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Zero Mass as a Borel Structure

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Abstract

The Lorentz group $\text{Lor}_{1,3} = \text{SO}_0(1,3)$ has two point fixgroups, namely $\text{SO}(3)$ for time-like translations and $\text{SO}_0(1,1) \times \mathbb{R}^2$ for light-like translations. However, for light-like translations, it is reasonable to consider a line fixgroup that leads to the Borel structure of the Lorentz group and provides appropriate helicities for massless particles. Therefore, whether a particle is massless or massive is not so much a physical question but rather a question of the underlying Lie group symmetry.

Keywords: solvable Lie group; Borel subgroup; massless particle states; chirality states

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1. Introduction

The basic structure of physics is defined via the representations of Lie groups and Lie algebras. Symmetry arises via both the external or space-like and internal or charge-like degrees of freedom and is represented in complex vector spaces regarding the noncompact–compact dichotomy of the relevant groups. The groups appear in two different varieties, semisimple and solvable, serving as the building blocks for all other Lie groups. Simple Lie groups regenerate themselves under commutation and generate semisimple groups via direct products. Solvable Lie groups do not regenerate themselves under commutation but are constructed in a stepwise way out of Abelian subgroups. Non-semisimple Lie groups are semidirect products of semisimple and solvable subgroups. Note that there is no complete classification for solvable Lie groups and, therefore, for non-semisimple Lie groups.

The subgroup structure of the symmetry group plays a basic role in Wigner’s definition of particles in the electroweak standard model, e.g., the isotropy subgroups of the Lorentz group and their corresponding coset manifolds. In particular, for massive particles, the point fixgroup $\text{SO}(3)$ is the maximal connected compact simple subgroup of the Lorentz group $\text{Lor}_{1,3} \equiv \text{SO}_0(1,3)$, and the corresponding time-like energy–momentum hyperboloid is $Y^3 = \text{SO}_3(1,3)/\text{SO}(3)$. In this paper, we show that, in contrast to this, for massless particles, one obtains the line fixgroup $\text{Bor}_{1,3}$ as being the maximal connected noncompact solvable subgroup of the Lorentz group,

$m \neq 0$	maximal	connected	compact	simple	point fixgroup SO_3
	\uparrow	\uparrow	\times	\times	
$m = 0$	maximal	connected	noncompact	solvable	line fixgroup $\text{Bor}_{1,3}$

Here, $\text{Bor}_{1,3} \subset \text{Lor}_{1,3}$ is the Borel subgroup of the Lorentz group, and at least one of the manifolds $\text{Lor}_{1,3} / \text{Bor}_{1,3}$ is projective ($= S^2$).



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Scanning the recent literature concerning Lie groups for massless particles, few articles are found. Massless particles are considered within the Poincaré group [1–5] but without any reference to the possible solvability (solubility) of the corresponding stabilizer subgroup. Therefore, our work in this field, in previous publications [6,7] and in this publication, is pioneering and is based mainly on handbooks. As the basis for a deeper understanding, we suggest Refs. [8–11], from which many of the ideas that led to the research described in this paper were derived. At this point, it is worth emphasizing that, as physicists, we do not apply any rigorous mathematical formalism, consisting of definitions, lemmata, prepositions, theorems, and corollaries, together with the corresponding proofs. Instead, we mention the components of our considerations in these mathematical terms, without providing proofs in most instances; these, instead, can be found in the references that we cite at various points, which also mark the ends of the corresponding theorems. An exception is Lemma 1, which is new and the main outcome of our analysis. This lemma on the SL_2 reconstruction of the proper Lorentz group can be considered as a corollary of what has previously been discussed.

This publication is organized as follows. In Section 2, we present basic facts about the proper Lorentz group, and we present three theorems that are the foundations of our work: the Chevalley theorem, the Lie–Kolchin theorem, and a theorem related to Borel. Having explained Wigner’s concept of a little group, in Section 3, we first deal with Wigner’s result for this, given by the Euclidean group $E(2)$. However, the character equation has an additional solution that leads to the Borel subgroup, dealt with in Section 4. In fact, there are a couple of these subgroups, with the union giving the proper Lorentz group, while the cut is the maximal torus. We show explicitly that each of these groups is generated by two elements of the minimal solvable algebra \mathfrak{sol}_2 , spanning the algebra of the Borel subgroup as a Kronecker sum. Turning to topology, in Section 5, we show that the quotient of the Lorentz group and the Borel subgroup is a projective variety. Section 6 deals with representations, presenting also our Lemma 1, focused on reconstructing the proper Lorentz group by two copies of the simplest noncompact group $SL_2(\mathbb{R})$. Via the Weinberg ansatz, in Section 7, we describe a connection back to physics. In Section 8, we present our conclusions.

2. Basics

Here, we give a brief description of the basic theorems and constructions of the theory of semisimple groups with applications to the proper Lorentz group [12–17]. The Minkowski representation $SO_0(1, 3)$ preserves the indefinite symmetric metric $\eta = \text{diag}(1, -1, -1, -1)$ in the real spacetime $\mathbb{R}^4 \ni x, y$,

$$x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = x^T \eta y.$$

As the defining representation of the causality-compatible Lorentz group,

$$\text{Lor}_{1,3} = \{\Lambda^T \eta \Lambda = \eta, \det \Lambda = 1, \Lambda^0_0 \geq 1, \Lambda \equiv (\Lambda^\mu_\nu) \in GL_4(\mathbb{R})\}$$

is parametrizable by six real parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$,

$$\Lambda(\omega) = \exp\left(-\frac{1}{2}\omega_{\mu\nu}e^{\mu\nu}\right) = \exp\left(\frac{1}{2}\omega_p \epsilon^{0pjk}e_{jk} + \sum \omega_{0p}e_{0p}\right),$$

where the first part is compact and the second part noncompact, and the domain of these six parameters is given by

$$D = \{\omega_{0p}, \omega_p \in \mathbb{R}, -\pi < \omega_p \leq \pi, p = 1, 2, 3\}$$

which is homeomorphic to $\mathbb{R}^3 \times \mathbb{P}_3$, where \mathbb{P}_3 is the three-dimensional projective space. The generators have the form

$$e_{jk} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{e}_j \vec{e}_k^T - \vec{e}_k \vec{e}_j^T \end{pmatrix}, \quad e_{0j} = \begin{pmatrix} 0 & \vec{e}_j^T \\ \vec{e}_j & 0_3 \end{pmatrix}, \quad (1)$$

where, in general, $[e_{\mu\nu}, e_{\rho\sigma}] = \eta_{\mu\rho}e_{\nu\sigma} + \eta_{\nu\sigma}e_{\mu\rho} - \eta_{\mu\sigma}e_{\nu\rho} - \eta_{\nu\rho}e_{\mu\sigma}$.

Here, \vec{e}_j , $j = 1, 2, 3$ is the Euclidean basis in \mathbb{R}^3 . Therefore, $\text{Lor}_{1,3}$ is a locally compact and doubly connected, path-connected, simple and reductive group with universal coverage $\text{SL}_2(\mathbb{C})$, i.e.,

$$\text{Lor}_{1,3} \cong \text{SL}_2(\mathbb{C}) / \mathbb{Z}_2 \cong \text{SO}(3, \mathbb{C}).$$

The last isomorphism means that the representations of $\text{Lor}_{1,3}$ may be seen as representations of the complex rotation group $\text{SO}(3, \mathbb{C})$. For completeness, note that the Lie algebra $\text{lor}_{1,3} \equiv \log \text{Lor}_{1,3}$ is the noncompact real form of

$$\text{so}(4, \mathbb{C}) = \text{sl}_2(\mathbb{C}) \oplus \text{sl}_2(\mathbb{C}).$$

To motivate what follows, it is instructive to look at Chevalley's theorem in the context of the defining representation of the Lorentz group, according to which $\text{Lor}_{1,3}$ acts on the flat spacetime or its dual energy–momentum space by a linear transformation,

$$\text{Lor}_{1,3} \ni \Lambda : \mathbb{E}_{1,3} \ni x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu \in \mathbb{E}_{1,3}.$$

Theorem 1 (Chevalley). *Let G be a linear algebraic group and $H \subset G$ a closed algebraic subgroup. Then, there is a rational representation $\phi : G \rightarrow \text{GL}(V)$ and a one-dimensional subspace $L \subset V$ such that*

$$H = \{g \in G : \phi(g)L = L\}. \quad (2)$$

Otherwise, if $\ell \in L$ spans the line L , i.e., $L = \mathbb{C}\ell$, then the equation

$$\phi(h)\ell = \chi(h)\ell \in L, \quad h \in H \quad (3)$$

defines a character $\chi \in \mathcal{X}(H)$ of H . This character is called the weight to the semi-invariant ℓ , and $\mathcal{X}(H) = \text{Mor}(H, \text{GL}_1)$. Note that, if $[H, H] = H$, then $\mathcal{X}(H) = \{0\}$, i.e., $\text{SL}_2(\mathbb{C})$ has no nontrivial characters and therefore the Lorentz group has no nontrivial characters [18].

Theorem 2 (Lie–Kolchin). *Let G be a connected solvable linear algebraic group, and let (ϕ, V) be a regular representation of G . Then, there exist characters $\chi_i \in \mathcal{X}(G)$, $i = 1, 2, \dots, n$ and a flag*

$$V = V_1 \supset V_2 \supset \dots \supset V_n \supset V_{n+1} = \{0\}$$

such that

$$(\phi(g) - \chi_i(g))V_i \subset V_{i+1}$$

for all $g \in G$. Taking $i = n$, one obtains the characteristic equation

$$\phi(g)V_n = \chi_n(g)V_n, \quad (4)$$

i.e., every solvable group has a common one-dimensional subspace $L \subset V$ [19,20].

Wigner's Little Group

What follows was proposed by Eugene Paul Wigner in 1939: for massive particles ($m \neq 0$), the point fixgroup (or little group) of the momentum $\vec{p} = (1, 0, 0, 0)$, and

$\lg(\overset{\circ}{p}) = \text{SO}(3) \subset \text{Lor}_{1,3}$ is maximal connected, compact and simple [4,5,13,14,16]. Proceeding purely mathematically, one finds that, for massless particles ($m = 0$), the little group is maximal connected, noncompact and solvable, resulting in a Borel subgroup $\text{Bor}_{1,3}$. By definition, a Borel subgroup of an algebraic group G is a maximal connected solvable subgroup.

Theorem 3. *Let G be a connected linear algebraic group. Then,*

1. G contains a Borel subgroup B ;
2. All other Borel subgroups of G are conjugate to B ;
3. The homogeneous manifold G/B is a projective variety;
4. $G = \bigcup_{g \in G} gBg^{-1}$, where B is a fixed Borel subgroup of G .

(see Theorem 11.4.7 in Ref. [21], p. 524).

Thus, every element $g \in G$ is contained in a Borel subgroup. As mentioned before, a solvable group has a semidirect structure. For the connected solvable group G , the set G_u of unipotent elements is a closed connected nilpotent subgroup of G . There exists a maximal torus $T_G \subset G$ and, for this, an exact sequence

$$e_G \rightarrow G_u \rightarrow G_u \rtimes T_G = G \rightarrow T_G \rightarrow e_G.$$

Since the maximal torus T_G and the maximal connected unipotent subgroup G_u of G are those for the Borel subgroup, one has the exact sequence

$$e_B \rightarrow B_u \rightarrow B_u \rtimes T_G = B \rightarrow T_G \rightarrow e_B. \quad (5)$$

and $B = N_G(B_u)$. Here, $G_u = \{g \in G : g = g_u\}$, where g_u is the unipotent component in the Jordan decomposition $g = g_s g_u$.

3. The Role of Mass

Before discussing the Borel subgroup in further mathematical detail, let us note some physical considerations regarding the kinematics. Taking the momentum four-vector to be p , with $p^2 = m^2 > 0$ the squared mass, for massive particles, one can move to the rest frame, where $p = (m, 0, 0, 0)^T$. The stabilizer subgroup or fixgroup is given by $\text{SO}(3)$, the three-dimensional rotations. However, if the particle is massless, such a move to the rest frame is no longer possible. According to Wigner's classification, the fixgroup is $E(2)$. This is the little group that Wigner indicates for massless particles like photons and (massless) neutrinos. For instance, for a momentum vector $p = (p_0, 0, 0, p_0)^T$ pointing in z direction, $E(2)$ consists of rotations about the z axis, translations orthogonal to it and reflections. However, what is not taken into account by this is the interchange of time and space components, which is obviously an additional symmetry transformation. Together with this additional transformation, the fixed point group is given by the Borel subgroup $\text{Bor}_{1,3}$.

Returning to mathematics, let $B \subset G$ be a Borel subgroup of G and V a finite-dimensional rational G -module. Then, the fixed points of B in V coincide with the fixed points of G . As the Lorentz group has no fixed points, for B , the character Equation (3) is the only one that can be solved. In contrast, for $\text{SO}(3)$, one has the commutant $[\text{SO}(3), \text{SO}(3)] = \text{SO}(3)$. Therefore, the character group $\mathcal{X}(\text{SO}(3))$ is trivial. The rotation group has no nontrivial characters, and the character Equation (3) is impossible to solve. Moreover, if V is an irreducible rotational G -module (let G be semisimple), then there is a unique Borel-stable one-dimensional subspace spanned by a maximal vector of some weight/character χ with multiplicity one.

The little group of Wigner is $E(2)$ is nonmaximal, connected, solvable and noncompact [13,14,22,23]. The nonmaximality is explained (or determined) by the requirement to be a point fixgroup of $(1, 0, 0, 1)^T$,

$$E(2) \cong \mathbb{R}^2 \rtimes \text{SO}(2).$$

In this semidirect product, the compact group $\text{SO}(2)$ acts on the abelian locally compact group $\exp \mathbb{R}^2$ by the multiplication rule

$$(x_1, R_1)(x_2, R_2) = (x_1 + R_1 x_2, R_1 R_2).$$

Every irreducible representation of $E(2)$ is equivalent either to a character of $\text{SO}(2)$ lifted to $E(2)$ or to an induced representation $\text{ind}_{\mathbb{R}^2}^{E(2)} \chi$, where χ is the nontrivial character of \mathbb{R}^2 . To avoid a continuum of helicity states, one has to require that, for physical states, the noncompact part of $E(2)$ is trivial in all representations, so the little group reduces to $\text{SO}(2)$. There are topological considerations that restrict the allowed values of the helicity to integers or half-integers. Thus, the helicity

$$\lambda = \frac{\vec{J} \cdot \vec{k}}{|\vec{k}|} = 0, \pm \frac{1}{2}, \pm 1, \dots$$

is Lorentz-invariant for massless particles with the total angular momentum \vec{J} .

As for any abelian group, the reducible representations of $\text{SO}(2)$ are one-dimensional. Therefore, according to Wigner's classification, the free massless particles have only a single degree of freedom and are characterized by the value λ of their helicity.

In nature, there are two classes of particles. The first class consists of particles that can exist in two helicity states $\pm \lambda$. Such a particle is defined as a representation of the parity-extended Poincaré group [1]. Since the electromagnetic interaction conserves parity, the photon is defined as the $\text{SO}(2)$ -doublet

$$\left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\text{SO}(2)} (+1) \oplus (-1) \oplus 2 \times (0) \xrightarrow{\text{parity}} (\pm 1) \oplus 2 \times (0).$$

The second class contains particles for which the parity is not defined, as the interactions that they are involved in violate parity. Such particles are the neutrinos that exist only with helicity $-\frac{1}{2}$ and antineutrinos with helicity $+\frac{1}{2}$.

4. The Borel Subgroup

The key observation in the preceding sections was the character Equation (3). Solving the character equation

$$B \overset{\circ}{p} = \chi(B) \overset{\circ}{p} \quad (6)$$

for the light-like standard vector $\overset{\circ}{p} = (1, 0, 0, 1)$ with $B \in \text{Lor}_{1,3}$, one obtains [6,7]

$$B^{(+)}(\vec{\beta}; \theta, \omega) = \begin{pmatrix} A & B^T \\ \vec{C} & D \end{pmatrix} \quad (7)$$

with

$$\begin{aligned} A &= \cosh \theta + \frac{1}{2} |\vec{\beta}|^2 e^{-\theta}, \\ \vec{B} &= \left(\sinh \theta - \frac{1}{2} |\vec{\beta}|^2 e^{-\theta} \right) \vec{e}_3 + \text{rot}_3 \omega \vec{\beta}, \\ \vec{C} &= \left(\sinh \theta + \frac{1}{2} |\vec{\beta}|^2 e^{-\theta} \right) \vec{e}_3 + e^{-\theta} \vec{\beta}, \\ D &= \left(\cosh \theta - 1 - \frac{1}{2} |\vec{\beta}|^2 e^{-\theta} \right) \vec{e}_3 \vec{e}_3^T - e^{-\theta} \vec{\beta} \vec{e}_3^T + (\mathbb{1}_3 + \vec{e}_3 \vec{\beta}^T) \text{rot}_3 \omega. \end{aligned} \quad (8)$$

Here, $\vec{\beta}^T = (\beta_1, \beta_2, 0) = \sum_{a=1}^2 \beta_a \vec{e}_a^T$, $\vec{e}_3^T = (0, 0, 1)$,

$$\text{rot}_3 \omega = \begin{pmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a consequence, the character is given by

$$\chi(B^{(+)}) = \frac{1}{2} \sum_{\mu, \nu=0,3} B^\mu{}_\nu = e^\theta.$$

The composition rule is

$$B^{(+)}(\vec{\beta}_1; \theta_1, \omega_1) B^{(+)}(\vec{\beta}_2; \theta_2, \omega_2) = B^{(+)}(\vec{\beta}_1 + e^{\theta_1} \text{rot}_3 \omega_1 \vec{\beta}_2; \theta_1 + \theta_2, \omega_1 + \omega_2) \quad (9)$$

and therefore

$$B^{(+)}(\vec{\beta}; \theta, \omega) = B_u^{(+)}(\vec{\beta}) \rtimes \mathcal{T}(\theta, \omega). \quad (10)$$

All such transformations $B^{(+)}(\vec{\beta}; \theta, \omega)$ with a noncompact parameter space for helicity and gauge, given by $\{\vec{\beta} \in \mathbb{R}^2, \theta > 0, 0 < \omega \leq \pi\}$, form the Borel subgroup

$$\text{Bor}_{1,3}^{(+)} = (\text{Bor}_{1,3}^{(+)})_u \rtimes \text{Tor}_{1,3} \subset \text{Lor}_{1,3}. \quad (11)$$

Here, $\text{Tor}_{1,3} = \text{SO}_0(1, 1) \times \text{SO}(2)$ is the maximal torus in $\text{Lor}_{1,3}$, and $(\text{Bor}_{1,3}^{(+)})_u$ is the unipotent radical of $\text{Bor}_{1,3}^{(+)}$.

The linearization of $\text{Bor}_{1,3}^{(+)}$ in the neighborhood of the identity

$$B^{(+)}(\vec{\beta}; \theta, \omega) = \mathbb{1}_4 + \beta_1 b_1 + \beta_2 b_2 + \theta b_0 + \omega b_3 = \mathbb{1}_4 - \frac{1}{2} \omega_{\mu\nu} e^{\mu\nu} \quad (12)$$

results in the Lie algebra $\text{bor}_{1,3}^{(+)} = \log \text{Bor}_{1,3}^{(+)}$, with

$$b_0 = e_{03}, \quad b_1 = e_{01} + e_{31}, \quad b_2 = e_{02} + e_{32}, \quad b_3 = e_{21}. \quad (13)$$

As a vector space $\text{span}_{\mathbb{R}}\{b_\mu\}_0^3$ endowed with commutation relations

$$[b_0, b_a] = b_a, \quad [b_3, b_a] = -\epsilon_{3ab} b_b, \quad a = 1, 2 \quad (14)$$

the Borel algebra reads

$$\text{bor}_{1,3}^{(+)} = (\text{bor}_{1,3}^{(+)})_u \rtimes \text{tor}_{1,3} = \text{so}(1, 1) \oplus \text{so}(2) \oplus \mathbb{R}^2, \quad (15)$$

where $(\text{bor}_{1,3}^{(+)})_u = \text{rad}_u^{(+)}$ is the Lie algebra corresponding to the unipotent radical.

Since the semisimple rank is $\text{rank}_{\text{ss}} \text{SO}(1, 3) = 2$, there exists a unique Borel subgroup $\text{Bor}_{1,3}^{(-)} \subset \text{Lor}_{1,3}$, called opposite $\text{Bor}_{1,3}$, such that [18,24–26]

$$\text{Bor}_{1,3}^{(-)} \cap \text{Bor}_{1,3}^{(+)} = \text{Tor}_{1,3} = \text{SO}_0(1, 1) \times \text{SO}(2)$$

and $\text{Lor}_{1,3} = \text{Bor}_{1,3}^{(-)} \cup \text{Bor}_{1,3}^{(+)}$.

To visualize the algebraic structure of $\text{bor}_{1,3}^{(+)}$, it is convenient to transform the basis to

$$t_0 = \frac{1}{2}(b_0 + ib_3), \quad t_+ = \frac{1}{2}(ib_1 - b_2), \quad [t_0, t_+] = t_+ \quad (16)$$

$$u_0 = \frac{1}{2}(b_0 - ib_3), \quad u_+ = \frac{1}{2}(-ib_1 - b_2), \quad [u_0, u_+] = u_+ \quad (17)$$

with $[t_{0,+}, u_{0,+}] = 0$. This leads to the Kronecker sum decomposition

$$\text{bor}_{1,3}^{(+)} = \text{sol}_2(e) \boxplus \text{sol}_2(f) := \text{sol}_2(e) \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \text{sol}_2(f). \quad (18)$$

The Kronecker sum decomposition is easily seen after applying the splitting map

$$\begin{aligned} t_0 &= \frac{1}{2}h \otimes \mathbb{1}_2, & t_+ &= (ie) \otimes \mathbb{1}_2 & \text{for } \text{sol}_2(e), \\ u_0 &= \mathbb{1}_2 \otimes \left(-\frac{1}{2}h\right), & u_+ &= \mathbb{1}_2 \otimes (if) & \text{for } \text{sol}_2(f). \end{aligned} \quad (19)$$

Here,

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is the natural Chevalley basis of $\text{sl}_2(\mathbb{C})$.

Again, the linearization of $\text{Bor}_{1,3}^{(-)}$ generates the basis of the underlying vector space as

$$k_0 = -e_{03}, \quad k_3 = e_{21}, \quad k_a = -e_{0a} + e_{3a}, \quad a = 1, 2 \quad (20)$$

with nonzero commutation relations

$$[k_0, k_a] = k_a, \quad [k_3, k_a] = -\epsilon_{3ab}k_b. \quad (21)$$

Thus,

$$\text{lor}_{1,3} = \text{bor}_{1,3}^{(-)} + \text{bor}_{1,3}^{(+)} = (\text{bor}_{1,3}^{(-)})_u \oplus (\text{bor}_{1,3}^{(+)})_u + \text{tor}_{1,3},$$

where $\text{tor}_{1,3} = \log \text{Tor}_{1,3} = \log \text{SO}_0(1, 1) \oplus \log \text{SO}(2)$.

The Kronecker sum decomposition of $\text{bor}_{1,3}^{(+)}$ into two fundamental solvable groups sol_2 is easily extended to an $\text{sl}_2(\mathbb{R})$ decomposition for the whole $\text{lor}_{1,3}$,

$$\text{lor}_{1,3} \cong \text{sl}_2(\mathbb{R})_e \boxplus \text{sl}_2(\mathbb{R})_f.$$

Using the splitting map, for $\text{sl}_2(\mathbb{R})_e$, one obtains

$$\begin{aligned} t_0 &= \frac{1}{2}(b_0 + ib_3) \rightarrow \left(\frac{1}{2}h\right) \otimes \mathbb{1}_2 \\ t_+ &= \frac{1}{2}(ib_1 - b_2) \rightarrow (ie) \otimes \mathbb{1}_2 \\ t_- &= \frac{1}{2}(-ib_1 - b_2) \rightarrow (if) \otimes \mathbb{1}_2 \end{aligned} \quad (22)$$

with commutation relations

$$[t_+, t_-] = -2t_0, \quad [t_0, t_\varepsilon] = \varepsilon t_\varepsilon, \quad \varepsilon = \pm$$

For $\mathfrak{sl}_2(\mathbb{R})_f$, one obtains

$$\begin{aligned} u_0 &= \frac{1}{2}(b_0 - ib_3) \rightarrow \mathbb{1}_2 \otimes \left(-\frac{1}{2}h\right) \\ u_+ &= \frac{1}{2}(-ib_1 - b_2) \rightarrow \mathbb{1}_2 \otimes (if) \\ u_- &= \frac{1}{2}(ik_1 - k_2) \rightarrow \mathbb{1}_2 \otimes (ie) \end{aligned} \quad (23)$$

with commutation relations

$$[u_+, u_-] = -2t_0, \quad [u_0, u_\varepsilon] = \varepsilon t_\varepsilon, \quad \varepsilon = \pm$$

and $[\mathfrak{sl}_2(\mathbb{R})_e, \mathfrak{sl}_2(\mathbb{R})_f] = 0$.

Returning from the defining matrix representation of the Borel algebra to the group matrix representation, by exponentiation, one obtains

$$\mathrm{tor}_{1,3} \ni \vartheta b_0 + \omega b_3 \xrightarrow{\exp} \exp(\vartheta b_0 + \omega b_3) \in \mathrm{Tor}_{1,3}$$

with

$$\begin{aligned} \exp(\vartheta b_0 + \omega b_3) &= \left(\mathbb{1}_4 + \sinh \vartheta b_0 + (\cosh \vartheta - 1)b_0^2 \right) \left(\mathbb{1}_4 + \sin \omega b_3 + (1 - \cos \omega)b_3^2 \right) = \\ &= \begin{pmatrix} \sqrt{1 + \vec{x}^T \vec{x}} & \vec{x}^T \\ \vec{x} & \mathrm{rot}_3 \omega \sqrt{\mathbb{1}_3 + \vec{x} \vec{x}^T} \end{pmatrix}, \quad \vec{x} = \sinh \vartheta \vec{e}_3. \end{aligned}$$

The Cartan–Killing form in the defining representation is indefinite,

$$(\vartheta b_0 + \omega b_3, \vartheta b_0 + \omega b_3) = 2(\vartheta^2 - \omega^2)$$

and

$$(\exp(\vartheta b_0 + \omega b_3), \exp(\vartheta b_0 + \omega b_3)) = 2(\cosh \vartheta + \cos \omega) \geq 0.$$

For the unipotent radical, one has

$$\mathrm{rad}_u^{(+)} \ni b = \beta_1 b_1 + \beta_2 b_2 \xrightarrow{\exp} \exp b = -\mathbb{1}_4 + \exp(\beta_1 b_1) + \exp(\beta_2 b_2).$$

Since the algebra $\mathrm{bor}_{1,3}^{(+)}$ is solvable, and its derived algebra is given by

$$\mathrm{Der} \mathrm{bor}_{1,3}^{(+)} = \mathrm{rad}_u^{(+)},$$

the Cartan–Killing form is identically zero, $(\mathrm{rad}_u^{(+)}, \mathrm{rad}_u^{(+)}) \equiv 0$. For $\mathrm{Rad}_u^{(+)}$, one obtains $(\exp b, \exp b) = 4$, so, obviously, $b \in \mathrm{rad}_u^{(+)}$ nilpotent and $\exp b \in \mathrm{Rad}_u^{(+)}$ unipotent.

5. The Quotients

Given a closed subgroup H of an algebraic group G , there is a smooth projection $\pi : G \rightarrow G/H$, where the fibers are precisely the cosets gH , $g \in G$. The projection π has a smooth local injection, given by the compatible section

$$\begin{array}{ccc} \gamma : G/H & \rightarrow & G \\ & \searrow \text{id} & \downarrow \pi \\ & & G/H \end{array}$$

such that $\pi \circ \gamma = \text{id}_{G/H}$.

As an example, we consider the Cartan decomposition $\mathfrak{g} = \log G = \log K \oplus \mathfrak{p}$, where $\log K$ is the maximal compact subalgebra of \mathfrak{g} , and the subspace $\mathfrak{p} = \mathfrak{g} \bmod \log K$ consists of the noncompact generators of \mathfrak{g} . Exponentiating the Lie algebra decomposition into the Lie group,

$$\begin{array}{ccccc} \mathfrak{g} = & \log K & \oplus & \mathfrak{p} \\ & \text{compact} & & \text{noncompact} \\ \exp \downarrow & \downarrow & & \downarrow \\ G = & K & \otimes & \exp \mathfrak{p} \\ & \text{compact} & & \text{coset} \\ & \text{subgroup} & & \text{representatives} \end{array}$$

one obtains the parametrization of the algebraic manifold $G/K = \exp \mathfrak{p}$.

The map of $K \times \mathfrak{p}$ onto G

$$K \times \mathfrak{p} \ni (k, X) \rightarrow k \exp X \in G$$

is a diffeomorphism into G , i.e., $G = K \times \exp \mathfrak{p}$.

Different choices of the section γ give different formulae for the coset representatives. For the Borel subgroup $B \subset G$, the factor set G/B is the largest homogeneous space for G , having the structure of a projective variety. Since G/B is complete, the Borel subgroup B has a fixed point in G/B .

5.1. The $SO(3)$ Parametrization

The Borel decomposition of the Lorentz group $\text{Lor}_{1,3} = \text{SO}_0(1, 3)$ is generated by the decomposition of the algebra $\mathfrak{lor}_{1,3} = \log \text{SO}_0(1, 3)$ in a natural way by reordering the usual parametrization

$$\begin{array}{ccc} \mathfrak{lor}_{1,3} \ni -\frac{1}{2}\omega_{\mu\nu}e^{\mu\nu} = & \underbrace{\omega_1 e_{31} + \omega_2 e_{32}}_{\text{coset representatives}} & + \underbrace{\beta_1 b_1 + \beta_2 b_2 + \vartheta e_{03} + \omega e_{21}}_{\text{bor}_{1,3}} \\ & \exp \downarrow & \downarrow \\ \text{Lor}_{1,3} \ni & \exp \sum_{a=1}^2 \omega_a e_{3a} & \times \text{Bor}_{1,3}^{(+)}(\beta_1, \beta_2; \vartheta, \omega) \end{array}$$

More precisely,

$$\begin{aligned} \mathbb{P}_{(2)} &\equiv \sum_{a=1}^2 \omega_a e_{3a} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{e}_3 \vec{\omega}_{(3)}^T - \vec{\omega}_{(3)} \vec{e}_3^T \end{pmatrix} \\ &\rightarrow \exp \mathbb{P}_{(2)} = \begin{pmatrix} 1 & \vec{0}^T & 0 \\ \vec{0} & \sqrt{1_2 - \vec{x}_{(2)} \vec{x}_{(2)}^T} & -\vec{x}_{(2)} \\ 0 & \vec{x}_{(2)}^T & \sqrt{1 - \vec{x}_{(2)}^T \vec{x}_{(2)}} \end{pmatrix}. \end{aligned}$$

Here,

$$\vec{x}_{(2)} = \frac{\sin \omega_{(2)}}{\omega_{(2)}} \vec{\omega}_{(2)}, \quad \vec{\omega}_{(2)} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \omega_{(2)}^2 = \omega_1^2 + \omega_2^2, \quad \vec{\omega}_{(3)} = \begin{pmatrix} \vec{\omega}_{(2)} \\ 0 \end{pmatrix}.$$

The Cartan–Killing inner product for representatives

$$(\mathbb{P}_{(2)}, \mathbb{P}_{(2)}) = \text{tr } \mathbb{P}_{(2)}^2 = -2\omega_{(2)}^2 < 0$$

is negative, i.e., the representatives are compact operators. For the representatives of the group coset, one has

$$(\exp \mathbb{P}_{(2)}, \exp \mathbb{P}_{(2)}) = 4 \cos^2 \omega_{(2)} > 0.$$

Therefore, the geometric manifold for the representatives of the group coset is compact. Moreover, the parametrization of the $\text{Bor}_{1,3}$ -classes can be given by the three-point $\vec{x} = (x_1, x_2, x_3)^T$ in \mathbb{R}^3 as

$$\vec{x}_{(2)} = \frac{\sin \omega_{(2)}}{\omega_{(2)}} \vec{\omega}_{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and $x_3 = \cos \omega_{(2)}$. Then,

$$\exp \mathbb{P}_{(2)} = \begin{pmatrix} 1 & \vec{0}^T & 0 \\ \vec{0}^T & \mathbb{1}_2 - \frac{1}{1+x_3} \vec{x}_{(2)} \vec{x}_{(2)}^T & -\vec{x}_{(2)} \\ 0 & \vec{x}_{(2)}^T & x_3 \end{pmatrix}$$

with $\det \exp \mathbb{P}_{(2)} = x_1^2 + x_2^2 + x_3^2 = 1$. Therefore, the SO-type coset representatives generate a compact factor set, the two-sphere in \mathbb{R}^3 [27],

$$\text{Lor}_{1,3} / \text{Bor}_{1,3}^{(+)} = \text{SO}(3) / \text{SO}(2) = S^2.$$

Using the Iwasawa decomposition

$$\begin{aligned} \text{Lor}_{1,3} &= \text{SO}(3) \times \text{SO}_0(1,1) \times \exp \mathbb{R}^2 \\ &= (\text{SO}(3) / \text{SO}(2)) \times \underbrace{\text{SO}(2) \times \text{SO}_0(1,1) \times \exp \mathbb{R}^2}_{\text{Bor}_{1,3}} \end{aligned}$$

and $\text{Bor}_{1,3}^{(+)} = (\text{SO}(2) \times \text{SO}_0(1,1)) \ltimes \exp \mathbb{R}^2$, one obtains the same result.

The projective coordinates for this parametrization are

$$-\infty < z_a = \frac{x_a}{\sqrt{1-x_1^2-x_2^2}} = \frac{x_a}{x_3} < \infty, \quad a = 1, 2.$$

As mentioned, different choices for the section γ provide different parametrizations.

5.2. The $\text{SO}(1, 2)$ Parametrization

Let

$$\begin{array}{ccc} \text{lor}_{1,3} \ni -\frac{1}{2} \omega_{\mu\nu} e^{\mu\nu} = \vartheta_1 e_{01} + \vartheta_2 e_{02} & + & \beta_1 b_1 + \beta_2 b_2 + \vartheta e_{03} + \omega e_{21} \\ \text{coset representatives} & & \text{bor}_{1,3}^{(+)} \\ \exp \downarrow & & \downarrow \\ \text{Lor}_{1,3} \ni \exp \vartheta_{(2)} & \times & \text{Bor}_{1,3}^{(+)} \end{array}$$

Here, $\vartheta_{(2)} = \sum_{a=1}^2 \vartheta_a e_{0a}$ is noncompact, as the Cartan–Killing form is positive,

$$(\vartheta_{(2)}, \vartheta_{(2)}) = 2\vartheta_{(2)}^2 = 2(\vartheta_1^2 + \vartheta_2^2) > 0.$$

The group representatives $\exp \vartheta_{(2)}$ are noncompact,

$$\mathrm{SO}_0(1,2) \ni \exp \vartheta_{(2)} = \begin{pmatrix} \sqrt{1 + \vec{y}_{(3)}^T \vec{y}_{(3)}} & \vec{y}_{(3)}^T \\ \vec{y}_{(3)} & \sqrt{1_3 + \vec{y}_{(3)} \vec{y}_{(3)}^T} \end{pmatrix}, \quad \vec{y}_{(3)} = \frac{\sinh \vartheta_{(2)}}{\vartheta_{(2)}} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ 0 \end{pmatrix},$$

as the Cartan–Killing form is positive,

$$(\exp \vartheta_{(2)}, \exp \vartheta_{(2)}) = 4 \cosh^2 \vartheta_{(2)} > 0.$$

The parametrization of the coset representations for the group can be chosen as the point $y = (y_0, y_1, y_2)$ in \mathbb{R}^3 or as the projective coordinates $z_a = y_a / y_0$, $a = 1, 2$ in the interior of the unit circle of the $y_0 = 1$ plane. In the first case, we have

$$y_0 = \cosh \vartheta_{(2)}, \quad y_a = \frac{\sinh \vartheta_{(2)}}{\vartheta_{(2)}} \vartheta_a, \quad a = 1, 2,$$

so that $y_0^2 - y_1^2 - y_2^2 = 1$, and the representatives are

$$Q(y_0, y_1, y_2) = \begin{pmatrix} y_0 & \vec{y}_{(3)}^T \\ \vec{y}_{(3)} & 1_3 + \frac{1}{1 + y_0} \vec{y}_{(3)} \vec{y}_{(3)}^T \end{pmatrix}$$

with $\det Q = (y_0^2 - y_1^2 - y_2^2) / (1 + y_0) = 1$ and $(Q, Q) = 4y_0^2 > 0$. As a consequence, the representatives are found on the noncompact on-shell hyperboloid

$$Y^2 = \mathrm{Lor}_{1,3} / \mathrm{Bor}_{1,3}^{(+)} = \mathrm{SO}_0(1,2) / \mathrm{SO}(2).$$

The projective coordinates (q_1, q_2) are

$$-1 < q_a = \frac{y_a}{\sqrt{1 + y_1^2 + y_2^2}} = \frac{y_a}{y_0} < 1, \quad a = 1, 2.$$

Therefore, the two examples for the coset representatives considered up to now are the time-like on-shell hyperboloid Y^2 and (as a compact partner) the sphere S^2 with the common compact subgroup given by $\mathrm{SO}(2) \subset \mathrm{Tor}_{1,3}$.

5.3. The Borel Parametrization

Finally, to be systematic, the Borel structure of the Lorentz group provides a constructive procedure to determine representatives of different cosets. To begin with, we recall the Borel decomposition

$$\mathrm{lor}_{1,3} = \mathrm{bor}_{1,3}^{(-)} \cup \mathrm{bor}_{1,3}^{(+)} = \mathrm{rad}_u^{(-)} \oplus (\mathrm{rad}_u^{(+)} \rtimes \mathrm{tor}_{1,3}) = \mathrm{rad}_u^{(+)} \oplus (\mathrm{rad}_u^{(-)} \rtimes \mathrm{tor}_{1,3}).$$

As underlying vector spaces, one has

$$\vec{\mathrm{lor}}_{1,3} = \mathrm{span}_{\mathbb{R}} \{e_{\mu\nu} = -e_{\nu\mu}\}_0^3 = \mathrm{span}_{\mathbb{R}} \{b_0, b_3, b_a, k_a\}_{a=1}^2 = \mathrm{span}_{\mathbb{R}} \{t_0, t_\varepsilon; u_0, u_\varepsilon\}_{\varepsilon=\pm 1}.$$

It is convenient to choose the basis of $\mathrm{rad}_u^{(-)}$ as

$$k_a = \begin{pmatrix} 0 & -\vec{e}_a^T \\ -\vec{e}_a & \vec{e}_3 \vec{e}_a^T - \vec{e}_a \vec{e}_3^T \end{pmatrix}.$$

Then,

$$\text{rad}_u^{(-)} \ni k = \kappa_1 k_1 + \kappa_2 k_2 = \begin{pmatrix} 0 & -\vec{\kappa}_{(3)}^T \\ -\vec{\kappa}_{(3)} & \vec{e}_3 \vec{\kappa}_{(3)}^T - \vec{\kappa}_{(3)} \vec{e}_3^T \end{pmatrix},$$

where

$$\vec{\kappa}_{(3)} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ 0 \end{pmatrix}, \quad \kappa_{(3)}^2 = \kappa_1^2 + \kappa_2^2,$$

and the Cartan–Killing product $(k, k) = \text{tr } k^2 = 0$. This is the second criterion for the solvability of $\text{bor}_{1,3}^{(-)}$: an algebra \mathfrak{g} is solvable if and only if its Cartan–Killing metric tensor is identically zero on its derived algebra $\mathcal{D}^1 \mathfrak{g}$.

The expression for the representatives of the group coset is obtained by exponentiation:

$$\begin{array}{ccc} \text{lor}_{1,3} \supset & \text{rad}_u^{(-)} & \oplus \text{bor}_{1,3}^{(+)} \\ & \exp \downarrow & \downarrow \\ \text{Lor}_{1,3} \supset & \text{Rad}_u^{(-)} & \times \text{Bor}_{1,3}^{(+)} \end{array}$$

From this, the representations of the group coset can be read as

$$\text{Rad}_u^{(-)} \ni \exp k = \mathbb{1}_4 + k + \frac{1}{2} k^2 = \begin{pmatrix} 1+t & -x & -y & t \\ -x & 1 & 0 & -x \\ -y & 0 & 1 & -y \\ -t & x & y & 1-t \end{pmatrix},$$

where $t = \frac{1}{2} \kappa_{(3)}^2$, $x = \kappa_1$, $y = \kappa_2$ and $2t - (x^2 + y^2) = 0$. One observes that the real parameters (t, x, y) describe the subspace of the noncompact elliptic paraboloid. The projective coordinates of this parametrization are

$$(p_1, p_2) = \left(\frac{x}{t}, \frac{y}{t} \right) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right), \quad p_1^2 + p_2^2 > 0.$$

The Cartan–Killing form for the representatives of the group coset is positive:

$$(\exp k, \exp k) = 4 > 0.$$

As a consequence, one has the following:

- $m \neq 0$: a rest frame exists for massive particles \Rightarrow
stabilizer subgroup is the point fixgroup $\text{SO}(3) \Rightarrow$ leading to spin;
- $m = 0$: no rest frame exists for massless particles \Rightarrow
stabilizer subgroup is the line fixgroup $\text{Bor}_{1,3} \Rightarrow$ leading to helicity.

Indeed, accepting the undulatory theory of light, the plane wave, as the most elementary type of wave, cannot be localized in space. Moreover, the characteristic Equation (3) suggests that the massless particle can be enclosed on a line. Therefore, the question of how a massless particle with energy E differs from the same particle with energy λE is a quantum-mechanical problem in the form of Planck's formula $E = h\nu$, rather than a problem of symmetry. In fact, the difference is a mathematical one:

- $m \neq 0$: semisimple compact;
- $m = 0$: solvable noncompact.

6. The Representations

In general, Lie algebras play their role in physics not as abstract algebras but through their representations that act on suitable representation spaces. For example, spin and helicity are determined by the stabilizer subgroups of $\text{Lor}_{1,3}$. For mathematical convenience, it is reasonable to consider representations in vector spaces over complex number fields. This is because, in physics, the concept of reducibility is of fundamental importance, and the mathematical structure of quantum mechanics works with complex Hilbert spaces [13–16,22,28–31].

A representation D of a real algebra \mathfrak{g} can be extended to a unique linear complex representation \hat{D} by the holomorphic extension

$$\hat{D}(A + iB) = D(A) + iD(B), \quad A, B \in \mathfrak{g}.$$

Although $\text{Lor}_{1,3} \cong \text{SO}(3, \mathbb{C})$, the process of holomorphic extension for $\text{lor}_{1,3}$ arises from the fact that $\text{lor}_{1,3}$ is the real form of $\text{so}_4(\mathbb{C})$,

$$\text{lor}_{1,3} \rightarrow \mathbb{C} \otimes_{\mathbb{R}} \text{lor}_{1,3} \cong \text{so}_4(\mathbb{C}) = \text{sl}_2(\mathbb{C}) \oplus \text{sl}_2(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \text{su}(2) \oplus \mathbb{C} \otimes_{\mathbb{R}} \text{su}(2).$$

This complexification of $\text{lor}_{1,3}$ provides the link between the real-valued Lorentz algebra $\text{lor}_{1,3}$ and the real-valued algebra $\text{su}(2)$, and, using this link, one can construct all the representations of $\text{lor}_{1,3}$.

Since a representation τ_m of $\text{so}(3)$ comes from a representation $D^{(m)}$ of $\text{SO}(3) \subset \text{Lor}_{1,3}$, the diagram

$$\begin{array}{ccc} \tau_m : \text{so}_3 \in x & \longrightarrow & \tau_m(x) \\ \exp \downarrow & & \downarrow \\ T^{(m)} : \text{SO}(3) \in \exp x & \longrightarrow & D^{(m)}(\exp x) = \exp \tau_m(x) \end{array}$$

has to be commutative. However, for odd m , there is no such representation $D^{(m)}$. In order to overcome this problem, one needs $\text{su}(2)$ to generate all the finite-dimensional representations of $\text{lor}_{1,3}$. The complex representations $T^{(k,l)}$ ($k, l = 0, \frac{1}{2}, 1, \dots$) may be obtained by holomorphic extension and Weyl's unitary trick,

$$\begin{aligned} \{e_{\mu\nu}\}_0^3 &\xrightarrow[\text{unitary trick}]{\text{Weyl's}} \left\{ \frac{1}{2} \left(-\frac{1}{2} \epsilon_{pnq} e_{nq} \pm i e_{0p} \right) \right\}_1^3 \\ &\xrightarrow[\text{map}]{\text{splitting}} \{D_p = m_p \boxplus m_p, B_p = -i(m_p \boxplus (-m_p))\}_1^3 \\ &\xrightarrow{T^{(k,l)}} \left\{ T^{(k,l)}(D_p) = D^{(k)}(m_p) \boxplus D^{(l)}(m_p), T^{(k,l)}(B_p) = -i(D^{(k)}(m_p) \boxplus (-D^{(l)}(m_p))) \right\}_1^3. \end{aligned} \quad (24)$$

Here, $\{m_p\}_1^3$ generate the algebra $\text{su}(2)$ and $D^{(k)}, k = 0, \frac{1}{2}, 1, \dots$ are the common representations of $\text{su}(2)$.

Any irreducible finite-dimensional representation of $\text{lor}_{1,3}$ is isomorphic to $T^{(k,l)}$ for some (k, l) . As a special case, there are two inequivalent fundamental representations from which all others can be obtained by reducing the tensor products. The two-dimensional spinor representation $(\frac{1}{2}, 0)$ is defined by the commutative diagram

$$\begin{array}{ccccc} \text{Lor}_{1,3} \ni \Lambda & E_{1,3} \ni p & \implies & \Lambda p & \\ T^{(1/2,0)} \downarrow & \sigma \downarrow & & \sigma \downarrow & \\ \text{SL}_2(\mathbb{C}) \ni T^{(1/2,0)}(\Lambda) \equiv A_\Lambda & \mathbb{H}_2 \ni \sigma(p) & \longrightarrow & A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p) & \end{array}$$

The representation $(0, \frac{1}{2})$ is defined as

$$\begin{array}{ccccc} \text{Lor}_{1,3} \ni \Lambda: & E_{1,3} \ni p & \implies & \Lambda p \\ T^{(0,1/2)} \downarrow & \tilde{\sigma} \downarrow & & \tilde{\sigma} \downarrow \\ \text{SL}_2(\mathbb{C}) \ni T^{(0,1/2)}(\Lambda) \equiv \tilde{A}_\Lambda: & \mathbb{H}_2 \ni \tilde{\sigma}(p) & \longrightarrow & \tilde{A}_\Lambda \tilde{\sigma}(p) \tilde{A}_\Lambda^\dagger = \tilde{\sigma}(\Lambda p) \end{array}$$

Here, $\tilde{\sigma}(p) = p^\mu \tilde{\sigma}_\mu$, $\sigma(p) = p^\mu \sigma_\mu$, with $\tilde{\sigma}_0 = \sigma_0 = \mathbb{1}_2$ and $\tilde{\sigma}_p = -\sigma_p$ ($p = 1, 2, 3$) being the Pauli matrices. Since the finite-dimensional representations of $\mathfrak{su}(2)$ are in one-to-one correspondence with those of $\mathfrak{sl}_2(\mathbb{R})$, and the Lie algebra $\mathfrak{lor}_{1,3}$ is noncompact, it is reasonable to define the representations of $\mathfrak{lor}_{1,3}$ in terms of the algebra $\mathfrak{sl}_2(\mathbb{R})$. Moreover, the most important technique by which to study the representations of linear noncompact groups is to reduce the problem to the subgroups isomorphic to the simplest noncompact group $\text{SL}_2(\mathbb{R})$. For example, $\mathfrak{lor}_{1,3}$ contains three such \mathfrak{sl} -isomorphic subalgebras but only one \mathfrak{su} -isomorphic subalgebra. The following lemma gives the \mathfrak{sl} -structure for $\mathfrak{lor}_{1,3}$.

Lemma 1. If $e_{\mu\nu} = -e_{\nu\mu}$, $\mu, \nu = 0, 1, 2, 3$ are defined by

$$\begin{aligned} e_{01} &= e_2 \boxplus (-e_2) \\ e_{02} &= -i(e_1 \boxplus (-e_1)) \\ e_{03} &= e_3 \boxplus (-e_3) \\ e_{31} &= e_1 \boxplus e_1 \\ e_{32} &= -i(e_2 \boxplus e_2) \\ e_{21} &= -i(e_3 \boxplus e_3), \end{aligned}$$

where $\{e_k\}_1^3$ are the generators of the algebra $\mathfrak{sl}_2(\mathbb{R})$,

$$[e_3, e_1] = e_2, \quad [e_3, e_2] = e_1, \quad [e_1, e_2] = -e_3,$$

then $e_{\mu\nu}$ generate the Lorentz algebra $\mathfrak{lor}_{1,3}$,

$$[e_{\mu\nu}, e_{\rho\sigma}] = \eta_{\mu\rho} e_{\nu\sigma} + \eta_{\nu\sigma} e_{\mu\rho} - \eta_{\mu\sigma} e_{\nu\rho} - \eta_{\nu\rho} e_{\mu\sigma}.$$

Applying this lemma, one can define the representations of $\mathfrak{lor}_{1,3}$ in terms of the holomorphic extensions of the irreducible representations $\pi^{(n)}$ of $\mathfrak{sl}_2(\mathbb{R})$ [24–26,32],

$$\begin{aligned} \pi^{(k,l)}(e_{01}) &= \pi^{(k)}(e_2) \boxplus (-\pi^{(l)}(e_2)) \\ \pi^{(k,l)}(e_{31}) &= \pi^{(k)}(e_1) \boxplus \pi^{(l)}(e_1) \\ \pi^{(k,l)}(e_{21}) &= -i(\pi^{(k)}(e_3) \boxplus \pi^{(l)}(e_3)) \end{aligned} \tag{25}$$

$(k, l = 0, \frac{1}{2}, 1, \dots)$. Here, $\pi^{(k)}$ is the standard representation of $\mathfrak{sl}_2(\mathbb{R})$,

$$\begin{aligned} \pi^{(k)}(e_1)|k, m\rangle &= \frac{1}{2}\rho_m^{(k)}|k, m+1\rangle - \frac{1}{2}\rho_{m-1}^{(k)}|k, m-1\rangle, \\ \pi^{(k)}(e_2)|k, m\rangle &= \frac{1}{2}\rho_m^{(k)}|k, m+1\rangle + \frac{1}{2}\rho_{m-1}^{(k)}|k, m-1\rangle, \\ \pi^{(k)}(e_3)|k, m\rangle &= m|k, m\rangle, \end{aligned} \tag{26}$$

with $\rho_m^{(k)} = \sqrt{(k+m+1)(k-m)}$, $m = -k, -k+1, \dots, k$.

In some contexts, it is more convenient to work with the \mathfrak{sl} -basis $e = e_1 + e_2$, $f = -e_1 + e_2$, and $h = 2e_3$. Via the lemma, the Borel algebras have the form

$$\begin{aligned} b_0 &= \frac{1}{2}(h \boxplus (-h)) \\ b_1 &= e \boxplus (-f) \\ b_2 &= -i(e \boxplus f) \\ b_3 &= -\frac{i}{2}(h \boxplus h) \end{aligned} \quad (27)$$

for $\mathfrak{bor}_{1,3}^{(+)}$ and

$$\begin{aligned} k_0 &= -\frac{1}{2}(h \boxplus (-h)) \\ k_1 &= (-f) \boxplus e \\ k_2 &= -i(f \boxplus e) \\ k_3 &= -\frac{i}{2}(h \boxplus h) \end{aligned} \quad (28)$$

for the opposite $\mathfrak{bor}_{1,3}^{(-)}$.

The $(2k+1)$ -dimensional representation $\pi^{(k)}$ of $\mathfrak{sl}_2(\mathbb{C})$ is

$$\begin{aligned} \pi^{(k)}(h)|k, m\rangle &= 2m|k, m\rangle \\ \pi^{(k)}(e)|k, m\rangle &= \rho_m^{(k)}|k, m+1\rangle \\ \pi^{(k)}(f)|k, m\rangle &= \rho_{m-1}^{(k)}|k, m-1\rangle \end{aligned} \quad (29)$$

with ρ and k as in Equation (26).

Theorem 4. Let $2k \in \mathbb{N}$ and let (π, V) be a simple representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $2k+1$. Then,

1. π is equivalent to $\pi^{(k)}$ for some k ;
2. the eigenvalues of $\pi^{(k)}(h)$ are $\{-2k, -2k-2, \dots, 2k\} = \text{Spec } \pi^{(k)}(h)$;
3. if $0 \neq v \in V$ satisfies $\pi^{(k)}(e)v = 0$, then $\pi^{(k)}(h)v = 2kv$, i.e., $\pi^{(k)}(h)$ and $\pi^{(k)}(e)$ have the common eigenvector $|k, k\rangle$;
4. if $0 \neq v \in V$ satisfies $\pi^{(k)}(f)v = 0$, then $\pi^{(k)}(h)v = -2kv$, i.e., $\pi^{(k)}(h)$ and $\pi^{(k)}(f)$ have the common eigenvector $|k, -k\rangle$.

(Theorem 19.2.5 in Ref. [33], p. 281).

As a matter of fact, points 3 and 4 generate/define the eigenvectors of the representation $\pi^{(k,l)}$, called the helicity states for $\mathfrak{bor}_{1,3}$. Using the \mathfrak{sl} -decomposition (27) of $\mathfrak{bor}_{1,3}$, one obtains

$$\begin{aligned} \pi^{(k,l)}(t_0) &= \frac{1}{2}\pi^{(k)}(h) \otimes \mathbb{1}_{2l+1} \\ \pi^{(k,l)}(t_+) &= i\pi^{(k)}(e) \otimes \mathbb{1}_{2l+1} \\ \pi^{(k,l)}(u_0) &= \mathbb{1}_{2k+1} \otimes \frac{1}{2}\pi^{(l)}(h) \\ \pi^{(k,l)}(u_+) &= \mathbb{1}_{2k+1} \otimes i\pi^{(l)}(f) \end{aligned}$$

and, with respect to the direct product basis $|k, l; m_k, m_l\rangle = |k, m_k\rangle \otimes |l, m_l\rangle$, where $-k \leq m_k \leq k$ and $-l \leq m_l \leq l$, one obtains the $2l + 1$ common eigenvectors for $\text{sol}_2(e)$, given by

$$\pi^{(k,l)}(t_0)|k, l; k, m_l\rangle = k|k, l; k, m_l\rangle, \quad \pi^{(k,l)}(t_+)|k, l; k, m_l\rangle = 0 \quad (30)$$

for $m_l = -l, -l + 1, \dots, l$. The $2k + 1$ common eigenvectors for $\text{sol}_2(f)$ are given by

$$\pi^{(k,l)}(u_0)|k, l; m_k, -l\rangle = l|k, l; m_k, -l\rangle, \quad \pi^{(k,l)}(u_+)|k, l; m_k, -l\rangle = 0 \quad (31)$$

for $m_k = -k, -k + 1, \dots, k$. Thus, the sol_2 -invariant subspaces of the representations of the proper Lorentz group, represented by the two components of the Kronecker sum as “left-handed” and “right-handed” states, lead to the concept of helicity. Accordingly, the (group-theoretical version of the) “Weinberg ansatz” is based on the concept of helicity.

7. The Weinberg Ansatz

Considering the states (m_k, m_l) as points on a lattice of dimension $(2k + 1) \times (2l + 1)$, according to Equations (30) and (31), the eigenstates of sol_2 are found with the values $m_k = k$ and $m_l = -l$, i.e., at the boundary of this lattice. This can be interpreted physically as a constraint on the spin degrees of freedom of the massless particle to only one of the helicity states. It is even possible to show that, for particles, this is the left-handed helicity state [7]. If, however, the particle is equal to its antiparticle, the full spectrum of helicity states is available. In general terms, this is formulated in Refs. [15,16] in the following way:

1. If a massless particle is equal to its antiparticle, it is described by the irreducible representation (k, k) of the proper orthochronous Lorentz group (Majorana case);
2. If a massless particle is not equal to its antiparticle, the particle is described by the irreducible representation $(k, 0)$ of the proper orthochronous Lorentz group, while the antiparticle is described by the irreducible representation $(0, k)$ of the proper orthochronous Lorentz group (Dirac case).

Note that the massless particle is defined via the Borel subgroup by the irreducible representation of the proper orthochronous Lorentz group, without the need to introduce space inversion.

7.1. The Majorana Case

For the representation (k, k) , the ansatz yields $2k + 1$ helicity states associated with $\text{sol}_2(e)$,

$$|k, k; k, -k + p\rangle, \quad p = 0, 1, \dots, 2k,$$

and $2k + 1$ helicity states associated with $\text{sol}_2(f)$,

$$|k, k; -k + p, -k\rangle, \quad p = 0, 1, \dots, 2k.$$

Since the state $|k, k; k, -k\rangle$ is twice and at the same time excluded by the condition

$$D_3^{(k,k)}|k, k; k, -k\rangle = 0, \quad B_3^{(k,k)}|k, k; k, -k\rangle = 2k|k, k; k, -k\rangle,$$

the particle with zero mass and helicity $\lambda = 2k$ has $4k$ helicity states. In particular, the defining representation $(\frac{1}{2}, \frac{1}{2})$ describes a massless particle with helicity 1 and two helicity states,

$$|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle \quad \text{and} \quad |\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}\rangle.$$

7.2. The Dirac Case

According to Weinberg, the Dirac case $(k, 0) \oplus (0, k)$ is a particular case of the general situation (k, l) with $l = 0$ or $k = 0$, respectively. For the representation $(k, 0)$, there exists only a single eigenvector $|k, 0; k, 0\rangle$ of the Borel algebra $\text{bor}_{1,3}(k, 0)$, i.e., only a single helicity state $|k, 0; k, 0\rangle$ with

$$t_0|k, 0; k, 0\rangle = k|k, 0; k, 0\rangle, \quad t_+|k, 0; k, 0\rangle = u_0|k, 0; k, 0\rangle = u_+|k, 0; k, 0\rangle = 0.$$

Similarly, the only helicity state for the representation $(0, k)$ is $|0, k; 0, -k\rangle$ with

$$u_0|0, k; 0, -k\rangle = k|0, k; 0, -k\rangle, \quad u_+|0, k; 0, -k\rangle = t_0|0, k; 0, -k\rangle = t_+|0, k; 0, -k\rangle = 0.$$

For example, the fundamental representation $\text{bor}_{1,3}(\frac{1}{2}, 0)$ can be expressed as

$$b_0(\frac{1}{2}, 0) = \frac{1}{2}h, \quad b_1(\frac{1}{2}, 0) = e, \quad b_2(\frac{1}{2}, 0) = -ie, \quad b_3(\frac{1}{2}, 0) = \frac{1}{2}ih.$$

The corresponding representation of the algebra $\text{sol}_2(e)$ has the form

$$t_0(\frac{1}{2}, 0) = \frac{1}{2}h, \quad t_+(\frac{1}{2}, 0) = ie$$

with $\text{sol}_2(f)$ being trivial. Therefore, in the case of the irreducible representation $(\frac{1}{2}, 0)$, there exists only a single solution $e_1 = (1, 0)^T$, i.e., a helicity state $\lambda = 1/2$, and this helicity state is equal to the solution of the Weyl equation

$$\bar{\sigma}_\mu p^\mu \psi(p) = 0.$$

In case of the representation $(0, \frac{1}{2})$, $\text{sol}_2(e)$ is trivial, and the nontrivial algebra $\text{sol}_2(f)$ is of the form

$$u_0(0, \frac{1}{2}) = -\frac{1}{2}h, \quad u_+(0, \frac{1}{2}) = -if$$

with only a single common eigenvector $e_1 = (0, 1)^T$.

8. Conclusions

With this work, we have delved into the rich solvable structure of the proper Lorentz group. As for a massless particle, the stabilizer subgroup of the momentum four-vector is given by the Borel subgroup as the maximal noncompact subgroup of the Lorentz group, of which the Lorentz group contains two copies. Thus, we can generate the Borel subgroup as a Kronecker sum of two copies of the simplest solvable algebra $\text{sol}_2 \subset \text{sl}_2$ and, correspondingly, the proper Lorentz group as a Kronecker sum of two copies of the simplest noncompact algebra sl_2 . This is formulated in Lemma 1. From our investigation in this paper, we conclude that, if there is a particle state with pure helicity or spin, the mass of this particle is zero and the stabilizer subgroup is the Borel subgroup, fixing the line of the light-like propagation. Therefore, at least for the electromagnetic field, the symmetry determines the dynamics.

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Abbreviations

The following abbreviations are used in this manuscript:

$\text{Lor}_{1,3}$	proper Lorentz group $\text{SO}_0(1,3)$
$\text{lor}_{1,3}$	proper Lorentz algebra $\text{so}_0(1,3)$
$\text{Bor}_{1,3}$	Borel subgroup of the Lorentz group $\text{SO}_0(1,3)$
$\text{bor}_{1,3}$	Borel subalgebra of the Lorentz algebra $\text{so}_0(1,3)$
$(\text{Bor}_{1,3})_u$	unipotent radical of $\text{Bor}_{1,3}$
$\text{Der } \mathfrak{g}$	derived Lie algebra of the Lie algebra \mathfrak{g}
$\text{rad}_u^{(\pm)}$	Lie algebra of the unipotent radical $(\text{bor}_{1,3})_u = \text{Der } \text{bor}_{1,3}^{(\pm)}$
$\text{Rad}_u^{(\pm)}$	exponential of $\text{rad}_u^{(\pm)}$
$\text{Tor}_{1,3}$	maximal torus group of the Lorentz group $\text{SO}_0(1,3)$
$\text{tor}_{1,3}$	maximal torus algebra of the Lorentz algebra $\text{so}_0(1,3)$
lg	Wigner's little group (point fixgroup)
\mathcal{T}	translational group
$\text{GL}_n(\mathbb{K})$	general linear group of dimension n over the field \mathbb{K}
$\text{SL}_n(\mathbb{K})$	special linear group of dimension n over the field \mathbb{K}
$\text{sl}_n(\mathbb{K})$	Lie algebra corresponding to $\text{SL}_n(\mathbb{K})$
sol_2	two-dimensional solvable Lie algebra

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