On-Shell Methods
— in Three and Six Dimensions —

by

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Abstract

In the past few years, on-shell analytic methods have played a pivotal role in gauge theory calculations. Since the initial success of these methods in Standard Model physics, considerable activity has led to development and application in supersymmetric gauge theories.

In particular, the maximally supersymmetric super Yang-Mills theory received much attention after the discovery of holographic dualities. Here, the spinor helicity formalism and on-shell superspace is described initially for four dimensions. The framework of general unitarity is shown to be a useful tool for calculating loop corrections of scattering amplitudes.

Once the foundation is laid, application in three and six dimensions is explored. In six dimensions the case of interest is a theory with (1,1) supersymmetry which captures the dynamics of five-branes in string theory. In this setup the one-loop superamplitude with four and five external particles is calculated and checked for consistency.

In three dimensions, the supersymmetric gauge theory that is supposed to describe the dynamics of M2-branes is considered. This particular theory is also related to M-theory via the holographic duality. The goal was to explore and determine the infra-red divergences of the theory. This was achieved by calculating the Sudakov form factor at two loops.

Declaration

I hereby declare that the material presented in this thesis is a representation of my own personal work, unless otherwise stated, and is a result of collaborations with Andreas Brandhuber, Ömer Gürdoğan, Daniel Koschade, Robert Mooney and Gabriele Travaglini.

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1 | Introduction

Quantum Electrodynamics (QED) turned out to be the most accurate scientific theory* of the 20th century. This was inextricably linked to the development of the path integral and 'Feynman diagrams'. It served as a model for constructing the theoretical framework that describes two additional fundamental interactions, the weak and the strong. Excluding gravity, electromagnetism and the two other forces have been successfully put under one framework: the Standard Model. The underlying principle is that each force is the consequence of an internal symmetry of nature. The different particles and the way they interact is described by quantum field theory, which is the only way quantum principles can be reconciled with that of special relativity.

The ramifications of the Standard Model have been approached experimentally by scattering particles and observing the products of these interactions and their behavior. The physical quantity that describes these interactions is the scattering cross section. It specifies the area within which one particle must hit the other for a specific process to take place; the product of the area where the particles collide times the quantum mechanical probability (density) that the given process will occur if the particles do collide. Such quantum mechanical probability densities always appear as the absolute value of a quantum mechanical amplitude squared, which in this case is referred to as a scattering amplitude.

Scattering amplitudes are conventionally calculated by means of perturbation theory using Feynman rules and diagrams. This has been studied so extensively

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*The agreement with experiment found this way is to within one part in a billion (10^{-8}).

that we now have a very good understanding of both the physical and mathematical details of this process. Despite being an extremely powerful method of performing calculations, Feynman diagrams are not panacea! Since the idea behind them is to sum over all possible diagrams of certain kinds (which correspond to possible interactions that might occur in a scattering process) the complexity of the calculation factorially increases with the number of particles. Moreover, when increased precision is required then the perturbative effects produce integrals whose complexities are also factorial in the number of particles. Apart from these complexities, there is also another non-trivial problem related to the general framework of Feynman's diagrams. Unphysical singularities were for many years an embarrassing feature of quantum field theories. Fortunately, this has been resolved in way that is not merely a computational trick but has a rather deep physical meaning in the spirit of Wilson's renormalization process.

In spite of the universality and the power of the remedy Feynman put forward in performing calculations for quantum field theories, we are forced to ignore information in exchange for more mechanical calculations. The best example of this is non-Abelian gauge theories. In these theories gauge invariance enforces a particular structure of three and four-point interactions. When following the standard method for calculating these, gauge invariance is ignored throughout the computation as the Feynman rules guarantee that the amplitude is going to be gauge invariant. However, this is something that can be used to bypass several steps and obtain amplitudes faster and more efficiently, in more compact forms and with fewer unphysical singularities.

Fortunately, there is a framework (or formalism) that enables us to represent on-shell scattering amplitudes of massless particles in in a more natural way: the *spinor-helicity* formalism. This framework enables us to take advantage of the fact that asymptotic states of zero mass are completely determined by their momentum and helicity. Thus, the basic ingredient (the S-matrix) of scattering amplitudes should be function of these variables alone. This redundancy in gauge field theory

formalism stems from the choice to make Lorentz invariance manifest resulting in a gauge redundancy that must be introduced to eliminate extra propagating degrees of freedom. This implies that external states are redundantly labeled by polarization vectors and the amplitudes obey non-trivial Ward identities.

The spinor-helicity formalism relieves us from having to mention anything about gauge symmetry and polarization vectors when writing down scattering amplitudes. Moreover, group covariance strongly constrains or even determines the form of the on-shell scattering amplitudes [1, 2]. Further motivation for this formalism is the perturbative expansion of a Yang-Mills theory, which has remarkable properties that are not manifest in terms of Feynman diagrams. The maximally helicity violating (MHV) amplitudes can be written in terms of a simple holomorphic or antiholomorphic function. This was first conjectured by Parke and Taylor, materialized in their celebrated formula [3, 4] and later proved by Berends and Giele [5]. In his seminal paper [6], Witten gave an interesting and fresh perspective on explaining these features of this class of amplitudes. It is now well established that the spinor helicity formalism is not simply a computational trick, but rather a way of representing amplitudes in their elementary, most physical form.

The efficiency of this formalism in four dimensions posed the obvious question of whether it is possible to benefit from such a description of physical observables in theories away from four dimensions. This lead to the development and application of this language in a variety of interesting theories including the ones that are the main focus of this thesis, namely $\mathcal{N} = (1, 1)$ six-dimensional gauge theory and ABJM theory that lives in three dimensions.

In more recent years, a rekindled interest in the structure of scattering amplitudes has produced a plethora of advancements in the field. The spinor helicity formalism combined with unitarity has brought forth an extremely useful framework to perform calculations but also played an integral part in unravelling new dualities and even trialities. In particular, the dual conformal symmetry that is non-manifest in the Lagrangian formulation of a theory yields constraints that can be used to determine

the structure of amplitudes and opened the way to integrability methods through the Yangian symmetry. It also plays an indispensable role in understanding the duality between amplitudes and Wilson loops [7, 8]. The Wilson loop-amplitude duality is also that is currently explored in the context of ABJM theory (for example see [9] and references therein). Moreover, the recently discovered duality between amplitudes and correlation functions [10, 11] established the triality between amplitudes, Wilson loops and correlation functions.

Many of the aforementioned developments were made possible or at least better understood through the spinor-helicity language combined with unitarity. Working with unitarity means that only on-shell gauge invariant objects enter the calculations. Factorization of amplitudes linked to unitarity is a general feature regardless of the theory specifics and the dimensionality of spacetime, allowing for recursion relations and loop calculations to be treated efficiently in a variety of cases. The recursion relation that was proved by Britto, Cachazo, Feng and Witten [12] allow for a simpler way to calculate tree amplitudes regardless of the number of external particles. By now it has been implemented to a wide variety of theories including string theory [13] but also in the theories that will be of interest in the present work; more specifically, in the $\mathcal{N} = (1, 1)$ six-dimensional theory that will be considered in Chapter 3 [13] as well as in ABJM theory [14] that will be the subject of Chapter 4. These recursion relations can be viewed as an application of unitarity at tree level.

Unitarity however is perhaps mostly appreciated as a tool to perform loop calculations. Via the optical theory (Section 2.4.1) one immediately simplifies the task of approaching a loop calculation. Substantial progress was made however when generalized unitarity matured. Under generalized unitarity, the loop calculation is broken down to a completely algebraic procedure of determining the coefficients of known integrals in terms of tree amplitudes. This, again, is a procedure applicable in theories beyond the well studied cases of four dimensional theories.

The fact that this set of modern tools have provided the means to gain new insights as well as effective ways of performing otherwise out-of-reach calculations

motivated the present body of work. Fascinating contemporary theories seem the perfect candidates to apply modern methods to study their properties and gain a better understanding. The outline of the thesis is the following. Chapter 2 presents the spinor helicity formalism and unitarity methods in the context of four-dimensioanal Yang-Mills theory and maximally supersymmetric Yang-Mills theory. These provide the foundation for later chapters. In Chapter 3 we venture to six dimensions and develop the one-loop calculation for up to a five-point amplitude and is based on [15]. Next, in Chapter 4 we approach three dimensions and in particular ABJM theory by calculating the two-loop Sudakov form factor as done in [16]. Finally, the concluding remarks offer an overview and summarize the findings in support of the idea that state-of-the-art calculational methods are applicable in a wide variety of areas and provide a framework capable of answering physically interesting questions.

2 | Foundations

This chapter focuses on the foundations of modern methods for calculating scattering amplitudes. Based on their properties, efficient ways of capturing the physics of a scattering process are developed. The goal is to describe a formalism that reduces the intricacy of computations. This is motivated by the fact that the end results can be strikingly simple whereas the underlying diagramatic calculations using Feynman diagrams can become rapidly complicated. What follows is formally developed in four dimensions of spacetime and will be involved with pure Yang-Mills theory as well as the maximally supersymmetric case. The tools that will be presented in this chapter have been extensively discussed an used in the literature. It is however important to note that especially the unitarity framework (Section 2.4) is valid in any number of spacetime dimensions. It is therefore directly applicable to theories in three and six dimensions that preoccupy the following chapters.

In a nutshell, what will be introduced here can be described as a set of strategies to dissect complicated amplitudes into simpler forms at both tree and loop level. The unitarity based deconstruction enables us to use on-shell tree amplitudes as building blocks for amplitudes of rather invovled processes. The properties of these on-shell amplitudes provide further restrictions and as a result calculations tend to simplify. A vital component for the success of this program is an uncluttered formalism that enables us to express amplitudes in their most minimal form. This is provided by the spinor-helicity formalism and the on-shell superspace for the supersymmetric case in four dimensions and will provide the basis for what follows.

2.1 Structure of Amplitudes

In the present section all the necessary tools are developed in order to describe the amplitudes in the simplest possible form. This starts by the color decomposition of the amplitude and then by applying the spinor-helicity formalism it is possible to write down the Parke-Taylor formula [3, 4] for the MHV amplitudes. The development of this formalism is the blueprint for further applications in the more exotic theories of the subsequent chapters.

2.1.1 Color ordering

The color decomposition of amplitudes is based on the idea that it should be possible to deconstruct the amplitudes into smaller gauge invariant objects by taking advantage of the group theory structure. These are then composed of Feynman diagrams with a fixed cyclic ordering of external legs (colored ordered Feynamn rules). Since their poles and cuts can only appear in kinematic invariants which are made out of cyclically adjacent sums of momenta these new objects are inherently simpler. A concise exposition of this aspect of amplitudes which is followed here, can be found in [17].

The first interesting implementation of this strategy would be on Quantum Chromodynamics (QCD). The group in this case is $\mathsf{SU}(3)$ but may be generalized in this discussion by considering $\mathsf{SU}(N_c)$. The particles of QCD can be divided to gluons and quarks, and each variety carry different types of indices. Quarks (and antiquarks) carry a color index $\mathbf{N}_c$ (and $\bar{\mathbf{N}}_c$) taking values $i(,\bar{j}) = 1, 2, \ldots, N_c$. Gluons on the other hand carry an adjoint index $a = 1, 2, \ldots, N_c^2 - 1$. The generators of $\mathsf{SU}(N_c)$ are traceless hermitian $N_c \times N_c$ matrices $((T^a)_i{}^{\bar{j}})$ in the fundamental representation. The Feynman rules require a factor of $(T^a)_i{}^{\bar{j}}$ for each gluon-quark-antiquark vertex and a group theory structure constant for each pure gluon three-vertex and a contracted product of two structure constants for a pure gluon four-vertex. Structure

constants encapsulate the group structure and can be expressed as

$$f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}\left[\left[T^a, T^b\right], T^c\right] \tag{2.1.1}$$

where the T^a's are normalized as

$$\left[T^a, T^b\right] = i\sqrt{2} f^{abc} T^c, \qquad \text{Tr}\left[T^a T^b\right] = \delta^{ab}. \tag{2.1.2}$$

Propagators are realized in color space as Kroenecker deltas that contract indices together. In other words there is a δ_{ab} for each gluon propagator and a $\delta_i{}^{\bar{j}}$ for each quark propagator. Uncontracted adjoint indices correspond to external gluons and uncontracted (un)barred indices correspond to (anti)quarks. By replacing all structure constants with the corresponding traces we end up with a long string of trace factors of the form $\text{Tr}\left(\ldots T^a \ldots\right) \text{Tr}\left(\ldots T^a \ldots\right) \ldots \text{Tr}\left(\ldots\right)$. To reduce the number of traces we can use the completeness relation

$$(T^a)_{i_1}{}^{\bar{j}_1} (T^a)_{i_2}{}^{\bar{j}_2} = \delta_{i_1}{}^{\bar{j}_2} \delta_{i_2}{}^{\bar{j}_1} - \frac{1}{N_c} \delta_{i_1}{}^{\bar{j}_1} \delta_{i_2}{}^{\bar{j}_2} \tag{2.1.3}$$

where summation over the gluon color index is understood. By applying the completeness relation (2.1.3) all the structure constants can be recast in terms of sums of single traces of generators. This process is colloquially known as "Fierzing". For the scattering of gluons a general amplitude may be written as

$$A_n^{\text{tree}} = g^{n-2} \sum_{\sigma_i \in \mathsf{S}_n/Z_n} \text{Tr}[T^{a_{\sigma_1}} T^{a_{\sigma_2}} \ldots T^{a_{\sigma_n}}] \mathcal{A}_n^{\text{tree}} (\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{2.1.4}$$

where now $\mathcal{A}_n(1, 2, \ldots, n)$ is a color-ordered partial amplitude that contains all the kinematic information and g is the gauge coupling constant ($g^2 = 4\pi\alpha_s$). The permutation of n objects is represented as a set in the form of the permutation group S_n and excluding the subset of cyclic permutations Z_n which leaves the trace unchanged. Thus, the full amplitude is obtained by summing over S_n/Z_n, the set of all non-cyclic permutations of the indices $[1, 2, \ldots, n]$.

Yet by performing this manipulation the hard work is merely postponed for later on. However, what is gained is that the partial amplitudes are simpler than the

full amplitude. This is due to $\mathcal{A}_n$ being a colored ordered object: it only receives contributions from diagrams with a particular cyclic ordering of the gluons*. For example the simplest case of three gluons is

$$A_3^{\text{tree}} = g \left\{ \text{Tr} \left[T^{a_1} T^{a_2} T^{a_3} \right] \mathcal{A}_3^{\text{tree}} (1,2,3) + \text{Tr} \left[T^{a_1} T^{a_3} T^{a_2} \right] \mathcal{A}_3^{\text{tree}} (1,3,2) \right\} \qquad (2.1.5)$$

while for the four-point amplitude this is:

$$\begin{aligned} A_4^{\text{tree}} = g \Big\{ &\text{Tr} \left[T^{a_1} T^{a_2} T^{a_3} T^{a_4} \right] \mathcal{A}_4^{\text{tree}} (1,2,3,4) + \text{Tr} \left[T^{a_1} T^{a_2} T^{a_4} T^{a_3} \right] \mathcal{A}_4^{\text{tree}} (1,2,4,3) \\ &+ \text{Tr} \left[T^{a_1} T^{a_3} T^{a_2} T^{a_4} \right] \mathcal{A}_4^{\text{tree}} (1,3,2,4) + \text{Tr} \left[T^{a_1} T^{a_4} T^{a_2} T^{a_3} \right] \mathcal{A}_4^{\text{tree}} (1,4,2,3) \\ &+ \text{Tr} \left[T^{a_1} T^{a_3} T^{a_4} T^{a_2} \right] \mathcal{A}_4^{\text{tree}} (1,3,4,2) + \text{Tr} \left[T^{a_1} T^{a_4} T^{a_3} T^{a_2} \right] \mathcal{A}_4^{\text{tree}} (1,4,3,2) \Big\} . \end{aligned}$$

$$(2.1.6)$$

If a $\mathsf{U}(N_c)$ theory is considered then we can take advantage of the fact that $\mathsf{U}(N_c) = \mathsf{SU}(N_c) \times \mathsf{U}(1)$. The $\mathsf{U}(1)$ generator is given by

$$\left(T_{\mathsf{U}(1)}^a \right)_i{}^{\bar{j}} = \frac{1}{\sqrt{N_c}} \delta_i{}^{\bar{j}} \qquad (2.1.7)$$

which should be added to the completeness relation (2.1.3) and thus canceling the $-1/N_c$ term. The $\mathsf{U}(1)$ gauge field (the photon) is colorless (i.e. it commutes with $\mathsf{SU}(N_c)$; $f^{a_{\mathsf{U}(1)}bc} = 0, \ \forall b, c$) and therefore does not couple directly to gluons. Of course, quarks are charged under $\mathsf{U}(1)$. Physically, this means that at each vertex the contribution coming from the $\mathsf{U}(1)$ gauge boson is subtracted.

2.1.2 Spinor Helicity formalism

In the previous section the richness of information, due to the color structure of an $\mathsf{SU}(N_c)$ theory, was traded to more (simple) diagrams. Now the number of diagrams will be increased once more, but this time the trade-off will be between diagrams and polarization vectors. This entails writing down amplitudes in terms of massless Weyl spinors which will lead to a better notation for the kinematics of massless

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*Specifically, they only depend on adjacent momenta.

particles. More information on spinors can be found in [18], while for the details of the formalism one may refer to [6, 17].

Consider the massless Dirac equation[†]:

$$p_\mu \gamma^\mu u(p) = \not{p} u(p) = 0 \tag{2.1.8}$$

where $u(p)$ is a four component spinor and γ^μ, $\mu = 0, \ldots, 3$ are the Dirac matrices that in the chiral representation take the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.9}$$

with σ^i being the standard Pauli matrices. The Dirac spinors can be written as two-component spinors (in the van der Waerden notation)

$$u(p) = \begin{pmatrix} u_\alpha(p) \\ \bar{u}^{\dot\alpha}(p) \end{pmatrix}. \tag{2.1.10}$$

The two-component spinors are called Weyl spinors and transform independently under Lorentz transformations. In other words, they are irreducible, massless representations of the Lorentz group. Thus, from a mathematical point of view, Weyl spinors may be considered as more "fundamental" than Dirac spinors.

Setting either of the Weyl spinors to zero in a Dirac spinor yields the eigenstates of γ^5:

$$\gamma^5 \begin{pmatrix} u_\alpha \\ 0 \end{pmatrix} = + \begin{pmatrix} u_\alpha \\ 0 \end{pmatrix} \tag{2.1.11}$$

$$\gamma^5 \begin{pmatrix} 0 \\ \bar{u}^{\dot\alpha} \end{pmatrix} = - \begin{pmatrix} 0 \\ \bar{u}^{\dot\alpha} \end{pmatrix} \tag{2.1.12}$$

which defines the positive and negative chirality spinors. In the massless case this can also be expressed as [20]:

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} u_\alpha = u_\alpha \tag{2.1.13}$$

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \bar{u}^{\dot\alpha} = -\bar{u}^{\dot\alpha}, \tag{2.1.14}$$

[†]Here the focus is clearly in four dimensions. The spinor heliciy formalism is described in full generality in higher dimensions in [19].

known as the Weyl equations. Making $\hbar$ explicit then gives

$$\vec{S} \cdot \hat{p}\, u_\alpha = \frac{\hbar}{2} u_\alpha \tag{2.1.15}$$

$$\vec{S} \cdot \hat{p}\, \bar{u}^{\dot\alpha} = -\frac{\hbar}{2} \bar{u}^{\dot\alpha} \tag{2.1.16}$$

where $\vec{S}(= \hbar\, \vec{\sigma}/2)$ is the spin operator and $\vec{S} \cdot \hat{p}$ is the *helicity operator*. The eigenvalue is the helicity of the state and corresponds to the component of the spin along the direction of the motion of the particle. In the massless limit, Weyl spinors are eigenstates of the helicity operators. For a massive particle helicity depends on the frame of reference, since one can always boost to a frame in which its momentum is in the opposite direction (but its spin is unchanged) [20]. Thus, Weyl spinors only have well-defined helicities in the massless limit.

As will be discussed in Section 2.1.3, in order to obtain non-zero MHV amplitudes we need to work with complex momenta. Therefore, it is useful to know that the complexified Lorentz group in four dimensions is locally isomorphic to $\mathsf{SO}(3, 1, \mathbb{C}) \cong \mathsf{SL}(2, \mathbb{C}) \times \mathsf{SL}(2, \mathbb{C})$ [6]. As will be discussed in later chapters the Lorentz group in six dimensions is $\mathsf{SO}(1, 5)$ and allows one to write vectors as $\mathsf{SU}^*(4)$ matrices [21]. In three dimensions the Lorentz group is simply $\mathsf{SL}(2, \mathbb{R})$. The little group in four dimensions is a rather trivial $\mathsf{U}(1)$ not yielding any useful restricitive conditions. Similarly trivial is the little group in three dimensions, a simple $\mathbb{Z}_2$ reflection. In six dimensions however, as will become apparent in Chapter 3, the little group is a more interesting [13] $\mathsf{SL}(2, \mathbb{C}) \times \mathsf{SL}(2, \mathbb{C})$.

Conventionally, we write a generic negative helicity spinor λ_α (or positive helicity $\tilde{\lambda}^{\dot\alpha}$) and we can raise (lower) the indices with $\varepsilon^{\alpha\beta}$ $\left(\varepsilon_{\dot\alpha\dot\beta}\right)$ [6]. Antisymmetric Lorentz-invariant scalar products are defined as

$$\langle \lambda\eta \rangle = \varepsilon^{\alpha\beta}\lambda_\alpha\eta_\beta = \varepsilon_{\alpha\beta}\lambda^\alpha\eta^\beta = \lambda^\alpha\eta_\alpha\,, \tag{2.1.17}$$

$$[\tilde{\lambda}\tilde{\eta}] = \varepsilon_{\dot\alpha\dot\beta}\tilde{\lambda}^{\dot\alpha}\tilde{\eta}^{\dot\beta} = \varepsilon^{\dot\alpha\dot\beta}\tilde{\lambda}_{\dot\alpha}\tilde{\eta}_{\dot\beta} = \tilde{\lambda}_{\dot\alpha}\tilde{\eta}^{\dot\alpha}. \tag{2.1.18}$$

This gives $\langle \lambda\eta \rangle = -\langle \eta\lambda \rangle$. A momentum vector p^μ can be written as a bi-spinor $p_{\alpha\dot\alpha}$ using one spinor index for each chirality. Using $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ Dirac

matrices can be written as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.1.19}$$

and for any spinor p_μ it is possible to define [6]

$$p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu \tag{2.1.20}$$

giving

$$p_{\alpha\dot{\alpha}} = p_0 \mathbb{1} + \vec{p} \cdot \vec{\sigma} \tag{2.1.21}$$

$$= \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \tag{2.1.22}$$

which leads to $p^\mu p_\mu = \det(p_{\alpha\dot{\alpha}}) = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} p_{\alpha\dot{\alpha}} p_{\beta\dot{\beta}}$ and in the case of a lightlike particle $\det(p_{\alpha\dot{\alpha}}) = 0$. However, for any 2×2 matrix the rank is at most two, and in this case this means $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} + \mu_\alpha \tilde{\mu}_{\dot{\alpha}}$. Given that the determinant is zero for lightlike vectors makes it viable to write

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} . \tag{2.1.23}$$

For two given Weyl spinors, λ and $\tilde{\lambda}$ there is a corresponding lightlike vector p but the opposite is not true since for a given complex p, λ and $\tilde{\lambda}$ are determined modulo the rescaling

$$(\lambda, \tilde{\lambda}) \to \left(z\lambda, z^{-1}\tilde{\lambda} \right), \ z \in \mathbb{C}^* . \tag{2.1.24}$$

The formula $p^\mu p_\mu = \det(p_{\alpha\dot{\alpha}})$ generalizes for any two vectors p and q as $p \cdot q = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} p_{\alpha\dot{\alpha}} q_{\beta\dot{\beta}}$. In the case of lightlike vectors which we can write as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ and $q_{\beta\dot{\beta}} = \mu_\beta \tilde{\mu}_{\dot{\beta}}$ gives[‡]

$$2\, p \cdot q = \langle \lambda\mu \rangle [\tilde{\lambda}\tilde{\mu}] . \tag{2.1.25}$$

[‡]We could also write: $(p+q)^2 = p^2 + q^2 + 2p \cdot q$ and since we are dealing with lightlike momenta $(p+q)^2 = 2p \cdot q$. Therefore, the product is often defined as $2p \cdot q = \langle \lambda\mu \rangle [\tilde{\lambda}\tilde{\mu}]$.

It is interesting to consider other spacetime signatures and the behavior of spinors as done in [6]. In a mostly minus Minkowski signature, demanding real momenta forces the spinors λ and $\tilde{\lambda}$ to be complex conjugates of each other:

$$\tilde{\lambda}_{\dot{\alpha}} = \pm \bar{\lambda}_{\alpha} \tag{2.1.26}$$

where the sign depends on whether p^{μ} has positive or negative energy. For complex momenta the spinors λ and $\tilde{\lambda}$ are independent complex variables. In a $(+ + --)$ signature, the spinors are real, independent two-component objects. The Lorentz group $(\mathsf{SO}(2,2))$ in this case is locally isomorphic to $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ [6].

For a given momentum p^{μ} additional information is needed to specify λ and consequently $\tilde{\lambda}$ in complexified Minkowski space with real $p_{\alpha\dot{\alpha}}$. This is equivalent to a choice of wavefunction for a spin $1/2$ particle of momentum p^{μ}. The Dirac equation can also be written in the following form [6]

$$i\sigma^{\mu}_{\alpha\dot{\alpha}} \frac{\partial \psi^{\alpha}}{\partial x^{\mu}} = 0 \,. \tag{2.1.27}$$

A plane wave $\psi^{\alpha} = \rho^{\alpha} e^{ip\cdot x}$ with a constant ρ^{α} obeys (2.1.27) if and only if $p_{\alpha\dot{\alpha}}\rho^{\alpha} = 0$ which directly implies that $\rho^{\alpha} = c\lambda^{\alpha}$. By the same reasoning for the positive-helicity spinors we can end up with the fermions wavefunctions of helicity $\pm 1/2$ as

$$\psi^{\dot{\alpha}} = \tilde{\lambda}^{\dot{\alpha}} e^{ix_{\beta\dot{\beta}}\lambda^{\beta}\tilde{\lambda}^{\dot{\beta}}}, \qquad \psi^{\alpha} = \lambda^{\alpha} e^{ix_{\beta\dot{\beta}}\lambda^{\beta}\tilde{\lambda}^{\dot{\beta}}}. \tag{2.1.28}$$

Since massless particles of spin ± 1 are being considered the usual method is to specify a polarization vector ϵ^{μ} in addition to their momentum and together with the constraint $p_{\mu}\epsilon^{\mu} = 0$. The polarization vector is subject to gauge invariance $\epsilon_{\mu} \to \epsilon_{\mu} + \omega p_{\mu}$ for constant ω. Up to a gauge transformation the constraint which is equivalent to the Lorentz gauge condition, is still satisfied.

In the case of a lightlike vector with momentum $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$ the polarization vectors can be written as [6]:

$$\epsilon^{+}_{\alpha\dot{\alpha}} = \frac{\mu_{\alpha}\tilde{\lambda}_{\dot{\alpha}}}{\langle \mu\lambda \rangle}, \qquad \epsilon^{-}_{\alpha\dot{\alpha}} = \frac{\lambda_{\alpha}\tilde{\mu}_{\dot{\alpha}}}{[\tilde{\mu}\tilde{\lambda}]} \,. \tag{2.1.29}$$

These obey the constraint $p_\mu \epsilon^\mu = 0$ since $\langle \lambda\lambda \rangle = 0$. The spinors μ and $\tilde{\mu}$ are arbitrary negative and positive chirality spinors respectively and the vectors ϵ^μ do not depend on them up to a gauge transformation. As long as gauge invariant quantities are being considered there is a freedom in choosing any reference spinor μ (or $\tilde{\mu}$). In fact, a clever choice of reference spinor can make many terms and diagrams vanish [17]. The structure of the polarization vectors also indicates how they rescale under the rescaling of the spinors (2.1.24):

$$\epsilon^-_{\alpha\dot\alpha} \to z^{+2}\epsilon^-_{\alpha\dot\alpha}, \qquad \epsilon^+_{\alpha\dot\alpha} \to z^{-2}\epsilon^+_{\alpha\dot\alpha}. \tag{2.1.30}$$

Finally, the following identities are simply presented, which are extremely useful when performing calculations with scattering amplitudes and can be proved using standard Dirac algebra [17]:

$$\langle p|\gamma^\mu|q]\langle r|\gamma_\mu|s] = -2\langle pr\rangle[qs] \qquad \text{(Fierz rearrangement)} \tag{2.1.31}$$

$$\langle pq\rangle\langle rs\rangle = \langle pr\rangle\langle qs\rangle + \langle ps\rangle\langle rq\rangle \qquad \text{(Schouten identity)}. \tag{2.1.32}$$

2.1.3 MHV amplitudes

This section presents the form of amplitudes as those emerge when complex momentum is considered. Interestingly, real momenta would deliver a zero but the formula presented by Parke and Taylor [3] yields a non-zero result for complex momenta. All the work done in the previous section allows us to unambiguously label each particle state by its momentum p_i, helicity h_i and color index a_i, as $|p_i, h_i, a_i\rangle$. The color decomposition of the amplitudes further reduces the explicit dependence of the amplitude to less external data. Remarkably, the idea can be summarized in the following relations (for $n > 3$):

$$\mathcal{A}(1^+, \ldots, n^+) = 0,$$

$$\mathcal{A}(1^+, \ldots, i^-, \ldots, n^+) = 0,$$

$$\mathcal{A}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \tag{2.1.33}$$

with ... denoting any number of positive helicity gluons. The last amplitude is known as "Maximally Helicity Violating" (MHV) amplitude.
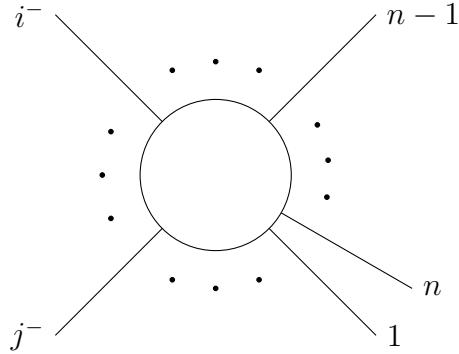
Figure 2.1   Generic MHV amplitude where only two particles have negative helicity.

The amplitudes with two negative helicity gluons are referred to as mostly plus MHV amplitudes or simply MHV and by applying parity we obtain the mostly plus MHV amplitudes or anti-MHV ($\overline{\text{MHV}}$) amplitudes and we write

$$\mathcal{A}(1^-, \ldots, i^+, \ldots, j^+, \ldots, n^-) = (-1)^n \, i \frac{[ij]^4}{[12][23] \ldots [n1]} \, . \tag{2.1.34}$$

The case for $n = 3$ is quite exceptional. Since we have $p_1 + p_2 + p_3 = 0$ and $p_i^2 = 0$, $\forall \, i$, it is true that $(p_1 + p_2)^2/2 = p_1 \cdot p_2 = p_3^2/2 = 0$. When considering a Lorentz signature, $p_i \cdot p_j = 0$ implies that all momenta are collinear and as a consequence the amplitude vanishes. Interestingly, when considering other signatures or simply complex momenta then the three point tree level amplitude is no longer a degenerate case. Thus, it is possible to write $2 \, p_i \cdot p_j = \langle \lambda_i \lambda_j \rangle [\tilde{\lambda}_i \tilde{\lambda}_j]$ and for each pair either $\langle \lambda_i \lambda_j \rangle = 0$ or $[\tilde{\lambda}_i \tilde{\lambda}_j] = 0$. Each condition is in effect a proportionality condition and at least one of them must be satisfied. This means that either all λ_i or all $\tilde{\lambda}_i$ are proportional. Therefore either the MHV or the $\overline{\text{MHV}}$ amplitudes are zero. The three-point amplitude is then given by

$$\mathcal{A}_3(1^-, 2^-, 3^+) = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \, . \tag{2.1.35}$$

In Section 2.3 a recursive relation is presented, which prescribes a possible way to construct an arbitrarily complicated tree level amplitude. The main building block will turn out to be exactly the three-point MHV (or anti-$\overline{\text{MHV}}$) amplitude.

In full generality, the Parke Taylor formula should be complemented with some more information about the amplitudes. First of all there is a prefactor of the coupling constant and there should also be a momentum conservation δ-function. The complete expression for a physical tree-level MHV amplitude with external particles r and s having negative helicity has the form [6]:

$$\mathcal{A}_n = g^{n-2}(2\pi)^4 \delta^4 \left(\sum_i^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha} \right) \frac{\langle \lambda_r \lambda_s \rangle^4}{\prod_{i=1}^n \langle \lambda_i \lambda_{i+1} \rangle} . \tag{2.1.36}$$

This expression of the amplitude is manifestly Lorentz invariant as it depends only Lorentz scalars and momentum is automatically conserved by the δ-function. Since the amplitude is linear in each polarization vector ϵ_j, it obeys the auxiliary condition for each j [6, 22]:

$$\left(\lambda_j^\alpha \frac{\partial}{\partial \lambda_j^\alpha} - \tilde{\lambda}_j^{\dot\alpha} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot\alpha}} \right) \mathcal{A}(\lambda_j, \tilde{\lambda}_j, h_j) = -2h_j \mathcal{A}(\lambda_j, \tilde{\lambda}_j, h_j) , \tag{2.1.37}$$

which is a consequence of the scaling of polarization vectors (2.1.30). This condition, along with Lorentz invariance fixes the structure of the amplitudes. It is also straightforward to check that for the massless case, the amplitude is exactly conformally invariant once the generators are expressed using spinors [6].

2.2 Superamplitudes in $\mathcal{N} = 4$ super Yang-Mills

Theories with maximal supersymmetry provide an arena for testing new ideas and developing computational tools. Due to the extended supersymmetry, and the constraints that it leads to, several steps of otherwise cumbersome calculations tend to simplify. Furthermore, the $\mathcal{N} = 4$ super Yang-Mills theory (sYM) is exceptional since it provides the first concrete example of a super-conformal theory that admits a holographic dual [23]. As a result, this theory has been extensively studied and

many of the tools and methods now employed for amplitudes calculations were first developed or refined within the context of $\mathcal{N}=4$ sYM.

The $\mathcal{N}=4$ on-shell supermultiplet is a convenient and compact way to represent all the particle content of the theory in a single object and applies for all the on-shell states. This is practically useful, as in such a setup the particle content is the largest possible compared to other four-dimensional supersymmetric theories. It will serve as a useful bookkeeping device of all the possible scattering amplitudes.

To this end one introduces Grassmann variables η^A (where $A \in \{1, \ldots, 4\}$), which belong to the fundamental representation of the R-symmetry group $\mathsf{SU}(4)$. The super-wavefunction $\Phi(p, \eta)$ has the following form [22]:

$$\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A \eta^B S_{AB}(p) + \frac{1}{3!}\eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p)$$
$$+ \frac{1}{4!}\eta^A \eta^B \eta^C \eta^D G^-(p) . \quad (2.2.1)$$

The first term of the super-wavefunction (2.2.1) has helicity $+1$ with the obvious notation. Therefore the entire right-hand side must have the same total helicity, whereas each state has decreasing helicity by a step of $-1/2$ so that G^- has helicity -1. Consequently, the variables η^A must carry a helicity of $+1/2$ to balance the total helicity of each term. The upper indices correspond to the fundamental representation of the R-symmetry group whereas the lower indices to the anti-fundamental.

The algebra for the supersymmetry generators and massless momentum $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ becomes [22]

$$\left\{ q_\alpha{}^A, \bar{q}_{\dot{\alpha}B} \right\} = \delta^A{}_B \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} . \quad (2.2.2)$$

As usual A, B are the $\mathsf{SU}(4)$ R-symmetry indices and as before α, $\dot{\alpha}$ are the $\mathsf{SU}(2)$ spinor indices in four dimensions. It is possible to decompose the supercharge q_α^A along two independent directions λ and μ in spinor space:

$$q_\alpha{}^A = \lambda_\alpha q_{(1)}{}^A + \mu_\alpha q_{(2)}{}^A \quad (2.2.3)$$

provided that $\langle \lambda \mu \rangle \neq 0$. It is also possible to provide an analogous decomposition for $\bar{q}_{\dot{\alpha}A}$. Using the above decomposition to rewrite the non-trivial anticommutator of the supersymmetry algebra (2.2.2) one concludes that the charges $q_{(2)}$ and $\tilde{q}_{(2)}$ anticommute among themselves and the other generators in this massless representation. They can therefore be set to zero. Thus, the supersymmetry algebra simplifies to

$$\left\{ q_{(1)}{}^A, \bar{q}_{(1)B} \right\} = \delta^A{}_B \, . \tag{2.2.4}$$

This is just a Clifford algebra for a space of $2\mathcal{N}$ dimensions and signature $(\mathcal{N}, \mathcal{N})$. Now we can drop the lower indices and realize the algebra in terms of the Grassmann variables

$$q^A = \eta^A \, , \qquad \bar{q}_B = \frac{\partial}{\partial \eta^B} \, . \tag{2.2.5}$$

Using this, the supercharges can now be written as $q_\alpha{}^A = \lambda_\alpha \eta^A$ and $\bar{q}_{\dot{\alpha}A} = \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^A}$ [22]. In these conventions the Grassmann variables η^A carry helicity of $+1/2$ and similarly $\bar{\eta}_A$ have helicity $-1/2$, which agree with the helicities construed from the structure of super-wavefunction (2.2.1) This representation of the algebra is *chiral*. It is completely equivalent to choose an *anti-chiral* representation, in which case the roles of q and $\bar{q}$ with respect to the Grassmann variables are interchanged.

The irreducible representations of such Clifford algebras are already known. Starting with the *Clifford vacuum* $|\Omega\rangle$ which satisfies [24]

$$q^A |\Omega\rangle = 0 \, , \qquad \forall \, A \in \{1, \dots, \mathcal{N}\} \, , \tag{2.2.6}$$

we can repeatedly act with $\bar{q}_A$. The Clifford vacuum carries quantum numbers corresponding to the momentum p and the helicity h, i.e. $|\Omega\rangle = |p, h\rangle$. It is now straightforward to understand and deduce the matter content of the $\mathcal{N} = 4$ supermultiplet (2.2.1) which is summarized in Table 2.1.

The superamplitude $\mathcal{A}_n$ for n external particles is defined as [22]

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \mathcal{A}_n \left(\Phi_1 \dots \Phi_n \right) \tag{2.2.7}$$

Particle type	States	Helicity	Multiplicity
Gauge Bosons	G^+	$+1$	1
	G^-	-1	1
Fermions	Γ_A	$+1/2$	4
	$\bar{\Gamma}^A$	$-1/2$	4
Scalars	S_{AB}	0	6

Table 2.1 The matter content of $\mathcal{N} = 4$ sYM and the respective helicities and multiplicities. The scalars S_{AB} in particular are subject to the reality condition $S_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{S}^{CD}$. A similar presentation of the matter content can be found in [25].

The reasoning behind this is simple: the super-wavefunction $\Phi(p, \eta)$ combines all the particles of the $\mathcal{N} = 4$ sYM into a single object. It is therefore a suitable object to construct amplitudes of all kinds of processes. The expression (2.2.7) is rather formal but the dependence of the amplitude on the super-wavefunctions implies a dependence on the spinors λ_i and $\tilde{\lambda}_i$ as well as the Grassmann variables η_i of all the external states. In this sense the scattering amplitudes of different particle types appear as component amplitudes A_n of $\mathcal{A}_n$. The strategy is to expand the superamplitude in powers of the Grassmann variables. Each term/component represents a scattering process involving the external states that correspond to the powers of η. For instance the expansion will contain terms like:

$$\mathcal{A}_n = (\eta_1)^4 A_n \left(G^- G^+ \ldots G^+\right) + \ldots + (\eta_1)^4 (\eta_2)^4 (\eta_3)^4 A_n (G^- G^- G^- G^+ \ldots G^+)$$
$$+ \frac{1}{3!}(\eta_1)^4 \eta_2^A \eta_2^B \eta_2^C \eta_3^E \epsilon_{ABCD} A_n \left(G^- \bar{\Gamma}_2^D \Gamma_{3E} G^+ \ldots G^+\right) \quad (2.2.8)$$

where we used the shorthand $(\eta_i)^4 = \frac{1}{4!}\eta_i^A \eta_i^B \eta_i^C \eta_i^D \epsilon_{ABCD}$. In this case the first term is the gluon component amplitude with one negative helicity gauge boson and $n-1$ positive helicity gauge bosons. This amplitude is of course zero for $n > 3$ but formally it is still a subamplitude in the Grassmann variable expansion.

Furthermore, the generic superamplitude $\mathcal{A}_n$ needs to be invariant under both the R-symmetry group transformations and supersymmetry transformations. The $\mathsf{SU}(4)$ invariance requirement restricts the form of the expansion to an inhomogeneous polynomial of degree $4n$ in the η_i^A's. Practically, the supersymmetry invariance is

enforced as a requirement that the superamplitude vanishes when acted upon with an infinitesimal supersymmetry generator. This automatically restricts the form of the superamplitude even further.

When dealing with a scattering process with n external particles the we denote the total momentum $P_{\alpha\dot\alpha}$, and the supersymmetry generators (often called *super-momenta*) as

$$P_{\alpha\dot\alpha} = \sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot\alpha}, \qquad Q_\alpha{}^A = \sum_{i=1}^{n} \lambda_{i\alpha}\eta_i^A, \qquad \bar{Q}_{\dot\alpha A} = \sum_{i=1}^{n} \tilde{\lambda}_{i\dot\alpha}\frac{\partial}{\partial\eta_i^A}. \qquad (2.2.9)$$

Supersymmetry invariance under $q_\alpha{}^A$ is translated to supermomentum conservation:

$$Q_\alpha{}^A \mathcal{A}_n = \sum_{i=1}^{n} \lambda_{i\alpha}\eta_i^A \mathcal{A}_n = 0. \qquad (2.2.10)$$

In the non-supersymmetric case, momentum conservation is implemented by a δ-function. Similarly, here, it can also be implemented by the introduction of a fermionic δ-function, hence the name *supermomentum*, defined as a product of the individual supercharge components [25],

$$\delta^{(8)}\left(Q_\alpha{}^A\right) = \prod_{A=1}^{4}\prod_{\alpha=1}^{2}\left(\sum_{i=1}^{n}\lambda_{i\alpha}\eta_i^A\right) \qquad (2.2.11)$$

Now the superamplitude takes a form which conveys more information and is manifestly invariant under $Q_\alpha{}^A$ supersymmetry transformations:

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}\left(P_{\alpha\dot\alpha}\right)\delta^{(8)}\left(Q_\alpha{}^A\right)\mathcal{P}_n(\lambda,\tilde{\lambda},\eta). \qquad (2.2.12)$$

Here, $\mathcal{P}_n$ is a polynomial in the Grassmann variables η_i. This generic form of the superamplitude is valid when $n \geq 4$. The three-point amplitude is exceptional again and demands complexification of momenta.

Requiring supersymmetry invariance of the superamplitude, actually restricts the structure of the polynomial, $\mathcal{P}_n$. Upon acting with the supersymmetry generators on the full superamplitude one gets

$$\bar{Q}_{\dot\alpha A}\mathcal{A}_n = \left(\sum_{i=1}^{n}\tilde{\lambda}_{i\dot\alpha}\frac{\partial}{\partial\eta_i^A}\delta^{(8)}\left(\lambda_{i\alpha}\eta_i^A\right)\right)\mathcal{P}_n + \delta^{(8)}(Q_\alpha{}^A)\left(\sum_{i=1}^{n}\tilde{\lambda}_{i\dot\alpha}\frac{\partial}{\partial\eta_i^A}\mathcal{P}_n\right) \qquad (2.2.13)$$

$$= 0. \qquad (2.2.14)$$

Momentum conservation automatically eliminates the first term whereas the second term imposes the constraint

$$\bar{Q}_{\dot{\alpha}A}\mathcal{P}_n = 0 \tag{2.2.15}$$

Given the fact that the amplitude $\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta)$ is a singlet under the $\mathsf{SU}(4)$ R-symmetry, the same must be true for the polynomial $\mathcal{P}_n$. A manifest way of ensuring that is true is by writing the polynomial as an expansion of terms that are of degree $4k$ in the Grassmann parameters η_i^A. This way, each term in the expansion is unavoidably invariant under the global $\mathsf{SU}(4)$. The expansion given in [22] is of the form

$$\mathcal{P}_n = \mathcal{P}_n^{(0)} + \mathcal{P}_n^{(4)} + \mathcal{P}_n^{(8)} + \cdots + \mathcal{P}_n^{(4n-16)}. \tag{2.2.16}$$

The upper indices of the terms in the expansion of the polynomial label the degree of Grassmann variables.

In order for the degree in η's to be determined it is necessary to consider the full structure of the superamplitude. Supersymmetry invariance has already introduced the fermionic δ-function of degree eight (8) in η's (but vanishing helicity). This is naturally the minimum degree in the fermionic variables of a superamplitude and accordingly corresponds to the first term of the expansion (2.2.16), $\mathcal{P}_n^{(0)}$ which unambiguously has zero degree in η. Recall, that when the super-wavefunction (2.2.1) was introduced in order to systematize the construction of any superamplitude, the helicity of η's was found to be $1/2$. It thus follows that $\mathcal{P}_n^{(0)}$ represents a subamplitude with a total helicity of $n - 4$. The next term then represents a NMHV amplitude with helicity $n - 6$ and so on. Thus, the last term organically corresponds to a $\overline{\text{MHV}}$ amplitude of total helicity $-(n - 4)$. The degree is $4n - 16$ in η as the degree of the amplitude has to be $4n - 8$ when the fermionic $\delta^{(8)}$-function is accounted for. Therefore the total degree of the superamplitude will be $4n - 8$ rather than $4n$ [22,25].

The fermionic degree of the polynomial is however not enough to fully resolve the form of each term in its expansion. Focusing on the MHV case, we should be able to exactly deduce the structure of the term. After the necessary integration over the Grassmann variables, the well known MHV tree-level amplitude should

surface. In full generality, simply by choosing only two gauge bosons (i and j) to be of negative helicity the factors $(\eta_i)^4$ and $(\eta_j)^4$ need to be extracted from the fermionic $\delta^{(8)}$-function, as clearly $\mathcal{P}_n^{(0)}$ simply cannot contribute any powers of η. The Grassmannian integrations will introduce a factor of $\langle ij \rangle^4$. Then, one simply has to compare with the known result of the Parke-Taylor (2.1.33) formula to extract

$$\mathcal{P}_n^{(0)} = (\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle)^{-1}, \tag{2.2.17}$$

which suggests that the MHV, n-point superamplitude for $n \geq 4$ is given by

$$\mathcal{A}_n^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)} \left(\sum_{i=1}^{n} \lambda_{i\alpha} \tilde{\lambda}_{i\dot\alpha} \right) \frac{\delta^{(8)} \left(\sum_{i=1}^{n} \lambda_{i\alpha} \eta_i^A \right)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \tag{2.2.18}$$

a result first presented by Nair [26] in this form.

However, as will become evident in the following discussions about recursion relations and unitarity, the above formula cannot provide the fundamental building block for loop calculations. It will be necessary to deal with three-point superamplitudes which so far have not been determined.

The kinematic constraints for the tree-level amplitudes lead to vanishing three-point (super)amplitudes for real momenta (see Section 2.1.3). Switching to complex momenta, the solutions lead to either a MHV or an $\overline{\text{MHV}}$ amplitude. By choosing $[ij] = 0$, the MHV $n = 3$ superamplitude takes the form:

$$\mathcal{A}_3^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)} \left(\sum_{i=1}^{3} \lambda_{i\alpha} \tilde{\lambda}_{i\dot\alpha} \right) \frac{\delta^{(8)} \left(\sum_{i=1}^{3} \lambda_{i\alpha} \eta_i^A \right)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \tag{2.2.19}$$

which is purely holomorphic in spinor variables and follows the form of the general case.

However, when choosing the product $\langle ij \rangle = 0$ to vanish, the result should be the $n = 3$ $\overline{\text{MHV}}$ superamplitude. The reasoning of the n-point case would suggest that it should be given by the last term in the expansion (2.2.16) of the polynomial $\mathcal{P}_n$. This would lead to the term $\mathcal{P}_3^{(-4)}$ of negative degree in η. For the $n = 3$ case the $\overline{\text{MHV}}$ is given by [2, 27]

$$\mathcal{A}_3^{\overline{\text{MHV}}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)} \left(\sum_{i=1}^{3} \lambda_{i\alpha} \tilde{\lambda}_{i\dot\alpha} \right) \frac{\delta^{(4)} \left(\eta_1 [23] + \eta_2 [31] + \eta_3 [12] \right)}{[12][23][31]}. \tag{2.2.20}$$

It is worth mentioning that now there is no fermionic $\delta^{(8)}$-function of the previous form which imposed supersymmetry invariance. The current fermionic $\delta^{(4)}$-function produces the correct total degree of η of a $\overline{\text{MHV}}$ amplitude which is $4n - 8$. This expression does nonetheless respect supersymmetry. In order to see this use the fermionic δ-function of (2.2.20) to solve for the fermionic variable η_1

$$\eta_1^A \lambda_{1\alpha} = \frac{-\eta_2^A[31] - \eta_3^A[12]}{[23]} \lambda_{1\alpha} \, . \tag{2.2.21}$$

These constraints affect the supersymmetry generator $Q_\alpha{}^A$ which now takes the form

$$Q_\alpha{}^A = \sum_{i=3}^{3} \lambda_{i\alpha} \eta_i^A = \eta_2^A \frac{\lambda_{1\alpha}[13] + \lambda_{2\alpha}[23]}{[23]} + \eta_3^A \frac{\lambda_{1\alpha}[21] + \lambda_{3\alpha}[23]}{[23]} \, . \tag{2.2.22}$$

Fortunately, due to momentum conservation enforced by the corresponding δ-function the supersymmetry generator vanishes and thus the $\overline{\text{MHV}}$ is invariant. In a similar fashion the $\bar{Q}_{A\dot\alpha}$ supersymmetry generator produces the following expression when acting on $\mathcal{A}_3^{\overline{\text{MHV}}}$

$$\bar{Q}_{A\dot\alpha} \mathcal{A}_3^{\overline{\text{MHV}}} \rightarrow \sum_{i=3}^{3} \tilde{\lambda}_i \frac{\partial}{\partial \eta_i} \left(\eta_1[23] + \eta_2[31] + \eta_3[12] \right) = \tilde{\lambda}_1[23] + \tilde{\lambda}_2[31] + \tilde{\lambda}_3[12] \tag{2.2.23}$$

which now vanishes due to the Schouten identity (2.1.32). Hence, both the MHV and $\overline{\text{MHV}}$ three-point superamplitudes as given above, are supersymmetry invariant in the $\mathcal{N} = 4$ case. It is now possible to use these expressions as fundamental blocks, either when trying to determine a tree amplitude recursively or when unitarity methods are applied for loop corrections.

2.3 Recursion relation - BCFW

In 2004 Britto, Cachazo, Feng [28] introduced, and then proved with Witten [29], a modus operandi for recursively calculating multi-gluon tree-level amplitudes. The aim of their procedure is to reproduce any tree-level amplitude as a sum over simpler terms. These terms are constructed from products of amplitudes of fewer particles 'connected' by a Feynman propagator. This factorization procedure can be reiterated until the amplitude is reduced to calculating three-point amplitudes connected

by the propagators. The nature of this procedure is such that a complex number is introduced as a parameter, making the use of Cauchy's theorem on Yang-Mills amplitudes possible. It is worth mentioning here that this work in other theories that are not restricted to four dimensions as this factorization is a general feature of scattering amplitudes.

The main idea behind the recursion relation is to shift the null momenta p_i and p_j of two external particles in an amplitude by shifting their spinors with a complex parameter z [29],

$$\lambda_j \to \lambda_j - z\lambda_i, \qquad \tilde{\lambda}_i \to \tilde{\lambda}_i + z\tilde{\lambda}_j. \tag{2.3.1}$$

This makes the amplitude a rational function of z and we write it as $\mathcal{A}(z)$. The particles i and j may have any helicity and respect momentum conservation for complex momenta:

$$p_i' + p_j' = \lambda_i \tilde{\lambda}_i' + \lambda_j' \tilde{\lambda}_j = \lambda_i \tilde{\lambda}_i + z\lambda_i \tilde{\lambda}_j + \lambda_j \tilde{\lambda}_j - z\lambda_i \tilde{\lambda}_j = p_i + p_j \tag{2.3.2}$$

such that $\mathcal{A}(z)$ is a well defined amplitude in $\mathbb{C}^4$. Therefore, Cauchy's theorem can be directly applied , and by assuming for now that $\mathcal{A}(z) \overset{z\to\infty}{\longrightarrow} 0$:

$$0 = \frac{1}{2\pi i} \oint_{C_\infty} \frac{dz}{z} \mathcal{A}(z) = \mathcal{A}(0) + \sum_{\text{poles } z_p} \frac{\text{Res}_p \mathcal{A}(z)}{z_p}. \tag{2.3.3}$$

The poles of $\mathcal{A}(z)$ appear exactly when a Feynman propagator of internal momenta goes on-shell and the residue is given by

$$\lim_{P_{k,m}^2 \to 0} \left[P_{k,m}^2 \mathcal{A} \right] = \sum_{h=\pm 1} \mathcal{A}(k, \ldots, m, -P_{k,m}^{-h}) \mathcal{A}(P_{k,m}^h, m+1, \ldots, k-1). \tag{2.3.4}$$

This splits the amplitude into the two physical subamplitudes. For $i \in \{k, \ldots, m\}$ and $j \notin \{k, \ldots, m\}$ (or vice versa) the propagator will be shifted and there will be a pole in z because

$$P_{k,m}'^2(z) = \left(P_{k,m} + z\lambda_i \tilde{\lambda}_j \right)^2 = P_{k,m}^2 + z\langle i | P_{k,m} | j] \tag{2.3.5}$$

which yields

$$z_{k,m} = -\frac{P_{k,m}^2}{\langle i | P_{k,m} | j]}. \tag{2.3.6}$$

Then the residue for this pole is

$$\lim_{z \to z_{k,m}} (z\mathcal{A}) = \frac{1}{\langle i|P_{k,m}|j]} \lim_{P^2_{k,m} \to 0} \left(P^2_{k,m}\mathcal{A}\right)$$

$$= \sum_{h=\pm 1} \frac{\mathcal{A}(k,\ldots,m,-P^{-h}_{k,m}(z_{k,m}))\mathcal{A}(P^h_{k,m}(z_{k,m}),m+1,\ldots,k-1)}{\langle i|P_{k,m}|j]},$$

$$(2.3.7)$$

which, combined with (2.3.3) gives

$$\mathcal{A} = \sum_{\substack{h=\pm 1 \\ k,m \in \mathcal{P}}} \frac{\mathcal{A}(k,\ldots,m,-P^{-h}_{k,m}(z_{k,m}))\mathcal{A}(P^h_{k,m}(z_{k,m}),m+1,\ldots,k-1)}{P^2_{k,m}}, \qquad (2.3.8)$$

where $\mathcal{P}$ is the set of all partitions such that $k \in \{j+1,\ldots,i\}$, $m \in \{i,\ldots,j-1\}$ and $k \neq m, m+2$.
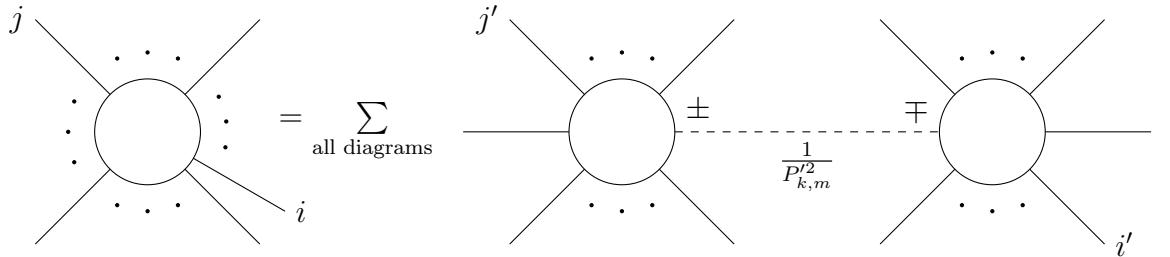
Figure 2.2   Representation of the BCFW equation (2.3.8).

In other words, we must sum over all internal momenta that are affected by the deformation (2.3.1). In [29] it is proved that $\mathcal{A}(z)$ is indeed well behaved (i.e. goes to zero) as $z \to \infty$.

These relations provide powerful tools for calculations due to their recursive nature. Using the basic amplitude in any theory (in the case of Yang-Mills the three point amplitude) it is possible to derive any n-point amplitude.

Given the computational power that comes with the recursion relation for amplitudes in Yang-Mills theory it seems as a rather appealing endeavor to try and develop similar techniques for other theories as well. Of particular interest of course is the maximally supersymmetric sYM in four dimensions. Indeed, in [27] and then [2] the

supersymmetric version of the recursion relation (2.3.8) is discussed in a systematic way. What follows is a brief description of the construction.

Similarly to the non-supersymmetric case the method is based in shifting two external momenta by a complex variable z according to

$$\lambda_j \to \lambda_j - z\lambda_i, \qquad \tilde{\lambda}_i \to \tilde{\lambda}_i + z\tilde{\lambda}_j, \tag{2.3.9}$$

which as discussed earlier respects momentum conservation. The supersymmetric Yang-Mills of course also requires some treatment of the supermomentum. The shift of the holomorphic spinor λ_j induces the following change in the supermomentum

$$q_j \to q_j - z\eta_j\lambda_i. \tag{2.3.10}$$

The Grassmannian spinor with label i shifts according to

$$\eta_i \to \eta_i + z\eta_j, \tag{2.3.11}$$

automatically preserving momentum conservation. The remaining steps of the construction of the recursion relation are essentially unchanged. The shifted tree-level superamplitude is again a rational function of the spinor variables and now a polynomial in the Grassmann parameters η_i^A. The superamplitude $\mathcal{A}(z)$ is an analytic function in z and once more only contains poles but no cuts over the complex plane. This renders the amplitudes suitable for the application of Cauchy's theorem.

The supersymmetric version of the recursion relation is

$$\mathcal{A}_n = \sum_{\text{poles } z_P} \int d^4\eta_{\hat{P}} \, \mathcal{A}_L(z_P) \frac{i}{P^2} \mathcal{A}_R(z_P). \tag{2.3.12}$$

The sum is over all simple poles and both subamplitudes are evaluated at these poles. The interesting difference compared to the non-supersymmetric version of BCFW is the assignment of helicities. The sum over all helicity configurations of the subamplitudes in (2.3.8) is now replaced by the integration over the Grassmann variable $\eta_{\hat{P}}$ which is assigned to the internal propagator. Physically this stems from the fact that the superamplitude already contains all possible particle and helicity configurations for a given number of external states.

Since the subamplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ are themselves supersymmetric they can be expanded in terms of the fermionic variables η_i just as in the expression (2.2.12). The Grassmann integration reduces the total power of η's on the RHS of the supersymmetric version of BCFW (2.3.12) by four. However, the helicity assignments of the full superamplitude is constrained by the fact that the superamplitude is completely determined and defined by the number of external states and its total helicity. Therefore, the sum of the total helicities of the two subamplitudes reduced by four should equal the total helicity of the recursively determined superamplitude $\mathcal{A}_n$.

There is one important point that is a requirement in the derivation and validity of the any version of the BCFW recursion relation that has not been carefully discussed so far. That is the vanishing of the (super)amplitude when $z \to \infty$, i.e. $\mathcal{A}(z) \xrightarrow{z \to \infty} 0$. In the papers that proposed and proved the recursion relation [28,29] the checks were on specific helicity configurations. However, an amplitude with an arbitrary helicity assignment will not generally vanish for large complex momenta. In fact, in [30] it is shown that the pure Yang-Mills amplitudes only vanish for large z under specific shifts. In particular, a choice of both the holomorphic (λ_j) and anti-holomorphic $(\tilde{\lambda}_i)$ spinors to be of positive helicities then the amplitude scales as $\propto 1/z$ and the entire bosonic amplitude vanishes at $z \to \infty$.

In a supersymmetric theory however a distinction between different helicity assignments for the superamplitudes is no longer necessary. Indeed in the original papers that discussed the behavior of superamplitudes in large complex momenta [2,27] it was shown that both in maximal sYM and maximal supergravity the amplitudes vanish for $z \to \infty$. Specifically, the superamplitudes in $\mathcal{N} = 4$ sYM scale as $\mathcal{A}(z) \propto 1/z$ and in $\mathcal{N} = 8$ supergravity as $\mathcal{M}(z) \propto 1/z^2$.

2.4 Unitarity

The next step in calculations in a quantum field theory is to consider quantum corrections to scattering processes. These manifest themselves as loop corrections

which we have ignored so far. In this section, on-shell methods which are related to the tree-level constructions will be considered. In particular, we will see how it is possible to reconstruct a scattering amplitude from its properties as a function over the complex plane.

2.4.1 The Optical Theorem

In any interacting quantum (field) theory a fundamental physical requirement is that probability must be conserved. This means that the S-matrix describing a scattering process must be unitary. The scattering amplitudes are interpreted as the transition matrix elements and therefore unitarity can be directly applied at the amplitude level. This leads to the *optical theorem* which mathematically relates the imaginary part of the amplitude to a sum over contributions from all intermediate particle states. Comprehensive treatment of this subject is offered in many quantum field theory textbooks (this section loosely follows [18]). For a more general discussion about analyticity in physics [31] is a rather pedagogical reference.

It is possible to "split" the S-matrix into a part that is physically uninteresting and represents the possibility of no interaction occurring and a part that captures all the details of possible interactions. This can be achieved simply by writing $S = 1 + iT$. Then, unitarity of the S-matrix implies the following for the part that captures interactions T,

$$S^\dagger S = \mathbb{1} \qquad \Leftrightarrow \qquad -i(T - T^\dagger) = T^\dagger T \,. \qquad (2.4.1)$$

Now, by considering the transition matrix elements between initial and final states the left hand side of (2.4.1) is twice the imaginary part of the T-matrix element. The T-matrix elements should represent the transition from an initial to a final particle state. By inserting a complete set of on-shell multiparticle states ($\sum_f |f\rangle\langle f| = 1$) in the right hand side, the optical theorem assumes the form

$$-i[A(a \to b) - A^*(b \to a)] = \sum_f \int d\text{LIPS}[A^*(b \to f)A(a \to f)] \,, \qquad (2.4.2)$$

where dLIPS is the n-particle Lorentz invariant phase space measure,

$$d\text{LIPS} = \prod_{i=1}^{n} \frac{d^4 q_i}{(2\pi)^4} \delta^{(+)}(q_i^2 - m_i^2)(2\pi)^4 \delta^{(4)}\left(p_a + p_b - \sum_i q_i\right) \qquad (2.4.3)$$

and the sum on the RHS runs over all possible intermediate states f.

The optical theorem in the form (2.4.2) states that the imaginary part of the amplitude can be obtained from the sum of phase space integrals of intermediate multi-particle states. The integrals are over products of scattering amplitudes which individually transform either the initial or the final particle states into the intermediate states. This may simplify the calculation of a scattering amplitude in the following way. It is possible to expand both sides in perturbation theory and then match the powers of the coupling constant. In the case of the full one-loop amplitude the RHS of (2.4.2) simplifies to the integrated product of two tree-level amplitudes.

The Feynman diagrams of a theory provide a better understanding of the imaginary part of an amplitude. As the Feynman rules dictate for the theory at hand, each diagrammatic contribution to the S-matrix element is purely real unless an internal propagator goes on shell. If this is the case, the $i\epsilon$ prescription for the virtual particle propagator generates an imaginary part. In order to demonstrate how this might appear, it is possible to allow for the amplitude $A(s)$ an analytic function of the variable $s = E_{\text{cm}}^2$, although s is physically a real variable. Now, let s_0 be the threshold energy for production of the lightest multiparticle state. If s is real and below s_0 the intermediate state cannot go on-shell, therefore $A(s)$ is real. Consequently, for real $s < s_0$ the following identity is true

$$A(s) = [A(s^*)]^*. \qquad (2.4.4)$$

Since each side of this equation is an analytic function of s it can be analytically continued to the entire complex s plane. Considering the real and imaginary parts of $A(s)$ separately for $s > s_0$ near the real axis one gets

$$\text{Re}\, A(s + i\epsilon) = \text{Re}\, A(s - i\epsilon), \qquad (2.4.5)$$

$$\text{Im}\, A(s + i\epsilon) = -\text{Im}\, A(s - i\epsilon). \qquad (2.4.6)$$

The key observation here is that the imaginary part of the amplitude is different above and below the real axis., having a branch cut which starts at the threshold energy s_0. The discontinuity of the analytic function $A(s)$ is given by

$$\mathrm{Disc}\, A(s) = 2i \,\mathrm{Im}\, A(s + i\epsilon)\,. \tag{2.4.7}$$

This demonstrates that the appearance of an imaginary part of $A(s)$ always requires a branch cut singularity. The $i\epsilon$ prescription indicates that the physical scattering amplitudes should be evaluated above the cut, i.e. at $s + i\epsilon$. The optical theorem then shows that the discontinuity of an amplitude is related to the sum of phase space integrals as in (2.4.2).

The entire procedure of calculating the discontinuity of an amplitude has been formalized in full generality by Cutkosky [32]. It is possible, by following the so-called Cutkosky rules to reconstruct the discontinuity. This has the form of a three-step procedure summarized as follows [18]:

1. Cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell.

2. For each cut, replace $1/(p^2 - m^2 + i\epsilon) \to -2\pi i \delta(p^2 - m^2)$ in each cut propagator, then perform the loop integrals.

3. Perfom the integration over the two-particle phase space integral.

This algorithm is the result of a series of papers [33–35] which finally lead to Cutkosky's generalization to multi-loop diagrams [32]. In the next section it will be argued that the cumbersome phase space integrals need not be performed and Feynman integrals can take their place.

2.4.2 Integral Basis

The optical theorem relates the discontinuity of an amplitude with a sum of phase-space integrals. The problem is that, in general, phase-space integrals can result

in rather involved calculations. There is however a way to avoid them using the approach of Bern, Dixon, Dunbar and Kosower (BDDK) [36, 37]. Following this procedure, unitarity can be applied directly at the level of amplitudes and completely bypass the use of Feynman diagrams. Now, instead of integrating over phase-space, the two δ-functions that force loop momenta to be on shell (cut) are replaced by propagators. This *reconstructs* Feynman integrals instead of producing phase-space integrals. This method, known as unitarity at the integrand level, allowed the authors of [36, 37] to construct many one-loop amplitudes in supersymmetric theories, including the n-point MHV amplitude for $\mathcal{N} = 1, 4$ sYM theories.

The analysis suggested by BDDK [36, 37] allows the identification of which integral functions can appear in the one-loop amplitude. In general, at one loop, all amplitudes in massless gauge theories can be written using a basis of certain integral functions

$$A_n^{\text{1-loop}} = \sum_{\mathcal{I}_i} c_i \mathcal{I}_i + \text{rational terms} \,. \tag{2.4.8}$$

The summation is over scalar integral-functions $\mathcal{I}_i$ which, in $d = 4$ are often denoted as boxes, triangles or bubbles, depending on the number of vertices (four for a box, etc.). It is also possible to obtain integrals with products of loop momenta in the numerator leading to tensor integrals. A further classification of the integrals is based on the clusters of external momenta. This depends on whether external momenta associated with each vertex is massless or not. Thus the boxes without entirely massless external momenta are of four types. One can similarly distinguish between three-mass, two-mass and one-mass triangles.

For the full amplitude, the unitarity-cut approach requires careful consideration of double cuts for each kinematic channel. However, this procedure will only produce only the so-called *cut-constructible* part of the amplitude[§]. This is exactly the part of the full amplitude that contains the discontinuities like (poly)logarithms. All the cut-free terms are unaccounted for. Thus, the BDDK double-cuts are ideal for

---

[§]The supersymmetric Yang-Mills theories with $\mathcal{N} = 1, 4$ are cut constructible [36]. The criterion is that the degree of the loop-momentum in the numerator polynomial of an n-point one-loop amplitude must be less than n.

Box type	vertex configuration
four-mass box $\mathcal{I}^{4m}$	$K_i^2 \neq 0, \forall i$
three-mass box $\mathcal{I}^{3m}$	$K_i^2 = 0$ for one i
two-mass easy $\mathcal{I}^{2me}$	$K_i^2 = K_{i+2}^2 = 0$
two-mass hard $\mathcal{I}^{2mh}$	$K_i^2 = K_{i+1}^2 = 0$
one-mass $\mathcal{I}^{1m}$	$K_i^2 = 0$ for three i's

Table 2.2 Box integrals.

constructing the full amplitude for supersymmetric theories where all rational terms are uniquely linked to the terms with discontinuities [36,37]. In non-supersymmetric theories however, the rational terms, to order $\mathcal{O}(\epsilon^0)$, in the expansion (2.4.8) are not linked to terms with discontinuities. Thus, two-particle cuts cannot be easily used for non-supersymmetric theories.

(a) A box.   **(b)** A triangle.   **(c)** A bubble.
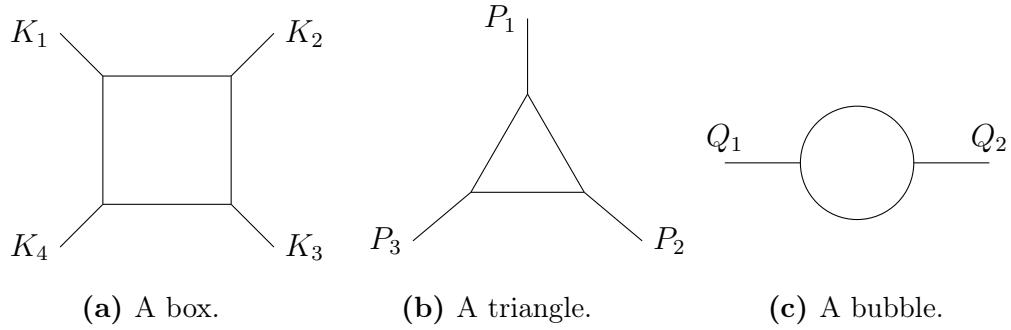
Figure 2.3 Integral functions that may appear in a massless gauge theory in four dimensions in a pictorial representation.

The aforementioned technical manipulations are in principle enough to compute amplitudes in various theories although the procedure requires tedious calculations. It is necessary to consider the cuts in all the different kinematic channels in order to obtain the full amplitude. Therein lies a non-trivial subtlety. The resulting integral function will have in principle, additional discontinuities of other channels. In other words, it is not straightforward to simply sum up all the contributions from the different channels to obtain the final expansion as this may lead to an over-counting

of some discontinuities. Moreover, upon the reconstruction of the Feynman integral, one is often left with tensor integrals. These require further manipulations in order to reduce them to scalar ones (such as the Passarino Veltman reduction [38]).

2.4.3 $d = 4$ Unitarity Cuts

The starting point for our considerations will be the fact that in $\mathcal{N} = 4$ super Yang-Mills theory, amplitudes contain only boxes [36, 37]

$$\mathcal{A}_{\mathrm{MHV}}^{\text{1-loop}}(i^-, j^-) = \sum_{a \in \text{boxes}} c_a I_a \,. \tag{2.4.9}$$

On both sides we can pick the discontinuity in P_{m_1, m_2}^2 by replacing the loop propagators between m_1 and $m_1 + 1$ and between $m_2 - 1$ and m_2 by delta functions in the square of the loop momenta. The left hand side of (2.4.9) produces an integral over the product of two tree amplitudes

$$-i\mathrm{Disc}\mathcal{A}_n^{\text{1-loop}} = \sum_{h, h'} \int d^D \ell_1 d^D \ell_2 \, \delta(\ell_1^2) \, \delta(\ell_2^2) \delta^{(D)}(\ell_1 - \ell_2 - P_{m_1, m_2}) \tag{2.4.10}$$

$$\times \mathcal{A}(m_2 + 1, \ldots, m_1 - 1, \ell_1^h, (-\ell_2)^{-h'}) \mathcal{A}(m_1, \ldots, m_2, \ell_2^{h'}, (-\ell_1)^{-h}) \tag{2.4.11}$$

as can be seen in Figure 2.4. To generate the Feynman integrals and avoid performing the phase space integrals one should replace the δ-functions associated with the cuts with propagators $\delta(\ell_i^2) \to 1/\ell_i^2$ in the above expression.

In order to have two non-vanishing tree amplitudes they must both be MHV. If i and j are both on the same side then $(h, h') = (1, -1)_{\mathrm{LHS}}$ or $(h, h') = (-1, 1)_{\mathrm{RHS}}$. If they are in the opposite sides, $h = h'$ can take any value that respects the structure of the $\mathcal{N} = 4$ supermultiplet and an extra summation is required. Interestingly, the summation over the $\mathcal{N} = 4$ multiplet gives the same result as if i and j are on the same side. Meaning that the integrand is written as

$$\mathcal{A}^{\text{tree}}(i^-, j^-) \frac{\langle (m_1 - 1)m_1 \rangle \langle \ell_1 \ell_2 \rangle \langle m_2(m_2 + 1) \rangle \langle \ell_2 \ell_1 \rangle}{\langle (m_1 - 1)\ell_1 \rangle \langle \ell_1 m_1 \rangle \langle m_2 \ell_2 \rangle \langle \ell_2(m_2 + 1) \rangle} \tag{2.4.12}$$
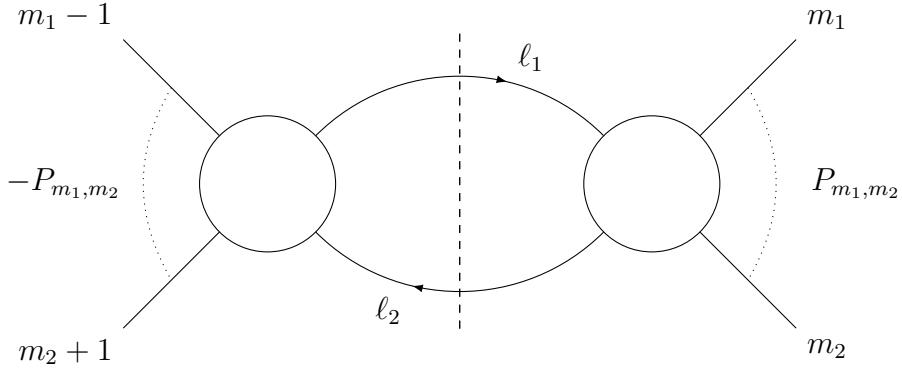
Figure 2.4 Double cut of the one-loop amplitude. Here, the dashed line is used to denote the cut of the two internal propagators.

The next step is to transform the denominator to scalar propagators by rewriting

$$\frac{1}{\langle \ell_i m_j \rangle} = \frac{[m_j \ell_i]}{2\ell_i \cdot m_j} = \frac{[m_j \ell_i]}{(\ell_i - m_j)^2} \, . \tag{2.4.13}$$

In Yang-Mills theory, a generic n-point amplitude may give an up to n-loop propagators and up to rank n tensor integrals

$$I_n[P(\ell^\mu)] = -i(4\pi)^2 \int \frac{d^D \ell}{(2\pi)^D} \frac{P(\ell^\mu)}{\ell^2 (\ell - K_1)^2 (\ell - K_1 - K_2)^2 \dots (\ell + K_n)^2} \, . \tag{2.4.14}$$

These integrals are usually reduced to a lower order tensor integral via a Passarino-Veltman reduction [38]. In the case of the $\mathcal{N} = 4$ sYM the box integrals can be reduced all the way to scalar integrals. An explicit result for the PV reduction of the linear pentagon integral that appears in six dimensions is given in Section 3.5.3 while more details are given in Appendix A.4.

2.4.4 Generalized Unitarity

In essence, the computation of a discontinuity across a cut in an amplitude has been computed by putting two propagators on shell. By sewing together two tree amplitudes to form the cut in a given channel only those integrals which have both propagators cut are obtained. This yields the coefficients only for those integrals. In order to obtain the full amplitude a careful consideration of cuts in all the other channels is required. However, if cuts on more propagators were allowed then fewer

terms would survive, and the calculation is immediately simplified by reducing loop amplitudes to tree amplitudes with fewer legs. This is what goes under the name of *generalized unitarity.* This procedure was first used in [12] to compute amplitudes in $\mathcal{N} = 4$ although the general idea was known since [39]. The best way to apply generalized unitarity is by having the right number of cuts as to isolate a single integral and a single coefficient.

It is natural to assume that there must be a better way to perform the calculation given that the final results are quite often of a simple form which is not obvious by the double-cut construction. Using the simple idea of cutting more propagators one finds a more direct way of calculating the coefficients in the integral basis expansion reducing the procedure to a purely algebraic excercise of manipulating tree amplitudes.

In the case of $\mathcal{N} = 4$ sYM where the only possible integrals are boxes, the quadruple cuts prove to be especially convenient. These cuts freeze the integrated momenta completely, reducing the calculation of the amplitude to a purely algebraic procedure. Moreover, it reduces the overlap between the different possible cuts and therefore simplifies the procedure of finding the integral coefficients. The extra on-shell conditions reduce the complexity of the PV reduction since fewer terms survive the procedure. The price for all this is the requirement of working with complex momenta. By cutting all loop momenta one is effectively sewing tree amplitudes. The tree-level amplitudes will often be three-point amplitudes (with the exception of the four-mass box $\mathcal{I}^{4m}$). On the one hand the number of tree amplitudes increases but especially the form of the three-point amplitude is suitably simple. As was discussed in Section 2.1.3 three-point amplitudes are non-zero when momentum is complexified.

2.4.5 Quadruple Cuts

The idea now is the simple, logical step of applying more cuts to *freeze* the loop momenta in four dimensions and obtain solutions. However, it is often the case that

real solutions are not allowed and the cut gives zero as a result thereby rendering the information useless. When turning to a $(- - ++)$ signature (or in complexified Minkowski space) this changes and the problem is indeed reduced to a purely algebraic one.

Considering a one-loop amplitude for which four internal lines are taken to go on-shell yields a box with a coefficient c_i is

$$c_i \int \frac{d^4 L}{(2\pi)^4} \frac{1}{(L^2 + i\epsilon)[(L + K_1)^2 + i\epsilon][(L + K_1 + K_2)^2 + i\epsilon][(L - K_4)^2 + i\epsilon]} \quad (2.4.15)$$

which can be cut to

$$c_i \sum_{\text{solutions}} \int d^4 L \, \delta(L^2) \, \delta[(L + K_1)^2] \, \delta[(L + K_2)^2] \, \delta[(L - K_4)^2]. \quad (2.4.16)$$

The momenta K_i are the clusters of momenta of Figure 2.3. The ϵ's are there to tame the infrared divergences that are inherent in the box integral. The clusters of momenta represent color-ordered momenta. In full generality, as already discussed the full one-loop amplitude may be decomposed in four different types of scalar box integrals. By adding the contributions from all the Feynman diagrams that cut the four internal lines the result is

$$\int d^4 L \sum_{\substack{\text{solutions,} \\ \text{states}}} \mathcal{A}(\ldots, -L, L + K_1)\mathcal{A}(\ldots, -L - K_1, L + K_1 + K_2) \quad (2.4.17)$$

$$\times \mathcal{A}(\ldots, -L - K_1 - K_2, L - K_4)\mathcal{A}(\ldots, -L + K_4, L) \quad (2.4.18)$$

$$\times \delta(L^2)\delta[(L + K_1)^2]\delta[(L + K_2)^2]\delta[(L - K_4)^2] \quad (2.4.19)$$

where by "states" we refer to the possibilities of different particles that may have those momenta, implying:

$$c_i = \frac{1}{2} \sum_{\substack{\text{solutions,} \\ \text{states}}} \mathcal{A}_A \mathcal{A}_B \mathcal{A}_C \mathcal{A}_D. \quad (2.4.20)$$

This means that the coefficient is given by products of four on-shell tree amplitudes in four dimensions. Physically, the quadruple cut can be depicted as a box with four tree amplitudes at the corners connected by loop propagators. Since this is the only

type of integral one may obtain, each quadruple cut singles out a unique box-function. Essentially the quadruple cut is just a specific choice of how the external states are distributed among the momenta clusters K_i. The four δ-functions have completely localized the integrand on the solutions of the on-shell conditions. It is then possible to remove the dependence of the loop momenta by algebraic manipulations and applying on-shell conditions. The remaining δ-functions can then be uplifted to the full Feynman propagators and produce the box integral function associated to the specific cut. The corresponding coefficient turns out to be simply the product of the tree-level amplitudes.

In more detail, the scalar box function must be purely kinematic object where all the helicity information of the external states is carried by the integral coefficients c_i. Thus, in the present setup, the coefficients are functions of the spinor variables λ_i and $\tilde{\lambda}_i$. The two relevant constraints that determine the coefficients are the on-shell conditions imposed by the cut and momentum conservation at each corner of the box. This reduces the loop integration to a discrete sum over two solutions:

$$\mathcal{S}_\pm : \qquad \ell_i^2 = 0 \,, \qquad\qquad \ell_{i+1}^\mu = \ell_i^\mu + K_i^\mu \,. \qquad (2.4.21)$$

The details of the explicit solutions can be found in [12]. Fortunately, in the application of quadruple cuts considered below the specific form of the solution is not needed. The constraints on momentum conservation are enough to simplify the loop momentum dependence at the level of the product of the tree-level amplitudes.

Consider the box coefficient of the one-loop four-point amplitude with a particular helicity configuration $\mathcal{A}_4^{\text{1-loop}}(1^-, 2^-, 3^+, 4^+)$ as depicted in Figure 2.5. The constraints imposed by cutting the loop momenta read:

$$\ell_1^2 = 0 \,, \qquad\qquad \ell_2^2 = (\ell_1 - p_1)^2 = 0 \,, \qquad (2.4.22)$$

$$\ell_3^2 = (\ell_1 - p_1 - p_2)^2 = 0 \,, \qquad\qquad \ell_4^2 = (\ell_1 + p_4)^2 = 0 \,. \qquad (2.4.23)$$

Now, take the loop momentum ℓ_1 to be of the form $\ell_1 = z\lambda_{\ell_1}\tilde{\lambda}_{\ell_1}$ where z is a constant to be fixed. Then by using the cut constraints one may choose a possible solution

for the loop momenta

$$\ell_1 = |4\rangle \frac{\langle 21 \rangle}{\langle 24 \rangle}[1|, \qquad \ell_2 = |2\rangle \frac{\langle 41 \rangle}{\langle 24 \rangle}[1|, \tag{2.4.24}$$

$$\ell_3 = |2\rangle \frac{\langle 43 \rangle}{\langle 42 \rangle}[3|, \qquad \ell_4 = |4\rangle \frac{\langle 23 \rangle}{\langle 42 \rangle}[3|. \tag{2.4.25}$$

Choosing the particular helicity configuration the upper left and the bottom right corners nest zero MHV amplitudes whereas the bottom left and upper right corners nest zero $\overline{\text{MHV}}$ amplitudes.
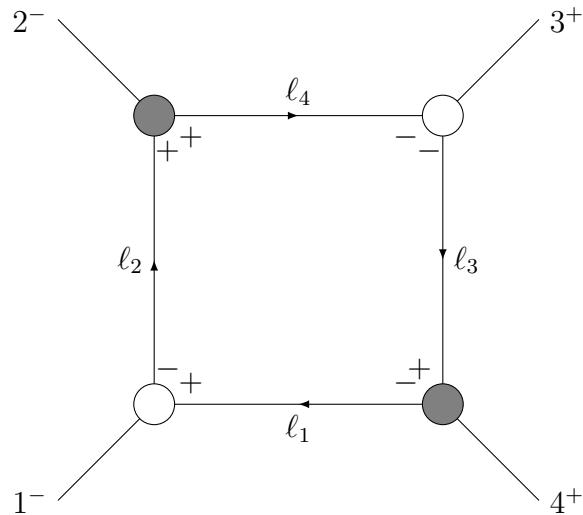

Figure 2.5 Four-point one-loop box. The darker corners represent $\overline{\text{MHV}}$ three-point subamplitudes.

A further comment is in order here. A complication arises in general when two adjacent amplitudes are both three-point amplitudes of the same type. The MHV and $\overline{\text{MHV}}$ amplitudes are defined for different kinematic configurations as discussed in Section 2.1.3. If they are both MHV then the kinematic configuration required is $\tilde{\lambda}_i \propto \tilde{\lambda}_\ell \propto \tilde{\lambda}_{i+1}$, or if they are both $\overline{\text{MHV}}$ it is necessary that $\lambda_i \propto \lambda_\ell \propto \lambda_{i+1}$. This leads to spinor products $[i, i+1] = 0$ or $\langle i, i+1 \rangle = 0$, which renders the invariant $s_{ii+1} = (p_i + p_{i+1})^2 = [i, i+1]\langle i, i+1 \rangle = 0$. Fortunately, for general kinematics[¶] this is never fulfilled. Therefore, this configuration cannot exist and only MHV $- \overline{\text{MHV}}$ pairs of three-point amplitudes can be appear in adjacent corners.

[¶]This is also true for the supersymmetric cases.

In the present case the non-vanishing arrangement of internal states is shown in Figure 2.5 which gives the contribution:

$$
\begin{aligned}
& \frac{1}{2} \frac{\langle 1\ell_2\rangle^3}{\langle \ell_2(-\ell_1)\rangle\langle(-\ell_1)\,1\rangle} \frac{[\ell_3(-\ell_2)]^3}{[(-\ell_2)2][2\,\ell_3]} \frac{\langle \ell_4(-\ell_3)\rangle^3}{\langle(-\ell_3)\,3\rangle\langle 3\,\ell_4\rangle} \frac{[(-\ell_4)\,4]^3}{[4\ell_1][\ell_1(-\ell_4)]} \\
= & \frac{1}{2} \frac{\langle 1\ell_2\rangle^3}{\langle 1\,\ell_1\rangle\langle \ell_1\ell_2\rangle} \frac{[\ell_2\ell_3]^3}{[2\,\ell_2][2\,\ell_3]} \frac{\langle \ell_3\ell_4\rangle^3}{\langle 3\,\ell_3\rangle\langle 3\,\ell_4\rangle} \frac{[4\,\ell_4]^3}{[\ell_4\ell_1][4\ell_1]} \\
= & \frac{1}{2} \frac{\langle 4\,1\rangle^2\langle 4\,3\rangle^2\langle 2\,3\rangle^2}{\langle 2\,4\rangle^6} \frac{\langle 1\,2\rangle^2}{\langle 1\,\ell_1\rangle\langle \ell_1\,2\rangle} \frac{[1\,3]^3}{[1\,2][2\,3]} \frac{\langle 2\,4\rangle^3}{\langle 2\,3\rangle\langle 2\,3\rangle} \frac{[3\,4]^3}{[3\,\ell_1][\ell_1 4]} \\
= & -\frac{1}{2} st\mathcal{A}^{\text{tree}},
\end{aligned}
$$

with s and t being the standard Mandelstam variables, and the last step occurs after repeated use of momentum conservation. Another solution can be obtained by exchanging $|\,\rangle \rightarrow |\,]$ and rearranging the helicities in the diagram such that $\mathcal{A}^{\text{1-loop}} = -st\mathcal{A}^{\text{tree}}I_4(1,2,3)$. Of course, in practice in the maximally supersymmetric one needs to sum over the full $\mathcal{N} = 4$ multiplet running in the loop and possibly more complicated amplitudes in the corners.

Generalized Unitarity for $\mathcal{N} = 4$ sYM

The first treatment of scattering amplitudes combining generalized unitarity and the superspace formalism was in [22, 27]. There the n-point MHV and NMHV super-amplitudes were calculated at one loop. Their results agreed with and confirmed previously obtained answers in [40, 41].

The anticipated expansion in the integral basis is

$$
\mathcal{A}_n^{\text{1-loop}} = \delta^{(4)}(P_{\alpha\dot\alpha}) \sum \left(c^{1m}\mathcal{I}^{1m} + c^{2\text{me}}\mathcal{I}^{2\text{me}} + c^{2\text{mh}}\mathcal{I}^{2\text{mh}} + c^{3m}\mathcal{I}^{3m} + c^{4m}\mathcal{I}^{4m} \right) .
\tag{2.4.26}
$$

Since now the focus is on the superamplitudes the (super)coefficients c^i consist of spinor variables and are also polynomials in the Grassmann variables η that "label" the different subamplitudes [22].

This approach takes advantage of the fact that the same integral functions appear as in the previous section, making the application of quadruple cuts rather

straightforward. Each cut singles out a specific box function and determines the cor-responding supercoefficient. The exceptional $n = 4$ case requires a single quadruple cut producing the zero-mass coefficient $c^{0\text{m}}$. This procedure determines the superco-efficients as a product of four tree-level superamplitudes.
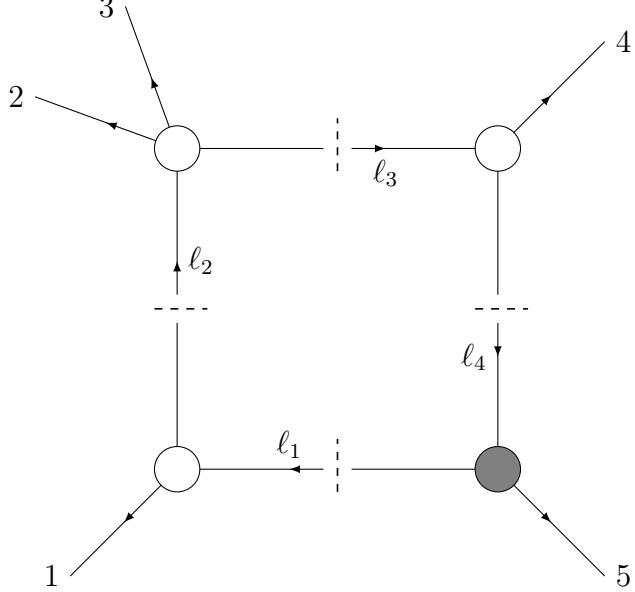

Figure 2.6   Five-point one-loop $\overline{\text{MHV}}$ superamplitude. Now there is only one gray corner that nests an $\overline{\text{MHV}}$ three-point amplitude whereas the remaining ones are all MHV.

In order to illustrate the procedure let us apply the method to the five-point $\overline{\text{MHV}}$ (or equivalently NMHV) one-loop superamplitude, employing Nair's formalism [26] and calculate the coefficient of one box integral. Firstly, it is useful to settle the kinematic conditions that are dictated by the delta functions that are imposed by the cut. Now, one cut solution is

$$\ell_1 = \lambda_{\ell_1}\tilde{\lambda}_1, \qquad \ell_2 = \lambda_{\ell_2}\tilde{\lambda}_1, \tag{2.4.27}$$

$$\ell_3 = \lambda_{\ell_3}\tilde{\lambda}_4, \qquad \ell_4 = \lambda_{\ell_4}\tilde{\lambda}_4. \tag{2.4.28}$$

Moreover, it is also true that $\ell_1 + p_5 = \ell_4$ and $\ell_3 + (p_2 + p_3) = \ell_2$ from momentum conservation. From now on the momenta of the second corner will be collectively referred to as $P = p_2 + p_3$ and $p_5 = \lambda_5\tilde{\lambda}_5 = Q$. The solutions now for this diagram

can be written as

$$\lambda_{\ell_1} = -\frac{Q|4]}{[14]}, \qquad \lambda_{\ell_4} = -\frac{Q|1]}{[41]} \tag{2.4.29}$$

$$\lambda_{\ell_3} = -\frac{P|1]}{[41]}, \qquad \lambda_{\ell_2} = \frac{P|4]}{[14]}. \tag{2.4.30}$$

In the maximally supersymmetric case then we have

$$\mathcal{A} = \int \prod_{i=1}^{4} d\eta_i \, \mathcal{A}^{(3)}(-\ell_1, 1, \ell_2) \mathcal{A}^{(4)}(-\ell_2, 2, 3, \ell_3) \mathcal{A}^{(3)}(-\ell_3, 4, \ell_4) \mathcal{A}^{(3)}(-\ell_4, Q, \ell_1) \tag{2.4.31}$$

where η are the usual Grassmann variables and an implicit averaging of the two solutions $\left(\frac{1}{2} \sum_{\mathcal{S}_\pm}\right)$ is assumed. Expanding the numerator of (2.4.31) and ignoring the overall momentum conservation δ-function for now gives

$$\mathsf{N} = \int \prod_{i=1}^{4} d\eta_i \, \big[\delta^{(8)}(-\eta_{\ell_1}\lambda_{\ell_1} + \eta_1\lambda_1 + \eta_{\ell_2}\lambda_{\ell_2}) \tag{2.4.32}$$

$$\times \, \delta^{(8)}(-\eta_{\ell_2}\lambda_{\ell_2} + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_{\ell_3}\lambda_{\ell_3}) \tag{2.4.33}$$

$$\times \, \delta^{(8)}(-\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 + \eta_{\ell_4}\lambda_{\ell_4}) \tag{2.4.34}$$

$$\times \, \delta^{(4)}(-\eta_{\ell_4}[5\ell_1] + \eta_5[\ell_1\ell_4] + \eta_{\ell_1}[\ell_45]) \big] \,. \tag{2.4.35}$$

Formally, the fermionic δ-function is defined in (2.2.11) as

$$\delta^{(8)}\left(Q_\alpha{}^A\right) := \prod_{A=1}^{4} \prod_{\alpha}^{1,2} \left(\sum_{i=1}^{n} \lambda_{i\alpha}\eta_i^A\right) = \prod_{A=1}^{4} \sum_{i,j} \eta_i^A \eta_j^A \langle ij \rangle \tag{2.4.36}$$

where in general, the index A depends on the number of supersymmetries.

The $\delta^{(4)}$-function gives

$$[5\ell_1]^4 \delta^{(4)}\left(\eta_{\ell_4} + \eta_5 \frac{[\ell_1\ell_4]}{[5\ell_1]} + \eta_{\ell_1} \frac{[\ell_45]}{[5\ell_1]}\right) \tag{2.4.37}$$

where the factor $[5\ell_1]^4$ is the Jacobian. The constraint

$$\eta_{\ell_4} = -\eta_5 \frac{[\ell_1\ell_4]}{[5\ell_1]} - \eta_{\ell_1} \frac{[\ell_45]}{[5\ell_1]} \tag{2.4.38}$$

is applied in the remaining $\delta^{(8)}$-functions. Thus, starting with (2.4.34), its argument becomes

$$-\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 + \left(-\eta_5\frac{[\ell_1\ell_4]}{[5\ell_1]} - \eta_{\ell_1}\frac{[\ell_45]}{[5\ell_1]}\right)\lambda_{\ell_4} \tag{2.4.39}$$

$$= -\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 - \eta_5\frac{[\ell_1|\langle\ell_4\ell_4]}{[5\ell_1]} - \eta_{\ell_1}\frac{[\ell_4\ell_4\rangle|5]}{[5\ell_1]} \tag{2.4.40}$$

but since $\ell_4 = \ell_1 + Q$,

$$-\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 - \frac{\eta_5[\ell_1|\ell_5 + \eta_{\ell_1}\ell_1|5]}{[5\ell_1]} \tag{2.4.41}$$

$$= -\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 + \eta_5\lambda_5 + \eta_{\ell_1}\ell_1 \tag{2.4.42}$$

such that $\delta^{(8)}(-\eta_{\ell_3}\lambda_{\ell_3} + \eta_4\lambda_4 + \eta_5\lambda_5 + \eta_{\ell_1}\ell_1)$. The first three δ-functions impose the following constraints on the supermomenta:

$$\begin{aligned}
-\eta_{\ell_1}\lambda_{\ell_1} &\quad +\eta_{\ell_2}\lambda_{\ell_2} &\quad +\ldots &\quad = 0\\
\ldots + &\quad -\eta_{\ell_2}\lambda_{\ell_2} &\quad +\eta_{\ell_3}\lambda_{\ell_3} &\quad = 0\\
\ldots + &\quad -\eta_{\ell_3}\lambda_{\ell_3} &\quad +\eta_{\ell_1}\lambda_{\ell_1} &\quad = 0
\end{aligned} \tag{2.4.43}$$

which can be plugged back into the δ-function (2.4.34) to isolate the external supermomenta

$$\delta^{(8)}(-\eta_{\ell_1}\lambda_{\ell_1} + \eta_1\lambda_1 + \eta_{\ell_2}\lambda_{\ell_2})\delta^{(8)}(-\eta_{\ell_2}\lambda_{\ell_2} + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_{\ell_3}\lambda_{\ell_3})\delta^{(8)}\left(\sum_{j\in\text{ext.}}\eta_j\lambda_j\right). \tag{2.4.44}$$

By doing so the ℓ dependence is eliminated from that particular δ-function and is no longer integrated over. Upon formal expansions the δ-functions can be written as

$$\prod_{A=1}^{4}\prod_{\alpha}^{1,2}(-\eta_{\ell_1}\lambda_{\ell_1} + \eta_1\lambda_1 + \eta_{\ell_2}\lambda_{\ell_2}) \;\rightarrow\; \prod_{A=1}^{4}\eta_{\ell_1}^A\left(\eta_1^A\lambda_{\ell_1}\lambda_1 + \eta_{\ell_2}^A\lambda_{\ell_1}\lambda_{\ell_2}\right) \tag{2.4.45}$$

$$= \prod_{A=1}^{4}\eta_{\ell_1}^A\left(\eta_1^A\langle\ell_11\rangle + \eta_{\ell_2}^A\langle\ell_1\ell_2\rangle\right) \tag{2.4.46}$$

for (2.4.32) and similarly for (2.4.33) we obtain

$$\prod_{B=1}^{4}\eta_{\ell_3}^B\left(\eta_{\ell_2}^B\langle\ell_3\ell_2\rangle + \eta_2^B\langle\ell_32\rangle + \eta_3^B\langle\ell_33\rangle\right). \tag{2.4.47}$$

Integrating over η_{ℓ_3} and η_{ℓ_1} gives

$$\prod_{A=1}^{4}\left(\eta_1^A\lambda_{\ell_1}\lambda_1 + \eta_{\ell_2}^A\lambda_{\ell_1}\lambda_{\ell_2}\right) \rightarrow \delta^{(4)}\left(\eta_1^A\lambda_{\ell_1}\lambda_1 + \eta_{\ell_2}^A\lambda_{\ell_1}\lambda_{\ell_2}\right) \tag{2.4.48}$$

and

$$\prod_{B=1}^{4}\left(\eta_{\ell_2}^B\langle\ell_3\ell_2\rangle + \eta_2^B\langle\ell_32\rangle + \eta_3^B\langle\ell_33\rangle\right) \rightarrow \delta^{(4)}\left(\eta_{\ell_2}^B\langle\ell_3\ell_2\rangle + \eta_2^B\langle\ell_32\rangle + \eta_3^B\langle\ell_33\rangle\right) \tag{2.4.49}$$

respectively. It is now possible to use the constraint from (2.4.48)

$$\langle\ell_2\ell_1\rangle^4\delta^{(4)}\left(\eta_{\ell_2} + \eta_1\frac{\langle\ell_11\rangle}{\langle\ell_2\ell_1\rangle}\right) \tag{2.4.50}$$

which plugged in (2.4.49) yields

$$\delta^{(4)}\left(-\eta_1\frac{\langle\ell_11\rangle}{\langle\ell_2\ell_1\rangle}\langle\ell_3\ell_2\rangle + \eta_2\langle\ell_32\rangle + \eta_3\langle\ell_33\rangle\right). \tag{2.4.51}$$

Returning to the solutions (2.4.30) in order to obtain the results of the contractions that appear in the δ-functions gives

$$-\langle 1\ell_1\rangle = \frac{\langle 1|Q|4]}{[41]} = \frac{\langle 1|5\rangle[54]}{[41]} \tag{2.4.52}$$

$$-\langle 2\ell_3\rangle = \frac{\langle 2|P|1]}{[41]} = \frac{\langle 2|(2+3)|1]}{[41]} = \frac{\langle 23\rangle[21]}{[41]} \tag{2.4.53}$$

$$-\langle 3\ell_3\rangle = \frac{\langle 3|(2+3)|1]}{[41]} = \frac{\langle 32\rangle[31]}{[41]} \tag{2.4.54}$$

and

$$\frac{\langle\ell_3\ell_2\rangle}{\langle\ell_2\ell_1\rangle} = \frac{\frac{[1|PP|4]}{[41]^2}}{\frac{[4|PQ|4]}{[41]^2}} \tag{2.4.55}$$

for which

$$[1|PP|4] = [1|\langle 23\rangle[23]|4] = \langle 23\rangle[23][14] \tag{2.4.56}$$

and

$$[4|PQ|4] = [4|(-4-1+Q)Q|4] = -[41]\langle 1Q\rangle[Q4]. \tag{2.4.57}$$

Using what has been calculated for the delta function (2.4.51) gives

$$\left(\frac{\langle 23\rangle}{[41]}\right)^4\delta^{(4)}\left(+\eta_1[23] + \eta_2[31] + \eta_3[21]\right). \tag{2.4.58}$$

finally producing expression

$$[5\ell_1]^4 \langle \ell_2 \ell_1 \rangle^4 \left(\frac{\langle 23 \rangle}{[41]} \right)^4 \delta^{(4)} \left(+\eta_1[23] + \eta_2[31] + \eta_3[21] \right) \delta^{(8)} \left(\sum_{j \in \text{ext.}} \eta_j \lambda_j \right). \qquad (2.4.59)$$

The solutions (2.4.30) are used to further simplify the expression to

$$\left(\frac{[51]\langle 15 \rangle [54]\langle 23 \rangle}{[41]} \right)^4 \delta^{(4)} \left(\eta_1[23] + \eta_2[31] + \eta_3[21] \right) \delta^{(8)} \left(\sum_{j \in \text{ext.}} \eta_j \lambda_j \right). \qquad (2.4.60)$$

The next step is to focus on the denominator of (2.4.31) which is again determined by the form of the tree amplitudes that reside at each corner of the box. The denominator D has the form

$$\mathsf{D} = \langle -\ell_1 1 \rangle \langle 1\ell_2 \rangle \langle \ell_2(-\ell_1) \rangle \, \langle -\ell_2 2 \rangle \langle 23 \rangle \langle 3\ell_3 \rangle \langle \ell_3(-\ell_2) \rangle \qquad (2.4.61)$$

$$\langle -\ell_3 4 \rangle \langle 4\ell_4 \rangle \langle \ell_4(-\ell_3) \rangle \, [\ell_4 5][5\ell_1][\ell_1\ell_4]. \qquad (2.4.62)$$

Using similar manipulations as before, it is possible to eliminate the dependence of the loop momenta. This basically involves applying momentum conservation in the brackets and employing the solutions (2.4.30). The denominator can therefore be rewritten as

$$\mathsf{D} = (\langle 15 \rangle [45])^3 \, ([15]\langle 45 \rangle)^3 \, \langle 23 \rangle^4 \frac{[12][23][34][45][51][14]}{[14]^8} \qquad (2.4.63)$$

$$= [12][23][34][45][51] \frac{(\langle 15 \rangle [51]\langle 45 \rangle [45])^3 \, \langle 23 \rangle^4}{[14]^8} \qquad (2.4.64)$$

It is now straightforward to combine it with the numerator (2.4.60) to yield the total prefactor

$$\mathsf{N} \times \mathsf{D} = \delta^{(4)} \left(\eta_1[23] + \eta_2[31] + \eta_3[21] \right) \delta^{(8)} \left(\sum_{j \in \text{ext.}} \eta_j \lambda_j \right) \frac{s_{15} t_{45}}{[12][23][34][45][51]\langle 45 \rangle^4} \qquad (2.4.65)$$

which is the correct coefficient for the box integral of the five-point one-loop $\overline{\text{MHV}}$ amplitude while the others are obtained by a cyclical permutation of the external states$^{\parallel}$.

$^{\parallel}$It is also necessary to average over the two possible solutions, i.e. $\frac{1}{2} \sum_{\mathcal{S}_\pm} (\cdots)$.

The full one-loop n-point MHV superamplitude has been calculated, first in [22], following an analogous procedure. A slightly different derivation with more detailed calculations can also be found in [42]. The final expression for a specific cut has a rather compact form:

$$\mathcal{A}_{n;1\text{-loop}}^{\text{MHV}} = \delta^{(4)}\left(P_{\alpha\dot\alpha}\right) \frac{\delta^{(8)}\left(\sum_{i=1}^n \eta_i \lambda_i\right)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\left[\sum_{s=3}^{n-1}\mathcal{I}_{1,2,s,s+1}\frac{1}{2}\sum_{\mathcal{S}_\pm}\left(P^2 Q^2 - st\right) + \text{cyclic}\right] \tag{2.4.66}$$

where $\mathcal{S}_\pm$ are the two solutions (2.4.21).
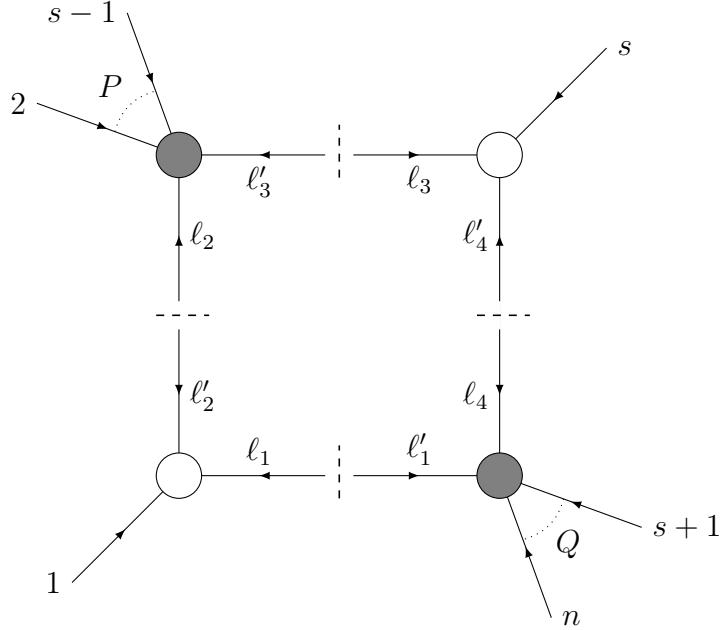
Figure 2.7 The n-point one-loop $\overline{\text{MHV}}$ superamplitude. The configuration allowed alternates between gray and white corners, that represent $\overline{\text{MHV}}$ multi-point and three-point MHV subamplitudes .

This concludes the introduction to the methods used in the following chapters for calculating amplitudes in more "exotic" theories than $\mathcal{N} = 4$ sYM in four dimensions. The spinor helicity formalism for six dimensions, relevant for the next chapter, is developed in Section 3.2.1 with further details and conventions given in Appendix A.1. The three dimensional spinor helicity formalism used in Chapter 4 is presented in Appendix B.1

3 | One-loop Amplitudes in Six-Dimensional (1,1) Theories

This chapter marks the departure from four dimensions. The focus now is on the $(1,1)$ supersymmetric theory in six dimensions. It will present our first example where the spinor helicity formalism and unitarity methods are applied in a theory other than the well studied $\mathcal{N} = 4$ sYM case. The findings of this chapter were first presented in [15].

3.1 Introduction

Several recent advancements in the study of scattering amplitudes in four dimensions have been made possible through the use of the spinor-helicity formalism. New dualities have been discovered and understood largely due to the structures that emerge when the amplitudes are expressed in their most physical form. Applying this strategy and employing unitarity methods in theories away from four dimensions is interesting in its own right. Furthermore, it is natural to assume that this may provide a framework that would promote advancements similar to those in four dimensions. To this end, the authors of [13] introduced a spinor helicity formalism in six dimensions while a discussion in arbitrary dimensions can be found in [19]. This, not only provides compact expressions of amplitudes but also allows the application of calculational tools developed for four-dimensional theories such as recursion relations [28, 29].

Similarly to four dimensions, maximally supersymmetric theories in six dimen-

sions are especially interesting. These arise as low-energy effective field theories on fivebranes in string theory or M-theory and come in two flavors: one with $(1, 1)$ and one with $(2, 0)$ supersymmetry. This embedding guarantees their UV completion. In particular, it is known that the (1,1) supersymmetric gauge theory in six dimensions is finite up to two loops [43]. Upon dimensional reduction on a two-torus both give rise to $\mathcal{N} = 4$ supersymmetry, but only $(2, 0)$ supersymmetric theories can give rise to superconformal symmetry [44].

Scattering superamplitudes in the $(1, 1)$ theory have been studied in [21,45]. More specifically, the tree-level three-, four- and five-point superamplitudes have been derived, as well as the one-loop four-point superamplitude, using the unitarity-based approach of [36, 37]. The description of the $(2, 0)$ theory in this framework is perhaps the most intriguing as it would illuminate the dynamics of M5-branes. The advantage of working with S-matrix elements is that it is possible to probe interactions even if the action is not available, as is the case for M-theory. Interestingly, the authors of [45] found that, under certain assumptions, the tree-level amplitudes identically vanish. This suggests that a superconformal interacting Lagrangian cannot be formulated using only $(2, 0)$ tensor multiplets, although additional degrees of freedom might resolve this problem [45].

An additional motivation to study higher-dimensional theories stems from the fact that QCD amplitudes in dimensional regularization naturally give rise to integral functions in higher dimensions, specifically in six and eight dimensions [46, 47]. A direct application of this formalism for the study of QCD one-loop amplitudes was performed in [48], where the six-dimensional spinors are expressed as four dimensional spinors with mass parameters.

This chapter presents the calculation of four- and five-point superamplitudes, in the maximally supersymmetric (1,1) theory, using two-particle as well as quadruple cuts at one loop. In particular, it is shown that the five-point superamplitude can be expressed solely in terms of a linear pentagon integral in six dimensions, which can be further reduced in terms of scalar pentagon and box functions. Because of the

non-chiral nature of the (1,1) on-shell superspace, this superamplitude contains all possible component amplitudes with five particles; in contradistinction with the four-dimensional case where one has to distinguish MHV and $\overline{\text{MHV}}$ helicity configurations.

The rest of this chapter is organized as follows. Section 3.2, provides a review of the spinor helicity formalism in six dimensions, and the on-shell (1,1) superspace, which is used to describe superamplitudes in the (1,1) theory. The expressions for the simplest tree-level amplitudes can be found in Section 3.3. These are used in Sections 3.4 and 3.5 for our calculations of one-loop amplitudes using (generalized) unitarity. In Section 3.4 the method is illustrated by re-deriving the four-point superamplitude using two-particle cuts as well as quadruple cuts. Next, Section 3.5 contains a detailed presentation of the derivation of the five-point superamplitude in the (1,1) theory from quadruple cuts. Finally, our result is subjected to several consistency checks using dimensional reduction to four dimensions in order to compare with the corresponding amplitudes in $\mathcal{N} = 4$ sYM. We also test some of the soft limits.

3.2 Background

This section begins with a brief review of the six-dimensional spinor helicity formalism developed in [13], which is required to present Yang-Mills scattering amplitudes in a compact form. It is then followed by a discussion of the on-shell (1,1) superspace description of amplitudes in maximally supersymmetric Yang-Mills which was introduced in [21].

3.2.1 Spinor helicity formalism in six dimensions

The key observation for a compact formulation of amplitudes in six-dimensional gauge and gravity theories is that, similarly to four dimensions, null momenta in six dimensions can be conveniently presented in a spinor helicity formalism, introduced in [13]. Firstly, one rewrites vectors of the Lorentz group $\mathsf{SO}(1,5)$ as antisymmetric

SU*(4) matrices

$$p^{AB} := p^\mu \tilde{\sigma}_\mu^{AB} \ , \tag{3.2.1}$$

using the appropriate Clebsch-Gordan symbols $\tilde{\sigma}_\mu^{AB}$, where $A, B = 1, \ldots, 4$ are fundamental indices of $\mathsf{SU}(4)$. Strictly speaking the $\mathsf{SO}(1,5)$ Lorentz group is isomorphic to the non-compact group $\mathsf{SU}^*(4)$. However, what is of practical interest here is the antisymmetric nature of the fundamental indices of $\mathsf{SU}^*(4)$. In what follows the *indicator will be dropped for simplicity. A detailed discussion of how the reality condition is imposed in six dimensional spinors can be found in [42] and in references therein.

One can similarly introduce*

$$p_{AB} := \frac{1}{2}\epsilon_{ABCD}p^{CD} := p^\mu \sigma_{\mu,AB} \ , \tag{3.2.2}$$

with $\sigma_{\mu,AB} := (1/2)\epsilon_{ABCD}\tilde{\sigma}_\mu^{CD}$. When $p^2 = 0$, it is natural to recast p^{AB} and p_{AB} as the product of two spinors as [13]

$$p^{AB} = \lambda^{Aa}\lambda_a^B \ , \tag{3.2.3}$$
$$p_{AB} = \tilde{\lambda}_A^{\dot{a}}\tilde{\lambda}_{B\dot{a}} \ .$$

Here $a = 1, 2$ and $\dot{a} = 1, 2$ are indices of the little group[†] $\mathsf{SO}(4) \simeq \mathsf{SU}(2) \times \mathsf{SU}(2)$, which are contracted with the usual invariant tensors ϵ_{ab} and $\epsilon_{\dot{a}\dot{b}}$. The expression for p given in (3.2.3) automatically ensures that p is a null vector, since

$$p^2 = -\frac{1}{8}\epsilon_{ABCD}\lambda_a^A\lambda_b^B\lambda_c^C\lambda_d^D\epsilon^{ab}\epsilon^{cd} = 0 \ . \tag{3.2.4}$$

This can be understood as follows: due to the antisymmetric contraction of the $\mathsf{SU}(2)$ indices, the bi-spinors are automatically antisymmetric in the $\mathsf{SU}(4)$ indices A and B. By equations (3.2.3), the 4×4 matrix p_i^{AB} has rank 2, so $p_i^2 \sim \epsilon_{ABCD}p_i^{AB}p_i^{CD}$ is zero, which satisfies the massless on-shell condition. The dot product of two null vectors p_i and p_j can also be conveniently written using spinors as

$$p_i \cdot p_j = -\frac{1}{4}p_i^{AB}p_{j;AB} \ . \tag{3.2.5}$$

---

[*]Our notation and conventions are outlined in Appendix A.1.
[†]Or $\mathsf{SL}(2, \mathbb{C}) \times \mathsf{SL}(2, \mathbb{C})$, if we complexify spacetime.

Lorentz invariant contractions of two spinors are expressed as

$$\langle i_a | j_{\dot{a}}] := \lambda_{i,a}^A \tilde{\lambda}_{j,A\dot{a}} = \tilde{\lambda}_{j,A\dot{a}} \lambda_{i,a}^A =: [j_{\dot{a}} | i_a\rangle \ . \tag{3.2.6}$$

Further Lorentz-invariant combinations can be constructed from four spinors using the $\mathsf{SU}(4)$ invariant ϵ tensor, as

$$\langle 1_a \, 2_b \, 3_c \, 4_d \rangle \quad := \quad \epsilon_{ABCD} \lambda_{1,a}^A \lambda_{2,b}^B \lambda_{3,c}^C \lambda_{4,d}^D \ , \tag{3.2.7}$$

$$[1_{\dot{a}} \, 2_{\dot{b}} \, 3_{\dot{c}} \, 4_{\dot{d}}] \quad := \quad \epsilon^{ABCD} \lambda_{1,A\dot{a}} \lambda_{2,B\dot{b}} \lambda_{3,C\dot{c}} \lambda_{4,D\dot{d}} \ .$$

This notation may be used to express compactly strings of six-dimensional momenta contracted with Dirac matrices, such as

$$\langle i_a | \hat{p}_1 \hat{p}_2 \ldots \hat{p}_{2n+1} | j_b \rangle := \lambda_{i,a}^{A_1} \, p_{1,A_1 A_2} \, p_2^{A_2 A_3} \ldots p_{2n+1,A_{2n+1} A_{2n+2}} \, \lambda_{j,b}^{A_{2n+2}} \ , \tag{3.2.8}$$

$$\langle i_a | \hat{p}_1 \hat{p}_2 \ldots \hat{p}_{2n} | j_{\dot{a}}] := \lambda_{i,a}^{A_1} \, p_{1,A_1 A_2} \, p_2^{A_2 A_3} \ldots p_{2n}^{A_{2n} A_{2n+1}} \, \tilde{\lambda}_{j,A_{2n+1} \dot{b}} \ .$$

Having discussed momenta, we now consider polarization states of particles. In four dimensions, these are associated to the notion of helicity. In six dimensions, physical states, and hence their wavefunctions, transform according to representations of the little group, and therefore carry $\mathsf{SU}(2) \times \mathsf{SU}(2)$ indices [13]. In particular, for gluons of momentum p defined as in (3.2.3) one has

$$\epsilon_{a\dot{a}}^{AB} := \lambda_a^{[A} \eta_b^{B]} \langle \eta_b | \lambda^{\dot{a}}]^{-1} \ , \tag{3.2.9}$$

or alternatively

$$\epsilon_{a\dot{a};AB} := \langle \lambda^a | \tilde{\eta}_{\dot{b}}]^{-1} \tilde{\eta}_{\dot{b}[A} \lambda_{\dot{a}B]} \ . \tag{3.2.10}$$

Here, η and $\tilde{\eta}$ are reference spinors, and the denominator is defined to be the inverse of the matrices $\langle q^b | p_{\dot{a}}]$ and $\langle p_a | q^{\dot{b}}]$, respectively.[‡]

It is amusing to make contact between six-dimensional spinors and momentum twistors [49], employed recently to describe amplitudes in four-dimensional conformal theories. There, one describes a point in (conformally compactified) Minkowski space as a six-dimensional null vector X, i.e. one satisfying $\eta_{ij} X^i X^j = 0$, with

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[‡]The reference spinors are chosen such that the matrices $\langle q^b | p_{\dot{a}}]$ and $\langle p_a | q^{\dot{b}}]$ are nonsingular.

$\eta = \mathrm{diag}(+---;+-)$. The conformal group $\mathsf{SO}(2,4)$ acts linearly on the X variables, and plays the role of the Lorentz group $\mathsf{SO}(1,5)$ acting on our six-dimensional momenta p. Furthermore, in contradistinction with the null six-dimensional momenta, the coordinate X are defined only up to non-vanishing rescalings. For (cyclically ordered) four-dimensional region momenta x_i, one defines the corresponding six-dimensional null X_i as $X_i = \lambda_i \wedge \lambda_{i+1}$, $X_j = \lambda_j \wedge \lambda_{j+1}$, and $X_i \cdot X_j = \langle i\, i+1\, j\, j+1 \rangle$.

3.2.2   (1,1) on-shell superspace

We will now review the on-shell superspace description of $(1,1)$ theories introduced in [21]. This construction is inspired by the covariant on-shell superspace formalism for four-dimensional $\mathcal{N} = 4$ sYM introduced by Nair in [26]. In the latter case, the $\mathcal{N} = 4$ algebra can be represented on shell as

$$\{q_\alpha^A, \tilde{q}_{B\dot\alpha}\} = \delta_B^A \lambda_\alpha \tilde{\lambda}_{\dot\alpha} , \tag{3.2.11}$$

where A, B are $\mathsf{SU}(4)$ R-symmetry indices and $\alpha, \dot\alpha$ are the usual $\mathsf{SU}(2)$ spinor indices in four dimensions. The supercharge q can be decomposed along two independent directions λ and μ as

$$q_\alpha^A = \lambda_\alpha q_{(1)}^A + \mu_\alpha q_{(2)}^A , \tag{3.2.12}$$

where $\langle \lambda\mu \rangle \neq 0$. A similar decomposition is performed for $\tilde{q}$. One can then easily see that the charges $q_{(2)}$ and $\tilde{q}_{(2)}$ anticommute among themselves and with the other generators, and can therefore be set to zero. The supersymmetry algebra becomes

$$\{q_{(1)}^A, \tilde{q}_{(1)B}\} = \delta_B^A . \tag{3.2.13}$$

Setting $q_{(1)}^A = q^A$ and $\tilde{q}_{(1)B} = \tilde{q}_B$, the Clifford algebra can be naturally realized in terms of Grassmann variables η^A, as

$$q^A = \eta^A , \qquad\qquad \tilde{q}_A = \frac{\partial}{\partial \eta^A} . \tag{3.2.14}$$

Note that this representation of the algebra is chiral. One could have chosen an anti-chiral representation, where the roles of q and $\tilde{q}$ in (3.2.14) are interchanged.

One can apply similar ideas to the case of the $\mathcal{N} = (1,1)$ superspace of the six-dimensional sYM theory. However, for this on-shell space the chiral and anti-chiral components do not decouple. To see this we start with the algebra

$$\{q^{AI}, q^{BJ}\} = p^{AB}\epsilon^{IJ} \, , \tag{3.2.15}$$

$$\{\tilde{q}_{AI'}, \tilde{q}_{BJ'}\} = p_{AB}\epsilon_{I'J'} \, ,$$

where A, B are the $\mathsf{SU}(4)$ Lorentz index and I, J and I', J' are indices of the R-symmetry group $\mathsf{SU}(2) \times \mathsf{SU}(2)$. As before, we decompose the supercharges as

$$q^{AI} = \lambda^{Aa}q^I_{(1)a} + \mu^{Aa}q^I_{(2)a} \, , \tag{3.2.16}$$

$$\tilde{q}_{BI'} = \tilde{\lambda}^{\dot{b}}_B \tilde{q}_{(1)\dot{b}I'} + \tilde{\mu}^{\dot{b}}_B \tilde{q}_{(2)\dot{b}I'} \, ,$$

with $\det(\lambda^{Aa}\tilde{\mu}^{\dot{a}}_A) \neq 0$ and $\det(\mu^{Aa}\tilde{\lambda}^{\dot{a}}_A) \neq 0$. Multiplying the supercharges in (3.2.16) by $\tilde{\lambda}_{A\dot{a}}$ and λ^B_b, respectively, and summing over the $\mathsf{SU}(4)$ indices, one finds that

$$\{q^I_{(2)a}, q^J_{(2)b}\} = 0 \, , \tag{3.2.17}$$

$$\{\tilde{q}_{(2)\dot{a}I'}, \tilde{q}_{(2)\dot{b}J'}\} = 0 \, .$$

One can thus set all the $q_{(2)}$ and $\tilde{q}_{(2)}$ charges equal to zero, so that $q^{AI} = \lambda^{Aa}q^I_{(1)a}$. The supersymmetry algebra then yields,

$$\{q^I_{(1)a}, q^J_{(1)b}\} = \epsilon_{ab}\epsilon^{IJ} \, ,$$

$$\{\tilde{q}_{(1)I'\dot{a}}, \tilde{q}_{(1)J'\dot{b}}\} = \epsilon_{\dot{a}\dot{b}}\epsilon_{I'J'} \, . \tag{3.2.18}$$

The realization of (3.2.18) in terms of anticommuting Grassmann variables is

$$q^{AI} = \lambda^{Aa}\eta^I_a \, , \qquad \tilde{q}_{AI'} = \tilde{\lambda}^{\dot{a}}_A \tilde{\eta}_{I'\dot{a}} \, . \tag{3.2.19}$$

In contrast to the four-dimensional $\mathcal{N} = 4$ sYM theory, the $\mathcal{N} = (1,1)$ on-shell superspace in six dimensions carries chiral and anti-chiral components. The field strength of the six-dimensional sYM theory transforms under the little group $\mathsf{SU}(2) \times \mathsf{SU}(2)$ and therefore carries both indices a and $\dot{a}$. Hence, one needs both η_a and $\tilde{\eta}_{\dot{a}}$ to describe all helicity states in this theory.
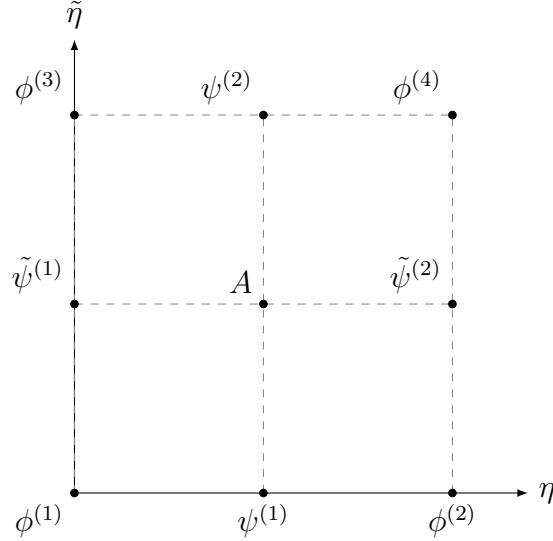
Figure 3.1 The component fields of the (1,1) superfield given in (3.2.21).

In order to describe only the physical components of the full six-dimensional sYM theory, one needs to truncate half of the superspace charges in (3.2.19) [21]. This is performed by contracting the R-symmetry indices with fixed two-component (harmonic) vectors, which effectively reduce the number of supercharges by a factor of two. The resulting truncated supersymmetry generators are then [21]

$$q^A = \lambda^{Aa}\eta_a\,, \qquad \tilde{q}_A = \tilde{\lambda}_A^{\dot{a}}\tilde{\eta}_{\dot{a}}\,. \qquad (3.2.20)$$

Using this on-shell superspace, one can neatly package all states of the theory into a six-dimensional analogue of Nair's superfield [26],

$$\Phi(p;\eta,\tilde{\eta}) = \phi^{(1)} + \psi_a^{(1)}\eta^a + \tilde{\psi}_{\dot{a}}^{(1)}\tilde{\eta}^{\dot{a}} + \phi^{(2)}\eta^a\eta_a + A_{a\dot{a}}\eta^a\tilde{\eta}^{\dot{a}} + \phi^{(3)}\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}}$$
$$+ \psi_a^{(2)}\eta^a\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}} + \tilde{\psi}_{\dot{a}}^{(2)}\tilde{\eta}^{\dot{a}}\eta^a\eta_a + \phi^{(4)}\eta^a\eta_a\tilde{\eta}^{\dot{a}}\tilde{\eta}_{\dot{a}}\,. \quad (3.2.21)$$

Here $\phi^{(i)}(p)$, $i = 1,\ldots,4$ are four scalar fields, $\psi^{(l)}(p)$ and $\tilde{\psi}^{(l)}(p)$, $l = 1,2$ are fermion fields and finally $A_{a\dot{a}}(p)$ contains the gluons. Upon reduction to four dimensions, $A_{a\dot{a}}$ provides, in addition to gluons of positive and negative (four-dimensional) helicity, the two remaining scalar fields needed to obtain the matter content of $\mathcal{N} = 4$ super

Yang-Mills.[§] A pictorial representation of the states in the (1,1) supermultiplet is given in Figure 3.1.

3.3 Tree-level amplitudes and their properties

3.3.1 Three-point amplitude

The smallest amplitude one encounters is the three-point amplitude. In four dimensions, and for real kinematics, three-point amplitudes vanish because $p_i \cdot p_j = 0$ for any of the three particles' momenta, but are non-vanishing upon spacetime complexification [28, 29]. In the six-dimensional spinor helicity formalism, the special three-point kinematics induces the constraint $\det\langle i_a | j_{\dot{a}}] = 0$, $i, j = 1, 2, 3$. This allows one to write (see Appendix (A.1.7))

$$\langle i_a | j_{\dot{b}}] = (-)^{\mathcal{P}_{ij}} u_{ia} \tilde{u}_{j\dot{b}} \ , \tag{3.3.1}$$

where we choose $(-)^{\mathcal{P}_{ij}} = +1$ for $(i, j) = (1, 2), (2, 3), (3, 1)$ and -1 for $(i, j) = (2, 1), (3, 2), (1, 3)$. One can also introduce the spinors w_a and $\tilde{w}_{\dot{a}}$ [13], defined as the inverse of u_a and $\tilde{u}_{\dot{a}}$,

$$u_a w_b - u_b w_a := \epsilon_{ab} \quad \Leftrightarrow \quad u^a w_a := -u_a w^a := 1 \ . \tag{3.3.2}$$

As stressed in [13] the w_i spinors are not uniquely specified since the choice $w_i' = w_i + b_i u_i$ is equally good. Momentum conservation suggests a further constraint that may be imposed in order to reduce this redundancy. This is used in various calculations throughout the present work. Specifically, for a generic three-point amplitude it is assumed that

$$|w_1 \cdot 1\rangle + |w_2 \cdot 2\rangle + |w_3 \cdot 3\rangle = 0 \ . \tag{3.3.3}$$

This is derived as follows. Starting from momentum conservation,

$$|1^{\dot{a}}][1_{\dot{a}}| + |2^{\dot{b}}][2_{\dot{b}}| + |3^{\dot{c}}][3_{\dot{c}}| = 0 = |1^a\rangle\langle 1_a| + |2^b\rangle\langle 2_b| + |3^c\rangle\langle 3_c| \tag{3.3.4}$$

[§]More details on reduction to four dimensions are provided in Section 3.5.5.

one may contract recursively with $\langle i_a |$. This will yield the identity $u_i^{\dot{a}}[i_{\dot{a}}| = u_j^{\dot{a}}[j_{\dot{a}}|$, $\forall\, i, j$ and similarly for the angle brackets. Now using the definition (3.3.2) it is possible to recast momentum conservation in the following form

$$|1 \cdot u_1\rangle \left(\langle w_1 \cdot 1| + \langle w_2 \cdot 2| + \langle w_3 \cdot 3| \right) - \left(|w_1 \cdot 1\rangle + |w_2 \cdot 2\rangle + |w_3 \cdot 3\rangle \right) \langle u_1 \cdot 1| = 0 \,. \tag{3.3.5}$$

The authors of [13] choose the more stringent constraint (3.3.3) which still allows for a residual redundancy: it is possible to make the shift $w_i \to w_i + b_i u_i$ where $\sum_i b_i = 0$. Since the b redundancy cannot be fully eliminated in this manner it is natural to ask what tensors can be constructed that remain invariant under a b change.

These are the so-called Γ and $\tilde{\Gamma}$ tenors that are given by

$$\Gamma_{abc} = u_{1a} u_{2b} w_{3c} + u_{1a} w_{2b} u_{3c} + w_{1a} u_{2b} u_{3c}, \tag{3.3.6}$$

$$\tilde{\Gamma}_{\dot{a}\dot{b}\dot{c}} = \tilde{u}_{1\dot{a}} \tilde{u}_{2\dot{b}} \tilde{w}_{3\dot{c}} + \tilde{u}_{1\dot{a}} \tilde{w}_{2\dot{b}} \tilde{u}_{3\dot{c}} + \tilde{w}_{1\dot{a}} \tilde{u}_{2\dot{b}} \tilde{u}_{3\dot{c}} \,.$$

One may then express the three-point tree-level amplitude for six-dimensional Yang-Mills theory as [13]

$$A_{3;0}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}) = i\Gamma_{abc}\tilde{\Gamma}_{\dot{a}\dot{b}\dot{c}} \,. \tag{3.3.7}$$

As shown in [21], this result can be combined with the $\mathcal{N} = (1, 1)$ on-shell superspace in six dimensions. The corresponding three-point tree-level superamplitude takes the simple form [21]

$$A_{3;0}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}) = i\, \delta(Q^A)\delta(\tilde{Q}_A)\delta(Q^B)\delta(\tilde{Q}_B)\delta(W)\delta(\tilde{W}) \,. \tag{3.3.8}$$

Here we have introduced the $\mathcal{N} = (1, 1)$ supercharges for the external states,

$$Q^A := \sum_{i=1}^{n} q_i^A = \sum_{i=1}^{n} \lambda_i^{Aa} \eta_{ia} \,, \qquad \tilde{Q}_A := \sum_{i=1}^{n} \tilde{q}_{iA} = \sum_{i=1}^{n} \tilde{\lambda}_{iA}^{\dot{a}} \tilde{\eta}_{i\dot{a}} \tag{3.3.9}$$

(with $n = 3$ in the three-point amplitude we are considering in this section). The quantities $W, \tilde{W}$ appear only in the special three-point kinematics case, and are given by

$$W := \sum_{i=1}^{3} w_i^a \eta_{ia} \,, \qquad \tilde{W} := \sum_{i=1}^{3} \tilde{w}_i^{\dot{a}} \tilde{\eta}_{i\dot{a}} \,. \tag{3.3.10}$$

Appendix A.2 presents an explicit proof of the (non manifest) invariance of the three-point superamplitude under supersymmetry transformations, and hence of the fact that the total supermomentum $Q^A = \sum_i q_i^A$ is conserved.

3.3.2 Four-point amplitude

The four-point tree-level amplitude in six dimensions is given by

$$A_{4;0}(1_{a\dot a}, 2_{b\dot b}, 3_{c\dot c}, 4_{d\dot d}) = -\frac{i}{st}\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot a}2_{\dot b}3_{\dot c}4_{\dot d}] \; , \qquad (3.3.11)$$

and was derived by using a six-dimensional version [13] of the BCFW recursion relations [28, 29]. The corresponding $\mathcal{N} = (1,1)$ superamplitude is [21]

$$A_{4;0}(1,\dots,4) = -\frac{i}{st}\delta^4(Q)\delta^4(\tilde Q) \; , \qquad (3.3.12)$$

where the $(1,1)$ supercharges are defined in (3.3.9). In (3.3.12) we follow [21] and introduce the fermionic δ-functions which enforce supermomentum conservation as

$$\begin{aligned}
\delta^4(Q)\delta^4(\tilde Q) &= \frac{1}{4!}\epsilon_{ABCD}\delta(Q^A)\delta(Q^B)\delta(Q^C)\delta(Q^D) \\
&\times \frac{1}{4!}\epsilon^{A'B'C'D'}\delta(\tilde Q_{A'})\delta(\tilde Q_{B'})\delta(\tilde Q_{C'})\delta(\tilde Q_{D'}) \\
&:= \delta^8(Q) \; .
\end{aligned} \qquad (3.3.13)$$

Hence, a $\delta^4(Q)$ sets $Q^A = 0$ whereas the $\delta^4(\tilde Q)$ sets $\tilde Q_A = 0$.

3.3.3 Five-point amplitude

The five-point tree-level amplitude was derived in [13] using recursion relations, and is equal to[¶]

$$A_{5;0}(1_{a\dot a}, 2_{b\dot b}, 3_{c\dot c}, 4_{d\dot d}, 5_{e\dot e}) = \frac{i}{s_{12}s_{23}s_{34}s_{45}s_{51}}\left(\mathcal{A}_{a\dot a b\dot b c\dot c d\dot d e\dot e} + \mathcal{D}_{a\dot a b\dot b c\dot c d\dot d e\dot e}\right) \qquad (3.3.14)$$

where the two tensors $\mathcal{A}$ and $\mathcal{D}$ are given by

$$\mathcal{A}_{a\dot a b\dot b c\dot c d\dot d e\dot e} = \langle 1_a|\hat p_2\hat p_3\hat p_4\hat p_5|1_{\dot a}]\langle 2_b 3_c 4_d 5_e\rangle[2_{\dot b}3_{\dot c}4_{\dot d}5_{\dot e}] + \text{cyclic permutations} \; , \qquad (3.3.15)$$

[¶] In Appendix A.5 the five-point amplitude (3.3.14) is reduced to four dimensions and found to be in agreement with the expected Parke-Taylor expression.

and

$$
\begin{aligned}
2\mathcal{D}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} &= \langle 1_a|(2\cdot\tilde{\Delta}_2)_{\dot{b}}]\langle 2_b 3_c 4_d 5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \langle 3_c|(4\cdot\tilde{\Delta}_4)_{\dot{d}}]\langle 1_a 2_b 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}] \\
&\quad + \langle 4_d|(5\cdot\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] - \langle 3_c|(5\cdot\tilde{\Delta}_5)_{\dot{e}}]\langle 1_a 2_b 4_d 5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] \\
&\quad - [1_{\dot{a}}|(2\cdot\Delta_2)_b\rangle\langle 1_a 3_c 4_d 5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] - [3_{\dot{c}}|(4\cdot\Delta_4)_d\rangle\langle 1_a 2_b 3_c 5_e\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}] \\
&\quad - [4_{\dot{d}}|(5\cdot\Delta_5)_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}5_{\dot{e}}] + [3_{\dot{c}}|(5\cdot\Delta_5)_e\rangle\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{a}}2_{\dot{b}}4_{\dot{d}}5_{\dot{e}}] \,.
\end{aligned}
$$
$$(3.3.16)$$

Here, the spinor matrices Δ and $\tilde{\Delta}$ are defined by

$$
\Delta_1 = \langle 1|\hat{p}_2\hat{p}_3\hat{p}_4 - \hat{p}_4\hat{p}_3\hat{p}_2|1\rangle, \quad \tilde{\Delta}_1 = [1|\hat{p}_2\hat{p}_3\hat{p}_4 - \hat{p}_4\hat{p}_3\hat{p}_2|1] \,, \tag{3.3.17}
$$

where the other quantities $\Delta_i, \tilde{\Delta}_i$ are generated by taking cyclic permutations on (3.3.17). The contraction between a Δ_i and the corresponding spinor λ_i^{Aa} is given by $\langle 1_a|(2\cdot\tilde{\Delta}_2)_{\dot{b}}] = \lambda_{1a}^A\tilde{\lambda}_{2A}^{\dot{a}'}[2_{\dot{a}'}|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2_{\dot{b}}]$.

The five-point superamplitude in the $\mathcal{N} = (1,1)$ on-shell superspace can also be calculated in a recursive fashion. It takes the form [21]

$$
\begin{aligned}
A_{5;0} \;&= i\,\frac{\delta^4(Q)\delta^4(\tilde{Q})}{s_{12}s_{23}s_{34}s_{45}s_{51}}\Bigg[&(3.3.18)\\
&\quad + \frac{3}{10}q_1^A\big[(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5) - (\hat{p}_2\hat{p}_5\hat{p}_4\hat{p}_3)\big]_A^B\tilde{q}_{2B} + \frac{3}{10}\tilde{q}_{1A}\big[(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5) - (\hat{p}_2\hat{p}_5\hat{p}_4\hat{p}_3)\big]_B^A q_2^B \\
&\quad + \frac{1}{10}q_3^A\big[(\hat{p}_5\hat{p}_1\hat{p}_2\hat{p}_3) - (\hat{p}_5\hat{p}_3\hat{p}_2\hat{p}_1)\big]_A^B\tilde{q}_{5B} + \frac{1}{10}\tilde{q}_{3A}\big[(\hat{p}_5\hat{p}_1\hat{p}_2\hat{p}_3) - (\hat{p}_5\hat{p}_3\hat{p}_2\hat{p}_1)\big]_B^A q_5^B \\
&\quad + q_1^A(\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5)_A^B\tilde{q}_{1B} + \text{cyclic permutations}\Bigg] \,,
\end{aligned}
$$

where the supercharges Q and $\tilde{Q}$ are defined in (3.3.9).

3.4   One-loop four-point amplitude

In this section we calculate the four-point one-loop amplitude using two-particle and four-particle cuts. As expected, the one-loop amplitude is proportional to the four-point tree-level superamplitude times the corresponding integral function.
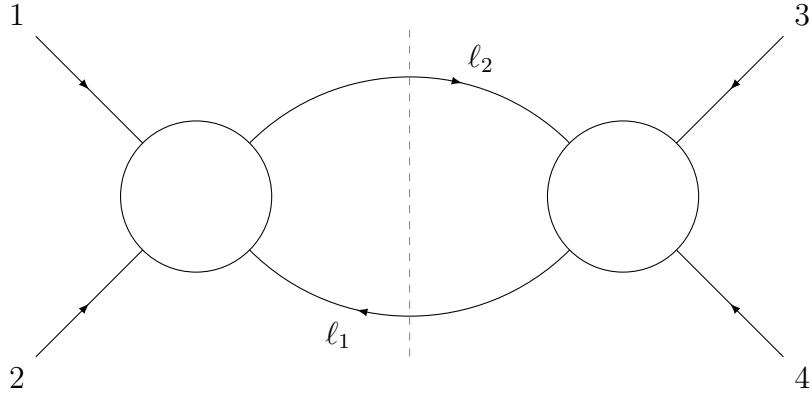
Figure 3.2 Double cut in the s-channel. The two internal cut-propagators, carrying momenta ℓ_1 and ℓ_2 set the two four-point subamplitudes on-shell. We identify $\ell_1 = \ell$ and $\ell_2 = \ell + p_1 + p_2$.

3.4.1 The superamplitude from two-particle cuts

As a warm-up exercise, we start by re-deriving the one-loop four-point superamplitude in six dimensions using two-particle cuts. This calculation was first sketched in [21]. Here, we will perform it in some detail while setting up our notations. We will then show how to reproduce this result using quadruple cuts.

We begin by considering the one-loop amplitude with external momenta $p_1, \ldots, p_4$, and perform a unitarity cut in the s-channel, see Figure 3.2. The s-cut of the one-loop amplitude is given by[||]

$$A_{4;1}\big|_{s\text{-cut}} = \int \frac{d^6\ell}{(2\pi)^6} \delta^+(\ell_1^2)\delta^+(\ell_2^2) \left[\prod_{i=1}^{2} \int d^2\eta_{\ell_i} d^2\tilde{\eta}_{\ell_i} \, A_{4;0}^{(L)}(\ell_1, 1, 2, -\ell_2) \, A_{4;0}^{(R)}(\ell_2, 3, 4, -\ell_1) \right] .$$

$$(3.4.1)$$

Plugging the expression (3.3.12) of the four-point superamplitude into (3.4.1), we get the following fermionic integral,

$$\prod_{i=1}^{2} \int d^2\eta_{\ell_i} d^2\tilde{\eta}_{\ell_i} \left(\frac{-i}{s_L t_L} \delta^4(\sum_L q_i) \delta^4(\sum_L \tilde{q}_i) \right) \left(\frac{-i}{s_R t_R} \delta^4(\sum_R q_i) \delta^4(\sum_R \tilde{q}_i) \right), \quad (3.4.2)$$

[||]See Appendix A for our definitions of fermionic integrals.

where the sums are over the external states of the left and right subamplitude in the cut diagram and the kinematical invariants are given by

$$t_L = (\ell_1 + p_1)^2 \,, \qquad t_R = (\ell_2 + p_3)^2 \,, \tag{3.4.3}$$

and

$$s_L = (p_1 + p_2)^2 = (p_3 + p_4)^2 = s_R = s \,. \tag{3.4.4}$$

Using supermomentum conservation it is possible remove the dependence of the loop-supermomenta on one side of the cut. For instance a $\delta^4(Q_R)$ sets $q_{\ell_1}^A = q_{\ell_2}^A + q_3^A + q_4^A$, which can be used in the remaining $\delta^4(Q_L)$ to write

$$\delta^4(\sum_L q_i) \to \delta^4(\sum_{\text{ext}} q_i) \equiv \delta^4(Q_{\text{ext}}) \,,$$
$$\delta^4(\sum_L \tilde{q}_i) \to \delta^4(\sum_{\text{ext}} \tilde{q}_i) \equiv \delta^4(\tilde{Q}_{\text{ext}}) \,. \tag{3.4.5}$$

Hence, (3.4.2) becomes

$$\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \prod_{i=1}^{2} \int d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \delta^4(\sum_R q_i)\delta^4(\sum_R \tilde{q}_i) \,. \tag{3.4.6}$$

To perform the integration, we need to pick two powers of η_{l_i} and two powers of $\tilde{\eta}_{l_i}^{\dot{a}}$. Expanding the fermionic δ-functions, we find one possible term with the right powers of Grassmann variables to be

$$\eta_{\ell_1 a}\eta_{\ell_1 b}\eta_{\ell_2 c}\eta_{\ell_2 d}\tilde{\eta}_{\ell_1 \dot{a}}\tilde{\eta}_{\ell_1 \dot{b}}\tilde{\eta}_{\ell_2 \dot{c}}\tilde{\eta}_{\ell_2 \dot{d}} \left[\epsilon_{ABCD}\lambda_{\ell_1}^{Aa}\lambda_{\ell_1}^{Bb}\lambda_{\ell_2}^{Cc}\lambda_{\ell_2}^{Dd} \,\epsilon^{EFGH}\tilde{\lambda}_{\ell_1 E}^{\dot{a}}\tilde{\lambda}_{\ell_1 F}^{\dot{b}}\tilde{\lambda}_{\ell_2 G}^{\dot{c}}\tilde{\lambda}_{\ell_2 H}^{\dot{d}} \right] \,. \tag{3.4.7}$$

Other combinations can be brought into that form by rearranging and relabeling indices. Integrating out the Grassmann variables gives

$$\left(\epsilon_{ABCD}\lambda_{\ell_1}^{Aa}\lambda_{\ell_1 a}^{B} \,\lambda_{\ell_2}^{Cb}\lambda_{\ell_2 b}^{D} \right) \left(\epsilon^{EFGH}\tilde{\lambda}_{\ell_1 E}^{\dot{a}}\tilde{\lambda}_{\ell_1 F\dot{a}} \,\tilde{\lambda}_{\ell_2 G}^{\dot{b}}\tilde{\lambda}_{\ell_2 H\dot{b}} \right) \,. \tag{3.4.8}$$

Hence, the two-particle cut reduces to

$$A_{4;1}\Big|_{\text{s-cut}} \propto (-1) \int \frac{d^6\ell}{(2\pi)^6} \delta^+(\ell_1^2)\delta^+(\ell_2^2) \left[\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \frac{\epsilon_{ABCD}\ell_1^{AB}\ell_2^{CD} \,\epsilon^{EFGH}\ell_{1EF}\ell_{2GH}}{s^2(\ell_1 + p_1)^2(\ell_2 + p_3)^2} \right] \,. \tag{3.4.9}$$

Next, we use (A.1.5) to rewrite

$$\epsilon_{ABCD}\, p_{\ell_1}^{AB} p_{\ell_2}^{CD}\, \epsilon^{EFGH} p_{\ell_1 EF} p_{\ell_2 GH} = 64\,(\ell_1 \cdot \ell_2)^2 \ . \tag{3.4.10}$$

Thus we obtain, for the one-loop superamplitude,

$$
\begin{aligned}
A_{4;1}\big|_{s\text{-cut}} &\propto -\,\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \int \frac{d^6\ell}{(2\pi)^6} \delta^+(\ell_1^2)\delta^+(\ell_2^2) \left[\frac{64\,(\ell_1 \cdot \ell_2)^2}{s^2(\ell_1 + p_1)^2(\ell_2 + p_3)^2} \right] \\
&= -\,16\,\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \int \frac{d^6\ell}{(2\pi)^6} \delta^+(\ell_1^2)\delta^+(\ell_2^2) \left[\frac{1}{(\ell_1 + p_1)^2(\ell_2 + p_3)^2} \right] \\
&= -\,16\,ist A_{4;0}(1,\dots,4)\,I_4(s,t)\big|_{s\text{-cut}} \ ,
\end{aligned}
\tag{3.4.11}
$$

where $A_{4;0}(1,\dots,4)$ is the tree-level four-point amplitude in (3.3.12), and

$$I_4(s,t) = \int \frac{d^6\ell}{(2\pi)^6} \left[\frac{1}{\ell_1^2 \ell_2^2 (\ell + p_1)^2 (\ell - p_4)^2} \right] . \tag{3.4.12}$$

The t-channel cut is performed in the same fashion and after inspecting it we conclude that

$$A_{4;1}(1,\dots,4) \ = \ st\, A_{4;0}(1,\dots,4)\, I_4(s,t) \ , \tag{3.4.13}$$

in agreement with the result of [21].

3.4.2 The superamplitude from quadruple cuts

We now move on to studying the quadruple cut of the one-loop four-point superamplitude, depicted in Figure 3.3. The loop momenta are defined as

$$\ell_1 = \ell, \qquad \ell_2 = \ell + p_1, \qquad \ell_3 = \ell + p_1 + p_2, \qquad \ell_4 = \ell - p_4 \ , \tag{3.4.14}$$

and all primed momenta ℓ_i' in Figure 3.3 are understood to flow in opposite direction to the ℓ_i's.

Four three-point tree-level superamplitudes enter the quadruple cut expression. Uplifting the cut by replacing cut with uncut propagators, we obtain, for the one-loop superamplitude,

$$
\begin{aligned}
A_{4;1} = \int \frac{d^6\ell}{(2\pi)^6} \Bigg[\prod_{i=1}^{4} \int d^2\eta_{l_i} d^2\tilde{\eta}_{l_i} \frac{1}{\ell_1^2} A_{3;0}(\ell_1, 1, \ell_2') \frac{1}{\ell_2^2} A_{3;0}(\ell_2, 2, \ell_3') \\
\times \frac{1}{\ell_3^2} A_{3;0}(\ell_3, 3, \ell_4') \frac{1}{\ell_4^2} A_{3;0}(\ell_4, 4, \ell_1') \Bigg] .
\end{aligned}
\tag{3.4.15}
$$
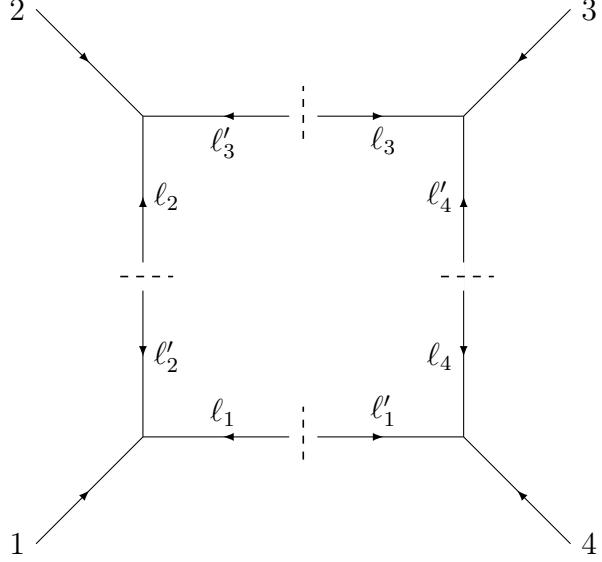
Figure 3.3 *The quadruple cut of a four-point superamplitude. The primed momenta ℓ'_i are defined as $\ell'_i := -\ell_i$.*

In the following we will discuss two different but equivalent approaches to evaluate the Grassmann integrals in (3.4.15).

Quadruple cut as reduced two-particle cuts

To begin with, we proceed in way similar to the case of a double cut. The idea is to integrate over two of the internal momenta, say ℓ_1 and ℓ_3 first, and treat ℓ_2 and ℓ_4 as fixed, i.e. external lines. In doing so the quadruple cut splits into two four-point tree-level superamplitudes, having the same structure as in case of the BCFW construction for the four-point tree-level superamplitude [21].

Let us start by focusing on the 'lower' part of the diagram first. Here we have two three-point superamplitudes connected by an internal (cut) propagator carrying momentum ℓ_1. Treating ℓ'_2 and ℓ_4 as external momenta (they are on-shell due to the cut) we can follow the procedure of a four-point BCFW construction. This involves rewriting fermionic δ-functions of both three-point amplitudes and integrating over $d^2\eta_{\ell_1} d^2\tilde{\eta}_{\ell_1}$, leading to the result

$$\delta^4(q_1 + q_{\ell'_2} + q_{\ell_4} + q_4)\delta^4(\tilde{q}_1 + \tilde{q}_{\ell'_2} + \tilde{q}_{\ell_4} + \tilde{q}_4)\, w^a_{\ell_1} w_{\ell'_1 a}\, \tilde{w}^{\dot{a}}_{\ell_1} \tilde{w}_{\ell'_1 \dot{a}}\,. \tag{3.4.16}$$

Note that the δ-functions are ensuring supermomentum conservation of the 'external momenta' and that we do not have an internal propagator with momentum ℓ_1 as in the recursive construction. Here, we get this propagator from uplifting the cut-expression for the one-loop amplitude. Furthermore, note that we do not have to shift any legs in order to use the BCFW prescription since the internal propagator is already on-shell due to the cut.

We may now perform the Grassmann integration over η_{ℓ_1} and $\tilde{\eta}_{\ell_1}$ in (3.4.16). Since the w-spinors are contracted we can simply use the spinor identity

$$w_{\ell_1}^a w_{\ell'_1 a} \tilde{w}_{\ell_1}^{\dot{a}} \tilde{w}_{\ell'_1 \dot{a}} = -s_{\ell_4 \ell'_2}^{-1} = -s_{14}^{-1}, \tag{3.4.17}$$

which is a direct generalization of the corresponding result from the BCFW construction (see also appendix A.3).

We can now turn to the 'upper' half of the cut-diagram. Following the description we derived above we get in a similar fashion after integrating over η_{ℓ_3} and $\tilde{\eta}_{\ell_3}$

$$\delta^4(q_{\ell_2} + q_2 + q_3 + q_{\ell'_4}) \delta^4(\tilde{q}_{\ell_2} + \tilde{q}_2 + \tilde{q}_3 + \tilde{q}_{\ell'_4}) w_{\ell_3}^a w_{\ell_3 a} \tilde{w}_{\ell'_3}^{\dot{a}} \tilde{w}_{\ell_3 \dot{a}}. \tag{3.4.18}$$

We also have

$$w_{\ell_1}^a w_{\ell'_1 a} \tilde{w}_{\ell_1}^{\dot{a}} \tilde{w}_{\ell'_1 \dot{a}} = -s_{\ell_2 \ell'_4}^{-1} = -s_{23}^{-1}. \tag{3.4.19}$$

Uplifting the quadruple cut, we get

$$A_{4;1} = \int \frac{d^6\ell}{(2\pi)^6} \int d^2\eta_{\ell_2} d^2\tilde{\eta}_{\ell_2} d^2\eta_{\ell_4} d^2\tilde{\eta}_{\ell_4} \left[\frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2} \frac{1}{s_{14} s_{23}} \right.$$

$$\times \delta^4(q_1 + q_{\ell'_2} + q_{\ell_4} + q_4) \delta^4(\tilde{q}_1 + \tilde{q}_{\ell'_2} + \tilde{q}_{\ell_4} + \tilde{q}_4)$$

$$\left. \times \delta^4(q_{\ell_2} + q_2 + q_3 + q_{\ell'_4}) \delta^4(\tilde{q}_{\ell_2} + \tilde{q}_2 + \tilde{q}_3 + \tilde{q}_{\ell'_4}) \right]. \tag{3.4.20}$$

Since $\ell'_i = -\ell_i$ we can use the constraints given by the $\delta^4(q_i)$ to eliminate the dependence of the remaining loop momenta in one of the sets of fermionic δ-functions and write it as a sum over external momenta only. The same argument holds for the Grassmann functions $\delta^4(\tilde{q}_i)$, and we find

$$A_{4;1} = \int \frac{d^6\ell}{(2\pi)^6} \int d^2\eta_{\ell_2} d^2\tilde{\eta}_{\ell_2} d^2\eta_{\ell_4} d^2\tilde{\eta}_{\ell_4} \left[\delta^4(Q_{\text{ext}}) \delta^4(\tilde{Q}_{\text{ext}}) \frac{1}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2} \frac{1}{s_{14} s_{24}} \right.$$

$$\left. \times \delta^4(q_{\ell_2} + q_2 + q_3 - q_{\ell_4}) \delta^4(\tilde{q}_{\ell_2} + \tilde{q}_2 + \tilde{q}_3 - \tilde{q}_{\ell_4}) \right], \tag{3.4.21}$$

where as before the Q^A_{ext} and $\tilde{Q}_{A\,\text{ext}}$ are the sums of all external supermomenta in η and $\tilde{\eta}$ respectively. The remaining integrations over η_{ℓ_2} and η_{ℓ_4} and their $\tilde{\eta}$-counterparts yield just as in the case of the two-particle cut

$$\epsilon_{ABCD}\,\ell_2{}^{AB}\ell_4{}^{CD}\,\epsilon^{EFGH}p_{\ell_2EF}p_{\ell_4GH} = 64\,(\ell_2 \cdot \ell_4)^2\,. \tag{3.4.22}$$

The product of the two loop momenta cancels with the factor

$$s_{14}s_{23} = 2(p_1 \cdot p_4)2(p_2 \cdot p_3) = 4(\ell'_2 \cdot \ell_4)(\ell_2 \cdot \ell'_4) = (-1)^2 4(\ell_2 \cdot \ell_4)^2. \tag{3.4.23}$$

so that our final result for the quadruple cut of the four-point superamplitude is

$$A_{4;1} \propto ist A_{4;0}(1,\ldots,4)\int \frac{d^6\ell}{(2\pi)^6}\left[\frac{16}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}\right]. \tag{3.4.24}$$

Hence we have shown that the quadruple cut gives the same structure as the two-particle cut discussed in Section 3.4.1.

Quadruple cut by Grassmann decomposition

In this section we will calculate the quadruple cut of the one-loop four-point superamplitude in an alternative fashion. Whereas in the last section we used the structure of the cut-expression to simplify the fermionic integrations, here we will explicitly perform the integrals by using constraints given by the δ-functions.

To perform the Grassmann integrations we work directly at the level of the three-point superamplitudes. The quadruple cut results in the following four on-shell tree-level amplitudes (see Figure 3.3)

$$A_3(\ell_1, 1, \ell'_2),\qquad A_3(\ell_2, 2, \ell'_3),\qquad A_3(\ell_3, 3, \ell'_4),\qquad A_3(\ell_4, 4, \ell'_1)\,. \tag{3.4.25}$$

Each of the three-point superamplitudes has the usual form [21]

$$A_{3,i} = i\left[\delta(Q^A_i)\delta(\tilde{Q}_{iA})\right]^2\delta(W_i)\delta(\tilde{W}_i)\,, \tag{3.4.26}$$

where $i = 1,\ldots,4$ labels the corners. The arguments of the δ-functions are

$$Q^A_i = q^A_{\ell_i} + q^A_i + q^A_{\ell'_{i+1}}\,,\qquad W_i = w^a_{\ell_i}\eta_{\ell_i a} + w^a_i\eta_{ia} + w^a_{\ell'_{i+1}}\eta_{\ell'_{i+1}a}\,, \tag{3.4.27}$$

with the identification $\ell_5 \equiv \ell_1$. Similar expressions hold for for $\tilde{Q}_{iA}$ and $\tilde{W}_i$. Note that since $\ell'_i = -\ell_i$ we find it convenient to define spinors with primed momenta ℓ'_i as

$$\lambda^A_{\ell'_i} = i\lambda^A_{\ell_i} \ , \qquad \tilde{\lambda}_{\ell'_i A} = \tilde{\lambda}_{\ell_i A} \ , \qquad \eta_{\ell'_i} = i\eta_{\ell_i} \ , \qquad \tilde{\eta}_{\ell'_i} = i\tilde{\eta}_{\ell_i} \ , \qquad (3.4.28)$$

which we will frequently use in the following manipulations.

We can use supermomentum conservation at each corner to reduce the number of δ-functions depending on the loop variables η_{ℓ_i} and $\tilde{\eta}_{\ell_i}$. There is a choice involved and we choose to remove the dependence of η_{ℓ_i} ($\tilde{\eta}_{\ell_i}$) from one copy of each $[\delta(Q^A_i)\delta(\tilde{Q}_{iA})]^2$. This yields for the Grassmann integrations

$$\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \int \prod_{i=1}^{4} d^2\eta_{\ell_i} d^2\tilde{\eta}_{\ell_i} \left[\delta(Q^A_i)\delta(\tilde{Q}_{iA})\delta(W_i)\delta(\tilde{W}_i) \right] \ . \qquad (3.4.29)$$

We can simplify the calculation by noticing that we have to integrate over 16 powers of Grassmann variables (8 powers of η and $\tilde{\eta}$ each) while at the same time we have 16 δ-functions in total. Therefore, when expanding the fermionic functions, each of them must contribute a power of Grassmann variables we are going to integrate over. Unless this is so, the result is zero. In other words, we can only pick the terms in the δ-functions that contribute an η_{ℓ_i} or $\tilde{\eta}_{\ell_i}$. This simplifies the structure considerably as we can drop all terms depending on external variables.

Equation (3.4.29) now becomes

$$\delta^4(Q_{\text{ext}})\delta^4(\tilde{Q}_{\text{ext}}) \int \prod_{i=1}^{4} d^2\eta_{\ell_i} d^2\tilde{\eta}_{\ell_i} \left[\delta(q^A_{\ell_i} - q^A_{\ell_{i+1}}) \, \delta(\tilde{q}_{\ell_i A} - \tilde{q}_{\ell_{i+1} A}) \right.$$
$$\left. \times \delta(w^a_{\ell_i}\eta_{\ell_i a} + iw^a_{\ell'_{i+1}}\eta_{\ell_{i+1} a})\delta(\tilde{w}^{\dot{a}}_{\ell_i}\tilde{\eta}_{\ell_i \dot{a}} + i\tilde{w}^{\dot{a}}_{\ell'_{i+1}}\tilde{\eta}_{\ell_{i+1} \dot{a}}) \right] \ . \qquad (3.4.30)$$

Notice that the w-spinors $w^a_{\ell'_{i+1}}$ are not identical to $w^a_{\ell_{i+1}}$.

Since the δ-functions only depend on the η_{ℓ_i} and $\tilde{\eta}_{\ell_i}$, we find convenient to decompose the integration variables as

$$\eta^a_{\ell_i} = u^a_{\ell_i}\eta^{\parallel}_{\ell_i} + w^a_{\ell_i}\eta^{\perp}_{\ell_i} \ , \qquad \tilde{\eta}^{\dot{a}}_{\ell_i} = \tilde{u}^{\dot{a}}_{\ell_i}\tilde{\eta}^{\parallel}_{\ell_i} + \tilde{w}^{\dot{a}}_{\ell_i}\tilde{\eta}^{\perp}_{\ell_i} \ , \qquad (3.4.31)$$

which implies

$$w_{\ell_i a}\eta^a_{\ell_i} = \eta^{\parallel}_{\ell_i} \ , \qquad u^a_{\ell_i}\eta_{\ell_i a} = \eta^{\perp}_{\ell_i} \ . \qquad (3.4.32)$$

Hence, we can rewrite the arguments of the δ-functions in the w-spinors as

$$iw^a_{\ell'_{i+1}}\eta_{\ell_{i+1}a} = iw^a_{\ell'_{i+1}}\left(u_{\ell_{i+1}a}\eta^{\parallel}_{\ell_{i+1}} + w_{\ell_{i+1}a}\eta^{\perp}_{\ell_{i+1}}\right) = \frac{i}{\sqrt{-s_{i,i+1}}}u^a_{\ell'_{i+1}}w_{\ell_{i+1}a}\eta^{\perp}_{\ell_{i+1}}$$

$$= \frac{i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{\ell_{i+1}} \, , \tag{3.4.33}$$

and similarly we have $i\tilde{w}^{\dot{a}}_{\ell'_{i+1}}\tilde{\eta}_{\ell_{i+1}\dot{a}} = \frac{i}{\sqrt{-s_{i,i+1}}}\tilde{\eta}^{\perp}_{\ell_{i+1}}$. Notice that we have used the fact that the $w'_{\ell_{i+1}}$ can be normalized such that they are proportional to the $u_{\ell_{i+1}}$ if the momenta fulfill the condition $\ell'_{i+1} = -\ell_{i+1}$. We give some more detail on such relations in Appendix A.3.2.

Using this, the δ-functions in the w-spinors become

$$\delta\left(-\eta^{\parallel}_{\ell_i} + \frac{i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{\ell_{i+1}}\right)\delta\left(-\tilde{\eta}^{\parallel}_{\ell_i} + \frac{i}{\sqrt{-s_{i,i+1}}}\tilde{\eta}^{\perp}_{\ell_{i+1}}\right) . \tag{3.4.34}$$

Next we proceed by integrating first over the $\eta^{\parallel}_{\ell_i}$ variables. This sets

$$\eta^{\parallel}_{\ell_i} = \frac{-i}{\sqrt{-s_{i,i+1}}}\eta^{\perp}_{\ell_{i+1}} \, , \tag{3.4.35}$$

with similar expressions for the $\tilde{\eta}^{\parallel}_{\ell_i}$. We then plug this into the remaining δ-functions of (3.4.30). First we notice that

$$\delta\left(\lambda^{Aa}_{\ell_i}\eta_{\ell_i a} + \lambda^{Aa}_{\ell'_{i+1}}\eta_{\ell'_{i+1}a}\right)\delta\left(\tilde{\lambda}^{\dot{a}}_{\ell_i A}\tilde{\eta}_{\ell_i \dot{a}} + \tilde{\lambda}^{\dot{a}}_{\ell'_{i+1}A}\tilde{\eta}_{\ell'_{i+1}\dot{a}}\right)$$

$$= \langle\ell^a_i|\ell'^{\dot{a}}_{i+1}]\eta_{\ell_i a}\tilde{\eta}_{\ell'_{i+1}\dot{a}} + \langle\ell'^a_{i+1}|\ell^{\dot{a}}_i]\eta_{\ell'_{i+1}a}\tilde{\eta}_{\ell_i \dot{a}} = -u^a_{\ell_i}\tilde{u}^{\dot{a}}_{\ell'_{i+1}}\eta_{\ell_i a}\tilde{\eta}_{\ell'_{i+1}\dot{a}} + u^a_{\ell'_{i+1}}\tilde{u}^{\dot{a}}_{\ell_i}\eta_{\ell'_{i+1}a}\tilde{\eta}_{\ell_i \dot{a}} \, . \tag{3.4.36}$$

The decomposition of the Grassmann spinors then yields

$$u^a_{\ell'_{i+1}}\eta_{\ell'_{i+1}a} = i\sqrt{-s_{i,i+1}}w^a_{\ell'_{i+1}}u_{\ell_{i+1}a}\eta^{\parallel}_{\ell_{i+1}} = -i\sqrt{-s_{i,i+1}}\,\eta^{\parallel}_{\ell_{i+1}} \tag{3.4.37}$$

and

$$\tilde{u}^{\dot{a}}_{\ell'_{i+1}}\tilde{\eta}_{\ell'_{i+1}\dot{a}} = -i\sqrt{-s_{i,i+1}}\,\tilde{\eta}^{\parallel}_{\ell_{i+1}} \, . \tag{3.4.38}$$

The remaining Grassmann integrations give

$$\int \prod_{i=1}^{4} d\eta_{\ell_i}^{\perp} d\tilde{\eta}_{\ell_i}^{\perp} \left[\delta\big(\lambda_{\ell_i}^{Aa}\eta_{\ell_i a} + \lambda_{\ell'_{i+1}}^{Aa}\eta_{\ell'_{i+1}a}\big) \delta\big(\tilde{\lambda}_{\ell_i A}^{\dot{a}}\tilde{\eta}_{\ell_i \dot{a}} + \tilde{\lambda}_{\ell'_{i+1}A}^{\dot{a}}\tilde{\eta}_{\ell'_{i+1}\dot{a}}\big) \right]$$

$$= \int \prod_{i=1}^{4} d\eta_{\ell_i}^{\perp} d\tilde{\eta}_{\ell_i}^{\perp} \left[i\sqrt{-s_{i,i+1}}\; \eta_{\ell_i}^{\perp}\tilde{\eta}_{\ell_{i+1}}^{\parallel} - i\sqrt{-s_{i,i+1}}\; \eta_{\ell_{i+1}}^{\parallel}\tilde{\eta}_{\ell_i}^{\perp} \right]$$

$$= \int \prod_{i=1}^{4} d\eta_{\ell_i}^{\perp} d\tilde{\eta}_{\ell_i}^{\perp} \left[\eta_{\ell_i}^{\perp}\tilde{\eta}_{\ell_{i+2}}^{\perp} - \eta_{\ell_{i+2}}^{\perp}\tilde{\eta}_{\ell_i}^{\perp} \right] , \qquad (3.4.39)$$

where we have used the solutions for $\eta_{\ell_i}^{\parallel}$ and $\tilde{\eta}_{\ell_i}^{\parallel}$ following (3.4.35). The integration is now straightforward, since the integrand is simply given by

$$\big(\eta_{\ell_1}^{\perp}\tilde{\eta}_{\ell_3}^{\perp} - \eta_{\ell_3}^{\perp}\tilde{\eta}_{\ell_1}^{\perp}\big)\big(\eta_{\ell_2}^{\perp}\tilde{\eta}_{\ell_4}^{\perp} - \eta_{\ell_4}^{\perp}\tilde{\eta}_{\ell_2}^{\perp}\big)\big(\eta_{\ell_3}^{\perp}\tilde{\eta}_{\ell_1}^{\perp} - \eta_{\ell_1}^{\perp}\tilde{\eta}_{\ell_3}^{\perp}\big)\big(\eta_{\ell_4}^{\perp}\tilde{\eta}_{\ell_2}^{\perp} - \eta_{\ell_2}^{\perp}\tilde{\eta}_{\ell_4}^{\perp}\big)$$

$$= 4\eta_{\ell_1}^{\perp}\tilde{\eta}_{\ell_3}^{\perp}\eta_{\ell_2}^{\perp}\tilde{\eta}_{\ell_4}^{\perp}\eta_{\ell_3}^{\perp}\tilde{\eta}_{\ell_1}^{\perp}\eta_{\ell_4}^{\perp}\tilde{\eta}_{\ell_2}^{\perp} . \qquad (3.4.40)$$

This yields

$$A_{4;1} \propto -4ist A_{4;0}(1,\ldots,4) \int \frac{d^6\ell}{(2\pi)^6} \left[\frac{1}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2} \right] , \quad (3.4.41)$$

recovering the expected result of [21] from two-particle cuts.

3.5   One-loop five-point superamplitude

We now move on to the one-loop five-point superamplitude and calculate its quadruple cuts. These cuts will reveal the presence of a linear pentagon integral, which will be reduced using standard Passarino-Veltman (PV) techniques [38] to a scalar pentagon plus scalar box integrals. Since the one-loop amplitude is in the maximally supersymmetric theory in six dimensions it is free of IR and UV divergences. Therefore, bubbles and triangles which would be UV divergent in six dimensions must be absent. It is for this reason that it will be enough to consider quadruple cuts, without having to inspect also triple and double cuts, which would be required if triangle and bubble functions were present.
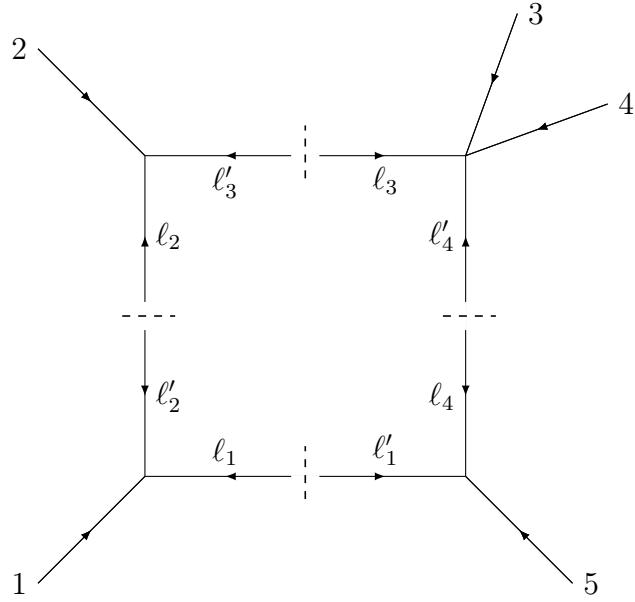
Figure 3.4   A specific quadruple cut of a five-point superamplitude. We choose to cut the legs such that we have the massive corner for momenta p_3, p_4.

3.5.1   Quadruple cuts

The quadruple cut has the structure

$$A_{5;1}\big|_{(3,4)\text{-cut}} = \int \frac{d^6\ell}{(2\pi)^6} \delta^+(\ell_1^2)\, \delta^+(\ell_2^2)\, \delta^+(\ell_3^2)\, \delta^+(\ell_4^2) \tag{3.5.1}$$
$$A_3(\ell_1, p_1, -\ell_2)\, A_3(\ell_2, p_2, -\ell_3)\, A_4(\ell_3, p_3, p_4, -\ell_4)\, A_3(\ell_4, p_5, -\ell_1)\,,$$

where the subscript $(3, 4)$ indicates where the massive corner is located, see Figure 3.4. In the following we will discuss this specific cut and all other cuts can be treated in an identical way.

From the three three-point superamplitudes and the four-point superamplitude, we have the following fermionic δ-functions,

$$\left[\delta(Q_1^A)\delta(\tilde{Q}_{1A})\right]^2 \delta(W_1)\delta(\tilde{W}_1) \left[\delta(Q_2^B)\delta(\tilde{Q}_{2B})\right]^2 \delta(W_2)\delta(\tilde{W}_2)$$
$$\times\, \delta^4(Q_3^C)\delta^4(\tilde{Q}_{3C}) \left[\delta(Q_4^D)\delta(\tilde{Q}_{4D})\right]^2 \delta(W_4)\delta(\tilde{W}_4)\,, \tag{3.5.2}$$

where the Q_i^A and the W_i are defined as sums over the supermomenta and products of w- and η-spinors respectively at a given corner (including internal legs). We may

now use the supermomentum constraints $Q_i^A = 0$ at all four corners and rewrite the $\delta^4(Q_3)\delta^4(\tilde{Q}_3)$ as a total δ^8 in the external momenta only,

$$\delta^4(Q_3) = \delta^4(Q_3 + Q_1 + Q_2 + Q_4) = \delta^4(Q_{\text{ext}}). \tag{3.5.3}$$

One is then left with the Grassmann integrations

$$\int \prod_{i=1}^{4} d^2\eta_{\ell_i} d^2\tilde{\eta}_{\ell_i} \left\{ \left[\delta(q_{\ell_1}^A + q_1^A - q_{\ell_2}^A)\delta(\tilde{q}_{\ell_1 A} + \tilde{q}_{1A} - \tilde{q}_{\ell_2 A}) \right]^2 \right.$$
$$\delta(w_{\ell_1}^a \eta_{\ell_1 a} + w_1^a \eta_{1a} + iw_{\ell_2'}^a \eta_{\ell_2 a})\delta(\tilde{w}_{\ell_1}^{\dot{a}} \tilde{\eta}_{\ell_1 \dot{a}} + \tilde{w}_1^{\dot{a}} \tilde{\eta}_{1\dot{a}} + i\tilde{w}_{\ell_2'}^{\dot{a}} \tilde{\eta}_{\ell_2 \dot{a}})$$
$$\left[\delta(q_{\ell_2}^B + q_2^B - q_{\ell_3}^B)\delta(\tilde{q}_{\ell_2 B} + \tilde{q}_{2B} - \tilde{q}_{\ell_3 B}) \right]^2$$
$$\delta(w_{\ell_2}^b \eta_{\ell_2 b} + w_2^b \eta_{2b} + iw_{\ell_3'}^b \eta_{\ell_3 b})\delta(\tilde{w}_{\ell_2}^{\dot{b}} \tilde{\eta}_{\ell_2 \dot{b}} + \tilde{w}_2^{\dot{b}} \tilde{\eta}_{2\dot{b}} + i\tilde{w}_{\ell_3'}^{\dot{b}} \tilde{\eta}_{\ell_3 \dot{b}})$$
$$\left[\delta(q_{\ell_4}^D + q_5^D - q_{\ell_1}^D)\delta(\tilde{q}_{\ell_4 D} + \tilde{q}_{5D} - \tilde{q}_{\ell_1 D}) \right]^2$$
$$\left. \delta(w_{\ell_4}^c \eta_{\ell_4 c} + w_5^c \eta_{5c} + iw_{\ell_1'}^c \eta_{1c})\delta(\tilde{w}_{\ell_4}^{\dot{c}} \tilde{\eta}_{\ell_4 \dot{c}} + \tilde{w}_5^{\dot{c}} \tilde{\eta}_{5\dot{c}} + i\tilde{w}_{\ell_1'}^{\dot{c}} \tilde{\eta}_{\ell_1 \dot{c}}) \right\} .$$
$$\tag{3.5.4}$$

Unfortunately, a decomposition as used for the quadruple cut of the four-point one-loop superamplitude is not immediately useful here. However, we notice that, due to the particular dependence of the δ-functions on the loop momenta ℓ_i, by removing a total δ^8 from the integrand one can restrict the dependence on the Grassmann variables η_{ℓ_3} and η_{ℓ_4} to six δ-functions each for this specific cut. This allows us to narrow the possible combinations of coefficients for, say, two powers of $\eta_{\ell_4 a}$ and two powers of $\tilde{\eta}_{\ell_4 \dot{a}}$. For example, two powers of $\eta_{\ell_4 a}$ can either come both from $\delta(Q_4^A)\delta(Q_4^B)$ or one from $\delta(Q_4^A)$ and one from[**] $\delta(W_4)$, and both possibilities needs to be appropriately contracted with the possible combinations from $\delta(\tilde{Q}_{4A})\delta(\tilde{Q}_{4B})\delta(\tilde{W}_4)$. If we choose both powers of $\eta_{\ell_4 a}$ from $\delta(Q_4^A)\delta(Q_4^B)$ we have a coefficient

$$\lambda_{\ell_4}^{Aa} \eta_{\ell_4 a} \lambda_{\ell_4}^{Bb} \eta_{\ell_4 b} , \tag{3.5.5}$$

which will be contracted at least by a $\tilde{\lambda}_{\ell_4 A}^{\dot{a}}$ or $\tilde{\lambda}_{\ell_4 B}^{\dot{a}}$ coming from the possible combinations for $\tilde{\eta}_{\ell_4 \dot{a}}$. As $\lambda_{ia}^A \tilde{\lambda}_{iA\dot{a}} = 0$, these terms vanish as shown in (3.2.4).

[**] This is similar to the recursive calculation of the five-point tree-level superamplitude in six dimensions, see also [21].

In conclusion, the only non-vanishing combination is

$$\lambda_{\ell_4}^{Aa}\eta_{\ell_4 a}\delta(\tilde{q}_{5A}-\tilde{q}_{\ell_1 A})\delta(q_5^B-q_{\ell_1}^B)\tilde{\lambda}_{\ell_4 B}^{\dot{a}}\tilde{\eta}_{\ell_4\dot{a}}w_{\ell_4}^b\eta_{\ell_4 b}\tilde{w}_{\ell_4'}^{\dot{b}}\tilde{\eta}_{\ell_4\dot{b}} \; . \tag{3.5.6}$$

The same argument holds for the expansion of the δ-functions depending on $\eta_{\ell_3 a}$ and $\tilde{\eta}_{\ell_3\dot{a}}$. Here, we only have to deal with additional signs and factors of i. We get for the expansion

$$(-1)\lambda_{\ell_3}^{Aa}\eta_{\ell_3 a}\delta(\tilde{q}_{\ell_2 A}+\tilde{q}_{2A})\delta(q_{\ell_2}^B+q_2^B)(-1)\tilde{\lambda}_{\ell_3 B}^{\dot{a}}\tilde{\eta}_{\ell_3\dot{a}}iw_{\ell_3'}^b\eta_{\ell_3 b}i\tilde{w}_{\ell_3'}^{\dot{b}}\tilde{\eta}_{\ell_3\dot{b}} \; . \tag{3.5.7}$$

This leads to the structure

$$\int\prod_{i=1}^{4}d^2\eta_{\ell_i}d^2\tilde{\eta}_{\ell_i}\Bigg\{ \left[\delta(q_{\ell_1}^A+q_1^A-q_{\ell_2}^A)\delta(\tilde{q}_{\ell_1 A}+\tilde{q}_{1A}-\tilde{q}_{\ell_2 A})\right]^2$$

$$\delta(w_{\ell_1}^a\eta_{\ell_1 a}+w_1^a\eta_{1a}+iw_{\ell_2'}^a\eta_{\ell_2 a})\delta(\tilde{w}_{\ell_1}^{\dot{a}}\tilde{\eta}_{\ell_1\dot{a}}+\tilde{w}_1^{\dot{a}}\tilde{\eta}_{1\dot{a}}+i\tilde{w}_{\ell_2'}^{\dot{a}}\tilde{\eta}_{\ell_2\dot{a}})$$

$$\lambda_{\ell_3}^{Cc}\eta_{\ell_3 c}\delta(\tilde{q}_{\ell_2 C}+\tilde{q}_{2C})\delta(q_{\ell_2}^D+q_2^D)\tilde{\lambda}_{\ell_3 D}^{\dot{c}}\tilde{\eta}_{\ell_3\dot{c}}(i)^2 w_{\ell_3'}^c\eta_{\ell_3 c}\tilde{w}_{\ell_3'}^{\dot{c}}\tilde{\eta}_{\ell_3\dot{c}}$$

$$\lambda_{\ell_4}^{Ed}\eta_{\ell_4 d}\delta(\tilde{q}_{5E}-\tilde{q}_{\ell_1 E})\delta(q_5^F-q_{\ell_1}^F)\tilde{\lambda}_{\ell_4 F}^{\dot{d}}\tilde{\eta}_{\ell_4\dot{d}}w_{\ell_4}^d\eta_{\ell_4 d}\tilde{w}_{\ell_4}^{\dot{d}}\tilde{\eta}_{\ell_4\dot{d}}\Bigg\} \; . \tag{3.5.8}$$

Notice that the six δ-functions of the first corner has not been expanded yet, therefore supermomentum conservation $Q_1^A=0$, $\tilde{Q}_1^A=0$ is still present. This constraint can be used to remove the dependence on $\eta_{\ell_2 a}$ in the third line of the above integrand, using $q_{\ell_2}^A=q_{\ell_1}^A+q_1^A$. The fermionic integral then becomes

$$\int\prod_{i=1}^{4}d^2\eta_{\ell_i}d^2\tilde{\eta}_{\ell_i}\Bigg\{ \left[\delta(q_{\ell_1}^A+q_1^A-q_{\ell_2}^A)\delta(\tilde{q}_{\ell_1 A}+\tilde{q}_{1A}-\tilde{q}_{\ell_2 A})\right]^2$$

$$\delta(w_{\ell_1}^a\eta_{\ell_1 a}+w_1^a\eta_{1a}+iw_{\ell_2'}^a\eta_{\ell_2 a})\delta(\tilde{w}_{\ell_1}^{\dot{a}}\tilde{\eta}_{\ell_1\dot{a}}+\tilde{w}_1^{\dot{a}}\tilde{\eta}_{1\dot{a}}+i\tilde{w}_{\ell_2'}^{\dot{a}}\tilde{\eta}_{\ell_2\dot{a}})$$

$$(i)^2\eta_{\ell_3 c}\tilde{\eta}_{\ell_3\dot{c}}\eta_{\ell_3 c'}\tilde{\eta}_{\ell_3\dot{c}'}\lambda_{\ell_3}^{Cc}\tilde{\lambda}_{\ell_3 D}^{\dot{c}}w_{\ell_3'}^{c'}\tilde{w}_{\ell_3'}^{\dot{c}'}\delta(\tilde{q}_{\ell_1 C}+\tilde{q}_{1C}+\tilde{q}_{2C})\delta(q_{\ell_1}^D+q_1^D+q_2^D)$$

$$\eta_{\ell_4 d}\tilde{\eta}_{\ell_4\dot{d}}\eta_{\ell_4 d'}\tilde{\eta}_{\ell_4\dot{d}'}\lambda_{\ell_4}^{Ed}\tilde{\lambda}_{\ell_4 F}^{\dot{d}}w_{\ell_4}^{d'}\tilde{w}_{\ell_4}^{\dot{d}'}\delta(\tilde{q}_{5E}-\tilde{q}_{\ell_1 E})\delta(q_5^F-q_{\ell_1}^F)\Bigg\} \; . \tag{3.5.9}$$

Just as before, only the first six δ-functions depend on $\eta_{\ell_2 a}$ and $\tilde{\eta}_{\ell_2\dot{a}}$ so we can expand straight away (noticing that this expansion yields yet another factor of $(i)^2$)

$$\int\prod_{i=1}^{4}d^2\eta_{\ell_i}d^2\tilde{\eta}_{\ell_i}\Bigg\{ (i)^2\eta_{\ell_2 b}\tilde{\eta}_{\ell_2\dot{b}}\eta_{\ell_2 b'}\tilde{\eta}_{\ell_2\dot{b}'}\lambda_{\ell_2}^{Ab}\tilde{\lambda}_{\ell_2 B}^{\dot{b}}w_{\ell_2'}^{b'}\tilde{w}_{\ell_2'}^{\dot{b}'}\delta(\tilde{q}_{\ell_1 A}+\tilde{q}_{1A})\delta(q_{\ell_1}^B+q_1^B)$$

$$(i)^2\eta_{\ell_3 c}\tilde{\eta}_{\ell_3\dot{c}}\eta_{\ell_3 c'}\tilde{\eta}_{\ell_3\dot{c}'}\lambda_{\ell_3}^{Cc}\tilde{\lambda}_{\ell_3 D}^{\dot{c}}w_{\ell_3'}^{c'}\tilde{w}_{\ell_3'}^{\dot{c}'}\delta(\tilde{q}_{\ell_1 C}+\tilde{q}_{1C}+\tilde{q}_{2C})\delta(q_{\ell_1}^D+q_1^D+q_2^D)$$

$$\eta_{\ell_4 d}\tilde{\eta}_{\ell_4\dot{d}}\eta_{\ell_4 d'}\tilde{\eta}_{\ell_4\dot{d}'}\lambda_{\ell_4}^{Ed}\tilde{\lambda}_{\ell_4 F}^{\dot{d}}w_{\ell_4}^{d'}\tilde{w}_{\ell_4}^{\dot{d}'}\delta(\tilde{q}_{5E}-\tilde{q}_{\ell_1 E})\delta(q_5^F-q_{\ell_1}^F)\Bigg\} \; . \tag{3.5.10}$$

One notes that, by expanding the fermionic δ-functions, the dependence on the Grassmann parameters $\eta_{\ell_1 a}$ and $\tilde{\eta}_{\ell_1 \dot{a}}$ has reduced to

$$\delta(\tilde{q}_{\ell_1 A} + \tilde{q}_{1A})\delta(q^B_{\ell_1} + q^B_1)\delta(\tilde{q}_{\ell_1 C} + \tilde{q}_{1C} + \tilde{q}_{2C})\delta(q^D_{\ell_1} + q^D_1 + q^D_2)\delta(\tilde{q}_{5E} - \tilde{q}_{\ell_1 E})\delta(q^F_5 - q^F_{\ell_1})$$
(3.5.11)

only. Expanding this further gives the sought-after coefficient of $\eta_{\ell_1 a}\eta_{\ell_1 b}\tilde{\eta}_{\ell_1 \dot{a}}\tilde{\eta}_{\ell_1 \dot{b}}$. The result (in an appropriate order of the Grassmann spinors) of the expansion of the six δ-functions in (3.5.11) is then given by

$$\tilde{\eta}_{\ell_1 \dot{a}}\tilde{\eta}_{\ell_1 \dot{b}}\left(-\tilde{\eta}_{1\dot{c}}\tilde{\lambda}^{\dot{a}}_{\ell_1 A}\tilde{\lambda}^{\dot{c}}_{1C}\tilde{\lambda}^{\dot{b}}_{\ell_1 E} - \tilde{\eta}_{2\dot{c}}\tilde{\lambda}^{\dot{a}}_{\ell_1 A}\tilde{\lambda}^{\dot{c}}_{2C}\tilde{\lambda}^{\dot{b}}_{\ell_1 E} - \tilde{\eta}_{5\dot{c}}\tilde{\lambda}^{\dot{a}}_{\ell_1 A}\tilde{\lambda}^{\dot{b}}_{\ell_1 C}\tilde{\lambda}^{\dot{c}}_{5E} + \tilde{\eta}_{1\dot{c}}\tilde{\lambda}^{\dot{c}}_{1A}\tilde{\lambda}^{\dot{a}}_{\ell_1 C}\tilde{\lambda}^{\dot{b}}_{\ell_1 E} \right)$$
$$\times \eta_{\ell_1 a}\eta_{\ell_1 b}\left(\eta_{5c}\lambda^{Ba}_{\ell_1}\lambda^{Db}_{\ell_1}\lambda^{Fc}_5 + \eta_{1c}\lambda^{Ba}_{\ell_1}\lambda^{Dc}_1\lambda^{Fb}_{\ell_1} + \eta_{2c}\lambda^{Ba}_{\ell_1}\lambda^{Dc}_2\lambda^{Fb}_{\ell_1} - \eta_{1c}\lambda^{Bc}_1\lambda^{Da}_{\ell_1}\lambda^{Fb}_{\ell_1} \right).$$
(3.5.12)

Having extracted the right powers of the Grassmann variables from all fermionic δ-functions, we can now integrate over the η_{l_i} and $\tilde{\eta}_{\ell_i}$. The integration is straightforward and yields,

$$\left(\tilde{\eta}_{1\dot{c}}[1^{\dot{c}}|\ell_3\rangle \cdot w_{\ell'_3} \; w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} - \tilde{\eta}_{1\dot{c}}[1^{\dot{c}}|\ell_2\rangle \cdot w_{\ell'_2} \; w_{\ell'_3} \cdot \langle\ell_3|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} \right.$$
$$\left. + \; \tilde{\eta}_{2\dot{c}}[2^{\dot{c}}|\ell_3\rangle \cdot w_{\ell'_3} \; w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} + \tilde{\eta}_{5\dot{c}}[5^{\dot{c}}|\ell_4\rangle \cdot w_{\ell_4} \; w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_3\rangle \cdot w_{\ell'_3} \right)$$
$$\times \left(\eta_{1c}\langle1^c|\ell_3\rangle \cdot \tilde{w}_{\ell'_3} \; \tilde{w}_{\ell'_2} \cdot [\ell_2|\hat{\ell}_1|\ell_4] \cdot \tilde{w}_{\ell_4} - \eta_{1c}\langle1^c|\ell_2\rangle \cdot \tilde{w}_{\ell'_2} \; \tilde{w}_{\ell'_3} \cdot [\ell_3|\hat{\ell}_1|\ell_4] \cdot \tilde{w}_{\ell_4} \right.$$
$$\left. + \; \eta_{2c}\langle2^c|\ell_3\rangle \cdot \tilde{w}_{\ell'_3} \; \tilde{w}_{\ell'_2} \cdot [\ell_2|\hat{\ell}_1|\ell_4] \cdot \tilde{w}_{\ell_4} + \eta_{5c}\langle5^c|\ell_4\rangle \cdot \tilde{w}_{\ell_4} \; \tilde{w}_{\ell'_2} \cdot [\ell_2|\hat{\ell}_1|\ell_3] \cdot \tilde{w}_{\ell'_3} \right).$$
(3.5.13)

Here we introduced the notation that $w_{\ell_i} \cdot \langle\ell_i| := w^a_{\ell_i}\langle\ell_{i,a}|$, and the $\hat{\ell}_i$ are slashed momenta, with e.g.

$$w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_3\rangle \cdot w_{\ell'_3} = w^a_{\ell'_2}\lambda^A_{\ell_2,a}\ell_{1,AB}\lambda^B_{\ell_3,b}w^b_{\ell'_3}.$$
(3.5.14)

Next, one rewrites the spinor expressions in (3.5.13) in terms of six-dimensional momenta, thereby removing any dependence on u- and w-spinors. An important observation to do so is the fact that the expressions depending on $\tilde{\eta}_1$ and/or η_1

antisymmetrize among themselves[††]. The result of these manipulations is

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{\ell}_1\hat{p}_5\hat{p}_2|1^c\rangle - \tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{\ell}_1\hat{p}_1|2^c\rangle + \tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\ell}_1\hat{p}_5\hat{p}_2|1^c\rangle$$

$$+\tilde{\eta}_{1\dot{c}}\eta_{5c}[1^{\dot{c}}|\hat{p}_2\hat{\ell}_1|5^c\rangle - \tilde{\eta}_{5\dot{c}}\eta_{1c}[5^{\dot{c}}|\hat{\ell}_1\hat{p}_2|1^c\rangle + \tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\ell}_1\hat{p}_5\hat{p}_1|2^c\rangle$$

$$+\tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}[5^{\dot{c}}|\hat{p}_1\hat{\ell}_1\hat{p}_2\hat{p}_1|5^c\rangle + \tilde{\eta}_{5\dot{c}}\eta_{2c}[5^{\dot{c}}|\hat{\ell}_1\hat{p}_1|2^c\rangle - \tilde{\eta}_{2\dot{c}}\eta_{5c}[2^{\dot{c}}|\hat{p}_1\hat{\ell}_1|5^c\rangle\,. \tag{3.5.15}$$

3.5.2   Final result (before PV reduction)
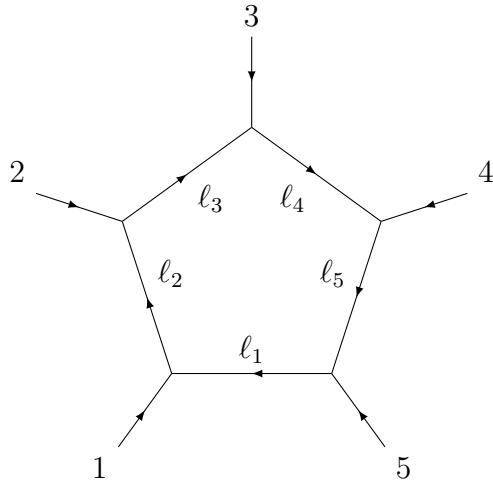
Figure 3.5   *A generic pentagon loop integral.*

Including all appropriate prefactors, our result for the five-point one-loop super-amplitude is expressed in terms of a single integral function, namely a linear pentagon integral. Explicitly,

$$A_{5;1} \;=\; \mathcal{C}_\mu\, I^\mu_{5,\ell_1}\,, \tag{3.5.16}$$

where

$$I^\mu_{5,\ell_1}(1,\dots,5) := \int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu_1}{\ell_1^2\ell_2^2\ell_3^2(p_3+\ell_3)^2\ell_5^2}\,, \tag{3.5.17}$$

[††]We give more details on these manipulations in Appendix A.3.2.

is the linear pentagon, and the coefficient $\mathcal{C}_\mu$ is given by

$$
\begin{aligned}
\mathcal{C}_\mu = \frac{1}{s_{34}} \Bigg\{ &\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{\sigma}_\mu\hat{p}_5\hat{p}_2|1^c\rangle - \tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{\sigma}_\mu\hat{p}_1|2^c\rangle + \tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_5\hat{p}_2|1^c\rangle \\
&+ \tilde{\eta}_{1\dot{c}}\eta_{5c}[1^{\dot{c}}|\hat{p}_2\hat{\sigma}_\mu|5^c\rangle - \tilde{\eta}_{5\dot{c}}\eta_{1c}[5^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_2|1^c\rangle + \tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}[2^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_5\hat{p}_1|2^c\rangle \\
&+ \tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}[5^{\dot{c}}|\hat{p}_1\hat{\sigma}_\mu\hat{p}_2\hat{p}_1|5^c\rangle + \tilde{\eta}_{5\dot{c}}\eta_{2c}[5^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_1|2^c\rangle - \tilde{\eta}_{2\dot{c}}\eta_{5c}[2^{\dot{c}}|\hat{\sigma}_\mu\hat{p}_1|5^c\rangle \Bigg\} . \qquad (3.5.18)
\end{aligned}
$$

The factor of $1/s_{34}$ and the additional propagator in the pentagon appearing in (3.5.16), are due to the prefactor of the four-point tree-level superamplitude entering the cut. We now proceed and summarize the result of the PV reduction of (3.5.17) in the next section.

3.5.3 Final result (after PV reduction)

In order to reduce the linear pentagon integral (3.5.17) to scalar integrals we use the Passarino-Velrman reduction method which is discussed in Appendix A.4. The linear pentagon can be decomposed on a basis of four independent momenta, as

$$
I_{5,\ell_1}^\mu(1,\ldots,5) = Ap_1^\mu + Bp_2^\mu + Cp_3^\mu + Dp_5^\mu . \qquad (3.5.19)
$$

The choice of $\{p_1, p_2, p_3, p_5\}$ as the basis vectors is the most convenient one due to the kinematical structure of the cut expression in (3.5.15). Contracting the linear pentagon with the basis momenta yields

$$
\begin{aligned}
2p_1 \cdot I_{5,\ell_1} &= \int \frac{d^D\ell}{(2\pi)^D} \frac{2p_1 \cdot \ell_1}{\prod_{i=1}^5 \ell_i^2} = I_{4,1} - I_{4,5} \overset{!}{=} Bs_{12} + Cs_{13} + Ds_{15} , \\
2p_2 \cdot I_{5,\ell_1} &= \int \frac{d^D\ell}{(2\pi)^D} \frac{2p_2 \cdot \ell_1}{\prod_{i=1}^5 \ell_i^2} = I_{4,2} - I_{4,1} - s_{12}I_5 \overset{!}{=} As_{12} + Cs_{23} + Ds_{25} , \\
2p_3 \cdot I_{5,\ell_1} &= \int \frac{d^D\ell}{(2\pi)^D} \frac{2p_3 \cdot \ell_1}{\prod_{i=1}^5 \ell_i^2} = I_{4,3} - I_{4,2} - (s_{12} + s_{23})I_5 \overset{!}{=} As_{13} + Bs_{23} + Ds_{35} , \\
2p_5 \cdot I_{5,\ell_1} &= \int \frac{d^D\ell}{(2\pi)^D} \frac{2p_5 \cdot \ell_1}{\prod_{i=1}^5 \ell_i^2} = I_{4,4} - I_{4,4} \overset{!}{=} As_{15} + Bs_{25} + Cs_{35} , \qquad (3.5.20)
\end{aligned}
$$

where we introduced the scalar integral functions I_5 for the pentagon and $I_{4,i}$ for the boxes. Here, the index i in $I_{4,i}$ labels the first leg of the massive corner for a

clockwise ordering of the external states as shown in Figure 3.6. In particular, the integral $I_{4,3}$ has the form

$$I_{4,3} := \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell_1^2(\ell_1+p_1)^2(\ell_1+p_1+p_2)^2(\ell_1+p_1+p_2+p_3+p_4)^2} \qquad (3.5.21)$$

The set of linear equations (3.5.20) can be solved to obtain the desired coefficients A, B, C and D:

$$
\begin{aligned}
A = \Delta^{-1} &\left\{ (s_{15}s_{23} - s_{13}s_{25}) \left[-I_{4,4}s_{23} + I_{4,5}s_{23} + [I_{4,2} - I_{4,3} + I_5(s_{13}+s_{23})]s_{25} \right] \right. \\
&+ \left[((I_{4,4}s_{12}s_{23} + (I_{4,1} - I_{4,2} + I_5 s_{12})s_{15}s_{23} \right. \\
&+ (I_{4,2}s_{12} - I_{4,3}s_{12} + I_{4,1}s_{13} - I_{4,2}s_{13} + 2I_5 s_{12}s_{13} + 2I_{4,1}s_{23} + I_5 s_{12}s_{23})s_{25} \\
&\left. \left. - I_{4,5}s_{23}(s_{12} + 2s_{25}) \right] s_{35} - s_{12}(I_{4,1} - I_{4,2} + I_5 s_{12})s_{35}^2 \right\}
\end{aligned} \qquad (3.5.22)
$$

$$
\begin{aligned}
B = \Delta^{-1} &\left\{ - \left[I_{4,4}s_{13} - I_{4,5}s_{13} - s_{15}(I_{4,2} - I_{4,3} + I_5(s_{13}+s_{23})) \right] (-s_{15}s_{23} + s_{13}s_{25}) \right. \\
&+ \left[I_{4,4}s_{12}s_{13} + s_{15}\left[I_{4,2}(s_{12} + 2s_{13}) - I_{4,1}(2s_{13} + s_{23}) - s_{12}(I_{43} + I_5 s_{13} - I_5 s_{23}) \right] \right. \\
&\left. - I_{41}s_{13}s_{25} + I_{45}(-s_{12}s_{13} + s_{15}s_{23} + s_{13}s_{25}) \right] s_{35} \\
&\left. + (I_{41} - I_{45})s_{12}s_{35}^2 \right\}
\end{aligned} \qquad (3.5.23)
$$

$$
\begin{aligned}
C = \Delta^{-1} &\left\{ -I_{41}s_{15}^2 s_{23} + I_{4,2}s_{15}^2 s_{23} - I_5 s_{12}s_{15}^2 s_{23} - 2I_{4,2}s_{12}s_{15}s_{25} \right. \\
&+ 2I_{4,3}s_{12}s_{15}s_{25} + I_{4,1}s_{13}s_{15}s_{25} - I_{4,2}s_{13}s_{15}s_{25} - I_5 s_{12}s_{13}s_{15}s_{25} - I_{4,1}s_{15}s_{23}s_{25} \\
&- 2I_5 s_{12}s_{15}s_{23}s_{25} + I_{4,1}s_{13}s_{25}^2 - I_{4,5}(s_{12}s_{15}s_{23} + s_{12}s_{13}s_{25} - s_{15}s_{23}s_{25} + s_{13}s_{25}^2) \\
&+ I_{4,5}s_{12}(s_{12} + s_{25})s_{35} + s_{12}\left((I_{4,1} - I_{4,2} + I_5 s_{12})s_{15} - I_{4,1}s_{25} \right) s_{35} \\
&\left. + I_{4,4}s_{12}(s_{15}s_{23} + s_{13}s_{25} - s_{12}s_{35}) \right\}
\end{aligned} \qquad (3.5.24)
$$

$$
\begin{aligned}
D = -I_5 + \Delta^{-1} &\left\{ -2I_{4,4}s_{12}s_{13}s_{23} + I_{4,2}s_{12}s_{15}s_{23} - I_{4,3}s_{12}s_{15}s_{23} + I_{4,1}s_{13}s_{15}s_{23} \right. \\
&- I_{4,2}s_{13}s_{15}s_{23} + 2I_5 s_{12}s_{13}s_{15}s_{23} + I_{4,1}s_{15}s_{23}^2 + I_5 s_{12}s_{15}s_{23}^2 + I_{4,2}s_{12}s_{13}s_{25} \\
&- I_{4,3}s_{12}s_{13}s_{25} - I_{4,1}s_{13}^2 s_{25} + I_{4,2}s_{13}^2 s_{25} - I_{4,1}s_{13}s_{23}s_{25} + I_5 s_{12}s_{13}s_{23}s_{25} \\
&- s_{12}(-I_{4,3}s_{12} - I_{4,1}s_{13} + I_{4,2}(s_{12} + s_{13}) + I_{4,1}s_{23} + I_5 s_{12}s_{23})s_{35} \\
&\left. + I_{4,5}s_{23}(-s_{15}s_{23} + s_{13}s_{25} + s_{12}(2s_{13} + s_{35})) \right\},
\end{aligned} \qquad (3.5.25)
$$

where Δ is the determinant of the "Gram" matrix (as defined in Appendix A.4) explicitly given by

$$\Delta = s_{15}^2 s_{23}^2 + (s_{13}s_{25} - s_{12}s_{35})^2 - 2s_{15}s_{23}(s_{13}s_{25} + s_{12}s_{35}) . \qquad (3.5.26)$$

It will however be more convenient to cast the one-loop five-point amplitude in the following form

$$A_{5;1} = \mathcal{C}^{(5)} I_5(1,\dots,5) + \sum_{i=1}^{5} \mathcal{C}^{(4,i)} I_{4,i}(1,\dots,5) , \qquad (3.5.27)$$

prioritizing the expansion over the integral basis.
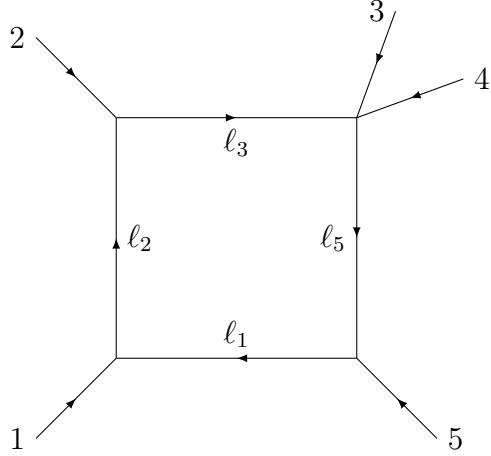
Figure 3.6 The five-point scalar integral $I_{4,3}$. It can be thought of as the result of "pinching out" the propagator between legs 3 and 4 of the pentagon.

Explicitly, the coefficients for the specific cut discussed in the previous section are given by

$$
\begin{aligned}
\mathcal{C}^{(5)/(4,3)} = {} & \tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}\Big(s_{12}[1^{\dot{c}}|\hat{p}_5\hat{p}_2|1^c\rangle A^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_3\hat{p}_5\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\Big) \\
& + \tilde{\eta}_{2\dot{c}}\eta_{2c}\frac{1}{s_{12}}\Big(s_{12}[2^{\dot{c}}|\hat{p}_5\hat{p}_1|2^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_5\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\Big) \\
& + \tilde{\eta}_{5\dot{c}}\eta_{5c}\frac{1}{s_{15}}\Big([5^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_2\hat{p}_1|5^c\rangle C^{(5)/(4,3)} + s_{15}[5^{\dot{c}}|\hat{p}_2\hat{p}_1|5^c\rangle D^{(5)/(4,3)}\Big) \\
& - \tilde{\eta}_{1\dot{c}}\eta_{2c}\frac{1}{s_{12}}\Big(s_{12}[1^{\dot{c}}|\hat{p}_2\hat{p}_5|2^c\rangle B^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_3\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\Big) \\
& + \tilde{\eta}_{2\dot{c}}\eta_{1c}\frac{1}{s_{12}}\Big(s_{12}[2^{\dot{c}}|\hat{p}_5\hat{p}_2|1^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3\hat{p}_5\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\Big) \\
& + \tilde{\eta}_{1\dot{c}}\eta_{5c}\Big([1^{\dot{c}}|\hat{p}_2\hat{p}_1|5^c\rangle A^{(5)/(4,3)} + [1^{\dot{c}}|\hat{p}_2\hat{p}_3|5^c\rangle C^{(5)/(4,3)}\Big) \\
& - \tilde{\eta}_{5\dot{c}}\eta_{1c}\Big([5^{\dot{c}}|\hat{p}_1\hat{p}_2|1^c\rangle A^{(5)/(4,3)} + [5^{\dot{c}}|\hat{p}_3\hat{p}_2|1^c\rangle C^{(5)/(4,3)}\Big) \\
& + \tilde{\eta}_{5\dot{c}}\eta_{2c}\Big([5^{\dot{c}}|\hat{p}_2\hat{p}_1|2^c\rangle B^{(5)/(4,3)} + [5^{\dot{c}}|\hat{p}_3\hat{p}_1|2^c\rangle C^{(5)/(4,3)}\Big) \\
& - \tilde{\eta}_{2\dot{c}}\eta_{5c}\Big([2^{\dot{c}}|\hat{p}_1\hat{p}_2|5^c\rangle B^{(5)/(4,3)} + [2^{\dot{c}}|\hat{p}_1\hat{p}_3|5^c\rangle C^{(5)/(4,3)}\Big) .
\end{aligned}
\qquad (3.5.28)
$$

Here, the variables $A^{(5)/(4,3)}, B^{(5)/(4,3)}, C^{(5)/(4,3)}$ and $D^{(5)/(4,3)}$ are the coefficients from the PV reduction of the scalar pentagon I_5 or box function $I_{4,3}$ respectively. For the scalar pentagon, we have

$$A^{(5)} = \Delta^{-1}(s_{15}s_{13}s_{23}s_{25} + s_{15}s_{25}s_{23}^2 - s_{13}s_{23}s_{25}^2 - s_{13}^2 s_{25}^2 + 2s_{12}s_{13}s_{25}s_{35}$$

$$+ s_{12}s_{23}s_{35}s_{15} + s_{12}s_{23}s_{25}s_{35} - s_{12}^2 s_{35}^2)$$

$$B^{(5)} = \Delta^{-1}s_{15}(s_{12}s_{23}s_{35} + s_{13}s_{23}s_{25} - s_{12}s_{13}s_{35} - s_{13}s_{23}s_{15} + s_{13}^2 s_{25} - s_{15}s_{23}^2)$$

$$C^{(5)} = \Delta^{-1}s_{12}s_{15}(s_{12}s_{35} - s_{15}s_{23} - s_{13}s_{25} - 2s_{23}s_{25})$$

$$D^{(5)} = \Delta^{-1}s_{12}s_{23}(s_{15}s_{23} + s_{13}s_{25} - s_{12}s_{35} + 2s_{15}s_{13}) \tag{3.5.29}$$

whereas for the coefficients of the box integral $I_{4,3}$ we find

$$A^{(4,3)} = \Delta^{-1}s_{25}(s_{13}s_{25} - s_{15}s_{23} - s_{12}s_{35})$$

$$B^{(4,3)} = \Delta^{-1}s_{15}(s_{15}s_{23} - s_{13}s_{25} - s_{12}s_{35})$$

$$C^{(4,3)} = \Delta^{-1}2s_{12}s_{15}s_{25}$$

$$D^{(4,3)} = \Delta^{-1}s_{12}(s_{12}s_{35} - s_{15}s_{23} - s_{13}s_{25}) \ . \tag{3.5.30}$$

Notice that for the final expression for the amplitude we have to collect the five box integrals $I_{4,i}$ with their respective coefficients which can be obtained by cyclic permutation of the states $(1, \dots, 5)$. Furthermore, we have to include one copy of the pentagon integral with its coefficient. The pentagon coefficient does not possess manifest cyclic symmetry, and each of the five quadruple cuts produces a different looking expression. However, the tests provided in the following section confirm that the pentagon coefficients have the expected cyclic symmetry.

3.5.4   Gluon component amplitude

In this section we extract from the one-loop five-point superamplitude its component where all external particles are six-dimensional gluons. This is useful since, dimensionally reducing this component amplitude to four dimensions, one can access the gluon MHV and $\overline{\text{MHV}}$ amplitudes of $\mathcal{N} = 4$ sYM.

In order to extract this component we have to integrate one power of η_i and $\tilde{\eta}_i$ for each external state, here denoted by $1_{a\dot{a}}$, $2_{b\dot{b}}$, $3_{c\dot{c}}$, $4_{d\dot{d}}$, and $5_{e\dot{e}}$. Doing this, one arrives at

$$
A_{5;1}\big|_{(3,4)\text{-cut}} \propto \int \frac{d^6\ell}{(2\pi)^6} \frac{1}{s_{34}} \delta^+(\ell_1^2)\delta^+(\ell_2^2)\delta^+(\ell_3^2)\delta^+(\ell_5^2) \frac{1}{(p_3+\ell_3)^2}
$$

$$
\times \left\{ \frac{1}{s_{12}}[1_{\dot{a}}|\hat{p}_2\hat{\ell}_1\hat{p}_5\hat{p}_2|1_a\rangle\langle 2_b3_c4_d5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \frac{1}{s_{12}}[2_{\dot{b}}|\hat{p}_1\hat{\ell}_1\hat{p}_5\hat{p}_1|2_b\rangle\langle 1_a3_c4_d5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\right.
$$

$$
+ \frac{1}{s_{15}}[5_{\dot{e}}|\hat{p}_1\hat{\ell}_1\hat{p}_2\hat{p}_1|5_e\rangle\langle 1_a2_b3_c4_d\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}]
$$

$$
- \frac{1}{s_{12}}[2_{\dot{b}}|\hat{p}_1\hat{\ell}_1\hat{p}_5\hat{p}_2|1_a\rangle\langle 2_b3_c4_d5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + \frac{1}{s_{12}}[1_{\dot{a}}|\hat{p}_2\hat{p}_5\hat{\ell}_1\hat{p}_1|2_b\rangle\langle 1_a3_c4_d5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]
$$

$$
- [1_{\dot{a}}|\hat{p}_2\hat{\ell}_1|5_e\rangle\langle 1_a2_b3_c4_d\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] + [5_{\dot{e}}|\hat{\ell}_1\hat{p}_2|1_a\rangle\langle 2_b3_c4_d5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}]
$$

$$
\left. - [5_{\dot{e}}|\hat{\ell}_1\hat{p}_1|2_b\rangle\langle 1_a3_c4_d5_e\rangle[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] + [2_{\dot{b}}|\hat{p}_1\hat{\ell}_1|5_e\rangle\langle 1_a2_b3_c4_d\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \right\}, \qquad (3.5.31)
$$

where ℓ_i, $i = 1, \ldots, 5$ are the five propagators in Figure 3.5. In the next section we perform the reduction to four dimension of (3.5.31), which will give us important checks on our result.

3.5.5   4D limit of the one-loop five-point amplitude

A series of nontrivial consistency checks on our six-dimensional five-point amplitude at one loop can be obtained by performing its reduction to four dimensions, and comparing it to the one-loop (MHV or $\overline{\text{MHV}}$) amplitude(s) directly calculated in four-dimensional $\mathcal{N} = 4$ sYM theory. In performing this reduction, we restrict any six-dimensional spinorial expression to four dimensions. As for the integral functions, we formally evaluate them in $6 - 2\epsilon$ dimensions. The four-dimensional limit is then obtained by simply replacing $\epsilon \to 1 + \epsilon$. Then, in order to perform the reduction to four dimensions of various six-dimensional spinorial quantities, we employ the results of [19] (see also [21]). There, it was found that the solutions to the Dirac equation with the external momenta living in a four-dimensional subspace,

i.e. $p = (p^0, p^1, p^2, p^3, 0, 0)$, can be written as

$$\lambda_a^A = \begin{pmatrix} 0 & \lambda_\alpha \\ \tilde{\lambda}^{\dot\alpha} & 0 \end{pmatrix}, \qquad \tilde{\lambda}_{A\dot a} = \begin{pmatrix} 0 & \lambda^\alpha \\ -\tilde{\lambda}_{\dot\alpha} & 0 \end{pmatrix}, \qquad (3.5.32)$$

where λ_α and $\tilde{\lambda}_{\dot\alpha}$ are the usual four-dimensional spinor variables. Hence, the Lorentz invariant, little group covariant quantities $\langle i_a | j_{\dot a}]$, $[i_{\dot a} | j_a \rangle$ become

$$\langle i_a | j_{\dot a}] = \begin{pmatrix} [ij] & 0 \\ 0 & -\langle ij \rangle \end{pmatrix}, \qquad [i_{\dot a} | j_a \rangle = \begin{pmatrix} -[ij] & 0 \\ 0 & \langle ij \rangle \end{pmatrix}. \qquad (3.5.33)$$

Here, we follow the standard convention of writing the four-dimensional spinor contractions as $\lambda_i^\alpha \lambda_{j\alpha} = \langle ij \rangle$ and $\tilde{\lambda}_{i\dot\alpha} \tilde{\lambda}_j^{\dot\alpha} = [ij]$.

The four-dimensional helicity group is a $\mathsf{U}(1)$ subgroup of the six-dimensional little group which preserves the structure of (3.5.32) and (3.5.33). In order to determine the (four-dimensional) helicity of a certain state in (3.2.21), a practical way to proceed is as follows. Each appearance of a dotted or undotted index equal to 1 (2) contributes an amount of $+1/2$ ($-1/2$) to the total four-dimensional helicity. As an example, consider the term $A_{a\dot a}$ in (3.2.21). States with $(a, \dot a) = (1, 1)$ correspond, upon reduction, to gluons with positive helicity and states with $(a, \dot a) = (2, 2)$ to gluons of negative helicity.

In the four-dimensional limit, the six-dimensional spinor brackets become[‡‡] [21]

$$-\langle i_+ | j_+] = [ij] = [i_+ | j_+ \rangle, \qquad \langle i_- | j_- \rangle = \langle ij \rangle = -[i_- | j_- \rangle, \qquad (3.5.34)$$
$$\langle i_- j_- k_+ l_+ \rangle = -\langle ij \rangle [kl], \qquad [i_- j_- k_+ l_+] = -\langle ij \rangle [kl],$$
$$\langle i_- j_+ k_- l_+ \rangle = +\langle ik \rangle [jl], \qquad [i_- j_+ k_- l_+] = +\langle ik \rangle [jl].$$

In the following we will use these identifications to check the four-dimensional limits of (3.5.31) for all MHV helicity assignments of the external gluons. As expected, we will always obtain the expected $\mathcal{N} = 4$ sYM result, i.e. the appropriate Parke-Taylor MHV prefactor multiplied by a four-dimensional one-loop box function.

[‡‡]Note that our definition of spinors of positive and negative helicities in four dimensions is opposite to that in [21], i.e. the spinor bracket $\langle \cdot , \cdot \rangle$ represents a product between spinors of negative helicity.

To begin with, we recall that upon four-dimensional reduction, a six-dimensional scalar pentagon reduces to five different box functions (plus terms vanishing in four dimensions) [50–52], and hence contributes to the coefficients of the relevant box functions. Schematically,

$$\mathcal{C}^{(5)}I_5 + \mathcal{C}^{(4,3)}I_{4,3} \xrightarrow{4D} \left[\mathcal{C}^{(5)} \frac{P^{(4,3)}}{2s_{12}s_{23}s_{34}s_{45}s_{51}} + \mathcal{C}^{(4,3)} \right] I_{4,3} \tag{3.5.35}$$

where

$$P^{(4,3)} = s_{12}s_{51}(s_{12}s_{23} - s_{12}s_{51} - s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{51}) \,, \tag{3.5.36}$$

when going to four dimensions. The authors of [50–52] show that the pentagon integral can be calculated using the recursive formula:

$$I_5^{D=6-2\varepsilon} \to \frac{1}{2} \sum_{i=1}^{5} c_i I_4^{(i)} \,, \tag{3.5.37}$$

where

$$c_i = \sum_{j=1}^{5} \mathcal{S}_{ij}^{-1} \tag{3.5.38}$$

and the $I_4^{(i)}$ boxes have a single massive external leg. The invariants $\mathcal{S}_{ij}$ are defined as

$$\mathcal{S}_{ij} \equiv m^2 - \frac{1}{2}p_{ij}^2 \,, \quad \text{with } p_{ii} \equiv 0 \,, \quad \text{and } p_{ij} = k_i + k_{i+1} + \cdots + k_{j-1} \,, \quad \text{for } i < j \,, \tag{3.5.39}$$

where they define k_i to be external momenta. In the present case all the masses are set to zero. The kinematic invariants $\mathcal{S}_{ij}$ are related to ours as

$$\mathcal{S}_{13} \to s_{12} \,, \quad \mathcal{S}_{24} \to s_{23} \,, \quad \mathcal{S}_{35} \to s_{34} \,, \quad \mathcal{S}_{41} \to s_{45} \,, \quad \mathcal{S}_{52} \to s_{51} \,. \tag{3.5.40}$$

Finally, we re-express invariants using only adjacent momenta, i.e.

$$s_{13} = s_{45} - s_{12} - s_{23} \tag{3.5.41}$$

$$s_{25} = s_{34} - s_{15} - s_{12} \tag{3.5.42}$$

$$s_{35} = s_{12} - s_{34} - s_{45} \,. \tag{3.5.43}$$

After some algebra one arrives at (3.5.35). Hence, upon dimensional reduction the coefficients of the PV reduction (3.5.29), (3.5.30) become (suppressing the box integral factor)

$$
\begin{aligned}
A &\rightarrow A^{(5)} \frac{P^{(4,3)}}{2s_{12}s_{23}s_{34}s_{45}s_{51}} + A^{(4,3)} &&= -\frac{s_{12}s_{15} - s_{15}s_{45} + s_{34}s_{45}}{2s_{23}s_{34}s_{45}}, && (3.5.44) \\
B &\rightarrow B^{(5)} \frac{P^{(4,3)}}{2s_{12}s_{23}s_{34}s_{45}s_{51}} + B^{(4,3)} &&= -\frac{s_{15}s_{12} - s_{15}s_{45}}{2s_{23}s_{34}s_{45}}, && \\
C &\rightarrow C^{(5)} \frac{P^{(4,3)}}{2s_{12}s_{23}s_{34}s_{45}s_{51}} + C^{(4,3)} &&= -\frac{s_{12}s_{15}}{2s_{23}s_{34}s_{45}}, && \\
D &\rightarrow D^{(5)} \frac{P^{(4,3)}}{2s_{12}s_{23}s_{34}s_{45}s_{51}} + D^{(4,3)} &&= -\frac{s_{12}}{2s_{34}s_{45}}\,. &&
\end{aligned}
$$

Let us now discuss specific helicity assignments by considering the configuration $(1^-, 2^-, 3^+, 4^+, 5^+)$ for the prefactor $A_{\mathrm{MHV}}|_{4D}$ of the integral function. In this case, after the PV reduction only the third term in (3.5.31) is non-vanishing. Hence, we have to consider the four-dimensional limit of

$$
\frac{1}{s_{34}s_{15}} \left([5_{\dot{e}}|\hat{p}_1\hat{p}_3\hat{p}_2\hat{p}_1|5_e\rangle C + [5_{\dot{e}}|\hat{p}_1\hat{p}_5\hat{p}_2\hat{p}_1|5_e\rangle D \right) \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}] . \qquad (3.5.45)
$$

Upon dimensional reduction, and using (3.5.34) the resulting contribution is

$$
\begin{aligned}
A_{\mathrm{MHV}}|_{4D} &= (s_{34}s_{15})^{-1} \left([51]^2 \langle 13 \rangle [23] \langle 12 \rangle C + [51]^2 \langle 15 \rangle [52] \langle 12 \rangle D \right) \langle 12 \rangle^2 [34]^2 \\
&= \frac{[51]^2 \langle 12 \rangle^2 [23] \langle 15 \rangle [21]}{2 \langle 23 \rangle [32] \langle 34 \rangle^2 [43]^2 \langle 45 \rangle [54] \langle 15 \rangle [51]} \left(\langle 13 \rangle [51] + [52] \langle 23 \rangle \right) \langle 12 \rangle^2 [34]^2 \\
&= \frac{[51] \langle 12 \rangle^4 [21]}{2 \langle 23 \rangle \langle 34 \rangle^2 \langle 45 \rangle [54]} \left(-\langle 23 \rangle [52] - [54] \langle 43 \rangle + [52] \langle 23 \rangle \right) \\
&= \frac{\langle 12 \rangle^4 [51][21]}{2 \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \\
&= \frac{s_{15}s_{12}}{2} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \qquad (3.5.46)
\end{aligned}
$$

where momentum conservation was used to write $\langle 13 \rangle [51] = -\langle 23 \rangle [52] - [54] \langle 43 \rangle$. Given the relation between the scalar box functions F_4 and the corresponding box integrals, $I_4 = 2F/(s_{12}s_{15})$, it is immediate to see that the kinematic factors in (3.5.46) cancel and the final result for the integral factor is the anticipated one:

$$
A(1^-, 2^-, 3^+, 4^+, 5^+)_{4D} = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} . \qquad (3.5.47)
$$

The fact that the form of the one-loop five-point amplitude upon reduction to four dimensions is precisely the well-known result is an expected, though highly non-trivial, outcome.

As mentioned above, we have performed checks for all external helicity configurations, finding in all cases agreement with the expected four-dimensional result. Further examples are presented in the Appendix A.5. We would like to highlight a particularly stringent test, namely that corresponding to the helicity configuration $(1^+, 2^+, 3^-, 4^-, 5^+)$, where all terms in (3.5.31) contribute to the four-dimensional reduction.

A final comment is in order here. It is known that collinear and soft limits put important constraints on tree-level and loop amplitudes in any gauge theory and in gravity [36]. In six dimensions, the lack of infrared divergences makes loop level factorization trivial, similarly to what happens to four-dimensional gravity because of its improved infrared behavior compared to four-dimensional Yang-Mills theory amplitudes. Therefore, the factorization properties we derive below from tree-level amplitudes will apply unmodified to one-loop amplitudes.

We now consider again the five-point amplitude (3.3.14) derived in [13], and take the soft limit where $p_1 \to 0$. A short calculation shows that

$$A^{(0)}_{5;a\dot{a}...} \to S_{a\dot{a}}(5,1,2)A^{(0)}_{4;...} \,, \tag{3.5.48}$$

where we find, for the six-dimensional soft function,

$$S_{a\dot{a}}(i,s,j) \;=\; \frac{\langle s_a|\hat{p}_j\hat{p}_i|s_{\dot{a}}]}{s_{is}s_{sj}} \,. \tag{3.5.49}$$

In (3.5.48) the dots stand for the little group indices of the remaining particles in the amplitude. Using the results in this section, it is also immediate to check that (3.5.49) reduces, in the four-dimensional limit, to the expected soft functions of [4]. As a final test on our five-point amplitude we have checked that the soft limits where legs 1, 2 or 5 become soft are all correct.

This provides an exhaustive set of checks of our result for the six-dimensional five-point superamplitude at one-loop.

4 | Two-loop Sudakov Form Factor in ABJM

This chapter focuses on amplitudes and form factors in three-dimensional $\mathcal{N} = 6$ Chern-Simons matter theory, also known as ABJM [53] and is based on [16]. This theory is closely related to the maximally supersymmetric theories constructed in [54, 55], and provides an interesting example of holographic duality in three dimensions.

4.1 Introduction

The interest in superconformal Chern-Simons theories was rekindled after the realization that it provided a description of the fundamental objects of the poorly understood M-theory [56]. A superconformal Chern-Simons theory with maximal supersymmetry was discovered by Bagger, Lambert and Gustavsson (BLG) [54, 55, 57] with the understanding that it described the dynamics of two interacting M2-branes [58, 59]. In order to generalize the descriptive power of the BLG theory to more than two interacting M2-branes, Aharony, Bergman, Jafferis and Maldacena (ABJM) [53] found a theory that is no longer maximally supersymmetric.

Besides the obvious physical motivation for studying ABJM theory, recent progress has revealed interesting structural similarities to the extensively studied $\mathcal{N} = 4$ super Yang-Mills (sYM). More specifically, the following have been discovered and studied in both theories:

- Dual Conformal Symmetry [60]

- Integrability [61–63]

- Duality with Wilson loops at four points [64–66]

- Uniform transcendentality of the two-loop four-point [67, 68]

- Six-point amplitude [69], color-kinematics duality [70].

Apart from the structural resemblance there is also interchangeability in computing methods between sYM and ABJM. The tools for computing amplitudes in four dimensions have been successfully modified to accommodate calculations in three dimensions. This has extended the usability of BCFW-type recursions relations [14], generalized unitarity [60, 71–73] and Grassmannian formulations [74] to address computational challenges in three dimensions.

With the broad aim of further exploring these similarities between ABJM and $\mathcal{N} = 4$ sYM, the present chapter focuses on the study of form factors in ABJM. Form factors are essentially an interpolation between fully on-shell quantities, i.e. scattering amplitudes, and correlation functions, which are inherently off shell. Remarkably, on-shell methods can be successfully applied in order to compute such quantities [75]. In general, form factors are obtained by acting with a local operator $\mathcal{O}(x)$ to the vacuum $|0\rangle$ and considering the overlap with a multi-particle state $\langle 1, \ldots, n|$.

In what follows we consider the Sudakov form factor of half-BPS operators, i.e. the overlap of a state created by an operator built from two scalars and a two-particle on-shell state. The study of the Sudakov form factor in $\mathcal{N} = 4$ sYM at one and two loops, was initiated by van Neerven in [76]. Recent studies of these quantities have produced a multitude of generalizations and extensions to one loop with more than two external on-shell particles [75, 77–79], and BPS operators with more than two scalars [77]. Applying generalized unitarity proved fruitful in calculating the two-loop, three-point form factor [80], as did the use of the symbol [81]. Both approaches provide examples that conform to the principle of maximal transcendentality that

was first observed in [82] for anomalous dimensions of composite operators. The calculation of the Sudakov form factor at three [83] and four [84] loops also showed that the result can be expressed in terms of uniformly transcendental integral functions. The above examples verify that much of the technology developed for the study of amplitudes also applies to the study of form factors. Interestingly, there are two important differences in the structures that appear in form factors. In particular, the absence of dual conformal symmetry and the appearance of non-planar integral topologies.

Massless gauge theories suffer from infrared singularities arising from soft and collinear loop momenta [85]. As explained in [85, 86] and summarized in [87] these singularities can be represented in terms of universal operators which in turn are related to the soft anomalous dimension entering the Sudakov form factor. In particular, in dimensional regularization, the Sudakov form factor obeys a differential equation, whose source term is the cusp anomalous dimension γ_K. Along with the constant of integration, these two functions control the infrared divergences to all orders for any amplitude in any massless gauge theory [87]. This provides a tool to predict the infrared divergent structure of loop amplitudes [85]. In the planar limit of $\mathcal{N} = 4$ sYM, the authors of [85] provide a general formula, which schematically reads:

$$\mathcal{M}_n = \prod_i \left(F(s_{i,i+1}, a_s, \varepsilon) \right)^{1/2} \times h_n(k_i, a_s, \varepsilon), \qquad (4.1.1)$$

where h_n is a hard remainder function. This formula captures the fact that IR divergences are universal and that are expressed through the Sudakov form factor $F(s_{i,i+1})$. A thorough treatment of Sudakov form factors can be found in [88]. In [89], Ashoke Sen provided a method to compute the on-shell Sudakov form factor in a non-Abelian theory to all orders of logarithms. Weinberg provides a diagrammatic treatment of the IR singulatities using form factors where he concludes that the anticipated cancellations in QCD are going to be complicated [90]. An outline of Weinberg's method to treat IR divergences using form factors can also be found in [18]. This property of the Sudakov form factor is expected to hold in other

dimensions, and in particular the relevant similarities between ABJM and $\mathcal{N} = 4$ sYM suggest that it is also valid in the present case of interest (a thorough discussion can be found in [67]).

The present chapter describes the calculation of the two-loop Sudakov form factor in ABJM, which is found to have a uniform degree of transcendentality and correctly captures the infrared divergences of two-loop amplitudes (also presented in [16]). Contrary to $\mathcal{N} = 4$ sYM the result is expressed as a single non-planar integral function with an untypical numerator. In $\mathcal{N} = 4$ sYM the two-loop form factor is split into a planar and a non-planar integral function with trivial numerators that are separately transcendental [76]. However, in ABJM the numerator of the integral function ensures two important properties: internal three-particle vertices cannot result in unphysical infrared divergences and the result is automatically transcendental. In fact, investigating several planar two-loop topologies shows that numerators that remove unphysical infrared divergences render the integrals transcendental. These planar topologies do not appear in the ABJM form factor but are ingredients of the form factors in ABJ which is also transcendental. The aforementioned properties are studied in detail in subsequent sections.

In [69], the calculation of two-loop amplitudes in ABJM, revealed constraints that are imposed on the integral functions stemming from the vanishing of the triple cuts which isolate three- and five-particle amplitudes. Here the vanishing triple cuts that involve only three-particle amplitudes guarantee the absence of unphysical infrared divergences. Simultaneously, they eliminate terms that would spoil the uniform transcendentality of the result. On a practical level, the vanishing triple cuts provide stringent constraints that one can use to check the consistency of the integral functions. This observation points to a familiar circle of questions about the choice of integral basis, based on physical as well as practical criteria. Demanding that the expansion coefficients are ϵ-independent, and that the integral functions are individually transcendental, seems to ensure desirable features. However, the underlying physical and mathematical criteria are not well understood. Form factors

in ABJ(M) provide a suitable setting to approach these issues.

After reviewing some general properties of ABJM amplitudes in Section 4.2, the calculation of the non-planar part of the one-loop four-point amplitude in this theory is presented in Section 4.3. Section 4.4 discusses the complete – planar plus non-planar – integrand of the four-point amplitude, which is a key ingredient in the construction of the Sudakov form factor. This quantity is derived firstly at one and then at two loops, using two- and three-particle cuts at the level of the integrand. As mentioned earlier, the result is expressed in terms of a single non-planar integral topology with a special numerator, whose properties are discussed in detail. In particular, certain three-vertex cuts are considered, which put strong constraints on the form of these numerators. Our result is then compared to the known infrared divergences of ABJM amplitude at two loops, where we find complete agreement. Finally, Section 4.5 presents three planar integral topologies which contribute to the ABJ form factor. Their properties are discussed and their maximally transcendental result is presented. Appendix B contains details and conventions on: spinors in three dimensions; half-BPS operators in ABJM; the one-loop box function in terms of which the four-point amplitude is expressed; and, on the reduction to master integrals of the integral topologies.

4.2   Scattering amplitudes in ABJM theory

The following provides a brief review of some key facts of the ABJM theory, and in particular of its tree amplitudes, which appear in the construction of loop amplitudes and form factors using unitarity [12, 36, 37, 91].*

4.2.1   Superamplitudes

Three-dimensional $\mathcal{N} = 6$ Chern-Simons matter theory [53] (or, in short, ABJM) is a quiver theory with gauge group $\mathsf{U}_k(N) \times \mathsf{U}_{-k}(N)$, where k and $-k$ are the

---

*In this paper we follow the conventions of Section 2 and Appendix A of [72] for the ABJM superamplitudes and the three-dimensional spinor helicity formalism, respectively.

Chern-Simons levels of the gauge fields A_μ and $\hat{A}_\mu$, respectively. The matter fields comprise four complex scalars ϕ^A and four fermions ψ_A^α, where $A = 1, \ldots, 4$ is a $\mathsf{SU}(4)$ R-symmetry index and $\alpha = 1, 2$ is a spin index. The fields $(\phi^A)_{\bar{j}}^i$ and $(\psi_A^\alpha)_{\bar{j}}^i$ transform in the bifundamental representation $(N, \bar{N})$ of the gauge group, while $(\bar{\phi}_A)_j^{\bar{i}}$ and $(\bar{\psi}_\alpha^A)_j^{\bar{i}}$ transform in the $(\bar{N}, N)$, with $i, \bar{j} = 1, \ldots, N$. An interesting variant of ABJM is the so-called ABJ theory, i.e. $\mathcal{N} = 6$ Chern-Simons theory with gauge group $\mathsf{U}_k(N) \times \mathsf{U}_{-k}(N')$. In this case $i = 1, \ldots, N$, $\bar{j} = 1, \ldots, N'$. Note that in the ABJ(M) theory the gauge fields are non-dynamical because of the topological nature of the Chern-Simons action, and hence they cannot appear as external states.

The momenta of the particles can be written efficiently in the three-dimensional spinor helicity formalism as

$$p_{\alpha\beta} := \lambda_\alpha \lambda_\beta \,, \qquad (4.2.1)$$

where λ_α are commuting spinors (Appendix B.1 provides further details on spinors in the present setup). The states of the ABJM theory can be packaged into two Nair superfields [26, 45],

$$\Phi(\lambda, \eta) = \phi^4(\lambda) + \eta^A \psi_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\phi^C(\lambda) + \frac{1}{3!}\epsilon_{ABC}\eta^A\eta^B\eta^C\psi_4(\lambda)\,, \quad (4.2.2)$$

$$\bar{\Phi}(\lambda, \eta) = \bar{\psi}^4(\lambda) + \eta^A\bar{\phi}_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\bar{\psi}^C(\lambda) + \frac{1}{3!}\epsilon_{ABC}\eta^A\eta^B\eta^C\bar{\phi}_4(\lambda)\,, \quad (4.2.3)$$

where η^A, $A = 1, 2, 3$ are Grassmann coordinates parameterizing an $\mathcal{N} = 3$ superspace. The superfields Φ and $\bar{\Phi}$ carry color indices $\Phi_{\bar{j}}^i$ and $\bar{\Phi}_j^{\bar{i}}$. Note that Φ is bosonic while $\bar{\Phi}$ is fermionic. This description breaks the $\mathsf{SU}(4)$ R-symmetry of the theory down to a manifest $\mathsf{U}(3)$.

Color-ordered partial amplitudes were introduced in [92], and we denote them as $\mathcal{A}(\bar{\Phi}_1, \Phi_2, \ldots, \Phi_n)$. An important feature of ABJ(M) is that any amplitude with an odd number of particles vanishes, as a simple consequence of gauge invariance. Invariance under translations and supersymmetry transformations ensures that amplitudes are proportional to $\delta^{(3)}(P)\delta^{(6)}(Q)$, where Q_α^A and $P_{\alpha\beta}$ are the total momentum and supermomentum of n particles, respectively:

$$P_{\alpha\beta} := \sum_{i=1}^n \lambda_{i,\alpha}\lambda_{i,\beta}\,, \qquad Q_\alpha^A := \sum_{i=1}^n \lambda_{i,\alpha}\eta_i^A\,. \qquad (4.2.4)$$
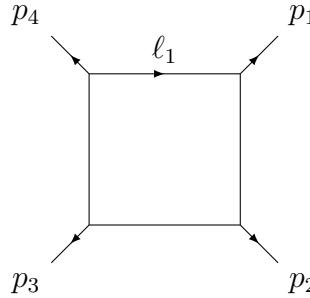
Figure 4.1   The one-loop box function in (4.2.7).

The first non-vanishing amplitude of the theory occurs at four points, and is the basic building block of higher-point amplitudes. At tree level it is given by the following compact expression [93],

$$\mathcal{A}_4^{(0)}(\bar{1}, 2, \bar{3}, 4) \; = \; i\, \frac{\delta^{(6)}(Q)\delta^{(3)}(P)}{\langle 1\,2\rangle\,\langle 2\,3\rangle}\,. \tag{4.2.5}$$

As usual, component amplitudes can be obtained by extracting the coefficient of the appropriate monomial in the η_i variables. For instance, in order to pick the component amplitude $A\big(\bar{\phi}_A(p_1), \phi^4(p_2), \bar{\phi}_4(p_3), \phi^A(p_4)\big)$ we need to expand the fermionic delta function $\delta^{(6)}(Q)$ and keep the term $(\eta_1)^1(\eta_2)^0(\eta_3)^3(\eta_4)^2\langle 1\,3\rangle\langle 3\,4\rangle^2$.

The one-loop color-ordered four-point superamplitude[†] was constructed in [67], and is equal to[‡]

$$\mathcal{A}^{(1)}(\bar{1}, 2, \bar{3}, 4) \; = \; i\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4)\, N\, I(1, 2, 3, 4)\,, \tag{4.2.6}$$

where $s_{ij} = (p_i + p_j)^2$. The one-loop integral $I(1, 2, 3, 4)$ is defined by

$$I(1, 2, 3, 4) \; := \; \int \frac{d^D\ell}{i\pi^{D/2}}\, \frac{s_{12}\,\mathrm{Tr}(\ell\, p_1\, p_4) \,+\, \ell^2\,\mathrm{Tr}(p_1\, p_2\, p_4)}{\ell^2(\ell - p_1)^2(\ell - p_1 - p_2)^2(\ell + p_4)^2}\,, \tag{4.2.7}$$

with $D = 3 - 2\epsilon$. Note that $\mathrm{Tr}(abc) = 2\epsilon(a, b, c) := 2\epsilon_{\mu\nu\rho}a^\mu b^\nu c^\rho$.

Explicit evaluation of the right-hand side of (4.2.6) shows that $\mathcal{A}^{(1)}(\bar{1}, 2, \bar{3}, 4)$ is of $\mathcal{O}(\epsilon)$, and hence vanishes in three dimensions [67]. This is consistent with the fact that all one-loop amplitudes in ABJM can be expanded in terms of one-loop triangle

---

[†] Here, and in what follows, we use the normalization $1/(i\pi^{D/2})$ per loop. In [67], the normalization is $1/(2\pi)^D$.

[‡] We suppress the Chern-Simons level k, which will be reinstated at the end of our two-loop calculation.

functions [71], as expected from dual conformal invariance. The vanishing of the four-point amplitude then follows since one-mass (and two-mass) triangles vanish in three dimensions. Very interestingly, the box function with the particular numerator in (4.2.7) is also dual conformal invariant, as was demonstrated in [67] using a five-dimensional embedding formalism. Furthermore, the expression for $\mathcal{A}^{(1)}(\bar{1}, 2, \bar{3}, 4)$ given in (4.2.6) is correct to all orders in the dimensional regularization parameter ϵ. In the following, the integrand of (4.2.7) will be a crucial ingredient in applying unitarity at the integrand level.

4.2.2   Color ordering at tree level

As is well known from experience in $\mathcal{N} = 4$ sYM, starting from two loops, cuts of form factors receive contributions from non-planar amplitudes which are as leading as those arising from the planar amplitudes (see for example [80,83]). For our present purposes, we will need the complete (planar and non-planar) one-loop amplitude in ABJM at four points. The complete tree amplitudes, denoted here by $\tilde{\mathcal{A}}$, are given by [63,92]

$$\tilde{\mathcal{A}}(\bar{1}, 2, \ldots, n) = \sum_{\mathcal{P}_n} \mathrm{sgn}(\sigma) \mathcal{A}^{(0)}\big(\sigma(\bar{1}), \sigma(2), \sigma(\bar{3}), \ldots, \sigma(n)\big) \big[\sigma(\bar{1}), \sigma(2), \sigma(\bar{3}), \ldots, \sigma(n)\big],$$
$$(4.2.8)$$

where $\mathcal{P}_n := (S_{n/2} \times S_{n/2})/C_{n/2}$ are permutations of n sites that only mix even (bosonic) and odd (fermionic) particles among themselves, modulo cyclic permutations by two sites. The function $\mathrm{sgn}(\sigma)$ is equal to -1 if σ involves an odd permutation of the odd (fermionic) sites, and $+1$ otherwise. $\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, \ldots, n)$ are color-ordered tree amplitudes, and we have also defined

$$\big[\bar{1}, 2, \bar{3}, \ldots, n\big] := \delta_{i_2}^{\bar{i}_1} \delta_{i_3}^{i_2} \delta_{i_4}^{\bar{i}_3} \cdots \delta_{i_1}^{i_n}.$$
$$(4.2.9)$$

For example, the n-point color-ordered amplitude under a cyclic permutation of their arguments by two sites is

$$\mathcal{A}^{(0)}(\bar{3}, \ldots, n, \bar{1}, 2) = (-1)^{(n-2)/2} \mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, \ldots, n),$$
$$(4.2.10)$$

which reflects the fact that odd sides are fermionic and even sites are bosonic. This marks another difference to the four dimensional color-ordered amplitude where the amplitude (2.2.18) has a manifest cyclic symmetry. Furthermore, in all tree-level Feynman diagrams, each external particle is connected to an antiparticle (and vice versa) by a fundamental color line and to another by an anti-fundamental color line as can be seen in Figure 4.2 [92]. This implies that a scattering process is non-vanishing only if there is a matching number of particles and anti-particles, i.e. only amplitudes with an even number of external particles are non-vanishing. This is another difference compared to the four dimensional case where there is no such constraint. For the superamplitude (4.2.5) the two-site cyclic property (4.2.10) is ensured by momentum conservation. For example, in the 4-point case

$$
\begin{aligned}
\mathcal{A}_4^{(0)}\big(\bar{1}, 2, \bar{3}, 4\big) &= i\,\frac{\delta^{(6)}(Q)\delta^{(3)}(P)}{\langle 3\,2\rangle\,\langle 2\,1\rangle} \\
&= i\,\frac{\delta^{(6)}(Q)\delta^{(3)}(P)}{-\langle 3\,4\rangle\,\langle 4\,1\rangle} \\
&= -\mathcal{A}_4^{(0)}\big(\bar{3}, 4, \bar{1}, 2\big)\,.
\end{aligned}
\tag{4.2.11}
$$

In the following we will just write $\big[1, 2, \cdots, n\big]$ without specifying if a particle is barred (i.e. fermionic) or un-barred (bosonic), with the understanding that the first entry in the bracket always represents a fermionic field.

As an example, consider the complete four-point amplitude at tree level. It includes the two color structures $[1, 2, 3, 4]$ and $[1, 4, 3, 2]$ (see Figure 4.2) and is given by the following expression:

$$
\tilde{\mathcal{A}}^{(0)}(\bar{1}, 2, \bar{3}, 4) = \mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4)\Big(\big[1, 2, 3, 4\big] - \big[1, 4, 3, 2\big]\Big)\,.
\tag{4.2.12}
$$

We have also used that

$$
\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) = \mathcal{A}^{(0)}(\bar{3}, 2, \bar{1}, 4)\,,
\tag{4.2.13}
$$

a fact that follows from (4.2.5).
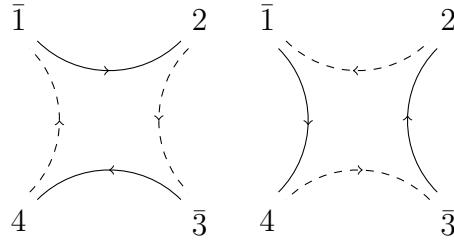
Figure 4.2 The two possible color orderings $[1, 2, 3, 4]$ and $[1, 4, 3, 2]$ appearing in the four-point tree-level amplitude (4.2.12).

4.2.3 Symmetry properties of amplitudes

It is useful to recall the following general relations for color-ordered amplitudes [71]:

$$\mathcal{A}^{(l)}(\bar{1}, 2, \bar{3} \cdots , n) = (-)^{\frac{n}{2}-1} \mathcal{A}^{(l)}(\bar{3}, 4, \cdots , \bar{1}, 2) , \qquad (4.2.14)$$

and

$$\mathcal{A}^{(l)}(\bar{1}, 2, \bar{3} \cdots , n) = (-)^{\frac{n(n-2)}{8}+l} \mathcal{A}^{(l)}(\bar{1}, n, \overline{n-1}, n-2, \cdots , \bar{3}, 2) . \qquad (4.2.15)$$

It should be noted that complete amplitudes should behave under exchange of any two particles as the spin-statistics theorem requires. In particular it is expected that

$$\tilde{\mathcal{A}}^{(l)}(\bar{1}, 2, \bar{3}, 4) = -\tilde{\mathcal{A}}^{(l)}(\bar{3}, 2, \bar{1}, 4) , \qquad (4.2.16)$$

at any loop order. This explains the presence of the crucial minus sign between the two possible color structures in (4.2.12).

4.2.4 Dual Conformal Symmetry

The structure of the color ordered amplitude revealed some differences between the three dimensional ABJM theory and the four dimensional $\mathcal{N} = 4$ sYM. An important similarity however, is the dual conformal symmetry is a symmetry of both theories. This was exploited in particular in [67] to construct an integral basis and calculate the two-loop four-point amplitude. As a result, the authors were able to construct the one-loop box function (4.2.7) which provides a starting point for the present calculation.

Dual conformal invariance was first studied in [94]. The first step is to define the dual variables

$$x_i^{ab} - x_{i+1}^{ab} = p_i^{ab}, \quad \text{and} \quad \theta_{iA}^a - \theta_{i+1,A}^a = q_{iA}^a, \tag{4.2.17}$$

where q is supermomentum and x is not a coordinate in spacetime but in the so-called dual space. Then, momentum and supermomentum δ-functions for a four-point superamplitude become

$$\delta^{(3)}(P)\delta^{(\mathcal{N})}(Q) \to \delta^{(3)}(x_1 - x_5)\delta^{(\mathcal{N})}(\theta_1^a - \theta_5^a). \tag{4.2.18}$$

Next, in any spacetime dimension, the dual conformal inversion $I[\ ]$ is defined as

$$I[x_i^\mu] = \frac{x_i^\mu}{x_i^2}, \quad I\big[|\theta_{iA}\rangle^{\dot a}\big] = \langle\theta_{iA}|_{\dot b}\frac{x_i^{\dot b a}}{x_i^2}, \tag{4.2.19}$$

$$I\big[[i|^a\big] = \frac{x_i^{\dot a b}}{x_i^2}|i]_b \qquad I\big[|i\rangle^{\dot a}\big] = \langle i|_{\dot b}\frac{x_{i+1}^{\dot b a}}{x_{i+1}^2}. \tag{4.2.20}$$

These dual coordinates can be used directly to denote known quantities and invariants, such as

$$x_{i,i+2}^2 = s_{i,i+1} = \langle i|i+1\rangle^2. \tag{4.2.21}$$

Therefore the inversion of a three-dimensional angle bracket is

$$I[\langle i|i+1\rangle] = \frac{\langle i|i+1\rangle}{\sqrt{x_i^2 x_{i+2}^2}}. \tag{4.2.22}$$

This can be applied directly to the four-point superamplitude (4.2.5) to give

$$I\left[\mathcal{A}_4^{(0)}(\bar{1},2,\bar{3},4)\right] = \sqrt{x_1^2 x_2^2 x_3^2 x_4^2}\,\mathcal{A}_4^{(0)}(\bar{1},2,\bar{3},4). \tag{4.2.23}$$

The structure for the n-point amplitude is actually the same [14]

$$I[\mathcal{A}_n] = \left(\prod_{i=1}^n \sqrt{x_i^2}\right)\mathcal{A}_n, \tag{4.2.24}$$

which shows that the tree-level superamplitudes in ABJM are dual conformal covariant with a uniform inversion weight of $1/2$ for each leg. Similarly in $\mathcal{N}=4$ sYM the superamplitudes also transform covariantly albeit with a different inversion weight

of 1 for each leg [94]. The combination of dual and ordinary superconformal symmetries forms an infinite dimensional Yangian algebra [92] which is similar to the Yangian symmetry of the planar $\mathcal{N} = 4$ sYM, a key feature of integrability.

Following the strategy of [95] of constructing dual conformal integrands in four dimensions by using a six-dimensional formalism, the authors of [67] constructed the one-loop box (4.2.7) now using a five-dimensional embedding formalism. The authors show that although a scalar numerator would not allow for a dual conformal integral of a four-point amplitude, it is possible to write a tensorial numerator that satisfies this symmetry. In particular they write it a five-dimensional notation [67] as

$$I_4^{1-L} = \int \mathcal{D}^3 X_5 \frac{4 \, \varepsilon(5,1,2,3,4)}{X_{51}^2 X_{52}^2 X_{53}^2 X_{54}^2} \, , \tag{4.2.25}$$

which in three dimensions is reduced to

$$I_4^{1-L} = \int \frac{d^3 x_5}{(2\pi)^3} \frac{2x_{51}^2 \varepsilon_{\mu\nu\rho} x_{21}^\mu x_{31}^\nu x_{41}^\rho + 2x_{31}^2 \varepsilon_{\mu\nu\rho} x_{51}^\mu x_{21}^\nu x_{41}^\rho}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \, . \tag{4.2.26}$$

This can be re-written in terms of momenta producing the box function (4.2.7).

4.3 One Loop Amplitude

4.3.1 Results

In this section we present our result for the complete four-point amplitude at one loop in ABJM. As mentioned earlier, this amplitude will be needed in order to construct the two-particle cuts of the two-loop form factor. The one-loop four-point amplitude is given by the sum of a planar and non-planar contribution:

$$\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4) \;=\; \mathcal{A}_{\mathrm{P}}^{(1)}(\bar{1},2,\bar{3},4) \;+\; \mathcal{A}_{\mathrm{NP}}^{(1)}(\bar{1},2,\bar{3},4) \, , \tag{4.3.1}$$

where

$$\mathcal{A}_{\mathrm{P}}^{(1)}(\bar{1},2,\bar{3},4) = i \, N \, \mathcal{A}^{(0)}(\bar{1},2,\bar{3},4) \, I(1,2,3,4) \Big(\big[1,2,3,4\big] + \big[1,4,3,2\big] \Big), \tag{4.3.2}$$

and

$$\mathcal{A}_{\mathrm{NP}}^{(1)}(\bar{1},2,\bar{3},4) = -2\,i\,\mathcal{A}^{(0)}(\bar{1},2,\bar{3},4)\Big[\big(I(1,2,3,4)-I(4,2,3,1)\big)[1,2][3,4]$$
$$-\big(I(2,3,4,1)-I(1,3,4,2)\big)[1,4][3,2]\Big]\,.$$

$$(4.3.3)$$

Note that the double-trace structure $[1,2]$ is

$$[1,2] \;=\; \delta_{i_2}^{\bar{i}_1}\delta_{i_1}^{i_2}\,. \qquad (4.3.4)$$

The complete one-loop amplitude can also be written in the following way,

$$\frac{\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)}{\mathcal{A}^{(0)}(\bar{1},2,\bar{3},4)} = i\Big\{I(1,2,3,4)\Big[N\big([1,2,3,4]+[1,4,3,2]\big)-2[1,2][3,4]-2[1,4][3,2]\Big]$$
$$+2\Big[I(4,2,3,1)[1,2][3,4]-I(1,3,4,2)[1,4][3,2]\Big]\Big\}. \quad (4.3.5)$$

4.3.2 Symmetry properties of the one-loop amplitude

Before discussing the derivation of (4.3.1), it is instructive to prove that $\mathcal{A}_{\mathrm{P}}^{(1)}$ and $\mathcal{A}_{\mathrm{NP}}^{(1)}$ are antisymmetric under the swap $\bar{1}\leftrightarrow\bar{3}$ (see (4.2.16)). In order to show this one needs to use (4.2.13) and the following relations satisfied by the one-loop box (4.2.7):

$$I(a,b,c,d) \;=\; -I(b,c,d,a)\,, \qquad I(a,b,c,d)=-I(c,b,a,d)\,. \qquad (4.3.6)$$

These relations state that by cyclically shifting the labels of the external legs of the box function (4.2.7) by one unit one picks a minus sign; and similarly if one swaps two non-adjacent legs. Both relations are straightforward to prove using the definition (4.2.7) of the box function. One then finds,

$$I(3,2,1,4) - I(4,2,1,3) = I(2,3,4,1) - I(1,3,4,2)\,,$$
$$I(2,1,4,3) - I(3,1,4,2) = I(1,2,3,4) - I(4,2,3,1)\,. \qquad (4.3.7)$$

Using (4.3.7) we get

$$\mathcal{A}_{\mathrm{P}}^{(1)}(\bar{1},2,\bar{3},4) = -\mathcal{A}_{\mathrm{P}}^{(1)}(\bar{3},2,\bar{1},4)\,,$$
$$\mathcal{A}_{\mathrm{NP}}^{(1)}(\bar{1},2,\bar{3},4) = -\mathcal{A}_{\mathrm{NP}}^{(1)}(\bar{3},2,\bar{1},4)\,. \qquad (4.3.8)$$

Notice the presence of a minus sign between the two non-planar color structure $[1,2][3,4]$ and $[1,4][3,2]$ appearing in the non-planar one-loop amplitude (4.3.3).

4.3.3   One-loop Amplitude

We now briefly outline the strategy for the derivation of the complete one-loop amplitude (4.3.1), which is very similar to that in $\mathcal{N} = 4$ sYM, see for example [96]. We consider the two-particle cuts of the complete one-loop amplitude, which are obtained by merging two tree-level amplitudes summed over all possible color structures and internal particle species. We will see that each cut can be re-expressed in terms of cuts of sums of box functions (4.2.7). The sum over internal species is (partially) performed via an integration over the Grassmann variables η_{ℓ_1} and η_{ℓ_2} associated to the cut momenta. If one of the particles crossing is bosonic and the other is fermionic we also have to add to this the same expression with $\ell_1 \leftrightarrow \ell_2$ – this is necessary only for the s- and t-cuts. For instance, the s-cut integrand of the one-loop amplitude is[§]

$$\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{s-\text{cut}} = \frac{1}{2} \int d^3 \eta_{\ell_1} d^3 \eta_{\ell_2} \tilde{\mathcal{A}}^{(0)}(\bar{1}, 2, -\bar{\ell}_2, -\ell_1) \times \tilde{\mathcal{A}}^{(0)}(\bar{3}, 4, \bar{\ell}_1, \ell_2) + \ell_1 \leftrightarrow \ell_2 \,.$$
(4.3.9)

The one-loop amplitude has cuts in the s-, t- and u-channels, for which we find the following integrands:

$$
\begin{aligned}
\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{s-\text{cut}} &= \frac{i}{2} \mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) \; c_s \, \mathcal{S}_{12} I(1, 2, 3, 4)|_{s-\text{cut}} \,, \\
\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{t-\text{cut}} &= \frac{i}{2} \mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) \; c_t \, \mathcal{S}_{23} I(1, 2, 3, 4)|_{t-\text{cut}} \,, \\
\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{u-\text{cut}} &= \frac{i}{2} \mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) \; c_u \, \mathcal{S}_{13} I(3, 1, 2, 4)|_{u-\text{cut}} \,, \quad \text{(4.3.10)}
\end{aligned}
$$

[§]For convenience we include here a factor of $\frac{1}{2}$ in the definition of the (symmetrized) cuts. In practice it means that we take the average of the two contributions in the s- and t-cuts, and multiply the u-cut with a symmetry factor as two identical (super)particles cross the cut.

where the color factors c_s, c_t, c_u are

$$c_s = N[1, 2, 3, 4] + N[1, 4, 3, 2] - 2[1, 2][3, 4],$$

$$c_t = N[1, 2, 3, 4] + N[1, 4, 3, 2] - 2[1, 4][3, 2],$$

$$c_u = 2[1, 2][3, 4] - 2[1, 4][3, 2], \tag{4.3.11}$$

which are the result of the contractions of the color factors of the tree subamplitudes written in the form of (4.2.12). Recall that by $\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4)$ we denote the color-ordered four-point amplitude. Furthermore, by $\mathcal{S}_{ab}I(a, b, c, d)|_{s_{ab}-\text{cut}}$, we indicate the s_{ab}-cut of the one-loop box function $I(a, b, c, d)$ in (4.2.7), symmetrized in the cut loop momenta ℓ_1 and ℓ_2, which are defined such that $\ell_1 + \ell_2 = p_a + p_b$,

$$\mathcal{S}_{12}I(1, 2, 3, 4)|_{s-\text{cut}} = \frac{s\,\text{Tr}(\ell_1 p_1 p_4)}{(\ell_1 - p_1)^2(\ell_1 + p_4)^2} + \ell_1 \leftrightarrow \ell_2,$$

$$\mathcal{S}_{23}I(1, 2, 3, 4)|_{t-\text{cut}} = \frac{(-t)\,\text{Tr}(\ell_1 p_1 p_2)}{(\ell_1 - p_1)^2(\ell_1 + p_2)^2} + \ell_1 \leftrightarrow \ell_2,$$

$$\mathcal{S}_{13}I(3, 1, 2, 4)|_{u-\text{cut}} = \frac{u\,\text{Tr}(\ell_2 p_3 p_4)}{(\ell_2 - p_3)^2(\ell_2 + p_4)^2} + \ell_1 \leftrightarrow \ell_2. \tag{4.3.12}$$

It is important to stress here that despite the simplified notation, the cut momenta ℓ_1 and ℓ_2 are different for the three distinct channels under considerations. For instance, $\ell_1 + \ell_2 = p_1 + p_2$ for the s-cut, while $\ell_1 + \ell_2 = p_2 + p_3$ in the t-cut and $\ell_1 + \ell_2 = p_1 + p_3$ in the u-cut. Recall that the symmetrization in the cut momenta in the s- and t-channel coefficients originates from summing over all possible particle species that can propagate on the cut legs, while in the u cut there is a single configuration allowed, and the result turns out to be automatically symmetric in ℓ_1 and ℓ_2. Explicit derivations of the cuts for each kinematic channel can be found in Appendix B.3.

Next the cuts are merged into box functions. For the planar structures $[1, 2, 3, 4]$ and $[1, 4, 3, 2]$ this is immediate as the only function consistent with the s- and t-cuts in (4.3.10) and vanishing u-cut is $I(1, 2, 3, 4)$. Hence, the corresponding planar amplitude is

$$i\,\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4)\,N\big([1, 2, 3, 4] + [1, 4, 3, 2]\big)\,I(1, 2, 3, 4), \tag{4.3.13}$$

thus arriving at the expression (4.3.2) for the planar part of the full one-loop ampli-
tude.[¶] For the non-planar terms $[1,2][3,4]$ and $[1,4][3,2]$ it is necessary to use the
results of Appendix B.4.2 and in particular (B.4.13), which is reproduced here,

$$\mathcal{S}_{ab}I(a,b,c,d)|_{s_{ab}-\text{cut}} = \mathcal{S}_{ab}I(a,b,d,c)|_{s_{ab}-\text{cut}}. \qquad (4.3.14)$$

Firstly, note that an immediate consequence of this result is that

$$\mathcal{S}_{23}I(2,3,4,1)|_{t-\text{cut}} - \mathcal{S}_{23}I(2,3,1,4)|_{t-\text{cut}} = 0, \qquad (4.3.15)$$

in other words the combination $I(2,3,4,1) - I(2,3,1,4)$, symmetrized in the loop
momenta ℓ_1 and ℓ_2, with $\ell_1 + \ell_2 = p_2 + p_3$, has a vanishing t-channel cut as ex-
pected for the coefficient of the $[1,2][3,4]$ color structure (see (4.3.11)). For the same
combination we find, using $I(2,3,4,1) = -I(1,2,3,4)$, the symmetrized s-cut

$$- \mathcal{S}_{12}I(1,2,3,4)|_{s-\text{cut}}, \qquad (4.3.16)$$

and similarly, for the symmetrized u-cut we obtain

$$\mathcal{S}_{13}I(3,1,4,2)|_{u-\text{cut}} = \mathcal{S}_{13}I(3,1,2,4)|_{u-\text{cut}}, \qquad (4.3.17)$$

where we have used $I(2,3,1,4) = -I(3,1,4,2)$ and (B.4.13), which allows us to swap
the last two legs on the symmetrized u-cut. Comparing with (4.3.10) and (4.3.11)
we can uniquely fix the coefficient of the non-planar structure $[1,2][3,4]$:

$$2\,i\,\mathcal{A}^{(0)}(\bar{1},2,\bar{3},4)\,[1,2][3,4]\left[I(2,3,4,1) - I(2,3,1,4)\right], \qquad (4.3.18)$$

or, using the first relation of (4.3.6),

$$- 2\,i\,\mathcal{A}^{(0)}(\bar{1},2,\bar{3},4)\,[1,2][3,4]\left[I(1,2,3,4) - I(4,2,3,1)\right]. \qquad (4.3.19)$$

One can proceed similarly for the coefficient of the other non-planar structure $[1,4][3,2]$,
arriving at the result quoted earlier in (4.3.3). Note that in that result we use the
freedom to rename loop momenta in order to eliminate the various symmetrizations
introduced by the operation $\mathcal{S}_{ab}$ above.

[¶]Note that at the level of the integral we can simply replace $\mathcal{S}_{12}I(1,2,3,4)$ by $2\,I(1,2,3,4)$.

4.4 The Sudakov form factor at one and two loops

We now move on to the form factors of gauge-invariant, single-trace scalar operators

$$\mathcal{O} = \text{Tr}\left(\phi^{A_1}\bar{\phi}_{B_1}\phi^{A_2}\bar{\phi}_{B_2}\dots\phi^{A_L}\bar{\phi}_{B_L}\right)\chi^{B_1\dots B_L}_{A_1\dots A_L}\,, \tag{4.4.1}$$

where A and B are indices of the **4** and $\bar{\mathbf{4}}$ representation of the R-symmetry group $SU(4)$. The operators (4.4.1) are half BPS if χ is a symmetric traceless tensor in all the A_i and B_i indices separately (see for example [61, 62]). For $L = 2$, the relevant operator is$^{\|}$

$$\mathcal{O}^A{}_B = \text{Tr}\left(\phi^A\bar{\phi}_B - \frac{\delta^A{}_B}{4}\phi^K\bar{\phi}_K\right)\,. \tag{4.4.2}$$

In the rest of the paper we will focus on the Sudakov form factor

$$\langle(\bar{\phi}_A)^{\bar{i}_1}_{i_1}(p_1)\,(\phi^4)^{i_2}_{\bar{i}_2}(p_2)|\text{Tr}(\bar{\phi}_A\phi^4)(0)|0\rangle \;:=\; [1,2]\ F(q^2)\,, \tag{4.4.3}$$

where $q := p_1 + p_2$ and $A \neq 4$, and we recall that $[1,2] := \delta^{\bar{i}_1}_{\bar{i}_2}\delta^{i_2}_{i_1}$. At tree level,

$$F^{(0)}(q^2) \;=\; 1\,. \tag{4.4.4}$$

We will now derive this quantity at one and two loops.

4.4.1 One-loop form factor in ABJM

At one loop it is possible to determine the integrand of the form factor from a single unitarity cut in the q^2 channel. As shown in Figure 4.3, on one side of the cut there is the Sudakov form factor and on the other side the complete four-point amplitude, both at tree level. The color-ordered tree amplitude is given in (4.2.5). Let us work out the color factor first. It is given by

$$\delta^{\bar{i}_{\ell_2}}_{\bar{i}_{\ell_1}}\delta^{i_{\ell_1}}_{i_{\ell_2}}(\delta^{\bar{i}_1}_{\bar{i}_2}\delta^{i_2}_{i_{\ell_1}}\delta^{\bar{i}_{\ell_1}}_{\bar{i}_{\ell_2}}\delta^{i_{\ell_2}}_{i_1} - \delta^{\bar{i}_1}_{\bar{i}_{\ell_2}}\delta^{i_2}_{i_1}\delta^{\bar{i}_{\ell_1}}_{\bar{i}_2}\delta^{i_{\ell_2}}_{i_{\ell_1}}) \;=\; (N'-N)\delta^{\bar{i}_1}_{\bar{i}_2}\delta^{i_2}_{i_1}\,. \tag{4.4.5}$$

Obviously, the one-loop form factor vanishes identically in ABJM theory, because in this case $N' = N$.

$^{\|}$More details on half-BPS operators, as well as conventions are discussed in Appendix B.2.
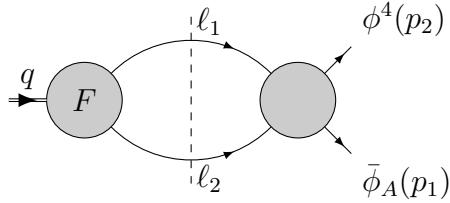
Figure 4.3 The q^2 cut of the Sudakov form factor. Note that the amplitude on the right-hand side of the cut is summed over all possible color orderings.

We now consider the kinematic part. Since the operator is built solely out of scalars, only the four-point scalar amplitude can appear in the cut. To match the particles of the tree amplitude in Figure 4.3, we pick the $(\eta_1)^1(\eta_{\ell_1})^3(\eta_{\ell_2})^2(\eta_2)^0$ component from the $\delta^6(Q)$ to write the q^2 cut of the one-loop form factor as:

$$\frac{\delta^{(6)}(Q)\big|_{(\eta_1)^1(\eta_{\ell_1})^3(\eta_{\ell_2})^2(\eta_2)^0}}{\langle 1\,2\rangle\langle 2\,\ell_1\rangle} = \frac{\langle \ell_1\,\ell_2\rangle^2\langle 1\,\ell_1\rangle}{\langle 1\,2\rangle\langle 2\,\ell_1\rangle} = \frac{\langle 1\,2\rangle\langle 1\,\ell_1\rangle}{\langle 2\,\ell_1\rangle} = -\frac{\mathrm{Tr}(\ell_1 p_1 p_2)}{2(\ell_1 \cdot p_2)}, \qquad (4.4.6)$$

which can be immediately lifted to a full integral as it is the only possible cut of the form factor. Thus we get,

$$F^{(1)}(q^2) = (N' - N)\int \frac{d^D\ell_1}{i\pi^{D/2}} \frac{\mathrm{Tr}(\ell_1 p_1 p_2)}{\ell_1^2\,(\ell_1 - p_2)^2(\ell_1 - p_1 - p_2)^2}. \qquad (4.4.7)$$

The integral in (4.4.7) is a linear triangle and is of $\mathcal{O}(\epsilon)$. Hence, we conclude that the one-loop Sudakov form factor in ABJ theory vanishes in strictly three dimensions. Moreover, the three-dimensional integrand vanishes in ABJM theory but is non-vanishing for $N \neq N'$ and can (and does) participate in unitarity cuts at two loops in ABJ theory. Note, that the vanishing of the one-loop form factors in ABJ(M) is consistent with the infrared finiteness of one-loop amplitudes in ABJ(M).

4.4.2   Two-loop form factor in ABJM

Next, we come to the computation of the two-loop Sudakov form factor. In order to construct an ansatz for its integrand we will make use of two-particle cuts, and fix potential remaining ambiguities with various three-particle cuts described in detail in this section.

Three-particle cuts are very useful because they receive contributions from planar as well as non-planar integral functions at the same time, and thus are particularly constraining. A special feature of ABJM theory is that all amplitudes with an odd number of external particles vanish and, as a consequence, all cuts involving such amplitudes are identically zero [69]. In our case this observation will be important for triple cuts, where three- and five-particle amplitudes would appear.

A particular type of such cuts, first considered in [69] in the context of loop amplitudes in ABJM, involves three adjacent cut loop momenta meeting at a three-point vertex. The vanishing of these cuts imposes strong constraints on the form of the loop integrands. This will discussed and exploited later in this section, where we will also make the intriguing observation that integral functions with numerators satisfying such constraints are transcendental and free of certain unwanted infrared divergences.

Two-particle cuts

We begin by considering the cut shown in Figure 4.4, which contains a tree-level Sudakov form factor merged with the integrand of the complete one-loop, four-point amplitude. The internal particle assignment is fixed and is determined by the particular operator we consider. The integrand of this cut is schematically given by

$$F^{(0)}(\bar{\ell}_2, \ell_1)[\ell_2, \ell_1] \; \tilde{\mathcal{A}}^{(1)}\big(\bar{\phi}_A(p_1), \phi^4(p_2), \bar{\phi}_4(-\ell_1), \phi^A(-\ell_2)\big) \;, \qquad (4.4.8)$$

where we picked the relevant component amplitude of the complete one-loop amplitude $\tilde{\mathcal{A}}^{(1)}$, given in (4.3.1), and we recall that the color factor $[a, b]$ is defined in (4.3.4).

The first step is to work out the color structures that will appear in the result. Firstly consider the planar amplitude (4.3.2) and combine it with the part of the non-planar amplitude (4.3.3) containing $I(1, 2, -\ell_1, -\ell_2)$. Intriguingly, by contracting this with the tree-level form factor (given in (4.4.3) and (4.4.4)) we obtain a vanishing

result:

$$\left(N\big([1,2,\ell_1,\ell_2]+[1,\ell_2,\ell_1,2]\big)-2[1,2][\ell_1,\ell_2]\right)[\ell_2,\ell_1]= \tag{4.4.9}$$

$$N\left(\delta_{\bar{i}_2}^{\bar{i}_1}\delta_{i\ell_1}^{i_2}\delta_{\bar{i}\ell_2}^{i\ell_1}\delta_{i_1}^{i\ell_2}+\delta_{\bar{i}\ell_2}^{\bar{i}_1}\delta_{i\ell_1}^{i\ell_2}\delta_{\bar{i}_2}^{\bar{i}\ell_1}\delta_{i_1}^{i_2}\right)\delta_{i\ell_2}^{i\ell_1}\delta_{\bar{i}\ell_1}^{\bar{i}\ell_2}-2\delta_{\bar{i}_2}^{\bar{i}_1}\delta_{i_1}^{i_2}\delta_{\bar{i}\ell_2}^{\bar{i}\ell_1}\delta_{i\ell_1}^{i\ell_2}\delta_{\bar{i}\ell_2}^{\bar{i}\ell_1}\delta_{i\ell_1}^{i\ell_2}= \tag{4.4.10}$$

$$N^2\delta_{\bar{i}_2}^{\bar{i}_1}\delta_{i_1}^{i_2}+N^2\delta_{\bar{i}_2}^{\bar{i}_1}\delta_{i_1}^{i_2}-2N^2\delta_{\bar{i}_2}^{\bar{i}_1}\delta_{i_1}^{i_2}=0 \tag{4.4.11}$$

We now consider the remaining contributions arising from the non-planar one-loop amplitude (4.3.3). There are two possible color contractions to consider,

$$c_{\mathrm{NP}}^{(1)}:=2\,[1,2][\ell_1,\ell_2][\ell_2,\ell_1]\;=\;2\,N^2[1,2]\;, \tag{4.4.12}$$

and

$$c_{\mathrm{NP}}^{(2)}:=2\,[\ell_1,2][1,\ell_2][\ell_2,\ell_1]\;=\;2\,[1,2]\;. \tag{4.4.13}$$

Note that (4.4.13) is subleading in the large N limit, and can be discarded in the large-N limit.
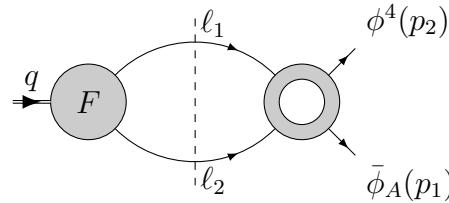

Figure 4.4 Tree-level form factor glued to the complete one-loop amplitude.

We now need to determine the coefficient of $c_{\mathrm{NP}}^{(1)}$. On the two-particle cut $\ell_1^2=\ell_2^2=0$ its integrand is given by the appropriate component tree-level amplitude (4.4.6) times a particular box integral (4.3.3):

$$\mathcal{C}_1^{(\mathrm{NP})}\big|_{s-\mathrm{cut}}:=\frac{1}{2}\frac{\langle12\rangle\langle1\ell_1\rangle}{\langle2\ell_1\rangle}\,I(-\ell_2,2,-\ell_1,1)+\ell_1\leftrightarrow\ell_2\;. \tag{4.4.14}$$

Recall that the expression must be symmetrized in order to include all particle species in the sum over intermediate on-shell states. Since $I(-\ell_2,2,-\ell_1,1)$ is antisymmetric

under $\ell_1 \leftrightarrow \ell_2$ the complete cut-integrand can be written as[**]

$$
\begin{aligned}
\mathcal{C}_1^{(\mathrm{NP})}\big|_{s-\mathrm{cut}} \;&:=\; \frac{1}{2}\left(\frac{\langle 12\rangle\langle 1\ell_1\rangle}{\langle 2\ell_1\rangle} - \frac{\langle 12\rangle\langle 1\ell_2\rangle}{\langle 2\ell_2\rangle}\right) I(-\ell_2, 2, -\ell_1, 1) \qquad (4.4.15)\\
&=\; -\frac{1}{2}\int \frac{d^D\ell_3}{i\pi^{D/2}} \frac{q^2\left[\mathrm{Tr}\,(p_1 p_2 \ell_1 \ell_3) - q^2 \ell_3^2\right]}{\ell_3^2\,(\ell_1-\ell_3)^2(p_1-\ell_3)^2(\ell_3-\ell_1+p_2)^2}\;.
\end{aligned}
$$

Summarizing, the two-particle cut indicates that the two-loop form factor is expressed in terms of a single crossed triangle with a particular numerator, represented in Figure 4.5,

$$
\mathbf{XT}(q^2) \;=\; \int \frac{d^D\ell_1 d^D\ell_3}{(i\pi^{D/2})^2} \frac{q^2\left[\mathrm{Tr}\,(p_1 p_2 \ell_1 \ell_3) - q^2 \ell_3^2\right]}{\ell_1^2\,\ell_2^2\,\ell_3^2\,(\ell_1-\ell_3)^2(p_1-\ell_3)^2(\ell_3-\ell_1+p_2)^2}\,, \qquad (4.4.16)
$$

so that

$$
\mathcal{C}_1^{(\mathrm{NP})} = -\frac{1}{2}\mathbf{XT}(q^2)\,. \qquad (4.4.17)
$$

For future convenience we will define

$$
\mathbf{xt} \;:=\; \frac{q^2\left[\mathrm{Tr}\,(p_1 p_2 \ell_1 \ell_3) - q^2 \ell_3^2\right]}{\ell_1^2\,\ell_2^2\,\ell_3^2\,(\ell_1-\ell_3)^2(p_1-\ell_3)^2(\ell_3-\ell_1+p_2)^2}\,. \qquad (4.4.18)
$$

The result of the evaluation of $\mathbf{XT}(q^2)$ is quoted in (4.4.27) and shows that this quantity has maximal degree of transcendentality. Before evaluating $\mathbf{XT}(q^2)$, we use triple cuts in order to confirm the correctness of the ansatz obtained from two-particle cuts.
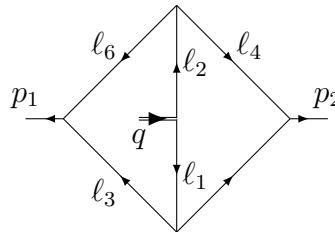

Figure 4.5 The crossed triangle integral arising from gluing a tree form factor with the complete one-loop four-point amplitude. The arrow in the middle denotes the location where the momentum $q = p_1 + p_2$ is injected. We call these integrals "crossed triangles" because they have the topology of the master integral (B.5.28). Note however that the latter integral is non-transcendental, while the particular numerator in (4.4.16) makes this integral transcendental.

[**]Similarly as done earlier for the complete one-loop amplitude, we include a factor of $1/2$ in the symmetrization.

Three-vertex cuts

To confirm the uplift of the two-particle cut to the integral (4.4.16), we will study additional cuts. We begin by focusing on three-point vertex cuts involving three adjacent legs meeting at a three-point vertex. These cuts were first examined in [69], where it was observed that they must vanish since there are no three-particle amplitudes in ABJM theory. Calling k_1, k_2 and k_3 the momenta meeting at the vertex, we have

$$k_1 + k_2 + k_3 = 0 \,, \qquad k_1^2 = k_2^2 = k_3^2 = 0 \,. \tag{4.4.19}$$

The conditions (4.4.19) imply that all spinors associated to these momenta are proportional, thus

$$\langle k_1 \, k_2 \rangle = \langle k_2 \, k_3 \rangle = \langle k_3 \, k_1 \rangle = 0 \,. \tag{4.4.20}$$

As an example consider the three-point vertex cut of $\mathbf{XT}(q^2)$ with momenta ℓ_2, ℓ_4 and $\ell_6 := \ell_2 - \ell_4$ (see Figure 4.5 for the labeling of the momenta). Importantly, the form factor is expected to vanish as the three momenta belonging to a three-point vertex become null. By rewriting the numerator of (4.4.16) using only cut momenta, it is immediately seen that it vanishes, since

$$
\begin{aligned}
\mathrm{Tr}\big[p_1 p_2 (p_1 - \ell_2)(p_1 - \ell_6)\big] - q^2(p_1 - \ell_6)^2 &= -\mathrm{Tr}\big[p_1 p_2 (p_1 - \ell_2)\ell_6\big] - q^2(p_1 - \ell_6)^2 \\
&= -\mathrm{Tr}(p_1 p_2 p_1 \ell_6) + 4(p_1 \cdot p_2)(p_1 \cdot \ell_6) = 0 \,,
\end{aligned}
\tag{4.4.21}
$$

where the fact that $\langle \ell_2 \ell_6 \rangle = 0$ was used to set $\mathrm{Tr}(p_1 p_2 \ell_2 \ell_6) = 0$. It is easy to see that all other three-vertex cuts of the integral (4.4.16) vanish in a similar fashion because of the particular form of its numerator.

Important consequences of these specific properties of the numerator of the integral function (4.4.16) are that the result is transcendental as we will show below and is free of unphysical infrared divergences related to internal three-point vertices. These divergences appear generically in three-dimensional two-loop integrals with internal three-vertices even if the external kinematics is massive (unlike in four dimensions) and it appears that master integrals with appropriate numerators to cancel these peculiar infrared divergences are a preferred basis for amplitudes and

form factors in ABJM. Related discussions in the context of ABJM amplitudes have appeared in [69, 97]. Note that for form factors we do not have dual conformal symmetry, which gives further constraints on the structure of the numerators of integral functions appearing in amplitudes.

Three-particle cuts

The remaining cut we will study is a triple cut of the type illustrated in Figure 4.6. These cuts may potentially detect additional integral functions which have no two-particle cuts at all, and are thus very important. Moreover, such cuts are sensitive to both planar and non-planar topologies. In this triple cut, a tree-level amplitude is connected to a tree-level form factor by three cut propagators. Due to the vanishing of amplitudes with an odd number of external legs in the ABJM theory, the triple cut in question vanishes. We will now check that the triple cut of the two-loop crossed triangle **XT** of (4.4.16), which we have detected using two-particle cuts, is indeed equal to zero.
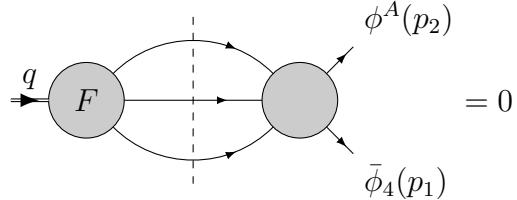

Figure 4.6 The (vanishing) three-particle cut of the two-loop Sudakov form factor.

To this end, we note that there are two possible ways to perform a triple-cut on **XT**, shown in Figures 4.7a and 4.7b. The cut loop momenta are called ℓ_2, ℓ_5 and ℓ_3 and satisfy

$$\ell_2 + \ell_5 + \ell_3 = p_1 + p_2 \,, \qquad \ell_2^2 = \ell_5^2 = \ell_3^2 = 0 \,. \tag{4.4.22}$$

We observe that these two cuts cannot be converted into one another by a simple relabeling of the cut momenta because of the non-trivial numerators. The A-cut

depicted in Figure 4.7a of the non-planar integrand is:

$$\mathbf{XT}\big|_{\text{3-p cut A}} = -q^2 \frac{\langle 1\,2\rangle}{\langle \ell_3\,\ell_5\rangle\langle \ell_5\,2\rangle\langle 1\,\ell_3\rangle} \; . \tag{4.4.23}$$

After a similar calculation, the B-cut of this integral, depicted in Figure 4.7b, turns out to be identical to the A-cut:

$$\mathbf{XT}\big|_{\text{3-p cut B}} = \mathbf{XT}\big|_{\text{3-p cut A}} = -q^2 \frac{\langle 1\,2\rangle}{\langle \ell_3\,\ell_5\rangle\langle \ell_5\,2\rangle\langle 1\,\ell_3\rangle} \; . \tag{4.4.24}$$

A quick way to establish the vanishing of the triple cuts consists in symmetrizing in the particle momenta p_1 and p_2, which is allowed since the Sudakov form factor is a function of q^2. This symmetrization gives

$$-\frac{q^2\langle 1\,2\rangle}{\langle \ell_3\,\ell_5\rangle}\left[\frac{1}{\langle \ell_5\,2\rangle\langle 1\,\ell_3\rangle} - \frac{1}{\langle \ell_5\,1\rangle\langle 2\,\ell_3\rangle}\right] = -\frac{q^4}{\langle 1|\ell_5|2] \, \langle 1|\ell_3|2]} \; . \tag{4.4.25}$$

This expression is symmetric in ℓ_5 and ℓ_3. In evaluating the triple cut one has to introduce a Jacobian proportional to $\epsilon(\ell_2, \ell_3, \ell_5)$ [69] which effectively makes this triple cut vanish upon integration. This implies that the complete answer for the two-loop form factor in ABJM is proportional $\mathbf{XT}(q^2)$ and no additional integral functions have to be introduced.

(a) The A-cut. (b) The B-cut.
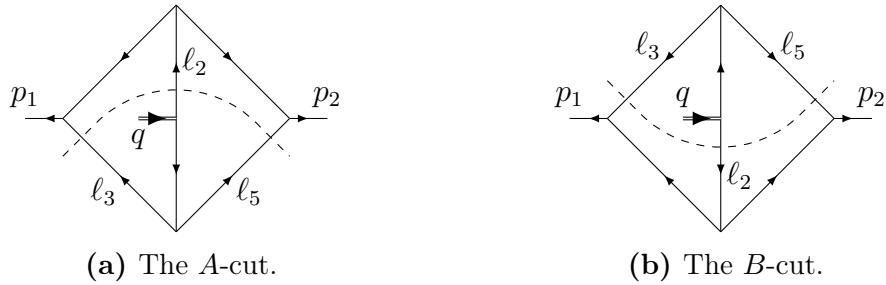
Figure 4.7   The two triple cuts of the crossed triangle, with $\ell_2 + \ell_3 + \ell_5 = q$. In the second figure we have relabeled the loop momenta in order to merge the two contributions.

Results and comparison to the two-loop amplitudes

Combining the information from the unitarity cuts discussed above, we conclude that the two-loop Sudakov form factor in ABJM is given by a single non-planar integral

$$F_{\text{ABJM}}(q^2) = -2\left(\frac{N}{k}\right)^2\left(-\frac{1}{2}\right)\mathbf{XT}(q^2)\,, \tag{4.4.26}$$

where $\mathbf{XT}(q^2)$ is defined in (4.4.16) and we have reintroduced the dependence on the Chern-Simons level k. The integral $\mathbf{XT}(q^2)$ can be computed by reduction to master integrals using integration by parts identities. The details of the reductions are provided in Appendix B.5. The expansion of the result in the dimensional regularization parameter ϵ can then be found using the expressions for the master integrals (B.5.25)–(B.5.28). Plugging these masters into the reduction (B.5.29), we arrive at

$$\mathbf{XT}(q^2) = \left(\frac{-q^2 e^{\gamma_E}}{\mu^2}\right)^{-2\epsilon} \left[\frac{\pi}{\epsilon^2} + \frac{2\pi \log 2}{\epsilon} - 4\pi \log^2 2 - \frac{2\pi^3}{3} + \mathcal{O}(\epsilon)\right], \quad (4.4.27)$$

where γ_E is the Euler-Mascheroni constant. One comment is in order here. We have derived (4.4.27) in a normalization where the loop integration measure is written as $d^D\ell(i\pi^{D/2})$. This should be converted to the standard one $d^D\ell(2\pi)^D$. At two loops, this implies that (4.4.27) has to be multiplied by a factor of $-1/(4\pi)^D$. The result in the standard normalization is then

$$\mathcal{F}_{\text{ABJM}}(q^2) = -\frac{1}{(4\pi)^3}\left(\frac{N}{k}\right)^2 \left(\frac{-q^2 e^{\gamma_E}}{4\pi\mu^2}\right)^{-2\epsilon} \left[\frac{\pi}{\epsilon^2} + \frac{2\pi \log 2}{\epsilon} - 4\pi \log^2 2 - \frac{2\pi^3}{3} + \mathcal{O}(\epsilon)\right].$$
$$(4.4.28)$$

We note that $\mathcal{F}(q^2)$ can be expressed more compactly by introducing a new scale

$$\mu'^2 := 8\pi\, e^{-\gamma_E}\mu^2\,, \quad (4.4.29)$$

in terms of which we get

$$\mathcal{F}_{\text{ABJM}}(q^2) = \frac{1}{64\pi^2}\left(\frac{N}{k}\right)^2 \left(\frac{-q^2}{\mu'^2}\right)^{-2\epsilon} \left[-\frac{1}{\epsilon^2} + 6\log^2 2 + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)\right], \quad (4.4.30)$$

which is our final result.

We now discuss two consistency checks that confirm the correctness of (4.4.30). Firstly, we recall that the Sudakov form factor captures the infrared divergences of scattering amplitudes. We now check that (4.4.30) matches the infrared poles of the four-point amplitude evaluated in [67,68]. Here we quote its expression as given in [68]:

$$\mathcal{A}_4^{(2)} = -\frac{1}{16\pi^2}\mathcal{A}_4^{(0)}\left[\frac{(-s/\mu'^2)^{-2\epsilon}}{4\epsilon^2} + \frac{(-t/\mu'^2)^{-2\epsilon}}{4\epsilon^2} - \frac{1}{2}\log^2\left(\frac{-s}{-t}\right) - 4\zeta_2 - 3\log^2 2\right],$$
$$(4.4.31)$$

where μ' is related to μ in the same way as in (4.4.29). Hence, the Sudakov form factor (4.4.30) is in perfect agreement with the form of the infrared divergences of (4.4.31). Secondly, we have also checked that the expansion of our result in terms of master integrals (i.e. the expansion of the two-loop non-planar triangle **XT** defined in (4.4.16)) is identical to that obtained from the Feynman diagram based result of [98]. This implies that the cut-based calculation of this paper and the Feynman diagram calculation of [98] agree to all orders in ϵ – even though we have been using cuts in strictly three dimensions.

4.5   Maximally transcendental integrals in 3d

As discussed in Section 4.4.2, the integrand **xt** – that appears in the Sudakov form factor in ABJM – has a particular numerator such that all the cuts which isolate a three-point vertex vanish. This property ensures that the integral **XT** has a uniform (and maximal) degree of transcendentality[††]. Failure to obey the triple-cut condition, for instance by altering the form of the numerator, would result in the appearance of new terms with lower degree of transcendentality. An extreme case is when the numerator is set to one, resulting in a scalar double triangle integral **sLT**. Clearly, this numerator cannot satisfy the vanishing of the three-vertex cut condition. The pole structure of this integral takes the form:

$$\mathbf{sLT}(q^2) \sim -\frac{3\pi}{2\epsilon} + \pi(-8 - 9\gamma_E + 6\log 2) + \mathcal{O}(\epsilon)\,, \qquad (4.5.1)$$

which has neither a uniform nor maximal degree of transcendentality. Further details on the evaluation of Mellin-Barnes integrals are provided in Section B.5.3.

In this section, we present further integrals that vanish in these three-particle cuts and have maximal degree of transcendentality. These integrals are expected to appear in the form factor of ABJ theory where cancellations between color factors such as that in (4.4.9), do not occur.

[††]It is noted here that π and the logarithm have a transcendental degree of 1 while ε has an assigned degree of -1.

Thus, when color factors are ignored, the cut that will provide the planar contribution contains the tree level Sudakov form factor and the integrand of the one-loop, four-point scattering amplitude as shown in Figure 4.4. There is a fixed internal particle configuration due to the operator inside the form factor. Using the relevant component of the one-loop superamplitude integrand (as derived in [67]):

$$\mathcal{A}\big((p_1)_{\bar{\phi}_A}, (p_2)_{\phi^4}, (-\ell_1)_{\bar{\phi}_4}, (-\ell_2)_{\phi_A}\big) =$$
$$\frac{\langle 1\ell_1\rangle\langle\ell_1\ell_2\rangle}{\langle 1\ell_2\rangle} \frac{2(\ell_3 - p_2)^2\epsilon(p_1,\ell_2,p_2) + 2s_{2\ell_1}\epsilon(\ell_3,1,2)}{(\ell_3 - p_2)^2(\ell_3 - p_2 - p_1)^2(\ell_3 - \ell_1)^2\ell_3^2} \quad (4.5.2)$$

where ℓ_3 is the loop momentum for the one-loop amplitude and as before $\epsilon(a,b,c) := \epsilon_{\mu\nu\rho}a^\mu b^\nu c^\rho$. The tree-level Sudakov form factor is normalized to 1, therefore the amplitude integrand alone constitutes the cut of the form factor.

In order to promote the cut expression to an integrand, the spinor-index traces are rearranged in the numerator to cancel the factor of $\langle 1\ell_2\rangle$. Then, the cut propagators $\frac{1}{\ell_1^2}$ and $\frac{1}{\ell_2^2}$ are introduced, with $\ell_2 = q - \ell_1$, to get:

$$\frac{-\mathrm{Tr}(p_1\ell_3 p_2\ell_1) + (\ell_1 - p_1)^2(\ell_3 - p_2)^2}{\ell_1^2\,(p_1 + p_2 - \ell_1)^2\,\ell_3^2\,(p_1 + p_2 - \ell_3)^2(\ell_1 - \ell_3)^2(\ell_3 - p_2)^2}. \quad (4.5.3)$$

This yields the following planar integral function

$$\mathbf{LT}(q^2) = \int\frac{d^D\ell_1 d^D\ell_3}{(i\pi^{D/2})^2}\frac{-q^2\,[\,\mathrm{Tr}(p_1\,\ell_3\,p_2\,\ell_1) - (\ell_1 - p_1)^2(\ell_3 - p_2)^2]}{\ell_1^2\,(p_1 + p_2 - \ell_1)^2\,\ell_3^2\,(p_1 + p_2 - \ell_3)^2(\ell_1 - \ell_3)^2(\ell_3 - p_2)^2}$$
$$= \left(\frac{-q^2 e^{\gamma_E}}{\mu^2}\right)^{-2\epsilon}\left[-\frac{\pi}{4\epsilon^2} - \frac{\pi\log 2}{\epsilon} + 2\pi\log^2 2 - \frac{5\pi^3}{8} + \mathcal{O}(\epsilon)\right],$$
$$(4.5.4)$$

which is shown in Figure 4.8a. It is easy to see that the three vertex cut $\{\ell_1, \ell_3, \ell_5\}$ vanishes, since on this cut the numerator can be placed in the form

$$\langle\ell_1\,1\rangle\langle\ell_3\,2\rangle\langle 1\,2\rangle\langle\ell_3\,\ell_1\rangle, \quad (4.5.5)$$

after using a Schouten identity. The numerator (4.5.5) vanishes because $\langle\ell_3\,\ell_1\rangle = 0$ on this cut.

A further property of (4.5.4) emerges when we consider particular triple cuts involving two adjacent massless legs, which in three dimensions are associated with

(a) The **LT** topology   **(b)** The **CT** topology   **(c)** The **FAN** topology
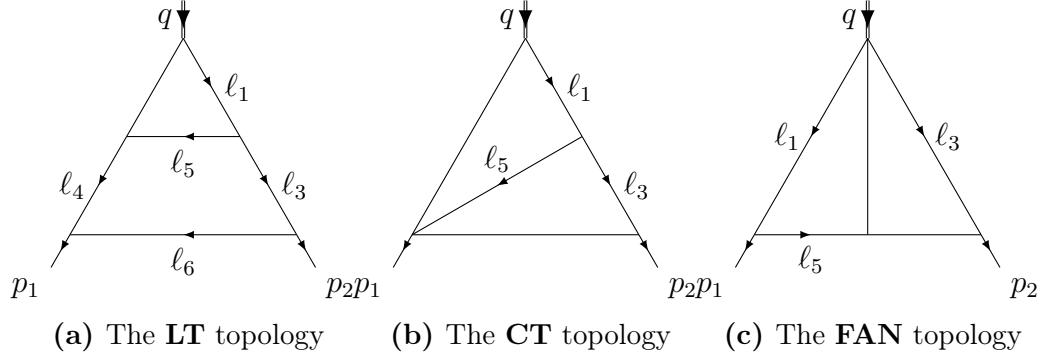
Figure 4.8   The three maximally transcendental integrals considered in (4.5.4), (4.5.9) and (4.5.11).

soft gluon exchange [69]. With reference to Figure 4.8a, we cut the three momenta ℓ_3, ℓ_6 and ℓ_4. The cut conditions $\ell_3^2 = \ell_6^2 = \ell_4^2 = 0$ together with the masslessness of p_1 and p_2 can only be satisfied if ℓ_6 becomes soft, that is

$$\ell_6 \to 0, \quad \ell_4 \to p_1, \quad \ell_3 \to p_2. \tag{4.5.6}$$

In this limit, the second term of (4.5.4) vanishes since $\ell_3 - p_2 = \ell_6 \to 0$. The first term becomes

$$-q^2 \frac{\mathrm{Tr}(p_1 \, \ell_3 \, p_2 \, \ell_1)}{8\epsilon(\ell_3, p_1, p_2)} \to -q^2 \frac{\langle 2|\ell_1|1\rangle}{4\langle 12\rangle}, \tag{4.5.7}$$

where $8\epsilon(\ell_3, p_1, p_2)$ is the Jacobian[‡‡]. After restoring the remaining propagators we are left with

$$\frac{2\epsilon(\ell_1, p_1, p_2)}{\ell_1^2(\ell_1 - p_2)^2(q - \ell_1)^2}, \tag{4.5.8}$$

which reproduces the one-loop integrand of the one-loop form factor, given earlier in (4.4.7).

Other examples of integrals with different topologies that satisfy the three-particle cut condition are depicted in Figures 4.8b and 4.8c. The definitions of the integrals

---

[‡‡]This Jacobian arises from re-writing the δ-functions of the cut momenta, $\ell_3^2 = \ell_4^2 = \ell_6^2 = 0$, in terms of p_1, p_2 and ℓ_6.

as well as their values are listed below:

$$\mathbf{CT}(q^2) = \int \frac{d^D\ell_1 d^D\ell_3}{(i\pi^{D/2})^2} \frac{\mathrm{Tr}(p_1, p_2, \ell_3, \ell_1)}{\ell_1^2 (p_1 + p_2 - \ell_1)^2 \ell_3^2 (\ell_1 - \ell_3)^2 (\ell_3 - p_2)^2} \tag{4.5.9}$$

$$= \left(\frac{-q^2 e^{\gamma_E}}{\mu^2}\right)^{-2\epsilon} \left[-\frac{\pi}{4\epsilon^2} + \frac{7\pi^3}{24} + \mathcal{O}(\epsilon)\right], \tag{4.5.10}$$

$$\mathbf{FAN}(q^2) = \int \frac{d^D\ell_1 d^D\ell_3}{(i\pi^{D/2})^2} \frac{\mathrm{Tr}(p_1, p_2, \ell_3, \ell_1)}{\ell_1^2 \ell_3^2 (p_1 + p_2 - \ell_1 - \ell_3)^2 (\ell_1 - p_1)^2 (\ell_3 - p_2)^2} \tag{4.5.11}$$

$$= \left(\frac{-q^2 e^{\gamma_E}}{\mu^2}\right)^{-2\epsilon} \left[-\frac{\pi}{4\epsilon^2} + \frac{7\pi^3}{24} + \mathcal{O}(\epsilon)\right]. \tag{4.5.12}$$

Note that the ϵ expansion of (4.5.9) and (4.5.11) agree up to $\mathcal{O}(1)$. It is simple to show that these integrals satisfy the properties discussed earlier, for example by setting $\{\ell_1, \ell_3, \ell_5\}$ on shell in **CT** and $\{\ell_1, p_1, \ell_5\}$ in **FAN** and similarly for all other possible three-vertex cuts.

The reductions of the integrals considered in this section in terms of scalar master integrals through IBP identities can be found in Appendix B.5.

5 | Concluding remarks

This thesis demonstrates how a spinor helicity formalism combined with unitarity can be used as an effective framework for performing calculations in various supersymmetric theories. The examples used are based on [15, 16] where unitarity was employed to perform calculations at one and two loops.

While standard field theory provides a recipe for safely performing blind calculations, when the inherent properties of amplitudes are ignored by the Feynman rules, the benefit of immense simplifications of the calculations is relinquished. In order to depart from the standard field theory routine of describing fundamental interactions, the thesis also investigates how group theory structure alone affects the form of the amplitudes. This is presented in Chapter 2 for the well studied $\mathcal{N} = 4$ case in the literature and also in six dimensions in Chapter 3 which relies on the study of [13].

A few of the most essential tools for performing calculations, such as recursion relations and generalized unitarity are thoroughly reviewed in Chapter 2. The scope was to present the techniques that have proved fruitful in attacking calculations as well as unraveling new structures in perturbative field theory in four dimensions. Along these lines, the results presented support the notion that the spinor-helicity formalism combined with supersymmetry and unitarity are not merely a computational trick, but indeed a way of representing and describing certain physical properties in the most natural available form. This is amplified by developments over the recent years, which suggest that this framework is indeed exceptional in discovering new dualities, symmetries and structures (for example: [6, 7, 25, 99]).

After the ground is laid, Chapters 3 and 4 focus on the application of unitarity

methods for performing calculations in six and three dimensions respectively. In particular, in Chapter 3, following a review on the spinor helicity formalism in six dimensions, superamplitudes with three, four and five external states are constructed and calculated at one loop. An intriguing fact is that the concepts and strategies already developed in four dimensions were successfully applied in the maximally supersymmetric $\mathcal{N} = (1, 1)$ six-dimensional theory. An intrinsic difference between the two cases is the fact that the $\mathcal{N} = (1, 1)$ sYM theory has non-chiral superspace. This immediately complicates the construction of superamplitudes. However, the explicit transformation of the particle states under the $\mathsf{SU}(2) \times \mathsf{SU}(2)$ little group reinstates some appealing structure in the form of the superamplitudes. Then, double and quadruple cuts are considered for one-loop superamplitudes with four and five external states. The most interesting case is the five-point one since it is expressed in terms of a linear pentagon integral function in six dimensions.

Finally, in Chapter 4 the calculation of Sudakov form factors in ABJM theory at two loops [16] is presented. Here, a similar pattern emerges. The structure of superamplitudes is strikingly similar to that of the $\mathcal{N} = 4$ sYM case. In particular, dual conformal symmetry [60], integrability, [61–63], duality with Wilson loops at four points [64–66], uniform transcendentality of the two-loop four-point [67,68] and six-point amplitude [69], color-kinematics duality [70] have made their appearance in both theories. The broad aim of this study was to further explore the similarities between ABJM and $\mathcal{N} = 4$ sYM. The result of the two-loop calculation captures the infrared divergences of scattering amplitudes and yields perfect agreement with the form of divergences of the two-loop four-point amplitude.

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Appendices

A | Calculations for Six-Dimensional Superamplitudes

In the present appendix there are several details concerning derivations and calculations used in several steps in the context of six-dimensional unitarity application. The first part lays the conventions and then an explicit proof of supersymmetry invariance of the three-point superamplitude for the six-dimensional $(1, 1)$ sYM theory is presented. Several spinor manipulations in six dimensions are discussed as the PV reduction of the linear pentagon that appeared in the one-loop five-point superamplitude.

A.1 Notation and conventions

In this appendix we collect some details on our normalizations and conventions. The total antisymmetric $\mathsf{SU}(2)$-invariant tensors are given by

$$\epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.1.1}$$

The Grassmann integration measure is defined as $d^2\eta = (1/2)d\eta^a d\eta_a = d\eta_2 d\eta_1$, such that

$$\int d^2\eta \left[\lambda^{Aa}\eta_a \, \lambda^{Bb}\eta_b \right] = - \left(\lambda^{Aa}\lambda^B_a \right). \tag{A.1.2}$$

The Clebsch-Gordan symbols are normalized as

$$\tilde{\sigma}_\mu^{AB} := \frac{1}{2}\epsilon^{ABCD}\sigma_{\mu,CD}\,, \tag{A.1.3}$$

with

$$\mathrm{Tr}(\sigma^\mu\tilde{\sigma}^\nu) = 4\eta^{\mu\nu}\,. \tag{A.1.4}$$

Using these relations, the scalar product of two vectors p and q can equivalently be expressed as

$$p \cdot q = -\frac{1}{4}p^{AB}q_{AB} = -\frac{1}{8}\epsilon_{ABCD}p^{AB}q^{CD}\,, \tag{A.1.5}$$

where $p^{AB} := p^\mu\tilde{\sigma}_\mu^{AB}$ and $p_{AB} := p^\mu\sigma_{\mu,AB}$.

Momentum conservation for three-point amplitudes implies that $p_i \cdot p_j = 0$, $i,j = 1,2,3$. In six dimensions, this condition is equivalent to [13]

$$\det\langle i|j]_{a\dot{a}} = 0 \tag{A.1.6}$$

where $\lambda_{ia}^A\tilde{\lambda}_{jA\dot{a}} := \langle i_a|j_{\dot{a}}]$ and we used $p_i^{AB} = \lambda_{ia}^A\lambda_i^{Ba}$ and $p_{iAB} = \tilde{\lambda}_{iA}^{\dot{a}}\tilde{\lambda}_{iB\dot{a}}$. Hence, (A.1.6) allows to recast the matrix $\langle i_a|j_{\dot{a}}]$ as a product of two spinors, as [13]

$$\langle i_a|j_{\dot{b}}] = (-)^{\mathcal{P}_{ij}}u_{ia}\tilde{u}_{j\dot{b}}\,, \tag{A.1.7}$$

where we choose $(-)^{\mathcal{P}_{ij}} = +1$ for $(i,j) = (1,2),(2,3),(3,1)$, and -1 for $(i,j) = (2,1),(3,2),(1,3)$.

A.2   Supersymmetry invariance of the three-point superamplitudes

Here we provide an explicit proof of the fact that the three-point superamplitude (3.3.8) is supersymmetric. We choose to decompose each variable η_i as

$$\eta_i^a = u_i^a\eta_i^{\|} + w_i^a\eta_i^\perp\,, \tag{A.2.1}$$

which is a convenient choice since $u_i^a w_{ia} = 1$. We also notice that, using this decomposition, we can recast the quantities W and $\tilde{W}$ defined in (3.3.10) entering the

expression of the three-point superamplitude, as

$$W = \sum_{i=1}^{3} \eta_i^{\parallel}, \qquad \tilde{W} = \sum_{i=1}^{3} \tilde{\eta}_i^{\parallel}. \tag{A.2.2}$$

The supersymmetry generators can then be written as

$$Q^A = \sum_i \lambda_i^{Aa} u_{ia} \eta_i^{\parallel} + \sum_i \lambda_i^{Aa} w_{ia} \eta_i^{\perp}. \tag{A.2.3}$$

A direct consequence of six-dimensional momentum conservation is the fact that the quantities $\lambda_i^{Aa} u_{ia}$ are i-independent, therefore we can rewrite (A.2.3) in several equivalent ways, one of which is

$$Q^A = (\lambda_1^{Aa} u_{1a})W + (\lambda_1^{Aa} w_{1a})(\eta_2^{\perp} - \eta_1^{\perp}) + (\lambda_2^{Aa} w_{2a})(\eta_3^{\perp} - \eta_1^{\perp}), \tag{A.2.4}$$

where W is given in (A.2.2), and the constraint on the w's (3.3.3) is used. Using the decomposition (A.2.4) it is very easy to prove that $Q^A A_3 = 0$. To this end, we first observe that the presence of a factor $\delta(W)\delta(\tilde{W})$ in (3.3.8) effectively removes the first term from the expression of (A.2.4), and we are left to prove that $Q_\perp^A :=$ $(\lambda_1^{Aa} w_{1a})(\eta_2^{\perp} - \eta_1^{\perp}) + (\lambda_2^{Aa} w_{2a})(\eta_3^{\perp} - \eta_1^{\perp})$ annihilates the amplitude. Specifically, we will show that

$$Q_\perp^A \left[\delta(Q^A)\delta(\tilde{Q}_A) \right]^2 = 0. \tag{A.2.5}$$

To begin with, we observe that

$$
\begin{aligned}
\delta(Q^A)\delta(\tilde{Q}_A) &= \sum_{i,j=1}^{3} \langle i_a | j_{\dot{a}}] \eta_i^a \tilde{\eta}_j^{\dot{a}} = \sum_{i,j=1}^{3} (-)^{\mathcal{P}_{ij}} u_{ia} \tilde{u}_{j\dot{a}} (1 - \delta_{ij}) \eta_i^a \tilde{\eta}_j^{\dot{a}} \\
&= \sum_{i,j=1}^{3} (-)^{\mathcal{P}_{ij}} (1 - \delta_{ij}) \eta_i^{\perp} \tilde{\eta}_j^{\perp} \\
&= \eta_1^{\perp} \tilde{\eta}_2^{\perp} - \eta_1^{\perp} \tilde{\eta}_3^{\perp} - \eta_2^{\perp} \tilde{\eta}_1^{\perp} + \eta_2^{\perp} \tilde{\eta}_3^{\perp} + \eta_3^{\perp} \tilde{\eta}_1^{\perp} - \eta_3^{\perp} \tilde{\eta}_2^{\perp}, \quad \text{(A.2.6)}
\end{aligned}
$$

where we have used (A.1.7). Using (A.2.6), one then finds (we drop the superscript $\perp$ in the following)

$$
\begin{aligned}
\left[\delta(Q^A)\delta(\tilde{Q}_A) \right]^2 =& -\eta_1\tilde{\eta}_2\eta_2\tilde{\eta}_1 + \eta_1\tilde{\eta}_2\eta_2\tilde{\eta}_3 + \eta_1\tilde{\eta}_2\eta_3\tilde{\eta}_1 + \eta_1\tilde{\eta}_3\eta_2\tilde{\eta}_1 - \eta_1\tilde{\eta}_3\eta_3\tilde{\eta}_1 + \eta_1\tilde{\eta}_3\eta_3\tilde{\eta}_2 \\
&- \eta_2\tilde{\eta}_1\eta_1\tilde{\eta}_2 + \eta_2\tilde{\eta}_1\eta_1\tilde{\eta}_3 + \eta_2\tilde{\eta}_1\eta_3\tilde{\eta}_2 + \eta_2\tilde{\eta}_3\eta_1\tilde{\eta}_2 + \eta_2\tilde{\eta}_3\eta_3\tilde{\eta}_1 - \eta_2\tilde{\eta}_3\eta_3\tilde{\eta}_2 \\
&+ \eta_3\tilde{\eta}_1\eta_1\tilde{\eta}_2 - \eta_3\tilde{\eta}_1\eta_1\tilde{\eta}_3 + \eta_3\tilde{\eta}_1\eta_2\tilde{\eta}_3 + \eta_3\tilde{\eta}_2\eta_1\tilde{\eta}_3 + \eta_3\tilde{\eta}_2\eta_2\tilde{\eta}_1 - \eta_3\tilde{\eta}_2\eta_2\tilde{\eta}_3.
\end{aligned}
$$
$$\tag{A.2.7}$$

Next, we calculate

$$\eta_1\left[\delta(Q^A)\delta(\tilde{Q}_A)\right]^2 \;=\; 2\eta^1\eta^2\eta^3(\tilde{\eta}^1\tilde{\eta}^3 - \tilde{\eta}^1\tilde{\eta}^2 - \tilde{\eta}^2\tilde{\eta}^3)\;, \qquad\qquad \text{(A.2.8)}$$

and furthermore we find that

$$\eta_2\left[\delta(Q^A)\delta(\tilde{Q}_A)\right]^2 = \eta_3\left[\delta(Q^A)\delta(\tilde{Q}_A)\right]^2 = \eta_1\left[\delta(Q^A)\delta(\tilde{Q}_A)\right]^2 . \qquad\qquad \text{(A.2.9)}$$

Inspecting the form of Q^A in (A.2.4) and using (A.2.9), we conclude that (A.2.5) holds, and therefore the three-point superamplitude is invariant under supersymmetry.

A.3   Useful spinor identities in six dimensions

In this appendix we collect identities between six-dimensional spinor variables that we have frequently used in the calculations presented in this paper.

We begin by quickly stating two basic relations for three-point spinors $u_{i,a}$ and $w_{i,a}$. For a general three-point amplitude in six dimensions we have [13]

$$u_i^a|i_a\rangle = u_j^b|j_b\rangle\;, \qquad\qquad \tilde{u}_i^{\dot{a}}|i_{\dot{a}}] = \tilde{u}_j^{\dot{b}}|j_{\dot{b}}]\;. \qquad\qquad \text{(A.3.1)}$$

We also have the constraints (3.3.3) on the w's and their $\tilde{w}$ counterparts, which are essentially a consequence of momentum conservation.

Next, we make use of relations between two three-point amplitudes, connected by an internal propagator, just as in the BCFW construction of the four-point amplitude. We give a pictorial representation of this in Figure A.1. We have defined the internal momenta ℓ and ℓ' to be incoming for the three-point amplitudes, giving the relation $\ell' = -\ell$. Since six-dimensional momenta are products of two spinors we can define

$$|\ell'_i\rangle = i|\ell_i\rangle\;, \qquad\qquad |\ell_i\rangle = (-i)|\ell'_i\rangle\;, \qquad\qquad \text{(A.3.2)}$$

and similarly for $\tilde{\lambda}$-spinors. Also note that we can normalize the spinors u_a, w_b of one three-point subamplitudes in Figure A.1 such that they are related to the spinors of
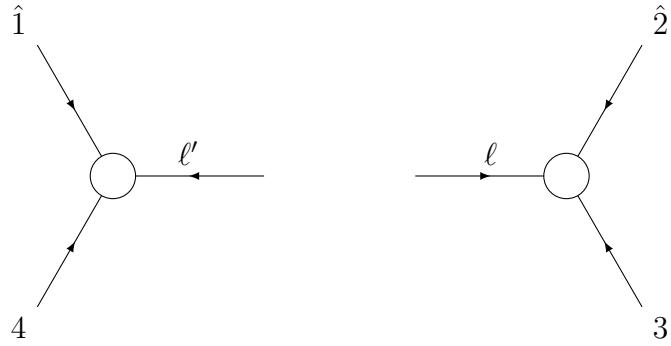
Figure A.1   The recursive construction of a four-point tree-level amplitude. The shifted legs are 1 and 2 and we have $\ell' = -\ell$ for the internal propagator.

the other subamplitude, yielding (see Appendix A.3.2)

$$w_{\ell_i a} = \frac{u_{\ell_i a}}{\sqrt{-s}} \;, \qquad\qquad w_{\ell_i a} = -\frac{u_{\ell'_i a}}{\sqrt{-s}} \;. \qquad (A.3.3)$$

Similar expressions hold for the spinors $\tilde{u}_{\dot{a}}, \tilde{w}_{\dot{b}}$. In the following we will be discussing several relations in the cases of the four- and five-point amplitudes.

A.3.1   Product of two u-spinors

In the calculation of the five-point cut-expression we encounter u-spinors belonging to the same external state and would like to remove them from the expression. Consider the object $u_{ia}\tilde{u}_{i\dot{a}}$ with states p_i and p_j belonging to the same three-point amplitude. We can write [21]

$$
\begin{aligned}
u_{ia}\tilde{u}_{i\dot{a}} &= u_{ia}\tilde{u}_{i\dot{b}}\delta_{\dot{a}}^{\dot{b}} = u_{ia}\tilde{u}_{i\dot{b}}\frac{\langle P_b|i^{\dot{b}}]}{\langle P_b|i^{\dot{a}}]} \\
&= u_{ia}\tilde{u}_{i\dot{b}}\langle P_b|i^{\dot{b}}]\langle P_b|i^{\dot{a}}]^{-1} = u_{ia}\tilde{u}_{i\dot{b}}[i^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} \\
&= u_{ia}\tilde{u}_{j\dot{b}}[j^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} = (-)^{\mathcal{P}_{ij}}\langle i_a|j_{\dot{b}}][j^{\dot{b}}|P_b\rangle\langle P^b|i_{\dot{a}}]\frac{1}{s_{iP}} \\
&= \frac{(-)^{\mathcal{P}_{ij}}}{s_{iP}}\langle i_a|\hat{p}_j\hat{p}_P|i_{\dot{a}}] \;, \qquad\qquad\qquad (A.3.4)
\end{aligned}
$$

where we have $(-)^{\mathcal{P}_{ij}} = +1$ for clockwise ordering of the states (i,j) for the three-point amplitude. Also, P is an arbitrary momentum. By the same series of manip-

ulations we can show that

$$u_{ia}\tilde{u}_{i\dot{a}} = u_{ib}\tilde{u}_{i\dot{a}}\delta_a^b = \frac{(-)^{\mathcal{P}_{ji}}}{s_{iP}}\langle i_a|\hat{p}_P\hat{p}_j|i_{\dot{a}}] \ . \tag{A.3.5}$$

Note that the difference between (A.3.4) and (A.3.5) is just a sign since $(-)^{\mathcal{P}_{ji}} = -(-)^{\mathcal{P}_{ji}}$.

A.3.2   The relation $w_\ell \cdot w_{\ell'} \ \tilde{w}_\ell \cdot \tilde{w}_{\ell'} = -s_{ij}^{-1}$

Here we provide an expression for the contraction between w- and $\tilde{w}$-spinors of two three-point amplitudes, connected by an internal propagator, originally encountered in the recursive calculation of the four-point tree amplitude in [13].

We start with expression A.3.4 and choose $i = 1, j = 4$ and $P = 2$, following Figure A.1. This yields

$$u_{1a}\tilde{u}_{1\dot{a}}s_{\hat{1}\hat{2}} = -\langle\hat{1}_a|\hat{p}_4\hat{p}_{\hat{2}}|\hat{1}_{\dot{a}}] \ . \tag{A.3.6}$$

However, we can also write

$$\begin{aligned}
\langle\hat{1}_a|\hat{p}_4\hat{p}_{\hat{2}}|\hat{1}_{\dot{a}}] &= -u_{\hat{1}a}\tilde{u}_4^{\dot{d}}[4_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] = -u_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}[l'_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] \\
&= (-i)u_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}[\ell_{\dot{d}}|\hat{2}^b\rangle\langle\hat{2}_b|\hat{1}_{\dot{a}}] = iu_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}\tilde{u}_{\ell\dot{d}}u_{\hat{2}}^b\langle\hat{2}_b|\hat{1}_{\dot{a}}] \\
&= iu_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}\tilde{u}_{\ell\dot{d}}u_\ell^b\langle\ell_b|\hat{1}_{\dot{a}}] = u_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}\tilde{u}_{\ell\dot{d}}u_\ell^b\langle\ell'_b|\hat{1}_{\dot{a}}] \\
&= -u_{\hat{1}a}\tilde{u}_{\ell'}^{\dot{d}}\tilde{u}_{\ell\dot{d}}u_\ell^b u_{\ell'b}\tilde{u}_{\hat{1}\dot{a}} = -u_{\hat{1}a}\tilde{u}_{\hat{1}\dot{a}}\tilde{u}_{\ell'}\cdot\tilde{u}_\ell \ u_\ell\cdot u_{\ell'} \ . \tag{A.3.7}
\end{aligned}$$

Comparing (A.3.6) and (A.3.7) we conclude

$$\tilde{u}_{\ell'}\cdot\tilde{u}_\ell \ u_{\ell'}\cdot u_\ell = -s_{12} \ , \tag{A.3.8}$$

since $s_{\hat{1}\hat{2}} = s_{12}$. Now we express the contractions of u-spinors in terms of w-spinors. As discussed in [13] we can deduce from (A.3.8) that

$$u_\ell\cdot w_{\ell'} = \tilde{u}_\ell\cdot\tilde{w}_{\ell'} = w_\ell\cdot u_{\ell'} = \tilde{w}_\ell\cdot\tilde{u}_{\ell'} = 0 \ , \tag{A.3.9}$$

by using the redundancy of the w-spinors under a shift $w_{la} \to w_{\ell a} + b_\ell u_{\ell a}$. Exploiting the defining relation between a spinor u_ℓ and its inverse w_ℓ and multiplying by $u_{\ell',a}$ and $w_{\ell',b}$ we have

$$u_\ell^a u_{\ell',a}w_\ell^b w_{\ell',b} - u_\ell^b w_{\ell',b}w_\ell^a w_{\ell',a} = u_{\ell',a}w_{\ell',b}\epsilon^{ab} \ . \tag{A.3.10}$$

Now, the second term on the RHS vanishes as stated in (A.3.9). Since $u_\ell \cdot u_{\ell'} \neq 0$, we have the relation

$$u_\ell \cdot u_{\ell'} \; w_\ell \cdot w_{\ell'} = 1 \quad \Leftrightarrow \quad u_\ell \cdot u_{\ell'} = \frac{1}{w_\ell \cdot w_{\ell'}} \; . \tag{A.3.11}$$

From this we can deduce that a spinor $w_{\ell a}/w_{\ell' b}$ is related to the spinor $u_{\ell' a}/u_{\ell b}$, respectively, and we can choose to normalize as in (A.3.3)

$$w_{\ell'_i a} = \frac{u_{\ell_i a}}{\sqrt{-s_{ij}}} \; , \qquad\qquad w_{\ell_i a} = -\frac{u_{\ell'_i a}}{\sqrt{-s_{ij}}} \; . \tag{A.3.12}$$

A.3.3   Spinor identities for the one-loop five-point calculation

Here we would like to outline some steps of the calculation which takes us from (3.5.13) to (3.5.15).

The basic idea is to express the result of the Grassmann integration as a sum of coefficients of factors $\tilde{\eta}_{i\dot{c}}\eta_{jc}$ with $i, j = 1, 2, 5$ for the $(3, 4)$-cut. It is then a matter of algebra to rewrite the coefficient of $\tilde{\eta}_{i\dot{c}}\eta_{jc}$ in such a way that any dependence on the three-point quantities w_{ℓ_i}, $w_{\ell'_i}$ and their counterparts in $\tilde{\eta}_{\ell_i}$ is removed. In the following we provide some explicit terms as examples.

Let us consider one of the terms of the product in (3.5.15), e.g.

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\left\{ \langle 1^c|\ell_3] \cdot \tilde{w}_{\ell'_3}\tilde{w}_{\ell'_2} \cdot [\ell_2|\hat{\ell}_1|\ell_4] \cdot \tilde{w}_{\ell_4} - \langle 1^c|\ell_2] \cdot \tilde{w}_{\ell'_2}\tilde{w}_{\ell'_3} \cdot [\ell_3|\hat{\ell}_1|\ell_4] \cdot \tilde{w}_{\ell_4} \right\}$$
$$\times \left\{ [1^{\dot{c}}|\ell_3\rangle \cdot w_{\ell'_3}w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} - [1^{\dot{c}}|\ell_2\rangle \cdot w_{\ell'_2}w_{\ell'_3} \cdot \langle\ell_3|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} \right\} \; . \tag{A.3.13}$$

The first thing one realizes is that the two factors in the brackets antisymmetrize among themselves. This can be seen by applying the normalization relations for the w-spinors related to the internal momenta

$$[1^{\dot{c}}|\ell_3\rangle \cdot w_{\ell'_3}w_{\ell'_2} \cdot \langle\ell_2|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4} = [1^{\dot{c}}|\ell_3\rangle \cdot w_{\ell'_3}\frac{u^a_{\ell_2}}{\sqrt{-s_{12}}}\langle\ell_{2a}|\hat{\ell}_1|\ell_4\rangle \cdot w_{\ell_4}$$
$$= \frac{1}{\sqrt{-s_{12}}}u^a_{l'_3}w^b_{l'_3}[1^{\dot{c}}|l_{3b}\rangle\langle l'_{3a}|\hat{\ell}_1|l_4\rangle \cdot w_{l_4} \tag{A.3.14}$$

where a similar relation is used for the second term of each bracket factor. Since

$$u^a_{\ell'_3}w^b_{\ell'_3} - u^b_{\ell'_3}w^a_{\ell'_3} = \epsilon^{ab} \; , \tag{A.3.15}$$

we can write (A.3.13) as

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\Big(\frac{1}{\sqrt{-s_{12}}}\Big)^2[1^{\dot{c}}|\ell_{3b}\rangle\epsilon^{ab}\langle\ell'_{3a}|\hat{\ell}_1|\ell_4\rangle\cdot w_{\ell_4}\,\langle 1^c|\ell_{3\dot{b}}]\epsilon^{\dot{a}\dot{b}}[\ell'_{3\dot{a}}|\hat{\ell}_1|\ell_4]\cdot\tilde{w}_{\ell_4}$$

$$=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{i2}{-s_{12}}[1^{\dot{c}}|\ell_3^a\rangle\langle\ell_{3a}|\hat{\ell}_1|\ell_4\rangle\cdot w_{\ell_4}\,\langle 1^c|\ell_3^{\dot{a}}][\ell_{3\dot{a}}|\hat{\ell}_1|\ell_4]\cdot\tilde{w}_{\ell_4}$$

$$=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{(-1)}{-s_{12}}[1^{\dot{c}}|\hat{\ell}_3\hat{\ell}_1|\ell_4\rangle\cdot w_{\ell_4}\,\langle 1^c|\hat{\ell}_3\hat{\ell}_1|\ell_4]\cdot\tilde{w}_{\ell_4}$$

$$=\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{\ell}_1|\ell_4\rangle\cdot w_{\ell_4}\,\tilde{w}_{\ell_4}\cdot[\ell_4|\hat{\ell}_1\hat{p}_2|1^c\rangle\,,\tag{A.3.16}$$

where we have used momentum conservation at the second corner, $\ell_3 = \ell_2 + p_2 = \ell_1 + p_1 + p_2$, in the last line.

The next step is to remove the dependence on the w-spinors. The following relation holds:

$$\hat{\ell}_1|\ell_{4a}\rangle w_{\ell_4}^a\tilde{w}_{\ell_4}^{\dot{a}}[\ell_{4\dot{a}}|\hat{\ell}_1 = \hat{p}_5|\ell_{4a}\rangle w_{\ell_4}^a\tilde{w}_{\ell_4}^{\dot{a}}[\ell_{4\dot{a}}|\hat{p}_5 = \hat{p}_5|\ell'_{1a}\rangle w_{\ell'_1}^a(-1)^2\tilde{w}_{\ell'_1}^{\dot{a}}[\ell'_{1\dot{a}}|\hat{p}_5$$

$$= \Big(\frac{1}{\sqrt{-s_{15}}}\Big)^2\hat{p}_5|\ell'_{1a}\rangle u_{\ell_1}^a\tilde{u}_{\ell_1}^{\dot{a}}[\ell'_{1\dot{a}}|\hat{p}_5 = \frac{1}{s_{15}}\hat{p}_5|1_a\rangle u_1^a\tilde{u}_1^{\dot{a}}[1_{\dot{a}}|\hat{p}_5\,.$$

$$\tag{A.3.17}$$

Using the result of (A.3.4) we arrive at the following string of momenta,

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}s_{15}s_{1P}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_1\hat{\ell}_1\hat{P}\hat{p}_1\hat{p}_5\hat{p}_2|1^c\rangle\,.\tag{A.3.18}$$

Choosing now $P = p_5$, after some rearrangement of the momenta we arrive at

$$\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}s_{15}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{p}_1\hat{\ell}_1\hat{p}_5\hat{p}_2|1^c\rangle\,.\tag{A.3.19}$$

This expression can be further simplified as follows: Since $\ell_1 = p_5 + \ell_4$ we have

$$\hat{\ell}_1\hat{p}_5 = \hat{\ell}_1(\hat{\ell}_1 - \hat{\ell}_4) = -(\hat{\ell}_4 + \hat{p}_5)\hat{\ell}_4 = -\hat{p}_5\hat{\ell}_1\,.\tag{A.3.20}$$

Permuting now the string of external momenta the final result for the coefficient becomes

$$-\,\tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{p}_5\hat{\ell}_1\hat{p}_2|1^c\rangle = \tilde{\eta}_{1\dot{c}}\eta_{1c}\frac{1}{s_{12}}[1^{\dot{c}}|\hat{p}_2\hat{\ell}_1\hat{p}_5\hat{p}_2|1^c\rangle\tag{A.3.21}$$

by rearranging the order of $\hat{p}_5$ and $\hat{\ell}_1$ again.

This algebraic procedure can then be similarly repeated to simplify all the other coefficients in the cut expression (3.5.13).

A.4   Passarino–Veltman reduction

This section will provide a more general discussion of the PV reduction which is used to reduce the linear pentagon integral of the one-loop five-point superamplitude (3.5.17). The general manipulations of reducing linear integrals will be presented, followed by an application using a linear triangle as an example.

Tensor integrals may generically appear in one-loop amplitude calculations in well known theories like QCD, gravity or $\mathcal{N} < 4$ YM. These are efficiently reduced to scalar integrals (i.e. no powers of loop-momenta in the numerator) using the so-called Passarino-Veltman reduction [38]. This approach is based on the fact that at one-loop, scalar products of loop momenta with external momenta can always be expressed as combinations of propagators.
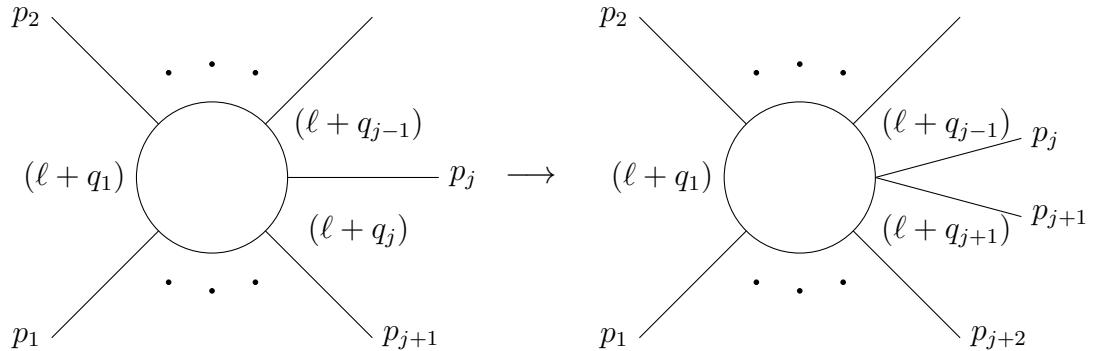

Figure A.2   This is the general form of the one-loop n-point integral that may have a numerator with non-zero powers of momenta. After the PV reduction the propagator $(\ell + p_j)^{-2}$ is "pinched out".

In general this method applies to an integral of n external particles with r powers of loop momenta in the numerator, and has the form

$$I_n \sim \int \frac{d^D \ell}{(2\pi)^D} \frac{f^{(r)}(\ell)}{(\ell^2 - m_0^2)((\ell + q_1)^2 - m_1^2)\dots((\ell + q_{n-1})^2 - m_{n-1}^2)}, \qquad (A.4.1)$$

where

$$q_k = \sum_{i=1}^{k} p_i. \qquad (A.4.2)$$

In the present discussion, the external momenta p_i will be considered as incoming and the loop integration is in $D = 4 - 2\varepsilon$ dimensions. However, external momenta will be strictly in four dimensions. In this notation, the integral with $f^{(0)} = 1$ is the scalar integral.

A linear integral has the following structure:

$$I_n(\ell^\mu) \sim \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu}{(\ell^2 - m_0^2)((\ell + q_1)^2 - m_1^2)\ldots((\ell + q_{n-1})^2 - m_{n-1}^2)} . \qquad \text{(A.4.3)}$$

The key step of the PV reduction method is identifying that the linear integral (A.4.3) can only be equal to a linear combination of the independent external momenta*. This can be written as

$$I_n(\ell^\mu) = \sum_{i=1}^{n-1} C_{n;i}\, p_i^\mu . \qquad \text{(A.4.4)}$$

The next step is to contract both sides of (A.4.4) with p_j^μ:

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell \cdot p_j}{(\ell^2 - m_0^2)((\ell + q_1)^2 - m_1^2)\ldots((\ell + q_{n-1})^2 - m_{n-1}^2)} = \sum_{i=1}^{n-1} C_{n;i}\Delta^{ij} , \qquad \text{(A.4.5)}$$

where $\Delta^{ij} = p_i \cdot p_j$ is the "Gram" matrix. Using the fact that $p_j = q_j - q_{j-1}$, the inner product $\ell \cdot p_j$ can be recast to what is also known as the Passarino-Veltman reduction formula:

$$2\ell \cdot p_j = \big((\ell + q_j)^2 - m_j^2\big) - \big((\ell + q_{j-1})^2 - m_{j-1}^2\big) + m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2 . \qquad \text{(A.4.6)}$$

This form of the numerator provides terms that cancel with the propagators labeled with the indices j and $j-1$. This eliminates any loop momentum dependence of the numerator and the integral is reduced to a scalar one. What is left is a set of $n-1$ linear equations (one for each p_j) that can be used to solve for the coefficients $C_{n;i}$:

$$2\sum_{i=1}^{n-1} C_{n;i}\Delta^{ij} = I_{n-1}^{(j)} - I_{n-1}^{(j-1)} + (m_j^2 - m_{j-1}^2 + q_j^2 + q_{j-1}^2)I_n . \qquad \text{(A.4.7)}$$

Now, the integrals that appear in the system of equations (A.4.7) are scalar integrals with n and $n-1$ internal propagators. $I_{n-1}^{(j)}$ denotes an integral that has the jth propagator canceled (or "pinched out").

---

*Momentum conservation ensures that only $n-1$ momenta are independent as $\sum_{i=1}^{n-1} p_i = p_n$.

The coefficients $C_{n;i}$ are therefore

$$C_{n;i} = \frac{1}{2} \sum_j \Delta_{ij}^{-1} \left(I_{n-1}^{(j)} - I_{n-1}^{(j-1)} + (m_j^2 - m_{j-1}^2 + q_j^2 + q_{j-1}^2)I_n \right) . \qquad \text{(A.4.8)}$$

One potential problem arises in kinematic regions where the determinant of the Gram matrix, $\det(\Delta^{ij}) = \Delta$ is equal to zero but this will not be considered here.

This concludes the general treatment of reducing linear integrals. It was shown that they can be written as

$$I_n(\ell^\mu) = \sum_{i=1}^{n-1} C_{n;i} \, p_i^\mu \qquad \text{(A.4.9)}$$

using the coefficients (A.4.8) that consist of kinematic invariants and scalar integrals of n or $n-1$ internal momenta. This is also the case for the linear pentagon that appears in Section 3.5. The PV reduction method can also be used for integrals with numerators of higher rank tensors. This further increases the degree of complexity and is not relevant to the examples considered here.
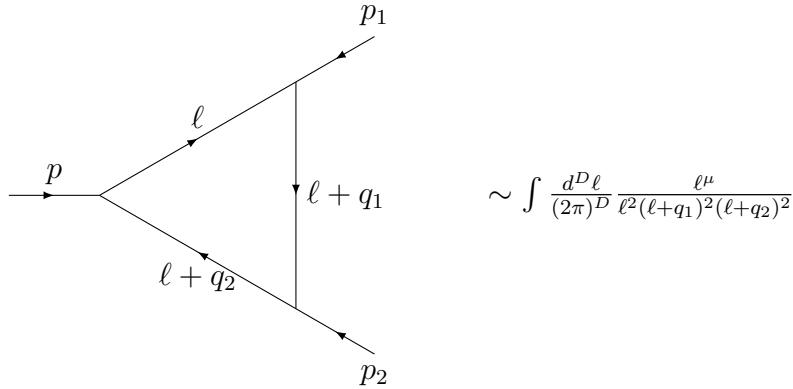

Figure A.3   A linear triangle.

As a concrete example we shall perform the PV reduction for a linear triangle integral (three-point function). The masses will be set to zero, as this is similar to the case of the linear pentagon of Section 3.5 and simplifies the calculations. However, external momentum are not considered to be on shell for now, i.e. $p_i^2 \neq 0$.

Momentum conservation dictates

$$p_1 + p_2 + p_3 = 0 \,. \tag{A.4.10}$$

The linear integral has the form

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^\mu}{\ell^2 (\ell + q_1)^2 (\ell + q_2)^2} = C_{3;1}\, p_1^\mu + C_{3;2}\, p_2^\mu \,, \tag{A.4.11}$$

where, as defined in (A.4.2), $q_1 = p_1$ and $q_2 = p_1 + p_2$. Following the procedure outlined above, we contract both sides with p_1^μ to obtain

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell \cdot p_1}{\ell^2 (\ell + q_1)^2 (\ell + q_2)^2} = C_{3;1}\, p_1^2 + C_{3,2}\, p_1 \cdot p_2 \,. \tag{A.4.12}$$

The inner product of the loop momenta with p_1 is

$$2\,\ell \cdot p_1 = (\ell + p_1)^2 - \ell^2 - p_1^2 \,, \tag{A.4.13}$$

which when plugged in (A.4.12) yields

$$C_{3;1}\, p_1^2 + C_{3;2}\, p_1 \cdot p_2 = \frac{1}{2} \left(I_2^{(1)} - I_2^{(0)} - p_1^2 I_3 \right) \,. \tag{A.4.14}$$

The scalar integrals that appear in (A.4.14) are defined, up to an overall factor, as:

$$I_2^{(1)} = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 \, (\ell + q_2)^2} \tag{A.4.15}$$

$$I_2^{(0)} = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell + q_1)^2 \, (\ell + q_2)^2} \,. \tag{A.4.16}$$

Next, contract both sides of (A.4.11) with p_2^μ to obtain

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell \cdot p_2}{\ell^2 (\ell + q_1)^2 (\ell + q_2)^2} = C_{3;1}\, p_2 \cdot p_1 + C_{3;2}\, p_2^2 \,. \tag{A.4.17}$$

The inner product of the loop momenta with p_2 is now

$$2\,\ell \cdot p_2 = (\ell + p_2)^2 - \ell^2 - p_2^2 \,, \tag{A.4.18}$$

which can be used in (A.4.17) to write it as

$$C_{3;1}\, p_2 \cdot p_1 + C_{3;2}\, p_2^2 = \frac{1}{2}\left(I_2^{(1)} - I_2^{(1)} - (2\, p_1 \cdot p_2 + p_2^2)I_3 \right) \qquad \text{(A.4.19)}$$

and the integral $I_2^{(2)}$ is defined as

$$I_2^{(2)} = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2\,(\ell + q_1)^2}\,. \qquad \text{(A.4.20)}$$

The Gram determinant is $\Delta = p_1^2\, p_2^2 - (p_1 \cdot p_2)^2$ and the solution for the coefficients

is

$$C_{3,1} = \Delta^{-1}\left[(p_2^2 + p_1 \cdot p_2)I_2^{(1)} - p_2^2 I_2^{(0)} - p_1 \cdot p_2 I_2^{(2)} \right.$$
$$\left. - \left(p_1^2\, p_2^2 - 2(p_1 \cdot p_2)^2 - p_2^2\,(p_1 \cdot p_2) \right) I_3 \right] \qquad \text{(A.4.21)}$$

$$C_{3;2} = \Delta^{-1}\left[(p_1^2 + p_1 \cdot p_2)I_2^{(1)} + p_1^2 I_2^{(2)} - p_1 \cdot p_2 I_2^{(0)} \right.$$
$$\left. - \left(p_1^2\, p_2^2 + p_1^2\,(p_1 \cdot p_2) \right) I_3 \right]\,. \qquad \text{(A.4.22)}$$

If external momenta are taken on shell, the result simplifies significantly:

$$C_{3,1} = (p_1 \cdot p_2)^{-1}\left(I_2^{(2)} - I_2^{(1)} - 2\,(p_1 \cdot p_2)I_3 \right) \qquad \text{(A.4.23)}$$

$$C_{3;2} = (p_1 \cdot p_2)^{-1}\left(I_2^{(0)} - I_2^{(1)} \right)\,. \qquad \text{(A.4.24)}$$

A.5 Reduction to four dimensions of six-dimensional Yang-Mills amplitudes

In this appendix we consider the four-dimensional limit of the four- and five-point tree-level amplitudes in pure Yang Mills theory, and provide detailed information of how the calculations of Section 3.5.5 are carried out.

We begin with the four-point amplitude of [13], given by (3.3.11). This can be also thought of as the extraction of $G_1^-, G_2^-, G_3^+, G_4^+$ from the superamplitude (3.3.12) by integrating over the measure $d\eta_{1-}d\tilde{\eta}_{1-}d\eta_{2-}d\tilde{\eta}_{2-}d\eta_{3+}d\tilde{\eta}_{3+}d\eta_{4+}d\tilde{\eta}_{4+}$. This will reduce down to a four-dimensional amplitude with helicities $(1^-, 2^-, 3^+, 4^+)$. Using (3.5.34),

the four-dimensional reduction of (3.3.11) yields

$$A_4^{(4d)} = -\frac{i}{st}\langle 12\rangle^2[34]^2 = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \, . \tag{A.5.1}$$

Next, we consider the five-point amplitude (3.3.14) and reduce to a four-dimensional helicity configuration $(1^+, 2^+, 3^+, 4^-, 5^-)$. For this case, only a few terms in (3.3.14) survive. The $\mathcal{A}$-tensor becomes

$$\begin{aligned}
\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} &= \langle 1_a|\hat{p}_2\hat{p}_3\hat{p}_4\hat{p}_5|1_{\dot{a}}]\langle 2_b3_c4_d5_e\rangle[2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \\
&+ \langle 2_b|\hat{p}_3\hat{p}_4\hat{p}_5\hat{p}_1|2_{\dot{b}}]\langle 3_c4_d5_e1_a\rangle[3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}1_{\dot{a}}] \\
&+ \langle 3_c|\hat{p}_4\hat{p}_5\hat{p}_1\hat{p}_2|3_{\dot{c}}]\langle 4_d5_e1_a2_b\rangle[4_{\dot{d}}5_{\dot{e}}1_{\dot{a}}2_{\dot{b}}] \, ,
\end{aligned} \tag{A.5.2}$$

which in the four-dimensional limit takes the form

$$\begin{aligned}
\mathcal{A}_{a\dot{a}b\dot{b}c\dot{c}d\dot{d}e\dot{e}} \quad \xrightarrow{4d} \quad &-[12]\langle 23\rangle[34]\langle 45\rangle[51] \times [23]^2\langle 45\rangle^2 \\
&-[23]\langle 34\rangle[45]\langle 51\rangle[12] \times [31]^2\langle 45\rangle^2 \\
&-[34]\langle 45\rangle[51]\langle 12\rangle[23] \times [12]^2\langle 45\rangle^2 \, .
\end{aligned} \tag{A.5.3}$$

For our specific helicity choice, the non-zero parts of the $\mathcal{D}$-tensor are those involving the Lorentz invariant brackets

$$\langle 2_b3_c4_d5_e\rangle[1_{\dot{a}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}] \quad \text{and} \quad [2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}5_{\dot{e}}]\langle 1_a3_c4_d5_e\rangle \, , \tag{A.5.4}$$

where both reduce to $[23]\langle 45\rangle[13]\langle 45\rangle$ in four dimensions, using (3.5.34). Each factor multiplies $\langle 1_a(2.\tilde{\Delta}_2)_{\dot{b}}]$ and $[1_{\dot{a}}(2.\Delta_2)_b\rangle$. The quantities Δ_i's that are of interest here take the form:

$$\Delta_2 = \langle 2|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2\rangle \, , \quad \text{and} \quad \tilde{\Delta}_2 = [2|\hat{p}_3\hat{p}_4\hat{p}_5 - \hat{p}_5\hat{p}_4\hat{p}_3|2] \, . \tag{A.5.5}$$

Expanding the expression of the first non-vanishing $\mathcal{D}$-term yields

$$\langle 1_a(2.\tilde{\Delta}_2)_{\dot{b}}] = \langle 1_a|2^{\dot{b}}][2_{\dot{b}}|3^c\rangle\langle 3_c|4^{\dot{d}}][4_{\dot{d}}|5^e\rangle\langle 5_e|2_{\dot{b}}] - \langle 1_a|2^{\dot{b}}][2_{\dot{b}}|5^e\rangle\langle 5_e|4^{\dot{d}}][4_{\dot{d}}|3^c\rangle\langle 3_c|2_{\dot{b}}] \tag{A.5.6}$$

The helicities of spinors $\langle 1_a|$ and $|2_{\dot{b}}]$ remain fixed (both positive in the present case), whereas the string of spinors within $\langle 1_a|$ and $|2^{\dot{b}}]$ is summed over. The indices should

be lowered, and this determines the overall sign of each product. This is achieved by using the antisymmetric ε tensor. Applying this to the first term of (A.5.6), and using (3.5.34) gives:

$$\varepsilon^{\dot{b}\dot{b}'}\varepsilon^{\dot{c}\dot{c}'}\varepsilon^{\dot{d}\dot{d}'}\varepsilon^{\dot{e}\dot{e}'}\langle 1_a|2_{\dot{b}'}][2_{\dot{b}}|3_{c'}\rangle\langle 3_c|4_{\dot{d}'}][4_{\dot{d}}|5_{e'}\rangle\langle 5_e|2_{\dot{b}}] = \tag{A.5.7}$$

$$\varepsilon^{-+}\varepsilon^{+-}\varepsilon^{-+}\varepsilon^{+-}\langle 1_+|2_+][2_-|3_-\rangle\langle 3_+|4_+][4_-|5_-\rangle\langle 5_+|2_+] = \tag{A.5.8}$$

$$[12]\langle 23\rangle[34]\langle 45\rangle[52] \tag{A.5.9}$$

By analogous manipulations it is possible to calculate the second term in (A.5.6) and also the $[1_{\dot{a}}(2.\Delta_2)_b\rangle$ term. The combined result is:

$$2\mathcal{D}_{a\dot{a}\dot{b}b\dot{c}c\dot{d}d\dot{e}e} \xrightarrow{4d} 2\left([12]\langle 23\rangle[34]\langle 45\rangle[52] - [12]\langle 25\rangle[54]\langle 43\rangle[32]\right)\times[13][23]\langle 45\rangle^2. \tag{A.5.10}$$

Now, the use of the Schouten identity and momentum conservation should reduce the number of terms, eventually to one single fraction. For example, the first term of the $\mathcal{D}$-term and the (A.5.3) term of the $\mathcal{A}$-term can be combined together and using the Schouten identity (2.1.32), the result is

$$[12]\langle 23\rangle[34][23]\langle 45\rangle^3\left([52][13] - [51][23]\right) = [12]\langle 23\rangle[34][23]\langle 45\rangle^3[53][12] \tag{A.5.11}$$

Similarly, adding the second $\mathcal{D}$-term to the second (A.5.3) $\mathcal{A}$-term and using momentum conservation gives:

$$-[12][23]\langle 34\rangle[45][13]\langle 45\rangle^2\left(\langle 52\rangle[23] + \langle 51\rangle[13]\right) = [12][23]\langle 34\rangle[45][13]\langle 45\rangle^2\langle 54\rangle[43] \tag{A.5.12}$$

It is now possible to combine (A.5.11) with the third of the $\mathcal{A}$-term (A.5.3) and get

$$[12]^2[34][23]\langle 45\rangle^3\left(-[53]\langle 32\rangle - [51]\langle 12\rangle\right) = [12]^2[34][23]\langle 45\rangle^3[54]\langle 42\rangle \tag{A.5.13}$$

Expression (A.5.13) needs to be added to (A.5.12) to give

$$[12][23][34][45]\langle 45\rangle^3\left([13]\langle 34\rangle - \langle 24\rangle[12]\right) = [12][23][34][45][51]\langle 45\rangle^4, \tag{A.5.14}$$

where momentum conservation was used. When (A.5.14) is divided by the string of invariants as they appear in (3.3.14), it gives the correct four-dimensional result:

$$i\frac{[12][23][34][45][51]\langle 45\rangle^4}{s_{12}s_{23}s_{34}s_{45}s_{51}} = i\frac{\langle 45\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}. \tag{A.5.15}$$

A similar procedure can be followed in order to obtain the four dimensional limit of the full one-loop five-point amplitude (3.5.31). In Section 3.5.5 the configuration where only particles 1 and 2 had negative helicities and all other positive. Here we will also consider the configuration where only the first and the fifth external particles are of negative helicity, $(1^-, 2^+, 3^+, 4^+, 5^-)$. In this case the only non-vanishing term is the second term in (3.5.31). Thus, we only need to focus on the four dimensional limit of

$$\frac{1}{s_{34}} \frac{1}{s_{12}} \left([2_{\dot b}|\hat p_1 \hat \ell_1 \hat p_5 \hat p_1|2_b\rangle B + [2_{\dot b}|\hat p_1 \hat p_3 \hat p_5 \hat p_1|2_b\rangle C \right) \langle 1_a 3_c 4_d 5_e\rangle [1_{\dot a} 3_{\dot c} 4_{\dot d} 5_{\dot e}], \qquad (A.5.16)$$

with

$$B = -\frac{s_{15}(s_{12} - s_{45})}{2 s_{23} s_{34} s_{45}}, \qquad\qquad C = -\frac{s_{12} s_{15}}{2 s_{23} s_{34} s_{45}}. \qquad (A.5.17)$$

For this specific helicity choice, and due to momentum conservation, expression (A.5.16) becomes

$$\frac{1}{s_{34}} \frac{1}{s_{12}} \left([12]^2 \langle 12\rangle [25]\langle 51\rangle B + [12]^2 \langle 13\rangle [35]\langle 51\rangle C \right) \langle 15\rangle^2 [34]^2$$

$$= \frac{[12]^2 \langle 15\rangle^3 s_{15} [34]^2}{2\, s_{34} s_{12} s_{23} s_{34} s_{45}} \left(\langle 12\rangle [52] s_{12} - \langle 12\rangle [52] s_{45} + \langle 13\rangle [53] s_{12} \right)$$

$$= \frac{[12]^2 \langle 15\rangle^3 [34]^2 s_{15}}{2\, s_{34} s_{12} s_{23} s_{34} s_{45}} [45]\langle 12\rangle \left([21]\langle 14\rangle + [25]\langle 54\rangle \right)$$

$$= -\frac{[12]^2 \langle 15\rangle^3 [34]^2 s_{15}}{2\, s_{34} s_{12} s_{23} s_{34} s_{45}} [45]\langle 12\rangle [23]\langle 34\rangle$$

$$= -\frac{[12]^2 \langle 15\rangle^3 [34]^2 s_{15} [45]\langle 12\rangle [23]\langle 34\rangle}{2\langle 34\rangle [43]\langle 12\rangle [21]\langle 23\rangle [32]\langle 34\rangle [43]\langle 45\rangle [54]}$$

$$= -\frac{s_{12} s_{15}}{2} \frac{\langle 15\rangle^3}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle}. \qquad (A.5.18)$$

Recall, that the box integral is of the form $I = 2F/s_{12} s_{15}$, giving expected the Parke-Taylor formula.

B | Spinors in Three Dimensions and Integral Properties

B.1 Spinor conventions in 3d

This section presents a stenographic summary of the spinor conventions used in Chapter 4 and are in agreement with the those that appear in [72]. The signature used is $(+, -, -)$ and real Pauli matrices relate momenta in vector and double-spinor notation

$$\sigma^0_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; , \; \sigma^1_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma^2_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; , \tag{B.1.1}$$

such that a generic, possibly off-shell momentum can be written as

$$P_{\alpha\beta} = p_\mu \sigma^\mu_{\alpha\beta} = \begin{pmatrix} E - p_y & -p_x \\ -p_x & E + p_y \end{pmatrix} \; . \tag{B.1.2}$$

Note that this is a symmetric matrix and hence any off-shell momentum can be written alternatively as the symmetrized product of two two-spinors ξ, μ as

$$P_{\alpha\beta} = \xi_{(\alpha}\mu_{\beta)} = \frac{1}{2}(\xi_\alpha\mu_\beta + \xi_\beta\mu_\alpha) \, . \tag{B.1.3}$$

Moreover, if we choose ξ and μ to be real then there is a rescaling invariance in this representation $\xi \to r\xi, \mu \to \mu/r$ with r a non-zero real number. Alternatively one can choose the spinor variables to be complex, but then they are related by complex conjugation $\mu = \overline{\xi}$ in order for the momenta to be real. This representation is invariant under $\xi \to e^{i\phi}\xi$, $\mu \to e^{-i\phi}\mu$.

For on-shell momenta $p_\mu p^\mu = \det p_{\alpha\beta} = 0$, and we simply set $\mu = \xi = \lambda$, which reduces the rank of the two-by-two matrix defined above and removes the rescaling invariance except for the reflection $\lambda \to -\lambda$. Therefore, we have

$$p_{\alpha\beta} = \lambda_\alpha \lambda_\beta \, , \tag{B.1.4}$$

and we note that for positive energy λ must be real, while for negative energy it is purely imaginary.

Spinor variables can be contracted in an $\mathsf{SL}(2, \mathbb{R})$ (Lorentz-)invariant fashion using the epsilon tensor $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ with $\epsilon_{12} = +1$, which is also used to raise and lower spinor indices. The fundamental invariant of two spinor variables λ and μ is defined as

$$\langle \lambda\mu \rangle = \epsilon_{\alpha\beta} \lambda^\alpha \mu^\beta \, , \tag{B.1.5}$$

in terms of which it is possible to write any Lorentz-invariant momentum vector contractions, two very common examples being

$$(p_1 + p_2)^2 = \langle 12 \rangle^2 \, ,$$
$$2\epsilon^{\mu\nu\rho} p_{1\mu} p_{2\nu} p_{3\rho} = \mathrm{Tr}(\sigma^\mu \sigma^\nu \sigma^\rho) p_{1\mu} p_{2\nu} p_{3\rho} = \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \, . \tag{B.1.6}$$

Here we have also introduced the short-hand notation $\langle \lambda_1 \lambda_2 \rangle \equiv \langle 12 \rangle$. Finally, we note that for a generic momentum written as in (B.1.3), one has

$$P^2 = -\frac{1}{4} \langle \xi \, \mu \rangle^2 \, . \tag{B.1.7}$$

Now, if $P = p_1 + p_2$, then in the notation of (B.1.3) it is $\xi = \lambda_1 + i\lambda_2$ and $\mu = \lambda_1 - i\lambda_2$, where $p_i := \lambda_i \lambda_i$, $i = 1, 2$.

B.2   Half-BPS operators

In this section we briefly recall how half-BPS operators are introduced in ABJM theory. Consider the variation of operators of the form $\mathrm{Tr}\left(\phi^I \bar\phi_J\right)$ with $I \neq J$. Setting for example $I = 1$ and $J = 4$, this expands to

$$\delta \mathrm{Tr}\left(\phi^1 \bar\phi_4\right) = \mathrm{Tr}\left(\delta\phi^1 \bar\phi_4 + \phi^1 \delta\bar\phi_4\right) . \tag{B.2.1}$$

Following [100], we use the transformations:

$$\delta\phi^I = i\,\omega^{IJ}\psi_J\,, \tag{B.2.2}$$

$$\delta\bar{\phi}_I = i\,\bar{\psi}^J\omega_{IJ}\,. \tag{B.2.3}$$

The ω_{IJ}'s are given in terms of the $(2+1)$-dimensional Majorana spinors, ϵ_i ($i = 1,\ldots,6$) which are the supersymmetry generators:

$$\omega_{IJ} = \epsilon_i(\Gamma^i)_{IJ}\,, \tag{B.2.4}$$

$$\omega^{IJ} = \epsilon_i\left((\Gamma^i)^*\right)^{IJ}\,, \tag{B.2.5}$$

that are antisymmetric in I, J. The 4×4 matrices Γ^i are given by:

$$\Gamma^1 = \sigma_2 \otimes 1_2\,, \qquad \Gamma^4 = -\sigma_1 \otimes \sigma_2\,, \tag{B.2.6}$$

$$\Gamma^2 = -i\sigma_2 \otimes \sigma_3\,, \qquad \Gamma^5 = \sigma_3 \otimes \sigma_2\,, \tag{B.2.7}$$

$$\Gamma^3 = i\sigma_2 \otimes \sigma_1\,, \qquad \Gamma^6 = -i1_2 \otimes \sigma_2\,, \tag{B.2.8}$$

and satisfy the following relations,

$$\{\Gamma^i, \Gamma^{j\dagger}\} = 2\delta_{ij}\,, \qquad (\Gamma^i)_{IJ} = -(\Gamma^i)_{IJ}\,, \tag{B.2.9}$$

$$\frac{1}{2}\epsilon^{IJKL}\Gamma^i_{KL} = -\left(\Gamma^{j\dagger}\right)^{IJ} = \left((\Gamma^i)^*\right)^{IJ}\,, \tag{B.2.10}$$

leading to

$$\left(\omega^{IJ}\right)_\alpha = \left((\omega_{IJ})^*\right)_\alpha\,, \qquad \omega^{IJ} = \frac{1}{2}\epsilon^{IJKL}\omega_{KL}\,. \tag{B.2.11}$$

Explicitly, ω_{IJ} is given by the following matrix:

$$\omega_{IJ} = \begin{pmatrix} 0 & -i\epsilon_5 - \epsilon_6 & -i\epsilon_1 - \epsilon_2 & \epsilon_3 + i\epsilon_4 \\ i\epsilon_5 + \epsilon_6 & 0 & \epsilon_3 - i\epsilon_4 & -i\epsilon_1 + \epsilon_2 \\ i\epsilon_1 + \epsilon_2 & -\epsilon_3 + i\epsilon_4 & 0 & i\epsilon_5 - \epsilon_6 \\ -\epsilon_3 - i\epsilon_4 & i\epsilon_1 - \epsilon_2 & -i\epsilon_5 + \epsilon_6 & 0 \end{pmatrix}\,. \tag{B.2.12}$$

The term $\phi^1\delta\bar{\phi}_4$ yields

$$\phi^1\delta\bar{\phi}_4 = \phi^1\left[-\bar{\psi}^1(\epsilon_3 + i\epsilon_4) + i\bar{\psi}^2(\epsilon_1 + i\epsilon_2) - i\bar{\psi}^3(\epsilon_5 + i\epsilon_6) + 0\right]\,. \tag{B.2.13}$$

Therefore, requiring $\phi^1 \delta \bar{\phi}_4 = 0$ the conditions are:

$$\epsilon_1 + i\epsilon_2 = \ 0\,,$$
$$\epsilon_3 + i\epsilon_4 = \ 0\,, \qquad\qquad\qquad (\text{B.2.14})$$
$$\epsilon_5 + i\epsilon_6 = \ 0\,,$$

which relate half of the generators with the other half by constraining the components $\omega_{4J} = 0$.

Note that because of the relations (B.2.11) which set components of the form ω_{4L} to zero, the entries ω^{IJ} with $I, J \in (1, 2, 3)$ automatically vanish implying that $\delta\phi^I = 0 \iff I \in (1, 2, 3)$. This procedure may be iterated to show that generally the operators $\text{Tr}\left(\bar{\phi}_I \phi^J\right)$ for $I \neq J$ are indeed half-BPS. In the present work the operators under consideration are of the type

$$\mathcal{O} = \text{Tr}\left(\phi^A \bar{\phi}_4\right), \qquad\qquad\qquad (\text{B.2.15})$$

where $A \neq 4$.

B.3   Box integral cuts

In Section 4.3.3 the different cuts of the box function (4.2.7) are used to construct the full one-loop amplitude. Here, we present how the cuts produce the different expressions for the box of Figure B.1. The color structure will be suppressed as it doesn't affect the results.

The first cut under consideration will be the s-cut. This is computed by (4.3.9)

$$\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{s-\text{cut}} = \frac{1}{2} \int d^3 \eta_{\ell_1} d^3 \eta_{\ell_2} \tilde{\mathcal{A}}^{(0)}(\bar{1}, 2, -\bar{\ell}_2, -\ell_1) \times \tilde{\mathcal{A}}^{(0)}(\bar{3}, 4, \bar{\ell}_1, \ell_2) + \ell_1 \leftrightarrow \ell_2\,,$$
$$(\text{B.3.1})$$

using $\ell_1 + \ell_2 = p_1 + p_2$ and taking the loop momenta to be on shell, $\ell_1^2 = 0 = \ell_2^2$. This yields

$$2\,\tilde{\mathcal{A}}^{(1)}(\bar{1}, 2, \bar{3}, 4)|_{s-\text{cut}} = \delta^{(3)}(P) \int d^3 \eta_{\ell_1} d^3 \eta_{\ell_2} \frac{\delta^{(6)}(Q_L)}{\langle 21 \rangle \langle 1\ell_1 \rangle} \frac{\delta^{(6)}(Q_R)}{\langle \ell_2 \ell_1 \rangle \langle \ell_1 4 \rangle} + \ell_1 \leftrightarrow \ell_2\,. \ (\text{B.3.2})$$

The δ-functions are expanded as

$$\delta^{(6)}(Q_L) = \prod_{A=1}^{3} \delta(q_1^{A\alpha} + q_2^{A\alpha} - q_{\ell_1}^{A\alpha} + q_{\ell_2}^{A\alpha})\delta(q_{1\alpha}^A + q_{2\alpha}^A - q_{\ell_1\alpha}^A + q_{\ell_2\alpha}^A) \tag{B.3.3}$$

$$\delta^{(6)}(Q_R) = \prod_{A=1}^{3} \delta(q_3^{A\alpha} + q_4^{A\alpha} + q_{\ell_1}^{A\alpha} - q_{\ell_2}^{A\alpha})\delta(q_{3\alpha}^A + q_{4\alpha}^A + q_{\ell_1\alpha}^A - q_{\ell_2\alpha}^A). \tag{B.3.4}$$

Doing the η-integrations gives

$$\int d^3\eta_{\ell_1}d^3\eta_{\ell_2}\delta^{(6)}(Q_L)\delta^{(6)}(Q_R) = -\delta^{(6)}(Q_{ext})\langle\ell_1\ell_2\rangle^3. \tag{B.3.5}$$

Therefore, the s-cut is equal to

$$2\,\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)|_{s-\text{cut}} = \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{\langle\ell_1\ell_2\rangle^3}{\langle 21\rangle\langle 1\ell_1\rangle\langle\ell_2\ell_1\rangle\langle\ell_1 4\rangle} + \ell_1 \leftrightarrow \ell_2 \tag{B.3.6}$$

which can be re-written as

$$2\,\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)|_{s-\text{cut}} = \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{\langle 12\rangle^2\langle 1\ell_1\rangle\langle\ell_1 4\rangle\langle 14\rangle}{\langle 21\rangle\langle 1\ell_1\rangle^2\langle 14\rangle\langle\ell_1 4\rangle^2} + \ell_1 \leftrightarrow \ell_2$$

$$= -i\mathcal{A}_{\text{tree}}\frac{\langle 12\rangle^2\langle\ell_1 1\rangle\langle 14\rangle\langle 4\ell_1\rangle}{(\ell_1 + p_4)^2(\ell_1 - p_1)^2} + \ell_1 \leftrightarrow \ell_2 \tag{B.3.7}$$

$$= i\mathcal{A}_{\text{tree}}\frac{s\,\text{Tr}(\ell_1 p_1 p_4)}{(\ell_1 + p_4)^2(\ell_1 - p_1)^2} + \ell_1 \leftrightarrow \ell_2,$$

where again $\text{Tr}(abc) = 2\epsilon(a,b,c) := 2\epsilon_{\mu\nu\rho}a^\mu b^\nu c^\rho$. Hence, the s-cut of the box is indeed

$$I(1,2,3,4)|_{s\text{-cut}} = \frac{s\,\text{Tr}(\ell_1 p_1 p_4)}{(\ell_1 + p_4)^2(\ell_1 - p_1)^2}. \tag{B.3.8}$$

Next we perform the same procedure for the t-cut. Now the cut loop momenta are renamed as $\ell_1 + \ell_2 = p_1 + p_4$.

$$2\,\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)|_{t-\text{cut}} = \int d^3\eta_{\ell_1}d^3\eta_{\ell_2}\tilde{\mathcal{A}}^{(0)}(\bar{3},\ell_2,\bar{\ell}_1,2) \times \tilde{\mathcal{A}}^{(0)}(\bar{1},-\ell_1,-\bar{\ell}_2,4) + \ell_1 \leftrightarrow \ell_2$$

$$= \delta^{(3)}(P)\int d^3\eta_{\ell_1}d^3\eta_{\ell_2}\frac{\delta^{(6)}(Q_L)}{\langle\ell_1 2\rangle\langle 23\rangle}\frac{\delta^{(6)}(Q_R)}{\langle 1\ell_1\rangle\langle\ell_1\ell_2\rangle} + \ell_1 \leftrightarrow \ell_2. \tag{B.3.9}$$

After the η-integrations

$$
\begin{aligned}
2\,\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)|_{t-\text{cut}} &= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{\langle\ell_1\ell_2\rangle^3}{\langle\ell_12\rangle\langle23\rangle\langle1\ell_1\rangle\langle\ell_1\ell_2\rangle} + \ell_1 \leftrightarrow \ell_2 \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{\langle14\rangle^2}{\langle\ell_12\rangle\langle23\rangle\langle1\ell_1\rangle} + \ell_1 \leftrightarrow \ell_2 \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{\langle14\rangle^2\langle12\rangle\langle1\ell_1\rangle\langle\ell_12\rangle}{\langle23\rangle\langle12\rangle\langle\ell_12\rangle^2\langle1\ell_1\rangle^2} + \ell_1 \leftrightarrow \ell_2 \quad\text{(B.3.10)} \\
&= i\mathcal{A}_{\text{tree}}\frac{-t\,\text{Tr}(\ell_1 p_1 p_2)}{(\ell_1-p_1)^2(\ell_1+p_2)^2} + \ell_1 \leftrightarrow \ell_2 \\
&= i\mathcal{A}_{\text{tree}}I(1,2,3,4)|_{t-\text{cut}} + \ell_1 \leftrightarrow \ell_2
\end{aligned}
$$

Finally, we perform the u-channel cut. Now the loop momenta to be cut are configured as $\ell_1 + \ell_2 = p_1 + p_3$. The starting point is similar but now there is a single particle configuration and there is no need to take the average of $\ell_1 \leftrightarrow \ell_2$

$$
\begin{aligned}
\tilde{\mathcal{A}}^{(1)}(\bar{1},2,\bar{3},4)|_{t-\text{cut}} &= \int d^3\eta_{\ell_1}d^3\eta_{\ell_2}\tilde{\mathcal{A}}^{(0)}(\bar{1},-\ell_1,\bar{3},-\ell_2)\times\tilde{\mathcal{A}}^{(0)}(\bar{\ell}_2,2,\bar{\ell}_2,4) \\
&= \delta^{(3)}(P)\int d^3\eta_{\ell_1}d^3\eta_{\ell_2}\frac{\delta^{(6)}(Q_L)}{\langle1\ell_2\rangle\langle\ell_23\rangle}\frac{\delta^{(6)}(Q_R)}{\langle\ell_22\rangle\langle2\ell_1\rangle} \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-\langle\ell_1\ell_2\rangle^3}{\langle1\ell_2\rangle\langle\ell_23\rangle\langle\ell_22\rangle\langle2\ell_1\rangle} \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-u^2\langle\ell_1\ell_2\rangle}{\langle13\rangle^2\langle1\ell_2\rangle\langle\ell_23\rangle\langle\ell_22\rangle\langle2\ell_1\rangle} \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-u^2\langle\ell_1\ell_2\rangle}{\langle31\rangle\langle3\ell_1\rangle\langle\ell_1\ell_2\rangle\langle\ell_23\rangle\langle\ell_24\rangle\langle4\ell_1\rangle} \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-u^2}{\langle31\rangle\langle3|\ell_1|4\rangle\langle3|\ell_2|4\rangle} \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-u^2}{\langle31\rangle\langle3|\ell_1+\ell_2|4\rangle}\left(\frac{1}{\langle3|\ell_2|4\rangle}+\frac{1}{\langle3|\ell_1|4\rangle}\right) \\
&= \delta^{(3)}(P)\delta^{(6)}(Q_{ext})\frac{-u}{\langle14\rangle\langle43\rangle}\left(\frac{\langle3\ell_2\rangle\langle\ell_24\rangle\langle43\rangle}{\langle3\ell_2\rangle^2\langle\ell_24\rangle^2}+\ell_1 \leftrightarrow \ell_2\right) \\
&= i\mathcal{A}_{\text{tree}}\frac{-u\,\text{Tr}(\ell_2 p_3 p_4)}{(\ell_2-p_3)^2(\ell_2+p_4)^2} + \ell_1 \leftrightarrow \ell_2\,.
\end{aligned}
$$

$$\text{(B.3.11)}$$

B.4   Properties of the box integral
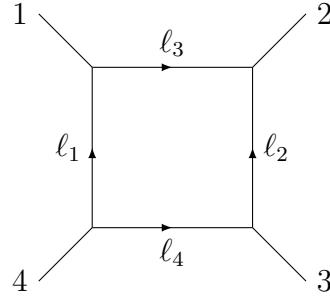
Figure B.1   Four-point one-loop box.

The box integral function (4.2.7) was constructed and used in [67]. Beyond the dual conformal symmetry presented in Section 4.2.4 it has several interesting properties that have been exploited in the present work. This section presents and proves (some of) these properties.

B.4.1   Rotation by $90°$

The first property we wish to discuss is what could be called a $\pi/2$ rotation symmetry. Focusing on the numerator of the box integrand,

$$N \; = \; s\,\mathrm{Tr}(\ell_1 p_1 p_4) + \ell_1^2\,\mathrm{Tr}(p_1 p_2 p_4), \tag{B.4.1}$$

we can eliminate ℓ_1 in favor of ℓ_3 and arrange to have only the external legs p_2, p_3, p_1 appear in the numerator. Using momentum conservation, we can re-write N as

$$N = (-t - u)\,\mathrm{Tr}\left((\ell_3 + 1)p_1(-p_1 - p_2 - p_3)\right) + (\ell_3 + p_1)\,\mathrm{Tr}\left(p_1 p_2(-p_1 - p_2 - p_3)\right)$$
$$= -\left[t\,\mathrm{Tr}(\ell_3 p_2 p_1) + \ell_3^2\,\mathrm{Tr}(p_2 p_3 p_4)\right] + \mathcal{R}\,,$$
$$\tag{B.4.2}$$

where

$$\mathcal{R} = s\,\mathrm{Tr}(\ell_3 p_3 p_1) - u\,\mathrm{Tr}(\ell_3 p_2 p_1) - 2(\ell_3 \cdot p_1)\mathrm{Tr}(p_2 p_3 p_1)\,. \tag{B.4.3}$$

In three dimensions the loop momentum ℓ_3 can be expressed as a function of the external momenta p_1, p_2, p_3 as

$$\ell_3 = \alpha p_1 + \beta p_2 + \gamma p_3\,, \tag{B.4.4}$$

where α, β, γ are arbitrary coefficients. When this identity is used in the expression for $\mathcal{R}$, we find that $\mathcal{R}$ vanishes identically zero in three dimensions. Hence

$$s \operatorname{Tr}(\ell_1 p_1 p_4) + \ell_1^2 \operatorname{Tr}(p_1 p_2 p_4) = -t \left(\operatorname{Tr}(\ell_3 p_2 p_1) + \ell_3^2 \operatorname{Tr}(p_2 p_3 p_4) \right). \tag{B.4.5}$$

It is also interesting to write down explicitly the s- and t-cut of the one-loop box. Starting from the expression of the box integral

$$I(1,2,3,4) := \int \frac{d^D \ell}{i\pi^{D/2}} \frac{N}{\ell^2 (\ell - p_1)^2 (\ell - p_1 - p_2)^2 (\ell + p_4)^2}, \tag{B.4.6}$$

with N given in (B.4.1), we first consider the s-cut of this function. This gives

$$I(1,2,3,4)|_{s\text{-cut}} = \frac{s \operatorname{Tr}(\ell_1 p_1 p_4)}{\ell_3^2 \ell_4^2}, \tag{B.4.7}$$

which upon using $\ell_3 = \ell_1 - p_1$ and $\ell_4 = -(\ell_1 + p_4)$ becomes

$$I(1,2,3,4)|_{s\text{-cut}} = \frac{s \langle 41 \rangle}{\langle 4\ell_1 \rangle \langle \ell_1 1 \rangle}. \tag{B.4.8}$$

Similarly the t-channel expression of the full integrand is

$$I(1,2,3,4) = \frac{t \operatorname{Tr}(\ell_3 p_2 p_1) + \ell_3^2 \operatorname{Tr}(p_2 p_3 p_1)}{\ell_1^2 \ell_2^2 \ell_3^2 \ell_4^2}. \tag{B.4.9}$$

The t-cut of $I(1,2,3,4)$ is immediately written using the three-dimensional identity (B.4.5),

$$I(1,2,3,4)|_{t\text{-cut}} = -\frac{t \operatorname{Tr}(\ell_3 p_2 p_1)}{\ell_1^2 \ell_2^2} \tag{B.4.10}$$

$$= \frac{t \langle 12 \rangle}{\langle 1\ell_3 \rangle \langle \ell_3 2 \rangle}.$$

Finally, if we re-introduce the tree-level amplitude prefactor $\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) = 1/(\langle 12 \rangle \langle 23 \rangle)$, we can write down the two cuts of the one-loop amplitude,

$$\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) \times I(1,2,3,4)|_{s\text{-cut}} = -\frac{\langle 34 \rangle}{\langle 4\ell_1 \rangle \langle \ell_1 1 \rangle}, \tag{B.4.11}$$

$$\mathcal{A}^{(0)}(\bar{1}, 2, \bar{3}, 4) \times I(1,2,3,4)|_{t\text{-cut}} = \frac{\langle 23 \rangle}{\langle 1\ell_3 \rangle \langle \ell_3 2 \rangle}. \tag{B.4.12}$$

B.4.2   An identity for the s-channel cuts of $I(1,2,3,4)$ and $I(1,2,4,3)$

We consider the s-channel cut of $I(1,2,3,4)$ and will symmetrize it in the cut loop momenta ℓ_1 and ℓ_2, where $\ell_1 + \ell_2 = p_1 + p_2$. The result we wish to show is that the symmetrized three-dimensional cuts of $I(1,2,3,4)$ and $I(1,2,4,3)$ are in fact identical:

$$I(1,2,3,4)|_{s\text{-cut}} \; + \; \ell_1 \leftrightarrow \ell_2 \; = \; I(1,2,4,3)|_{s\text{-cut}} \; + \; \ell_1 \leftrightarrow \ell_2 \;. \qquad (B.4.13)$$

In order to do so, we use (B.4.8) to write

$$I(1,2,3,4)|_{s\text{-cut}} \; = \; s\frac{\langle 41 \rangle}{\langle 4|\ell_1|1 \rangle} \qquad (B.4.14)$$

Again using (B.4.8) this time for the s-cut of $I(1,2,4,3)$ one can write,

$$I(1,2,4,3)|_{s\text{-cut}} \; = \; \frac{s\langle 31 \rangle}{\langle 3\ell_1 \rangle \langle \ell_1 1 \rangle} \;. \qquad (B.4.15)$$

Next we compare (B.4.14) to (B.4.15):

$$
\begin{aligned}
I(1,2,3,4)|_{s\text{-cut}} - I(1,2,4,3)|_{s\text{-cut}} &= s\left(\frac{\langle 41 \rangle}{\langle 4\ell_1 \rangle \langle \ell_1 1 \rangle} - \frac{\langle 31 \rangle}{\langle 3\ell_1 \rangle \langle \ell_1 1 \rangle} \right) \\
&= \langle \ell_1\ell_2 \rangle \left(\frac{\langle 41 \rangle \langle \ell_1\ell_2 \rangle}{\langle 4\ell_1 \rangle \langle \ell_1 1 \rangle} - \frac{\langle 31 \rangle \langle \ell_1\ell_2 \rangle}{\langle 3\ell_1 \rangle \langle \ell_1 1 \rangle} \right),
\end{aligned} \qquad (B.4.16)
$$

where we used the fact that $s = \langle \ell_1\ell_2 \rangle^2$. Now, use the Schouten identity (2.1.32) on the numerators to obtain:

$$
\begin{aligned}
I(1,2,3,4)|_{s\text{-cut}} - I(1,2,4,3)|_{s\text{-cut}} &= \langle \ell_1\ell_2 \rangle \left(\frac{\langle 4\ell_1 \rangle \langle 1\ell_2 \rangle - \langle 4\ell_2 \rangle \langle 1\ell_1 \rangle}{\langle 4\ell_1 \rangle \langle \ell_1 1 \rangle} \right. \\
&\qquad\qquad \left. - \frac{\langle 3\ell_1 \rangle \langle 1\ell_2 \rangle - \langle 3\ell_2 \rangle \langle 1\ell_1 \rangle}{\langle 3\ell_1 \rangle \langle \ell_1 1 \rangle} \right) \\
&= \langle \ell_1\ell_2 \rangle \left(\frac{\langle 4\ell_2 \rangle}{\langle \ell_1 4 \rangle} - \frac{\langle 3\ell_2 \rangle}{\langle \ell_1 3 \rangle} \right).
\end{aligned} \qquad (B.4.17)
$$

Next, we add the terms symmetrized in $\ell_1 \leftrightarrow \ell_2$:

$$
\begin{aligned}
&I(1,2,3,4)|_{s\text{-cut}} - I(1,2,4,3)|_{s\text{-cut}} + \ell_1 \leftrightarrow \ell_2 \\
&\qquad\qquad = \langle \ell_1\ell_2 \rangle \left(\frac{\langle 4\ell_2 \rangle}{\langle \ell_1 4 \rangle} - \frac{\langle 4\ell_1 \rangle}{\langle \ell_2 4 \rangle} - \frac{\langle 3\ell_2 \rangle}{\langle \ell_1 3 \rangle} + \frac{\langle 3\ell_1 \rangle}{\langle \ell_2 3 \rangle} \right), \qquad (B.4.18)
\end{aligned}
$$

and consider the pairs

$$\frac{\langle\ell_1\ell_2\rangle\langle4\ell_2\rangle}{\langle\ell_14\rangle} + \frac{\langle\ell_1\ell_2\rangle\langle3\ell_1\rangle}{\langle\ell_23\rangle}, \quad\text{and}\quad -\frac{\langle\ell_1\ell_2\rangle\langle4\ell_1\rangle}{\langle\ell_24\rangle} - \frac{\langle\ell_1\ell_2\rangle\langle3\ell_2\rangle}{\langle\ell_13\rangle}. \tag{B.4.19}$$

Starting with the first one and applying momentum conservation $\ell_1 + \ell_2 = p_3 + p_4$,

$$\begin{aligned}
\frac{\langle4\ell_2\rangle\langle\ell_2\ell_1\rangle}{\langle4\ell_1\rangle} + \frac{\langle3\ell_1\rangle\langle\ell_1\ell_2\rangle}{\langle\ell_23\rangle} &= \frac{\langle43\rangle\langle3\ell_1\rangle}{\langle4\ell_1\rangle} + \frac{\langle34\rangle\langle4\ell_2\rangle}{\langle\ell_23\rangle} \\
&= \frac{\langle43\rangle\langle\ell_23\rangle\langle3\ell_1\rangle + \langle34\rangle\langle4\ell_1\rangle\langle4\ell_2\rangle}{\langle4\ell_1\rangle\langle\ell_23\rangle} \\
&= \frac{\langle43\rangle}{\langle4\ell_1\rangle\langle\ell_23\rangle}(\langle\ell_2|3|\ell_1\rangle + \langle\ell_2|4|\ell_1\rangle) \\
&= \frac{\langle43\rangle}{\langle4\ell_1\rangle\langle\ell_23\rangle}(\langle\ell_2|3+4|\ell_1\rangle) \\
&= \frac{\langle43\rangle}{\langle4\ell_1\rangle\langle\ell_23\rangle}(\langle\ell_2|\ell_1 + \ell_2|\ell_1\rangle) \\
&= 0,
\end{aligned} \tag{B.4.20}$$

and similarly for the pair

$$-\frac{\langle\ell_1\ell_2\rangle\langle4\ell_1\rangle}{\langle\ell_24\rangle} - \frac{\langle\ell_1\ell_2\rangle\langle3\ell_2\rangle}{\langle\ell_13\rangle} = 0. \tag{B.4.21}$$

This concludes the proof of (B.4.13).

B.5   Details on the evaluation of integrals

The integral in the result (4.4.26) can be reduced to a set of four scalar, single-scale, master integrals using integration by parts identities and the `FIRE` package [101] for `Mathematica`. In this appendix we present the details of this reduction as well as the values of these master integrals. For an outline of the general ideas and methodology of expressing Feynman multiloop integrals in terms of a basis of master integrals see [102].

It is possible to classify Feynman integrals according to their topology and this is determined by the number of propagators. For the maximal number of propagators each one is raised to a power a_i. When certain propagators are absent, which effectively means that the corresponding indices a_i are set to zero, sub-topologies are

being considered. The `FIRE` package applies a set of integration by parts (IBP) identities for each topology. These identities are a consequence of Poincaré invariance and relate integrals with different values of a_i. The integration by parts relations stem from the derivatives of the integrands [103]:

$$\int \dots \int d^D \ell_1 d^D \ell_2 \dots \frac{\partial}{\partial \ell_i} \left(p_j \frac{1}{E_1^{a_1} \dots E_n^{a_n}} \right) = 0 \,, \tag{B.5.1}$$

where p_j can be either external or loop momentum. The integrand of the Feynman integral $F(a_1, \dots, a_n)$ is a scalar integrand, a result of a tensor reduction. A numerator that is an irreducible polynomial is expressed as a combination of propagators E_i raised to a negative power. The derivatives of any scalar product of the form $k_i \cdot k_j$ or $k_i \cdot q_j$ are expressed linearly in terms of the propagators. The IBP relations assume the following form:

$$\sum \alpha_i F(a_1 + b_{i,1}, \dots, a_n + b_{i,n}) = 0 \,, \tag{B.5.2}$$

where $b_{i,j}$ are fixed integers and α_i's are polynomials in a_j. By substituting possible combinations of $(a_1, a_2, \dots, a_n)$ one obtains a potentially large number of relations. The `FIRE` package systematizes this reduction procedure.

As a demonstration of the procedure we will apply it on the integral with the topology of the one-loop triangle as in Figure A.3. Now the integral will be of the general form

$$I\{a_1, a_2, a_3\} = \int d^D \ell \frac{1}{E_1^{a_1} E_2^{a_2} E_3^{a_3}} \,. \tag{B.5.3}$$

Similarly to Appendix A.4 external momenta will be denoted as p_1 and p_2 and now will be taken to be on-shell, $p_i^2 = 0$, with zero masses. The choice for the propagators is identical to that of Appendix A.4

$$E_1 = \ell^2 \tag{B.5.4}$$

$$E_2 = (\ell + p_1)^2 \tag{B.5.5}$$

$$E_3 = (\ell + p_1 + p_2)^2 \,. \tag{B.5.6}$$

The first identity that can be written down is simply

$$\int d^D \ell \frac{\partial}{\partial \ell_\mu} \frac{p_{1;\mu}}{E_1^{a_1} E_2^{a_2} E_3^{a_3}} = 0 \,. \tag{B.5.7}$$

The derivatives of the propagators can be expressed in terms of the propagators themselves

$$p_{1;\mu}\frac{\partial}{\partial \ell_\mu}E_1 = 2p_1 \cdot \ell = E_2 - E_1 \tag{B.5.8}$$

$$p_{1;\mu}\frac{\partial}{\partial \ell_\mu}E_2 = 2p_1 \cdot \ell + 2p_1^2 = E_2 - E_1 \tag{B.5.9}$$

$$p_{1;\mu}\frac{\partial}{\partial \ell_\mu}E_3 = 2p_1 \cdot \ell + 2p_1^2 + 2p_1 \cdot p_2 = E_2 - E_1 + s\,, \tag{B.5.10}$$

where $s = 2p_1 \cdot p_2$ as usual. Now the first identity (B.5.7) reads

$$\int d^d\ell \frac{1}{E_1^{a_1} E_2^{a_2} E_3^{a_3}}\left(a_1 - a_2 - a_1\frac{E_2}{E_1} + a_2\frac{E_1}{E_2} + a_3\frac{E_1}{E_3} - a_3\frac{E_2}{E_3} - a_3\frac{s}{E_3}\right) = 0\,. \tag{B.5.11}$$

It is common to introduce operators $\mathbf{i}^+$ and $\mathbf{i}^-$ such that

$$\mathbf{1}^+\mathbf{3}^- I\{a_1, a_2, a_3\} = I\{a_1 + 1, a_2, a_3 - 1\}\,. \tag{B.5.12}$$

Then, the first IBP identity can be written as

$$\left(a_1 - a_2 - a_1\mathbf{1}^+\mathbf{2}^- + a_2\mathbf{2}^+\mathbf{1}^- + a_3\mathbf{3}^+\mathbf{1}^- - a_3\mathbf{3}^+\mathbf{2}^- - a_3 s\mathbf{3}^+\right) I\{a_1, a_2, a_3\} = 0\,. \tag{B.5.13}$$

The remaining identities are obtained by differentiating while taking $p_j = p_{2;\mu}$ and $p_j = \ell_\mu$ respectively:

$$\left[a_2 - a_3 + a_1(\mathbf{1}^+\mathbf{2}^- - \mathbf{1}^+\mathbf{3}^-) - a_2\mathbf{2}^+\mathbf{3}^- + a_3\mathbf{3}^+\mathbf{2}^- + a_1 s\mathbf{1}^+\right] I\{a_1, a_2, a_3\} = 0 \tag{B.5.14}$$

$$\left[d - 2a_1 - a_2 - a_3 - a_2\mathbf{2}^+\mathbf{1}^- - a_3\mathbf{3}^+\mathbf{1}^- + a_3 s\mathbf{3}^+\right] I\{a_1, a_2, a_3\} = 0\,. \tag{B.5.15}$$

It is now possible to use the identities to reduce integrals of a certain topology to a linear combination of simpler integrals. For example, taking $a_1 = a_2 = a_3 = 1$ the integral $I\{2, 1, 1\}$, can be re-written in the following way using the second IBP identity (B.5.14)

$$sI\{2, 1, 1\} = I\{2, 1, 0\} + I\{1, 2, 0\} - I\{2, 0, 1\} - I\{1, 0, 2\}\,. \tag{B.5.16}$$

It is now useful to employ certain simple identities that are true for this topology. For example, by letting $\ell \to -\ell - p_1$ it is trivial to show that $E_1^{a_1} E_2^{a_2} \to E_1^{a_2} E_2^{a_1}$ proving that $I\{a_1, a_2, 0\} = I\{a_2, a_1, 0\}$. Similarly one can show that $I\{a_1, 0, a_3\} = I\{a_3, 0, a_1\}$. In this way, the expression (B.5.16) becomes

$$\frac{s}{2}I\{2,1,1\} = I\{2,1,0\} - I\{2,0,1\}. \tag{B.5.17}$$

For $\{a_1, a_2, a_3\} = \{2, 1, 0\}$ in (B.5.13) and $\{a_1, a_2, a_3\} = \{1, 0, 1\}$ in (B.5.15) we get:

$$I\{2,1,0\} = I\{3,0,0\} \tag{B.5.18}$$

$$-sI\{2,0,1\} = (D-3)I\{1,0,1\} - I\{2,0,0\}. \tag{B.5.19}$$

Since the theory under consideration is massless, the tadpole integrals $I\{3,0,0\}$ and $I\{2,0,0\}$ are zero and the original integral is immediately reduced to

$$I\{2,1,1\} = \frac{2(D-3)}{s^2}I\{1,0,1\}. \tag{B.5.20}$$

The integral $I\{1,0,1\}$ is irreducible and is called a Master Integral. For a massive theory, master integrals would include tadpole integrals. The idea is that every integral of this topology can be reduced to the master integrals that are simpler than the original by a similar procedure.

The IBP method described above was used to simplify the integrals that appeared in Section 4.5. It is this process that is systematized by the `FIRE` package. However, before the package can be used, it is necessary to recast the integrals in a different form. Consider first the $\mathbf{LT}(q^2)$ integral (4.5.4). In order to express the integrand entirely in terms of propagators (inverse or not), the set of propagators E_i in Table B.1 are chosen:

E_1	E_2	E_3	E_4	E_5	E_6	E_7
ℓ_1^2	$(p_1 + p_2 - \ell_1)^2$	ℓ_3^2	$(p_1 + p_2 - \ell_3)^2$	$(\ell_1 - \ell_3)^2$	$(\ell_3 - p_2)^2$	$(\ell_1 - p_1)^2$

Table B.1 Propagator expressions that are used to express the integrals of maximal transcendentality.

Using the operators defined in (B.5.12) the integral $\mathbf{LT}(q^2)$ is written as

$$
\begin{aligned}
\mathbf{LT}(q^2) = \Bigg(&-\frac{q^2}{2}\mathbf{1}^-\mathbf{3}^-\mathbf{6}^+ + \frac{q^2}{2}\mathbf{1}^- + \frac{q^4}{2}\mathbf{1}^-\mathbf{6}^+ - \frac{q^2}{2}\mathbf{2}^-\mathbf{4}^-\mathbf{6}^+ + \frac{q^2}{2}\mathbf{2}^- + \frac{q^4}{2}\mathbf{2}^-\mathbf{6}^+ \\
&+ \frac{q^2}{2}\mathbf{3}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^4}{2}\mathbf{3}^-\mathbf{6}^+ + \frac{q^2}{2}\mathbf{3}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^4}{2}\mathbf{4}^-\mathbf{6}^- - \frac{q^4}{2}\mathbf{5}^-\mathbf{6}^+ \\
&- \frac{q^4}{2} - \frac{q^4}{2}\mathbf{6}^+\mathbf{7}^- - \frac{q^6}{2}\mathbf{6}^+ \Bigg) G_P(1,1,1,1,1,0,0)
\end{aligned}
$$

$$(\text{B.5.21})$$

where $G_P(1,1,1,1,1,0,0)$ is the integral with the propagators of B.1. For the $\mathbf{XT}(q^2)$ integral (4.4.16) the procedure is similar. Now $E_4 = (p_1 - \ell_1 + \ell_3)^2$ and the rest of the propagators are the same as for $\mathbf{LT}(q^2)$ shown in Table B.1. The integral $\mathbf{XT}(q^2)$ assumes the form

$$
\begin{aligned}
\mathbf{XT}(q^2) = \Bigg(&\frac{q^2}{2}\mathbf{1}^-\mathbf{2}^-\mathbf{6}^+ + \frac{q^2}{2}\mathbf{1}^-\mathbf{3}^-\mathbf{6}^+ - \frac{q^2}{2}\mathbf{1}^- - \frac{q^2}{2}\mathbf{1}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^2}{2}\mathbf{2}^-\mathbf{4}^-\mathbf{6}^+ \\
&- \frac{q^2}{2}\mathbf{2}^-\mathbf{5}^-\mathbf{6}^+ - \frac{q^2}{2}\mathbf{2}^-\mathbf{6}^+\mathbf{7}^- - \frac{q^2}{2}\mathbf{3}^-\mathbf{6}^+\mathbf{7}^- - \frac{q^2}{2}\mathbf{4}^-\mathbf{6}^+\mathbf{7}^- - \frac{q^4}{2}\mathbf{3}^-\mathbf{6}^+ \\
&- \frac{q^4}{2}\mathbf{4}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^2}{2}\mathbf{5}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^2}{2}\mathbf{7}^- + \frac{q^2}{2}\mathbf{6}^+\mathbf{7}^-\mathbf{7}^- \\
&+ \frac{q^4}{2}\mathbf{6}^+\mathbf{7}^- \Bigg) G_{NP}(1,1,1,1,1,0,0)\,.
\end{aligned}
$$

$$(\text{B.5.22})$$

The $\mathbf{CT}(q^2)$ integral (4.5.9) can be expressed using the same propagators as $\mathbf{LT}(q^2)$ (Table B.1) and the same planar integral G_P. It can be written as

$$
\begin{aligned}
\mathbf{CT}(q^2) = \Bigg(&\frac{1}{2}\mathbf{1}^-\mathbf{2}^-\mathbf{3}^-\mathbf{6}^+ - \frac{1}{2}\mathbf{1}^-\mathbf{4}^- + \frac{q^2}{2}\mathbf{1}^-\mathbf{4}^-\mathbf{6}^+ - \frac{q^2}{2}\mathbf{2}^-\mathbf{4}^-\mathbf{4}^-\mathbf{6}^+ + \frac{1}{2}\mathbf{2}^-\mathbf{4}^- \\
&+ \frac{q^2}{2}\mathbf{2}^-\mathbf{4}^-\mathbf{6}^+ - \frac{1}{2}\mathbf{3}^-\mathbf{4}^-\mathbf{6}^+\mathbf{7}^- + \frac{q^2}{2}\mathbf{2}^-\mathbf{3}^-\mathbf{6}^+ + \frac{1}{2}\mathbf{4}^-\mathbf{4}^-\mathbf{6}^+\mathbf{7}^- - \frac{q^2}{2}\mathbf{4}^-\mathbf{4}^-\mathbf{6}^+ \\
&- \frac{q^2}{2}\mathbf{4}^-\mathbf{5}^-\mathbf{6}^+ - \frac{q^2}{2}\mathbf{4}^- - \frac{q^2}{2}\mathbf{4}^-\mathbf{6}^+\mathbf{7}^- - \frac{q^2}{2}\mathbf{4}^-\mathbf{6}^+ \Bigg) G_P(1,1,1,1,1,0,0)\,.
\end{aligned}
$$

$$(\text{B.5.23})$$

Finally, for the $\mathbf{FAN}(q^2)$ integral (4.5.11) we use a different set of propagators shown in Table B.2.

E_1	E_2	E_3	E_4	E_5	E_6	E_7
$(\ell_1 - p_1)^2$	ℓ_1^2	ℓ_3^2	$(\ell_3 - p_2)^2$	$(p_1 + p_2 - \ell_1 - \ell_3)^2$	$(\ell_3 + \ell_1)^2$	$(\ell_1 + p_2)^2$

Table B.2   Propagator expressions for the **FAN** topology.

The **FAN**(q^2) integral is then written as

$$
\mathbf{FAN}(q^2) = \Bigg(\frac{1}{2}\mathbf{1^-2^-} - \frac{1}{2}\mathbf{1^-3^-} + \frac{1}{2}\mathbf{1^-4^-} - \frac{1}{2}\mathbf{1^-7^-} - \frac{1}{2}\mathbf{2^-5^-}
$$
$$
+ \frac{1}{2}\mathbf{2^-6^-} - \frac{1}{2}\mathbf{2^-7^-} + \frac{1}{2}\mathbf{3^-7^-} - \frac{q^2}{2}\mathbf{3^-} - \frac{1}{2}\mathbf{4^-7^-}
$$
$$
+ \frac{1}{2}\mathbf{7^-} - \frac{1}{2}\mathbf{6^-7^-} + \frac{q^2}{2}\mathbf{6^-} + \frac{1}{2}\mathbf{7^-7^-} - \frac{q^2}{2}\mathbf{7^-} \Bigg) G_{\text{FAN}}(1,1,1,1,1,0,0) .
$$

$$(\text{B.5.24})$$

Using the above expressions of the integrals as input for `FIRE` , they are automatically reduced by a recursive use IBP identities to combinations of master integrals. The outcome of these manipulations is shown in Appendix B.5.2.

B.5.1   Two-loop master integrals in three dimensions

The master integrals that appear at two loops, in particular in the reduction of our result (4.4.26), are given in $D = 3 - 2\epsilon$ dimensions by the following expressions:

$$
\mathbf{SUNSET}(q^2) = \text{—}\!\!\bigcirc\!\!\text{—} = -\left(\frac{-q^2}{\mu^2} \right)^{-2\epsilon} \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)^3 \Gamma(2\epsilon)}{\Gamma\left(\frac{3}{2} - 3\epsilon\right)}
\tag{B.5.25}
$$

$$
\mathbf{TRI}(q^2) = \text{—}\!\!\bigcirc\!\!\text{—} = -(-q^2)^{-1} \left(\frac{-q^2}{\mu^2} \right)^{-2\epsilon} \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)^2 \Gamma(-2\epsilon) \Gamma\left(\frac{3}{2} + \epsilon\right) \Gamma(2 + 2\epsilon)}{\epsilon(1 + 2\epsilon)^2 \Gamma\left(\frac{1}{2} - 3\epsilon\right)}
\tag{B.5.26}
$$

$$
\mathbf{GLASS}(q^2) = \text{—}\!\!\bigcirc\!\!\bigcirc\!\!\text{—} = (-q^2)^{-1} \left(\frac{-q^2}{\mu^2} \right)^{-2\epsilon} \frac{\Gamma\left(\frac{1}{2} - \epsilon\right)^4 \Gamma\left(\frac{1}{2} + \epsilon\right)^2}{\Gamma(1 - 2\epsilon)^2}
\tag{B.5.27}
$$

$$
\mathbf{TrianX}(q^2) = \quad \text{—}\!\!\prec\!\!\text{—} = (-q^2)^{-3} \left(\frac{-q^2}{\mu^2} \right)^{-2\epsilon} e^{-2\gamma_E \epsilon} \left[\frac{4\pi}{\epsilon^2} + \frac{\pi(3 + 8\log 2)}{\epsilon} \right.
$$
$$
- \frac{2\pi}{3}\left(81 + 4\pi^2 + 6\log 2 \left(4\log 2 - 9\right) \right) + \frac{\pi}{6}\left(-\pi^2(7 + 40\log 2) \right.
$$
$$
\left. \left. + 8\big(69 + 6\log 2 + 2\log^2 2(8\log 2 - 27) - 113\zeta_3\big) \right)\epsilon + \mathcal{O}(\epsilon) \right],
$$

$$(\text{B.5.28})$$

where the conventions for the integration measure agree with those of [104] . The first

three of these integrals are planar and their expressions in all orders in ϵ can be easily

obtained by first computing their Mellin-Barnes representations most conveniently

using the `AMBRE` package [105] and then performing the Mellin-Barnes integrations

using the MB tools, in particular `MB.m` [104] and `barnesroutines.m` by David

Kosower. The expansion around $\epsilon = 0$ of the **TRI** and **GLASS** topologies has

uniform degree of transcendentality, while this is not the case for the **SUNSET** and

TrianX topologies.

B.5.2   Reduction to master integrals

Here we present the reductions of the integral (4.4.16) that appears in our result

(4.4.26) in terms of the master integrals (B.5.25)–(B.5.28) of the previous section:

$$\mathbf{XT}(q^2) = \frac{7(D-3)(3D-10)(3D-8)}{2(D-4)^2(2D-7)}\mathbf{SUNSET}(q^2) \tag{B.5.29}$$
$$+ (-q^2)\frac{5(D-3)(3D-10)}{2(D-4)(2D-7)}\mathbf{TRI}(q^2) + (-q^2)^3\frac{D-4}{4(2D-7)}\mathbf{TrianX}(q^2)\,.$$

Note that the GLASS topology does not appear in $\mathbf{XT}(q^2)$. Two other integrals we

have considered are:

$$\mathbf{LT}(q^2) = \frac{8-3D}{D-3}\,\mathbf{SUNSET}(q^2) + q^2\left(\mathbf{GLASS}(q^2) - \mathbf{TRI}(q^2)\right)\,, \tag{B.5.30}$$
$$\mathbf{CT}(q^2) = \mathbf{FAN}(q^2) = \left(\frac{1}{4\epsilon} - \frac{3}{2}\right)\mathbf{SUNSET}(q^2)\,. \tag{B.5.31}$$

B.5.3   Mellin-Barnes Integrals

This section provides further details on the Mellin-Barnes representation of integrals

as well as the method used to calculate them. The description of the method relies

on [104, 106, 107].

The key step in this procedure is to recast a Feynman integral into a Mellin-

Barnes representation. This relies on the following identity:

$$\frac{1}{(A+B)^\nu} = \frac{1}{2\pi i\Gamma(\nu)}\int_{-i\infty}^{i\infty} dz\frac{A^z}{B^{\nu+z}}\Gamma(-z)\Gamma(\nu+z)\,. \tag{B.5.32}$$

The contour needs to be chosen in such a way that the poles of $\Gamma(\nu+z)$ are separated from the poles of $\Gamma(-z)$ and should ensure that the arguments of the Γ-functions are all positive [106]. This is required in order to apply Barnes' lemmas that are valid only under this condition [104]. Through Barnes' lemmas, it is possible to simplify the integrals to such an extent that would allow for an analytical result. These are

Lemma B.5.1 (Barnes' first lemma).

$$\int_{-i\infty}^{i\infty} dz\, \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$
(B.5.33)

Lemma B.5.2 (Barnes' second lemma).

$$\int_{-i\infty}^{i\infty} dz\, \Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z) =$$
$$\frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)} \quad \text{(B.5.34)}$$

In order to apply the identity (B.5.32), the integral is usually rewritten using Feynman parameters. In the Mellin-Barnes representation the integration over Feynman parameters is performed using

$$\int_0^1 \prod_{i=1}^N dx_i x_i^{q_i-1} \delta\left(1 - \sum_j x_j\right) = \frac{\Gamma(q_1)\dots\Gamma(q_N)}{\Gamma(q_1+q_2+\dots+q_N)}$$
(B.5.35)

This is best illustrated by an example. The steps of a Mellin-Barnes transformation will be presented explicitly on the integral

$$\mathrm{LT}_{\mu\nu} = \int d^D\ell_1 \int d^D\ell_3 \frac{(\ell_1)_\mu(\ell_3)_\nu}{(\ell_1-\ell_3)^2(\ell_1-q)^2\ell_1^2(\ell_3-p_1)^2(\ell_3-q)^2\ell_3^2}\,.$$
(B.5.36)

This integral could emerge from a form factor with the structure:

$$F_{\bar\phi_4\phi^A} = q^2(2p_1^{[\mu}p_2^{\nu]} + q^2 g^{\mu\nu})\mathrm{LT}_{\mu\nu}(q^2,\epsilon)$$
(B.5.37)

The first step in order to evaluate this integral is to separate the ℓ_1 and ℓ_3 integrations:

$$\mathrm{LT}_{\mu\nu} = \int d^D\ell_3 \frac{(\ell_3)_\nu}{(\ell_3-p_1)^2(\ell_3-q)^2\ell_3^2} \int d^D\ell_1 \frac{(\ell_1)_\mu}{(\ell_1-\ell_3)^2(\ell_1-q)^2\ell_1^2}$$
(B.5.38)

Following the examples on page 37 of [108] it is possible to conveniently express the ℓ_1 integral in terms of $D + 2$-dimensional integrals:

$$\int d^D \ell_1 \frac{(\ell_1)_\mu}{(\ell_1 - \ell_3)^2(\ell_1 - q)^2\ell_1^2} = (\ell_3)_\mu \int d^{D+2}\ell_1 \frac{1}{[(\ell_1 - \ell_3)^2]^2 (\ell_1 - q)^2\ell_1^2} \qquad \text{(B.5.39)}$$
$$+ q_\mu \int d^{D+2}\ell_1 \frac{1}{(\ell_1 - \ell_3)^2 [(\ell_1 - q)^2]^2 \ell_1^2}$$

The starting point will be the integrand of the five-dimensional integral which is modified using Feynman parameters

$$\frac{1}{(\ell_1 - \ell_3)^4 (\ell_1 - q)^2 \ell_1^2} = \int_0^1 dx\, dy\, dz\, \frac{(4-1)!\, x\, \delta\, (x + y + z - 1)}{D^4} \qquad \text{(B.5.40)}$$

with

$$D = x\, (\ell_1 - \ell_3)^2 + y\, (\ell_1 - q)^2 + z\ell_1^2 = \ell_1^2 - 2\ell_1(x\ell_3 + yq) + x\ell_3^2 + yq^2 \,. \qquad \text{(B.5.41)}$$

We perform the shift

$$\ell_1' = \ell_1 - (x\ell_3 + yq) \qquad \text{(B.5.42)}$$

and then D becomes

$$D = \ell_1^2 - 2xy\, \ell_3 \cdot q - (x^2 - x)\, \ell_3^2 - (y^2 - y)\, q^2 \,. \qquad \text{(B.5.43)}$$

Now the $D + 2$ integral is explicitly

$$I_{\ell_1} = 3! \int d^{D+2}\ell_1 \int_0^1 dx\, dy\, \frac{x}{(\ell_1^2 - 2xy\, \ell_3 \cdot q - (x^2 - x)\, \ell_3^2 - (y^2 - y)\, q^2)^4} \,. \qquad \text{(B.5.44)}$$

The next step is to switch to a Mellin-Barnes representation which, converts the integral (B.5.44) to*.

$$I_{\ell_1} = \frac{1}{(2\pi i)^3} \int d^{D+2}\ell_1 \int_0^1 dx\, dy \int_{-i\infty}^{+i\infty} dz_1 dz_2 dz_3 \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_{123} + 4)$$
$$\times \frac{(-1)^{z_{123}} 2^{z_1} (\ell_3 \cdot q)^{z_1} x^{z_1+1} (x^2 - x)^{z_2} y^{z_1} (y^2 - y)^{z_3} \ell_3^{2z_2} q^{2z_3}}{(\ell_1^2)^{4+z_{123}}} \,. \qquad \text{(B.5.45)}$$

---

*With $z_{123} = z_1 + z_2 + z_3$.

Explicitly the Mellin-Barnes steps are:

$$
\tilde{I}_{\ell_1} = 3! \int_0^1 dx\, dy \frac{x}{\left(\ell_1^2 - 2xy\,\ell_3 \cdot q - (x^2 - x)\,\ell_3^2 - (y^2 - y)\,q^2\right)^4}
$$
$$
= \frac{1}{2\pi i} \frac{3!}{\Gamma(4)} \int_0^1 dx\, dy \int_{-i\infty}^{+i\infty} dz_1 \Gamma(4 + z_1)\Gamma(-z_1) \frac{x\,(-2xy\,\ell_3 \cdot q)^{z_1}}{\left(\ell_1^2 - (x^2 - x)\,\ell_3^2 - (y^2 - y)\,q^2\right)^{4+z_1}}
$$
$$\tag{B.5.46}$$

$$
= \frac{1}{(2\pi i)^2} \int_0^1 dx\, dy \int_{-i\infty}^{+i\infty} dz_1 dz_2 \frac{\Gamma(4 + z_1)}{\Gamma(4 + z_1)}\Gamma(-z_1)\Gamma(4 + z_{12})\Gamma(-z_2)
$$
$$
\times \frac{(-2xy\,\ell_3 \cdot q)^{z_1}\, x\left[-(x^2 - x)\,\ell_3^2\right]^{z_2}}{\left(\ell_1^2 - (y^2 - y)\,q^2\right)^{4+z_{12}}} \tag{B.5.47}
$$

$$
= \frac{1}{(2\pi i)^3} \int_0^1 dx\, dy \int_{-i\infty}^{+i\infty} dz_1 dz_2 dz_3 \frac{\Gamma(4 + z_1 + z_2)}{\Gamma(4 + z_1 + z_2)}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(4 + z_{123})
$$
$$
\times \frac{(-2xy\,\ell_3 \cdot q)^{z_1}\, x\left[-(x^2 - x)\,\ell_3^2\right]^{z_2}\left[-(y^2 - y)\,q^2\right]^{z_3}}{\left(\ell_1^2\right)^{4+z_{123}}} \tag{B.5.48}
$$

Now the integration over Feynman parameters gives:

$$
\int_0^1 y^{z_1}\left(y^2 - y\right)^{z_3} dy = \frac{(-1)^{z_3}\Gamma\left(1 + z_3\right)\Gamma\left(2 + z_1 + z_3\right)}{\Gamma\left(3 + z_1 + 2z_3\right)} = \frac{(-1)^{z_3}}{B\left(1 + z_3, 2 + z_1 + z_3\right)}
$$
$$\tag{B.5.49}$$

while for x one gets

$$
\int_0^1 x^{z_1+1}\left(x^2 - x\right)^{z_2} dx = \frac{(-1)^{z_2}\Gamma\left(1 + z_2\right)\Gamma\left(2 + z_1 + z_2\right)}{\Gamma\left(3 + z_1 + 2z_2\right)} = \frac{(-1)^{z_2}}{B\left(1 + z_2, 2 + z_1 + z_2\right)}
$$
$$\tag{B.5.50}$$

which can be substituted back in eq. (B.5.45). The B-function is related to the Γ-functions as

$$
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}\,. \tag{B.5.51}
$$

The procedure presented above is automated through the use of the AMBRE package. The next step would be to find the integration contours and perform the analytic continuation which involves several interesting subtleties (see for example [106]), that

will not be discussed here. This procedure is also automated through the `MB.m` package [104]. The package finds the contours of integration. It is then possible to merge the resulting integrals by linearly adding the integrals with the same contour, suppress the integrals that vanish at a user-defined order of expansion or simply expand them. Once the contours are determined the user may perform the analytic continuation through a single command. Simplifying the integrals using Barnes' lemmas is most conveniently done using `barnesroutines.m` by David Kosower. This may split the answer in a part that is analytic and a part that requires numerical integration. In cases when numerical integration is necessary it is still possible to reconstruct the answer in terms of logarithms, polylogarithms etc. This can be achieved using the `PSLQ` algorithm (available as a `Mathematica` package) which determines integer relations between real numbers[†].

As an application of this procedure, the maximally transcendental integrals in Section 4.5 can be computed directly. This bypasses the expansion in terms of master integrals and directly verifies that the structure of the numerators ensures maximal and uniform transcendentality. When the numerators are expanded there are several resulting integrals that need to be evaluated. Some of them produce results that include terms that spoil the transcendentality requirements. However, these terms always cancel out to produce the results presented in Section 4.5. Although this procedure is more direct, it is uneconomical as the benefit of reusable master integrals for each topology is lost and therefore requires the evaluation of more integrals. It is however useful as a check and was performed for the **LT** (4.5.4) and **CT** (4.5.9) integrals.

The numerator of the **LT** integral (4.5.4) produces a total of seven integrals that need to be computed. Recall that $\frac{1}{2}\text{Tr}(p_1, \ell_3, p_2, \ell_1) = (p_1 \cdot \ell_3)(p_2 \cdot \ell_1) - (p_1 \cdot p_2)(\ell_3 \cdot \ell_1) + (p_1 \cdot \ell_1)(\ell_3 \cdot p_2)$. The second term of the numerator (4.5.4) produces another four integrals when the on-shell condition is applied to the external momenta. These can be computed directly using the procedure outlined above. The result is (overall

---

[†]Details can be found here `http://mathworld.wolfram.com/PSLQAlgorithm.html`

normalization factors are suppressed):

$$
\left(-\pi + \frac{\pi}{2\epsilon} - 2\pi \log 2\right) + \left(\pi + \frac{\pi}{2\epsilon}\right) + \left(\frac{3\pi}{2} + \frac{\pi}{4\epsilon} - \pi \log 2\right)
$$
$$
+ \left(-7\pi + \frac{\pi}{4\epsilon^2} + \frac{5\pi^3}{8} - 2\pi \log^2 2\right) + \left(\frac{\pi}{4\epsilon} + \pi \left(\frac{1}{2} + \log 2\right)\right)
$$
$$
+ \pi \left(2 + \frac{1}{\epsilon}\right) + \left(2\pi - \frac{\pi}{\epsilon}\right) \rightarrow \left(-\frac{\pi}{4\epsilon^2} - \frac{\pi \log 2}{\epsilon} + 2\pi \log^2 2 - \frac{5\pi^3}{8} + \mathcal{O}(\epsilon)\right),
$$
$$
\tag{B.5.52}
$$

where each term in brackets is the result of an individual integral. It is noteworthy that any terms of lower transcendentality that appear in some integrals cancel out completely. This approach clearly demonstrates the cancellations occurring due to each individual term of the numerator and further justifies the particular structure that it has.

It is however possible to manipulate the integrals in a collective manner. For the **CT** integral, the three terms of the numerator are now treated collectively by the package. When the integrals with the same contour are merged and analytically continued it is possible to simplify the result further through Barnes' lemmas. This results in a purely analytical part A, and a part that needs to be numerically integrated, N. The analytical part is

$$
A = \frac{1}{24} \left(-\frac{6}{\epsilon^2} + 5\pi^2 + 12 \left(-8 + 4\log^2 2 + 4\gamma_E \log 2\right)\right) \tag{B.5.53}
$$

while, with the help of `PSLQ`, the numerical part assumes the form

$$
N = -\frac{1}{12}\pi \left(-48 - \pi^2 + 24\gamma_E \log 2 + 24\log^2 2\right). \tag{B.5.54}
$$

When the analytical and numerical part are added together the result is the same as in Section 4.5,

$$
\mathbf{CT} \sim A + N = -\frac{\pi}{4\epsilon^2} + \frac{7\pi^3}{24} + \mathcal{O}(\epsilon). \tag{B.5.55}
$$

This concludes the verification of the **LT** (4.5.4) and **CT** (4.5.9) integrals through a different method of evaluation.

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