{}_j$	${V_\mu}^i{}_j$
${\omega_m}^{ab}$	${\omega_\mu}^{ab}$	$-{\omega_\mu}^{ab}$	$-{\omega_\mu}^{ab}$
$\mathrm{i}\,{\phi_m}^i$	${\phi_\mu}^i$	$-{\phi_\mu}^i+\frac{1}{3}T_{ab}\gamma^{ab}{\psi_\mu}^i$	$2\,{\phi_\mu}^i-\frac{2}{3}v_{ab}\gamma^{ab}{\psi_\mu}^i$
${f_m}^a$	$-{f_\mu}^a+\frac{1}{3}\psi_{\mu i}\gamma^a\chi^i$	$-{f_\mu}^a+\frac{1}{3}{\psi_\mu}^i\gamma^a\chi_i$	$-{f_\mu}^a+\frac{\mathrm{i}}{24}{\psi_\mu}^i\gamma^a\chi_i$
W^{ab}	$-4T^{ab}$	$\frac{16}{3}T^{ab}$	$\frac{4}{3}v^{ab}$
χ^i	χ^i	χ^i	$\frac{1}{32}\chi^i$
D	D	D	$\frac{1}{16}(D-\frac{8}{3}v^{ab}v_{ab})$
$R(Q)_{ab}{}^i$	$\frac{1}{2}R(Q)_{ab}{}^i$	$\frac{1}{2}R(Q)_{ab}{}^i$	$R(Q)_{ab}{}^i$
$R(M)_{ab}{}^{cd}$	$R(M)_{ab}{}^{cd}$	$-R(M)_{ab}{}^{cd}$	$-R(M)_{ab}{}^{cd}$
$R(J)_{ab}{}^i{}_j$	$-\frac{1}{2}R(\mathcal{V})_{ab\,j}{}^i$	$-R(V)_{ab}{}^i{}_j$	$R(U)_{ab}{}^i{}_j$
$\mathrm{i}\,R(S)_{ab}{}^i$	$\frac{1}{2}R(S)_{ab}{}^i$	$-\frac{1}{2}R(S)_{ab}{}^i+\frac{1}{6}T_{cd}\gamma^{cd}R(Q)_{ab}{}^i$	$R(S)_{ab}{}^i-\frac{1}{3}v_{cd}\gamma^{cd}R(Q)_{ab}{}^i$
$R(K)_{ab}{}^c$	$-R(K)_{ab}{}^c+\frac{1}{3}R(Q)_{abi}\gamma^c\chi^i$	$-R(K)_{ab}{}^c+\frac{1}{3}R(Q)_{ab}{}^i\gamma^c\chi_i$	$-R(K)_{ab}{}^c+\frac{\mathrm{i}}{24}R(Q)_{ab}{}^i\gamma^c\chi_i$

Table B.1: Conventions for Weyl multiplet

de Wit and Katmadas	Bergshoeff et al.	Fujita et al.
$\varepsilon^i=2\,\xi^i$	$\varepsilon^i=2\,\xi^i$	$\varepsilon^i=\xi^i$
$\eta'^i=2\mathrm{i}\eta^i$	$\eta'^i=-2\mathrm{i}\eta^i+\frac{2}{3}T_{ab}\gamma^{ab}\xi^i$	$\eta'^i=\mathrm{i}\eta^i+\frac{1}{3}v_{ab}\gamma^{ab}\xi^i$
$\Lambda'^a_K=-\Lambda^a_K+\frac{2}{3}\xi_i\gamma^a\chi^i$	$\Lambda'^a_K=-\Lambda^a_K+\frac{2}{3}\xi^i\gamma^a\chi_i$	$\Lambda'^a_K=-\Lambda^a_K+\frac{\mathrm{i}}{24}\xi^i\gamma^a\chi_i$

Table B.2: Conventions for $\delta_Q+\delta_S+\delta_K$

Our conventions	de Wit and Katmadas	Bergshoeff et al.	Fujita et al.
W	σ	σ	M
λ^i	Ω^i	$-\psi^i$	$-2\Omega^i$
X^{ij}	$2Y^{ij}$	$2Y^{ij}$	$2Y^{ij}$
v_m	W_μ	$-A_\mu$	W_μ
f_{mn}	$F_{\mu\nu}$	$-F_{\mu\nu}$	$F_{\mu\nu}$

Table B.3: Conventions for vector multiplet

Our conventions	de Wit and Katmadas	Bergshoeff et al.	Fujita et al.
G^{ij}	$\frac{1}{2}L^{ij}$	$\frac{1}{2}L^{ij}$	$\frac{1}{2}L^{ij}$
φ^i	$-\frac{\mathrm{i}}{2}\varphi^i$	$-\frac{\mathrm{i}}{2}\varphi^i$	$-\frac{\mathrm{i}}{2}\varphi^i$
F	N	N	$\frac{1}{2}N$
H^a	$\hat{E}^a$	$-E^a$	$\frac{1}{2}E^a$

Table B.4: Conventions for linear multiplet

B.2 5D $N=1$ conformal superspace identities

The Lorentz generators act on the superspace covariant derivatives $\nabla_A=(\nabla_a,\nabla_\alpha^i)$ in the following way

$$[M_{ab},M_{cd}]=2\eta_{c[a}M_{b]d}-2\eta_{d[a}M_{b]c}\ , \tag{B.2.1a}$$

$$[M_{ab},\nabla_c]=2\eta_{c[a}\nabla_{b]}\ , \tag{B.2.1b}$$

$$[M_{\alpha\beta},\nabla_\gamma^i]=\varepsilon_{\gamma(\alpha}\nabla_{\beta)}^i\ , \tag{B.2.1c}$$

where $M_{\alpha\beta}=1/2(\Sigma^{ab})_{\alpha\beta}M_{ab}$. The $\mathrm{S}U(2)_R$ and dilatation generators satisfy

$$[J^{ij},J^{kl}]=\varepsilon^{k(i}J^{j)l}+\varepsilon^{l(i}J^{j)k}\ ,\quad [J^{ij},\nabla_\alpha^k]=\varepsilon^{k(i}\nabla_\alpha^{j)} \tag{B.2.1d}$$

$$[\mathbb{D},\nabla_a]=\nabla_a\ ,\quad [\mathbb{D},\nabla_\alpha^i]=\frac{1}{2}\nabla_\alpha^i\ . \tag{B.2.1e}$$

The Lorentz and $\mathrm{S}U(2)_R$ generators act on the special conformal generators $K_A=(K_a,S_{\alpha i})$ according to the rules

$$[M_{ab},K_c]=2\eta_{c[a}K_{b]}\ ,\quad [M_{\alpha\beta},S_\gamma^i]=\varepsilon_{\gamma(\alpha}S_{\beta)}^i\quad [J^{ij},S_\alpha^k]=\varepsilon^{k(i}S_\alpha^{j)}\ , \tag{B.2.1f}$$

while the dilatation generator acts on K_A as

$$[\mathbb{D},K_a]=-K_a\ ,\quad [\mathbb{D},S_{\alpha i}]=-\frac{1}{2}S_{\alpha i}\ . \tag{B.2.1g}$$

Among themselves, the generators K_A obey the only nontrivial anti-commutation relation

$$\{S_\alpha^i,S_\beta^j\}=-2\mathrm{i}\,\varepsilon^{ij}(\Gamma^c)_{\alpha\beta}K_c\ . \tag{B.2.1h}$$

The algebra of K_A with ∇_A is given by

$$[K_a,\nabla_b]=2\eta_{ab}\mathbb{D}+2M_{ab}\ , \tag{B.2.1i}$$

$$[K_a,\nabla_\alpha^i]=\mathrm{i}(\Gamma_a)_\alpha{}^\beta S_\beta^i\ , \tag{B.2.1j}$$

$$\{S_{\alpha i},\nabla_\beta^j\}=2\varepsilon_{\alpha\beta}\delta_i^j\mathbb{D}-4\delta_i^jM_{\alpha\beta}+6\varepsilon_{\alpha\beta}J_i{}^j\ , \tag{B.2.1k}$$

$$[S_{\beta i},\nabla_a]=\mathrm{i}(\Gamma_a)_\beta{}^\alpha\nabla_{\alpha i}-\frac{1}{4}W_a{}^b(\Gamma_b)_\beta{}^\alpha S_{\alpha i}+\frac{\mathrm{i}}{8}(\Gamma_a\Gamma^b)_\beta{}^\gamma X_{\gamma i}K_b-\frac{\mathrm{i}}{4}W_{ab\beta i}K^b\ . \tag{B.2.1l}$$

The anticommutator of two spinor derivatives, $\{\nabla_\alpha^i, \nabla_\beta^j\}$, has the following non-zero torsion and curvatures

$$\mathscr{T}_{\alpha\beta}^{i\,j\,c} = 2\mathrm{i}\varepsilon^{ij}(\Gamma^c)_{\alpha\beta} \ , \tag{B.2.2a}$$

$$\mathscr{R}(M)_{\alpha\beta}^{i\,j\,cd} = 2\mathrm{i}\varepsilon^{ij}\varepsilon_{\alpha\beta}W^{cd} + \mathrm{i}\varepsilon^{ij}(\Gamma_b)_{\alpha\beta}\tilde{W}^{bcd} \ , \tag{B.2.2b}$$

$$\mathscr{R}(S)_{\alpha\beta}^{i\,j\,\gamma k} = \frac{3\mathrm{i}}{4}\varepsilon^{ij}\varepsilon_{\alpha\beta}X^{\gamma k} + \mathrm{i}\varepsilon^{ij}\delta^\gamma_{[\alpha}X^k_{\beta]} \ , \tag{B.2.2c}$$

$$\begin{aligned}\mathscr{R}(K)_{\alpha\beta}^{i\,j\,c} = &-\frac{\mathrm{i}}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}\nabla^bW_b{}^c + \frac{\mathrm{i}}{2}\varepsilon^{ij}(\Gamma^a)_{\alpha\beta}\nabla^d\tilde{W}_{da}{}^c - \frac{\mathrm{i}}{32}\varepsilon^{ij}(\Gamma^c)_{\alpha\beta}Y \\ &+ \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon_{\alpha\beta}\tilde{W}_c{}^{de}W_{de} + \frac{\mathrm{i}}{2}\varepsilon^{ij}(\Gamma^a)_{\alpha\beta}\Big(W_{ad}W^{cd} - \frac{3}{16}W^{bd}W_{bd}\delta_a{}^c\Big) \ .\end{aligned} \tag{B.2.2d}$$

The non-vanishing torsion and curvatures in the spinor-vector commutator $[\nabla_b, \nabla_\alpha^i]$ are:

$$\mathscr{T}_{b\alpha k}^{\ i\ \gamma} = \frac{1}{4}\delta_k^i\Big(3(\Gamma_b)_\alpha{}^\beta W_\beta{}^\gamma - W_\alpha{}^\beta(\Gamma_b)_\beta{}^\gamma\Big) \ , \tag{B.2.3a}$$

$$\mathscr{R}(D)_{b\alpha}^{\ i} = -\frac{1}{4}(\Gamma_b)_\alpha{}^\gamma X_\gamma^i \ , \tag{B.2.3b}$$

$$\mathscr{R}(J)_{b\alpha}^{\ i\ jk} = -\frac{3}{4}(\Gamma_b)_\alpha{}^\gamma\varepsilon^{i(j}X_\gamma^{k)} \ , \tag{B.2.3c}$$

$$\mathscr{R}(M)_{b\alpha}^{\ i\ cd} = -(\Gamma_b)_\alpha{}^\gamma W^{cdi}{}_\gamma - \frac{1}{4}\varepsilon_b{}^{cdef}W_{ef}{}^i_\alpha + \frac{1}{2}\delta_b^{[c}(\Gamma^{d]})_\alpha{}^\gamma X_\gamma^i \ , \tag{B.2.3d}$$

$$\begin{aligned}\mathscr{R}(S)_{b\alpha}^{\ i\ \gamma j} = &\frac{1}{16}X_{cd}{}^{ij}(\Sigma^{cd}\Gamma_b - 2\Gamma_b\Sigma^{cd})_\alpha{}^\gamma \\ &- \frac{3\mathrm{i}}{8}\varepsilon^{ij}\nabla_{[b}W_{cd]}(\Sigma^{cd})_\alpha{}^\gamma - \frac{\mathrm{i}}{8}\varepsilon^{ij}\nabla_dW^{dc}(\Sigma_{cb})_\alpha{}^\gamma \\ &+ \frac{3\mathrm{i}}{16}\varepsilon^{ij}\nabla^dW_{db}\,\delta_\alpha^\gamma - \frac{\mathrm{i}}{8}\varepsilon^{ij}\nabla^c\tilde{W}_{cb}{}^d(\Gamma_d)_\alpha{}^\gamma \\ &+ \frac{\mathrm{i}}{16}\varepsilon^{ij}\tilde{W}^{cde}W_{de}(\Sigma_{cb})_\alpha{}^\gamma - \frac{3\mathrm{i}}{32}\varepsilon^{ij}\tilde{W}_{bde}W^{de}\delta_\alpha{}^\gamma \\ &+ \frac{\mathrm{i}}{4}\varepsilon^{ij}W_{bd}W^{cd}(\Gamma_c)_\alpha{}^\gamma - \frac{3\mathrm{i}}{64}\varepsilon^{ij}W^{cd}W_{cd}(\Gamma_b)_\alpha{}^\gamma \ ,\end{aligned} \tag{B.2.3e}$$

$$\begin{aligned}\mathscr{R}(K)_{b\alpha}^{\ i\ c} = &\frac{1}{6}(\Gamma^c)_\alpha{}^\beta\nabla^dW_{db}{}^i_\beta + \frac{1}{12}(\Gamma_b)_\alpha{}^\beta\nabla^dW_d{}^{ci}{}_\beta + \frac{1}{6}\nabla_\alpha{}^\beta W_b{}^{ci}{}_\beta - \frac{1}{24}\varepsilon_b{}^{cde\hat{f}}\nabla_dW_{e\hat{f}}{}^i_\alpha \\ &+ \frac{1}{8}(\Gamma^c)_\alpha{}^\beta\nabla_bX_\beta^i + \frac{1}{64}W^{de}(3\Gamma_b\Sigma_{de}\Gamma^c - \Sigma_{de}\Gamma_b\Gamma^c)_\alpha{}^\beta X_\beta^i \\ &- \frac{1}{48}\tilde{W}_{bde}(\Gamma^c)_\alpha{}^\beta W^{dei}{}_\beta + \frac{1}{8}\delta_b{}^cW^{de}W_{de}{}^i_\alpha \\ &+ \frac{1}{12}(\Sigma_b{}^c)_\alpha{}^\beta W_{de}{}^i_\beta W^{de} - \frac{1}{12}W^{de}(\Sigma_{de})_\alpha{}^\beta W_b{}^c{}_\beta{}^i \\ &+ \frac{13}{48}W_{bd}W^{dci}{}_\alpha + \frac{11}{48}W_{bd}{}^i_\alpha W^{dc} - \frac{13}{96}(\Gamma_b)_\alpha{}^\beta W_{de}{}^i_\beta\,\tilde{W}^{dec} \ .\end{aligned} \tag{B.2.3f}$$

The commutator of two vector derivatives $[\nabla_a, \nabla_b]$ has the following non-zero torsion and curvatures:

$$\mathscr{T}_{ab}{}_i^\alpha = -\frac{\mathrm{i}}{2}W_{ab}{}_i^\alpha \ , \tag{B.2.4a}$$

$$\mathscr{R}(J)_{ab}{}^{ij} = -\frac{3\mathrm{i}}{4}X_{ab}{}^{ij} \ , \tag{B.2.4b}$$

$$\mathscr{R}(M)_{ab}{}^{cd} = -\frac{1}{4}(\Sigma_{ab})^{\alpha\beta}(\Sigma^{cd})^{\gamma\delta}\Big(\mathrm{i}W_{\alpha\beta\gamma\delta} + 3W_{(\alpha\beta}W_{\gamma\delta)}\Big) \ , \tag{B.2.4c}$$

$$\mathscr{R}(S)_{ab}{}^i_\alpha = -\frac{1}{2}\nabla_\alpha{}^\beta W_{ab\beta}{}^i - \frac{1}{2}(\Gamma_{[a})_\alpha{}^\beta\nabla^cW_{b]c\beta}{}^i$$

$$- \frac{1}{8} W_\alpha{}^\beta W_{ab\beta}{}^i + \frac{1}{16} (\Sigma_{ab})_\alpha{}^\beta W^{cd} W_{cd\beta}{}^i + \frac{3}{8} W^c{}_{[a} W_{b]c\alpha}{}^i \ , \tag{B.2.4d}$$

$$\mathscr{R}(K)_{ab}{}^c = \frac{1}{4} \nabla_d \mathscr{R}(M)_{ab}{}^{cd} - \frac{\mathrm{i}}{16} W_{ab}{}^\alpha_j (\Gamma^c)_\alpha{}^\beta X^j_\beta - \frac{\mathrm{i}}{8} W_{d[a}{}^\alpha_j (\Gamma_{b]})_\alpha{}^\beta W^{cd\,j}_\beta$$
$$+ \frac{\mathrm{i}}{8} W_{ad}{}^\alpha_i (\Gamma^c)_\alpha{}^\beta W_b{}^d{}_\beta{}^i \ . \tag{B.2.4e}$$

The $S-$supersymmetry transformation of the descendant superfields X^i_α, $W_{\alpha\beta\gamma}{}^i$, $W_{\alpha\beta\gamma\delta}$, $X_{\alpha\beta}{}^{ij}$, and Y of the super-Weyl tensor is given as follows:

$$S_{\alpha i} W_{\beta\gamma\delta}{}^j = 6 \delta^j_i \varepsilon_{\alpha(\beta} W_{\gamma\delta)} \ , \qquad S_{\alpha i} X^j_\beta = 4 \delta^j_i W_{\alpha\beta} \ ,$$
$$S_{\alpha i} W_{\beta\gamma\delta\rho} = 24 \varepsilon_{\alpha(\beta} W_{\gamma\delta\rho) i} \ , \qquad S_{\alpha i} Y = 8\mathrm{i} X_{\alpha i} \ ,$$
$$S_{\alpha i} X_{\beta\gamma}{}^{jk} = -4 \delta^{(j}_i W_{\alpha\beta\gamma}{}^{k)} + 4 \delta^{(j}_i \varepsilon_{\alpha(\beta} X^{k)}_{\gamma)} \ . \tag{B.2.5}$$

Appendix C

$6D\ N = (1,0)$ conformal superspace

In this appendix we collect results about $6D\ N = (1,0)$ conformal superspace [31] in the traceless frame of [32, 69] focusing on the ingredients relevant to our discussion in Chapter 6. We adhere to the notations and conventions of [69].

C.1 $6D\ N = (1,0)$ conformal superspace identities

The Lorentz generators act on the superspace covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha^i)$ as

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} \,, \tag{C.1.1}$$

$$[M_{ab}, \nabla_c] = 2\eta_{c[a}\nabla_{b]} \,, \tag{C.1.2}$$

$$[M_\alpha{}^\beta, \nabla_\gamma^k] = -\delta_\gamma^\beta \nabla_\alpha^k + \frac{1}{4}\delta_\alpha^\beta \nabla_\gamma^k \,, \tag{C.1.3}$$

where $M_\alpha{}^\beta = -\frac{1}{4}(\gamma^{ab})_\alpha{}^\beta M_{ab}$. The $SU(2)_R$ and dilatation generators satisfy

$$[J^{ij}, J^{kl}] = \varepsilon^{k(i}J^{j)l} + \varepsilon^{l(i}J^{j)k} \,, \quad [J^{ij}, \nabla_\alpha^k] = \varepsilon^{k(i}\nabla_\alpha^{j)} \,, \tag{C.1.4}$$

$$[\mathbb{D}, \nabla_a] = \nabla_a \,, \quad [\mathbb{D}, \nabla_\alpha^i] = \frac{1}{2}\nabla_\alpha^i \,. \tag{C.1.5}$$

The Lorentz and $SU(2)_R$ generators act on the special conformal generators $K^A = (K^a, S_i^\alpha)$ as

$$[M_{ab}, K^c] = 2\delta_{[a}^c K_{b]} \,, \quad [M_\alpha{}^\beta, S_k^\gamma] = \delta_\alpha^\gamma S_k^\beta - \frac{1}{4}\delta_\alpha^\beta S_k^\gamma \,, \quad [J^{ij}, S_k^\gamma] = \delta_k^{(i} S^{\gamma j)} \,, \tag{C.1.6}$$

while the dilatation generator acts on K^A as

$$[\mathbb{D}, K^a] = -K^a \,, \quad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha \,. \tag{C.1.7}$$

Among themselves, the generators K_A obey the only nontrivial anti-commutation relation

$$\{S_i^\alpha, S_j^\beta\} = -2\mathrm{i}\varepsilon_{ij}(\tilde{\gamma}_c)^{\alpha\beta}K^c \,. \tag{C.1.8}$$

The algebra of K^A with ∇_A is given by

$$[K_a, \nabla_b] \;=\; 2\eta_{ab}\mathbb{D} + 2M_{ab} \,, \tag{C.1.9}$$

$$
\begin{aligned}
[K^a,\nabla^i_\alpha] &= -\mathrm{i}(\gamma^a)_{\alpha\beta}S^{\beta i}\ , && \text{(C.1.10)}\\
\{S^\alpha_i,\nabla^j_\beta\} &= 2\delta^\alpha_\beta\delta^j_i\mathbb{D}-4\delta^j_i M_\beta{}^\alpha+8\delta^\alpha_\beta J_i{}^j\ , && \text{(C.1.11)}\\
[S^\alpha_i,\nabla_b] &= -\mathrm{i}(\tilde\gamma_b)^{\alpha\beta}\nabla_{\beta i}+\frac{1}{10}W_{bcd}(\tilde\gamma^{cd})^\alpha{}_\gamma S^\gamma_i-\frac14 X^\alpha_i K_b\\
&\quad+\Big[\frac14(\tilde\gamma_{bc})^\alpha{}_\beta X^\beta_i+\frac12(\gamma_{bc})_\beta{}^\gamma X_{\gamma i}{}^{\beta\alpha}\Big]K^c\ . && \text{(C.1.12)}
\end{aligned}
$$

The anticommutator of two spinor derivatives, $\{\nabla^i_\alpha,\nabla^j_\beta\}$, has the following non-zero torsion and curvatures

$$
\begin{aligned}
\mathscr{T}_{\alpha\beta}^{i\,j\,c} &= 2\mathrm{i}\varepsilon^{ij}(\gamma^c)_{\alpha\beta}\ , && \text{(C.1.13a)}\\
\mathscr{R}(M)_{\alpha\beta}^{i\,j\,cd} &= 4\mathrm{i}\varepsilon^{ij}(\gamma_a)_{\alpha\beta}W^{acd}\ , && \text{(C.1.13b)}\\
\mathscr{R}(S)_{\alpha\beta\gamma}^{i\,j\,k} &= -\frac32\varepsilon^{ij}\varepsilon_{\alpha\beta\gamma\delta}X^{\delta k}\ , && \text{(C.1.13c)}\\
\mathscr{R}(K)_{\alpha\beta c}^{i\,j} &= \mathrm{i}\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\left(\frac14\eta_{ac}Y-\nabla^b W_{abc}+W_a{}^{ef}W_{cef}\right)\ . && \text{(C.1.13d)}
\end{aligned}
$$

The non-zero torsion and curvatures in the commutator $[\nabla_a,\nabla^j_\beta]$ are:

$$
\begin{aligned}
\mathscr{T}_{a\beta k}^{\,j\,\gamma} &= -\frac12(\gamma_a)_{\beta\delta}W^{\delta\gamma}\delta^j_k\ , && \text{(C.1.14a)}\\
\mathscr{R}(\mathbb{D})_{a\beta}^{\,j} &= -\frac{\mathrm{i}}{2}(\gamma_a)_{\beta\gamma}X^{\gamma j}\ , && \text{(C.1.14b)}\\
\mathscr{R}(M)_{a\beta}^{\,j\,cd} &= \mathrm{i}\delta_a^{[c}(\gamma^{d]})_{\beta\gamma}X^{\gamma j}-\mathrm{i}(\gamma_a{}^{cd})_{\gamma\delta}X_\beta^{j\gamma\delta}+2\mathrm{i}(\gamma_a)_{\beta\gamma}(\gamma^{cd})_\delta{}^\rho X_\rho^{j\gamma\delta}\ , && \text{(C.1.14c)}\\
\mathscr{R}(J)_{a\beta}^{\,j\,kl} &= 2\mathrm{i}(\gamma_a)_{\beta\gamma}X^{\gamma(k}\varepsilon^{l)j}\ , && \text{(C.1.14d)}\\
\mathscr{R}(S)_{a\beta\gamma}^{\,j\,k} &= -\frac{\mathrm{i}}{4}(\gamma_a)_{\beta\delta}Y_\gamma{}^{\delta jk}+\frac{3\mathrm{i}}{20}(\gamma_a)_{\gamma\delta}Y_\beta{}^{\delta jk}-\frac{\mathrm{i}}{8}(\gamma_a)_{\beta\delta}\nabla_{\gamma\rho}W^{\delta\rho}\varepsilon^{jk}\\
&\quad+\frac{\mathrm{i}}{40}(\gamma_a)_{\gamma\delta}\nabla_{\beta\rho}W^{\delta\rho}\varepsilon^{jk}-\frac{\mathrm{i}}{8}(\gamma_a)_{\delta\varepsilon}\varepsilon_{\beta\rho\tau\gamma}W^{\delta\rho}W^{\varepsilon\tau}\varepsilon^{jk}\ , && \text{(C.1.14e)}\\
\mathscr{R}(K)_{a\beta c}^{\,j} &= \frac{\mathrm{i}}{4}(\gamma_c)_{\beta\gamma}\nabla_a X^{\gamma j}-\frac{\mathrm{i}}{4}(\gamma_{acd})_{\gamma\delta}\nabla^d X_\beta^{j\gamma\delta}+\frac{\mathrm{i}}{3}(\gamma_a)_{\beta\delta}(\gamma_{cd})_\rho{}^\gamma\nabla^d X_\gamma^{j\delta\rho}\\
&\quad-\frac{\mathrm{i}}{8}(\gamma_a)_{\beta\gamma}(\gamma_c)_{\delta\rho}W^{\gamma\delta}X^{\rho j}+\frac{5\mathrm{i}}{12}(\gamma_a)_{\beta\rho}(\gamma_c)_{\gamma\varepsilon}W^{\gamma\delta}X_\delta^{j\rho\varepsilon}\\
&\quad+\frac{\mathrm{i}}{4}(\gamma_a)_{\gamma\rho}(\gamma_c)_{\beta\varepsilon}W^{\gamma\delta}X_\delta^{j\rho\varepsilon}-\frac{\mathrm{i}}{2}(\gamma_a)_{\gamma\rho}(\gamma_c)_{\delta\varepsilon}W^{\gamma\delta}X_\beta^{j\rho\varepsilon}\ . && \text{(C.1.14f)}
\end{aligned}
$$

The commutator of two vector derivatives, $[\nabla_a,\nabla_b]$, has the following non-vanishing torsion and curvatures:

$$
\begin{aligned}
\mathscr{T}_{ab}{}^\gamma_k &= (\gamma_{ab})_\beta{}^\alpha X_{\alpha k}{}^{\beta\gamma}\ , && \text{(C.1.15a)}\\
\mathscr{R}(M)_{ab}{}^{cd} &= Y_{ab}{}^{cd}=\frac14(\gamma_{ab})_\gamma{}^\alpha(\gamma^{cd})_\delta{}^\beta Y_{\alpha\beta}{}^{\gamma\delta}\ , && \text{(C.1.15b)}\\
\mathscr{R}(J)_{ab}{}^{kl} &= \frac12(\gamma_{ab})_\delta{}^\gamma Y_\gamma{}^{\delta kl}=Y_{ab}{}^{kl}\ , && \text{(C.1.15c)}\\
\mathscr{R}(S)_{ab}{}^k_\gamma &= -\frac{\mathrm{i}}{3}(\gamma_{ab})_\delta{}^\alpha\nabla_{\gamma\beta}X_\alpha^{k\beta\delta}-\frac{\mathrm{i}}{6}(\gamma_{abc})_{\alpha\beta}\nabla^c X_\gamma^{k\alpha\beta}-\frac{\mathrm{i}}{6}\varepsilon_{\gamma\beta\varepsilon\rho}(\gamma_{ab})_\delta{}^\rho W^{\alpha\beta}X_\alpha^{k\delta\varepsilon}\ , && \text{(C.1.15d)}\\
\mathscr{R}(K)_{abc} &= \frac14\nabla^d Y_{abcd}+\frac{\mathrm{i}}{3}X_\alpha^{k\beta\gamma}X_{\beta k}{}^{\alpha\delta}(\gamma_{abc})_{\gamma\delta}+\mathrm{i}(\gamma_{ab})_\varepsilon{}^\alpha(\gamma_c)_{\gamma\delta}X_\alpha^{k\beta\gamma}X_{\beta k}{}^{\delta\varepsilon}
\end{aligned}
$$

$$+\frac{\mathrm{i}}{4}X^{\alpha k}X_{\beta k}{}^{\gamma\delta}(\gamma_{ab})_\gamma{}^\beta(\gamma_c)_{\alpha\delta}\ . \tag{C.1.15e}$$

Recall the descendant superfields $X^{\alpha i}$, $X_{\alpha i}{}^{\beta\gamma}$, Y, $Y_\alpha{}^{\beta kl}$, $Y_{\alpha\beta}{}^{\gamma\delta}$ (and equivalently $Y_{ab}{}^{cd}$), were defined in (2.2.90) and (2.2.91). They transform under S-supersymmetry as [31]

$$\begin{aligned} S_i^\alpha X^{\beta j} &= \frac{8\mathrm{i}}{5}\delta_i^j W^{\alpha\beta}\ , \qquad S_i^\alpha X_\beta^{j\gamma\delta} = -\mathrm{i}\delta_i^j\delta_\beta^\alpha W^{\gamma\delta} + \frac{2\mathrm{i}}{5}\delta_i^j\delta_\beta^{(\gamma}W^{\delta)\alpha}\ , &\text{(C.1.16a)}\\ S_k^\gamma Y_\alpha{}^{\beta ij} &= -\delta_k^{(i}\left(16X_\alpha^{j)\gamma\beta} - 2\delta_\alpha^\beta X^{\gamma j)} + 8\delta_\alpha^\gamma X^{\beta j)}\right)\ , &\text{(C.1.16b)}\\ S_j^\rho Y_{\alpha\beta}{}^{\gamma\delta} &= 24\left(\delta^\rho_{(\alpha}X_{\beta)j}{}^{\gamma\delta} - \frac{1}{3}\delta^{(\gamma}_{(\alpha}X_{\beta)j}{}^{\delta)\rho}\right)\ , \qquad S_i^\alpha Y = -4X_i^\alpha\ . &\text{(C.1.16c)}\end{aligned}$$

By using (2.2.89) and the previous definitions, one can derive spinor covariant derivative acting on these descendants of the super-Weyl tensor and can be found in [32, 69] in the traceless frame. Here we only provide the relations which are useful for our analysis, which are

$$\begin{aligned} \nabla_\alpha^i X^{\beta j} &= -\frac{2}{5}Y_\alpha{}^{\beta ij} - \frac{2}{5}\varepsilon^{ij}\nabla_{\alpha\gamma}W^{\gamma\beta} - \frac{1}{2}\varepsilon^{ij}\delta_\alpha^\beta Y\ , &\text{(C.1.17a)}\\ \nabla_\alpha^i X_\beta^{j\gamma\delta} &= \frac{1}{2}\delta_\alpha^{(\gamma}Y_\beta{}^{\delta)ij} - \frac{1}{10}\delta_\beta^{(\gamma}Y_\alpha{}^{\delta)ij} - \frac{1}{2}\varepsilon^{ij}Y_{\alpha\beta}{}^{\gamma\delta} - \frac{1}{4}\varepsilon^{ij}\nabla_{\alpha\beta}W^{\gamma\delta}\\ &\quad + \frac{3}{20}\varepsilon^{ij}\delta_\beta^{(\gamma}\nabla_{\alpha\rho}W^{\delta)\rho} - \frac{1}{4}\varepsilon^{ij}\delta_\alpha^{(\gamma}\nabla_{\beta\rho}W^{\delta)\rho}\ . &\text{(C.1.17b)}\end{aligned}$$