Departamento de Física de Partículas

NON-PERTURBATIVE METHODS IN NON-LINEAR FIELD THEORIES AND THEIR SUPERSYMMETRIC EXTENSIONS

José Manuel Fernández Queiruga

Santiago de Compostela, Xuño de 2013.

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Tese presentada para optar ao grao de Doutor en Física por:

José Manuel Fernández Queiruga

Compostela, Xuño de 2013

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CERTIFICAN: que a memoria titulada *Non-perturbative methods in non-linear field theories and their supersymmetric extensions* foi realizada, baixo a nosa dirección, por José Manuel Fernández Queiruga, no departamento de Física de Partículas desta Universidade e constitúe o traballo de Tese que presenta para optar ao grao de Doutor en Física.

Asinado:

Joaquín Sánchez Guillén **Christoph Adam**

Compostela, Xuño de 2013.

á miña familia

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Chapter 1

Introduction

1.1 Motivation

Despite its non-renormalizability, field theories with kinetic terms with powers higher than two (usually called K field theories) arise naturally in many areas in theoretical physics as effective field theories. Moreover, if we focus on a specific non-linear phenomenon of special importance, the topological solitons, and we want to ensure its stability in higher dimensions, we have two natural ways. One possibility consists of the inclusion of gauge fields. Known examples are the Abelian Higgs or the Chern-Simons Higgs models in $2+1$ dimensions, the BPS monopole in $3+1$... These models have a rich topological structure which can be exploited via their supersymmetric extensions (remember the relation between topological charges and central extensions of the supersymmetric algebra). The other possibility consists of the addition of higher derivative terms, leading us again to K field theories (therefore, it is natural to think on supersymmetric extensions of these models because of the intimate relation between supersymmetry and topology).

The study of K field theories is interesting in itself (from a formal point of view), but, in addition, it has multiple applications. The canonical example of a K field theory is the Skyrme model (SkM). Other K field models have direct applications in cosmology, for example, they are used to describe phenomena like K-inflation [2] or K-essence [3].

Moreover, these theories include new features studied in for example [4], [5], [7], [33] and [51]. Of course, in the context of cosmology (in the inflationary epoch) the supersymmetric extension of such theories becomes relevant and arises naturally.

Relevant phenomena associated to the supersymmetric versions of K field theories (Galileons, ghost condensates, DBI inflation) have been studied in [135] and in lower dimensions in [132]. More applications of these models can be found in [31], [60] and [59]. The existence of topological defects with compact support (compactons), a usual feature of this kind of theories, have been discovered in [36].

In general, topological defects resulting from K field theories are quite different from the corresponding ones of the standard theories [5], [74] and [12]. However, under certain conditions both defects can share the same energy density and profile (Doppelgänger effect [51]), in this case we say that the theories are twin-like. This feature makes them interesting in a wide range of applications of K field theories.

In the framework of K field theories, the SkM is singled out especially due to its rich structure and applications. The most popular application of the SkM is found in strong interaction and nuclear physics ([102], [103], [150], [106]), for which it was formulated. In this field, the SkM is interpreted as a low energy effective model for QCD when the number of colors becomes large, [109] and [108]. Its supersymmetric extensions in $3+1$ dimensions have been studied in [16] and [15] (in its $\mathbb{C}P^1$ restriction).

On the other hand, the baby Skyrme model (bSkM) is a non-linear theory with topological solitons in $2+1$ dimensions and $\mathbb{S}^2$ as target space [156], [81],and [82]. Topologically speaking, this model is similar to the SkM (for example, solitons in this model are labeled by a winding number), moreover, the fact that it is almost impossible to obtain analytic solutions directly from SkM justifies the study of its restrictions to simpler models, in our case the bSkM. This model is interesting in itself and has its specific applications for example to Hall ferromagnets [85] or spin textures [86] and [87]. Moreover, the analysis of supersymmetric extensions of the bSkM can be useful as a model for the study of general properties of supersymmetric topological solitons, an issue widely discussed throughout this thesis.

1.2 Content of this thesis

It this Ph.D. thesis, supersymmetric extensions of non-linear field theories are investigated, in particular, supersymmetric extensions of the so-called K field theories. Moreover, features of the specific solutions of this models inherited from supersymmetry are analyzed. For example, BPS solutions, energy bounds or central charges of the SUSY models are directly related to the topological charges.

We will see that supersymmetry provides enough structure to determine systematically first-order BPS equations. We will pay special attention to the SkM in lower dimensions (the so-called bSkM) which is a paradigmatic example of a theory with higher derivatives. We will show how supersymmetry constrains in $2 + 1$ dimensions the coexistence of quadratic, quartic and potential terms. It is even more interesting to see what happens if you try to reconcile the bSkM with $N = 2$ extended supersymmentry. In this case, the quadratic term is absent due to supersymmetry, but we can add a potential which depends on the metric of the target manifold. It will be shown how to obtain systematically the BPS equations for gauged and ungauged models, from the corresponding SUSY transformations. These results are interesting by themselves, but moreover, taking into account the dimensional reduction from $N = 1$ in $d = 3 + 1$ to $N = 2$ in $d = 2 + 1$, we can extend our lower dimension results with extended supersymmetry to three dimensional space with ordinary supersymmetry.

In a slightly different way of investigation with K field theories we will find the conditions that ensure the existence of twin models, even up to equivalence between fluctuation spectra. These results provide a kind of dictionary of a correspondence between K and standard theories. This correspondence allows us to investigate, in a specific range, more complex theories (in this case, K field theories) in terms of standard theories, which are generally simpler. This construction is then extended to supersymmetric models.

Finally, symmetries and solutions of the BPS Skyrme model are analyzed. The BPS Skyrme model is a Skyrme type model in $3+1$ dimensions which includes a sextic term in derivatives. The BPS bound for the energy as well as different solutions preserving symmetries of subgroups of the group of Volume Preserving Diffeomorphisms are calculated.

1.3 Structure of this thesis

This Ph.D. thesis is organized as follows:

- Chapters 2 to 4 are dedicated to present general features of supersymmetry which will be necessary in the following chapters. In chapter 5 a brief presentation of classical results about complex geometry and its relation with supersymmetry is given.
- Chapter 6 includes a basic introduction to the SkM and some relevant properties.
- In chapter 7 a first example of a supersymmetric extension of K field theories is presented. We will see in this section why supersymmetric extensions of K field theories are not at all trivial.
- In chapter 8 we present a supersymmetric extension of K field theories of the form $\mathcal{L} = \sum_i \alpha_i X^i + V(\phi)$ where $X = \partial_\mu \phi \partial^\mu \phi$ and $V(\phi)$ is a potential. General properties and solutions are analyzed.
- In chapter 9 domain wall solutions and BPS bounds of K field models of the form $\mathcal{L} = \sum_i \alpha_i X^i + V(\phi)$ are calculated. Concretely, we demonstrate that all the domain wall solutions which exist for this family of theories are BPS solutions and that the corresponding BPS energy reappears as a central charge in the SUSY algebra.

- In chapter 10 algebraic conditions which imply the existence of the so-called twin-like models are found.

- In chapter 11 the algebraic conditions which imply the existence of the twin models are generalized to imply the equivalence of the linear fluctuation spectra between the corresponding solutions.

- In chapter 12 an explicit $N = 1$ supersymmetric extension of the baby Skyrme model is presented, and different consequences of the supersymmetrization are analyzed.

- In chapter 13, in a first step an $N = 1$ SUSY extension of a the gauge bSkM is presented. Then we analyze a $N = 2$ SUSY extensions of both gauged and ungauged bSkM. Moreover, the relation between Bogomol'nyi equations and extended supersymmetry is studied more generally. A general scheme that generates Bogomolny equations in $2+1$ dimensions is found both for general gauged and ungauged theories with $N = 2$ supersymmetry.

- Chapter 14 is devoted to the study of the BPS Skyrme model (A Skyrme type model consisting of a sextic term in derivatives in 3+1 dimensions). The symmetries of Volume Preserving Diffeomorphisms symmetry are used to calculated solutions.

- Chapter 15 contains a brief summary of the results obtained along this thesis.

- Finally, chapter 16 contains the main conclusions of this work.

Chapter 2

SUSY N=1 d=1+1 and d=2+1

For this chapter we will follow the conventions of [18]. $N = 1$ supersymmetry in $2+1$-dimensional (and also in $1+1$-dimensional) space is specially simple, because in this case the Lorentz group is $SL(2,\mathbb{R})$ (instead of $SL(2,\mathbb{C})$) and the fundamental representation acts on real (Majorana) 2-component spinors. We can lower or rise spinor indices with the totally antisymmetric symbol, $C_{\alpha\beta} = -i\epsilon_{\alpha\beta}$, with $\epsilon_{12} = 1$, i.e.:

$$\psi_\alpha = \psi^\beta C_{\beta\alpha} \quad , \quad \psi^\alpha = C^{\alpha\beta}\psi_\beta \tag{2.1}$$

where the Majorana spinor $\psi^\alpha = (\psi^+, \psi^-)$. To denote a general coordinate on superspace we use the short notation $z = (x^\mu, \theta^\alpha)$, where the first components correspond to usual space-time coordinates and the second ones to the anticommutative part of the superspace:

$$\{\theta^\alpha, \theta^\beta\} = 0 \rightarrow (\theta^i)^2 = 0 \tag{2.2}$$

2.1 Berezin integration

At the end of the day we will need to integrate over anticommuting variables (Grassmann variables) in order to obtain the supersymmetric invariant actions. In this section we introduce the concept of integration over anticommuting objects (Berezin integration). The main two ingredients are the linearity and the invariance under translations in the Grassman variable. Suppose that our superspace has only one anticommuting variable, then, the dimension of the Grassmann part of the space is exactly one. If we are working in 3-dimensional Minkowski space we denote the corresponding superspace as $\mathbb{R}^{3|1}$. The most general superfunction that we can construct is:

$$\Phi(\theta) = a + \theta b \tag{2.3}$$

where $a : M^3 \to N$, and b is an anticommuting field, being M^3 the 3-dimensional Minkowski space and N whatever manifold. Imposing translation invariance we have

$$\int d\theta \Phi(\theta) = \int d\theta \Phi(\theta + \eta) \tag{2.4}$$

or equivalently:

$$\int d\theta b\eta = 0 \tag{2.5}$$

by linearity:

$$\int d\theta 1 = 0 \tag{2.6}$$

Now integrating again $\Phi(\theta)$:

$$\int d\theta \Phi(\theta) = b \int d\theta \theta \tag{2.7}$$

and what we need now is to fix the normalization of the integral, for example we can take:

$$\int d\theta \theta = 1 \tag{2.8}$$

finally the result for the total inegral is:

$$\int d\theta \Phi(\theta) = b \tag{2.9}$$

Note that:

$$\int d\theta \quad \equiv \quad \frac{d}{d\theta} \tag{2.10}$$

Now it is straightforward to generalize this integration to Grassmann algebras of arbitrary dimension. Let $\{\theta^1, ..., \theta^N\}$ be anticommuting variables:

$$\{\theta^\alpha, \theta^\beta\} = 0 \tag{2.11}$$

From this we can construct a Grassmann algebra of dimension 2^N. A generic basis has the following form:

$$f(\theta^1, ..., \theta^N) = f_0 + \sum f_i\theta^i + \sum f_{ij}\theta^i\theta^j + ... + f_{1,2,..,N}\theta^1\theta^2...\theta^N \tag{2.12}$$

We must be careful with the order of $\theta's$ in the integration, because a minus sign appears if the integration variable appears in an odd position w.r.t. the number of anticommuting variables, i.e.:

$$\int d\theta^i\theta^1...\theta^i...\theta^n = (-1)^{i+1}\theta^1...\hat{\theta^i}...\theta^n \tag{2.13}$$

(in this case the superindex labelled the position of the variable in the product). One interesting and fundamental feature of the Berezin integration is that after integration over all Grassmann space only the component with the highest order in $\theta's$ survives, for the previous expression:

$$\int d\theta^N...d\theta^1(f_0+\sum f_i\theta^i+\sum f_{ij}\theta^i\theta^j+...+f_{1,2,..,N}\theta^1\theta^2...\theta^N) = f_{1,2,..,N} \tag{2.14}$$

2.2 Superfields

In order to construct the correct algebra we need to grade the Poincarè algebra by introducing the generators of supersymmetry (Q_α). The commutation relations involving translations and Q_α are (assuming no central extension):

$$\begin{aligned} [P_{\mu\nu}, P_{\rho\sigma}] &= 0 \\ \{Q_\mu, Q_\nu\} &= 2P_{\mu\nu} \\ [Q_\mu, P_{\nu\rho}] &= 0 \end{aligned} \tag{2.15}$$

where a single index is interpreted as a spinor index and a double index means: $A_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} A_\mu$. The algebra (2.15) can be realized on superfields (functions depending on both space-time coordinates and Grassmann coordinates) in terms of:

$$P_{\mu\nu} = i\partial_{\mu\nu} \quad , \quad Q_\mu = i(\partial_\mu - i\theta^\nu \partial_{\mu\nu}) \tag{2.16}$$

If we have $N = 1$ supersymmetry in either $1 + 1$ or $2 + 1$ dimensions the supersymmetric generators are two-component spinors. This implies that our superspace will be $\mathbb{R}^{d|2}$, i.e. the most general superfield we can costruct in these dimensions is:

$$\Phi(x, \theta) = \phi(x) + \theta^\alpha \psi_\alpha(x) - \theta^2 F(x) \tag{2.17}$$

(with $\theta^2 = i\theta^+\theta^-$). $\phi(x)$ is a real scalar field, $\psi_\alpha(x)$ a real two-component Majorana spinor and $F(x)$ a real auxiliary field. As we will see, usually F is non-dynamical and we can eliminate it using its (usually algebraic) equations of motion, but it is necessary to have it in the superfield formulations to compensate the bosonic and fermionic degrees of freedom. The supersymmetric derivative is defined to be:

$$D_\alpha = \partial_\alpha + i\theta^\nu \partial_{\mu\nu}. \tag{2.18}$$

From a general supersymmetric transformation it is easy to obtain the corresponding transformations on the components:

$$\delta\Phi(x, \theta) = i\epsilon^\alpha Q_\alpha \Phi(x, \theta) = -\epsilon^\alpha (\partial_\alpha - i\theta^\beta \partial_{\alpha\beta}) \Phi(x, \theta) \tag{2.19}$$

or

$$\delta\Phi(x, \theta) = \delta\phi(x) + \theta^\alpha \delta\psi_\alpha(x) - \theta^2 \delta F(x). \tag{2.20}$$

Now equating powers of θ:

$$\delta\phi(x) = \epsilon^{\alpha}\psi_{\alpha}(x) \quad (2.21)$$
$$\delta\psi_{\alpha}(x) = \epsilon^{\beta}(C_{\alpha\beta}F(x) + i\partial_{\alpha\beta}\phi(x)) \quad (2.22)$$
$$\delta F(x) = -i\epsilon^{\alpha}\partial_{\alpha}^{\beta}\psi_{\beta}. \quad (2.23)$$

It is straightforward to verify that this supersymmetric algebra in components closes:

$$[\delta(\epsilon), \delta(\eta)] = -2i\epsilon^{\alpha}\eta^{\beta}\partial_{\alpha\beta} \quad (2.24)$$

or supersymmetric transformations are the "square root" of the space-time translations. The question now is: how to construct supersymmetric invariant actions? And the answer in $2+1$ and $1+1$ dimensions and $N = 1$ supersymmetry is simple : Everything constructed in terms of superfields is supersymmetric!

Let us analyze general actions in 3 dimensions:

$$S = \int d^3xd^2\theta f(\Phi, D_{\alpha}\Phi, \partial_{\mu}\phi, ...) \quad (2.25)$$

In superfield formalism a supersymmetric actions is obtained by integrating over all the Grassmann space. The action (2.25) is invariant under SUSY transformations if:

$$\delta S = \int d^3xd^2\theta\delta f(\Phi, D_{\alpha}\Phi, \partial_{\mu}\phi, ...) = \int d^3x\partial_{\mu}J^{\mu} \quad (2.26)$$

i.e. if the variation of the integrand is a 3-divergence. But if we remember the form of the supersymmetric charge:

$$Q_{\alpha} = i\frac{\partial}{\partial\theta^{\alpha}} + \theta^{\beta}\partial_{\beta\alpha} \quad (2.27)$$

then:

$$\delta S = \int d^3xd^2\theta\epsilon^{\alpha}(i\frac{\partial}{\partial\theta^{\alpha}} + \theta^{\beta}\partial_{\beta\alpha})f(\Phi, D_{\alpha}\Phi, \partial_{\mu}\phi, ...) \quad (2.28)$$

and the term corresponding to the derivative w.r.t. θ vanishes after the integration over the grassmann space, while the space-time derivative is itself a boundary term. In order to simplify all the superfield calculations we will use the following property of the integration w.r.t anticommuting variables:

$$\int d^2\theta \Sigma(x,\theta) \equiv D^2 \Sigma(x,\theta)| \tag{2.29}$$

where $D^2 = \frac{1}{2} D^\alpha D_\alpha$ and "$|$" means to set all $\theta' s$ equal to zero after the derivation. Also the following identities will be useful:

$$D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\beta\alpha} D^2 \tag{2.30}$$

$$D^\alpha D_\beta D_\alpha = 0 \tag{2.31}$$

$$\{D^2, D_\alpha\} = 0 \tag{2.32}$$

$$D^2 D_\alpha = i\partial_{\alpha\beta} D^\beta \tag{2.33}$$

$$(D^2)^2 = \Box \tag{2.34}$$

Next, let us fix the gamma matrix conventions. We want to choose a representation where the components of the Majorana spinor are real. This may be achieved by choosing an imaginary, hermitian $\beta \equiv \gamma^0$ and hermitian, real $\alpha_k \equiv \beta\gamma^k$. Concretely, we choose (the σ_i are the Pauli matrices)

$$\beta = \sigma_2\,, \quad \alpha_1 = -\sigma_1\,, \quad \alpha_2 = -\sigma_3 \quad \Rightarrow \quad \gamma^0 = \sigma_2\,, \quad \gamma^1 = i\sigma_3\,, \quad \gamma^2 = -i\sigma_1. \tag{2.35}$$

This choice of gamma matrices enables us to introduce the"barred spinor" notation of [19], [20]. The introduction of a second notation may seem a bit artificial, but it turns out that some calculations (especially the rather lengthy ones of section 4 of chapter 7) are significantly simpler in this second notation. We define the barred spinor

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi^{\mathrm{T}} \gamma^0 \quad \Rightarrow \quad \bar{\psi}_\alpha = \psi_\beta (\sigma_2)_{\beta\alpha}. \tag{2.36}$$

It may be checked easily that the barred spinor is identical to the spinor with upper components in the notation of [18], $\bar{\psi}_\alpha \equiv \psi^\alpha = i(\psi_2, -\psi_1)$, where $\psi_\alpha = (\psi_1, \psi_2)$. The main advantage of the barred spinor notation is that all spinor indices are lower and we may dispense with the spinor metric. There is one possible source of confusion related to the use of two different notations, which we resolve by introducing a further bar. The problem is that the gamma matrices should be objects with two lower indices in the barred spinor notation, whereas they should be objects with one lower and one upper spinor index in the spinor metric notation of [18]. That is to say,

$$(\gamma^\mu\psi)_\alpha \equiv \gamma^\mu{}_\alpha{}^\beta\psi_\beta \equiv \bar{\gamma}^\mu{}_{\alpha\beta}\psi_\beta \tag{2.37}$$

where summation over repeated indices is assumed in both cases. Here,

$$\gamma^0{}_\alpha{}^\beta \equiv \bar{\gamma}^0{}_{\alpha\beta} \equiv (\sigma_2)_{\alpha\beta} \tag{2.38}$$

etc., and obviously $\bar{\gamma}^\mu{}_{\alpha\beta}$ belongs to the barred spinor notation and should not be confused with $\gamma^\mu{}_{\alpha\beta} = \gamma^\mu{}_\alpha{}^\gamma C_{\gamma\beta}$.

In the barred spinor notation, the spinorial expressions assume a simpler and more familiar form, like

$$\bar{\chi}\psi = \bar{\chi}_\alpha\psi_\alpha = \chi^\alpha\psi_\alpha\,, \quad \bar{\psi}\psi = \bar{\psi}_\alpha\psi_\alpha = \psi^\alpha\psi_\alpha = 2\psi^2 \tag{2.39}$$

or

$$\bar{\psi}\partial\!\!\!/\psi \equiv \bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}_\alpha\bar{\gamma}^\mu{}_{\alpha\beta}\partial_\mu\psi_\beta = \psi^\alpha\gamma^\mu{}_\alpha{}^\beta\partial_\mu\psi_\beta. \tag{2.40}$$

In the barred spinor notation, the scalar superfield reads

$$\Phi(x,\theta) = \phi(x) + \bar{\theta}\psi(x) - \frac{1}{2}\bar{\theta}\theta F(x). \tag{2.41}$$

For example, we can obtain a supersymmetric model consisting of one Klein-Gordon plus Dirac field from an action in terms of superfields:

$$S_k = \frac{1}{2}\int d^3x d^2\theta D^\alpha\Phi D_\alpha\Phi = \frac{1}{2}\int d^3x D^2[D^\alpha\Phi D_\alpha\Phi] \tag{2.42}$$

and using the previous identities:

$$S_k = \frac{1}{2}\int d^3x \left(D^2D^\alpha\Phi D_\alpha\Phi + D^\alpha\Phi D^2D_\alpha\Phi + D^\alpha D^\beta\Phi D_\alpha D_\beta\Phi\right) \tag{2.43}$$

or in components (only the bossnic sector):

$$S_k = \frac{1}{2}\int d^3x \left(F^2 - i\psi^\alpha\partial_{\alpha\beta}\psi^\beta + \partial_\mu\phi\partial^\mu\phi\right) \tag{2.44}$$

In this case the equation of motion for auxiliary field F is $F = 0$, and the on-shell action is finally:

$$S_{k,on-shell} = \frac{1}{2}\int d^3x \left(-i\psi^\alpha \partial_{\alpha\beta}\psi^\beta + \partial_\mu\phi\partial^\mu\phi\right) \tag{2.45}$$

What is interesting now is how to include a potential, and in lower dimension and $N = 1$ supersymmetry the solution is trivial: Every function without derivatives works as a potential:

$$S_p = \int d^3x d^2\theta W(\Phi) = \int d^3x (W''(\phi)\psi^2 + W'(\phi)F) \tag{2.46}$$

Now adding (2.43) and (2.46) and eliminating the auxiliary field:

$$S_k + S_p = \frac{1}{2}\int d^3x \left(-i\psi^\alpha \partial_{\alpha\beta}\psi^\beta + \partial_\mu\phi\partial^\mu\phi - W'(\phi)^2 + W''(\phi)\psi^2\right) \tag{2.47}$$

i.e. we have obtained the standard kinetic terms for bosons and fermions plus a semidefinite positive potential and a coupling between bosonic and fermionic fileds. Having in mind the previous example, it seems that to generate a supersymmetric action it is enough to replace the scalar field in the bosonic action with a superfield and a space-time derivative with a superderivative.

$$\partial_\mu\phi\partial^\mu\phi \longrightarrow D^\alpha\Phi D_\alpha\Phi \tag{2.48}$$

But unfortunately this is not true in general. Supersymmetric higher derivative terms are another story. For example, if we try to generate a quartic term in derivatives with this rule:

$$(\partial_\mu\phi\partial^\mu\phi)^2 \longrightarrow (D^\alpha\Phi D_\alpha\Phi)^2 \tag{2.49}$$

we obtain:

$$S|_{\psi=0} = \int d^3x d^2\theta (D^\alpha\Phi D_\alpha\Phi)^2|_{\psi=0} = 0 \tag{2.50}$$

i.e. this action has no bosonic sector. In the following chapters we will show how to generate this kind of actions.

2.3 Susy, solitons and Bogomolny in two dimensions

It is well known that supersymmetric algebras are modified by the existence of solitons. Following Witten and Olive [44] we present a naive example where the relation of supersymmetry with soliton states and Bogomolny bounds is shown:

The supersymmetric form of a scalar field theory in two dimensions is:

$$L = \int d^2x[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\bar{\psi}i\partial\!\!\!/\psi - \frac{1}{2}V^2(\phi) - \frac{1}{2}V'(\psi)\bar{\psi}\psi] \tag{2.51}$$

The supercurrent associated with the supersymmetry transformation is:

$$J^\mu = (\partial_\nu\phi)\gamma^\nu\gamma^\mu\psi + iV(\phi)\gamma^\mu\psi \tag{2.52}$$

And from the 0-component of the this current we can write explicitly the supersymmetric charges. In chiral components are:

$$Q_+ = \int dx|(\partial_0\phi + \partial_1\phi)\psi_+ - V(\phi)\psi_-| \tag{2.53}$$

$$Q_- = \int dx|(\partial_0\phi - \partial_1\phi)\psi_- + V(\phi)\psi_+| \tag{2.54}$$

These supercharges satisfy the algebra (2.15) in chiral components, i.e.:

$$\{Q_+, Q_-\} = 0 \tag{2.55}$$

$$Q_+^2 = P_+ \tag{2.56}$$

$$Q_-^2 = P_- \tag{2.57}$$

But considering the first of this relations carefully and keeping the surface terms what we obtain is (from (2.53) and (2.54)):

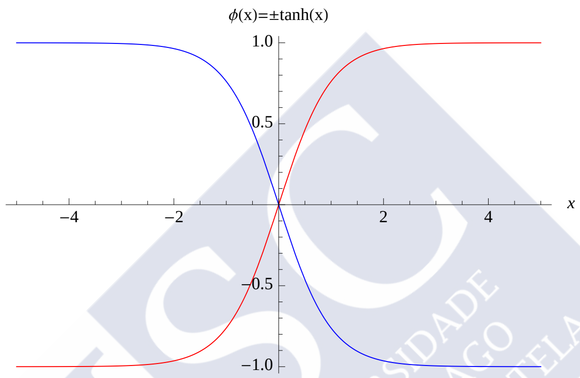
$$\{Q_+, Q_-\} = 2\int dx V(\phi)\frac{\partial\phi}{\partial x} \tag{2.58}$$

or equivalently, taking $W'(\phi) = V(\phi)$:

$$\{Q_+, Q_-\} = 2\int dx \frac{\partial}{\partial x} W(\phi) \tag{2.59}$$

and we obtain an integral of a divergence that must vanish, but in a non trivial solionic state this is not neccesarly zero.

Let V be the corresponding function for $\lambda\phi^4$ theory, $V(\phi) = -\lambda(\phi^2 - 1)$. Then the potential energy is $\lambda^2(\phi^2-1)^2$. This theory has two ground states corresponding to $\phi = \pm 1$. $W(\phi) = \lambda\phi - \frac{1}{3}\lambda\phi^3$. The bosonic non-trivial solitons for this theory are :



and therefore:

$$T := \{Q_+, Q_-\} = \int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}(2\lambda\phi - \frac{2}{3}\lambda\phi^3) = \frac{8}{3}\lambda \tag{2.60}$$

and what we see is that this quantity T depends basically on the topological structure of the vacuum and will be related to the usual topological charge. Moreover, the correct supersymmetric algebra is now:

$$\{Q_+, Q_-\} = T \tag{2.61}$$
$$Q_+^2 = P_+ \tag{2.62}$$
$$Q_-^2 = P_- \tag{2.63}$$

From this relations it is easy to find:

$$P_+ + P_- = T + (Q_- - Q_-)^2 \tag{2.64}$$
$$P_+ + P_- = -T + (Q_+ - Q_-)^2 \tag{2.65}$$

but $(Q_\pm - Q_-)^2 \geq 0$, so $P_+ + P_- \geq T$. If we think in a particle at rest of mass M then $P_+ = P_- = M$, then eqns. (2.65) implies:

$$M \geq \frac{1}{2}|T| \tag{2.66}$$

And now we can ask: When is (2.66) saturated? To answer this question we can analyze first the energy of the model:

$$\begin{aligned} E &= \frac{1}{2}\int[(\frac{\partial\phi}{\partial t})^2 + (\frac{\partial\phi}{\partial x})^2 + V(\phi)^2]dx = && (2.67)\\ &= \frac{1}{2}\int[(\frac{\partial\phi}{\partial t})^2 + (\frac{\partial\phi}{\partial x} \mp V(\phi))^2]dx \pm \int V(\phi)\frac{\partial\phi}{\partial x}dx \geq && (2.68)\\ &\geq |\int V(\phi)\frac{\partial\phi}{\partial x}dx| && (2.69)\end{aligned}$$

the inequality for E turns into an equality iff:

$$\frac{\partial\phi}{\partial t} = 0 \quad , \quad \frac{\partial\phi}{\partial x} = \pm V(\phi) \tag{2.70}$$

Now if we go back to supersymmetry, the condition for a state $|\alpha\rangle$ to saturate (2.66) is $(Q_+ + Q_-)|\alpha\rangle = 0$ or $(Q_+ - Q_-)|\alpha\rangle = 0$. But this condition is automatically satisfied (taking into account (2.54)) if:

$$\frac{\partial\phi}{\partial t} = 0 \quad , \quad \frac{\partial\phi}{\partial x} = \pm V(\phi) \tag{2.71}$$

the same condition we obtained before! Generalizing this result we can conclude that supersymmetry provides a systematic way to obtain Bogomol'nyi solutions. The strategy in principle seems to be simple:

1. Supersymmetric extension of the corresponding bosonic model.
2. Calculation of supercharges.
3. Try to find a solution which annihilates certain combination of supercharges.

Chapter 3

SUSY N=2 d=1+1 and d=2+1

3.1 Introduction

In this section we will study extended supersymmetry in 2−dimensional space-time following the conventions of [174]. Of course it is interesting by itself, but, moreover, it constitutes the dimensional reduction from $N = 1$, $d = 3+1$ to $N = 2$, $d = 1+1$ and therefore we can say that we can translate the results in this dimension to $3+1$ space-time with one supersymmetry. In this chapter we present $N = 2$ SUSY in $1+1$ dimensions, but, due tot he similarity of the spinor representation, it is formally equivalent to $N = 2$ in $2+1$ dimensions. In chapter 13 we will fix the notation for this dimension. Note that, in this case, the Grassmann space has twice the number of Grassmann variables (4 in this case):

$$\theta^+, \quad \theta^-, \quad \bar{\theta}^+, \quad \bar{\theta}^- \tag{3.1}$$

satisfying the usual anti-commuting algebra. A general superfield is a function defined in superspace:

$$\begin{aligned} G(x^0, x^1, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-) &= g_0(x^0, x^1) + \theta^+ g_+(x^0, x^1) + && (3.2) \\ &+ \theta^- g_-(x^0, x^1) + \bar{\theta}^+ \bar{g}_+(x^0, x^1) + ... && (3.3) \\ &+ \theta^+ \theta^- g_{+-}(x^0, x^1) + ... && (3.4) \end{aligned}$$

Note that a general superfield has 16 terms. In analogy with $N = 1$ supersymmetry we introduce the the supersymmetry generators:

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm \tag{3.5}$$

$$\bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm \tag{3.6}$$

Where $\partial_\pm$ are the derivatives in light-cone coordinates:

$$\partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right) \tag{3.7}$$

As usual we introduce the set of superderivatives:

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm \tag{3.8}$$

$$\bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm \tag{3.9}$$

satisfying the following anticommutation relations:

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm \tag{3.10}$$

$$\{D_\pm, \bar{D}_\pm\} = 2i\partial_\pm. \tag{3.11}$$

For centrally extended $N = 2$ superalgebras we have:

$$\{Q^L_\alpha, Q^M_\beta\} = -2i\delta^{LM}\partial_{\alpha\beta} + T\epsilon^{LM}\epsilon_{\alpha\beta} \tag{3.12}$$

(with $L, M = 1, 2$). Due to the dimension of the Grassmann space, it will be a little bit more difficult to construct supersymmetric actions. First of all, we need to constrain our general superfields, so we define a superfield Φ satisfying the following equation,

$$\bar{D}_\pm\Phi = 0 \tag{3.13}$$

which is called *chiral superfield*. The complex conjugate of the previous equation is

$$D_{\pm}\bar{\Phi} = 0 \tag{3.14}$$

and this superfield $\bar{\Phi}$ is called *anti-chiral superfield.* We can also define superfields Σ with twisted conditions, for example:

$$\bar{D}_{+}\Sigma = D_{-}\Sigma = 0 \tag{3.15}$$

called twisted chiral superfield, but we will not use it.

3.2 Supersymmetric actions

We will repeat the same formalism as before but taking into account the new Grassmann space. From the form of supercharges it is obvious that any action constructed in terms of general superfields and superderivatives and integrated over all the Grassmann space is supersymmetric, i.e.:

$$S = \int d^2x d^4\theta H(\Phi, \bar{\Phi}, D_{\pm}\Phi ...) = \int d^2x d\theta^+ d\theta^- d\bar{\theta}^+ \bar{\theta}^- H(\Phi, \bar{\Phi}, D_{\pm}\Phi ...). \tag{3.16}$$

We see that

$$\delta S = \int d^2x \partial_\mu F^\mu \tag{3.17}$$

where δ is the supersymmetric transformation

$$\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+. \tag{3.18}$$

Taking into account that the first part of $Q's$ is a derivative w.r.t. θ, the contribution of this terms after integration over $\theta's$ gives zero and the second term of $Q's$ is directly a derivative, this shows (3.17). As we will see later, this kind of integrals (*D-terms*) give us kinetic terms, so we need another integration in order to generate potentials. Suppose that we only integrate in half the Grassmann space, for example:

$$S_P = \int d^2x d^2\theta W(\Phi) = \int d^2x d\theta^- d\theta^+ W(\Phi)|_{\bar{\theta}^\pm = 0} \tag{3.19}$$

where Φ is a chiral superfield. This kind of terms are called *F-terms*. Let's check that this action is supersymmetric. First of all, we will restrict to the component ϵ_+ of δ (the calculation is the same for ϵ_-):

$$\delta|_{\epsilon_+} S_P = \int d^2x d\theta^- d\theta^+ \left(\frac{\partial}{\partial \theta^-} + i\bar{\theta}^- \partial_- \right) W(\Phi) \tag{3.20}$$

The second term vanishes because we put $\bar{\theta}^\pm = 0$ and the first vanishes after θ-integration. But if we look now at the term involving $\bar{\epsilon}_+$:

$$\delta|_{\bar{\epsilon}_+} S_P = \int d^2x d\theta^- d\theta^+ \left(-\frac{\partial}{\partial \bar{\theta}^-} - i\theta^- \partial_- \right) W(\Phi) \tag{3.21}$$

in principle we can not guarantee that it is a total derivative, but using the following relation

$$\bar{Q}_\pm = D_\pm - 2i\theta^\pm \partial_\pm, \tag{3.22}$$

we obtain

$$\delta|_{\bar{\epsilon}_+} S_P = \int d^2x d\theta^- d\theta^+ \bar{\epsilon}_+ \left(\bar{D}_- - 2i\theta^- \partial_- \right) W(\Phi). \tag{3.23}$$

Now it is clear that the second term is a total derivative in x^μ and the first term is zero because Φ is chiral,

$$\bar{D}_- W(\Phi) = W'(\Phi) \bar{D}_- \Phi = 0 \tag{3.24}$$

since $\bar{D}_- \Phi = 0$. We have seen that, in order to construct susy F-terms (which generate the potentials) we have to use chiral superfields integrating over chiral anti-commuting variables ($\theta^\pm$) or anti-chiral superfields integrating over anti-chiral anti-commuting coordinates ($\bar{\theta}^\pm$).

The way to generate supersymmetric actions is different from the one with $N = 1$, for example, to generate a standard action we need a D-term of the form:

$$S_D = \int d^2x d^4\theta \bar{\Phi} \Phi \tag{3.25}$$

being Φ and $\bar{\Phi}$ chiral and antichiral superfields. After the expansion of the product we need only to integrate in $\theta's$:

$$S_D = \int d^2x \left(\partial_\mu \bar{\phi} \partial^\mu \phi + i\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + i\bar{\psi}_+(\partial_0 - \partial_1)\psi_+ + \bar{F}F\right) \quad (3.26)$$

Performing the same calculation for the F-term with superpotential W we get

$$S_F = \int d^2x \left(W'F + \bar{W}'\bar{F} - W''\psi_+\psi_- - \bar{W}''\bar{\psi}_+\bar{\psi}_-\right). \quad (3.27)$$

We can add these actions, and after the elimination of the auxiliary field we obtain

$$\begin{aligned} S_D + S_F &= \int d^2x(\partial_\mu \bar{\phi} \partial^\mu \phi + i\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + \\ &+ \bar{\psi}_+(\partial_0 - \partial_1)\psi_+ - W''\psi_+\psi_- - \bar{W}''\bar{\psi}_+\bar{\psi}_- - |W'|^2). \quad (3.28) \end{aligned}$$

We observe in this actions small but fundamental differences w.r.t the $N = 1$ action. First of all, the kinetic part has been built in terms of a real combination of complex superfields (this is a Kähler potential) and this condition constrains the possible $N = 2$ supersymmetric actions. The second part is that the superpotential is coming from the F-term which is a sum of holomorphic plus antiholomorphic functions of the superfields, and this fact constrains again the possibilities. It is always possible to reduce an $N = 2$ model to $N = 1$ by restriction of the superspace. Obviously the other way is not true in general, but only under certain conditions: if we accomodate Majorana spinors of a $N = 1$ scalar multiple into complex spinors and there exists a $U(1)$ symmetry under the fermion rotation $\psi \to e^{i\alpha}\psi$ it is possible to accomodate $N = 1$ supermultiples in $N = 2$ supermultiplets.

3.3 Gauge invariant $N = 2$ actions

We can think on the lagrangian

$$L = \int d^4\theta \bar{\Phi}\Phi. \quad (3.29)$$

If we generalize the usual phase rotation $\phi \to e^{i\alpha}\phi$ which leads to gauge invariance, in terms of superfields we have the natural generalization $\Phi \to e^{iA}\Phi$, being A a chiral superfield, which sends chiral fields to chiral fields. The integrand of (3.29) is not invariant under such a transformation:

$$\bar{\Phi}\Phi \to \bar{\Phi}e^{-i\bar{A}+iA}\Phi, \tag{3.30}$$

but if we introduce a real superfield V that transforms as

$$V \to V + i(\bar{A} - A) \tag{3.31}$$

when

$$\Phi \to e^{iA}\Phi \tag{3.32}$$

then a gauge invariant lagragian under transformations (3.31) and (3.32) can be written as

$$L = \int d^4\theta\bar{\Phi}e^V\Phi. \tag{3.33}$$

The real superfield in the Wess-Zumino gauge (equivalent to $V^3 = 0$) is expessed in the form:

$$\begin{aligned} V &= \theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0 + v_1) + i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) \\ &+ i\theta^-\theta^+(\theta^-\lambda_- + \theta^+\lambda_+) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} + \theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D \end{aligned} \tag{3.34}$$

where v_μ is the gauge field, σ is a complex field, λ is a Dirac fermions and D is a real auxiliary field. If we have chiral superfields coupled to vector superfields, the supersymmetric transformations of the chiral fields are modified because of the gauge symmetry, in the present case, the supersymmetric transformation for $\Phi = (\phi, \psi, F)$ and $V = (v_\mu, \lambda, \sigma, D)$ are:

$$\begin{aligned}
\delta\phi &= i\bar{\varepsilon}_\pm\lambda_\pm + i\varepsilon_\pm\bar{\lambda}_\pm && (3.35)\\
\delta\psi_+ &= i\bar{\varepsilon}_-(D_0+D_1)\phi + \varepsilon_+ F\varepsilon_+\bar{\sigma}\phi && (3.36)\\
\delta\psi_- &= -i\varepsilon + -(D_0-D_1)\phi + \varepsilon_- F + \bar{\varepsilon}_-\sigma\phi && (3.37)\\
\delta F &= -i\varepsilon + -(D_0-D_1)\psi_+ - i\bar{\varepsilon}_-(D_0+D_1)\psi_- + && (3.38)\\
&+ \varepsilon_+\bar{\sigma}\psi_- + \bar{\varepsilon}_-\sigma\psi_+ + i(\bar{\varepsilon}_-\bar{\lambda}_+ - \bar{\varepsilon}_+\bar{\lambda}_-)\phi\\
\delta v_\pm &= i\bar{\varepsilon}_\pm\lambda_\pm + i\varepsilon_\pm\bar{\lambda}_\pm && (3.39)\\
\delta\sigma &= i\bar{\varepsilon}_+\lambda_- - i\varepsilon_-\bar{\lambda}_+ && (3.40)\\
\delta D &= -\bar{\varepsilon}_+\partial_-\lambda_+ - \bar{\varepsilon}_-\partial_+\lambda_- + \varepsilon_+\partial_-\bar{\lambda}_+ + \varepsilon_-\partial_+\bar{\lambda}_- && (3.41)\\
\delta\lambda_+ &= i\varepsilon_+(D+iv_{\mu\nu}) + 2\varepsilon_-\partial_+\bar{\sigma} && (3.42)\\
\delta\lambda_- &= i\varepsilon_-(D-iv_{\mu\nu}) + 2\varepsilon_+\partial_-\sigma && (3.43)
\end{aligned}$$

with $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. This supersymmetric transformation will lead us to BPS equations of the gauged bSkM in chapter 13. Again the $N=2$ and gauge invariant lagrangians are constructed like always, for example, if we gauge the lagrangian (3.29) we obtain:

$$\begin{aligned}
L_{k,gauged} &= \int d^4\theta\bar{\Phi}e^V\Phi = && (3.44)\\
&= -D^\mu\bar{\phi}D_\mu\phi + i\bar{\psi}_-(D_0+D_1)\psi_- + i\bar{\psi}_+(D_0-D_1)\psi_+ +\\
&+ D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 - \bar{\psi}_-\sigma\psi_+ - \bar{\psi}_+\bar{\sigma}\psi_- - i\bar{\phi}\lambda_-\psi_+ +\\
&+ i\bar{\phi}\lambda_+\psi_- + i\bar{\psi}_+\bar{\lambda}_-\phi - i\bar{\psi}_-\bar{\lambda}_+\phi
\end{aligned}$$

The super Yang-Mills lagragian in 1+1 dimensions is constructed in terms of the superfield strength $\Sigma := \bar{D}_+D_-V$ as:

$$\begin{aligned}
L_{YM} &= -\frac{1}{2e^2}\int d^4\theta\bar{\Sigma}\Sigma = && (3.45)\\
&= \frac{1}{2e^2}\left(-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- + i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ + v_{01}^2 + D^2\right)
\end{aligned}$$

(e^2 is the gauge coupling). In chapter 13 we will use the generalization of (3.44) with a general Kähler potential which will allow us to construct the $N=2$ bSkM.

Chapter 4

SUSY N=1 d=3+1

4.1 Introduction

Although we will not use explicitly supersymmetry in $3+1$ dimensions along this thesis, we include here a brief introduction that will allow us to see the parallelism between this and $N=2$ supersymmetry in $1+1$ or $2+1$ dimensions. This connection via dimensional reduction ensures, in particular, that our results in $2+1$ dimensions and extended supersymmetry can be extended to the ordinary space. We follow for this chapter [160] and [161].

4.2 Superspace

We first define the superspace coordinates:

$$z = (x^{\mu}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}) \tag{4.1}$$

where x^{μ} are ordinary space-time coordinates and θ and $\bar{\theta}$ are two component Grassmann variables which transforms as Weyl spinors. We define susy transformation in superspace:

$$(x^{\mu}, \theta, \bar{\theta}) \rightarrow (x^{\mu} + i\theta\sigma^{\mu}\bar{\xi} - i\xi\sigma^{\mu}\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) \tag{4.2}$$

In this relation ξ and $\bar{\xi}$ are Weyl spinors and describe translations in superspace for the Grassmann coordinates. We can write this transformation in terms of the generators as:

$$\delta_\xi = \xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha} \tag{4.3}$$

with:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \tag{4.4}$$

$$\bar{Q}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{4.5}$$

we have the algebra:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{4.6}$$

in the general expression for extended SUSY in 3 + 1-dimensions the central charges are included: $\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2i\delta^{IJ}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu + \delta_{\alpha\beta}U^{IJ} + (\gamma_5)_{\alpha\beta}V^{IJ}$, where $U^{IJ} = -U^{JI}$ and $V^{IJ} = -V^{JI}$ are the central charges.

4.3 Superfields

A general superfield can be expanded in components like:

$$\begin{aligned} F(x,\theta,\bar\theta) &= f(x) + \theta\psi(x) + \bar\theta\bar\psi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \\ &+ \theta\sigma^\mu\bar\theta v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\chi(x) + \\ &+ \theta\theta\bar\theta\bar\theta d(x) \end{aligned} \tag{4.7}$$

We define again the set of superderivates which allows us to define chiral and anti-chiral superfields:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \tag{4.8}$$

$$\bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{4.9}$$

A superfield Φ satisfying the condition $\bar{D}\Phi = 0$ is called *chiral cuperfield* while a superfield $\Phi^\dagger$ satisfying the condition $D\Phi^\dagger = 0$ is called *anti-chiral.*

To obtain explicitly the components of chiral and antichiral superfields we can act with D_α and $\bar{D}_{\dot{\alpha}}$ in (4.8) and solve the equations

$$D_\alpha F(x,\theta,\bar{\theta}) = 0 \tag{4.10}$$
$$\bar{D}_{\dot{\alpha}} F(x,\theta,\bar{\theta}) = 0 \tag{4.11}$$

but this is a little bit tedious. To write the explicit form in components we introduce the chiral coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{4.12}$$

which satisfies the condition $\bar{D}_{\dot{\alpha}} y^\mu = 0$. If the superfield Φ is written in terms of this coordinate the chirality condition is automatically satisfied,

$$\bar{D}_{\dot{\alpha}}\Phi(y,\theta) = 0 \tag{4.13}$$

We may now write

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + (\theta\theta)F(y) \tag{4.14}$$

and reexpanding again we obtain

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A(x) + \\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta}\sigma^\mu\bar{\theta} + (\theta\theta)F(x).\end{aligned} \tag{4.15}$$

We can impose other conditions on the superfields to obtain different constraint superfields. For example, to obtain a *vector superfield* $V(x,\theta,\bar{\theta})$ we impose the reality condition $V(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta})^\dagger$. This new superfield can be written in components as:

$$\begin{aligned}V(x,\theta,\bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}(\theta\theta)(M(x)+iN(x)) - \\ &- \frac{i}{2}(\bar{\theta}\bar{\theta})(M(x)-iN(x)) - \theta\sigma^\mu\bar{\theta}v_\mu(x) + i(\theta\theta)\bar{\theta}[\bar{\lambda}(x) + \\ &+ \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)] - i(\bar{\theta}\bar{\theta})\theta[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}] + \\ &+ \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D(x) + \frac{1}{2}\Box C(x)).\end{aligned} \tag{4.16}$$

Here the bosonic components C, D, M, N and v_μ are real. One can simplify this superfield through some "covariant" constraint. We will see that the superfield transformation

$$V(z) \to V(z) + \Lambda(z) + \lambda^\dagger(z) \tag{4.17}$$

is the supersymmetric version of an abelian gauge transformation (here $\Lambda(z)$ and $\Lambda^\dagger(z)$ are chiral and antichiral superfields, respectively, with components $(A(x), \psi(x), F(x))$). The gauge transformations of the component fields of $V(x, \theta, \bar{\theta})$ are given as:

$$\begin{aligned}
C(x) &\to C(x) + A(x) + A(x)^\star && (4.18)\\
\chi(x) &\to \chi(x) - i\sqrt{2}\psi(x) && (4.19)\\
M(x) + iN(x) &\to M(x) + iN(x) - 2iF(x) && (4.20)\\
v_\mu(x) &\to v_\mu(x) - i\partial_\mu(A(x) + A(x)^\star) && (4.21)\\
\lambda(x) &\to \lambda(x) && (4.22)\\
D(x) &\to D(x) && (4.23)
\end{aligned}$$

For a special gauge transformation

$$\begin{aligned}
C(x) &= -(A(x) + A(x)^\star) && (4.24)\\
\chi(x) &= i\sqrt{2}\psi(x) && (4.25)\\
M(x) + iN(x) &= 2iF(x), && (4.26)
\end{aligned}$$

and the vector superfield is reduced to

$$\begin{aligned}
V(x, \theta, \bar{\theta}) &= -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i(\theta\theta)\bar{\theta}\bar{\lambda}(x) - i(\bar{\theta}\bar{\theta})\theta\lambda(x) + && (4.27)\\
&+ \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x). && (4.28)
\end{aligned}$$

This special gauge (4.26) is known as the *Wess-Zumino gauge*. And, what about supersymmetric gauge actions? The procedure is the same as in the previous chapter,

$$S_k = \int d^4xd^4\theta\bar{\Phi}e^V\Phi \tag{4.29}$$

which in terms of the component fields reads

$$\begin{aligned} S_k &= \int d^4x[D^\mu\bar{\phi}D_\mu\phi + i\psi\sigma^\mu D_\mu\bar{\psi} + \bar{F}F + \\ &+ D\bar{\phi}\phi + \sqrt{2}\lambda\psi\bar{\phi} + \sqrt{2}] \end{aligned} \tag{4.30}$$

where D_μ is the covariant derivative. Defining $W_\alpha = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}D_\alpha V$ the super Yang-Mills action can be written as an F-term,

$$\begin{aligned} S_{YM} &= \int d^4xd^2\theta W^\alpha W_\alpha + h.c. = \\ &= \int d^4x[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2] \end{aligned} \tag{4.31}$$

where $F_{\mu\nu} = \partial_\mu\partial_\nu - \partial_\nu\partial_\nu$ is the field strength for the abelian gauge field. We see what happens in $1+1$ dimensions and extended supersymmetry, the actions there are essentially equal except because two components of the gauge field (in $3+1$) are absorbed in the complex scalar field σ. In $2+1$ dimension the only difference is that σ is real. It is because of this reason that we say that $d = 3{+}1$, $N = 1$ and $d = 2{+}1$, $N = 1$ are almost equivalent.

Chapter 5

Geometry and Supersymmetry

5.1 Introduction

In this chapter we will present briefly the relation between supersymmetry and geometry. We use the results of *Álvarez-Gaumé* and *Freedman* [68] and also [69] and [70], to discuss the relation between the complex structure of the target space manifold for *bosonic non-linear σ-model* and the number of supersymmetries which this model can allow. We start with 2-dimensional σ-models.

5.2 Relation between complex geometry and SUSY

Given a n-dimensional Riemannian manifold with metric $g_{ij}(\Phi^k)$ one can define a supersymmetric σ-model with $N = 1$ supersymmetry. The superfield action is

$$S[\Phi] = \frac{1}{4i} \int d^2x d^2\theta g_{ij}(\Phi^k) \bar{D}\Phi^i D\Phi^j, \tag{5.1}$$

where Φ^k is a real scalar superfield,

$$\Phi^k(x,\theta) = \phi^k(x) + \bar{\theta}\psi^k(x) + \frac{1}{2}\bar{\theta}\theta F^k(x). \tag{5.2}$$

After integration in θ and the elimination of the auxiliary field, we obtain, in terms of physical fields,

$$S[\phi,\psi]=\frac{1}{2}\int d^2x\{g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j+ig_{ij}(\phi)\bar{\psi}^i\gamma^\mu D_\mu\psi^j+\frac{1}{6}R_{iklj}(\bar{\psi}^i\psi^l)(\bar{\psi}^k\psi^j)\} \tag{5.3}$$

with the covariant derivative $D_\mu\psi^k=\partial_\mu\psi^k+\Gamma^k_{ji}\partial_\mu\phi^j\psi^k$. The action is invariant under the following supersymmetric transformations,

$$\delta\phi^k = \bar{\varepsilon}\psi^k \tag{5.4}$$
$$\delta\psi^k = -i\partial\!\!\!/\phi^k\varepsilon-\Gamma^k_{ji}(\bar{\varepsilon}\psi^j)\psi^i \tag{5.5}$$

We want to study the possibility of additional supersymmetric invariances of the action (5.3). First of all, we know that the action is invariant under (5.5) and also under reparametrizations of the target manifold M:

$$\phi'^k = \phi'^k(\phi) \tag{5.6}$$
$$\psi'^k = \frac{\partial\phi'^k}{\partial\phi'^j}\psi^j \tag{5.7}$$

It can be checked that the most general Ansatz for SUSY transformation rules which is consistent with dimensional arguments, and Lorentz and parity invariance is

$$\delta\phi^k = f^k_j\bar{\varepsilon}\psi^j \tag{5.8}$$
$$\delta\psi^k = -ih^k_j\partial\!\!\!/\psi^j\varepsilon-S^k_{ji}(\bar{\varepsilon}\psi^j)\psi^i- \tag{5.9}$$
$$- V^k_{ji}(\bar{\varepsilon}\gamma^\mu\psi^j)\gamma_\mu\psi^i-P^k_{ji}(\bar{\varepsilon}\gamma_5\psi^j)\gamma_5\psi^i \tag{5.10}$$

Commutations with diffeomorphisms implies that f, h, V and P are tensors. We require that the action (5.3) be stationary under the variations (5.10). Absence of linear term in ψ of δS requires the conditions

$$g_{ik}f^k_j = g_{jk}h^k_i \tag{5.11}$$
$$\nabla_k f^i_j = 0, \tag{5.12}$$

then f_j^k is covariantly constant. Let us suppose now the general rules (5.10) obey the supersymmetric algebra

$$\{Q^a, \bar{Q}^b\} = 2\delta^{ab}\not{P}, \tag{5.13}$$

which implies

$$f_j^i h_k^j = \delta_k^i, \tag{5.14}$$

and these relations plus the previous one between h and f allow us to write

$$g_{ij} f_k^i f_l^j = g_{kl}. \tag{5.15}$$

We now assume that there are several supersymmetries with covariantly constant tensors $f_j^{(a)i}$. Then (5.13) implies

$$f^{(a)} f^{(b)-1} + f^{(b)} f^{(a)-1} = 2\delta^{ab}. \tag{5.16}$$

Assuming that one of these transformations is the original ($f_j^{(0)i} = \delta_j^i$) and $b = 0$ we have

$$f_k^{(a)i} f_j^{(a)k} = -\delta_j^i. \tag{5.17}$$

We collect these properties for the tensor f,

$$\nabla_k f_j^i = 0 \tag{5.18}$$
$$g_{ij} f_k^i f_l^j = g_{kl} \tag{5.19}$$
$$f_k^i f_j^k = -\delta_j^i. \tag{5.20}$$

The third one implies that the dimension of M is even. From the three relations it follows that M can be covered smoothly with complex coordinate charts $(z^\alpha, z^{\bar{\alpha}})$ such that transition functions in overlapping coordinate patches are holomorphic. In complex coordinates the line element ds^2 can be written as:

$$ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha dz^{\bar{\beta}}; \tag{5.21}$$

the two form

$$F = ig_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \quad (5.22)$$

is closed which implies locally

$$g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} K(z, \bar{z}), \quad (5.23)$$

where $K(z, \bar{z})$ is the Kähler potential. Thus a supersymmetric σ-model on a Riemann manifold M admits a second supersymmetry if and only if M is Kähler. It is possible to extend these results to more dimensions or more supersymmetry. In the following table we summarize some of these results:

dimension	SUSY	manifold
d=2	N=1	no restriction on M
	N=2	M is Kähler
	N=4	M is hyper-Kähler
d=4	N=1	M is Kähler
	N=2	M is hyper-Kähler
	N=4	No extension exists
		(bosonic sector with spin 1) .

We emphasize that these geometric constraints hold for SUSY extensions of σ-models, the inclusion of gauge fields changes the geometry in general. This relationship between supersymmetry and geometry will be important in Chapter 13.

Chapter 6

The Skyrme model

Let us analyze first the *sine-Gordon* model in order to suggest a kind of generalization to the Skyrme model. To do this, we need two components ϕ_0 and ϕ_1 which are functions on the $(1+1)$-dimensional space-time with the constraint

$$\phi_0^2 + \phi_1^2 = 1, \tag{6.1}$$

then it is possible to fulfill this constraint by setting

$$\phi_0 = \cos\alpha(x,t) \quad , \quad \phi_1 = \sin\alpha(x,t). \tag{6.2}$$

Now substituting this explicit form for the fields in the original Lagrangian proposed by Skyrme to describe nucleon fields interacting with pseudo scalar meson field,

$$\mathcal{L} = \frac{1}{2}\sum_{\rho}[(\partial_\mu\phi_\rho)^2 + \frac{1}{2}k^2\phi_\rho^4] + \bar{\psi}[i\gamma^\mu\partial_\mu + g(\phi_0 + i\gamma_5\tau\cdot\phi)]\psi \tag{6.3}$$

we obtain the lagrangian of the sine-Gordon model,

$$\mathcal{L}_{SG} = \frac{1}{2}[(\partial_t\alpha)^2 - (\partial_x\alpha)^2] - k^2(1-\cos\alpha) \tag{6.4}$$

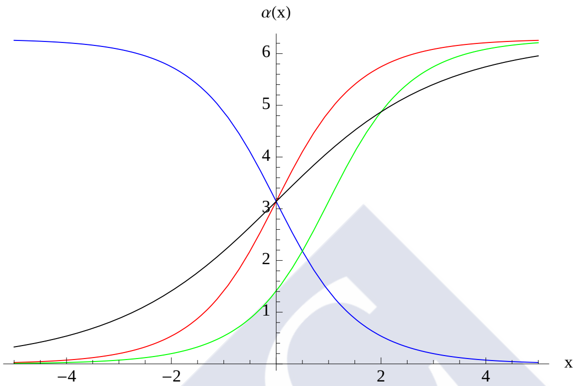
with the corresponding Euler-Lagrange equation

$$\partial_x^2\alpha - \partial_t^2 - k^2\sin\alpha = 0 \tag{6.5}$$

with solutions of the form (2π-solitons)

$$\alpha(x) = 4\tan^{-1}[\exp\{\pm k(x - x_0)\}] \tag{6.6}$$

with the following profile



There are solutions interpolating all different neighboring values of the vacuum.

6.0.1 The topological charge

All solutions of the model satisfy the following boundary condition

$$\alpha(x) \to 0(mod 2\pi) \quad as \quad |x| \to \infty. \tag{6.7}$$

In this family of solutions satisfying these boundary conditions we can distinguish between solutions where $\alpha(x)$ takes the zero value at both boudaries $x = \pm\infty$, on the other hand we have $\alpha(-\infty) = 0$ and $\alpha(\infty) = 2\pi$. Those solutions are not transformable into each other by any continuous transformation, hence the space of solutions with the boundary condition (6.7) is split into distinct connected components. Since a continuous deformation could be regarded as an evolution of the classical system, we can assign to these solutions a'characteristic" which does not change its value under time evolution. Now taking into account the existence of the conserved current

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\alpha \tag{6.8}$$

such that $\partial_\mu J^\mu = 0$ independent of the equations of motion, we can define the topological charge

$$Q = \int_{-\infty}^{\infty} dx J^0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \frac{\partial \alpha}{\partial x} = \frac{1}{2\pi}]\alpha(+\infty) - \alpha(-\infty)] \tag{6.9}$$

which is exactly the quantity (winding number, Chern-Pontryagin number...) which labels the different sectors where solutions live. If we write our fields in the form $\phi = \phi_0 + i\phi_1$ or $\phi(x) = \exp(i\alpha(x))$, the constraint (6.1) can be written as

$$\phi(x)\phi(x)^\star = 1 \tag{6.10}$$

and the integrand of the second term in (6.9) as

$$\frac{\partial \alpha}{\partial x} = -i\phi^\star \frac{\partial \phi}{\partial x} \tag{6.11}$$

such that $\phi^\star = \phi^{-1}$. We can try to construct an analog of this kind of models in (3+1) dimensions. The boundary condition at infinity compactifies $\mathbb{R}^3$ onto the three sphere $\mathbb{S}^3$, which is topologycally isomorphic to $SU(2)$. Hence we can use a quarternionic representation of $SU(2)$. Our fields now are:

$$U(x,t) = \phi_0(x,t) + i\tau \cdot \phi(x,t) \tag{6.12}$$

where $x \in \mathbb{R}^3$ and τ are the Pauli matrices. The constraint (6.1) in terms of U is rewritten as

$$U \cdot U^\dagger = \mathbb{1}. \tag{6.13}$$

To ensure the compactification $\mathbb{R}^2 \cup \infty \backsimeq \mathbb{S}^3$ the field satisfies

$$U(x) \to \mathbb{1} \quad as \quad |x| \to \infty. \tag{6.14}$$

We can write now a straightforward generalization on the condition (6.11) in terms of the quaternionic field

$$R_\mu = U^{-1}\partial_\mu U = i\tau_a R^a_\mu, \tag{6.15}$$

this field must satisfy the Maurer-Cartan structural equations of the zero curvature conditions,

$$\partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0. \tag{6.16}$$

These conditions are necessary and sufficient conditions for the reconstruction of the field U in terms of BR_μ. At this point we can write down the lagrangian proposed by Skyrme as a low energy effective theory for QCD, becoming exact as the number of quark colors becomes large,

$$L = \frac{\varepsilon}{4\pi^2}\{\kappa^2 R^a_\mu R^a_\mu - \frac{1}{2}[(R^a_\mu R^a_\mu)^2 - (R^a_\mu R^a_\nu)^2]\}, \tag{6.17}$$

or in terms of the quaternionic field,

$$L = \frac{F_\pi^2}{16} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{32e^2} Tr[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2. \tag{6.18}$$

This is the lagrangian corresponding to the *Skyrme model* (with $\lambda = 2/F_\pi$ and $\varepsilon = (\sqrt{2}e)^{-1}$). We will analyze different features of this model in this chapter. After scaling the parameters the Euler-Lagrange equations of motion can be written as

$$\partial_\mu \left(R^\mu + \frac{1}{4}[R^\nu, [R_\nu, R^\mu]] \right) = 0. \tag{6.19}$$

If one restricts to static fields, then the Skyrme energy functional derived from the Lagrangian is

$$E = \frac{1}{12\pi^2} \int d^3x \{ -\frac{1}{2} Tr(/R_i R_i) - \frac{1}{16} Tr([R_i, R_j][R_i, R_j])\}. \tag{6.20}$$

Now the boundary condition at infinity implies a one-point compactification of space, so that $U : \mathbb{S}^3 \mapsto \mathbb{S}^3$, where the domain $\mathbb{S}^3$ is to be identified with $\mathbb{R}^3 \cup \{\infty\}$.The homotopy group $\pi_3(\mathbb{S}^3)$ is $\mathbb{Z}$, which implies that maps between 3-spheres fall into homotopy classes indexed by an integer. This integer B is also the degree of the map U and has the explicit representation

$$B = -\frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} Tr(R_i R_j R_k). \tag{6.21}$$

As B is a topological invariant, it is conserved under continuous deformations of the field, including time evolution. It is this conserved topological charge which Skyrme identified with baryon number. But the existence of this invariant is not enough to ensure the existence of stable topological solitons. Note that the energy decomposes into two contributions, quadratic and quartic in derivatives $E = E_2 + E_4$. Under a rescaling of the spatial coordinates $x \to \mu x$. the energy becomes

$$E(\mu) = \frac{1}{\mu} E_2 + \mu E_4. \tag{6.22}$$

We see that these two terms scale in an opposite way, leading to a minimal value of $E(\mu)$ for a finite $\mu \neq 0$. From this discussion is now clear why the σ-model (consisting only on the quadratic term) does not support stable solitons. Any term with 4 or more derivatives can cure this problem but the Skyrme term is the unique expression of minimal degree (4 in this dimension) which is Lorentz invariant and for which the resulting equations of motion are second order in the time derivative. However, notice that the antisymmetric sextic contribution, of topological origin, also satisfies this consistency requirement, this fact will be important in our work.

6.1 The Baby Skyrme model

Faddeev suggested [173] that stable closed strings may exist as topological solitons of an $O(3)$ σ-model modified by terms with higher derivatives. The following model realizes this idea. If we restrict the Skyrme field to the 2-sphere, $\mathbb{S}^2$, the usual $SU(2)$ target space, the field corresponding to the SkM is a real three-component vector of unit length, ϕ^a, with $\phi^a \phi^a = 1$. This field is related to the original Skyrme field via $U = i\phi^a \tau^a$ (where τ^a are the Pauli matrices). Subtsituting this into the Skyrme lagrangian results in

$$L = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\varepsilon^{abc} \partial_\mu \phi^b \partial_\nu \phi^c)^2. \tag{6.23}$$

The above lagrangial is known as Skyrme-Faddev lagrangian. We can generalize this lagrangian by adding an additional higher derivative term following [169], [168]

$$L = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\varepsilon^{abc}\partial_\mu \phi^b \partial_\nu \phi^c)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial^\mu \phi^a)^2. \qquad (6.24)$$

Now if we go to $2+1$ dimensions, the basic field of the reduced model maps the three-dimensional Minkowski space M^3 onto $\mathbb{S}^2$,

$$\phi : M^3 \mapsto \mathbb{S}^2. \qquad (6.25)$$

Now, after the addition of a potential term depending on the third component of the field, we obtain the class of baby Skyrme models we shall consider along this thesis,

$$\begin{aligned} L &= \frac{\lambda_2}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{\lambda_4}{4}(\varepsilon^{abc}\partial_\mu \phi^b \partial_\nu \phi^c)^2 + \\ &+ \frac{\tilde{\lambda}_4}{4}(\partial_\mu \phi^a \partial^\mu \phi^a)^2 - \lambda_0 V(\phi_3) \end{aligned} \qquad (6.26)$$

with $\phi^a\phi^a = 1$. The relevant homotopy group is $\pi_2(\mathbb{S}^2) = \mathbb{Z}$ which implies that maps between 2-spheres fall into homotopy classes indexed by an integer (like for the original Skyrme model). The degree of such maps Q can be written explicitly as:

$$Q = \frac{1}{4\pi}\int dxdy \epsilon_{abc}\phi^a \partial_x \phi^b \partial_y \phi^c \qquad (6.27)$$

After the stereographic projection

$$\phi^a = (u + u^\star, -i(u - u^\star), (|u|^2 - 1)/(1 + |u|^2)) \qquad (6.28)$$

the above lagrangian can be written as

$$\begin{aligned} L &= 2\lambda_2 \frac{\partial_\mu u \partial^\mu u^\star}{(1+|u|^2)^2} + 2\lambda_4\{\frac{(\partial_\mu u)^2(\partial_\nu u^\star)^2}{(1+|u|^2)^4} + \\ &+ (\frac{2\tilde{\lambda}_4}{\lambda_4} - 1)\frac{(\partial_\mu u \partial^\mu u^\star)^2}{(1+|u|^2)^4} + V((|u|^2-1)/(1+|u|^2))\}. \end{aligned} \qquad (6.29)$$

We will see that the $N = 1$ supersymmetrization of this model if possible for any $\tilde{\lambda}_4$, because each term can be extended separately. However, $N = 2$ extended SUSY constrains the parameters of the model enforcing the condition $\tilde{\lambda}_4 = 0$.

Chapter 7

First try: N=1 SUSY K field models

7.1 Introduction

In this chapter we analyze a first example for the supersymmetric extension of K field theories. This extension is quite simple and will constitute a pedagogical example to illustrate different problems related with such supersymmetric theories. In section 7.2.1, we briefly discuss the standard scalar supersymmetric kink. Then, in Section 7.2.2, we introduce a different supersymmetric extension of the (non-supersymmetric) scalar kink. This extension is on-shell, i.e., the scalar field equation agrees with the field equation of the non-supersymmetric scalar theory, whereas the bosonic part of the action is not equal to the action of the non-supersymmetric theory. It is this second supersymmetric extension which can be easily generalized to K theories. In Section 7.3.1, we briefly describe the class of scalar K field theories we want to consider, whereas in Section 7.3.2 we introduce the supersymmetric extensions of these K field theories, analogously to what we did in Section 7.2.2 for the standard scalar kink. In Section 7.4, we investigate the issue of central extensions of the SUSY algebra in a kink background for our new supersymmetric extensions of K field theories. However, in the last section we show the problem of this extension (in the sense that it contains ghosts), although we will solve this problem in the next chapter.

7.2 Two versions for the supersymmetric kink

7.2.1 The standard supersymmetric kink

For comparison with later results, let us first briefly review the standard supersymmetric kink theories. The simplest scalar superfield action is

$$S = \int d^3x d^2\theta \left[-\frac{1}{4} D^\alpha \Phi D_\alpha \Phi + P(\Phi) \right] = \int d^3x D^2 \left[-\frac{1}{4} D^\alpha \Phi D_\alpha \Phi + P(\Phi) \right] \Big| \tag{7.1}$$

where use was made of the fact that Grassmann integration is equivalent to Grassmann differentiation. Performing the derivatives explicitly and setting θ^α to zero at the end results in

$$S = \int d^3x \Big[\frac{1}{2} F^2 + \frac{1}{2} i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}P''(\phi)\bar{\psi}\psi + P'(\phi)F \Big]. \tag{7.2}$$

Finally, eliminating the auxiliary field via its field equation $F = -P'$ we get the standard supersymmetric action

$$S = \int d^3x \Big[-\frac{1}{2}(P'(\phi))^2 + \frac{1}{2} i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}P''(\phi)\bar{\psi}\psi \Big]. \tag{7.3}$$

Here, $P(\phi)$ is the prepotential which provides both the potential $V = (1/2)P'^2$ and the Yukawa type interaction $Y = P''$ with the fermion. observe the presence of the factors (1/2) in the fermionic part of the action in the conventions used here.

The Euler–Lagrange equations for the action (7.3) are

$$\partial_\mu\partial^\mu\phi + V'(\phi) = 0 \tag{7.4}$$

for the scalar and

$$i\bar{\gamma}^\mu_{\alpha\beta}\partial_\mu\psi_\beta + Y\psi_\alpha = 0 \tag{7.5}$$

for the spinor field. With our gamma matrix conventions we may now specialize the Euler–Lagrange equations to 1+1 dimensions and to the static case of kink or soliton solutions and the corresponding fermionic zero mode equation. Concretely we get

$$\phi_{xx} - V'(\phi) = 0 \quad \Rightarrow \quad \frac{1}{2}\phi_x^2 = V \tag{7.6}$$

for the scalar field and

$$\mp\partial_x\psi^\pm + Y\psi^\pm = 0 \tag{7.7}$$

for the fermion field (here $\psi_\alpha = (\psi^+, \psi^-)$). If V has more than one vacuum, then there exist finite energy solutions (kinks) of Eq. (7.6) which interpolate between different vacua. In the background of such a kink, one of the fermionic zero mode equations (7.7) generically has a normalizable solution (e.g. $(\psi^+, 0)$), whereas the second equation has a non-normalizable solution (e.g. $(0, \psi^-)$). We remark for later use that if we apply, e.g., the minus Dirac operator to the plus Dirac (zero mode) equation then we get

$$(\partial_x + Y)(-\partial_x + Y)\psi^+ = (-\partial_x^2 + Y'\phi_x + Y^2)\psi^+ = (-\partial_x^2 + V'')\psi^+ \tag{7.8}$$

where we used $Y = P''$, $V = (1/2)P'^2$ and $\phi_x = \sqrt{2V} = P'$. Further, the normalizable solution (zero mode) ψ^+ is just the derivative of the kink, $\psi^+ = \epsilon\phi_x$ (here ϵ is a Grassmann-valued constant):

$$(-\partial_x + Y)\psi^+ = \epsilon(-\partial_x + P'')\phi_x = -\epsilon(\phi_{xx} - P''P') = -\epsilon(\phi_{xx} - V') = 0. \tag{7.9}$$

This is a consequence of both supersymmetry, which implies that the bosonic and fermionic zero modes (=zero energy solutions of the linear fluctuation equations) in the kink background are the same, and of the translational symmetry of the kink, which implies that the bosonic zero mode is the derivative of the kink.

7.2.2 A new supersymmetric action

Now let us introduce a new supersymmetric action by simply supersymmetrizing (in the sense of replacing scalar fields by superfields) the bosonic part of the above action (7.3). This bosonic part reads

$$S_{\text{bos}} = \int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) \tag{7.10}$$

so let us introduce the action

$$S = \int d^3x d^2\theta \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)\right) \tag{7.11}$$

where

$$\mathcal{X} \equiv \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \theta^\alpha\partial_\mu\phi\partial^\mu\psi_\alpha - \theta^2\left(\partial_\mu\phi\partial^\mu F + \frac{1}{2}\partial_\mu\psi^\alpha\partial^\mu\psi_\alpha\right) \tag{7.12}$$

is a genuine superfield like Φ itself. In components this action reads

$$\begin{aligned}\int d^3x D^2(\mathcal{X} - V(\Phi))| \;&=\; \int d^3x(\partial_\mu\phi\partial^\mu F + \frac{1}{2}\partial_\mu\psi^\alpha\partial^\mu\psi_\alpha - \\ &-\; \frac{1}{2}V''(\phi)\psi^\alpha\psi_\alpha - V'(\phi)F).\end{aligned} \tag{7.13}$$

In this action, derivatives act on the auxiliary field F, so its field equation is no longer algebraic. Nevertheless, this field remains auxiliary in a certain sense, as we shall see in a moment. The field F only appears linearly in the above action, therefore it disappears from its own Euler–Lagrange equation. Indeed, varying w.r.t. F gives the equation

$$\partial_\mu\partial^\mu\phi + V'(\phi) = 0, \tag{7.14}$$

i.e., the standard field equation of the scalar field. In other words, F essentially is a Lagrange multiplier which enforces the standard scalar field equation.

The Euler–Lagrange equation for the fermion field is

$$\partial_\mu\partial^\mu\psi_\alpha - V''\psi_\alpha = 0, \tag{7.15}$$

which is not exactly equal to the Dirac equation of the standard theory. However, the two theories share the same zero modes in a kink background, i.e., the same static, one-dimensional solutions. Indeed, the restriction of this equation to one-dimensional, static configurations is identical to Eq. (7.8).

observe that the auxiliary field F does not show up in the two above equations for ϕ and ψ_α, i.e., there is no backreaction of F on the evolution of ϕ and ψ_α. In precisely this sense F still is an auxiliary field. The field F may in principle be calculated from the Euler–Lagrange equation for ϕ,

$$\partial_\mu\partial^\mu F + V''F + \frac{1}{2}V'''\bar{\psi}\psi = 0 \tag{7.16}$$

once ϕ and ψ have been determined, but due to the auxiliary nature of F in the sense explained above we treat these solutions as physically irrelevant.

7.3 The supersymmetric extended models

7.3.1 K field theories with kinks

Firstly, let us briefly introduce the K field theories we want to discuss here. The class of bosonic Lagrangians we consider read

$$S_{\mathrm{K,bos}} = \int d^3x \left(f(X) - V(\phi) \right), \quad X \equiv \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \tag{7.17}$$

where f is an at the moment arbitrary function of its argument. Several physical conditions (positivity of the energy, global hyperbolicity, well-defined Cauchy problem) may impose further restrictions on f. The resulting Euler–Lagrange equation is

$$\partial_\mu \left(f'(X)\partial^\mu \phi \right) + V'(\phi) = 0. \tag{7.18}$$

For convenience, let us display two explicit examples of these K theories. A first example is the purely quartic model

$$L = X|X| - \frac{3}{4}\lambda^2(\phi^2 - 1)^2 \tag{7.19}$$

which has the compact kink solutions

$$\phi(x) = \begin{cases} -1 & x \leq -\frac{\pi}{2\sqrt{\lambda}} \\ \sin\sqrt{\lambda}x & -\frac{\pi}{2\sqrt{\lambda}} \leq x \leq \frac{\pi}{2\sqrt{\lambda}} \\ 1 & x \geq \frac{\pi}{2\sqrt{\lambda}}, \end{cases} \tag{7.20}$$

see [12]. Here the absolute value symbol in the kinetic term is irrelevant for static (compact kink) solutions, but is important to guarantee positivity of the energy in the full, time-dependent system.

The second example is specifically designed such that the resulting K theory still has the standard ϕ^4 kink as a solution (of course, the dynamics will be different from the standard ϕ^4 theory). These K theories have been originally introduced in [58], and we just briefly repeat their construction. Indeed, for a general K field theory of the type (7.17), the field equation for a static, one-dimensional (kink) solution may be integrated once to give

$$f - 2f'X = V \tag{7.21}$$

where $X = -\frac{1}{2}\phi_x^2$ in the static one-dimensional case. For the ϕ^4 kink $f = X$, $V = \frac{1}{2}(1-\phi^2)^2$, we get the equation $\phi_x^2 = (1-\phi^2)^2$, and the kink solution is $\phi = \tanh x$. For general f we may still assume that $\phi_x^2 = (1-\phi^2)^2$ (i.e., the standard ϕ^4 kink solution) and use this condition to determine the corresponding potential in Eq. (7.21). A specific example of this type is provided by $f = X + \alpha X^2$ where α is a real parameter. Assuming $-2X = \phi_x^2 = (1-\phi^2)^2$ and using Eq. (7.21) to determine the potential, one gets the Lagrangian density

$$L = X + \alpha X^2 - \frac{1}{2}(1-\phi^2)^2 - \frac{3}{4}\alpha(1-\phi^2)^4. \tag{7.22}$$

other choices for the potential are, of course, possible, but in general they do not lead to closed, analytic expressions for the corresponding kink solutions.

7.3.2 The supersymmetric extensions

In complete analogy with what we did in section 2.B we now supersymmetrize the K field action of the above section in the sense of replacing the scalar field by a superfield. Doing this, we get the supersymmetric action

$$S_{\mathrm{K,SUSY}} = \int d^3x d^2\theta \left(f(\mathcal{X}) - V(\Phi)\right) \tag{7.23}$$

where $\mathcal{X}$ is defined in Eq. (7.12). In components this action reads

$$\begin{aligned} S_{\mathrm{K,SUSY}} &= \int d^3x \mathcal{L}_{\mathrm{K,SUSY}} = \int d^3x D^2 \left(f(\mathcal{X}) - V(\Phi)\right)| = \\ &= \int d^3x [\frac{1}{2}f''(X)\partial_\mu\phi\partial^\mu\psi^\alpha\partial_\nu\phi\partial^\nu\psi_\alpha + \\ &+ f'(X)(\partial_\mu\phi\partial^\mu F + \frac{1}{2}\partial_\mu\psi^\alpha\partial^\mu\psi_\alpha) \\ &- \frac{1}{2}V''(\phi)\psi^\alpha\psi_\alpha - V'(\phi)F]. \end{aligned} \tag{7.24}$$

The field equation for ϕ is again provided by the Euler–Lagrange equation w.r.t. the auxiliary field F. Explicitly it reads

$$\partial_\mu \left(f'(X)\partial^\mu\phi\right) + V'(\phi) = 0 \tag{7.25}$$

and is, therefore, identical to Eq. (7.18). The Euler–Lagrange equation for the spinor field is

$$\partial_\mu \left(f''(X)\partial^\nu\phi\partial_\nu\psi_\alpha\partial^\mu\phi + f'(X)\partial^\mu\psi_\alpha\right) + V''(\phi)\psi_\alpha = 0. \tag{7.26}$$

We remark that again the auxiliary field F does not couple to either the scalar or the spinor field and may be treated as auxiliary or unphysical in this sense.

Finally, let us demonstrate that the fermionic zero mode in a kink background continues to be the derivative of the kink. For a static, one-dimensional scalar field $\phi(x)$ the Euler–Lagrange equation (7.25) reads

$$-\partial_x(f'(X)\phi_x) + V'(\phi) = 0 \tag{7.27}$$

or, with the help of $X_x = -\phi_x\phi_{xx}$,

$$f''\phi_x^2\phi_{xx} - f'\phi_{xx} + V' = 0. \tag{7.28}$$

on the other hand, the Euler–Lagrange equation (7.26) for a static spinor $\psi_\alpha(x)$ in a kink background $\phi(x)$ reads

$$\partial_x(f''\phi_x^2\psi_{\alpha,x} - f'\psi_{\alpha,x}) + V''\psi_\alpha = 0 \tag{7.29}$$

and is identically satisfied for a spinor $\psi_\alpha = \epsilon_\alpha\phi_x$ where ϵ_α is a constant spinor. Indeed, inserting this spinor in the above equation we get

$$\epsilon_\alpha\partial_x(f''\phi_x^2\phi_{xx} - f'\phi_{xx} + V') = 0 \tag{7.30}$$

i.e., just the x derivative of the kink equation (7.28).

7.4 Supercurrent and SUSY algebra

It is a well-known fact that a standard supersymmetric scalar field theory in 1+1 dimensions has a centrally extended SUSY algebra if it supports topological soliton solutions (kinks) [22], where the central charges are related to the topological charges of the solitons. Here we want to investigate whether this phenomenon continues to hold in the case of the supersymmetric extensions of K field theories introduced in the last section. The SUSY transformations of the fields are

$$\begin{aligned} \delta\phi = \bar{\epsilon}\psi \quad &, \quad \delta\psi = -i\gamma^\mu \epsilon \partial_\mu \phi - \epsilon F \\ \delta F = i\bar{\epsilon}\gamma^\mu \partial_\mu \psi \quad &, \quad \delta\bar{\psi} = i\bar{\epsilon}\gamma^\mu \partial_\mu \phi - \bar{\epsilon} F. \end{aligned} \tag{7.31}$$

The supersymmetric K field Lagrangian related to the action (7.24) transforms under the SUSY transformations by the following total derivative

$$\delta \mathcal{L}_{\text{K,SUSY}} = i\bar{\epsilon}\partial_\mu [f'(X)\partial_\nu \phi \gamma^\mu \partial^\nu \psi - V'(\phi)\gamma^\mu \psi] \equiv \partial_\mu J_2^\mu \tag{7.32}$$

where the following relations are useful for the calculation,

$$\bar{\epsilon}\psi = \bar{\psi}\epsilon\,, \quad \bar{\epsilon}\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\epsilon\,, \tag{7.33}$$

$$\bar{\epsilon}\gamma^\mu\gamma^\nu\psi = \bar{\epsilon}\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\psi = \bar{\psi}\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} - \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\epsilon\,. \tag{7.34}$$

The part of the SUSY Noether current related directly to the field variations is

$$\begin{aligned} J_1^\mu \;&\equiv\; \left(\delta\phi\frac{\partial}{\partial(\partial_\mu\phi)} + \delta F\frac{\partial}{\partial(\partial_\mu F)} + \delta\psi\frac{\partial}{\partial(\partial_\mu\psi)} + \delta\bar{\psi}\frac{\partial}{\partial(\partial_\mu\bar{\psi})}\right)\mathcal{L}_{\text{K,SUSY}} \\ &=\; \bar{\epsilon}f'(X)[\partial^\mu F\psi - F\partial^\mu\psi + i\partial^\mu\phi\partial\!\!\!/\psi + i\partial\!\!\!/\phi\partial^\mu\psi] \\ &+\; \bar{\epsilon}f''(X)\Big(\frac{1}{2}(\partial^\mu\bar{\psi}\partial_\nu\phi\partial^\nu\psi + \partial^\nu\bar{\psi}\partial_\nu\phi\partial^\mu\psi)\psi + \\ &\quad\; \partial^\mu\phi[(\partial_\nu\phi\partial^\nu F + \frac{1}{2}\partial_\nu\bar{\psi}\partial^\nu\psi)\psi + i\partial^\nu\phi\partial\!\!\!/\phi\partial_\nu\psi - F\partial_\nu\phi\partial^\nu\psi]\Big) \\ &+\; \frac{1}{2}\bar{\epsilon}f'''(X)\partial^\mu\phi(\partial_\lambda\phi\partial^\lambda\bar{\psi}\partial_\nu\phi\partial^\nu\psi)\psi \end{aligned} \tag{7.35}$$

(here in the first line it is understood that the field insertions should be made exactly at the positions where the corresponding field derivatives act), and the full SUSY Noether current is

$$J^\mu_{\text{SUSY}} = J_1^\mu - J_2^\mu \equiv \bar{\epsilon}\mathcal{J}^\mu \equiv \bar{\epsilon}_\alpha \mathcal{J}^\mu_\alpha \tag{7.36}$$

where we introduced some notation at the r.h.s. It may be checked by a lengthy but straight forward calculation that this current is conserved on-shell.

For an evaluation of the SUSY algebra it is useful to study the simpler case $f(X) = X$ (the model of Section 2.C) first. The current in this case is

$$\mathcal{J}^\mu_\alpha = \partial^\mu F\psi_\alpha - F\partial^\mu\psi_\alpha + i\partial^\mu\phi(\partial\!\!\!/\psi)_\alpha + i(\partial\!\!\!/\phi\partial^\mu\psi)_\alpha - i\partial_\nu\phi(\gamma^\mu\partial^\nu\psi)_\alpha + iV'(\gamma^\mu\psi)_\alpha \tag{7.37}$$

and the correct field equal time (anti) commutators are

$$[\phi(x), \dot{F}(y)] = i\delta(x-y)\ , \quad [F(x), \dot{\phi}(y)] = i\delta(x-y) \tag{7.38}$$

$$\{\psi_\alpha(x), \dot{\bar{\psi}}_\beta(y)\} = i\delta_{\alpha\beta}\delta(x-y)\ , \quad \{\dot{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = -i\delta_{\alpha\beta}\delta(x-y). \tag{7.39}$$

The bosonic commutators are obvious from the action (7.13), whereas the anticommutators are obvious up to an overall sign. An easy way to check that our sign choice is right is to observe that with this sign choice the correct SUSY transformations of the fields are produced, i.e.,

$$[i\epsilon Q, \phi_n] = \delta\phi_n \qquad \phi_n = (\phi, \psi, \bar{\psi}, F) \tag{7.40}$$

where

$$Q_\alpha = \int dx \mathcal{J}^0_\alpha. \tag{7.41}$$

For the SUSY anticommutator $\{\mathcal{J}^0_\alpha(x), \bar{Q}_\beta\}$ we find after another lengthy calculation

$$\{\mathcal{J}^0_\alpha(x), \bar{Q}_\beta\} = 2T^0{}_\nu(\bar{\gamma}^\nu)_{\alpha\beta} + 2i(\bar{\gamma}^5)_{\alpha\beta}V'\phi' \tag{7.42}$$

(remember $(\gamma^\mu\epsilon)_\alpha \equiv \bar{\gamma}^\mu{}_{\alpha\beta}\epsilon_\beta \equiv \gamma^\mu{}_\alpha{}^\beta\epsilon_\beta$ in the barred spinor and spinor metric notations, respectively, where ϵ is an arbitrary spinor; further, $\gamma^5 = \gamma^0\gamma^1$). The corresponding energy momentum tensor is

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu\phi\partial^\nu F + \partial^\nu\phi\partial^\mu F + \frac{1}{2}\left(\partial^\mu\bar{\psi}\partial^\nu\psi + \partial^\nu\bar{\psi}\partial^\mu\psi\right) - \\ & g^{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda F + \frac{1}{2}\partial_\lambda\bar{\psi}\partial^\lambda\psi - \frac{1}{2}V''\bar{\psi}\psi - V'F\right). \end{aligned} \tag{7.43}$$

It is interesting to contrast this result with the corresponding one for a standard theory like the one in Section 2.B (where the energy-momentum tensor is different, of course),

$$\{\mathcal{J}^0_\alpha(x), \bar{Q}_\beta\} = 2T^0{}_\nu(\gamma^\nu)_{\alpha\beta} + 2i(\gamma^5)_{\alpha\beta}P'\phi'. \tag{7.44}$$

The result looks formally almost identical, with the only difference that in the second term at the r.h.s. the prepotential P appears instead of the potential V itself. This difference is, however, important. Indeed, in the standard case a further integration $\int dx$ leads to the SUSY algebra with central extension,

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\mathcal{P}_\nu(\gamma^\nu)_{\alpha\beta} + 2i(\gamma^5)_{\alpha\beta}(P(\phi_+) - P(\phi_-)) \tag{7.45}$$

where $\phi_\pm = \phi(x = \pm\infty)$, and $\mathcal{P}_\nu$ is the momentum operator. For a kink, $\phi_+ \neq \phi_-$, and also $P(\phi_+)$ and $P(\phi_-)$ are different, so a central extension appears in the SUSY algebra in a kink background.

For the anticommutator (7.42), on the other hand, a further integral leads to

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\mathcal{P}_\nu(\gamma^\nu)_{\alpha\beta} + 2i(\gamma^5)_{\alpha\beta}(V(\phi_+) - V(\phi_-)) = 2\mathcal{P}_\nu(\gamma^\nu)_{\alpha\beta} \tag{7.46}$$

because $\phi_\pm$ must take vacuum values, and $V(\phi)$ is zero by definition for a vacuum value. Therefore, for the theory of Section 2.C there is *no* central extension in the SUSY algebra in a kink background.

It remains to calculate the SUSY algebra for the supersymmetric K field theories of Section 3.B. For this purpose it is useful to introduce the canonical momenta from the variations of the Lagrangian

$$\begin{aligned} \frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_\mu \phi)} &= f'''(X)\partial^\mu\phi\partial_\lambda\phi\partial^\lambda\bar{\psi}\partial_\nu\phi\partial^\nu\psi + \\ &+ f''(X)\left(\partial^\mu\bar{\psi}\partial_\nu\phi\partial^\nu\psi + \partial_\nu\phi\partial^\nu\bar{\psi}\partial^\mu\psi + \partial^\mu\phi\partial_\nu\phi\partial^\nu F\right) + \\ &+ f'(X)\partial^\mu F \end{aligned} \tag{7.47}$$

$$\frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_\mu F)} = f'(X)\partial^\mu\phi \tag{7.48}$$

$$\frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_\mu \bar{\psi}_\alpha)} = \frac{1}{2}\left(f''(X)\partial^\mu\phi\partial_\nu\phi\partial^\nu\psi_\alpha + f'(X)\partial^\mu\psi_\alpha\right) \tag{7.49}$$

$$\frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_\mu \psi_\alpha)} = \frac{1}{2}\left(f''(X)\partial^\mu\phi\partial_\nu\phi\partial^\nu\bar{\psi}_\alpha + f'(X)\partial^\mu\bar{\psi}_\alpha\right). \tag{7.50}$$

For the bosonic fields we have directly

$$\Pi_\phi \equiv \frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_0 \phi)}, \quad \Pi_F \equiv \frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial(\partial_0 F)}, \tag{7.51}$$

whereas for the fermi fields we have to take into account that ψ and $\bar{\psi}$ are not independent, i.e,

$$\bar{\epsilon}_\alpha (\Pi_\psi)_\alpha \equiv (\Pi_{\bar{\psi}})_\alpha \epsilon_\alpha = \bar{\epsilon}_\alpha \frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial_0 \bar{\psi}_\alpha} + \frac{\partial \mathcal{L}_{\mathrm{K,SUSY}}}{\partial_0 \psi_\alpha} \epsilon_\alpha \tag{7.52}$$

for an arbitrary spinor ϵ. It follows that e.g.

$$(\Pi_\psi)_\alpha = f''(X) \dot{\phi} \partial_\nu \phi \partial^\nu \bar{\psi}_\alpha + f'(X) \dot{\bar{\psi}}_\alpha, \tag{7.53}$$

and the SUSY charge density is

$$\mathcal{J}^0_\alpha = \psi_\alpha \Pi_\phi + i(\partial\!\!\!/ \psi)_\alpha \Pi_f + i(\partial\!\!\!/ \phi \Pi_\psi)_\alpha - F(\Pi_\psi)_\alpha - i\partial_\nu \phi (\gamma^0 \partial^\nu \psi)_\alpha - iV'(\phi)(\gamma^0 \psi)_\alpha . \tag{7.54}$$

Finally, the equal time (anti) commutators are

$$[\phi(x), \Pi_\phi(y)] = i\delta(x-y) \ , \quad [F(x), \Pi_F(y)] = i\delta(x-y) \tag{7.55}$$

$$\{\psi_\alpha(x), (\Pi_\psi)_\beta(y)\} = i\delta_{\alpha\beta}\delta(x-y) \ , \quad \{\bar{\psi}_\alpha(x), (\Pi_{\bar{\psi}})_\beta(y)\} = -i\delta_{\alpha\beta}\delta(x-y) \tag{7.56}$$

(the anticommutators for ψ and $\bar{\psi}$ are of course not independent). For the SUSY charge and charge density algebra we find again Eq. (7.42). The SUSY algebra in a kink background, therefore, again contains *no* central extension. The energy-momentum tensor is, of course, different from the one in Eq. (7.43). Its explicit expression is rather long and not particularly illuminating, therefore we do not display it here.

7.5 Problems of the extension

Remember that, in fact, all these constructions are explicitly supersymmetric, because, first of all, we are promoting bosonic fields to superfields. We leave space-time derivatives unchanged and do not promote them to superderivates as is usually done. Space-time derivatives, however, are anticommutators of superderivatives and, therefore, map superfields into superfields,

$$\{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta}. \tag{7.57}$$

Then our scheme for this supersymmetrization is the following,

$$L_{bos} = L(\phi, \partial_\mu \phi, ...) \longrightarrow L_{SUSY} = L(\Phi, \partial_\mu \Phi, ...). \tag{7.58}$$

Although the bosonic sector in the SUSY version is not the same we had in the original bosonic model, the variation of the action w.r.t the auxiliary field F generates the e.o.m. of the bosonic field, remember the trivial example

$$L_{bos} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}(P'(\phi))^2 \tag{7.59}$$

with e.o.m.

$$\partial_\mu \partial^\mu \phi + P'(\phi)P''(\phi) = 0 \tag{7.60}$$

and the corresponding extension

$$L_{SUSY} = \partial_\mu \phi \partial^\mu F + \frac{1}{2}\partial_\mu \psi^\alpha \partial^\mu \psi_\alpha - \frac{1}{2}V''(\phi)\psi^\alpha \psi_\alpha - V'(\phi)F. \tag{7.61}$$

Variation w.r.t. F implies:

$$\partial_\mu \partial^\mu \phi + V'(\phi) = 0. \tag{7.62}$$

Now, making the right choice for the superpotential, i.e., such that $V'(\phi) = P'(\phi)P''(\phi)$, we have the same equation. Up to here nothing new. Another interesting observation is that this scheme is absolutely general (at least for scalar field theories), we always obtain the equation of motion of the original model from the variation of the susy model w.r.t. the auxiliary field. Then, where are the problems? If we have a look at (7.61), we see that F becomes dynamical, and if we change the field like

$$\phi = A + B \tag{7.63}$$

$$F = A - B, \tag{7.64}$$

we can rewrite the lagrangian as

$$\begin{aligned} L_{SUSY,ghost} &= \partial_\mu A \partial^\mu A - \partial_\mu B \partial^\mu B + \frac{1}{2}\partial_\mu \psi^\alpha \partial^\mu \psi_\alpha - \\ &- \frac{1}{2}V''(\phi)\psi^\alpha \psi_\alpha - V'(\phi)(A - B), \end{aligned} \tag{7.65}$$

and the field B constitutes a ghost which allows to have infinitely negative energy. Still, the study of these models has been instructive in understanding the new structures and difficulties in SUSY extensions of K field theories, because we were able to go rather far in the explicit calculation and even determine the complete SUSY algebra with its central extensions. In the following chapters we propose different extensions to avoid the problem mentioned above with the ghost field.

Chapter 8

N=1 SUSY extension of K field theories

After having displayed the inherent complications related with the supersymmetrization of K field theories, we propose in this chapter a possible SUSY extension of these models with a detailed analysis of solitonic solutions and exact calulations of energies. This chapter consists of a paper published in [95].

Supersymmetric K field theories and defect structures

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Abstract: We construct supersymmetric K field theories (i.e., theories with a non-standard kinetic term) in 1+1 and 2+1 dimensions such that the bosonic sector just consists of a nonstandard kinetic term plus a potential. Further, we study the possibility of topological defect formation in these supersymmetric models. Finally, we consider more general supersymmetric K field theories where, again, topological defects exist in some cases.

8.1 Introduction

Topological defects are of fundamental importance in a wide range of physical theories. Both in particle theory and in condensed matter physics, topological defects may exist as stable, particle-like excitations above the ground state of a theory. In some cases, states containing topological defects are even energetically preferred over the homogeneous state, such that the true ground state of the system is a condensate or lattice of topological defects. Another field where topological defects are deemed relevant is cosmology. On the one hand, topological defects are crucial in inflationary scenarios, where they may form domain walls separating different vacua of some primordial fields in the symmetry-breaking phase. As a consequence, it is widely believed that a pattern of these topological defects might be responsible for the structure formation in the very early universe, see e.g. [23], [24], [25]. On the other hand, topological defects also play an important role in the so-called brane-world scenario, where it is assumed that the visible universe is a 3+1 dimensional subspace “brane”) in a higher-dimensional bulk universe. The brane may be either strictly 3+1 dimensional “thin brane”) or have a small but nonzero extension also in the additional dimensions “thick brane”). In the latter, thick brane case, these branes are normally topological defects in the higher-dimensional bulk space [26], [27], [28], [29], [30] [9]. In all these cosmological applications, the relevant topological defects are usually solutions of some effective field theories of one or several scalar fields. The scalar field theories may either consist of the standard kinetic term of the scalar fields plus a potential, in which case the specific properties of the defects are related to the properties of the potential. Or one may relax the condition on the kinetic term and allow for more general field theories with a Lagrangian depending both on the fields and their first derivatives. These so-called K field theories have been increasing in importance during the last years, beginning with the observation about a decade ago that they might be relevant for the solution of some problems in cosmology, like K-inflation [2] and K-essence [3]. K field theories have found their applications in cosmology [4], [5], [7], [33], [51], and they introduce some qualitatively new phenomena, like the formation of solitons with compact support, so-called compactons [35] -

[40].

If K field theories turn out to be relevant for the cosmological problems described above at sufficiently early times (e.g., in the inflationary epoch) and/or sufficiently small scales, then the question of supersymmetric extensions of these theories naturally arises (see, e.g., [41], [42]). Here the situation is quite different for theories supporting topological defects with standard kinetic term (possibly coupled to gauge fields), on the one hand, and K field theories, on the other hand. Standard scalar field theories (for co-dimension one defects), the abelian Higgs (or Chern–Simons Higgs) models (for co-dimensions two defects), the t'Hooft-Polyakov monopole theory (for co-dimension 3 defects) and pure Yang–Mills theory (for co-dimension 4 defects) are all well-known to allow for supersymmetric extensions [43], [44], [45], [46], [129], and these supersymmetric extensions have been studied intensively over the last decades.

On the contrary, much less is known about supersymmetric extensions of K field theories supporting topological solitons. To the best of our knowledge, the problem of supersymmetric extensions was first investigated in relation to the Skyrme model [1], which is one of the best-known theories supporting topological solitons and possessing a non-standard kinetic term. Concretely, the supersymmetric extensions of a S^2 (or CP(1)) restriction of the Skyrme model (the so-called Skyrme–Faddeev–Niemi (SFN) model [48]) were investigated in [15] and in [16]. In both papers, a formulation of the SFN model was used where the CP(1) restriction of the Skyrme model is achieved via a gauging of the third, unwanted degree of freedom. As a result, the SFN model is expressed by two complex scalar fields and an undynamical gauge field, which are then promoted to two chiral superfields and a real vector superfield in the Wess–Zumino gauge, respectively. The result of the analysis is that the SFN model in its original form cannot be supersymmetrically extended by these methods. Instead, the supersymmetric extension contains further terms already in the bosonic sector, and also the field equations of the bosonic fields are different. Recently, we were able to show, using methods similar to the ones employed in the present article, that the baby Skyrme model in 2+1 dimensions does allow for a supersymmetric extension [49]. Quite recently, the investigation of the problem of possible supersymmetric extensions of scalar K field theories has gained momentum, [17], [50], [135], [136]. Here, [17] and [50] studied supersymmetric extensions of K field theories in 1+1 and in 2+1 dimensions, whereas the investigations of [135] and [136] are for 3+1 dimensional K theories, and related to some concrete cosmological applications (ghost condensates and Galileons).

It is the purpose of the present article to introduce and study a large class of supersymmetric extensions of scalar K field theories as well as their static topological defect solutions. The supersymmetric field theories we construct exist both in 1+1 and in 2+1 dimensional Minkowski space, due to the similarity of the spin structure in these two spaces. The topological defect solutions we study, on the other hand, all will belong to the class of defects in 1+1 dimensions (kinks), or to co-dimension one defects in a more general setting. Concretely, in Section 8.2 we introduce a set of supersymmetric Lagrangians which we shall use as "building blocks" for the specific supersymmetric Lagrangians we want to construct. We find that it is possible to construct supersymmetric Lagrangians such that their bosonic sectors just consist of a generalized kinetic term plus a potential term. We also investigate stability issues (energy positivity and the null energy condition). In Section 8.3, we investigate topological defect solutions of the theories introduced in Section 8.2. We find that there exist two classes of solutions, namely the so-called "generic" ones, which exist for a whole class of Lagrangians, and "specific" ones which depend on the specific Lagrangian under consideration. As these non-linear theories are rather uncommon, we discuss one prototypical example of the "specific" solutions in some detail. In this example, it results that all specific topological kink solutions belong to the class $\mathcal{C}^1$ of continuous functions with a continuous first derivative. We then briefly discuss some further examples, where both compact solitons and $\mathcal{C}^\infty$ functions may be found among the specific solutions. In Section 8.4, we introduce and study a more general class of supersymmetric Lagrangians, where the bosonic sector no longer can be expressed as a sum of a generalized kinetic term and a potential. We also comment on the relation of our results with the results of Bazeia, Menezes, and Petrov [17]. Finally, Section 8.5 contains a discussion of our results.

8.2 Supersymmetric models

8.2.1 Conventions

Our supersymmetry conventions are based on the widely used ones of [18], where our only difference with their conventions is our choice of the Minkowski space metric $\eta_{\mu\nu} = \text{diag}(+,-,-)$ (or its restriction to 1+1 dimensions, where appropriate). All sign differences between this paper and [18] can be traced back to this difference. Concretely, we use the superfield

$$\Phi(x,\theta) = \phi(x) + \theta^\gamma \psi_\gamma(x) - \theta^2 F(x) \tag{8.1}$$

where ϕ is a real scalar field, ψ_α is a fermionic two-component Majorana spinor, and F is the auxiliary field. Further, θ^α are the two Grassmann-valued superspace coordinates, and $\theta^2 \equiv (1/2)\theta^\alpha\theta_\alpha$. Spinor indices are risen and lowered with the spinor metric $C_{\alpha\beta} = -C^{\alpha\beta} = (\sigma_2)_{\alpha\beta}$, i.e., $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \psi^\beta C_{\beta\alpha}$. The superderivative is

$$D_\alpha = \partial_\alpha + i\theta^\beta \partial_{\alpha\beta} = \partial_\alpha - i\gamma^\mu{}_\alpha{}^\beta \theta_\beta \partial_\mu \tag{8.2}$$

and obeys the following useful relations ($D^2 \equiv \frac{1}{2} D^\alpha D_\alpha$):

$$D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\alpha\beta} D^2 \quad ; \quad D^\beta D_\alpha D_\beta = 0 \quad ; \quad (D^2)^2 = -\Box \tag{8.3}$$

$$D^2 D_\alpha = -D_\alpha D^2 = i\partial_{\alpha\beta} D^\beta \quad ; \quad \partial^{\alpha\gamma}\partial_{\beta\gamma} = -\delta_\beta^\gamma \Box \tag{8.4}$$

and

$$D^2 = \frac{1}{2}\partial^\alpha \partial_\alpha - i\theta^\alpha \partial_{\beta\alpha}\partial^\beta + \theta^2 \Box \tag{8.5}$$

$$D^2\Phi = F - i\theta^\gamma \partial_{\delta\gamma}\psi^\delta + \theta^2 \Box\phi \tag{8.6}$$

$$D^2(\Phi_1\Phi_2) = (D^2\Phi_1)\Phi_2 + (D^\alpha\Phi_1)(D_\alpha\Phi_2) + \Phi_1 D^2\Phi_2 \tag{8.7}$$

The components of superfields can be extracted with the help of the following projections

$$\phi(x) = \Phi(z)|, \quad \psi_\alpha(x) = D_\alpha\Phi(z)|, \quad F(x) = D^2\Phi(z)|, \tag{8.8}$$

where the vertical line | denotes evaluation at $\theta^\alpha = 0$.

8.2.2 Lagrangians

For the models we will construct we need the following superfields

$$D^\alpha\Phi D_\alpha\Phi = 2\psi^2 - 2\theta^\alpha(\psi_\alpha F + i\psi^\beta\partial_{\alpha\beta}\phi) + 2\theta^2(F^2 - i\psi^\alpha\partial_{\alpha\beta}\psi^\beta + \partial_\mu\phi\partial^\mu\phi) \quad (8.9)$$

$$\begin{aligned} D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi &= \\ &= 2\partial_\mu\phi\partial^\mu\phi - \theta^\gamma\partial^{\alpha\beta}\psi_\gamma\partial_{\alpha\beta}\phi + \theta^2\partial^{\alpha\beta}\phi\partial_{\alpha\beta}F + \\ &+ \frac{1}{2}\theta^2\partial^{\alpha\beta}\psi^\gamma\partial_{\alpha\beta}\psi_\gamma + F^2 - 2iF\theta^\gamma\partial_{\delta\gamma}\psi^\delta + 2\theta^2F\Box\phi \\ &+ \theta^2\partial_\delta{}^\gamma\psi^\delta\partial_{\beta\gamma}\psi^\beta) \end{aligned} \quad (8.10)$$

$$D^2\Phi D^2\Phi = F^2 - 2iF\theta^\gamma\partial_{\delta\gamma}\psi^\delta + 2\theta^2F\Box\phi + \theta^2\partial_\delta{}^\gamma\psi^\delta\partial_{\beta\gamma}\psi^\beta \quad (8.11)$$

as well as their purely bosonic parts (we remark that all spinorial contributions to the lagrangians we shall consider are at least quadratic in the spinors, therefore it is consistent to study the subsector with $\psi_\alpha = 0$)

$$(D^\alpha\Phi D_\alpha\Phi)_{\psi=0} = 2\theta^2(F^2 + \partial^\mu\phi\partial_\mu\phi) \quad (8.12)$$

$$(D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi)_{\psi=0} = 2(F^2 + \partial^\mu\phi\partial_\mu\phi) + 4\theta^2(F\Box\phi - \partial_\mu\phi\partial^\mu F) \quad (8.13)$$

$$(D^2\Phi D^2\Phi)_{\psi=0} = F^2 + 2\theta^2F\Box\phi. \quad (8.14)$$

Next, let us construct the supersymmetric actions we want to investigate. A supersymmetric action always is the superspace integral of a superfield. Further, due to the Grassmann integration rules $\int d^2\theta = 0$, $\int d^2\theta\theta_\alpha = 0$, $\int d^2\theta\theta^2 = -1 = D^2\theta^2$, the corresponding Lagrangian in ordinary space-time always is the θ^2 component of the superfield. Besides, we are mainly interested in the bosonic sectors of the resulting theories, therefore we shall restrict to the purely bosonic sector in the sequel. We will use the following supersymmetric Lagrangian densities (in ordinary space-time) as building blocks,

$$\begin{aligned} (\mathcal{L}^{(k,n)})_{\psi=0} &= -\left(D^2[(\frac{1}{2}D^\alpha\Phi D_\alpha\Phi)(\frac{1}{2}D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi)^{k-1}(D^2\Phi D^2\Phi)^n]|\right)_{\psi=0} \\ &= (F^2 + \partial_\mu\phi\partial^\mu\phi)^k F^{2n} \end{aligned} \quad (8.15)$$

where $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. The idea now is to choose certain linear combinations of the $\mathcal{L}^{(k,n)}$ with specific properties. We observe that a general linear combination contains terms where powers of the auxiliary field F couple to the kinetic term $\partial_\mu\phi\partial^\mu\phi$. But there exists a specific linear combination where these mixed terms are absent, namely

$$(\mathcal{L}^{(k)})_{\psi=0} \equiv (\mathcal{L}^{(k,0)})_{\psi=0} - \binom{k}{1}(\mathcal{L}^{(k-1,1)})_{\psi=0} + \binom{k}{2}(\mathcal{L}^{(k-2,2)})_{\psi=0} + \dots \quad (8.16)$$
$$\dots + (-1)^{k-1}\binom{k}{k-1}(\mathcal{L}^{(1,k-1)})_{\psi=0} = (\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}.$$

The Lagrangians we want to consider are linear combinations of the above, where we also want to include a potential term, because we are mainly interested in topological solitons and defect solutions. That is to say, we add a prepotential $P(\Phi)$ to the action density in superspace which, in ordinary space-time, induces the bosonic Lagrangian density $(D^2P)| = P'(\phi)F$ (here the prime denotes a derivative w.r.t. the argument ϕ). Therefore, the class of Lagrangians we want to consider is

$$\begin{aligned}\mathcal{L}_b^{(\alpha,P)} &= \sum_{k=1}^{N}\alpha_k(\mathcal{L}^{(k)})_{\psi=0} + P'F \\ &= \sum_{k=1}^{N}\alpha_k[(\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}] + P'(\phi)F \qquad (8.17)\end{aligned}$$

where the lower index b means "bosonic" (we only consider the bosonic sector of a supersymmetric Lagrangian), and the upper index α should be understood as a multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of coupling constants. Further, N is a positive integer. In a next step, we eliminate F via its algebraic field equation

$$\sum_{k=1}^{N}(-1)^{k-1}2k\alpha_kF^{2k-1} + P'(\phi) = 0. \qquad (8.18)$$

For a given function $P(\phi)$ this is, in general, a rather complicated algebraic equation for F. However, we made no assumption yet about the functional dependence of P, therefore we may understand this equation in a second, equivalent way: we assume that F is an arbitrary given function of ϕ, which in turn determines the prepotential $P(\phi)$. This second way of interpreting Eq. (8.18) is more useful for our purposes. Eliminating the resulting $P'(\phi)$

we arrive at the Lagrangian density

$$\mathcal{L}_b^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k [(\partial^\mu \phi \partial_\mu \phi)^k - (-1)^{k-1}(2k-1)F^{2k}] \tag{8.19}$$

where now $F = F(\phi)$ is a given function of ϕ which we may choose freely depending on the theory or physical problem under consideration.

8.2.3 Energy considerations

We would like to end this section with some considerations on the positivity of the energy. The energy density corresponding to the Lagrangian (8.19) is (in 1+1 dimensions and with $\dot{\phi} = \partial_t \phi$, $\phi' = \partial_x \phi$)

$$\mathcal{E}_b^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k \left((\dot{\phi}^2 - \phi'^2)^{k-1}((2k-1)\dot{\phi}^2 + \phi'^2) + (-1)^{k-1}(2k-1)F^{2k} \right). \tag{8.20}$$

This expression is obviously positive semi-definite if only the α_k with odd k are nonzero and positive. It remains positive semi-definite if both the lowest (usually $k = 1$) and the highest value of k ($k = N$) with a nonzero and positive α_k are odd, provided that the intermediate α_k for even k are not too large. For a given value of N, inequalities for the coefficients α_k guaranteeing positive semi-definiteness of the energy density can be derived without difficulties.

A second, less restrictive condition which is deemed sufficient to guarantee stability is the so-called null energy condition

$$T_{\mu\nu} n^\mu n^\nu \geq 0 \tag{8.21}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and n^μ is an arbitrary null vector. For general Lagrangians $\mathcal{L}(X, \phi)$ where $X \equiv \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$, the energy momentum tensor reads

$$T_{\mu\nu} = \mathcal{L}_{,X} \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \tag{8.22}$$

(here $\mathcal{L}_{,X}$ is the X derivative of $\mathcal{L}$), and the null energy condition simply is $\mathcal{L}_{,X} \geq 0$. For our specific class of Lagrangians (8.19), the null energy condition therefore reads

$$(\mathcal{L}_b^{(\alpha,F)})_{,X} = \sum_{k=1}^{N} 2k\alpha_k(\partial_\mu\phi\partial^\mu\phi)^{k-1} \geq 0. \tag{8.23}$$

Again, this condition is automatically satisfied if only α_k for odd k are nonzero, or if the α_k for even k obey certain restrictions.

Remark: in [50] a class of models based on the superfield (8.82) of Section 4.2 were introduced. These models satisfy neither energy positivity nor the null energy condition. They support, nevertheless, topological kink solutions, and their energy densities can be expressed as the squares of the corresponding supercharges. A more complete analysis of these models which would resolve the issue of stability is, therefore, an open problem at the moment which requires further investigation.

8.3 Solutions

The Euler-Lagrange equation for the Lagrangian density (8.19) is

$$\sum_{k=1}^{N} 2k\alpha_k \left(\partial_\mu[(\partial_\nu\phi\partial^\nu\phi)^{k-1}\partial^\mu\phi] + (-1)^{k-1}(2k-1)F^{2k-1}F_{,\phi}\right) = 0 \tag{8.24}$$

which, in 1+1 dimensions and for static (time-independent) fields simplifies to

$$\sum_{k=1}^{N} 2k(-1)^{k-1}\alpha_k \left(-\partial_x(\phi_{,x}^{2k-1}) + (2k-1)F^{2k-1}F_{,\phi}\right) = 0. \tag{8.25}$$

8.3.1 Generic static solutions

First of all, we want to demonstrate that, due to the restrictions imposed by supersymmetry, this equation has a class of static, one-dimensional solutions which are completely independent of the coefficients α_k, which we shall call the "generic" solutions. Indeed, if we impose equation (8.25) for each k (i.e., for each term in the sum) independently, the resulting equation is

$$\partial_x(\phi_{,x}^{2k-1}) = (2k-1)F^{2k-1}F_{,\phi} \tag{8.26}$$

or, after multiplying by $\phi_{,x}$ and dividing by $(2k-1)$,

$$\phi_{,x}^{2k-1}\phi_{,xx} = F^{2k-1}F_{,\phi}\phi_{,x} \tag{8.27}$$

which may be integrated to $\phi_{,x}^{2k} = F^{2k}$ and, therefore, to the k independent solution

$$\phi_{,x} \equiv \phi' = \pm F. \tag{8.28}$$

That is to say, these solutions only depend on the choice of $F = F(\phi)$, but do not depend on the α_k and, therefore, exist for an infinite number of theories defined by different values of the α_k. Depending on the choice for $F(\phi)$, the static solutions may be topological solitons. E.g. for the simple choice $F = 1-\phi^2$, the solution of (8.28) is just the well-known ϕ^4 kink solution $\phi(x) = \tanh(x-x_0)$ where x_0 is an integration constant (the position of the kink). As another example, for $F = \sqrt{|1-\phi^2|}$, we get the compacton solution

$$\phi(x) = \begin{cases} -1 & x - x_0 \le -\frac{\pi}{2} \\ \sin(x-x_0) & -\frac{\pi}{2} \le x - x_0 \le \frac{\pi}{2} \\ 1 & x - x_0 \ge \frac{\pi}{2} \end{cases} \tag{8.29}$$

where, again, x_0 is an integration constant.

The energy of a generic supersymmetric kink solution may be calculated with the help of the first order formalism, which has the advantage that an explicit knowledge of the kink solution is not needed for the determination of its energy (for details on the first order formalism we refer to [32]). All that is needed is the field equation of a generic solution $\phi' = \pm F$ (we shall choose the plus sign corresponding to the kink, for concreteness). The idea now is to separate a factor ϕ' in the energy density with the help of the field equation, because this allows to rewrite the base space integral of the energy functional as a target space integral with the help of the relation $\phi' dx \equiv d\phi$. Concretely, we get for the energy density of the generic kink solution

$$\begin{aligned} \mathcal{E} &= \sum_{k=1}^{N}(-1)^{k-1}\alpha_k(\phi'^{2k} + (2k-1)F^{2k}) = \sum_{k=1}^{N}(-1)^{k-1}\alpha_k 2kF^{2k} \\ &= \phi'\sum_{k=1}^{N}(-1)^{k-1}\alpha_k 2kF^{2k-1} \equiv \phi' W_{,\phi} \end{aligned} \tag{8.30}$$

where $W_{,\phi}$ and its ϕ integral $W(\phi)$ are understood as functions of ϕ. For the energy this leads to

$$E = \int_{-\infty}^{\infty} dx \phi' W_{,\phi} = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi W_{,\phi} = W(\phi(\infty)) - W(\phi(-\infty)). \tag{8.31}$$

As indicated, all that is needed for the evaluation of this energy is the root $\phi' = F(\phi)$ and the asymptotic behaviour $\phi(\pm\infty)$ of the kink. We remark that the integrating function $W(\phi)$ of the first order formalism is identical to the prepotential $P(\phi)$,

$$W(\phi) = P(\phi) \tag{8.32}$$

as is obvious from Eq. (8.18). This is exactly as in the case of the standard supersymmetric scalar field theory with the standard, quadratic kinetic term. It also remains true for the class of models introduced and studied in [17], as we shall discuss in some more detail in Section 8.4.2 Both for the standard supersymmetric scalar field theories and for the models introduced in [17] it is, in fact, possible to rewrite the energy functional for static field configurations in a BPS form, such that both the first order field equations for static fields and the simple, topological expressions $E = P(\phi(\infty)) - P(\phi(-\infty))$ for the resulting energies are a consequence of the BPS property of the energy functional (for the models introduced in [17] we briefly recapitulate the BPS property of static kink solutions in Section 8.4.2). on the contrary, for the models introduced in the previous section there is no obvious way to rewrite them in a BPS form, despite the applicability of the first order formalism, because the energy functional contains, in general, many more than two terms (just two terms are needed to complete a square and arrive at the BPS form).

on the other hand, for the additional, specific solutions of the theories of Section 8.2 to be discussed in the following two subsections, the relation $W = P$ is no longer true, although it is still possible to calculate the energies of the specific solutions with the help of the first order formalism.

8.3.2 Specific solutions: an example

Next, we want to study whether in addition to the solutions $\phi_{,x}^2 = F^2$, which do not depend on the specific Lagrangian (i.e., on the coefficients α_k), there exist further (static) solutions which do depend on the Lagrangian. Both the existence of such additional solutions and their properties (e.g., being topological solitons) will depend on the Lagrangian, therefore the results will be less general and have to be discussed separately for each model. So, let us select a specific Lagrangian (specific values for the α_k) as an example. Concretely, we want to study the simplest case which gives rise to a potential with several vacua and obeys certain additional restrictions (positivity of the energy). Positivity of the energy requires that both the highest and the lowest nonzero α_k are for odd k, so we choose nonzero α_3 and α_1 for the simplest case. Further, we want that the potential factorizes and gives rise to several vacua, so we choose the concrete example $\alpha_3 = \frac{1}{5}$, $\alpha_2 = \frac{2}{3}$, and $\alpha_1 = 1$, which gives rise to the Lagrangian density

$$\mathcal{L}_b^{ex} = \frac{1}{5}(\partial_\mu\phi\partial^\mu\phi)^3 + \frac{2}{3}(\partial_\mu\phi\partial^\mu\phi)^2 + \partial_\mu\phi\partial^\mu\phi - F^6 + 2F^4 - F^2 \qquad (8.33)$$

where, indeed, the potential in terms of F factorizes, $F^6 - 2F^4 + F^2 = F^2(1-F^2)^2$. Next, we want to assume the simplest relation between F and ϕ, namely $F^2(\phi) = \phi^2$. The resulting Lagrangian is

$$\mathcal{L}_b^{ex} = \frac{1}{5}(\partial_\mu\phi\partial^\mu\phi)^3 + \frac{2}{3}(\partial_\mu\phi\partial^\mu\phi)^2 + \partial_\mu\phi\partial^\mu\phi - \phi^2(1-\phi^2)^2. \qquad (8.34)$$

We already know that it gives rise to the static solutions

$$(\phi_{,x})^2 = \phi^2 \Rightarrow \phi(x) = \exp \pm (x - x_0). \qquad (8.35)$$

These solutions have infinite energy and are not solitons. We want to investigate whether there exist additonal solutions and, specifically, whether there exist topological solitons. The potential has the three vacua $\phi = (0, 1, -1)$, therefore topological solitons (static solutions which interpolate between these vacua) are not excluded. We shall find that these solitons exist in the space $\mathcal{C}^1$ of continuous functions with a continuous first derivative, but not in the spaces $\mathcal{C}^n$ (with continuous first n derivatives) for $n > 1$.

The once-integrated field equation for static solutions (with the integration constant set equal to zero, as required by the finiteness of the energy) reads ($\phi' \equiv \phi_{,x}$)

$$\phi'^6 - 2\phi'^4 + \phi'^2 = \phi^6 - 2\phi^4 + \phi^2 \qquad (8.36)$$

and obviously has the solutions (8.35). For a better understanding of further solutions the following observations are useful. Firstly, for a fixed value $x = \tilde{x}$ of the independent variable x, the field ϕ and its derivative ϕ' have to obey the equation

$$V(\phi) = V(\phi') = c \tag{8.37}$$

where c is a real, positive constant (or zero) and

$$V(\lambda) \equiv \lambda^6 - 2\lambda^4 + \lambda^2 = \lambda^2(1-\lambda^2)^2 \tag{8.38}$$

is the potential (see Figure (8.1)). In general, the equation $V(\lambda) = c$ has six solutions $\lambda = \pm\lambda_i(c), i = 1\ldots 3$. In other words, if we choose the initial condition $\phi(\tilde{x}) = \tilde{\phi}$, then $\phi'(\tilde{x})$ is not uniquely determined (as would be the case for a linear first order equation) and may take any of the six values $\pm\lambda_i(c)$ such that $V(\pm\lambda_i(c)) = V(\tilde{\phi}) = c$. obviously, the choice $\phi'(\tilde{x}) = \pm\tilde{\phi}$ leads to the exponential solutions (8.35), whereas other choices will lead to additional solutions.

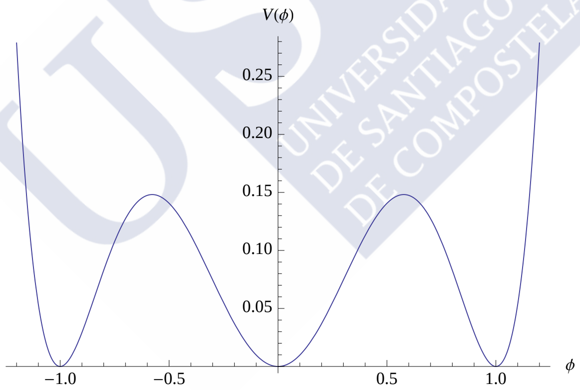

Figure 8.1: The potential $V(\phi) = \phi^6 - 2\phi^4 + \phi^2$.

Secondly, the field equation (8.36) leads to the following equation for the second derivative

$$\phi'' = \frac{(3\phi^4 - 4\phi^2 + 1)\phi}{3\phi'^4 - 4\phi'^2 + 1} \tag{8.39}$$

where the numerator is zero for all critical points (minima and maxima) $\phi = (0, \pm 1, \pm\frac{1}{\sqrt{3}})$ of the potential $V(\phi)$, whereas the denominator is zero for the critical points $\phi' = (\pm 1, \pm\frac{1}{\sqrt{3}})$. For later convenience we also remark that at the two local maxima $\phi = \pm\frac{1}{\sqrt{3}}$ the potential takes the value $V(\pm\frac{1}{\sqrt{3}}) = \frac{4}{27}$ and that the equation $V(\lambda) = \frac{4}{27}$ has the two further solutions $\lambda = \pm\frac{2}{\sqrt{3}}$ which are not critical points.

We observe that the equation $V(\phi') = V(\phi)$ can in fact be solved algebraically for ϕ' and leads to the solutions

$$\phi' = \pm\phi \tag{8.40}$$

(that is, the exponential solutions (8.35)) and to the four further solutions

$$\phi' = \pm\frac{1}{2}\left(\phi \pm \sqrt{4 - 3\phi^2}\right). \tag{8.41}$$

For this last expression, reality of ϕ' requires that $|\phi| \leq \frac{2}{\sqrt{3}}$. The resulting integral for ϕ

$$\int_0^\phi \frac{d\tilde{\phi}}{\pm(\phi \pm \sqrt{4 - 3\phi^2})} = \frac{1}{2}(x - x_0) \tag{8.42}$$

may be resolved explicitly, providing an implicit solution $x - x_0 = H(\phi)$ where, for each choice of signs, $H(\phi)$ is a combination of logaritms and inverse trigonometric functions. The explicit expressions for H are, however, rather lenghty and not particularly illuminating, therefore we prefer to continue our discussion with a combination of qualitative arguments and numerical calculations. We want to remark, however, that the graphs of the numerical solutions shown in the figures below agree exactly with the graphs of the analytic solutions (8.42) (we remind the reader that for the graph of a function the implicit solution is sufficient).

For the qualitative discussion, we now assume that we choose an "initial value" at a given point $\tilde{x}$. Due to translational invariance we may choose this point at zero $\tilde{x} = 0$, i.e. $\phi(0) = \phi_0$, without loss of generality. For $0 < |\phi_0| < \frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}} < |\phi_0| < 1$ and $1 < |\phi_0| < \frac{2}{\sqrt{3}}$, $\phi'(0)$ may take any of the six real solutions of the equation $V(\phi') = V(\phi_0)$. Further, $\phi''(0)$ as well as all higher order derivatives at $x = 0$ are uniquely determined by linear equations, as we shall see in a moment. Therefore, for these "initial conditions" ϕ_0, there exist indeed the six solutions (8.35) and (8.42). For $|\phi_0| > \frac{2}{\sqrt{3}}$, only the two solutions $\phi'(0) = \pm\phi_0$ of the equation $V(\phi') = V(\phi_0)$ are real, therefore only the two exponential solutions (8.35) exist. At the critical points $\phi_0 = 0, \pm 1$ and $\phi_0 = \pm\frac{1}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}$ the situation is slightly more complicated (strictly speaking $\phi_0 = \pm\frac{2}{\sqrt{3}}$ are not critical points, because $V'(\pm\frac{2}{\sqrt{3}}) \neq 0$; however, $\phi_0 = \pm\frac{2}{\sqrt{3}}$ provides the same level height of the potential like the critical points $\phi_0 = \pm\frac{1}{\sqrt{3}}$, that is, $V(\pm\frac{1}{\sqrt{3}}) = V(\pm\frac{2}{\sqrt{3}}) = \frac{4}{27}$, therefore these points play a special role in the analysis, too). In order to understand what happens it is useful to insert the Taylor expansion about $x = 0$,

$$\phi(x) = \sum_{k=0}^{\infty} f_k x^k \tag{8.43}$$

into the field equation $V(\phi) - V(\phi') = 0$, which, up to second order, reads

$$\begin{aligned} 0 &= f_0^2(1-f_0^2)^2 - f_1^2(1-f_1^2)^2 + & (8.44)\\ & [2f_0(1-4f_0^2+3f_0^4)f_1 - 4f_1(1-4f_1^2+3f_1^4)f_2]x + & (8.45)\\ & [(1-12f_0^2+15f_0^4)f_1^2 + 2f_0(1-4f_0^2+3f_0^4)f_2 - & \\ & -4(1-12f_1^2+15f_1^4)f_2^2 - 6f_1(1-4f_1^2+3f_1^4)f_3]x^2 + \ldots & (8.46) \end{aligned}$$

It can be inferred easily that for generic values of f_0 and f_1 (values which are not critical points), f_2 is determined uniquely by a linear equation from the term of order x^1. on the other hand, if f_1 takes a critical value, then the coefficient multiplying f_2 in the order x^1 term is zero, and f_2 is determined, instead, by a quadratic equation coming from the term of order x^2. These points will be important in the following, because precisely at these points we may join different solutions such that the resulting solution belongs to the class $\mathcal{C}^1$ of continuous functions with a continuous first derivative. Specifically, we find the following possible values for f_1 and f_2 for a given, critical f_0 (we only consider the cases $f_0 \geq 0$ because of the obvious symmetry $\phi \to -\phi$ of the theory). For $f_0 = 0$

$$(f_0, f_1, f_2) = (0,0,0) \text{ or } (0, \pm 1, \pm\frac{1}{4}) \tag{8.47}$$

where the first case corresponds to the trivial vacuum solution $\phi \equiv 0$, and the second case corresponds to the four solutions (8.42). The exponential solutions (8.35) are obviously incompatible with the "initial condition" $\phi(0) = 0$ (the vacuum solution $\phi(x) = 0$ can be understood as a limiting case of the two exponential solutions for infinite integration constant x_0). Next, for $f_0 = 1$

$$(f_0, f_1, f_2) \;=\; (1,0,0) \;\text{ or }\; (1, \pm 1, \pm\frac{1}{2}) \tag{8.48}$$

where the first case corresponds to the trivial vacuum solution $\phi \equiv 1$. The second case consists of the exponential solutions (two solutions) and of two of the four solutions (8.42). The other two are incompatible with the "initial conditions". For $f_0 = \frac{1}{\sqrt{3}}$ we get

$$(f_0, f_1, f_2) \;=\; (\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{2\sqrt{3}}) \;\text{ or }\; (\frac{1}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}, 0) \tag{8.49}$$

where the first case contains both the two exponential solutions and two of the four solutions (8.42), and the second case corresponds to the other two solutions (8.42). Finally, for $f_0 = \frac{2}{\sqrt{3}}$ we find

$$(f_0, f_1, f_2) \;=\; (\frac{2}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \;\text{ or }\; (\frac{2}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \infty) \tag{8.50}$$

where the first case provides the two exponentail solutions (8.35), whereas the second case shows that the solutions (8.42) run into a singularity when $|f_0| = \frac{2}{\sqrt{3}}$.

Now let us study some of these cases in more detail. Concretely, we investigate the case $(f_0, f_1, f_2) = (0, 1, \frac{1}{4})$. Firstly, for negative x, $\phi(x)$ diminishes from $\phi(0) = 0$ towards -1, and $\phi'(x)$ diminishes from $\phi'(0) = 1$ towards 0, such that for a fixed value of x ϕ and ϕ' have the same height on the graph of V, see Fig. (8.1). If x is sufficiently negative such that $\phi(x)$ is close to its vacuum value -1 and $\phi'(x)$ is close to zero, the field equation may be linearized about the vacuum -1, and it follows easily that the vacuum is approached exponentially, like $\phi(x) \sim -1 + \exp(4x)$ (remember that x is negative). In other words, for negative x the solution behaves like a nice kink or topological soliton and does not reach the vacuum value -1 for finite x. For positive x, in a first instant both $\phi(x)$ and $\phi'(x)$ grow till they reach the values $\phi(x_1) = \frac{1}{\sqrt{3}}$ and $\phi'(x_1) = \frac{2}{\sqrt{3}}$ for some $x_1 > 0$. At this point $\phi''(x_1) = 0$ therefore ϕ' may change direction in a smooth way. For $x > x_1$, ϕ continues to grow while ϕ' shrinks till they reach the values $\phi(x_2) = \frac{2}{\sqrt{3}}$ and $\phi'(x_2) = \frac{1}{\sqrt{3}}$ for some $x = x_2$. At this point $\phi''(x_2) = \infty$, and the solution

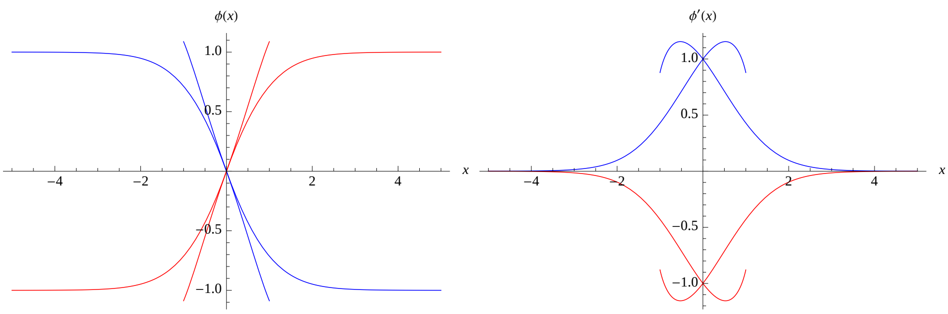

Figure 8.2: For the "initial condition" $\phi(0) = 0$ all the five solutions (including the trivial solution $\phi \equiv 0$) $\phi(x)$ (left figure) and the first derivatives $\phi'(x)$ (right figure). The singularity at $\phi(x_2) = \pm\frac{2}{\sqrt{3}}$, $\phi'(x_2) = \pm\frac{1}{\sqrt{3}}$ for some x_2, where the integration breaks down, is clearly visible.

hits a singularity. A numerical integration confirms these findings, see Figure (8.2).

There exists, however, the possibility to form a topological soliton or kink solution in the class $\mathcal{C}^1$ of continuous functions with continuous first derivatives by simply joining the solution $(f_0, f_1, f_2) = (0, 1, \frac{1}{4})$ for negative x with the solution $(f_0, f_1, f_2) = (0, 1, -\frac{1}{4})$ for positive x. Indeed, both $\phi(0)$ and $\phi'(0)$ agree, so the resulting solution is $\mathcal{C}^1$. Further, ϕ' in the second case diminishes for positive x because $\phi''(0)$ is negative. Therefore, $\phi(x)$ approaches 1 and ϕ' approaches 0 for large positive x, and a linearized analysis reveals that in that region $\phi(x) \sim 1 - \exp(-4x)$. As a consequence, the solution obtained by the joining procedure behaves exactly like a kink interpolating between the vacuum -1 at $x = -\infty$ and the vacuum $+1$ at $x = \infty$. For the corresponding result of a numerical integration, see Figure (8.3).

Finally, let us discuss the possibility to form a kink in the class of $\mathcal{C}^1$ functions which interpolates, e.g., between the vacuum 0 and the vacuum 1. For this purpose, we should join the solution $(f_0, f_1, f_2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}})$ for $x < 0$ with the solution $(f_0, f_1, f_2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}})$ for $x > 0$. Indeed, the solution $(f_0, f_1, f_2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}})$ is just the exponential solution $\exp x$ and behaves well (approaches 0 exponentially) for negative x. For the solution $(f_0, f_1, f_2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}})$, on the other hand, both $\phi(0)$ and $\phi'(0)$ are equal

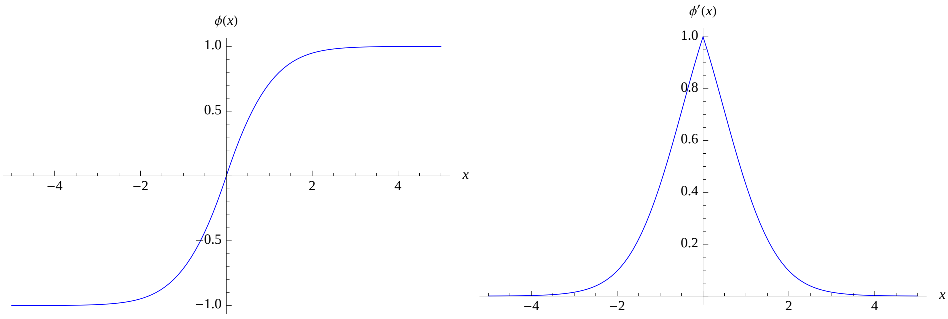

Figure 8.3: For the "initial condition" $\phi(0) = 0$, the kink solution interpolating between $\phi = -1$ and $\phi = 1$ (left figure) and its first derivative (right figure).

to $\frac{1}{\sqrt{3}}$ at $x = 0$. For increasing x, $\phi(x)$ increases and $\phi'(x)$ decreases until they get close to 1 and 0, respectively. But near these values, again, a linearized analysis applies and tells us that ϕ behaves like $\phi(x) \sim 1 - \exp(-4x)$. Therefore, the solution produced by the joining procedure describes a kink which interpolates between the vacuum $\phi = 0$ at $x = -\infty$ and the vacuum $\phi = 1$ at $x = \infty$. The general solution for the initial condition $\phi(0) = \frac{1}{\sqrt{3}}$ is displayed in Figure (8.4), and the kink solution is shown in Figure (8.5).

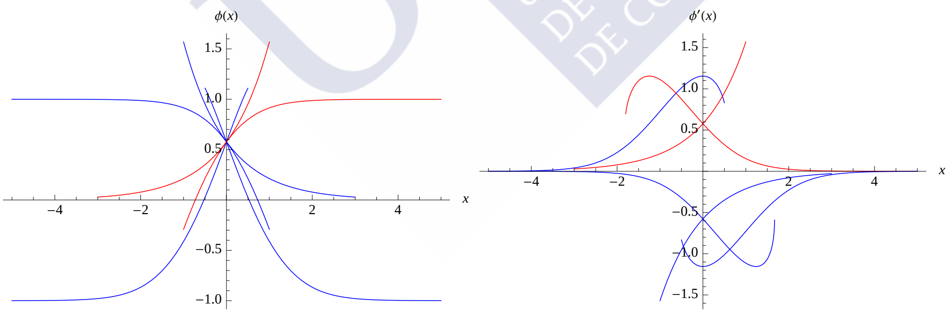

Figure 8.4: For the "initial condition" $\phi(0) = \frac{1}{\sqrt{3}}$ all six solutions $\phi(x)$ (left figure) and the first derivatives $\phi'(x)$ (right figure). Again, the singularities at $\phi(x_2) = \pm\frac{2}{\sqrt{3}}$, $\phi'(x_2) = \pm\frac{1}{\sqrt{3}}$ for some x_2 for the non-exponential solutions are clearly visible.

The remaining kink and antikink solutions which we have not discussed explicitly may be easily found with the help of the obvious symmetries $x \to -x$ and $\phi \to -\phi$. We remark that from the point of view of the

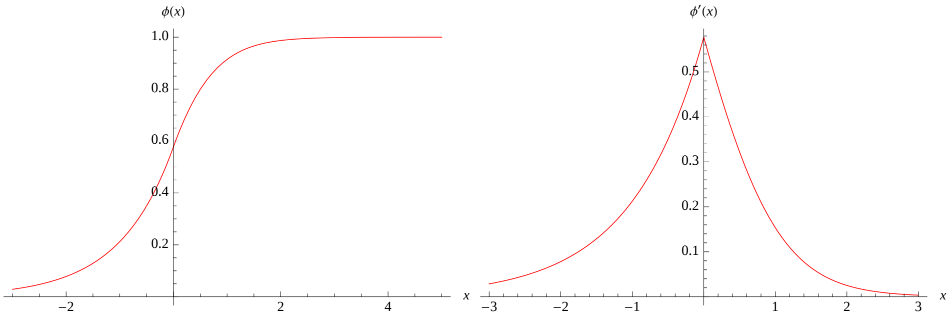

Figure 8.5: For the "initial condition" $\phi(0) = \frac{1}{\sqrt{3}}$, the kink solution interpolating between $\phi = 0$ and $\phi = 1$ (left figure) and its first derivative (right figure).

variational calculus, solutions in the $\mathcal{C}^1$ class of functions are perfectly valid. They provide well-defined energy densities and, therefore, well-defined critical points of the energy functional. Whether they are acceptable from a physics point of view depends, of course, on the concrete physical problem under consideration.

Finally, we want to calculate the energies of the kinks constructed by the joining procedure described above. These energies can again be calculated exactly and do not require the knowledge of the explicit solutions $\phi(x)$ but, instead, just the knowledge of the six roots (8.40), (8.41) of the first order equation $V(\phi) = V(\phi')$, Eq. (8.36). Indeed, with the help of the first order equation we find for the energy density for static solutions

$$\mathcal{E} = -\mathcal{L}_b^{ex} = \frac{1}{5}\phi'^6 - \frac{2}{3}\phi'^4 + \phi'^2 + \phi^2(1-\phi^2)^2 = \frac{6}{5}\phi'^6 - \frac{8}{3}\phi'^4 + 2\phi'^2 \quad (8.51)$$

and for the energy

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \mathcal{E} = \int_{-\infty}^{\infty} dx \phi' (\frac{6}{5}\phi'^5 - \frac{8}{3}\phi'^3 + 2\phi') \\ &\equiv \int_{-\infty}^{\infty} dx \phi' W_{,\phi} = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi W_{,\phi} = W(\phi(\infty)) - W(\phi(-\infty)) \quad (8.52) \end{aligned}$$

where

$$W_{,\phi}(\phi) = \frac{6}{5}\phi'^5 - \frac{8}{3}\phi'^3 + 2\phi' \quad (8.53)$$

(and its ϕ integral $W(\phi)$) must be understood as a function of ϕ which results when evaluating the above expression for one of the six roots (8.40), (8.41) for ϕ'. Here we use again the first order formalism of [32] to which refer for a detailed discussion.

In our case, the kinks are constructed by joining two different solutions, therefore the expression for the energy is slightly more complicated and reads

$$E = W^{(2)}(\phi(\infty)) - W^{(2)}(\phi(0)) + W^{(1)}(\phi(0)) - W^{(1)}(\phi(-\infty)) \tag{8.54}$$

where $W^{(2)}$ and $W^{(1)}$ are the functions which result from evaluating the expression (8.53) for the two different roots ϕ' which form the specific kink solution, and from performing the corresponding ϕ integrals. The joining may be done at any point x_0 in base space (because of translational invariance) but we chose $x_0 = 0$ in our specific examples, therefore the joining point in target space is $\phi(0)$.

Concretely, for the soliton of Figure 3 which interpolates between $\phi(-\infty) = -1$ and $\phi(\infty) = 1$, with joining point $\phi(0) = 0$, the correct roots are

$$\phi < 0 \, : \, \phi' = \frac{1}{2}(\phi + \sqrt{4 - 3\phi^2}) \, , \qquad \phi > 0 \, : \, \phi' = -\frac{1}{2}(\phi - \sqrt{4 - 3\phi^2}). \tag{8.55}$$

Further, positive and negative ϕ regions give exactly the same contribution to the total energy, therefore the soliton energy can be calculated to be

$$\begin{aligned} E_{(-1,1)} &= 2 \int_{-1}^{0} d\phi \left(\frac{6}{5} [\frac{1}{2}(\phi + \sqrt{\Delta})]^5 - \frac{8}{3} [\frac{1}{2}(\phi + \sqrt{\Delta})]^3 + \phi + \sqrt{\Delta} \right) \\ &= 2 \left[\frac{\phi^6}{10} - \frac{\phi^4}{12} + \left(\frac{\phi}{10} + \frac{11}{60}\phi^3 - \frac{\phi^5}{10} \right) \sqrt{\Delta} + \frac{2}{3\sqrt{3}} \arcsin \left(\frac{\sqrt{3}}{2}\phi \right) \right]_{\phi=-1}^{0} \\ &= 2 \left(\frac{1}{6} + \frac{2\pi}{9\sqrt{3}} \right) = \frac{1}{3} + \frac{4\pi}{9\sqrt{3}}. \end{aligned} \tag{8.56}$$

For the kinks which interpolate between 0 and ± 1, we choose the one which interpolates between $\phi(-\infty) = -1$ and $\phi(\infty) = 0$, because then we may use exactly the same solution as above for the region between $\phi = -1$ and $\phi = -\frac{1}{\sqrt{3}}$. For ϕ between $-\frac{1}{\sqrt{3}}$ and 0, the correct root is $\phi' = -\phi$, therefore we get for the total energy of this kink

$$\begin{aligned} E_{(-1,0)} &= [\frac{\phi^6}{10} - \frac{\phi^4}{12} + \left(\frac{\phi}{10} + \frac{11}{60}\phi^3 - \frac{\phi^5}{10}\right)\sqrt{4-3\phi^2} + \\ &+ \frac{2}{3\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}\phi\right)]_{\phi=-1}^{-\frac{1}{\sqrt{3}}} + \\ &+ \left[-\frac{1}{5}\phi^6 + \frac{2}{3}\phi^4 - \phi^2\right]_{\phi=-\frac{1}{\sqrt{3}}}^{0} \\ &= \frac{1}{6} + \frac{2\pi}{9\sqrt{3}} - \frac{7}{45} - \frac{\pi}{9\sqrt{3}} + \frac{4}{15} = \frac{5}{18} + \frac{\pi}{9\sqrt{3}}. \end{aligned} \tag{8.57}$$

The remaining kinks, obviously, have the same energy. We remark that $E_{(-1,1)} > 2E_{(-1,0)}$. Therefore, the kink interpolating between -1 and 1 probably is unstable against the decay into one kink interpolating between -1 and 0 plus one kink interpolating between 0 and $+1$. Establishing this conjecture would, however, require a numerical integration of the time-dependent system, which is beyond the scope of the present article.

8.3.3 Further examples of specific solutions

In this subsection we shall discuss two more examples which are similar to the theory studied in the last subsection. In the first example, the main difference is that the kinks no longer approach their vacuum values in an exponential fashion. Instead, two of the three vacua are approached compacton-like (i.e. the field takes the corresponding vacuum value already for finite x), whereas the third vacuum is approached in a power-like way (concretely like $\phi \sim x^{-1}$). In the second example, we will find that there exists a specific kink solution which belongs to the class of $\mathcal{C}^\infty$ functions. In both examples, we use the same values for the α_i like in (8.33). Besides, these examples are similar in many respects to the one discussed above in detail, so the discussion which follows can be much shorter. Also the resulting soliton energies can be calculated analytically, using exactly the same method like in the above example, therefore we do not repeat this calculation.

In the first example, we choose for F

$$F = \sqrt{|1-\phi^2|}. \tag{8.58}$$

We already know that this model leads to the α-independent static field equations

$$\phi' = \pm F = \pm\sqrt{|1-\phi^2|} \tag{8.59}$$

which have the compacton solutions (8.29) (the corresponding anti-compacton solutions for the minus sign). The once integrated static field equation is

$$\phi'^6 - 2\phi'^4 + \phi'^2 = |1-\phi^2|^3 - 2(1-\phi^2)^2 + |1-\phi^2| = \phi^4|(1-\phi^2)| \tag{8.60}$$

and might lead to further solutions, as in the previous subsection. Indeed, the potential $\tilde{V} = \phi^4|1-\phi^2|$ has the three vacua $\phi = 0, \pm 1$. Further, the potential $\tilde{V}$ behaves like $\tilde{V} \sim |\delta\phi|$ near the two vacua ± 1 (i.e., for $\phi = \pm(1-\delta\phi)$ and small $\delta\phi$), whereas it behaves like $\tilde{V} \sim \delta\phi^4$ near the vacuum 0 (i.e. for $\phi = \delta\phi$ and small $\delta\phi$). The asymptotic field equations for $\delta\phi$ are $\delta\phi'^2 \sim |\delta\phi|$ near the two vacua ± 1, with the asymptotic compacton-like solution

$$\delta\phi \sim \frac{1}{2}(x-x_0)^2 \quad \text{for} \quad x \le x_0\ ; \quad \delta\phi = 0 \quad \text{for} \quad x > x_0 .$$

The asymptotic field equation near the vacuum 0 is $\delta\phi'^2 \sim \delta\phi^4$ with the solutions

$$\delta\phi = \pm\frac{1}{x-x_0} + o\left(\frac{1}{x}\right), \quad |x| \to \infty$$

and, therefore, the algebraic, power-like localization announced above. It only remains to determine whether it is possible to join a solution with this asymptotic behaviour to a compacton-like solution, forming a kink of the semi-compacton type, which interpolates, e.g., between the vacuum $\phi(-\infty) = 0$ (with a power-like approach) and the vacuum $\phi(x_1) = 1$ (where x_1 is the compacton boundary). For this purpose we note that Eq. (8.60) has, in addition to the two roots (8.59), the four roots

$$\phi' = \pm\frac{1}{\sqrt{2}}\sqrt{1+\phi^2+\sqrt{1+2\phi^2-3\phi^4}} \tag{8.61}$$

$$\phi' = \pm\frac{1}{\sqrt{2}}\sqrt{1+\phi^2-\sqrt{1+2\phi^2-3\phi^4}}. \tag{8.62}$$

The solution with the right behaviour (i.e., approaching the vacuum $\phi = 0$) is the lower one, Eq. (8.62). Now we just have to determine whether it is possible to join this solution with the compacton solution such that both

ϕ and ϕ' coincide at the joining point. The result is that this joining is indeed possible, as may be checked easily. For the kink interpolating between $\phi(-\infty) = 0$ and $\phi(x_1) = 1$, e.g., the joining point is $\phi(x_0) = \sqrt{\frac{2}{3}}$, $\phi'(x_0) = \frac{1}{\sqrt{3}}$ where, as always, the joining point x_0 in base space is arbitrary. Further, for joining point x_0, the compacton boundary of the semi-compacton is at $x_1 = x_0 + \frac{\pi}{2} - \arcsin\sqrt{\frac{2}{3}}$.

We remark that in this example all kink solutions (both the compactons and the semi-compactons) are solutions in the $\mathcal{C}^1$ class of functions, because the second derivative of the field at the compacton boundary is not uniquely determined (its algebraic equation has three solutions, corresponding to the vacuum, the compacton, and a third solution with infinite energy, respectively). For the semi-compacton, the second derivative of the field at the joining point obviously is not uniquely determined, as well, analogously to the kinks formed by the joining procedure in the previous subsection.

For the second example we choose

$$F = 1 - \phi^2 \tag{8.63}$$

which leads to the following first order equation

$$\phi'^6 - 2\phi'^4 + \phi'^2 = (1-\phi^2)^6 - 2(1-\phi^2)^4 + (1-\phi^2)^2 = \phi^4(1-\phi^2)^2(2-\phi^2)^2. \tag{8.64}$$

Therefore, the resulting potential $\tilde{\tilde{V}} = \phi^4(1-\phi^2)^2(2-\phi^2)^2$ has the five vacua $\phi^2 = (0, 1, 2)$. Further, the four vacua $\phi = \pm 1$ and $\phi = \pm\sqrt{2}$ are approached quadratically and will, therefore, lead to the usual exponential kink tail. The vacuum $\phi = 0$, on the other hand, is approached with a fourth power, $\tilde{\tilde{V}} \sim \delta\phi^4$, and will lead to an algebraic, power-like tail, like in the last example. Concretely, we find near $\phi = 0$, $\delta\phi'^2 \sim 4\delta\phi^4$ and therefore

$$\delta\phi \sim \pm\frac{1}{2(x - x_0)}, \quad |x| \to \infty. \tag{8.65}$$

The six roots of Eq. (8.64) are

$$\phi' = \pm(1 - \phi^2) \tag{8.66}$$

(which is just the first order equation for the standard ϕ^4 kink) and the four additional equations

$$\phi' = \pm\frac{1}{2}\left(1-\phi^2+\sqrt{1+6\phi^2-3\phi^4}\right) \quad (8.67)$$

$$\phi' = \pm\frac{1}{2}\left(1-\phi^2-\sqrt{1+6\phi^2-3\phi^4}\right). \quad (8.68)$$

Here, equation (8.67) describes solutions which approach the two vacua $\phi = \pm\sqrt{2}$, whereas equation (8.68) describes solutions which approach the vacuum $\phi = 0$. By joining different solutions, we may create kinks in the class $\mathcal{C}^1$ which interpolate between any two different vacua of the model, as we did in the previous two examples. Two of the kinks belong, however, to the class of $\mathcal{C}^\infty$ functions. The first $\mathcal{C}^\infty$ kink is just the standard ϕ^4 kink interpolating between the two vacua $\phi = -1$ and $\phi = 1$, and it respresents a generic solution of the model. The second $\mathcal{C}^\infty$ kink is the one interpolating between $\phi = -\sqrt{2}$ and $\phi = \sqrt{2}$, as we want to demonstrate now. Indeed, equation (8.67) describes a solution which approaches the two vacua $\phi = \pm\sqrt{2}$. Further, this equation (we choose the root with the plus sign, for concreteness) is completely regular in the interval $-\sqrt{2} \le \phi \le \sqrt{2}$ (the equation develops singularities at the two points $\phi^2 = 1 + \frac{2}{\sqrt{3}}$, but these points are outside the interval where the kink takes its values). Therefore, we expect that this equation should describe a smooth $\mathcal{C}^\infty$ kink which interpolates between the two vacua. Both an exact, implicit integration and a numerical integration precisely confirm this expecation. We display the result of a numerical integration in Fig. (8.6).

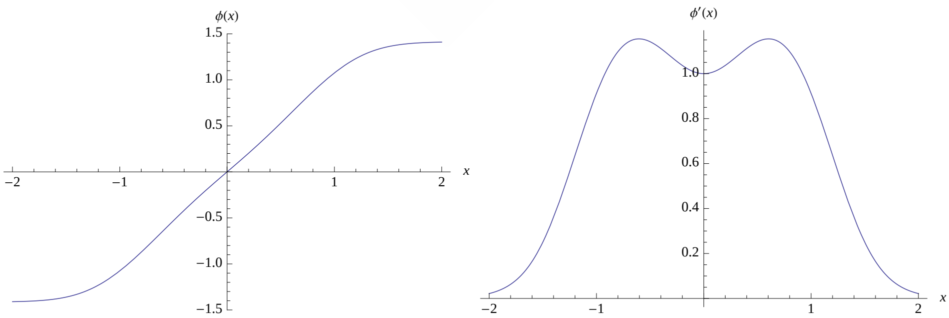

Figure 8.6: The specific, regular kink solution interpolating between the two vacua $\phi = -\sqrt{2}$ and $\phi = \sqrt{2}$ (left figure) and its first derivative (right figure).

We remark that this kink has the interesting feature that its first derivative (and, therefore, also the energy density) has a local minimum at the position of the kink center, whereas the two local maxima are slightly displaced to the right and left of the center. We further remark that this example demonstrated explicitly that $\mathcal{C}^\infty$ kinks may exist not only among the generic solutions but also among the specific solutions of our supersymmetric K field theories (which was not obvious in the other two examples studied so far).

8.4 Further models

8.4.1 Field-dependent α_k

Here we want to construct further supersymmetric K field theories based on the observation that the models introduced in the last section remain supersymmetric when the factors α_k multiplying each power of the kinetic term depend on ϕ instead of being constants. Indeed, a superfield which is just an arbitrary function $\alpha(\Phi)$ of the basic superfield Φ has the superspace expansion in the bosonic sector $\psi = 0$

$$(\alpha(\Phi))_{\psi=0} = \alpha(\phi) - \theta^2\alpha'(\phi)F \tag{8.69}$$

(the prime $\alpha'(\phi)$ denotes the derivative w.r.t. the argument ϕ). If this superfield is multiplied by the superfield $(D^\alpha\Phi D_\alpha\Phi)_{\psi=0}$ which only has a θ^2 component in the bosonic sector, then in the product only the θ-independent component of $\alpha(\Phi)$ contributes,

$$(\alpha(\Phi))_{\psi=0}| = \alpha(\phi) \tag{8.70}$$

and the multiplication with the superfield $\alpha(\Phi)$ corresponds to a multiplication with the ordinary field $\alpha(\phi)$ of the Lagrangian densities (8.15), i.e., to the new building blocks

$$- \quad (\mathcal{L}^{(k,n)})_{\psi=0} = \tag{8.71}$$

$$= \quad \left(D^2[\alpha^{(k,n)}(\Phi)(\frac{1}{2}D^\alpha\Phi D_\alpha\Phi)(\frac{1}{2}D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi)^{k-1}(D^2\Phi D^2\Phi)^n]| \right)_{\psi=0} =$$

$$= \quad -\alpha^{(k,n)}(\phi)(F^2 + \partial_\mu\phi\partial^\mu\phi)^k F^{2n} \tag{8.72}$$

where again $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. The cancellation of the mixed terms $(\partial_\mu\phi\partial^\mu\phi)^i F^{2j}$ may again be achieved by calculating the sum analogous to (8.17) provided all the $\alpha^{(k,n)}$ in the sum are equal. Linear combinations of these Lagrangians are therefore

$$\mathcal{L}_b^{(\alpha,P)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}] + P'(\phi)F \quad (8.73)$$

just like in Section 8.2, but now with ϕ dependent coefficients $\alpha_k(\phi)$. Also, the field equation for the auxiliary field F is the same and leads to the Lagrangian

$$\mathcal{L}_b^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k - (-1)^{k-1}(2k-1)F^{2k}] \quad (8.74)$$

where $F = F(\phi)$ is an arbitrary function of ϕ, like in (8.19), but now with field dependent coefficients $\alpha_k(\phi)$. This result provides us with a new class of supersymmetric K field theories where now different powers of the kinetic term may be multiplied by functions of the scalar field.

Now we shall discuss an explicit example, where we choose the functions $\alpha_k(\phi)$ and $F(\phi)$ such that the resulting model possesses a simple defect solution. Concretely we choose the nonzero α_k

$$\alpha_3 = \frac{1}{24}\,, \quad \alpha_2 = \frac{1}{4}(1-\phi^2)^2\,, \quad \alpha_1 = \frac{1}{2}[1+(1-\phi^2)^4]. \quad (8.75)$$

For non-constant $\alpha_k(\phi)$ positivity of the energy and the null energy condition become slightly more involved. For the moment we only consider the null energy condition, which is satisfied for this specific model. Indeed, the resulting Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{24}[(\partial_\mu\phi\partial^\mu\phi)^3 - 5F^6] + \frac{1}{4}(1-\phi^2)^2[(\partial_\mu\phi\partial^\mu\phi)^2 + 3F^4] + \quad (8.76)\\ &+ \frac{1}{2}[1+(1-\phi^2)^4](\partial_\mu\phi\partial^\mu\phi - F)\end{aligned}$$

and for $\mathcal{L}_X$ we get

$$\mathcal{L}_X = X^2 + 2(1-\phi^2)^2 X + (1-\phi^2)^4 + 1 = (X + (1-\phi^2)^2)^2 + 1 > 0 \quad (8.77)$$

(remember $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$), so the null energy condition holds. In order to have simple defect solutions we now choose for F

$$F^2 = (1-\phi^2)^2 \tag{8.78}$$

such that the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{24}[8X^3 - 5(1-\phi^2)^6] + \frac{1}{4}(1-\phi^2)^2[4X^2 + 3(1-\phi^2)^4] + \\ &+ \frac{1}{2}[1+(1-\phi^2)^4][2X - (1-\phi^2)^2]. \end{aligned} \tag{8.79}$$

The once integrated field equation for static solutions is equivalent to the condition that the one-one component of the energy momentum tensor is constant (see, e.g. [32, 17]). Further, for finite energy solutions this constant must be zero,

$$T_{11} = \mathcal{L} - 2X\mathcal{L}_X = 0 \tag{8.80}$$

where now $X = -\frac{1}{2}\phi'^2$ because ϕ is a static configuration. For the concrete example this equation reads

$$-\frac{5}{3}X^3 - 3(1-\phi^2)^2X^2 - [1+(1-\phi^2)^4]X + \frac{1}{24}(1-\phi^2)^6 - \frac{1}{2}(1-\phi^2)^2 = 0. \tag{8.81}$$

It may be checked easily that this equation is solved by $X = -\frac{1}{2}(1-\phi^2)^2$, i.e., $\phi'^2 = (1-\phi^2)^2$ which is just the field equation of the ϕ^4 kink with the solution $\phi(x) = \pm\tanh(x - x_0)$. Therefore, our concrete example has the standard ϕ^4 kink as a defect solution (it was specifically chosen to have this solution). Due to the nonlinear character of the above field equation, the model probably has more solutions (like the ones of Section 8.3), but this issue is beyond the scope of the present paper.

Finally, let us remark that, although for the above model (i.e., the choice (8.75) for the α_k) the energy density is not positive semi-definite, it is easy to find a small variation of the model with positive semi-definite energy density. All one has to do is to increase the relative size of α_3 and (or) α_1 as compared to α_2. Choosing, for example, α_1 and α_2 as in (8.75), and $\alpha_3 = \frac{1}{3}$, the resulting energy density is positive semi-definite for arbitrary $F(\phi)$, as may be shown easily. If we further choose $F^2 \sim (1-\phi^2)^2$ then the resulting potential will have at least the two vacua $\phi = \pm 1$, and a linearization of the model near these two vacua shows that the vacua are approached exponentially, like in the case of the standard kink. Therefore, the model most likely supports kinks which interpolate between the two vacua, although the explicit kink solutions will be more complicated.

8.4.2 The models of Bazeia, Menezes and Petrov

The supersymmetric K field theories of Bazeia, Menezes and Petrov (BMP) [17] are based on the superfield

$$\partial_\mu \Phi \partial^\mu \Phi = \partial_\mu \phi \partial^\mu \phi + 2\theta^\alpha \partial_\mu \phi \partial^\mu \psi_\alpha - 2\theta^2 \partial_\mu \phi \partial^\mu F - \theta^2 \partial_\mu \psi^\alpha \partial^\mu \psi_\alpha . \tag{8.82}$$

Indeed, the bosonic component of the superfield $D_\alpha \Phi D^\alpha \Phi$ only consists of a term proportional to θ^2, therefore multiplying this superfield by an arbitrary function of the above superfield (8.82), $f(\partial_\mu \Phi \partial^\mu \Phi)$, only the theta independent term $f(\partial_\mu \phi \partial^\mu \phi)$ will contribute, leading to the Lagrangian

$$\mathcal{L}_{\rm BMP} = -\left(D^2 [f(\partial_\mu \Phi \partial^\mu \Phi) \frac{1}{2} D_\alpha \Phi D^\alpha \Phi]| \right)_{\psi=0} = f(\partial_\mu \phi \partial^\mu \phi)(F^2 + \partial_\mu \phi \partial^\mu \phi). \tag{8.83}$$

obviously, these Lagrangians produce a coupling of the auxiliary field F with the kinetic term $\partial_\mu \phi \partial^\mu \phi$. on the other hand, the auxilliary field only appears quadratically, implying a linear (algebraic) field equation for F.

First of all, we want to remark that for functions $f(\partial_\mu \phi \partial^\mu \phi)$ which have a Taylor expansion about zero, the same bosonic Lagrangians may be constructed from the building blocks (8.15) of Section 8.2 by taking a different linear combination (the fermionic parts of the corresponding Lagrangians will in general not coincide)

$$\begin{aligned} \mathcal{L}^{(k)}_{\rm BMP} &\equiv (\mathcal{L}^{(k,0)})_{\psi=0} - \binom{k-1}{1} (\mathcal{L}^{(k-1,1)})_{\psi=0} + \binom{k-1}{2} (\mathcal{L}^{(k-2,2)})_{\psi=0} + \ldots \\ \ldots &+ (-1)^{k-1} \binom{k-1}{k-1} (\mathcal{L}^{(1,k-1)})_{\psi=0} = \\ &= (\partial^\mu \phi \partial_\mu \phi + F^2)(\partial_\mu \phi \partial^\mu \phi)^{k-1}. \end{aligned} \tag{8.84}$$

We may easily recover the Lagrangian (8.83) by taking linear combinations of these,

$$\begin{aligned} \mathcal{L}_{\rm BMP} &= \sum_{k=1}^{\infty} \beta_k \mathcal{L}_{\rm BMP}^{(k)} = (F^2 + \partial_\mu\phi\partial^\mu\phi) \sum_k \beta_k (\partial_\mu\phi\partial^\mu\phi)^{k-1} \equiv \\ &\equiv (F^2 + \partial_\mu\phi\partial^\mu\phi) f(\partial_\mu\phi\partial^\mu\phi). \end{aligned} \tag{8.85}$$

In order to have more interesting solutions, BMP added a potential term, as we did in Section 8.2. The resulting theories can, in fact, be analyzed with methods very similar to the ones employed in the previous sections. Concretely, they studied the Lagrangians

$$\mathcal{L}_{\rm BMP}^{(P)} = f(\partial_\mu\phi\partial^\mu\phi)(F^2 + \partial_\mu\phi\partial^\mu\phi) + P'(\phi)F \tag{8.86}$$

which after eliminating the auxiliary field F using its algebraic field equation

$$F = -\frac{P'}{2f} \tag{8.87}$$

becomes

$$\mathcal{L}_{\rm BMP}^{(P)} = f(\frac{P'^2}{4f^2} + \partial_\mu\phi\partial^\mu\phi) - \frac{P'^2}{2f} = \partial_\mu\phi\partial^\mu\phi f - \frac{P'^2}{4f} \tag{8.88}$$

where we suppressed the arguments of P and f in the last expression to improve readability. The X derivative of this Lagrangian is

$$(\mathcal{L}_{\rm BMP}^{(P)})_{,X} = f_{,X}\left(\frac{P'^2}{4f^2} + 2X\right) + 2f \tag{8.89}$$

(please remember that $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ and $f = f(2X)$ such that $f_{,X} = 2f'$). The null energy condition already imposes rather nontrivial restrictions on the function f. A sufficient condition is $f \geq 0$, $f_{,X} \geq 0$ and $f \geq |Xf_{,X}|$ as may be checked easily. Finally, the once integrated field equation (8.80) for static fields, after a simple calculation, leads to

$$\frac{1}{4f^2}(f + 2Xf_{,X})(8Xf^2 + P'^2) = 0 \tag{8.90}$$

where now $X = -\frac{1}{2}\phi'^2$ and $f = f(-\phi'^2)$ and, therefore, to the two equations

$$2\phi'(x)f(-\phi'(x)^2) = \pm P'(\phi) \tag{8.91}$$

where we reinserted the arguments in the last expression for the sake of clarity. For some choices of f and P these equations lead to defect solutions. Finally, in the models of BMP the energy of a kink may be calculated with the help of the first order formalism first introduced in [32], in close analogy to the calculations presented in Section 8.3.1. It also remains true that, like in Section 8.3.1, the prepotential $P(\phi)$ is equal to the integrating function $W(\phi)$. The energy functional for static configurations (but *without* the use of the field equation) may, in fact, be re-written in a BPS form (exactly like for the standard supersymmetric scalar field theory), from which both the first order equations and the equality $P = W$ follow immediately. Indeed, the energy functional may be written like

$$E^{(P)}_{\mathrm{BMP}} = \int dx \left(\phi'^2 f + \frac{P'^2}{4f} \right) = \int dx \left(\frac{1}{4f}(2\phi' f \mp P')^2 \pm \phi' P' \right) \tag{8.92}$$

and for a solution to the first order (or BPS) equation (8.91) (we take the plus sign for definiteness) the resulting energy is therefore

$$E^{(P)}_{\mathrm{BMP}} = \int_{-\infty}^{\infty} dx \phi' P' = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi P' = P(\phi(\infty)) - P(\phi(-\infty)) \tag{8.93}$$

which proves the above statement. For a more detailed discussion we refer to [17] (we remark that BMP use a slightly different notation in [17]: they use the notation h instead of P for the prepotential, and their X is defined like $X = \partial_\mu \phi \partial^\mu \phi$ instead of the definition $X = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$ used in the present paper and in [32]).

8.5 Discussion

We developed and described a method to construct general supersymmetric scalar K field theories in 1+1 and 2+1 dimensions. Among these theories, we found a large class of models which, in the purely bosonic sector, consist of a generalized kinetic term plus a potential, where the vacuum structure of the potential is crucial for the determination of the topological defect solutions, similarly to the standard case (i.e., kinetic term $\sim \partial_\mu \phi \partial^\mu \phi$). Due to the enhanced nonlinearity of these supersymmetric K field models there are, nevertheless, some significant differences, like different roots of the first order field equations leading to a larger number of kink solutions, or the possibility to join different solutions forming additional kinks in the space $\mathcal{C}^1$ of continuous functions with a continuous first derivative [1]. These results are new and are by themselves interesting, broadening the range of applicability of supersymmetry to a new class of field theories and, at the same time, enhancing our understanding of these field theories.

As far as possible applications are concerned, the natural arena seems to be the area of cosmology, as already briefly mentioned in the introduction. Indeed, if these scalar field theories are interpreted as effective theories which derive from a more fundamental theory with supersymmetry (like string theory), then it is natural to study the supersymmetric versions of the effective models. If, in addition, the defect formation and phase transition (e.g. from a symmetry breaking phase to a symmetric phase) relevant for cosmological considerations occur at time or energy scales where supersymmetry is still assumed unbroken, then also the defect solutions of the *supersymmetric* effective field theories are the relevant ones.

At this point, several questions show up. The first one is the inclusion of fermions. It is, e.g., expected that, as a consequence of supersymmetry and translational invariance, there should exist a fermionic zero mode for each kink background where, in addition, the fermionic zero mode is equal to the spatial derivative of the kink. This fact has already been confirmed explicitly in some supersymmetric K field theories [17], [50]. The inclusion of fermions in the Lagrangians studied in the present article does not present any difficulty on a fundamental level, the only practical obstacle being that, for purely combinatorical reasons, the resulting expressions will be rather lengthy. A second question is whether the SUSY algebra in a kink background contains central extensions related to the topological charge of the kink, as happens for the standard supersymmetric kink [44]. Again, this prob-

[1] Whether such $\mathcal{C}^1$ solutions are physically relevant depends, of course, on the concrete physical system under consideration, but we want to emphasize that if the supersymmetric scalar field theory is interpreted as an *effective* theory then the $\mathcal{C}^1$ solutions should be taken into account. In this case, the spike in the field derivative (and in the energy density) of a $\mathcal{C}^1$ solution will, in any case, be resolved by the true UV degrees of freedom.

lem has already been studied for some supersymmetric K field theories [50]. A further question concerns the issue of quantization. If the SUSY K field theories are interpreted as *effective* theories, as would be appropiate, e.g., in a cosmological context, then already the classical model contains some relevant information of the underlying quantum theory, like spontaneous symmetry breaking or the existence of topological defects. In this context, the quantization of quadratic fluctuations about the topological defect is the correct procedure to obtain further information about the underlying theory. Independently, one may, nevertheless, pose the problem of a full quantization of the supersymmetric K field theory, where the enhanced degreee of nonlinearity certainly implies further complications. one may ask, e.g.,. whether the additional, non-quadratic kinetic terms may be treated perturbatively, like the non-quadratic terms of the potential in the standard case. The answer will certainly depend on the space-time dimension. A related question is whether supersymmetry simplifies the task by taming possible divergences, as happens in the standard case. Here it is interesting to note that, even at the classical level, supersymmetry implies some restrictions on possible Lagrangians which are visible already in the bosonic sector. Indeed, as is obvious e.g. from Eq. (8.19), there exists a relation between the kinetic and the potential terms (this relation is responsible for the existence of the so-called generic solutions). one wonders what this relation implies for the quantum theory, e.g., in the form of Ward-like identities. These and related questions will be investigated in future publications.

Finally, we think that the supersymmetric models we found present some independent mathematical interest of their own, given their high degree of non-linearity, on the one hand, and the possibility to obtain rather precise information on their solutions (e.g. all kink solutions together with their exact energies), on the other hand. Further investigations in this direction (e.g., time-dependent solutions, or the stability of topological solitons) will be pursued, as well.

Chapter 9

BPS bounds in N=1 K field theories

Continuing the line of analysis of supersymmetric K field theories presented in the previous chapter, we demonstrate that all domain wall solutions of such models are, in fact, BPS solutions. Moreover, in this chapter a first analysis of the supersymmetric algebra is done, finding that the central charge of the SUSY algebra coincides with the one for standard models. This chapter consists of a paper published in [96].

Supersymmetric K field theories and defect structures

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Abstract: We demonstrate that in the supersymmetric extensions of a class of generalized (or K) field theories introduced recently, the static energy satisfies a BPS bound in each topological sector. Further, the corresponding soliton solutions saturate the bound. We also find strong indications that the BPS bound shows up in the SUSY algebra as a central extension, as is the

case in the well-known supersymmetric field theories with standard kinetic terms.

9.1 Introduction

If a quantum field theory is assumed to be applicable to physical processes at arbitrary energy scales, then both its field contents and possible terms contributing to the Lagrangian are quite constrained, mainly by the requirement of renormalizability. Recently, however, a different point of view has gained support, where the field theory under consideration is interpreted as a low-energy effective field theory which, at sufficiently high energies, is superseded by a more fundamental theory (string theory being the most prominent proposal). In this effective field-theory interpretation, the presence of non-renormalizable terms in the lagrangian just indicates the existence of a natural cutoff in the effective field theory, beyond which calculations within the effective field theory framework are no longer trustworthy, and effects of the fundamental theory have to be taken into account. The effective field theory point of view, therefore, allows to consider a much broader class of Lagrangians, which may, in a first instance, be rather general functions of the fields and their space-time derivatives. Allowing for higher than first derivatives in the Lagrangian, however, may introduce some further problems like, e.g., the necessity to introduce ghosts, so it is natural to consider a class of generalized field theories given by Lagrangians which depend in a Poincare-invariant way on the fields and on their first derivatives. Specifically, a broader class of kinetic terms, generalizing the standard quadratic kinetic terms, may be considered. These theories with generalized kinetic terms (termed K field theories) have been studied with increasing effort in the last years, especially in the context of cosmology, where they might resolve some problems like inflation or late time acceleration (K-inflation [2] or K-essence [3]). Another relevant issue in cosmology is the formation of (topological or non-topological) defects [23], [27], [28], [30], [29], [26], [9] where, again, K field theories allow for a much richer phenomenology [4], [33], [51], [36], [10], [11], [39], [40]. Specifically, the formation of domain walls is described by effectively 1+1 dimensional theories [5], [7], [31], [32], [12], with possible applications to the structure formation in the early universe. In this context, the problem of supersymmetric extensions of K field theories emerges naturally. Indeed, if the fundamental theory (e.g., string theory) is supersymmetric, and if some of the supersymmetry is assumed unbroken even for the effective field theory in a regime of not too low energy (e.g., in the very early universe [41], [42], [52], [53]), then it is an important question whether the resulting supersymmetric effective field theory can be described, at all, in the context of K field theories. The investigation of this problem has been resumed very recently. Concretely, in [54], supersymmetric (SUSY) extensions of some 3+1 dimensional K field theories with cosmological relevance (ghost condensates, galileons, DBI inflation) have been investigated, whereas the SUSY extensions of some lower-dimensional theories relevant, e.g., for domain wall formation, have been studied in [17], [64], [56].

If SUSY extensions of some K field theories can be constructed, and if these SUSY K field theories support topological defect solutions, then the following very important questions arise immediately: are the topological defects BPS solutions? And, if so, are they invariant under part of the SUSY transformations? Further, if the defect solutions can be classified by a topological charge, does this charge reappear in the SUSY algebra as a central extension? All these interrelated features are well-known to show up in SUSY field theories with standard kinetic terms [43], [44], [45], [46], [129], and SUSY allows, in fact, to better understand both the existence and the structure of BPS solutions. Analogous results for SUSY K field theories would, therefore, be very important for a better understanding of these theories. It is the purpose of the present paper to investigate this question for a large class of SUSY K field theories in 1+1 dimensions.

Concretely, in [56] we introduced a class of SUSY K field theories and studied their domain wall solutions, but in that paper we were not able to determine whether these topological defects were of the BPS type. As a consequence, all the related questions listed above could not be adressed, either. In the present paper we shall close these loopholes. In Section 9.2, we briefly review the class of SUSY K field theories we consider and, in a next step, demonstrate the BPS property of all their domain wall solutions. In Section 9.3, then, we demonstrate that the domain wall (kink) solutions are invariant under part of the SUSY transformations, and that they show up in the SUSY algebra as central extensions. We also briefly discuss the same issue for the class of models originally introduced in [17]. Finally, Section 9.4 contains our conclusions.

9.2 The BPS bound

9.2.1 The models

The present paper continues the investigation of the models introduced in [56], therefore we use the same conventions as in that reference, to which we refer for details. The field theories we consider exist in 1+1 dimensional Minkowski space, and we use the metric convention $ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dx^2$. Furhter, we use the superfield ($\theta^2 = \frac{1}{2}\theta^\alpha\theta_\alpha$)

$$\Phi(x,\theta) = \phi(x) + \theta^\gamma\psi_\gamma(x) - \theta^2 F(x), \tag{9.1}$$

and for the spinor metric to rise and lower spinor indices we use $C_{\alpha\beta} = -C^{\alpha\beta} = (\sigma_2)_{\alpha\beta}$. For the gamma matrices we choose a representation where the components of the Majorana spinor are real. Concretely, we choose (the σ_i are the Pauli matrices)

$$\gamma^0 = \sigma_2\,, \quad \gamma^1 = i\sigma_3\,, \quad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1. \tag{9.2}$$

Further, the superderivative is

$$D_\alpha = \partial_\alpha + i\theta^\beta\partial_{\alpha\beta} = \partial_\alpha - i\gamma^\mu{}_\alpha{}^\beta\theta_\beta\partial_\mu \tag{9.3}$$

and allows to extract the components of an arbitrary superfield via ($D^2 \equiv \frac{1}{2}D^\alpha D_\alpha$):

$$\phi(x) = \Phi(x,\theta)|, \quad \psi_\alpha(x) = D_\alpha\Phi(x,\theta)|, \quad F(x) = D^2\Phi(x,\theta)|, \tag{9.4}$$

(the vertical line | denotes evaluation at $\theta^\alpha = 0$). A Lagrangian always is the θ^2 component of a superfield, so it may be calculated from the corresponding superfield via the projection $D^2|$.

Attempts to find supersymmetric extensions of field theories with non-standard kinetic terms typically face the problem that the auxiliary field couples to derivatives or becomes dynamical. Recently, however, we found linear combinations of superfields such that the auxiliary field F still obeys an algebraic field equation and, in the bosonic sector, only couples to the scalar field ϕ and not to derivatives [56]. The construction uses the following superfields as building blocks,

$$\mathcal{S}^{(k,n)} = (\frac{1}{2}D^\alpha\Phi D_\alpha\Phi)(\frac{1}{2}D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi)^{k-1}(D^2\Phi D^2\Phi)^n \tag{9.5}$$

where $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. The right linear combinations are

$$\mathcal{S}^{(k)} \equiv \sum_{n=0}^{k-1}(-1)^n \binom{k}{n} \mathcal{S}^{(k-n,n)} \tag{9.6}$$

and arbitrary linear combinations of these expressions, each one multiplied by an arbitrary real function $\alpha_k(\Phi)$ of the superfield Φ, are permitted. In addition, we may include a superpotential $-P(\Phi)$. That is to say, we define the superfield

$$\mathcal{S}^{(\alpha,P)} \equiv \sum_{k=1}^{N} \alpha_k(\Phi)\mathcal{S}^{(k)} - P(\Phi) \tag{9.7}$$

(here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multiindex of scalar functions), then the bosonic sector (i.e., with the fermions set equal to zero, $\psi_\alpha = 0$) of the corresponding Lagrangian,

$$\mathcal{L}_{\mathrm{b}}^{(\alpha,P)} \equiv \left(-D^2\mathcal{S}^{(\alpha,P)}|\right)_{\psi=0} \tag{9.8}$$

(b stands for "bosonic") reads explicitly

$$\mathcal{L}_b^{(\alpha,P)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}] - P'(\phi)F \tag{9.9}$$

and, as announced, F only appears algebraically and does not couple to derivatives, see [56] for details.

In a next step, we should eliminate F via its algebraic field equation

$$\sum_{k=1}^{N}(-1)^{k-1}2k\alpha_k(\phi)F^{2k-1} - P'(\phi) = 0 \tag{9.10}$$

which, however, for a given $P(\phi)$ is a rather complicated equation for F with several solutions. It is, therefore, more natural to assume a given on-shell value $F = F(\phi)$ for F and interpret the above equation as a defining equation for the corresponding superpotential P. Eliminating the resulting $P'(\phi)$ we arrive at the Lagrangian density

$$\mathcal{L}_b^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k - (-1)^{k-1}(2k-1)F^{2k}] \tag{9.11}$$

where now $F = F(\phi)$ is a given function of ϕ which we may choose freely depending on the system we want to study. The $\alpha_k(\phi)$, too, are functions which we may choose freely, but they should obey certain restrictions in order to guarantee, e.g., positivity of the energy, or the null energy condition (NEC), see [56] for details. Next, we have to briefly discuss the field equations. For a general Lagrangian $\mathcal{L}(X,\phi)$ where $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}(\dot{\phi}^2 - \phi'^2)$, the Euler–Lagrange equation reads

$$\partial_\mu(\mathcal{L}_{,X}\partial^\mu\phi) - \mathcal{L}_{,\phi} = 0, \tag{9.12}$$

and the energy momentum tensor is

$$T_{\mu\nu} = \mathcal{L}_{,X}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{9.13}$$

For static configurations $\phi = \phi(x)$, $\phi' \equiv \partial_x\phi$, only two components of the energy momentum tensor are nonzero,

$$T_{00} = \mathcal{E} = -\mathcal{L} \tag{9.14}$$

$$T_{11} = \mathcal{P} = \mathcal{L}_{,X}\phi'^2 + \mathcal{L} \tag{9.15}$$

where $\mathcal{E}$ is the energy density and $\mathcal{P}$ is the pressure. Further, for static configurations the Euler–Lagrange equation may be integrated once to give

$$-2X\mathcal{L}_{,X} + \mathcal{L} = \phi'^2\mathcal{L}_{,X} + \mathcal{L} \equiv \mathcal{P} = 0 \tag{9.16}$$

(in general, there may be an arbitrary, nonzero integration constant at the r.h.s. of Eq. (9.16), but the condition that the vacuum has zero energy density sets this constant equal to zero). For the Lagrangian (9.11), we, therefore, get the once integrated static field equation

$$\sum_{k=1}^{N}(2k-1)(-1)^{k-1}\alpha_k(\phi)\left(\phi'^{2k} - F^{2k}\right) = 0. \tag{9.17}$$

In a first step, it is useful to interpret this equation as an algebraic, polynomial equation for ϕ' of order $2N$. It obviously has the two solutions (roots)

$$\phi' = \pm F(\phi) \tag{9.18}$$

which are independent of the $\alpha_k(\phi)$, therefore we call them"generic" roots. In addition, in general it will have $2N-2$ further roots

$$\phi' = \pm R_i(\phi) \; , \quad i = 2, \dots, N \tag{9.19}$$

(we set $R_1 = F$), which depend both on $F(\phi)$ and on $\alpha_k(\phi)$. We, therefore, call them "specific" roots.

9.2.2 Kink solutions

In a next step, we now interpret the roots $\phi' = \pm R_i(\phi)$ as first order differential equations and want to understand under which conditions their solutions may be topological solitons (kinks and antikinks). A first condition is that the potential term in the Lagrangian (9.11),

$$V^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k(\phi)(-1)^{k-1}(2k-1)F^{2k} \tag{9.20}$$

must have at least two vacua, i.e., field values $\phi = \phi_{0,l}$ such that $V(\phi_{0,l}) = 0$, where $l = 1, \dots, L$ and $L \geq 2$. Now we will make some simplifying assumptions. The functions $\alpha_k(\phi)$ should have no singularities, i.e., $|\alpha_k(\phi)| < \infty$ for $|\phi| < \infty$, such that no kinetic term gets artificially enhanced. Further, the standard kinetic term should never vanish, i.e., $\alpha_1 > 0 \; \forall \; \phi$. Under these assumptions, the standard kinetic term dominates in the vicinity of the vacua, and the standard asymptotic analysis for kink solutions applies. A kink (antikink) is a static solution $\phi_k(x)$ which interpolates between two vacua, $\phi_k(\pm\infty) \equiv \phi_\pm \in \{\phi_{0,l}\}$, where for a kink it holds that $\phi_+ > \phi_-$, whereas for an antikink $\phi_- > \phi_+$. We shall assume in what follows that the signs of all the roots R_i have been chosen such that $\phi' = +R_i$ corresponds to the kink (if this equation has a kink solution, at all), and $\phi' = -R_i$ corresponds to the antikink.

A necessary condition for a root $R_i(\phi)$ to provide a kink solution is that it must have two zeros at two different vacua, i.e., $R_i(\phi_\pm) = 0$. This is a nontrivial condition because, generically, roots may have no or one zero, as well, with the only condition that the total number of zeros of all the roots coincides with the number of vacua of the potential, including multiplicities. In other words, both the existence of a sufficient number of vacua and the existence of roots with two zeros requires some finetuning of the functions F and α_k. The simplest way to achieve this finetuning is via symmetry considerations. If, for instance, F and all the α_k are symmetric under the reflection $\phi \to -\phi$, then all the roots R_i inherit this symmetry. If, therefore, a root has a zero $\phi_{0,l}$ then it has the second zero $-\phi_{0,l}$, by construction. The only additional finetuning required in this case is that the potential must have at least one vacuum at $\phi \neq 0$.

The generic root $\phi' = F$ will lead to a kink solution if the function F has at least two zeros, which obviously provide the corresponding vacua in the potential, see Eq. (9.20). We shall call the resulting kink solutions "generic kinks". If we choose, e.g., $F = 1 - \phi^2$, then all models with this F (i.e., for arbitrary α_k) will have the standard ϕ^4 kink $\phi_k = \tanh(x - x_0)$ (here x_0 is an integration constant reflecting translational invariance). Depending on the α_k, these models may have further kink solutions, based on some of the specific roots R_i, $i = 2, \ldots, N$. If these kinks exist, we shall call them "specific kinks".

We remark that for different roots which only have one zero each, but for different vacuum values, it is sometimes still possible to construct kink solutions interpolating between the two vacua in the space $\mathbf{C}^1$ of continuous functions with a continuous first derivative. Indeed, if two different roots R_i and R_j with two different zeros have a common range of values $\phi \in [\phi_<, \phi_>]$ between the two vacua, then we may form a kink solution in the space $\mathbf{C}$ of continuous functions with a discountinuous first derivative by joining the two local solutions at any value in the common range (the joining point x_0 in base space is arbitrary due to translational invariance). If, in addition, the equation $R_i(\phi) = R_j(\phi)$ has a solution ϕ_s in the common range, then the derivatives of the two local solutions coincide at this point, and we may form a kink solution in the space $\mathbf{C}^1$ by joining the two local solutions at ϕ_s.

Let us point out that if we require kinks to be solutions of the corresponding variational problem, then solutions in the space $\mathbf{C}^1$ are perfectly valid. They lead to well-defined energy densities and, therefore, provide well-defined critical points of the corresponding energy functional. For more details and some explicit examples, we refer to [56].

9.2.3 Kink energies and BPS bounds

In a next step, we want to study the energies of kinks. The energy density for the Lagrangian (9.11) is

$$\mathcal{E}_b^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k(\phi) \left((\dot{\phi}^2 - \phi'^2)^{k-1}((2k-1)\dot{\phi}^2 + \phi'^2) + (-1)^{k-1}(2k-1)F^{2k} \right) \tag{9.21}$$

and, for static configurations,

$$\mathcal{E} = \sum_{k=1}^{N} (-1)^{k-1} \alpha_k(\phi) \left(\phi'^{2k} + (2k-1)F^{2k} \right). \tag{9.22}$$

With the help of Eq. (9.17), for kink solutions this may be expressed like

$$\mathcal{E} = \sum_{k=1}^{N} (-1)^{k-1} 2k\alpha_k(\phi)\phi'^{2k} = \phi' \sum_{k=1}^{N} (-1)^{k-1} 2k\alpha_k(\phi)\phi'^{2k-1} \equiv \phi' w(\phi, \phi') \tag{9.23}$$

where the last expression is especially useful for the calculation of the corresponding energies. Indeed, for the energy calculation we should now replace ϕ' in $w(\phi, \phi')$ by the root R_i which corresponds to the kink solution, and interpret the resulting function of ϕ as the ϕ derivative of another function. That is to say, we define an integrating function $W_i(\phi)$ for each root R_i via

$$W_{i,\phi} \equiv w(\phi, R_i(\phi)) = \sum_{k=1}^{N} (-1)^{k-1} 2k\alpha_k(\phi) R_i^{2k-1}, \tag{9.24}$$

then the kink energy is

$$E = \int dx \phi' W_{i,\phi} = \int d\phi W_{i,\phi} = W_i(\phi_+) - W_i(\phi_-). \tag{9.25}$$

For the calculation of the kink energy we, therefore, do not have to know the kink solution. We just need the root and the two vacuum values $\phi_\pm$ of the kink. For the $\mathbf{C}^1$ kinks described above which are constructed by joining local solutions for two different roots R_i and R_j, we need the two corresponding integrating functions and the joining point ϕ_s. The energy then results in

$$E = W_j(\phi_+) - W_j(\phi_s) + W_i(\phi_s) - W_i(\phi_-). \tag{9.26}$$

Until now, the energy considerations have been for arbitrary roots, but now we shall see that the generic root $R_1 \equiv F$ apparently plays a particular role. Firstly, the integrating function of the generic root is just the superpotential, $W_1 = P$. Indeed, we find

$$W_{1,\phi} = \sum_{k=1}^{N} (-1)^{k-1} 2k\alpha_k(\phi) F^{2k-1} \equiv P'(\phi) \tag{9.27}$$

see Eq. (9.10). Secondly, if the generic root has a kink solution, then this solution is, in fact, a BPS solution and saturates a BPS bound, as we want to demonstrate now. In general, an energy density has a BPS bound if it may be written off-shell (i.e. without using the static Euler-Lagrange equation) as

$$\mathcal{E} = (PSD)(\phi, \phi') + t(x) \tag{9.28}$$

where (PSD) is a positive semi-definite function of ϕ and ϕ', and $t(x)$ is a topological density, i.e., a total derivative whose integral only depends on the boundary values $\phi_\pm$. Further, a soliton solution (a kink ϕ_k) is of the BPS type, i.e., saturates the BPS bound if the positive semi-definite function is zero when evaluated for the kink, $(PSD)(\phi_k, \phi_k') = 0$. In our case, the possible topological terms are the expressions $\phi' W_{i,\phi}$ for the different roots. In any case, a possible topological term must be linear in ϕ' in order to be a total derivative (we emphasize, again, that the BPS form (9.28) must be valid off-shell, i.e., it is not legitimate to replace ϕ' by a root R_i or vice versa). Let us now demonstrate that the energy density may be expressed in BPS form (9.28) for the generic topological term $t = \phi' W_{1,\phi} \equiv \phi' P_{,\phi}$, and that the corresponding positive semi-definite function is zero precisely for the generic kink, i.e., for $\phi' = F$. Indeed, we find for the difference $\mathcal{E} - t$ for the generic topological term

$$\begin{aligned}
\mathcal{E} - \phi' P_{,\phi} &= \sum_{k=1}^{N} (-1)^{k-1} \alpha_k(\phi) \left(\phi'^{2k} + (2k-1)F^{2k} - 2k\phi' F^{2k-1} \right) = \\
&= (\phi' - F)^2 S(\phi', F) \equiv \\
&\equiv (\phi' - F)^2 \sum_{k=1}^{N} (-1)^{k-1} \alpha_k(\phi) H_k(\phi', F) \qquad (9.29)
\end{aligned}$$

where

$$H_k(\phi', F) \equiv \sum_{i=1}^{2k-1} i \phi'^{2k-1-i} F^{i-1}. \qquad (9.30)$$

Before proving this algebraic identity, we want to make some comments. The above result implies a genuine BPS soliton provided that the positive semi-definite function is zero only iff ϕ obeys the corresponding generic kink equation $\phi' = F$. This implies that $S(\phi', F)$ must be strictly positive for any nontrivial field configuration (for the trivial vacuum $\phi' = 0$ and $F = 0$ it holds that $S(0, 0) = 0$), i.e., $S(a, b) > 0$ unless $a = 0$ and $b = 0$. This inequality, indeed, holds for each individual term $H_k(a, b)$, i.e., $H_k(a, b) > 0$ unless $a = 0$ and $b = 0$ (the proof requires two complete inductions, therefore we relegate it to appendix A). The inequality $S(a, b) > 0$ for the complete function S, therefore, implies some restrictions on the functions $\alpha_k(\phi)$ (one possible choice is that the α_k are zero for even k and positive semi-definite for odd k, but there are less restrictive choices). This is similar to the conditions of positivity of the energy density, or the NEC, which, too, imply some restrictions on the α_k, (again, α_k zero for even k and positive semi-definite for odd k is a possible choice), and we shall assume in the sequel that the α_k obey these restrictions (i.e., the restrictions resulting from the condition $S > 0$, and either positivity of the energy density or the NEC; these restrictions are probably related, but we shall not investigate this problem further and assume the two restrictions independently). Now let us prove the algebraic identity between Eq. (9.29) and Eq. (9.29). This follows from the following identities (we set $\phi' = a$, $F = b$)

$$\begin{aligned} & a^{2k} + (2k-1)b^{2k} - 2kab^{2k-1} && (9.31) \\ = \; & (a-b)\left(a^{2k-1} + a^{2k-2}b + a^{2k-3}b^2 + \ldots + ab^{2k-2} - (2k-1)b^{2k-1}\right) \\ = \; & (a-b)^2\left(a^{2k-2} + 2a^{2k-3}b + 3a^{2k-4}b^2 + \ldots + (2k-1)b^{2k-2}\right) \\ \equiv \; & (a-b)^2 H_k(a,b) && (9.32) \end{aligned}$$

where the equality of adjacent lines may be checked easily.

So we found, indeed, that generic kinks (if they exist) saturate a BPS bound, whereas up to now we could not make a comparable statement about additional "specific" kinks. This special role played by the generic kink solution is not surprising from the point of view of the supersymmetric extension, because only the generic kink obeys the simple equation $\phi' = F$, and only the generic kink has a topological charge which may be expressed in terms of the superpotential. on the other hand, the special character of the generic kink *is* surprising from the point of view of the purely bosonic theory

$$\mathcal{L}_b = \sum_{k=1}^{N} \alpha_k(\phi)(\partial^\mu\phi\partial_\mu\phi)^k - V(\phi) \tag{9.33}$$

(with given α_k and a given potential V), whose once-integrated static field equation just leads to the $2N$ roots

$$\phi' = \pm R_i(\phi) \; , \quad i = 1, \ldots, N \tag{9.34}$$

without distinguishing them in terms of an auxiliary field or a superpotential. The resolution of the puzzle may be understood if we express the once-integrated static field equation both in terms of the potential and in terms of the on-shell auxiliary field,

$$\sum_{k=1}^{N}(2k-1)(-1)^{k-1}\alpha_k(\phi)\left(\phi'^{2k} - F^{2k}\right) = \sum_{k=1}^{N}(2k-1)(-1)^{k-1}\alpha_k(\phi)\phi'^{2k} - V = 0. \tag{9.35}$$

Up to now we assumed a given $F(\phi)$ which lead to the two generic roots $\phi' = F$ and the remaining, specific roots. But now we may interpret this equation in a different way. We may treat only V and the α_k as given and try to find all the solutions for F of the equation

$$\sum_{k=1}^{N}(2k-1)(-1)^{k-1}\alpha_k(\phi)F^{2k} = V. \tag{9.36}$$

obviously, the solutions are just the roots $F_i = R_i(\phi)$ (see Eq. (9.17)), and the corresponding first order equations now just read $\phi' = \pm F_i$. We remark that different on-shell choices F_i for the auxiliary field F lead to different superpotentials and, therefore, to different supersymmetric extensions. As a result, the resolution of the puzzle is that one given bosonic theory allows for N different supersymmetric extensions such that each kink solution is the generic solution of its corresponding supersymmetric extension. As a consequence, the energy density allows for BPS bounds for all kink solutions. The existence of several BPS bounds for one and the same energy density may seem surprising, but the different bounds exist, of course, in different topological sectors (i.e., for different boundary values), so there is no contradiction. Finally, all topological charges (i.e., all BPS energies) are now given in terms of the corresponding superpotentials. Indeed, we calculate (see Eqs. (9.10), (9.23) and (9.24))

$$W_{i,\phi}(\phi) = w(\phi, R_i(\phi)) = w(\phi, F_i) = P'(F_i(\phi)) \equiv P_i'(\phi). \tag{9.37}$$

We remark that from a practical point of view it is still useful to choose a specific on-shell $F(\phi)$, because in this way we may choose simple functions with simple kink solutions. For generic α_k and a generic V, on the other hand, the resulting roots will usually be quite complicated.

9.3 SUSY algebra and central extensions

From now on, we will, again, restrict to a fixed supersymmetric extension, i.e., to fixed, given α_k, a fixed, given on-shell auxiliary field $F(\phi)$ and the corresponding superpotential given by Eq. (9.10). The SUSY transformations of the fields read

$$\delta\phi = \epsilon^\alpha\psi_\alpha \;, \quad \delta\psi_\alpha = -i(\gamma^\mu)_\alpha{}^\beta\epsilon_\beta\partial_\mu\phi - \epsilon_\alpha F \;, \quad \delta F = i\epsilon^\alpha(\gamma^\mu)_\alpha{}^\beta\partial_\mu\psi_\beta \tag{9.38}$$

(where $\epsilon_\alpha = (\epsilon_1, \epsilon_2)$ are the Grassmann-valued SUSY transformation parameters, and $\epsilon^\alpha = (i\epsilon_2, -i\epsilon_1)$), or more explicitly

$$\begin{aligned} \delta\phi &= i\,(\epsilon_2\psi_1 - \epsilon_1\psi_2) \\ \delta F &= i\left(\epsilon_2(\psi_1' - \dot{\psi}_2) - \epsilon_1(\dot{\psi}_1 - \psi_2')\right) \\ \delta\psi_1 &= \epsilon_1(\phi' - F) - \epsilon_2\dot{\phi} \\ \delta\psi_2 &= \epsilon_1\dot{\phi} - \epsilon_2(\phi' + F). \end{aligned} \tag{9.39}$$

obviously, for a generic kink solution ($\dot{\phi} = 0, \phi' = F, \psi_\alpha = 0$) the SUSY transformation restricted to $\epsilon_2 = 0$ is zero, whereas for a generic antikink the restriction $\epsilon_1 = 0$ gives zero.

On the other hand, the SUSY transformations of the fields are generated by the SUSY generators $Q = \epsilon^\alpha Q_\alpha$ via the commutators $\delta\phi = [iQ, \phi]$, etc., where Q should be determined from the Noether current of the SUSY transformations, and the commutators are evaluated with the help of the canonical (anti-)commutation relations of the fields. The supercharges Q_α are known to obey the algebra

$$\{Q_\alpha, Q^\beta\} = 2\Pi_\nu(\gamma^\nu)_\alpha{}^\beta + 2iZ(\gamma^5)_\alpha{}^\beta \tag{9.40}$$

or, explicitly,

$$\begin{aligned} Q_1^2 &= \Pi_0 + Z \\ Q_2^2 &= \Pi_0 - Z \\ \{Q_1, Q_2\} &= 2\Pi_1 \end{aligned} \tag{9.41}$$

where the curly bracket is the anti-commutator, $\Pi_\nu = (\Pi_0, \Pi_1)$ are the energy and momentum operators, and Z is a possible central extension which the SUSY algebra may contain. An explicit calculation of the operators which appear in the SUSY algebra requires the knowledge of the Noether current and the canonical momenta and, therefore, of the complete SUSY Lagrangian, including the fermionic terms, which, in general, is quite complicated. If we only want to determine the central charge, however, it is enough to evaluate the SUSY algebra for a specific field configuration, because the central charge is essentially a number (it commutes with all operators) and, therefore, must take the same value for all field configurations within a given topological sector. We now evaluate the SUSY algebra for a generic kink solution and make the reasonable assumption that not only the restricted SUSY transformation (i.e., the action of the corresponding SUSY charge on the fields) for a generic kink is zero, but that the corresponding SUSY charge itself is zero when evaluated for the generic kink. As we know the energy of the kink, this allows then to determine the central charge. Concretely, for the kink the corresponding charge is Q_2, and we get

$$Q_2^2 = 0 = E_k - Z = P(\phi_+) - P(\phi_-) - Z \quad \Rightarrow \quad Z = P(\phi_+) - P(\phi_-), \tag{9.42}$$

where P is the superpotential, and $\phi_\pm$ are the asymtopic values of the kink. For the antikink, Q_1 is zero, and we find $Z = P(\phi_-) - P(\phi_+)$. We remark that for positive semi-definite energy densities the resulting restrictions on the functions α_k imply that the central extension Z is always positive, because $P' \geq 0$ for the kink, and $P' \leq 0$ for the antikink, as follows from the energy density (9.22) and the defining equation for P', Eq. (9.10). We, therefore, found exactly the same result for the central extension as in the case of the SUSY extension of a standard scalar field theory with a quadratic kinetic term for the boson field.

9.3.1 Central extensions for the models of Bazeia, Menezes and Petrov

Here we want to demonstrate that the same central extensions of the SUSY algebra in terms of the superpotential may be found for another class of supersymmetric K field theories, originally introduced by Bazeia, Menezes and Petrov (BMP) [17]. They are based on the superfield

$$\mathcal{S}_{\text{BMP}} = f(\partial_\mu \Phi \partial^\mu \Phi) \frac{1}{2} D_\alpha \Phi D^\alpha \Phi \tag{9.43}$$

and lead to the bosonic Lagrangian

$$\mathcal{L}_{\text{BMP}} = f(\partial_\mu \phi \partial^\mu \phi)(F^2 + \partial_\mu \phi \partial^\mu \phi). \tag{9.44}$$

Here, the Lagrangian produces a coupling of the auxiliary field F with the kinetic term $\partial_\mu \phi \partial^\mu \phi$, but, on the other hand, the auxiliary field only appears quadratically, implying a linear (algebraic) field equation for F. The same bosonic Lagrangians may, in fact, be constructed from the building blocks (9.5) of Section 9.2 by taking a different linear combination (the fermionic parts of the corresponding Lagrangians will in general not coincide)

$$\mathcal{S}^{(k)}_{\rm BMP} \equiv \sum_{n=0}^{k-1} (-1)^n \binom{k-1}{n} \mathcal{S}^{(k-n,n)} \tag{9.45}$$

leading to the bosonic Lagrangians

$$\mathcal{L}^{(k)}_{\rm BMP} = (F^2 + \partial_\mu \phi \partial^\mu \phi)(\partial_\mu \phi \partial^\mu \phi)^{k-1}. \tag{9.46}$$

We may easily recover the Lagrangian (9.44) by taking linear combinations of these,

$$\begin{aligned} \mathcal{L}_{\rm BMP} &= \sum_{k=1}^{\infty} \beta_k \mathcal{L}^{(k)}_{\rm BMP} = (F^2 + \partial_\mu \phi \partial^\mu \phi) \sum_k \beta_k (\partial_\mu \phi \partial^\mu \phi)^{k-1} \equiv \\ &\equiv (F^2 + \partial_\mu \phi \partial^\mu \phi) f(\partial_\mu \phi \partial^\mu \phi). \end{aligned} \tag{9.47}$$

Adding a superpotential, the resulting bosonic Lagrangians are

$$\mathcal{L}^{(P)}_{\rm BMP} = f(\partial_\mu \phi \partial^\mu \phi)(F^2 + \partial_\mu \phi \partial^\mu \phi) - P'(\phi) F, \tag{9.48}$$

or, after eliminating the auxiliary field F using its algebraic field equation

$$F = \frac{P'}{2f}, \tag{9.49}$$

$$\mathcal{L}^{(P)}_{\rm BMP} = f \cdot (\frac{P'^2}{4f^2} + \partial_\mu \phi \partial^\mu \phi) - \frac{P'^2}{2f} = f \cdot \partial_\mu \phi \partial^\mu \phi - \frac{P'^2}{4f}. \tag{9.50}$$

The energy functional for static configurations may be written in a BPS form. Indeed,

$$E^{(P)}_{\rm BMP} = \int dx \left(\phi'^2 f + \frac{P'^2}{4f} \right) = \int dx \left(\frac{1}{4f} (2\phi' f \mp P')^2 \pm \phi' P' \right) \tag{9.51}$$

and for a solution to the first order (or BPS) equation

$$2\phi'(x) f(-\phi'^2) = P' \tag{9.52}$$

(we take the plus sign for a kink) the resulting energy is

$$E_{\rm BMP}^{(P)} = \int_{-\infty}^{\infty} dx \phi' P' = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi P' = P(\phi_+) - P(\phi_-). \qquad (9.53)$$

Finally, from Eq. (9.49) for F and the BPS equation (9.52) it follows that the equation $\phi' = F$ still holds for a kink solution and, therefore, the restricted SUSY transformation with only ϵ_1 nonzero is, again, zero when evaluated for the kink. We conclude that the central charge in the SUSY algebra is, again, given by the topological term

$$Z = |P(\phi_+) - P(\phi_-)| \qquad (9.54)$$

for this class of models.

9.4 Conclusions

In this paper we carried further the investigation of a class of SUSY K field theories originally introduced in [56]. Concretely, we demonstrated that all the domain wall solutions which exist for this class of field theories are, in fact, BPS solutions. Further, these BPS solutions are invariant under part of the SUSY transformations. We also found strong indications (based on a very reasonable assumption) that the topological charges carried by the domain wall solutions show up in the SUSY algebra as central extensions. That is to say, the situation we found is exactly equivalent to the case of standard SUSY theories with BPS solitons, despite the much more complicated structure of the SUSY K field theories investigated here. Let us emphasize, again, that from an effective field theory point of view, K field theories are as valid as field theories with a standard kinetic term, and there exists no reason not to consider them. Even one and the same topological defect with some given, well-known physical properties may result either from a theory with a canonical kinetic term, or from a certain related class of K field theories (so-called noncanonical twins of the standard, canonical theory), [51], [57]. K field theories should, therefore, be considered on a par with standard field theories in all situations where they cannot be excluded a priori. This implies that also the study of their possible SUSY extensions is a valid and relevant subject. Structural investigations of the type employed in the present paper are, then, important steps towards a better understanding of these supersymmetric generalized field theories with nonstandard kinetic terms.

Appendix A

We want to prove that

$$a^{2k-2} + 2a^{2k-3}b + \ldots + (2k-1)b^{2k-2} > 0 \quad \forall \quad k \tag{9.55}$$

unless $a = 0$ and $b = 0$. For $a = 0$, $b \neq 0$, and for $a \neq 0$, $b = 0$ this is obvious, so we may restrict to the case $a \neq 0$ and $b \neq 0$. In this case, we may divide by b^{2k-2}, so that we have to prove ($x \equiv a/b$)

$$f_k(x) \equiv x^{2k-2} + 2x^{2k-3} + \ldots + (2k-1) > 0 \tag{9.56}$$

which we do by complete induction. obviously, the statement is true for $k = 1$: $f_1(x) = x^2 + 2x + 3 = (x+1)^2 + 2 > 0$. Now we assume that it holds for f_k and calculate f_{k+1}. We get

$$f_{k+1}(x) = x^{2k} + 2(x^{2k-1} + x^{2k-2} + \ldots + 1) + f_k(x) \equiv g_k(x) + f_k(x) \tag{9.57}$$

and the statement is true if $g_k(x) \geq 0 \; \forall \; k$. This, again, we prove by induction. obviously, it is true for $k = 1$: $g_1(x) = x^2 + 2x + 2 \geq 0$. For g_{k+1} we calculate

$$g_{k+1}(x) = x^{2k}(x+1)^2 + g_k(x) \tag{9.58}$$

and it is obviously true that $g_k(x) \geq 0 \;\; \Rightarrow \;\; g_{k+1}(x) \geq 0$ and, therefore, $f_k(x) > 0 \; \Rightarrow \; f_{k+1}(x) > 0$, which is what we wanted to prove.

Chapter 10

Twin-like models and SUSY

In the previous chapters we accomplished the SUSY extension of general K field theories and we also analyzed different properties. In this chapter, we show the algebraic conditions that a K field theory must verify to have a twin model, i.e. a standard theory whose solutions have the same profile and the same energy density. This property would provide a method to study in more detail such theories. This chapter consists of a paper published in [97].

An algebraic construction of twin-like models

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Abstract: If the generalized dynamics of K field theories (i.e., field theories with a non-standard kinetic term) is taken into account, then the possibility of so-called twin-like models opens up, that is, of different field theories which share the same topological defect solution with the same energy density. These twin-like models were first introduced in Phys. Rev. D**82**, 105006 (2010), Ref. [51], where the authors also considered possible cosmological implications and gave a geometric characterization of twin-like models. A further analysis of the twin-like models was accomplished in Phys. Rev. D**84**, 045010 (2011) , Ref. [58], with the help of the first order formalism, where

also the case with gravitational self-interaction was considered. Here we show that by combining the geometric conditions of Ref. [51] with the first order formalism of [58], one may easily derive a purely algebraic method to explicitly calculate an infinite number of twin field theories for a given theory. We determine this algebraic construction for the cases of scalar field theories, supersymmetric scalar field theories, and self-gravitating scalar fields. Further, we give several examples for each of these cases.

10.1 Introduction

There exist wide classes of classical non-linear field theories which support topological defect solutions. These topological defect solutions typically have their energy densities concentrated in a certain finite region of space and are stable, where their stability is related to topological properties of the base and target spaces. Here, a nontrivial topological structure in base space (e.g., an effective compactification) is usually induced by the requirement of finite energy. These topological defects have found applications in many fields of physics, and in particular may have important applications in the field of cosmology. On the one hand, they may be relevant for structure formation in the early universe, and for its resulting evolution. Indeed, if the very early universe passed through a phase transition from a symmetric to a symmetry-breaking phase, then in the broken phase topological defects may have formed and influenced the distribution of matter and energy, see e.g. [23], [24], [25]. On the other hand, there exists the idea that the whole visible universe might be just a topological defect in some higher-dimensional bulk space, the so-called brane-world scenario. The brane (i.e., our universe) in this scenario may be either strictly 3+1 dimensional ("thin brane") or have a small but nonzero extension also in the additional dimensions ("thick brane"). In the latter, thick brane case, these branes are normally topological defects in the higher-dimensional bulk space [26], [27], [28], [29], [30], [9]. In these applications, the relevant topological defects are usually solutions of some (effective or fundamental) scalar field theory, where the theory may either be of the standard type (standard kinetic term plus a potential) or of a more general type where the Lagrangian density may be a general function of the fields and their first derivatives. These generalized theories where the kinetic term does not have to be of the standard form (so-called K field theories) have already found some applications, beginning with the observation about a decade ago that they might be relevant for the solution of some problems in cosmology, like K-inflation [2] and K-essence [3]. Further applications of K fields to cosmological issues may be found, e.g., in [59], [7], [4], [5], [60], [31] [33], whereas other, more formal or mathematical aspects of K field theories, like the existence of topological defects with compact support (so-called compactons) have been studied, e.g., in [35] -[40]. Well-posedness of the K field system and the issue of signal propagation in K field backgrounds has been investigated, e.g., in [4] and, recently, in [62].

The larger class of models allowed by generalized K field theories introduces further scales into the system under consideration via additional dimensionful couplings, therefore the resulting topological defects are, in general, quite different from their standard counterparts, see e.g. [5], [32], [12]. Quite recently it has been found, however, that there exists the possibility that a topological defect of a non-standard K field theory perfectly mimics a defect of a standard field theory by coinciding with the standard solution both in the profile (i.e., in the defect solution itself) and in the corresponding energy density [51]. These coinciding solutions with their coinciding energy densities were dubbed twin or Doppelgänger defects in Ref. [51]. The shape (profile) of a defect together with its energy density are the physically most relevant properties of a defect in a cosmological setting, therefore the possibility of these twins implies that, e.g., the influence of a pattern of K defects on the evolution of the universe could be mimicked by its standard twin, or vice versa. More generally, all measurable physical properties which are determined by the field profile and the energy density are indistinguishable between the K field theory and its standard twin. A more refined analysis shows, however, that there remain some differences between two twin-like models. The spectrum of linear fluctuations about the K field theory and its standard twin, for instance, are in general different [51], [58]. The authors of [51] discussed the example of a Dirac–Born–Infeld (DBI) type twin of a standard field theory in some detail, motivated by string theory considerations. They also gave a geometric characterization which possible twins of a standard theory have to obey and concluded from these that there exist, in general, infinitely many K field twin models for a given standard scalar field theory. The study of twin-like models was carried further in Ref. [58], where the authors employed the first order formalism in their analysis. They also considered the case with gravitational backreaction in 4+1 dimensions, where their results are of direct relevance for the brane world scenario. Further, they gave explicit examples of all cases they considered.

It is the purpose of the present acticle to derive a purely algebraic method for the construction of K field twins of a given scalar field theory which does not require knowledge of either the defect solution or its energy density. This algebraic construction may be found by combining the geometric characterization of twins of Ref. [51] with the first order formalism of [58] and allows to explicitly calculate an infinite number of twin field theories for any given scalar field theory.

Our paper is organized as follows. In section 10.2 we briefly review the first order formalism and the geometric characterization of twins. Then we explain the algebraic construction of twin models and give several explicit examples among which the examples of Refs. [51] and [58] can be found. We also briefly discuss stability issues (energy positivity and the null energy condition (NEC)). In Section 10.3 we repeat the same analysis for supersymmetric K field twins of supersymmetric scalar field theories. Here, one important pillar of the construction is, of course, the fact that supersymmetric K field theories exist at all, which has been demonstrated recently [63], [17], [64] (for supersymmetric K field theories in 3+1 dimensions see [135], [136]). Defects of supersymmetric theories may be of cosmological relevance if the formation of these defects occurs at time or energy scales when supersymmetry is still unbroken. In section 10.4 we consider the case of a self-gravitating scalar field in arbitrary dimensions, where the defects are of the wall type (i.e., still co-dimension one defects, like in the previous sections). We again derive the purely algebraic construction of K field twins of a self-gravitating standard field theory with topological defect solutions and provide several examples. In 3+1 dimensions these are just the defect solutions which are required for cosmological considerations, but now with the gravitational backreaction taken into account. In 4+1 dimensions the defects are the ones relevant for the brane world picture, where we also rederive the example already given in [58]. Section 10.5 contains a discussion of our results.

10.2 Twin-like models

10.2.1 Generalized K fields and first order formalism

The first order formalism for generalized K fields has been developed, e.g., in [32], to which we refer for a more detailed discussion. Here we just review those aspects which we shall need in the subsequent discussion. For a general Lagrangian $\mathcal{L}(X,\phi)$ where $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}(\dot{\phi}^2 - \phi'^2)$, the energy momentum tensor reads

$$T_{\mu\nu} = \mathcal{L}_{,X}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{10.1}$$

and the Euler–Lagrange equation is

$$\partial_\mu(\mathcal{L}_{,X}\partial^\mu\phi) - \mathcal{L}_{,\phi} = 0 \tag{10.2}$$

For static configurations $\phi = \phi(x)$, $\phi' \equiv \partial_x\phi$, the nonzero components of the energy momentum tensor are

$$\begin{aligned} T_{00} &= \mathcal{E} = -\mathcal{L} \qquad (10.3) \\ T_{11} &= \mathcal{P} = \mathcal{L}_{,X}\phi'^2 + \mathcal{L} \qquad (10.4) \end{aligned}$$

where $\mathcal{E}$ is the energy density and $\mathcal{P}$ is the pressure. The static Euler–Lagrange equation reads

$$(\mathcal{L}_{,X}\phi')' + \mathcal{L}_{,\phi} = 0 \tag{10.5}$$

and, after multiplication with ϕ', may be integrated once to give

$$-2X\mathcal{L}_{,X} + \mathcal{L} = \phi'^2\mathcal{L}_{,X} + \mathcal{L} \equiv \mathcal{P} = c \tag{10.6}$$

where c is an integration constant. For our purposes the only acceptable value of this constant is zero for the following reason. All the models we shall consider will have one or several (constant) vacuum values $\phi = \phi_{0i}$,

$i = 1, ..., n$, where the energy density takes its minimum value, and this minimum value is equal to zero (this may always be achieved by adding a constant to the Lagrangian). Further, static finite energy solutions (kinks) have to approach vacuum values for $|x| \to \infty$, which implies that for these finite energy solutions c in the above equation must be zero in the same limits. But c is a constant, so it is zero everywhere. Therefore, the once integrated field equation for static fields (or zero pressure condition) in our case reads ($\phi'^2 = -2X$)

$$-2X\mathcal{L}_{,X} + \mathcal{L} = 0. \tag{10.7}$$

Eq. (10.7) is a nonlinear first order oDE, but sometimes it is preferable to view it just as an algebraic equation for ϕ' with one or several (N) pairs of roots

$$(\phi_i')^2 = f_i(\phi)^2 \quad \Rightarrow \quad \phi_i' = \pm f_i(\phi)\ , \quad i = 1 \dots N \tag{10.8}$$

as solutions. A kink solution will, in general, be the solution to one of these roots (when viewed as a first order oDE), or it may even be the result of joining different solutions in a smooth way.

It is one of the virtues of the first order formalism that the knowledge of the roots (10.8) together with the asymptotic values (i.e., vacuum values) $\phi_\pm \equiv \phi_k(\pm\infty)$ of the kink solution $\phi_k(x)$ is sufficient for the calculation of the kink energy, i.e., one does not need the explicit solution $\phi_k(x)$. The important point is that with the help of the corresponding root, the energy density of a kink may be viewed as a function of either only ϕ or only ϕ'. This allows one to separate a factor ϕ' from the energy density which, together with the base space differential dx in the energy functional, may be traded for a target space differential according to $d\phi = dx\phi'$. The remainder must, of course, be interpreted as a function of ϕ only. Explicitly, the energy reads

$$E = \int_{-\infty}^{\infty} dx\mathcal{E} \equiv \int_{-\infty}^{\infty} dx\phi' W_{,\phi} = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi W_{,\phi} = W(\phi(\infty)) - W(\phi(-\infty)) \tag{10.9}$$

where $W_{,\phi}$ (and its ϕ integral $W(\phi)$) must be interpreted as a function of ϕ only, which results upon replacing ϕ' by its corresponding root $f_i(\phi)$ in the above expression.

For theories with a standard kinetic term X and a potential $V(\phi)$,

$$\mathcal{L}_s = X - V, \tag{10.10}$$

the integrated static field equation simply is

$$-X - V = 0 \quad \Rightarrow \quad \phi'^2 = 2V \tag{10.11}$$

with the two roots $\phi' = \pm\sqrt{2V}$. If the potential V has at least two vacua (which we assume from now on), then there will exist, in general, finite energy solutions of Eq. (10.11) which interpolate between different vacua (kinks), and the two roots correspond to kink and antikink, respectively. The static energy density for the standard theory is

$$\mathcal{E}_s = -X + V = \frac{1}{2}\phi'^2 + V \tag{10.12}$$

and for a kink solution it may be written as

$$\mathcal{E}_s|_{\phi_k} = (-X + V)|_{\phi_k} = 2V(\phi_k) = -2X|_{\phi_k} \tag{10.13}$$

where $\phi_k(x)$ is the kink solution under consideration, and the notation $|_{\phi_k}$ means that the expression (in general, a function of ϕ and ϕ'), is evaluated at the kink solution $\phi = \phi_k(x)$. Finally, the energy of a kink in this standard case simply is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx\phi'^2 = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi\phi' = \int_{\phi(-\infty)}^{\phi(\infty)} d\phi(\pm\sqrt{2V}) = \\ &= \pm[W_V(\phi(\infty)) - W_V(\phi(-\infty))] \end{aligned} \tag{10.14}$$

where

$$W_{V,\phi} = \sqrt{2V} \tag{10.15}$$

and the explicit expression for W_V depends, of course, on V. The two signs correspond to kink and antikink, respectively.

10.2.2 Twin or doppelgaenger defects

In [51] the authors observed the possibility of twin-like models within the class of generalized K field theories, that is, of field theories which share the same kink solution with the same energy density with a given standard field theory. They discussed a Dirac-Born-Infeld (DBI) like example in some detail where, however, the DBI term is multiplied by a target space geometric factor, because a pure DBI theory cannot be the twin of a standard field theory. Then they derived a necessary and sufficient geometrical condition which a second field theory $\mathcal{L}_2$ has to obey in order to be the twin of a given field theory $\mathcal{L}_1$. From their geometric description they already concluded that there exist, in principle, infinitely many twin theories for a given standard scalar field theory. We shall review this geometric construction in a first step, because we will find that combining it with the first order formalism provides us with a simple and purely algebraic method to explicitly calculate an infinite number of twin models for any given field theory. The authors of [51] demonstrated that if the theory $\mathcal{L}_1$ has a kink solution $\phi_k(x)$ with energy density $\mathcal{E}_k(x)$, then a necessary and sufficient condition for a second theory $\mathcal{L}_2$ to have the same kink solution with the same energy density is that both $\mathcal{L}$ and $\mathcal{L}_{,X}$ agree when evaluated for the kink solution, that is,

$$\mathcal{L}_1|_{\phi_k} = \mathcal{L}_2|_{\phi_k} \tag{10.16}$$

$$\mathcal{L}_{1,X}|_{\phi_k} = \mathcal{L}_{2,X}|_{\phi_k}. \tag{10.17}$$

obviously, the first condition implies that the energy densities are equal, see Eq. (10.3). Further, the first order equation Eq. (10.7) holds for $\mathcal{L}_1$ by assumption, then the two conditions (10.16) and (10.17) imply that Eq. (10.7) is an identity for $\mathcal{L}_2$. It follows that the two conditions (10.16) and (10.17) are sufficient for $\mathcal{L}_2$ to be a twin of $\mathcal{L}_1$. That the two conditions are necessary follows easily from the fact that the two equations (10.3) and (10.7) are linear in $\mathcal{L}$ and $\mathcal{L}_X$.

From what has been said above, it might appear that for the explicit construction of a twin model $\mathcal{L}_2$ for a given theory $\mathcal{L}_1$ it is necessary to know an explicit kink solution ϕ_k of the theory $\mathcal{L}_1$, and to use this kink in the evaluation of possible twin models $\mathcal{L}_2$, which would render calculations rather cumbersome. But this is, in fact, not true. The important point is that the lagrangian densities are functions of the target space variables ϕ and ϕ' only, therefore it is sufficient to implement the root $\phi' = \pm f_i(\phi)$ which leads to the kink (or antikink) solution under consideration. Further, we shall use the fact that all lagrangians we consider depend on ϕ' only via $X = -\frac{1}{2}\phi'^2$ (for static configurations), so that the above conditions transform into

$$\mathcal{L}_1|_{2X=-f_i^2} = \mathcal{L}_2|_{2X=-f_i^2} \tag{10.18}$$

$$\mathcal{L}_{1,X}|_{2X=-f_i^2} = \mathcal{L}_{2,X}|_{2X=-f_i^2} \tag{10.19}$$

where $f_i(\phi)$ is a known root (10.8) of the theory $\mathcal{L}_1$ leading to a kink solution. The above conditions are purely algebraic conditions in the target space variables ϕ and X and do not involve the base space variable x or explicit knowledge of a kink solution $\phi_k(x)$ at all.

Up to now we allowed for completely general lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ to emphasize the general character of the procedure. Now, however, we will concentrate on the case of a standard lagrangian $\mathcal{L}_1 = \mathcal{L}_s = X - V$ for concreteness, so the problem consists in finding possible twins to standard scalar field theories. Here $V(\phi)$ is a positive semi-definite potential with at least two vacua (zeros) such that kink solutions exist. The two roots for kink and antikink may be combined into $X = -V$, and the above conditions read (we write $\mathcal{L}$ for $\mathcal{L}_2$)

$$\mathcal{L}|_{X=-V} = -2V \tag{10.20}$$

$$\mathcal{L}_{,X}|_{X=-V} = 1. \tag{10.21}$$

Again, these two conditions are purely algebraic and allow an easy calculation of twin models, as we shall see in the next section.

10.2.3 Examples of twin models

As a first class of twin models let us consider the class of Lagrangians

$$\mathcal{L} = \sum_{k=1}^{K} f_k(\phi)X^k - U(\phi) \tag{10.22}$$

where the kinetic terms X^k are multiplied by functions of ϕ in a sigma-model like fashion. Here, the condition

$$\mathcal{L}_{,X}|_{X=-V} = \sum_{k=1}^{K} k f_k(\phi)(-V)^{k-1} \equiv 1 \tag{10.23}$$

imposes one condition on the functions $f_k(\phi)$. one may, for instance, choose arbitrary f_k for $k \geq 2$, then the above condition determines f_1 in terms of the remaining f_k and V. We remark that it is not possible to choose all f_k constant, but if at least one f_k has a nontrivial ϕ dependence then the above condition can always be fulfilled. The second condition

$$\mathcal{L}|_{X=-V} = \sum_{k=1}^{K} f_k(\phi)(-V)^k - U(\phi) \equiv -2V(\phi), \tag{10.24}$$

in turn, determines $U(\phi)$ in terms of the f_k and V.

One question to ask is whether the resulting twin models constitute viable field theories on their own, that is, whether they obey certain stability requirements like energy positivity or the null energy condition (NEC). Here we shall mainly be concerned with the NEC, because *i*) it is deemed sufficient for stability, *ii*) it is weaker than the condition of positivity of the energy density and *iii*) it is easier to implement for the class of models we study in this paper. The NEC in general is the condition that

$$n^\mu n^\nu T_{\mu\nu} \geq 0 \tag{10.25}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and n^μ is an arbitrary null vector. For the class of models $\mathcal{L}(X,\phi)$ the NEC simply reads

$$\mathcal{L}_{,X} \geq 0. \tag{10.26}$$

It is, in general, not completely trivial to reconcile the NEC with the two twin conditions (10.23) and (10.24), but it is easy to find certain special classes of models where the NEC holds by construction.

A first class of models which obeys both the NEC and the condition (10.23) by construction is given by field theories which obey

$$\mathcal{L}_{,X} = Kf(\phi)(X+V)^{K-1} + 1 \tag{10.27}$$

where f is an arbitrary, positive semi-definite function $f(\phi) \geq 0$ and K is an odd integer. The resulting Lagrangian (i.e., X integral) is

$$\mathcal{L} = f(\phi)(X+V)^K + X - \tilde{U}(\phi) \tag{10.28}$$

(where the integration "constant" $\tilde{U}(\phi)$ is an arbitrary function of ϕ), and the second twin condition (10.24) requires $\tilde{U} = V$ such that the class of twin Lagrangians reads

$$\mathcal{L} = f(\phi)(X+V)^K + X - V \ , \quad K = 3, 5, \dots \tag{10.29}$$

As a concrete example, we may e.g. choose $f = 1$ and $K = 3$ which results in the Lagrangian

$$\mathcal{L} = \frac{1}{3}X^3 + VX^2 + (V^2+1)X + \frac{1}{3}V^3 - V \tag{10.30}$$

which shares both the kink solution $\phi' = \pm\sqrt{2V}$ and the corresponding energy density with the standard scalar model $\mathcal{L}_s = X - V$. A second class of twin models obeying the NEC may be constructed from the equation

$$\mathcal{L}_{,X} = f^{1-K}(X+V+f)^{K-1} \tag{10.31}$$

(where $f = f(\phi) \geq 0$, and K is an odd integer) with Lagrangian

$$\mathcal{L} = \frac{f^{1-K}}{K}(X+V+f)^K - \tilde{U}. \tag{10.32}$$

Here the second twin condition leads to $\tilde{U} = 2V + (f/K)$ and, therefore, to the Lagrangian

$$\mathcal{L} = \frac{f^{1-K}}{K}(X+V+f)^K - 2V - \frac{f}{K}. \tag{10.33}$$

Next, let us describe another class of examples of twin models, different from the power expansion in X of Eq. (10.22). We start from the ansatz

$$\mathcal{L} = f(\phi)g(X) - U(\phi) \tag{10.34}$$

and calculate

$$\mathcal{L}_{,X} = f(\phi)g'(X) \tag{10.35}$$

and the NEC leads to the conditions

$$f \geq 0 \ , \quad g' \geq 0. \tag{10.36}$$

Further, the two twin conditions lead to $f(\phi) = (g'(-V))^{-1}$ and $U = 2V + (g(-V)/g'(-V))$ and, therefore, to the Lagrangian

$$\mathcal{L} = \frac{g(X)}{g'(-V)} - \frac{g(-V)}{g'(-V)} - 2V. \tag{10.37}$$

Among this class we may easily recover the DBI type example originally presented and discussed in [51]. Indeed, choosing for the kinetic function $g(X)$ the DBI type expression

$$g(X) = -\sqrt{1 - 2X} \tag{10.38}$$

we calculate

$$g'(X) = \frac{1}{\sqrt{1-2X}}\,, \quad f(\phi) = \sqrt{1+2V}\,, \quad U = 2V - (1+2V) = -1 \tag{10.39}$$

and the resulting Lagrangian is

$$\mathcal{L} = -\sqrt{1+2V}\sqrt{1-2X} + 1. \tag{10.40}$$

It is obvious from the derivation that the nontrivial target space geometry factor $f(\phi) = \sqrt{1+2V}$ is necessary for this DBI type action to be the twin of a standard scalar field theory, as announced above.

10.3 Supersymmetric twin models

To begin with, let us remind that a standard scalar field theory $\mathcal{L}_s = X - V$ with a positive semi-definite potential $V \geq 0$ may always be viewed as the purely bosonic sector of a supersymmetric scalar field theory. Indeed, before the elimination of the auxiliary field F the bosonic sector of the supersymmetric standard scalar field theory reads

$$\mathcal{L}_s = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + F^2) - FP_s'(\phi) \tag{10.41}$$

where $P_s(\phi)$ is the prepotential (also sometimes called superpotential) of the standard SUSY scalar field theory. Elimination of the auxiliary field F with the help of its algebraic field equation $F = P_s'$ leads to the lagrangian

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}P_s'^2 \tag{10.42}$$

which is just the standard scalar field lagrangian with the identification

$$V = \frac{1}{2} P_s'^2 \geq 0. \tag{10.43}$$

This observation leads to the obvious question whether there exist supersymmetric K field theory twins for the supersymmetric standard field theories. For this purpose, in a first instance we have to know whether there exist supersymmetric scalar K field theories at all. The answer is that these supersymmetric K theories do exist. Some classes of examples have been introduced and studied in [63], [17] (these theories exist both in 1+1 and in 2+1 dimensional Minkowski space, due to the similar spin structure in the two spaces), and we shall use some of these examples for the construction of our supersymmetric K field twins. In [63] a class of supersymmetric models was introduced such that their purely bosonic sector before the elimination of the auxiliary field reads

$$\mathcal{L}^{(\alpha,P)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}] - P'(\phi)F. \tag{10.44}$$

Next, the auxiliary field F should be eliminated via its algebraic field equation

$$\sum_{k=1}^{N}(-1)^{k-1}2k\alpha_k F^{2k-1} - P'(\phi) = 0 \tag{10.45}$$

which in general is, however, a rather complicated algebraic equation for F. As no assumption was made yet about the functional dependence of P, this equation may be understood in a second, equivalent way: one assumes that F is an arbitrary given function of ϕ, which in turn determines the prepotential $P(\phi)$. This second way of interpreting Eq. (10.45) is more useful for our purposes. Eliminating the resulting $P'(\phi)$ we arrive at the Lagrangian density

$$\mathcal{L}^{(\alpha,F)} = \sum_{k=1}^{N} \alpha_k(\phi)[(\partial^\mu\phi\partial_\mu\phi)^k - (-1)^{k-1}(2k-1)F^{2k}] \tag{10.46}$$

where now $F = F(\phi)$ is a given function of ϕ which may be chosen freely depending on the theory or physical problem under consideration. This class of lagrangians is exactly of the type (10.22), therefore the conditions for being the twin of a standard theory are exactly analogous to the conditions (10.24) and (10.23). The restrictions implied by supersymmetry (i.e., the requirement to express the "potential function " $U(\phi)$ in Eq. (10.22) in terms of $F(\phi)$), nevertheless, impose some additional restrictions, as we want to show now. Indeed, the second twin condition $L_{,X}|_{2X=-F_s^2} = 1$ leads to

$$\sum 2k\alpha_k(-F_s^2)^{k-1} = 1, \tag{10.47}$$

where we introduced the function $F_s(\phi)$ of the standard SUSY theory, i.e., the auxiliary field F of the standard theory evaluated at its field equation via

$$2V(\phi) \equiv P_s'^2(\phi) \equiv F_s^2(\phi) \tag{10.48}$$

for convenience. The first twin condition $L|_{2X=-F_s^2} = -F_s^2$ then leads to

$$\begin{aligned} L|_{2X=-F_s^2} &= \sum_k \alpha_k((-F_s^2)^k - (-1)^{k-1}(2k-1)F^{2k}) \\ &= \sum_k \alpha_k((-F_s^2)^k - (-F^2)^k - 2kF^2(-F^2)^{k-1}) \\ &\equiv -F_s^2 \end{aligned}$$

where we used (10.47) in the last step. This condition is solved by

$$F = \pm F_s. \tag{10.49}$$

As we shall see in a concrete example below, this is typically the only acceptable solution, therefore supersymmetry seems to imply the additional relation $F = F_s$ for the algebraic solutions of the auxiliary fields of standard and K field twin theories.

Again, the NEC is not automatic in these theories, but a more specific class of examples which obeys the NEC by construction may be given, analogous to the last subsection. Concretely, we give some examples starting from the X derivative

$$\mathcal{L}_{,X} = 2K(2X + 2V)^{K-1} + 1 \tag{10.50}$$

(we write $2X$ instead of X in order to be as close as possible to the notation used in Ref. [63] and in Eq. (10.46); K is an odd integer). The resulting Lagrangian is

$$\mathcal{L} = (2X + 2V)^K + X - \tilde{U}(\phi) \tag{10.51}$$

and obeys the NEC and the twin condition $\mathcal{L}_{,X}|_{X=-V} = 1$ by construction.

For a more concrete example, let us now assume $K = 3$ which leads to the Lagrangian

$$\mathcal{L} = (2X)^3 + 6V(2X)^2 + (12V^2 + \frac{1}{2})(2X) + 8V^3 - \tilde{U} \tag{10.52}$$

and therefore to

$$\alpha_3 = 1 \ , \quad \alpha_2 = 6V \ , \quad \alpha_1 = 12V^2 + \frac{1}{2} \tag{10.53}$$

and to the Lagrangian

$$\mathcal{L} = (2X)^3 - 5F^6 + 6V((2X)^2 + 3F^4) + (12V^2 + \frac{1}{2})(2X - F^2) \tag{10.54}$$

which explicitly is of the form (10.46) (we replaced the arbitrary integration"constant" $\tilde{U}(\phi)$ by the required F terms). Now the second twin condition $\mathcal{L}|_{X=-V} = -2V$ leads to

$$5F^6 - 18VF^4 + (12V^2 + \frac{1}{2})F^2 + 8V^3 - V = 0 \tag{10.55}$$

which may be viewed as a third order algebraic equation for F^2. The only acceptable (i.e., real and positive) solution is

$$F^2 = 2V \equiv F_s^2 \tag{10.56}$$

and leads to the Lagrangian

$$\mathcal{L} = (2X)^3 - 40V^3 + 6V((2X)^2 + 12V^2) + (12V^2 + \frac{1}{2})(2X - 2V) \tag{10.57}$$

which is the desired supersymmetric twin of the standard Lagrangian. As already remarked for the more general class of examples above, it holds that also the (algebraic) field equations for the auxiliary fields coincide, see Eq. (10.49). This equality is *not* a further condition, but a consequence of the twin conditions and supersymmetry.

Another class of supersymmetric theories has the following purely bosonic sector (before the elimination of the auxiliary field F) [17]

$$\mathcal{L} = g(\phi)f(X)(F^2 + 2X^2) - P'(\phi)F \tag{10.58}$$

where f and g are arbitrary, fixed functions of their arguments and for the moment we just assume $g \geq 0$. The algebraic field equation for the auxiliary field F has the solution

$$F = \frac{P'}{2gf} \tag{10.59}$$

and leads to the Lagrangian

$$\mathcal{L} = 2Xgf - \frac{P'^2}{4gf} \equiv 2g\left(Xf - \frac{h}{f}\right) \tag{10.60}$$

where

$$h(\phi) \equiv \frac{P'^2}{8g^2}. \tag{10.61}$$

The X derivative of this lagrangian is

$$\mathcal{L}_{,X} = 2g\left(f + Xf_{,X} + h\frac{f_{,X}}{f^2}\right). \tag{10.62}$$

A sufficient condition for the NEC consists in the following inequalities

$$f + Xf_{,X} \geq 0\,, \quad f_{,X} \geq 0 \tag{10.63}$$

but we have not been able to find a function f which obeys these inequalities. There exists, however, another possiblity to obey the NEC, and for this possibility we found solutions. Concretely, assume that

$$f + Xf_{,X} \geq 1 \tag{10.64}$$

and that further

$$\left|\frac{f_{,X}}{f^2}\right| \leq 1 \tag{10.65}$$

and

$$h \leq 1 \tag{10.66}$$

then the NEC holds. A specific function f obeying these conditions is

$$f = 1 + X^2 \tag{10.67}$$

which indeed leads to

$$f + Xf_{,X} = 1 + 3X^2 \geq 1\,, \quad \left|\frac{f_{,X}}{f^2}\right| = \frac{2|X|}{(1+X^2)^2} \leq 1. \tag{10.68}$$

We will study this explicit example in what follows. We remark that, as we shall see, the condition $h \leq 1$ leads to restrictions on possible potentials V, so if we insist on the NEC we may construct twins of the type considered here only for standard theories with certain potentials. For the specific choice $f = 1 + X^2$ the lagrangian and its X derivative read

$$\mathcal{L} = 2g\left(X + X^3 - \frac{h}{1+X^2}\right) \tag{10.69}$$

$$\mathcal{L}_{,X} = 2g\left(1 + 3X^2 + h\frac{2X}{(1+X^2)^2}\right). \tag{10.70}$$

The twin condition $\mathcal{L}_{,X}|_{X=-V} = 1$ leads to the equation

$$h = \frac{(1-V^2)^2}{2V}\left(1 + 3V^2 - \frac{1}{2g}\right) \tag{10.71}$$

and the second twin condition $\mathcal{L}|_{X=-V} = -2V$ leads, together with Eq. (10.71), to the solution

$$\frac{1}{2g} = 1 + X^2 \tag{10.72}$$

which, in turn, leads to

$$h = V(1+V^2)^2. \tag{10.73}$$

Now, the NEC requires $h \leq 1$ which obviously restricts possible potentials V. An example of a potential which is compatible with this condition is

$$V = \frac{1}{2}\frac{(1-\phi^2)^2}{(1+\phi^2)^2} \tag{10.74}$$

as may be checked easily. Further, this potential has the same vacuum structure as the standard ϕ^4 potential $V = (1/2)(1-\phi^2)^2$, so it will lead to similar kink solutions.

We want to end this section with the remark that the auxiliary field F, when evaluated at the kink equation $X = -V$, again coincides with the auxiliary field of the standard supersymmetric theory $F_s^2 = 2V$. Indeed, from Eq. (10.59) we infer that

$$F^2 = \frac{P'^2}{4g^2 f^2} = \frac{2h}{f^2} \tag{10.75}$$

which depends both on ϕ and on X. But evaluating it for the kink equation leads to $f|_{X=-V} = 1 + V^2$, which together with the solution $h = V(1+V^2)^2$ just leads to

$$F^2|_{X=-V} = \frac{2V(1+V^2)^2}{(1+V^2)^2} = 2V \equiv F_s^2 \tag{10.76}$$

which is, again, identical to the field equation of the auxiliary field for the standard supersymmetric scalar field theory.

10.4 Self-gravitating twins

Here we want to study the existence of twins of the standard scalar field theory fully coupled to gravity, that is, K field theories which give rise to exactly the same defect solution, energy density, and induced metric than the standard scalar field theory with self-gravitation fully taken into account. We shall find that the situation is completely equivalent to the Minkowski space case in that, again, there exist two purely algebraic "twin conditions" which allow to calculate twins of self-gravitating standard scalar field theories. The only differences will be that i) the "on-shell" condition for a defect is no longer $X = -V$ but, instead, $X = -(1/2)W_{,\phi}{}^2$, where the relation between W and V is slightly more complicated than in the flat space case; and ii) the "on-shell" value which the Lagrangian has to take will be different, as well, i.e., $\mathcal{L}|_{X=-(1/2)W_{,\phi}{}^2} = -W_{,\phi}{}^2 + c_d W^2$ instead of $\mathcal{L}|_{X=-V} = -2V$ (here c_d is a numerical coefficient which depends on the dimension d of space-time; in principle, it also depends on the gravitational constant κ and vanishes in the limit $\kappa \to 0$, but we shall choose units such that $\kappa = 1$ in the following).

Before starting the detailed calculations, some remarks are in order. The topological defect solution in flat Minkowski space may be either viewed as a kink solution in 1+1 dimensions or as a co-dimension one domain wall solution in higher dimensions. Both the defect solution and its energy density per length unit in the direction perpendicular to the wall do not depend on the dimension. This is no longer true once the gravitational self-interaction is taken into account. In 1+1 dimensions there is no gravitational interaction, because the Einstein tensor is identically zero, and for higher dimensions $d > 2$ the Einstein equations depend on the dimension d, therefore also the self-gravitating defect solutions will depend on d. Here we shall discuss the case for general d, but probably the two cases $d = 4$ and $d = 5$ are the most interesting ones. $d = 4$ is the dimension of our universe, at least at a macroscopic scale, so the resulting defects of the standard theory and its twins may be viewed just as domain walls in the universe. The case $d = 5$ is especially interesting in relation to the braneworld scenario, where our universe is identified with the domain wall, and the direction perpendicular to the domain wall is identified with a fifth direction or coordinate which is invisible due to the resulting warped geometry in five dimensions, which essentially confines all physics to the three dimensional domain wall or brane (four dimensional brane world hypersurface). As already stated, the $d = 5$ case was studied in [58], and we shall build on the results of that paper. Another remark concerns the possibility in flat space to express the linear energy density of a defect solution with the help of the integrating function W as $\mathcal{E} = \phi' W_{,\phi}$. Using the static field equation for a defect (10.7), this relation may be re-expressed like

$$\mathcal{E} = -\mathcal{L} = \phi'^2 \mathcal{L}_{,X} = \phi' W_{,\phi} \quad \Rightarrow \quad \phi' \mathcal{L}_{,X} = W_{,\phi} \tag{10.77}$$

and this last form is the most useful one for our purposes, because it may be generalized directly to the case with gravity, as we shall see below.

If the Einstein–Hilbert action is normalized as

$$S_{\text{EH}} = \frac{1}{\kappa} \int d^d x \sqrt{|\mathrm{g}|} \mathcal{R} \tag{10.78}$$

(where g is the determinant of the metric g_{MN}, $M, N = 0, \ldots, d-1$, $\mathcal{R}$ is the curvature scalar, and $\kappa = 4\pi G$ where G is Newton's constant), then the Einstein equation is

$$G_{MN} = 2\kappa T_{MN} \tag{10.79}$$

where

$$T_{MN} = \nabla_M \phi \nabla_N \phi \mathcal{L}_{,X} - \mathrm{g}_{MN} \mathcal{L} \tag{10.80}$$

is the energy-momentum tensor. Further, ∇_M is the covariant derivative, and

$$X \equiv \frac{1}{2} \nabla_M \phi \nabla^M \phi. \tag{10.81}$$

We shall choose length units such that $\kappa = 1$, therefore the Einstein equation we use reads

$$G_{MN} = 2T_{MN}. \tag{10.82}$$

For the self-gravitating defect solution we use the ansatz for the metric

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{10.83}$$

where $x^M = (x^\mu, y)$, y is the coordinate for the direction perpendicular to the domain wall (or brane), and $\eta_{\mu\nu} = \text{diag}(+, -\ldots, -)$ is the Minkowski metric in $d-1$ dimensions. Further, we assume that $\phi = \phi(y)$ only depends on the y coordinate. With this ansatz, the expression for X reduces to the same expression like in the flat space case, $X = -(1/2)(\partial_y\phi)^2 \equiv -(1/2)\phi'^2$.

The Einstein equations for this ansatz reduce to two independent equations for $A(y)$ and $\phi(y)$, and they depend on the dimensions d of space-time. Explicitly, they read

$$\frac{(d-1)(d-2)}{2}A'^2 + (d-2)A'' = 2\mathcal{L} \tag{10.84}$$

$$\frac{(d-1)(d-2)}{2}A'^2 = -4X\mathcal{L}_{,X} + 2\mathcal{L} \tag{10.85}$$

which may be resolved for A'^2 and A'',

$$A'' = \frac{4}{d-2}X\mathcal{L}_{,X} \tag{10.86}$$

$$A'^2 = \frac{4}{(d-1)(d-2)}(\mathcal{L} - 2X\mathcal{L}_{,X}). \tag{10.87}$$

The field equation for the scalar field ϕ is not an independent equation, but rather a consequence of the Einstein equations therefore we do not display it here. The first order formalism for static domain walls now consists in introducing an integrating function or superpotential $W = W(\phi)$ proportional to $-A'$ [65], [66], [67], [32], [58]. The right choice is

$$A' = -\frac{2}{d-2}W(\phi), \tag{10.88}$$

and inserting it into Eq. (10.86) leads to

$$A'' = -\frac{2}{d-2}W_{,\phi}\phi' = \frac{4}{d-2}(-\frac{1}{2}\phi'^2)\mathcal{L}_{,X} \quad \Rightarrow \quad W_{,\phi} = \phi'\mathcal{L}_{,X} \tag{10.89}$$

exactly as in the flat space case. In order to find the twin conditions which twin models of the standard Lagrangian $\mathcal{L}_s = X - V$ should obey, we first have to solve the Einstein equations for the standard Lagrangian. obviously, the first integral for the standard Lagrangian is

$$\phi' = W_{,\phi} \quad \Rightarrow \quad X = -\frac{1}{2} W_{,\phi}{}^2 \tag{10.90}$$

just like in the flat space case. This implies that the first twin condition for a K field lagrangian is just $\mathcal{L}_{,X}|_{X=-(1/2)W_{,\phi}{}^2} = 1$, in close analogy to the flat space case (although it is no longer true that $(1/2)W_{,\phi}{}^2 = V$, as we shall see in a moment). In order to find the relation between V and W we just insert the standard Lagrangian, the ansatz for A' and the first integral for the standard Lagrangian into the second Einstein equation (10.87) and find

$$A'^2 \equiv \frac{4}{(d-2)^2} W^2 = \frac{4}{(d-1)(d-2)}(X - V - 2X) = \frac{4}{(d-1)(d-2)}(\frac{1}{2}W_{,\phi}{}^2 - V) \tag{10.91}$$

$$\Rightarrow \quad V = \frac{1}{2} W_{,\phi}{}^2 - \frac{d-1}{d-2} W^2. \tag{10.92}$$

We remark that the first, $W_{,\phi}$ term is exactly like in the flat space case, whereas the second, W term is the correction due to gravity and depends on the dimension d. Inserting this result back into $\mathcal{L}_s$ leads to

$$\mathcal{L}_s|_{X=-(1/2)W_{,\phi}{}^2} = -W_{,\phi}{}^2 + \frac{d-1}{d-2} W^2 \tag{10.93}$$

and the resulting twin conditions for a general Lagrangian $\mathcal{L}$ to be the twin of a standard Lagrangian $\mathcal{L}_s = X - V$ are therefore

$$\mathcal{L}_{,X}|_{X=-(1/2)W_{,\phi}{}^2} = 1 \tag{10.94}$$

$$\mathcal{L}|_{X=-(1/2)W_{,\phi}{}^2} = -W_{,\phi}{}^2 + \frac{d-1}{d-2} W^2 \tag{10.95}$$

where the relation between the integrating function W and the potential V is given in (10.92). These relations are, again, purely algebraic and do not require the explicit knowledge of a defect solution.

We remark that solving Eq. (10.92) for a given potential V is, in general, quite difficult. It is simpler to choose an integrating function (or superpotential) W and determine the resulting potential V. In addition, by choosing an adequate W, it is also easy to assure that the simple equation $\phi' = \pm W_{,\phi}$ does support topological defect solutions.

Finally, let us present some explicit examples. As a first example, we choose the Lagrangian (10.28) of Section 2, but with V replaced by $\frac{1}{2}W_{,\phi}{}^2$ (where $f(\phi)$ is an arbitrary, nonnegative function),

$$\mathcal{L} = f(\phi)(X + \frac{1}{2}W_{,\phi}{}^2)^K + X - \tilde{U}(\phi)$$

which, by construction, obeys both the NEC and the twin condition (10.94). The second twin condition (10.95) determines $\tilde{U}$ to be

$$\tilde{U} = \frac{1}{2}W_{,\phi}{}^2 - \frac{d-1}{d-2}W^2 \equiv V, \tag{10.96}$$

just like in the case without gravity.

For a second class of examples, we use the ansatz (as in Section 2)

$$\mathcal{L} = f(\phi)g(X) - U(\phi). \tag{10.97}$$

The first twin condition (10.94) leads to

$$f(\phi) = \frac{1}{g'(-(1/2)W_{,\phi}{}^2)} \tag{10.98}$$

and the second twin condition (10.95) results in

$$U(\phi) = W_{,\phi}{}^2 + \frac{g(-(1/2)W_{,\phi}{}^2)}{g'(-(1/2)W_{,\phi}{}^2)} - \frac{d-1}{d-2}W^2 \tag{10.99}$$

For the specific, DBI type example $g(X) = -\sqrt{1-2X}$ we, therefore, get the Lagrangian

$$\mathcal{L} = -\sqrt{1 + W_{,\phi}{}^2}\sqrt{1-2X} + 1 + \frac{d-1}{d-2}W^2 \tag{10.100}$$

which, for $d = 5$, precisely coincides with the example presented in [58]. We remark that for the case with selfgravitation the authors of [58] use the definition $W_{,\phi} = \frac{1}{2}\phi'\mathcal{L}_{,X}$ for the integrating function (or superpotential) W, which differs by a factor two from the definition $W_{,\phi} = \phi'\mathcal{L}_{,X}$ employed here (but also in Ref. [58] for the case without gravity).

10.5 Discussion

In this article, we have derived a simple and purely algebraic method for the construction of K field twins of a standard scalar field theory, that is, of K field models which share the same topological defect with the same energy density with a given standard scalar field theory. This method may be derived for the cases of non-supersymmetric field theories, supersymmetric field theories and for self-gravitating fields. Further, we gave several examples for all these cases. The interest of these twin models lies in the fact that the field profile together with the energy density are the most relevant physical data of a defect which makes the twins almost indistinguishable from their standard counterparts in many situations. A pattern of defects in the very early universe will look the same irrespective of whether it is formed by defects of a standard theory or of a K field twin. The spectrum of linear fluctuations, on the other hand, is in general different between the standard theory and its twins [51], [58], so small differences will set in once dynamics (i.e., time dependence) is taken into account. A similar question is related to the behaviour of additional matter fields (e.g., fermion fields) coupled to twin defects. In the non-supersymmetric case there exist different possibilities to couple fermions to each field theory, therefore general statements cannot be made. The situation is different, however, for supersymmetric (SUSY) K field twins of standard SUSY scalar field theories. Here, the first important piece of information is, of course, the existence of SUSY K field theories [63], [17], [49], [136]. We want to point out again that defects of supersymmetric theories are the relevant ones to study if the symmetry breaking and the subsequent defect formation in the early universe occur at an energy scale where supersymmetry is still intact (e.g., at the end of inflation). For supersymmetric twins of the standard supersymmetric scalar field theories, it is interesting to observe that these SUSY twin models not only share the defect solution and its energy density with the standard theory. Also the (algebraic) field equation for the auxiliary field in the kink background is identical for the standard theory and the twin. Another interesting problem of these SUSY theories concerns, of course, the inclusion of fermions which we have set equal to zero in our discussion. In general, the fermionic sectors of the standard theory and the twin will be different, like the bosonic sectors. Standard and twin theory will, however, share some common features in the fermionic sector, too. They will, e.g., share the same fermionic zero mode in the background of the same kink solution. This is a consequence of translational invariance, on the one hand, which implies that both theories in the kink background have the same bosonic zero mode (or Goldstone mode) equal to the derivative of the kink. The second ingredient is, of course, supersymmetry, which guarantees that each bosonic Goldstone mode is paired by a fermionic zero mode which is, again, equal to the derivative of the kink field.

A final issue is the existence of twin models when the gravitational backreaction is taken into account, i.e., of twin defects sharing the same field profile, energy density and induced metric. Already for defect structures in cosmology (i.e., in the early universe) the full self-gravitating case should, in principle, be taken into account, although in many circumstances a Minkowski space calculation is sufficient. In the brane world scenario taking into account the full self-gravitating solution is mandatory. We found that, again, there exists a simple algebraic method to calculate infinitely many K field twins of a standard self-gravitating scalar field theory and gave several examples. We emphasize that for the 4+1 dimensional case relevant for the brane world scenario an example of a self-gravitating twin has already been given in [58] with the help of the first order formalism.

It was the main aim of the present article to shed more light on the existence of K field twin defects and the mathematical structures behind them, on the one hand, and to provide a simple calculational tool for the construction and study of twin-like models, on the other hand. We want to point out that, whenever K field theories cannot be excluded on purely theoretical grounds, they have to be considered on a par with the standard field theories as an immediate consequence of the existence of twin defects, because for twin-like models their most relevant physical manifestations are completely indistinguishable. This is the case, e.g., for effective field theories resulting from the integration of UV degrees of freedom, where higher kinetic terms are naturally induced.

Chapter 11

More on twin-like models

After the general framework which provides the algebraic conditions that twin-like models must verify, in this chapter we show that it is possible to add more purely algebraic constraints to the lagranian of the twin-like model to ensure that both linear fluctuation spectra coincide. The interesting result is that a semiclassical quantization about the topological defect provides the same results for the standard field theory and its K field twins. This chapter consists of a paper published in [98].

Twinlike models with identical linear fluctuation spectra

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Abstract: Recently, the possibility of so-called twinlike field theories has been demonstrated, that is, of different field theories which share the same topological defect solution with the same energy density. Further, purely algebraic conditions have been derived which the corresponding Lagrangians have to obey in order that the field theories be twins of each other. A further diagnostical tool which, in general, allows to distinguish the topological defects of a given theory from the corresponding defects of its twins is the spectrum of linear fluctuations about these defects. Very recently, however,

explicit examples of twin theories have been constructed such that not only their shapes and energy densities coincide, but also their linear fluctuation spectra are the same. Here we show that, again, there exist purely algebraic conditions for the Lagrangian densities which imply that the corresponding field theories are twins and that the fluctuation spectra about their defects coincide. These algebraic conditions allow to construct an infinite number of twins with coinciding fluctuation spectra for a given theory, and we provide some explicit examples. The importance of this result is related to the fact that coinciding defects with coinciding energy densities and identical fluctuation spectra are almost indistinguishable physically, that is, indistinguishable in a linear or semiclassical approximation. This implies that the measurable physical properties of a kink, in general, do not allow to determine the theory which provides the kink uniquely. Instead, in principle an infinite number of possible theories has to be considered.

11.1 Introduction

One of the most fertile concepts in theoretical physics in the last decades has been the concept of topological defects or topological solitons (see e.g. [162]). They are ubiquitous in condensed matter systems and, besides this, are deemed relevant for the cosmology of the early universe. Topological defects may, for instance, contribute to the structure formation in the very early universe (e.g., during or at the end of inflation) [23]-[25]. A topological soliton is, in general, a static solution of the Euler–Lagrange equations of the given field theory with finite energy which obeys nontrivial boundary conditions. Further, the stability of the topological soliton against transitions to the vacuum is guaranteed by the fact that a deformation to the vacuum configuration with trivial boundary conditions would require to change the field in an infinite volume and, therefore, cost an infinite amount of energy. The relevant data characterizing the physical properties of a soliton are, first of all, its shape or profile (i.e., the soliton solution itself), and its energy density. Additional important information is contained in the so-called spectrum of linear fluctuations about the topological defect. In order to determine this spectrum, one calculates the fluctuations about the soliton up to second order in the action (or up to first order in the Euler–Lagrange equations). For the fluctuation field then one introduces a temporal Fourier decomposition, which results in a stationary second order equation of the Schrödinger type. The (in general, infinitely many) solutions of this equation together with the allowed frequencies constitute the spectrum of linear fluctuations. The first relevant information contained in the spectrum of linear fluctuations is linear stability. For a stable soliton, the spectrum should contain no negative mode (i.e., no imaginary frequency). Another aspect where the fluctuation spectrum is important is the issue of semiclassical quantization in the presence of solitons [72] (for an easy to follow discussion see [73]). Concretely, the discrete part of the fluctuation spectrum describes some excited states of the soliton or, equivalently, soliton-meson bound states. Here by "meson" we mean a fluctuation field which is Gaussian in the leading approximation and obeys the boundary conditions of the vacuum configuration. Further, the continuous part of the spectrum describes soliton-meson scattering.

The discussion so far has been for general soliton models, but now we want to restrict to the case of a real scalar field in 1+1 dimensions. The standard scalar field theory in 1+1 dimensions is

$$\mathcal{L}_s = X - U(\phi) \quad , \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{11.1}$$

and we shall require that U is nonnegative,

$$U(\phi) \geq 0 \quad \forall\phi \tag{11.2}$$

This theory may support topological solitons (kinks) provided that the potential U has at least two vacua, i.e., there exist at least two (constant) values $\phi = \phi_i$ such that $U(\phi_i) = 0$. A kink is a static solution $\phi_k(x)$ which, in general, interpolates between two adjacent vacua, i.e., $\phi_k(-\infty) = \phi_i$, $\phi_k(\infty) = \phi_{i+1}$. The corresponding static kink equation is ($\phi' \equiv \partial_x\phi$)

$$\frac{1}{2}\phi'^2 \equiv -X = U \tag{11.3}$$

with the two roots (for kink and antikink)

$$\phi' = \pm\sqrt{2U}. \tag{11.4}$$

The kink equation (11.3) results from the static second order Euler–Lagrange equation by performing one integration, where the integration constant must be set equal to zero in order to satisfy the kink boundary conditions. Finally, the linear fluctuation equation in the kink background may be derived by inserting the decomposition $\phi(t,x) = \phi_k(x) + \eta(t,x)$ and the temporal Fourier decomposition $\eta(t,x) = \cos(\omega t)\eta(x)$ into the Euler–Lagrange equation and keeping terms linear in η. Explicitly, the linear fluctuation equation reads ($U_{,\phi} \equiv \partial_\phi U$, etc.)

$$-\eta'' = (\omega^2 - U_{,\phi\phi}|_{\phi_k})\eta \tag{11.5}$$

where the notation $|_{\phi_k}$ means that the expression has to be evaluated for the kink solution. The solutions of this Schrödinger type equation together with the allowed frequencies ω determine the spectrum of linear fluctuations in this case.

Up to now the logical line of reasoning has been to begin with a field theory and to derive from this starting point the topological defect (kink) and its properties. Now we want to see whether and how far this logical arrow can be reversed. That is to say, we start with a kink solution together with its properties, like energy density and linear fluctuation spectrum, and we want to know whether or to which degree we may recover the theory which gives rise to this defect solution with its properties. The answer depends on the class of Lagrangians we are willing to admit. For a standard scalar field theory (11.1), the kink solution itself is already sufficient to recover the Lagrangian, i.e., the potential, by inverting the solution $\phi = \phi_k(x) \Rightarrow x = x_k(\phi)$ and by inserting the resulting expression into the kink equation,

$$\phi'^2(x) = \phi'^2(x_k(\phi)) \equiv 2U(\phi), \tag{11.6}$$

which determines $U(\phi)$. on the other hand, the situation will be different if we allow for a more general class of Lagrangians. Concretely, we want to admit Lagrangians which are general functions of both ϕ and $X \equiv (1/2)\partial_\mu\phi\partial^\mu\phi$. There are several reasons which make these theories with a generalized kinetic term (the so-called K field theories) worth considering. First of all, K field theories have been applied already to some problems in cosmology, like inflation (so-called K-inflation [2]), late time acceleration (so-called K-essence [3]), or in the brane world scenario [31], [7], [74]. Secondly, generalized kinetic terms may serve to stabilize static field configurations, evading thereby the Derrick theorem and allowing the existence of soliton solutions. The third and probably strongest case in favor of K field theories is related to the fact that in many circumstances scalar field theories are interpreted as effective field theories which result from the integration of UV degrees of freedom of some more fundamental theory. In this case of an effective field theory, higher powers of derivatives are induced naturally, and therefore they have to be taken into account. In this paper we are specifically interested in K field theories whose topological defects coincide with the standard ones, but let us mention, nevertheless, that K field theories in general give rise to a much richer phenomenology of possible topological defects [5], [61], like, e.g. solitons with compact support (so-called compactons) [35] - [40]. other more mathematical aspects of K field theories have been discussed, e.g., in [4] and in [62].

For the generalized dynamics of K field theories (i.e., for general Lagrangians $\mathcal{L}(X,\phi)$) it was found recently [51] that different field theories may exist which share the same topological defect with the same energy density. The coinciding kinks with their coinciding energy densities were dubbed twin or Doppelgänger defects in [51], and the models which give rise to these identical kink solutions are called twinlike models. The investigation of twinlike models was carried further in [75] and in [76]. Specifically, in [76] it was demonstrated that there exist purely algebraic necessary and sufficient conditions for a Lagrangian $\mathcal{L}(X,\phi)$ to be the twin of a standard theory $\mathcal{L}_s = X - U$. As these conditions are algebraic, they do not require the knowledge of the topological defect solution and, therefore, allow the simple construction of an infinite number of twins for any given standard field theory supporting topological defects. Very recently, in [77] explicit examples of K field theories were found which not only are twin models of standard field theories, but where also the fluctuation spectra of the standard defect and its K field twins coincide, making the standard defect and its twins almost completely indistingushable physically. This implies that the measurable physical properties of a kink, in general, do not allow to determine the theory which provides the kink uniquely. Instead, in principle an infinite number of possible theories has to be considered.

It is the purpose of the present paper to show that, again, there exist purely algebraic conditions for a Lagrangian density which imply that the corresponding field theory is the twin of a standard scalar field theory *and* that the fluctuation spectra about their defects coincide. Further, these algebraic conditions allow to explicitly construct an infinite number of twins with coinciding fluctuation spectra for any given standard field theory. Concretely, in Sec. 11.2 we briefly review some known facts about twinlike models which we need. In Sec. 11.3, we derive the algebraic conditions for coinciding fluctuation spectra and provide some explicit examples. Further we discuss the relation of our results with the examples of Ref. [77]. Finally, Sec. 11.4 contains our conclusions.

11.2 Twinlike models

The algebraic twin conditions require the first order form of the static field equations, so let us briefly review this issue (for more details see, e.g., [76], [32]). For a general Lagrangian $\mathcal{L}(X,\phi)$ where $X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}(\dot\phi^2 - \phi'^2)$, the Euler–Lagrange equation reads

$$\partial_\mu(\mathcal{L}_{,X}\partial^\mu\phi) - \mathcal{L}_{,\phi} = 0. \tag{11.7}$$

Further, the energy momentum tensor is

$$T_{\mu\nu} = \mathcal{L}_{,X}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{11.8}$$

which, for static configurations $\phi = \phi(x)$, $\phi' \equiv \partial_x\phi$, simplifies to

$$T_{00} = \mathcal{E} = -\mathcal{L} \tag{11.9}$$

$$T_{11} = \mathcal{P} = \mathcal{L}_{,X}\phi'^2 + \mathcal{L} \tag{11.10}$$

where $\mathcal{E}$ is the energy density and $\mathcal{P}$ is the pressure. The static Euler–Lagrange equation may be integrated once to give

$$-2X\mathcal{L}_{,X} + \mathcal{L} \equiv \mathcal{P} = 0. \tag{11.11}$$

The general first integral allows for a nonzero constant on the r.h.s. (nonzero pressure), but the boundary conditions for finite energy field configurations require this constant to be zero (zero pressure condition). For a standard field theory $\mathcal{L}_s = X - U$, the energy density and pressure read

$$\mathcal{E}_s = -X + U = \frac{1}{2}\phi'^2 + U \tag{11.12}$$

$$\mathcal{P}_s = -X - U = \frac{1}{2}\phi'^2 - U, \tag{11.13}$$

and for a kink solution ϕ_k obeying $\phi_k'^2 = 2U$ these simplify to

$$\mathcal{E}_s|_{\phi_k} = -2X|_{\phi_k} = 2U|_{\phi_k} \tag{11.14}$$

$$-\mathcal{P}_s = X + U \equiv 0. \tag{11.15}$$

obviously, a K field theory will be the twin of a standard theory (i.e., have the same kink solution ϕ_k with the same energy density) if both $\mathcal{E}$ and $\mathcal{P} \equiv 0$ agree when evaluated for the kink solution. A necessary and sufficient condition for the K field Lagrangian is [51]

$$\mathcal{L}|_{\phi_k} = -2U \tag{11.16}$$

$$\mathcal{L}_{,X}|_{\phi_k} = 1, \tag{11.17}$$

as may be checked easily. Now the important point is that the first order form $\phi'^2 = -2X = 2U$ of the static kink equation may be interpreted as an algebraic equation involving the variables X and ϕ on which the K field Lagrangian depends. As a consequence, the evaluation condition $|_{\phi_k}$ may be replaced by the purely algebraic condition $|_{X=-U}$, leading to the so-called algebraic twin conditions [76]

$$\mathcal{L}|_{X=-U} \equiv \mathcal{L}| = -2U \tag{11.18}$$

$$\mathcal{L}_{,X}|_{X=-U} \equiv \mathcal{L}_{,X}| = 1 \tag{11.19}$$

(here and below the evaluation of an expression at $X \equiv -(1/2)\phi'^2 = -U$ (and its prolongations, when required) will always be denoted by the vertical line $|$, and will be called on-shell condition or on-shell evaluation frequently).

11.3 The algebraic conditions

11.3.1 The fluctuation equation

We start from the Euler–Lagrange equation (11.7) and insert the decomposition

$$\phi(t,x) = \phi_k(x) + \eta(t,x) \tag{11.20}$$

where ϕ_k is the kink solution and η is the fluctuation field. In first order in η we find

$$\begin{aligned} &\partial_\mu \left(\mathcal{L}_{,X}\partial^\mu \eta + \mathcal{L}_{,XX}\partial_\nu\phi_k\partial^\nu\eta\partial^\mu\phi_k + \mathcal{L}_{,X\phi}\eta\partial^\mu\phi_k\right) - \\ - \quad &\mathcal{L}_{,\phi\phi}\eta - \mathcal{L}_{,X\phi}\partial_\mu\phi_k\partial^\mu\eta = 0. \end{aligned} \tag{11.21}$$

Now we use the fact that ϕ_k only depends on x, and the ansatz for the fluctuation field

$$\eta(t,x) = \cos(\omega t)\eta(x) \tag{11.22}$$

and get

$$\begin{aligned} &\left(-\mathcal{L}_{,X}\eta' + \mathcal{L}_{,XX}(\phi_k')^2\eta' - \mathcal{L}_{,X\phi}\phi_k'\eta\right)' - \\ - \quad &\mathcal{L}_{,\phi\phi}\eta + \mathcal{L}_{,X\phi}\phi_k'\eta' - \omega^2\mathcal{L}_{,X}\eta = 0 \end{aligned} \tag{11.23}$$

or, more explicitly

$$\begin{aligned} &-(\mathcal{L}_{,X} + 2X\mathcal{L}_{,XX})\eta'' - \\ - \quad &(\mathcal{L}_{,X\phi} + 2X\mathcal{L}_{,XX\phi} - \phi_k''(3\mathcal{L}_{,XX} + 2X\mathcal{L}_{,XXX}))\phi_k'\eta' = \\ = \quad &\left(\omega^2\mathcal{L}_{,X} + \mathcal{L}_{,\phi\phi} - 2X\mathcal{L}_{,X\phi\phi} + \phi_k''(\mathcal{L}_{,X\phi} + 2X\mathcal{L}_{,XX\phi})\right)\eta. \end{aligned} \tag{11.24}$$

This expression should now be evaluated for the defect solution ϕ_k, i.e., implementing the on-shell condition $X| = -U$ and its first prolongation (that is, the original second order static field equation) $\phi''| \equiv \phi_k'' = U_{,\phi}$. Inserting these on-shell expressions above produces an expression containing U and its derivative, whereas the variables of $\mathcal{L}$ and its derivatives are X $(= -U)$ and ϕ. The problem is that for a general potential U the algebraic relation between ϕ and U is undetermined, so we would have to treat each potential separately, losing thereby some of the generality of the algebraic method.

The obvious alternative is to assume that the Lagrangian depends on ϕ only via the potential U, that is, $\mathcal{L} = \mathcal{L}(X, U)$. With

$$\mathcal{L}_{,\phi} = \mathcal{L}_{,U} U_{,\phi} \ , \quad \mathcal{L}_{,\phi\phi} = \mathcal{L}_{,UU} U_{,\phi}^2 + \mathcal{L}_{,U} U_{,\phi\phi} \tag{11.25}$$

we may rewrite the fluctuation equation like

$$\begin{aligned} - & \quad (\mathcal{L}_{,X} + 2X\mathcal{L}_{,XX})\,\eta'' - \\ - & \quad ((\mathcal{L}_{,XU} + 2X\mathcal{L}_{,XXU})U_{,\phi} - \phi_k''(3\mathcal{L}_{,XX} + 2X\mathcal{L}_{,XXX}))\phi_k'\eta' = \\ = & \quad (\omega^2\mathcal{L}_{,XU} + \mathcal{L}_{,U}U_{,\phi,\phi} - 2X\mathcal{L}_{,XUU}U_{,\phi}^2 - 2X\mathcal{L}_{,XU}U_{,\phi\phi} + \\ + & \quad \phi_k''(\mathcal{L}_{,XU} + 2X\mathcal{L}_{,XXU})U_{,\phi})\eta \end{aligned}$$

or, after implementing the on-shell conditions

$$X| = -U \ , \quad \phi''| = \phi_k'' = U_{,\phi}, \tag{11.26}$$

like

$$\begin{aligned} - & \quad (\mathcal{L}_{,X} + 2X\mathcal{L}_{,XX})\,|\,\eta'' - \\ - & \quad [(\mathcal{L}_{,XU} - 3\mathcal{L}_{,XX} + 2U(\mathcal{L}_{,XXX} - \mathcal{L}_{,XXU})]\,|\,U_{,\phi}\phi_k'\eta' = \\ = & \quad [\omega^2\mathcal{L}_{,X} + U_{,\phi}^2(\mathcal{L}_{,UU} + \mathcal{L}_{,XU} + 2U(\mathcal{L}_{,XUU} - \mathcal{L}_{,XXU})) + \\ + & \quad U_{,\phi\phi}(\mathcal{L}_{,U} + 2U\mathcal{L}_{,XU})]\,|\,\eta \end{aligned} \tag{11.27}$$

This expression should now be compared with the fluctuation equation of the standard case,

$$-\eta'' = (\omega^2 - U_{,\phi\phi}|)\eta. \tag{11.28}$$

Comparing the standard and generalized fluctuation equations for a twin defect solution, and taking into account the twin condition $\mathcal{L}_{,X}| = 1$, we find that a sufficient condition for the equality of the two fluctuation equations is provided by the following on-shell conditions

$$\mathcal{L}_{,XX}| = 0 \tag{11.29}$$

$$[\mathcal{L}_{,XU} + 2U(\mathcal{L}_{,XXX} - \mathcal{L}_{,XXU})]| = 0 \tag{11.30}$$

$$[\mathcal{L}_{,UU} + \mathcal{L}_{,XU} + 2U(\mathcal{L}_{,XUU} - \mathcal{L}_{,XXU})]| = 0 \tag{11.31}$$

and

$$(\mathcal{L}_{,U} + 2U\mathcal{L}_{,XU})| = -1. \tag{11.32}$$

These conditions are, again, purely algebraic conditions which the Lagrangian has to obey. If a Lagrangian obeys these conditions and the two twin conditions (11.18), (11.19), then it not only shares the same twin defect with the standard Lagrangian, but also the spectra of linear fluctuations about the defects coincide.

11.3.2 Examples

It is easy to understand that there must exist infinitely many Lagrangians for each U which obey these conditions. Indeed, if the Lagrangian $\mathcal{L}(X,U)$ is interpreted as a function of two independent variables X and U, then the six twin and linear fluctuation conditions are just conditions which the first few Taylor coefficients of $\mathcal{L}$ must obey "on the diagonal", i.e., for $X = -U$. In a next step, let us construct, as a first example, a class of infinitely many Lagrangians which obey these conditions. These Lagrangians were, in fact, already introduced in [76] as examples of twins of the standard Lagrangian without noticing that they also give rise to coinciding fluctuation spectra. The class of Lagrangians is given by

$$\mathcal{L}^{\text{ex1}} = \sum_{i=3,5,\ldots}^{2N+1} f_i(U)(X+U)^i + X - U \ , \quad f_i(U) \geq 0 \tag{11.33}$$

where the f_i are arbitrary nonnegative functions of their argument. The restriction to odd i implies that the above Lagrangian obeys the null energy condition (NEC) and, therefore, defines a healthy (stable) field theory. We remark that this restriction may be relaxed without violating the NEC provided that the f_i for even i obey certain inequalities, but here we restrict to odd i for reasons of simplicity. It is easy to check that the above Lagrangian obeys

$$\mathcal{L}^{\text{ex1}}| = -2U; , \quad \mathcal{L}^{\text{ex1}}_{,X}| = 1 \tag{11.34}$$

i.e., the twin conditions, as well as

$$\mathcal{L}^{\text{ex1}}_{,XX}| = \mathcal{L}^{\text{ex1}}_{,XU}| = \mathcal{L}^{\text{ex1}}_{,UU}| = 0 \ , \quad \mathcal{L}^{\text{ex1}}_{,U}| = -1 \tag{11.35}$$

and

$$\mathcal{L}^{\text{ex1}}_{,XXX}| = \mathcal{L}^{\text{ex1}}_{,XXU}| = \mathcal{L}^{\text{ex1}}_{,XUU}| = 6f_3. \tag{11.36}$$

Further, these conditions obviously imply the "fluctuation conditions" (11.29) - (11.32), therefore the class of Lagrangians (11.33) not only are twins of the standard Lagrangian $\mathcal{L}_s = X - U$ (i.e. they share the same kink solution with the same energy density), but also the linear fluctuation spectra about the kink solutions coincide.

We remark that it is obvious from the above derivation that the restriction to $f_i = f_i(U)$ in the above class of examples is not necessary, and we may in fact allow for functions $f_i = f_i(\phi) \geq 0$ without changing our results.

Another class of examples is provided by the power series expansion

$$\mathcal{L}^{\text{ex2}} = \sum_{i=0,j=0}^{M,N} a_{ij} X^i (X+U)^j - 2U \tag{11.37}$$

where the twin and fluctuation conditions lead to

$$a_{0j} = 0 \; \forall \, j \; , \quad a_{10} = 1 \; , \quad a_{1j} = 0 \; , j = 1 \ldots N \; , \quad a_{2j} = 0 \; \forall \, j. \tag{11.38}$$

It is again possible to satisfy the NEC by imposing the corresponding conditions (inequalities) on the nonzero coefficients a_{ij}.

For a more systematic search for examples it is useful to perform the following transformation of variables,

$$Y = X + U, \quad Z = U \qquad \Rightarrow \qquad \partial_X = \partial_Y, \quad \partial_U = \partial_Y + \partial_Z \tag{11.39}$$

where the evaluation condition now means evaluation at $Y = 0$, i.e., $| \equiv |_{Y=0}$. Shifting, in addition, the lagrangian by $2U$,

$$\tilde{\mathcal{L}} = \mathcal{L} + 2U \tag{11.40}$$

the two twin conditions and the first fluctuation condition read

$$\tilde{\mathcal{L}}| = 0 \tag{11.41}$$

$$\tilde{\mathcal{L}}_{,Y}| = 1 \tag{11.42}$$

and

$$\tilde{\mathcal{L}}_{,YY}| = 0 \tag{11.43}$$

and, taking these into account, the remaining fluctuation conditions become

$$\left(\tilde{\mathcal{L}}_{,Z} + 2Z\tilde{\mathcal{L}}_{,YZ}\right)| = 0 \tag{11.44}$$

$$\left(\tilde{\mathcal{L}}_{,YZ} - 2Z\tilde{\mathcal{L}}_{,YYZ}\right)| = 0 \tag{11.45}$$

and

$$\left[2\tilde{\mathcal{L}}_{,YZ} + \tilde{\mathcal{L}}_{,ZZ} + 2Z(\tilde{\mathcal{L}}_{,YYZ} + \tilde{\mathcal{L}}_{,YZZ})\right]| = 0. \tag{11.46}$$

As an application, let us study the Dirac–Born–Infeld (DBI) type theory which was first introduced in [51] as an example for a K field twin,

$$\begin{aligned}\tilde{\mathcal{L}}^{\rm DBI} &= -\sqrt{1+2U}\sqrt{1-2X} + \sum_i f_i(U)(X+U)^i \\ &= -\sqrt{1+2Z}\sqrt{1-2Y+2Z} + \sum_i f_i(Z)Y^i \end{aligned} \tag{11.47}$$

where the task consists in determining the coefficient functions $f_i(Z) = f_i(U)$ such that all the twin and fluctuation conditions are satisfied. After some calculation one finds that the two twin conditions (11.41), (11.42) and the first fluctuation condition (11.43) lead to

$$f_0 = 1 + 2Z\ , \quad f_1 = 0\ , \quad f_2 = \frac{1}{2}\frac{1}{1+2Z} \tag{11.48}$$

whereas the remaining fluctuation conditions are satisfied identically precisely for the above solutions for f_0, f_1 and f_2. We conclude that the DBI type Lagrangian

$$\mathcal{L}^{\rm DBI} = -\sqrt{1+2U}\sqrt{1-2X} + 1 + \frac{1}{2}\frac{1}{1+2U}(X+U)^2 \tag{11.49}$$

is a twin of the standard Lagrangian $X - U$ with coinciding linear fluctuation spectra about the common (twin) defect solution. The above DBI type Lagrangian as it stands does not obey the NEC, but we are allowed to add, e.g., a cubic term $f_3(X+U)^3$ without altering the twin or fluctuation conditions. It may be checked easily that, e.g., for functions $f_3(U)$ obeying the inequality $f_3 \geq [1/(3(1+2U)^2)]$, the resulting Lagrangian does obey the NEC.

Obviously, our algebraic method may be used without difficulty to produce more examples of K field twins with coinciding linear fluctuation spectra.

11.3.3 The examples of Bazeia and Menezes

In their recent paper [77], Bazeia and Menezes introduced a class of Lagrangians given by the following ansatz,

$$\mathcal{L}^{\rm BM} = -UF(Y) \ , \quad Y \equiv -\frac{X}{U} \tag{11.50}$$

where F is an arbitrary function of its argument. This ansatz may be justified by the observation that both the twin conditions (11.18), (11.19) and the fluctuation conditions (11.29) - (11.32) are compatible with a Lagrangian which is a homogeneous function of degree one in its two variables X and U. The Lagrangian in (11.50) obviously is such a homogeneous function of degree one. For the partial derivatives w.r.t X and U we get

$$\mathcal{L}^{\rm BM}_X = F' \ , \quad \mathcal{L}^{\rm BM}_{XX} = -\frac{F''}{U} \ , \quad \mathcal{L}^{\rm BM}_{XXX} = \frac{F'''}{U^2} \tag{11.51}$$

$$\mathcal{L}^{\rm BM}_U = -F - \frac{X}{U}F' \ , \quad \mathcal{L}^{\rm BM}_{UU} = -\frac{X^2}{U^3}F'' \tag{11.52}$$

and

$$\mathcal{L}^{\rm BM}_{XU} = \frac{X}{U^2}F'' \ , \quad \mathcal{L}^{\rm BM}_{XUU} = -2\frac{X}{U^3}F'' + \frac{X^2}{U^4}F''' , \quad \mathcal{L}^{\rm BM}_{XXU} = \frac{F''}{U^2} - \frac{X}{U^3}F'''. \tag{11.53}$$

These expressions should now be evaluated on-shell, i.e., for $X = -U$, and inserted into the twin and fluctuation conditions. We shall find that the homogeneity of the ansatz (11.50) not only is compatible with these conditions, but also leads to a considerable simplification for the fluctuation conditions. First of all, for the twin conditions we find

$$\mathcal{L}^{\rm BM}| = -UF(1) = -2U \quad \Rightarrow \quad F(1) = 2 \tag{11.54}$$

and

$$\mathcal{L}^{\rm BM}_{,X}| = F'(1) = 1 \tag{11.55}$$

where the on-shell condition $X = -U$ implies that the function $F(Y)$ and its derivatives are evaluated at $Y = 1$. For the fluctuation conditions we find that condition (11.31) is satisfied identically without providing a further restriction, whereas the remaining conditions lead to

$$\mathcal{L}^{\rm BM}_{XX}| = -\frac{F''(1)}{U} = 0 \tag{11.56}$$

$$[\mathcal{L}_{,XU} + 2U(\mathcal{L}_{,XXX} - \mathcal{L}_{,XXU})]| = -\frac{2}{U}F''(1) = 0 \tag{11.57}$$

and

$$(\mathcal{L}_{,U} + 2U\mathcal{L}_{,XU})| = -F(1) + F'(1) - 2F''(1) = -1 - 2F''(1) = -1 \tag{11.58}$$

where we used the two twin conditions in the last expression. In other words, for the ansatz of Bazeia and Menezes, all four fluctuation conditions just boil down to the simple condition

$$F''(1) = 0. \tag{11.59}$$

Finally, Bazeia and Menezes gave the following explicit example (one-parameter family of Lagrangians)

$$F(Y) = 1 + Y + \frac{\alpha}{3}(1-Y)^3 \quad \Rightarrow \quad \mathcal{L}^{BM,\alpha} = X - U + \frac{\alpha}{3U^2}(X+U)^3 \tag{11.60}$$

where α is a real, positive constant. This example belongs, in fact, to the first class of examples discussed in the previous subsection. Concretely it is of the type (11.33) for the choice

$$f_3(U) = \frac{\alpha}{3U^2}\,, \quad f_i = 0 \quad \text{for} \quad i > 3. \tag{11.61}$$

11.4 Conclusions

In this article we demonstrated that for every standard scalar field theory $\mathcal{L}_s = X - U(\phi)$ which supports a topological defect (a kink), there exist infinitely many generalized (or K) field theories $\mathcal{L}(X,\phi)$ ("twins" of the standard field theory) which support the same kink with the same energy density and with the same spectrum of linear fluctuations about the kink. Further, we gave a simple and explicit algebraic method to construct these twins of the standard scalar field theory with identical linear fluctuation spectra. As stated, some first examples of such twinlike models with coinciding kink solutions, energy densities and linear fluctuation spectra have been given already in [77]. K field twin defects with coinciding linear fluctuation spectra are almost completely indistinguishable from their standard counterparts and, as a consequence, the K field theories giving rise to them have to be considered on a par with the standard field theories in all situations where K field theories cannot be excluded on theoretical grounds. In particular, in the context of effective field theories, where higher kinetic terms are induced naturally, the topological defects formed in K field theories should be taken as seriously as their standard field theory twins, because they give rise to almost exactly the same physics. In this context, an observation of special interest is related to the fact that the coinciding linear fluctuation spectra imply that a semiclassical quantization about the topological defect provides the same results for the standard defect and its K field twins. This not only facilitates specific physical properties of the K field defect, but, more generally, provides us with a first partial result on the quantization of K field theories, which, in general, is a still unsolved and probably quite difficult problem.

Finally, let us briefly comment on possible generalizations and future work. A first issue is the inclusion of fermions and the supersymmetric extension of K field twins. Supersymmetric (SUSY) extensions of scalar K field theories have been found recently [78], [63], [64], and some examples of SUSY K field twins of standard SUSY theories have been given already in [76]. Here, one interesting question obviously is what the coinciding fluctuation spectra in the twin kinks imply for the SUSY fermions. Another interesting generalization concerns the issue of twins of topological defects in higher dimensions, like, e.g., vortices, monopoles, or skyrmions, possibly after a symmetry reduction (e.g. to spherical symmetry) of the Lagrangian or Euler–Lagrange equations. The case of vortices in generalized abelian Higgs models has been investigated in the very recent paper [79], where the authors do find twins of standard vortices. Certainly these issues are worth further investigation.

Chapter 12

N=1 SUSY extension of the BSkM

This chapter is devoted to a detailed analysis of an outstanding example of a K field theory, the baby Skyrme Model, in its supersymmetric version. The need for supersymmetric extensions of this model can be justified in two ways: if the baby Skyrme Model is an effective model (or possibly a toy model for a more realist effective model) of a fundamental theory (QCD in the case of Skyrme model), its supersymmetric extension arises naturally. On the other hand, the supersymmetric structure provides a powerful tool in the analysis of different aspects of the underlying theory. We will see in this chapter how supersymmetry constrains the model, avoiding for example a BPS baby Skyrme model (quartic term plus potential). This chapter consists of a paper published in [99].

$N = 1$ supersymmetric extension of the baby Skyrme model

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Abstract: We construct a method to supersymmetrize higher kinetic terms and apply it to the baby Skyrme model. We find that there exist $N = 1$ supersymmetric extensions for baby Skyrme models with arbitrary potential.

12.1 Introduction

The interest in topological soliton models has been rising ever since their discovery, both because of their rich intrinsic mathematical structure and due to a large field of possible applications, ranging from particle physics to condensed matter systems. One interesting question concerning topological soliton models is whether they allow for supersymmetric extensions and whether other mathematical properties of (some of) the models, like the existence of Bogomolny bounds and corresponding BPS solutions, may be related to the supersymmetric extensions and their properties, like central extensions in the corresponding SUSY algebra. In 1+1 dimensions, simple scalar field theories consisting of a standard kinetic and potential term support topological solitons if the potential allows for more than one vacuum. Further, it has been known for a long time that these simple models allow for supersymmetric extensions [43], and that the corresponding SUSY algebra has a central extension where the central charge is related to the topological charge of the soliton [44]. In higher dimensions, on the other hand, as a result of the Derrick theorem simple scalar field theories do not support, in principle, topological solitons and, therefore, one has to introduce more structure.

One possibility consists in the inclusion of gauge fields, and it is well-known that the resulting theories, like the abelian Higgs or the Chern–Simons

Higgs models in 2+1 dimensions, the BPS monopole model in 3+1 dimensions, or pure Yang–Mills theory in 4+0 dimensions, allow for supersymmetric extensions and that their topological charges are reflected in the central extensions of the corresponding SUSY algebras [45], [46], [129], [44].

Another possibility to circumvent the Derrick theorem in higher dimensions is to allow for non-standard kinetic terms, usually higher (than second) powers of first derivatives in the Lagrangian. The probably best-known model of this type which allows for topological solitons is the Skyrme model [1] in 3+1 dimensions with the group SU(2) as the field (target) space. Much less is known about supersymmetric extensions of this second type of topological soliton models. The supersymmetric extensions of a S^2 (or CP(1)) restriction of the Skyrme model (the so-called Skyrme–Faddeev–Niemi (SFN) model) were investigated in [15] and in [16]. In both papers, a formulation of the SFN model was used where the CP(1) restriction of the Skyrme model is achieved via a gauging of the third, unwanted degree of freedom. As a result, the SFN model is expressed by two complex scalar fields and an undynamical gauge field, which are then promoted to two chiral superfields and a real vector superfield in the Wess–Zumino gauge, respectively. The result of the analysis is that the SFN model as it stands cannot be supersymmetrically extended by these methods. Instead, the supersymmetric extension contains further terms already in the bosonic sector, and also the field equations of the bosonic fields are different.

In a different line of development, more general field theories with a non-standard kinetic term - so-called K theories - have been studied with increasing effort during the last years, beginning with the observation about a decade ago of their possible relevance for the solution of some problems in cosmology (k-inflation [2] and k-essence [3]). K field theories have found their applications in cosmology [4] - [9], and they introduce some qualitatively new phenomena, like the formation of solitons with compact support, so-called compactons [35] - [40]. Quite recently, investigations of the problem of possible supersymmetric extensions of these K field theories have been resumed [17], [50], [135], [136]. Here, [17] and [50] studied supersymmetric extensions of K field theories in 1+1 and in 2+1 dimensions, whereas the investigations of [135] and [136] are for 3+1 dimensional K theories, and with some concrete cosmological applications (ghost condensates and Galileons) in mind.

It is the purpose of this letter to explicitly construct an $N = 1$ supersymmetric extension of the baby Skyrme model. The baby Skyrme model is a model supporting topological solitons in 2+1 dimensions, with a S^2 target space [156], [81], [82]. For some recent results see e.g., [83], [84]. Its field contents and its Lagrangian are like the ones of the SFN model, but the topology is more similar to the Skyrme model (solitons are classified by a winding number, not by a linking number like in the SFN model). The baby Skyrme model serves, on the one hand, as a simpler toy model to study general features of topological solitons. Its supersymmetric extensions will, therefore, be interesting for the general understanding of the role of supersymmetry in topological soliton models, as well. On the other hand, the baby Skyrme model has found some applications, especially in condensed matter physics, e.g. for the description of quantum Hall ferromagnets [85] or of spin textures [86], [87]. The supersymmetric extension method we use is, in fact, similar to the methods used in [135], [136], but adapted to the case of 2+1 dimensions with its specific spin representation of the Lorentz group and its specific SUSY algebra. We shall find that our supersymmetric extension method may be applied to each term in the baby Skyrmion Lagrangian separately, which explains why it may be applied to arbitrary baby Skyrme models, in principle even allowing for the addition of further terms which do not belong to the standard baby Skyrme models.

12.2 Supersymmetric baby Skyrme models

The class of baby Skyrme models we shall consider in this letter is given by the Lagrangian

$$L = \frac{\lambda_2}{2} L_2 + \frac{\lambda_4}{4} L_4 + \frac{\tilde{\lambda}_4}{4} \tilde{L}_4 + \lambda_0 L_0 \qquad (12.1)$$

where the λ_i are coupling constants and the L_i are (the subindices refer to the number of derivatives)

$$L_2 = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \qquad (12.2)$$

(the standard nonlinear sigma model term),

$$L_4 = -(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 \tag{12.3}$$

(the Skyrme term),

$$\tilde{L}_4 = (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 \tag{12.4}$$

(another quartic term), and

$$L_0 = -V(\phi_3) \tag{12.5}$$

is a potential term which is usually assumed to depend only on the third component ϕ_3 of the field. The three-component field vector $\vec{\phi}$ obeys the constraint $\vec{\phi}^2 = 1$. The term $\tilde{L}_4$ is absent in the baby Skyrme model (i.e., $\tilde{\lambda}_4 = 0$), but this term is considered in some extensions of the model, especially in the corresponding model in one dimension higher (the SFN model in 3+1 dimensions), see, e.g., [88]. Further, we shall see that our supersymmetric extension can be applied to each term separately, therefore we include the $\tilde{L}_4$ term in the discussion for the sake of generality.

The field theories we consider exist in 2+1 dimensional Minkowski space, and our supersymmetry conventions are based on the widely used ones of [18], where our only difference with their conventions is our choice of the Minkowski space metric $\eta_{\mu\nu} = \text{diag}(+,-,-)$. All sign differences between this paper and [18] can be traced back to this difference. We introduce three $N = 1$ real scalar superfields, i.e.

$$\Phi^i(x,\theta) = \phi^i(x) + \theta^\alpha \psi^i_\alpha(x) - \theta^2 F^i(x), \quad i = 1,2,3 \tag{12.6}$$

where ϕ^i are three real scalar fields, ψ^i_α are fermionic two-component Majorana spinors, and F^i are the auxiliary fields. Further, θ^α are the two Grassmann-valued superspace coordinates, and $\theta^2 \equiv (1/2)\theta^\alpha\theta_\alpha$. Spinor indices are risen and lowered with the spinor metric $C_{\alpha\beta} = -C^{\alpha\beta} = (\sigma_2)_{\alpha\beta}$, i.e., $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \psi^\beta C_{\beta\alpha}$.

The components of superfields can be extracted with the help of the following projections

$$\phi(x) = \Phi(z)|, \quad \psi_\alpha(x) = D_\alpha\Phi(z)|, \quad F(x) = D^2\Phi(z)|, \tag{12.7}$$

where the superderivative is

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\gamma^\mu{}_\alpha{}^\beta\theta_\beta\partial_\mu \equiv \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\alpha\beta} \quad , \qquad D^2 \equiv \frac{1}{2}D^\alpha D_\alpha \tag{12.8}$$

and the vertical line | denotes evaluation at $\theta^\alpha = 0$.

The problem now consists in finding the supersymmetric extensions $\mathcal{L}_i$ of all the contributions L_i to the Lagrangian (12.1). For the non-linear O(3) sigma model term L_2 this supersymmetric extension was found long ago in [89], [90]. one simply chooses the standard SUSY kinetic term $D^2(-\frac{1}{2}D^\alpha\Phi^i D_\alpha\Phi^i)|$ for the lagrangian $\mathcal{L}_2$ and imposes the constraint $\vec{\phi}^2 = 1$ on the superfield, i.e., $\vec{\Phi}^2 = 1$, which in components reads

$$\phi^i\cdot\phi^i = 1 \tag{12.9}$$

$$\phi^i\cdot\psi^i_\alpha = 0 \tag{12.10}$$

$$\phi^i\cdot F^i = \frac{1}{2}\bar{\psi}^a\psi^a \tag{12.11}$$

or, in the purely bosonic sector with $\psi = 0$

$$\phi^i\cdot\phi^i = 1 \tag{12.12}$$

$$\phi^i\cdot F^i = 0. \tag{12.13}$$

It may be checked easily that the constraint $\vec{\Phi}^2 = 1$ is invariant under the $N = 1$ SUSY transformations

$$\delta\phi^i = \epsilon^\alpha\psi^i_\alpha \ , \quad \delta\psi^i_\alpha = -i\partial_\alpha{}^\beta\epsilon_\beta\phi^i - \epsilon_\alpha F^i \ , \quad \delta F^i = i\epsilon^\beta\partial_\beta{}^\alpha\psi^i_\alpha. \tag{12.14}$$

Remark: the fact that the constraint $\vec{\Phi}^2 = 1$ provides just one real constraint in superspace makes it appear natural to consider just $N = 1$ supersymmetry. It turns out, nevertheless, that the supersymmetric O(3) nonlinear sigma model possesses an extended $N = 2$ supersymmetry, which is not completely obvious in the $N = 1$ SUSY formalism, see [109]. In fact, all nonlinear sigma models with a Kähler target space metric have the $N = 2$ supersymmetry [91].

Our task now is to find the ($N = 1$) SUSY extensions of the remaining terms in the Lagrangian. As we are mainly interested in the bosonic sector of the resulting theory we shall set the spinor fields equal to zero, $\psi^i_\alpha = 0$, in the following. We remark that all spinorial contributions to the lagrangian we shall consider are at least quadratic in the spinors, therefore it is consistent to study the subsector with $\psi^i_\alpha = 0$. The following superfields (we display them for $\psi^i_\alpha = 0$) are useful for our considerations,

$$
\begin{aligned}
(D^\alpha \Phi^i D_\alpha \Phi^j)_{\psi=0} &= 2\theta^2 (F^i F^j + \partial^\mu \phi^i \partial_\mu \phi^j) && (12.15)\\
(D^\beta D^\alpha \Phi^i D_\beta D_\alpha \Phi^j)_{\psi=0} &= 2(F^i F^j + \partial^\mu \phi^i \partial_\mu \phi^j) + \\
& \quad 2\theta^2 (F^i \Box \phi^j + F^j \Box \phi^i - \partial_\mu \phi^i \partial^\mu F^j && (12.16)\\
& - \partial_\mu \phi^j \partial^\mu F^i) \\
(D^2 \Phi^i D^2 \Phi^j)_{\psi=0} &= F^i F^j + \theta^2 (F^i \Box \phi^j + F^j \Box \phi^i). && (12.17)
\end{aligned}
$$

We observe that both the product of Eq. (12.15) with Eq. (12.17) and the product of Eq. (12.15) with Eq. (12.17) contain terms of the type $F^2(\partial\phi)^2$, so by choosing the right linear combination we may cancel these unwanted terms. Concretely, we propose the following supersymmetric Lagrangians (remember that $D^2\theta^2 = \int d^2\theta\theta^2 = -1$)

$$
(\mathcal{L}_2)_{\psi=0} = -\frac{1}{2}[D^2(D^\alpha \Phi^i D_\alpha \Phi^i)|]_{\psi=0} = F^i F^i + \partial^\mu \phi^i \partial_\mu \phi^i \qquad (12.18)
$$

$$
\begin{aligned}
(\tilde{\mathcal{L}}_4)_{\psi=0} &= [D^2(D^\alpha \Phi^i D_\alpha \Phi^i)(D^2 \Phi^j D^2 \Phi^j - \frac{1}{4} D^\beta D^\alpha \Phi^j D_\beta D_\alpha \Phi^j)|]_{\psi=0} \\
&= -(F^i)^2 (F^j)^2 + (\partial_\mu \phi^i)^2 (\partial_\nu \phi^j)^2 && (12.19)
\end{aligned}
$$

$$
\begin{aligned}
& (\mathcal{L}_4)_{\psi=0} = \\
- & \frac{1}{2}\epsilon_{ijk}\epsilon_{i'j'k}[D^2(D^\alpha \Phi^i D_\alpha \Phi^{i'} D^2 \Phi^j D^2 \Phi^{j'} + D^\alpha \Phi^j D_\alpha \Phi^{j'} D^2 \Phi^i D^2 \Phi^{i'})|]_{\psi=0} \\
+ & \frac{1}{8}\epsilon_{ijk}\epsilon_{i'j'k}[D^2(D^\alpha \Phi^i D_\alpha \Phi^{i'} D^\gamma D^\beta \Phi^j D_\gamma D_\beta \Phi^{j'} + \\
+ & D^\alpha \Phi^j D_\alpha \Phi^{j'} D^\gamma D^\beta \Phi^i D_\gamma D_\beta \Phi^{i'})|]_{\psi=0} \\
= & \epsilon_{ijk}\epsilon_{i'j'k}(F^i F^{i'} F^j F^{j'} - \partial_\mu \phi^i \partial^\mu \phi^{i'} \partial_\nu \phi^j \partial^\nu \phi^{j'}) = \\
- & (\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 \qquad (12.20)
\end{aligned}
$$

and for the potential term, as usual

$$
(\mathcal{L}_0)_{\psi=0} = [D^2 P(\Phi_3)|]_{\psi=0} = F_3 P'(\phi_3) \qquad (12.21)
$$

where P is the prepotential and the prime denotes derivation w.r.t. its argument ϕ_3. The resulting bosonic lagrangian is

$$\begin{aligned}(\mathcal{L})_{\psi=0} &= \frac{\lambda_2}{2}[(\vec{F})^2 + \partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}] + \frac{\tilde{\lambda}_4}{4}[(\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi})^2 - ((\vec{F})^2)^2] \\ &- \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 + \lambda_0 F_3 P' + \mu_F(\vec{F}\cdot\vec{\phi}) + \\ &+ \mu_\phi(\vec{\phi}^2 - 1)\end{aligned} \tag{12.22}$$

where μ_F and μ_ϕ are Lagrange multipliers enforcing the constraints (12.13) and (12.12).

From now on, we restrict to the standard baby Skyrme SUSY extension with $\tilde{\lambda}_4 = 0$ so that the term $\tilde{\mathcal{L}}_4$ is absent. In this restricted case, the (algebraic) field equation for the field $\vec{F}$ is

$$\lambda_2 F^i + \lambda_0\delta^{i3}P'(\phi_3) + \mu_F\phi^i = 0. \tag{12.23}$$

Multiplying by $\vec{\phi}$ we find for the Lagrange multiplier

$$\mu_F = -\lambda_0\phi_3 P' \tag{12.24}$$

and for the auxiliary field $\vec{F}$

$$F^i = \frac{\lambda_0}{\lambda_2}(\phi_3\phi^i - \delta^{i3})P' \tag{12.25}$$

and, therefore, for the bosonic Lagrangian

$$\begin{aligned}(\mathcal{L})_{\psi=0} &= \frac{\lambda_2}{2}[(\vec{F})^2 + \partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}] - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 + \lambda_0 F_3 P' + \mu_\phi(\vec{\phi}^2 - 1) \\ &= \frac{\lambda_2}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 - \frac{\lambda_0^2}{2\lambda_2}(1-\phi_3^2)P'^2 + \\ &+ \mu_\phi(\vec{\phi}^2 - 1).\end{aligned} \tag{12.26}$$

This is exactly the standard (non-supersymmetric) baby Skyrme model with the potential term given by

$$V(\phi_3) = \frac{\lambda_0}{2\lambda_2}(1-\phi_3^2)P'^2(\phi_3). \tag{12.27}$$

obviously, all positive semi-definite potentials $V(\phi_3)$ may be obtained by an appropriate choice for the prepotential $P(\phi_3)$.

Remark: the relation between prepotential and potential differs slightly (by the additional factor $(1-\phi_3^2)$) from the standard SUSY relation between prepotential and potential, due to the constrained nature of the superfield $\vec{\Phi}$.

Remark: The baby Skyrme model has a Bogomolny bound in terms of the topological charge (winding number) of the scalar field $\vec{\phi}$, but nontrivial solutions in general do not saturate this bound. There exist, however, two limiting cases where nontrivial solutions do saturate a Bogomolny bound and solve the corresponding first order Bogomolny equations. one might wonder whether these limiting cases allow for the supersymmetric extension discussed in this letter, as well. The first limiting case is the case of the pure o(3) sigma model where both the potential and the quartic (Skyrme) term are absent, and, as discussed above, it is well-known that this case has a supersymmetric extension. Concerning the second case, it has been found recently that the model without the quadratic O(3) sigma model term (i.e., $\lambda_2 = 0$) originally introduced in [92], has nontrivial Bogomolny solutions and, further, an infinite number of symmetries and conservation laws [93], [94]. Given the close relation between Bogomolny solutions and supersymmetry, one might expect that this limiting case should have the supersymmetric extension, too, but this is, in fact, not true. The field equation (12.23) for $\vec{F}$ for the case $\lambda_2 = 0$ reads

$$\lambda_0 \delta^{i3} P'(\phi_3) + \mu_F \phi^i = 0.$$

It does not contain $\vec{F}$ at all, so $\vec{F}$ itself is a Lagrange multiplier in this case. For a nontrivial field configuration $\vec{\phi}$, the only solution of this equation is $\mu_F = 0$ and $\lambda_0 = 0$, therefore the potential term is absent. We conclude that the model consisting only of the quartic Skyrme term $\mathcal{L}_4$ does allow for a supersymmetric extension, whereas the model consisting of both the quartic Skyrme term and the potential term $\mathcal{L}_0$ does not allow for the supersymmetric extension discussed in this letter.

Remark: we also calculated the full Lagrangian with the spinors included. The contributions from $\mathcal{L}_2$ and $\mathcal{L}_0$ are just the standard spinor kinetic term and the Yukawa-type coupling term, respectively. The contribution from the Skyrme term $\mathcal{L}_4$, on the other hand, is quite long (it consists of 17 more terms) and not particularly illuminating, therefore we do not display it here.

12.3 Summary

We described a method to calculate the supersymmetric extensions of higher kinetic terms (K field theories) and applied it to the baby Skyrme model. We found that the baby Skyrme model has a supersymmetric extension which preserves the form of the original (non-supersymmetric) baby Skyrme Lagrangian in the bosonic sector for arbitrary potential. This possibility to supersymmetrize the baby Skyrme model seems to have gone unnoticed up to now, probably because of some inherent difficulties in the supersymmetrization of higher K terms. Indeed, in general supersymmetric extensions of higher kinetic terms tend to render the "auxiliary" field dynamical, or at least to couple it to field derivatives, which in turn drastically changes the behaviour of the field theory under consideration. Also, higher kinetic terms tend to jeopardize the energy balance between bosonic and fermionic degrees of freedom characteristic for standard SUSY theories. We remark that topological soliton models already at the classical or semiclassical level describe relevant degrees of freedom as low-energy limits of more complete quantum field theories in the ultraviolet. As a consequence, the possibility to directly supersymmetrize these topological soliton models is certainly of interest despite the fact that, at this moment, we are not aware of a direct physical application of the baby Skyrme model where supersymmetry is assumed to play a role. In addition, the possibility to construct supersymmetric extensions is an interesting mathematical property of a topological soliton model like the baby Skyrme model and might be useful for a better understanding of its theoretical structure. Interestingly, we found that the limiting case of the baby Skyrme model without the quadratic linear sigma model term (that is, the model consisting of $\mathcal{L}_4$ and $\mathcal{L}_0$), does not allow for the supersymmetric extension in spite of its infinitely many exact Bogomolny solutions and its infinitely many symmetries [93].

A further problem of interest concerns the possibility to apply the supersymmetric extension method presented in this letter to further K field theories. In 1+1 and 2+1 dimensions the supersymmetric extension is rather straight forward and may be applied to a quite general class of K field theories. A more detailed discussion of these issues will be published elsewhere. In 3+1 dimensions, on the other hand, the class of K field theories which admit a supersymmetric extension might be more restricted. There, the simplest superfield is the chiral superfield with a *complex* scalar field in the bosonic sector, which implies some restrictions on the field contents of theories amenable to supersymmetric extensions. Nevertheless, it might be possible to supersymmetrize topological soliton models in 3+1 dimensions by first choosing a field contents in accordance with the requirements of 3+1 dimensional supersymmetry, and by then introducing the constraints necessary for the reduction of the degrees of freedom to the soliton model one wants to investigate. We finally remark that, as already stated, supersymmetric extensions for some K field theories in 3+1 dimensions with applications in cosmology have been studied recently in [135], [136], using analogous methods.

Chapter 13

Extended SUSY and BPS solutions

The schemes of supersymmetrization are not uniques, and in this chapter another $N = 1$ SUSY extension of the bSM in presented which allows the BPS baby Skyrme model, this suggests directly a possible hidden second supersymmetry which is explicitly constructed. At this point, as already stated, due to the dimensional reduction $N = 1, d = 3 + 1 \longleftrightarrow N = 2, d = 2 + 1$ we can extend our results to $d = 3 + 1$ finding analog result as in [16] and [15]. We explore also gauged and ungauged SUSY extensions, both $N = 1$ and $N = 2$, and analyze their Bogomolnyi equations. A general method to obtain Bogonolnyi equations from extended supersymmetry in $d = 2 + 1$ is presented. This chapter consists of a paper published in [100].

$N = 1$ supersymmetric extension of the baby Skyrme model

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Abstract: We continue the investigation of supersymmetric extensions of baby Skyrme models in $d = 2 + 1$ dimensions. In a first step, we show that the CP(1) form of the baby Skyrme model allows for the same $N = 1$ SUSY extension as its O(3) formulation. Then we construct the $N = 1$ SUSY extension of the gauged baby Skyrme model, i.e., the baby Skyrme model coupled to Maxwell electrodynamics. In a next step, we investigate the issue of $N = 2$ SUSY extensions of baby Skyrme models. We find that *all* gauged and ungauged submodels of the baby Skyrme model which support BPS soliton solutions allow for an $N = 2$ extension such that the BPS solutions are one-half BPS states (i.e., annihilated by one-half of the SUSY charges). In the course of our investigation, we also derive the general BPS equations for completely general $N = 2$ supersymmetric field theories of (both gauged and ungauged) chiral superfields, and apply them to the gauged nonlinear sigma model as a further, concrete example.

13.1 Introduction

The Skyrme model [1] is a nonlinear field theory in 3+1 dimensional Minkowski space which supports topological soliton solutions. Its field variables take values in SU(2) which, together with an one-point compactification of the base space $\mathbb{R}^3 \rightarrow S^3$ implied by the condition of finite energy, leads to the classification of field configurations by an integer-valued winding number or topological degree. The most important application of the Skyrme model is in the field of nuclear and strong interaction physics [102], [106], [107], [150], [103], [151], [162]. In this context, the Skyrme model (or some of its generalizations) is interpreted as a low energy effective field theory which may be justified from the underlying fundamental theory (QCD), e.g., by invoking some large N_c (number of colors) arguments [108], [109]. In this interpretation, the primary fields of the effective theory are related to mesons (e.g. pions in the SU(2) case), whereas baryons and nuclei are described by the topological solitons of the theory, and baryon number is identified with the topological degree of the corresponding soliton.

The baby Skyrme model was introduced originally as a planar analogue of the three dimensional Skyrme model [110]-[118], although it has found its own applications, e.g., in condensed matter physics [119] or in brane cosmology [120]. Its target space is simplified accordingly, as well (S^2 instead of the $SU(2)$ target space of the Skyrme model), such that static field configurations again can be classified by a winding number. Like the original version of the Skyrme model, as proposed by Skyrme, also the Lagrangian of the baby Skyrme model consists of a kinetic term quadratic in first derivatives (the $O(3)$ nonlinear sigma model term) and a quartic kinetic term (the analogue of the Skyrme term). Further, for the baby Skyrme model, the inclusion of a potential term is obligatory for the existence of static finite energy solutions. The specific form of this potential term is, however, quite arbitrary, and different potentials have been studied [110]-[118]. The Skyrme model, too, allows for the addition of a potential (not obligatory in that case) or of some further terms like, e.g., the square of the topological current, which is sextic in first derivatives. In any case, the presence of higher derivative terms (”non-standard kinetic terms”) in Skyrme-type models is necessary for the existence of topological solitons. In addition, in both models the energies of static configurations can be bound from below by a Bogomol’nyi bound (a multiple of the topological degree), but generic soliton solutions do not saturate this bound. It is, however, possible both for the baby Skyrme model [121]-[124] and for a generalized Skyrme model [125], [126], [101], [159] (i.e., a generalization of the original model proposed by Skyrme) to find certain submodels such that their topological soliton solutions saturate the corresponding Bogomol’nyi bound, that is, they are of the BPS type and obey certain first-order BPS equations.

At this point, it is useful to compare the properties of Skyrme-type theories with those of the abelian and nonabelian Higgs models with their vortex-type or monopole-type solitons (see e.g. [162]). The topology of these solitons is different, because now the one-point compactification of the base space is not assumed, and the scalar fields "Higgs fields") may be classified by a topological degree related to their winding about the sphere at spatial infinity. As mentioned already, this behaviour leads to an infinite energy due to the presence of angular gradients in the kinetic energy density with a rather slow decay for large distances. The well-known way to remedy this problem is via the coupling of the Higgs field to a gauge field such that the unwanted angular gradients are converted into pure gauge configurations and do not contribute to the energy. If the standard kinetic terms for the gauge fields (Maxwell or Yang-Mills terms) are added, we just arrive at the abelian Higgs model or the t'Hooft-Polyakov (nonabelian) Higgs model, respectively. In spite of the different topology of the corresponding solitons, these theories share many properties with the Skyrme-type ones. For the Higgs theories, too, the energies of static configurations can be bound from below by Bogomol'nyi bounds where, however, solitons of generic theories do not saturate the bounds. Again, submodels can be found (usually, by a judicious choice of the Higgs potential) whose solitons (vortices or monopoles) are of the BPS type and saturate the bound.

There exists, however, one aspect where the two classes of theories are apparently rather different, namely the issue of supersymmetry. The Higgs-type theories are well-known to possess $N = 1$ supersymmetric (SUSY) extensions. Further, the submodels with BPS solitons even allow for an $N = 2$ SUSY extension such that the BPS soliton solutions are, in fact, one-half BPS states in the sense of SUSY, that is, field configurations which are annihilated by one-half of the SUSY charges (see, e.g., [43] - [45]). The construction of these SUSY extensions is facilitated by the fact that the kinetic terms both for the Higgs and for the gauge fields are of the standard form (quadratic in derivatives), because the SUSY extensions of these kinetic terms are well-known. On the other hand, until recently not much was known about the SUSY extensions of Skyrme-type theories, where the presence of nonstandard kinetic terms is mandatory. To the best of our knowledge, the first investigations of SUSY extensions of Skyrme-type theories were performed in [15], [16]. Concretely, the authors studied possible SUSY extensions of the so-called Skyrme-Faddeev-Niemi (SFN) model, which has exactly the field content of the baby Skyrme model, but in 3+1 dimensions (in 3 spatial dimensions the potential term is not mandatory and is usually omitted). In this model field configurations are no longer classified by a winding number but, instead, by a linking number (the Hopf index). In both papers, the authors treated the SFN model as a CP(1) restriction of the original Skyrme model, where the elimination of the third, unwanted degree of freedom is achieved by transforming it into pure gauge via the introduction of a non-dynamical gauge field. One consequence of this procedure is that the Skyrme term (which is non-standard) may be expressed as the standard Maxwell term of the non-dynamical gauge field. As a consequence, the resulting action only contains standard kinetic terms (the nonlinear sigma model term for the CP(1) field and the Maxwell term) and standard SUSY techniques may be used. The result of these investigations is that the original SFN model cannot be extended to a SUSY theory by these methods. Any SUSY extension achieved in this way contains additional terms already in the bosonic sector.

The investigation of SUSY extensions of genuinely nonstandard kinetic terms has been resumed only recently [132]-[139] (see also [140]-[143] for related discussions), where this rising interest is partly owed to the fact that field theories with non-standard kinetic terms may be instrumental in the resolution of some enigmas of cosmology [2]-[9]. Concretely, in [64] we demonstrated that the baby Skyrme model in the O(3) formulation does have a $N = 1$ SUSY extension for arbitrary non-negative potential. It turned out, however, that a submodel supporting BPS solitons (the so-called BPS baby Skyrme model, where the non-linear sigma model term is suppressed) cannot be supersymmetrically extended by the methods of that paper. It is the purpose of the present paper to go much further in the analysis of SUSY extensions of baby Skyrme models where, among other issues, the puzzle just mentioned will be resolved in the course of the investigation.

Our paper is organized as follows. In Section 13.2 we introduce the baby Skyrme models and fix some notation. In Section 13.3 we give our conventions for $N = 1$ SUSY in 2+1 dimensions. In Section 13.4 we discuss the $N = 1$ SUSY extension of the baby Skyrme model in the CP(1) formulation. In Section 13.5 we introduce the $N = 1$ SUSY extension of the gauged baby Skyrme model [144], i.e., the baby Skyrme model coupled to Maxwell electrodynamics in the standard way. In Section 13.6 we give our conventions for extended $N = 2$ SUSY. In Section 13.7 we attempt to find an $N = 2$ extension of the SUSY baby Skyrme model. We find that, in addition to the well-known Kähler potential term giving rise to the non-linear sigma model, we have to introduce a further term into the lagrangian superfield. This further term has the surprising effect that, after the substitution of the auxiliary fields via their field equations, not only the quartic (Skyrme) term is produced in the bosonic sector but, at the same time, the quadratic (nonlinear sigma model) term is eliminated for arbitrary values of the Kähler potential. Besides, a potential term depending on the Kähler metric is automatically induced in this process. In other words, in the purely bosonic sector we find precisely the BPS baby Skyrme model consisting of the Skyrme term and a potential, but without the sigma model term. Due to the absence of this sigma model term, we may choose arbitrary Kähler metrics and, therefore, arbitrary non-negative potentials. So in this case the potential is induced by the Kähler metric and *not* by a superpotential. A superpotential term is, in fact, forbidden in this construction. In Section 13.8 we discuss the issue of BPS (or Bogomol'nyi) equations for the BPS baby Skyrme models from the point of view of $N = 2$ SUSY. Concretely, in a first step we derive the equation for one-half BPS states for a *completely general* $N = 2$ chiral superfield. Then we apply the resulting equation to the SUSY BPS baby Skyrme model and find that its one-half BPS states are precisely the BPS solutions of the BPS baby Skyrme model [121]-[124]. In Section 13.9 we introduce the $N = 2$ SUSY extension of the gauged baby Skyrme model. Again, the procedure implies the absence of the (gauged) quadratic sigma model term, and we find the gauged BPS baby Skyrme model [145]. For this model, a BPS bound and BPS solitons have been found recently, where the construction of the BPS bound implied the introduction of a certain "superpotential" $\mathcal{W}$ which is related to the potential $\mathcal{V}$ by a first order differential equation ("superpotential equation"). We find that, again, the BPS solitons are one-half BPS states of the $N = 2$ SUSY extension, and the "superpotential equation" may be understood from the fact that both the "superpotential" $\mathcal{W}$ and the potential $\mathcal{V}$ are derived from a certain Kähler potential. In Section 13.10 we apply our methods to the gauged nonlinear sigma model, which is known to possess BPS solitons for a certain choice of potential [146]. It follows easily from our general construction that this model has an $N = 2$ SUSY extension and that the BPS solitons are one-half BPS states. In this case, the sigma model term and, therefore, the Kähler metric, have a fixed, given form, so, as a result, also the potential (which is again a function of the Kähler metric) is fixed. Finally, Section 13.11 contains our conclusions.

13.2 The baby Skyrme model

The field variables of the baby Skyrme model take values in the two-sphere, so it is naturally parametrized by a three component unit vector field $\vec{n}(x)$, where $\vec{n}^2 = 1$. The lagrangian density is a sum of three terms,

$$L^{bS} = L_2 + L_4 + L_0 \tag{13.1}$$

where L_2 is the sigma model term

$$L_2 = \frac{\lambda_2}{4}(\partial_\mu \vec{n})^2, \tag{13.2}$$

L_4 is the Skyrme term

$$L_4 = -\frac{\lambda_4}{8}(\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 \equiv -\frac{\lambda_4}{16}K_\mu^2, \tag{13.3}$$

where K^μ is the topological current

$$K^\mu = \epsilon^{\mu\nu\rho}\vec{n} \cdot (\partial_\nu \vec{n} \times \partial_\rho \vec{n}), \tag{13.4}$$

such that

$$k = (1/8\pi)\int d^2x K^0, \quad k \in \mathbb{Z} \tag{13.5}$$

is the winding number (topological degree) of the map $\vec{n}$. Finally, L_0 is the potential term

$$L_0 = -\lambda_0 \mathcal{V}(\vec{n}). \tag{13.6}$$

The lagrangian density has dimensions of $\frac{[\text{action}]}{[\text{length}]^2[\text{time}]}$ or, equivalently, $\frac{[\text{energy}]}{[\text{length}]^2}$. Further, we shall assume natural units where the velocity of light is equal to one such that [length] = [time]. Extracting a common energy scale E_0 we may write the Lagrangian density like

$$L = E_0 \left(\frac{\nu^2}{4}(\partial_\mu \vec{n})^2 - \frac{\lambda^2}{8}(\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 - \mu^2 \mathcal{V}(\vec{n}) \right) \tag{13.7}$$

where now ν is dimensionless, and λ and μ^{-1} have the dimension of length. A nonzero ν may always be set equal to one, $\nu = 1$, by an appropiate choice of the energy scale E_0. We shall, therefore, assume $\nu = 1$ or $\nu = 0$ in what follows, depending on whether the term L_2 is present or absent. Besides, all energies will be measured in units of E_0, which is equivalent to setting $E_0 = 1$, what we assume from now on. In a next step, we shall introduce dimensionless coordinates via $x^\mu = l_0 y^\mu$ (here, l_0 is a universal length scale) which are more appropriate for SUSY calculations, where we continue, however, to use the symbols x^μ (instead of y^μ) for the new, dimensionless coordinates. For nonzero λ, we may always choose $l_0 = \lambda$. Choosing, in addition, length units such that $l_0 = 1$, we get again the lagrangian (13.7) where, now, both ν and λ take the values 1 or 0 (depending on whether the corresponding terms are present or absent), and μ is a dimensionless coupling constant. But we have not yet made any assumption on the form of $\mathcal{V}$, therefore we may always reabsorb this constant into the definition of the potential. Doing so, our lagrangian density for the full baby Skyrme model (with all terms present) now reads

$$L = \left(\frac{1}{4}(\partial_\mu \vec{n})^2 - \frac{1}{8}(\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 - \mathcal{V}(\vec{n}) \right), \tag{13.8}$$

which is the dimenionless lagrangian density in the O(3) formulation of the baby Skyrme model. In the following, however, we shall need the model in the CP(1) formulation, where the field variable is parametrized by a complex scalar field $u(x)$ related to $\vec{n}$ by stereographic projection,

$$\vec{n} = \frac{1}{1+|u|^2}(u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2). \tag{13.9}$$

In terms of the field u (i.e., in CP(1) formulation), the dimensionless lagrangian density reads

$$L^{bS}_{CP^1} = L_2 + L_4 + L_0 \tag{13.10}$$

where

$$L_2 = \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+u\bar{u})^2} \tag{13.11}$$

is the nonlinear sigma-model term,

$$L_4 = -\frac{1}{(1+u\bar{u})^4}[(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\nu \bar{u} \partial^\nu \bar{u})] \tag{13.12}$$

is the "Skyrme" term quartic in first derivatives, and

$$L_0 = -\mathcal{V}(u\bar{u}) \tag{13.13}$$

is the potential term. From now on, we assume that $\mathcal{V}$ only depends on the modulus (squared) of u (i.e., only depends on n_3 in the O(3) formulation), which implies that the potential does not completely break the SU(2) target space symmetry (the O(3) symmetry in the O(3) formulation) of $L_2 + L_4$, but leaves a U(1) subgroup (the phase transformation $u \to e^{i\lambda}u$) intact. This is of special importance if we want to couple the Skyrme field u to the U(1) gauge field of electrodynamics.

It is sometimes useful to consider the slightly more general class of models given by

$$L_2 = g(u,\bar{u})\partial_\mu u \partial^\mu \bar{u}, \tag{13.14}$$

$$L_4 = -h(u,\bar{u})[(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\nu \bar{u} \partial^\nu \bar{u})] \tag{13.15}$$

where the original baby Skyrme model corresponds to the choice

$$g(u,\bar{u})^2 = h(u,\bar{u}) = \frac{1}{(1+u\bar{u})^4}. \tag{13.16}$$

Geometrically, g and h may be interpreted as the target space metric and the (square of the) target space area density, respectively.

13.3 $N = 1$ supersymmetry in $d = 2+1$ dimensions

We use the Minkowski space metric $\eta_{\mu\nu} = \text{diag}(+,-,-)$. Then, an $N = 1$ real scalar superfield is given by

$$\Phi(z) = \phi(x) + \theta^\alpha \psi_\alpha(x) - \theta^2 F(x), \tag{13.17}$$

where the coordinate z stands collectively for (x^μ, θ_α), ϕ is a real scalar field, ψ_α is a fermionic two-component Majorana spinor, and F is the auxiliary field. Further, θ^α are the two Grassmann-valued superspace coordinates, and $\theta^2 \equiv (1/2)\theta^\alpha\theta_\alpha$. The components of a superfield can be extracted with the help of the following projections

$$\phi(x) = \Phi(z)|, \quad \psi_\alpha(x) = D_\alpha\Phi(z)|, \quad F(x) = D^2\Phi(z)|, \tag{13.18}$$

where the superderivative is

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\gamma^\mu{}_\alpha{}^\beta\theta_\beta\partial_\mu \tag{13.19}$$

$$D^2 \equiv \frac{1}{2}D^\alpha D_\alpha \tag{13.20}$$

and the vertical line | denotes evaluation at $\theta^\alpha = 0$. From here it is easy to construct supersymmetric lagrangian, which are just the θ integrals of general superfields, that is, general functions of the basic superfields and their superderivatives, i.e.:

$$L_{N=1} = \int d^2\theta\mathcal{L}(\Phi^i, D^\alpha\Phi^j, ...). \tag{13.21}$$

It is also possible to construct $N = 1$ complex superfields by combining real ones.

13.4 $N = 1$ CP(1) baby Skyrme model

13.4.1 $N = 1$ extension

We will construct a $N = 1$ supersymmetric extension of the model (13.10). In a first step, we need the basic $N = 1$ superfields

$$\Phi^1 = \phi^1 + \theta^\alpha\psi^1_\alpha - \theta^2F^1, \quad \Phi^2 = \phi^2 + \theta^\alpha\psi^2_\alpha - \theta^2F^2. \tag{13.22}$$

Taking into account that $u \in \mathbb{C}$ and $\phi^i \in \mathbb{R}$, we introduce the following combinations for the new superfields U and $\bar{U}$:

$$U = \Phi^1 + i\Phi^2 \tag{13.23}$$

$$\bar{U} = \Phi^1 - i\Phi^2 \tag{13.24}$$

such that

$$U| = \phi^1 + i\phi^2 \equiv u \qquad (13.25)$$

$$\bar{U}| = \phi^1 - i\phi^2 \equiv \bar{u}. \qquad (13.26)$$

Similarly we define

$$\chi_\alpha \equiv \psi^1_\alpha + i\psi^2_\alpha, \qquad F \equiv F^1 + iF^2 \qquad (13.27)$$

$$\bar{\chi}_\alpha \equiv \psi^1_\alpha - i\psi^2_\alpha, \qquad \bar{F} \equiv F^1 - iF^2. \qquad (13.28)$$

With these complex combinations of real superfields we can now generate the quadratic term. Considering only the bosonic sector, we find

$$L_2|_{bos} = \frac{1}{2}\int d^2\theta g(U,\bar{U})D^\alpha U D_\alpha \bar{U}|_{\chi=0} = g(u,\bar{u})(F\bar{F} + \partial_\mu u \partial^\mu \bar{u}) \qquad (13.29)$$

that is, the quadratic term of the baby Skyrme model plus a term quadratic in the auxiliary field F. For the quartic term we need two contributions, which for the moment we write without their target space area factors h. The first one is

$$\tilde{L}_{4a} = \int d^2\theta[D^\alpha U D_\alpha U + D^\alpha \bar{U} D_\alpha \bar{U}][D^2 U D^2 U + D^2\bar{U} D^2 \bar{U} - \qquad (13.30)$$

$$- \frac{1}{4}(D^\alpha D^\beta U D_\alpha D_\beta U + D^\alpha D^\beta \bar{U} D_\alpha D_\beta \bar{U})] \qquad (13.31)$$

and its bosonic part results in

$$\tilde{L}_{4a}|_{bos} = (F^2+\bar{F}^2)^2 - (\partial_\mu u)^2(\partial_\nu u)^2 - (\partial_\mu \bar{u})^2(\partial_\nu \bar{u})^2 - 2(\partial_\mu u)^2(\partial_\nu \bar{u})^2 \qquad (13.32)$$

For the second contribution we define $A_1 = U$ and $A_2 = \bar{U}$, then the other part for the quartic Lagrangian is

$$\tilde{L}_{4b} = \sum_{ij}\int d^2\theta[D^\alpha A^i D_\alpha A^j][D^2 A^i D^2 A^j - \frac{1}{4}(D^\alpha D^\beta A^i D_\alpha D_\beta A^j)] \qquad (13.33)$$

and the bosonic part results in

$$\tilde{L}_{4b}|_{bos} = (F^2+\bar{F}^2)^2 - (\partial_\mu u)^2(\partial_\nu u)^2 - (\partial_\mu \bar{u})^2(\partial_\nu \bar{u})^2 - 2(\partial_\mu u \partial^\mu \bar{u})^2 \qquad (13.34)$$

finally

$$\tilde{L}_4|_{bos} = -\frac{1}{2}(\tilde{L}_{4a}|_{bos} - \tilde{L}_{4b}|_{bos}) = (\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u)^2(\partial_\nu \bar{u})^2. \qquad (13.35)$$

We remark for later use that in this specific linear combination, together with the unwanted terms depending on $\partial_\mu u$, also the auxiliary fields F and $\bar{F}$ have disappeared.

Including now the $h(u,\bar{u})$ factor, we get

$$L_{4a} = \int d^2\theta h(U,\bar{U})[D^\alpha U D_\alpha U + D^\alpha \bar{U} D_\alpha \bar{U}][D^2 U D^2 U + \quad (13.36)$$

$$+ D^2\bar{U}D^2\bar{U} - \frac{1}{4}(D^\alpha D^\beta U D_\alpha D_\beta U + D^\alpha D^\beta \bar{U} D_\alpha D_\beta \bar{U})], \quad (13.37)$$

$$L_{4b} = \sum_{ij} \int d^2\theta h(U,\bar{U})[D^\alpha A^i D_\alpha A^j][D^2 A^i D^2 A^j - \frac{1}{4}(D^\alpha D^\beta A^i D_\alpha D_\beta A^j)], \quad (13.38)$$

and the final result for the quartic term has the following form,

$$L_4|_{bos} = -h(u,\bar{u})[(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u)^2(\partial_\nu \bar{u})^2] \quad (13.39)$$

or

$$L_4|_{bos} = -\frac{1}{(1+u\bar{u})^4}[(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u)^2(\partial_\nu \bar{u})^2]. \quad (13.40)$$

As usual, in $N=1$ SUSY a potential term results from a (real) superfield $\mathcal{U}(U,\bar{U})$ called superpotential, which only depends on the basic superfields U and $\bar{U}$,

$$L_{\mathcal{U}} = \int d^2\theta \mathcal{U}(U,\bar{U}) \quad (13.41)$$

with the bosonic part

$$L_{\mathcal{U},bos} = \mathcal{U}_u F + \mathcal{U}_{\bar{u}} \bar{F}. \quad (13.42)$$

Taking into account (13.29) and (13.42), the equations of motion for the auxiliary fields are

$$g(u,\bar{u})\bar{F} + \mathcal{U}_u = 0, \quad g(u,\bar{u})F + \mathcal{U}_{\bar{u}} = 0 \quad (13.43)$$

or

$$\bar{F} = -\frac{\mathcal{U}_u}{g(u,\bar{u})}, \quad F = -\frac{\mathcal{U}_{\bar{u}}}{g(u,\bar{u})}. \quad (13.44)$$

Inserting these values in the total Lagrangian we obtain

$$L_{tot} = \frac{1}{(1+u\bar{u})^2}\partial_\mu u \partial^\mu \bar{u} - \frac{1}{(1+u\bar{u})^4}[(\partial_\mu u \partial^\mu \bar{u})^2 - \quad (13.45)$$

$$- (\partial_\mu u \partial^\mu u)(\partial_\nu \bar{u} \partial^\nu \bar{u})] - (1+u\bar{u})^2 \mathcal{U}_u \mathcal{U}_{\bar{u}} \quad (13.46)$$

and, therefore, precisely the lagrangian density (13.10) of the baby Skyrme model with the potential

$$\mathcal{V}(u,\bar{u}) = (1+u\bar{u})^2 \mathcal{U}_u \mathcal{U}_{\bar{u}}. \tag{13.47}$$

For potentials $\mathcal{V}(u\bar{u})$ with the residual U(1) symmetry we have to assume that also $\mathcal{U} = \mathcal{U}(U\bar{U})$.

What is interesting here is that we cannot eliminate the quadratic term. Setting $g(u,\bar{u}) = 0$ at the end leads to

$$\frac{\mathcal{U}_u \mathcal{U}_{\bar{u}}}{g(u,\bar{u})} \to \infty, \tag{13.48}$$

and starting without the quadratic term from the beginning has the consequence that the auxiliary fields only appear linearly in the lagrangian from the superpotential, $L(F,\bar{F}) \sim \mathcal{U}_u F + \mathcal{U}_{\bar{u}} \bar{F}$. They act, therefore, like Lagrange multipliers enforcing the "constraints" $\mathcal{U}_u = \mathcal{U}_{\bar{u}} = 0$. We conclude that, although the quartic term L_4 alone can be supersymmetrically extended by the methods of this section, this is not true for the BPS baby Skyrme model $L_4 + L_0$. We shall find in the next section, however, that we may find more general $N = 1$ extensions which are capable of producing the BPS Skyrme model in its bosonic sector. Later on, we will see that (reflecting its BPS nature) the BPS baby Skyrme model even allows for an $N = 2$ SUSY extension. In both cases, the potential term L_0 is *not* induced by a superpotential but, instead, by the target space metric g or by a Kähler potential related to g.

To summarize the results of this section, for the CP(1) version of the baby Skyrme model we found exactly the same $N = 1$ SUSY extension as for its O(3) version [64]. The two versions are, of course, classically equivalent. The SUSY extensions, however, require the introducion of fermions which must be treated as quantum objects to provide the correct SUSY algebra. The equivalence of the two SUSY extensions is, therefore, not completely obvious, but turns out to be true.

13.4.2 More general $N = 1$ extensions

The $N = 1$ SUSY extension of the previous section allows for certain generalizations, among which also the SUSY extension of the BPS baby Skyrme model can be found. Later (in Section 13.7) we shall even find that the BPS baby Skyrme model allows for an $N = 2$ extension. Concretely, let us define the following lagrangians which generalize the quartic lagrangian of the previous subsection,

$$\begin{aligned}\tilde{L}_\lambda &= \int d^2\theta \tilde{\mathcal{L}}_\lambda \equiv \int d^2\theta \{ D^\alpha \Phi D_\alpha \Phi^\dagger [D^2\Phi D^2 \Phi^\dagger - \\ &- \lambda D^\alpha D^\beta \Phi D_\alpha D_\beta \Phi^\dagger]\} \end{aligned} \quad (13.49)$$

$$\begin{aligned}\tilde{L}_\mu &= \int d^2\theta \tilde{\mathcal{L}}_\mu \equiv \int d^2\theta \{ D^\alpha \Phi D_\alpha \Phi [D^2\Phi^\dagger D^2 \Phi^\dagger - \\ &- \mu D^\alpha D^\beta \Phi^\dagger D_\alpha D_\beta \Phi^\dagger]\}. \end{aligned} \quad (13.50)$$

Here, λ and μ are real parameters. In components, and for the bosonic sector only, we get

$$\tilde{L}_\lambda = (F\bar{F})^2(2-4\lambda) + (F\bar{F})(\partial_\mu u \partial^\mu \bar{u})(2-8\lambda) - 2\lambda(\partial_\mu u \partial^\mu \bar{u})^2 \quad (13.51)$$

$$\begin{aligned}\tilde{L}_\mu &= (F\bar{F})^2(2-4\mu) + \bar{F}^2(\partial_\mu u \partial^\mu u)(2-4\mu) - 4\mu F^2 \partial_\mu \bar{u} \partial^\mu \bar{u} - \\ &- 4\mu(\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u}). \end{aligned} \quad (13.52)$$

It follows that

$$\begin{aligned}\mathfrak{Re}[\tilde{L}_\mu] &= 2(F\bar{F})^2(1-2\mu) + \bar{F}^2(\partial_\mu u \partial^\mu u)(1-4\mu) + \\ &+ F^2(\partial_\mu \bar{u} \partial^\mu \bar{u})(1-4\mu) - 4\mu(\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u}) \end{aligned} \quad (13.53)$$

and, specifically for $\mu = 1/4$,

$$\mathfrak{Re}[\tilde{L}_\mu]|_{\mu=\frac{1}{4}} = (F\bar{F})^2 - (\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u}) \quad (13.54)$$

A general linear combination of the two lagrangians is

$$\begin{aligned}\delta \mathfrak{Re}[\tilde{L}_\mu]|_{\mu=\frac{1}{4}} + \frac{\rho}{2}\tilde{L}_\lambda &= (F\bar{F})^2(\delta + \rho - 2\rho\lambda) + \\ &+ \rho(F\bar{F})(\partial_\mu u \partial^\mu \bar{u})(1-4\lambda) - 2\rho\lambda(\partial_\mu u \partial^\mu \bar{u})^2 - \\ &- \delta(\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u}) \end{aligned} \quad (13.55)$$

where δ and ρ are real coefficients. Two choices for these parameters are of special interest, namely

$$\tilde{L}_4^{(1)} \equiv \left(\delta \mathfrak{Re}[\tilde{L}_\mu] + \frac{\rho}{2}\tilde{L}_\lambda \right)_{\mu=\frac{1}{4}, \lambda=\frac{1}{4}, \rho=-2, \delta=1} = (\partial_\mu u \partial^\mu \bar{u})^2 - |\partial_\mu u \partial^\mu u|^2 \quad (13.56)$$

and

$$\tilde{L}_4^{(2)} \equiv \left(\delta\mathfrak{Re}[\tilde{L}_\mu] + \frac{\rho}{2}\tilde{L}_\lambda\right)_{\mu=\frac{1}{4},\lambda=0,\rho=2,\delta=-1} = 2(F\bar{F})(\partial_\mu u\partial^\mu\bar{u}) + (F\bar{F})^2 + |\partial_\mu u\partial^\mu u|^2. \quad (13.57)$$

In a next step, we introduce, again, the target space area density $h(u,\bar{u})$. This is done by multiplying the lagrangian densities in superspace by the corresponding superfield $h(U,U^\dagger)$ exactly like above, that is (where, again, we only consider the bosonic sector)

$$\tilde{L}_\lambda = \int d^2\theta\tilde{\mathcal{L}}_\lambda \quad \Rightarrow \quad L_\lambda = \int d^2\theta h(U,U^\dagger)\tilde{\mathcal{L}}_\lambda = h(u,\bar{u})\tilde{L}_\lambda \quad (13.58)$$

(and the same for L_μ). The reason for this is that each superderivative $D_\alpha\Phi$ is linear in θ in the bosonic sector, and both $\mathcal{L}_\lambda$ and $\mathcal{L}_\mu$ are quadratic in $D_\alpha\Phi$ (i.e., quadratic in θ in the bosonic sector), therefore all superfields multiplying them only contribute with their $\theta=0$ component. For the two quartic lagrangians $L_4^{(1)} = h(u,\bar{u})\tilde{L}_4^{(1)}$ and $L_4^{(2)} = h(u,\bar{u})\tilde{L}_4^{(2)}$ we get

$$L_4^{(1)} = h(u,\bar{u})\left((\partial_\mu u\partial^\mu\bar{u})^2 - |\partial_\mu u\partial^\mu u|^2\right) \quad (13.59)$$

and

$$L_4^{(2)} = h(u,\bar{u})\left(2(F\bar{F})(\partial_\mu u\partial^\mu\bar{u}) + (F\bar{F})^2 + |\partial_\mu u\partial^\mu u|^2\right). \quad (13.60)$$

The first expression (13.59) precisely coincides with the lagrangian (13.39), therefore this choice of parameters just reproduces the $N=1$ extension of the previous section. In order to understand the significance of $L_4^{(2)}$, it is useful to add it to the quadratic lagrangian (13.29) of the previous section to obtain

$$L = L_2 + L_4^{(2)} = g(u,\bar{u})(\partial^\mu u\partial_\mu\bar{u} + F\bar{F}) + h(u,\bar{u})\left(2(F\bar{F})(\partial_\mu u\partial^\mu\bar{u}) + (F\bar{F})^2 + |\partial_\mu u\partial^\mu u|^2\right). \quad (13.61)$$

Now we solve for the auxiliary fields $F,\bar{F}$. on the one hand, we find the trivial solution $F=\bar{F}=0$ which leads to the lagrangian

$$L = g(u,\bar{u})(\partial^\mu u\partial_\mu\bar{u}) + h(u,\bar{u}) + |\partial_\mu u\partial^\mu u|^2. \quad (13.62)$$

This lagrangian contains higher than second powers of time derivatives, and we shall not consider it further in this paper. on the other hand, we find the nontrivial solution

$$F\bar{F} = -\partial_\mu u \partial^\mu \bar{u} - \frac{g(u,\bar{u})}{2h(u,\bar{u})} \tag{13.63}$$

and, after substituting back into the lagrangian,

$$L = h(u,\bar{u})[(\partial^\mu u \partial_\mu u)(\partial^\mu \bar{u} \partial_\mu \bar{u}) - (\partial^\mu u \partial_\mu \bar{u})^2] - \frac{g(u,\bar{u})^2}{4h(u,\bar{u})}. \tag{13.64}$$

For the choice $h = (1 + u\bar{u})^{-4}$, this is precisely the lagrangian of the BPS baby Skyrme model, where the quadratic term has disappeared, provided that we identify the potential with

$$\mathcal{V}(u,\bar{u}) = \frac{g(u,\bar{u})^2}{4h(u,\bar{u})}. \tag{13.65}$$

As the quadratic term has disappeared, we are free to choose any function $g(u,\bar{u})$ we like and may, in this manner, produce the potentials we want. We emphasize that in this model the potential does not come from a superpotential but, instead, from the "target space metric" $g(u,\bar{u})$. Including a superpotential would result in a complicated fourth-order equation for the auxiliary field F, and the resulting lagragians would be completely different from the baby Skyrme model. We stop the discussion of the $N = 1$ SUSY extension of the BPS baby Skyrme model at this point, because later we will find that this model allows, in fact, for an $N = 2$ extension, such that also its BPS equations may be derived from $N = 2$ SUSY (see Sections 13.7, 13.8).

13.5 Gauged $N = 1$ CP(1) baby Skyrme model

In order to construct the gauged version of the $N = 1$ $\mathbb{C}P^1$ baby-Skyrme model we need and extra superfield containing the gauge field A_μ and a Majorana fermion λ_α (in this case the photon and the photino field). This superfield, which we call Γ_α, has the following decomposition,

$$\Gamma_\alpha = i\theta^\beta (\gamma^\mu)_{\beta\alpha} A_\mu - 2\theta^2 \lambda_\alpha. \tag{13.66}$$

In addition, we need the same complex superfield as above (constructed from two $N=1$ real superfields)

$$U(x) = u(x) + \theta^\alpha \chi_\alpha(x) - \theta^2 F(x) \tag{13.67}$$

where $u(x)$ and $F(x)$ are complex fields and $\chi_\alpha(x)$ is a Dirac fermion. Now it is easy to see that promoting the superderivative D_α to a covariant superderivative $\mathcal{D}_\alpha$,

$$\mathcal{D}^\alpha = D^\alpha + ie\Gamma^\alpha \tag{13.68}$$
$$\mathcal{D}_\alpha = D_\alpha - ie\Gamma_\alpha \tag{13.69}$$
$$\tag{13.70}$$

and adding the Maxwell term, the model is automatically gauged. In close analogy to the ungauged case, the quadratic term for the gauged model is

$$L_2^g = \int d^2\theta g(U^\dagger, U) \mathcal{D}^\alpha U^\dagger \mathcal{D}_\alpha U \tag{13.71}$$

and the bosonic (i.e., $\chi_\alpha = \lambda_\alpha = 0$) sector results in

$$L_2^g|_{bos} = g(\bar{u}, u)(D^\mu \bar{u} D_\mu u + F\bar{F}), \tag{13.72}$$

where

$$D_\mu u = \partial_\mu u + ieA_\mu u, \quad D_\mu \bar{u} = \partial_\mu \bar{u} - ieA_\mu \bar{u}. \tag{13.73}$$

Analogously, we find for the gauged quartic term

$$\begin{aligned} L_{4a}^g &= \int d^2\theta h(U^\dagger, U)[\mathcal{D}^\alpha U \mathcal{D}_\alpha U + \mathcal{D}^\alpha U^\dagger \mathcal{D}_\alpha U^\dagger][\mathcal{D}^2 U \mathcal{D}^2 U + \\ &+ \mathcal{D}^2 U^\dagger \mathcal{D}^2 U^\dagger - \frac{1}{4}(\mathcal{D}^\alpha \mathcal{D}^\beta U \mathcal{D}_\alpha \mathcal{D}_\beta U + \mathcal{D}^\alpha \mathcal{D}^\beta U^\dagger \mathcal{D}_\alpha \mathcal{D}_\beta U^\dagger)] \end{aligned} \tag{13.74}$$

$$\begin{aligned} L_{4a}^g|_{bos} &= h(\bar{u}, u)((F^2 + \bar{F}^2)^2 - (D_\mu u)^2 (D_\nu u)^2 \\ &- (D_\mu \bar{u})^2 (D_\nu \bar{u})^2 - 2(D_\mu u)^2 (D_\nu \bar{u})^2) \end{aligned} \tag{13.75}$$

and, after again defining $A_1 = U$ and $A_2 = U^\dagger$,

$$L_{4b}^g = \sum_{ij} \int d^2\theta h(U^\dagger, U)[\mathcal{D}^\alpha A^i \mathcal{D}_\alpha A^j][\mathcal{D}^2 V^i \mathcal{D}^2 V^j - \frac{1}{4}(\mathcal{D}^\alpha \mathcal{D}^\beta A^i \mathcal{D}_\alpha \mathcal{D}_\beta A^j)], \tag{13.76}$$

$$\begin{aligned} L^g_{4b}|_{bos} &= h(\bar{u},u)((F^2+\bar{F}^2)^2 - (D_\mu u)^2(D_\nu u)^2 - \\ &- (D_\mu \bar{u})^2(D_\nu \bar{u})^2 - 2(D_\mu u D^\mu \bar{u})^2) \end{aligned} \quad (13.77)$$

and finally

$$\begin{aligned} L^g_4|_{bos} &= -\frac{1}{2}(L^g_{4a}|_{bos} - L^g_{4b}|_{bos}) = \\ &- h(\bar{u},u)\left((D_\mu u D^\mu \bar{u})^2 - (D_\mu u)^2(D_\nu \bar{u})^2\right). \end{aligned} \quad (13.78)$$

In addition, we need the Maxwell term which is generated in terms of the spinor superfield only,

$$L_M = \frac{1}{8}\int d^2\theta \mathcal{D}^\beta \mathcal{D}^\alpha \Gamma_\beta \mathcal{D}^\gamma \mathcal{D}_\alpha \Gamma_\gamma. \quad (13.79)$$

Now we choose $g(\bar{u},u) = 1/(1+\bar{u}u)^2$, $h(\bar{u},u) = 1/(1+\bar{u}u)^4$. Putting all these terms together and eliminating the auxiliary fields we obtain in the bosonic sector

$$L^g_{tot} = \frac{1}{(1+u\bar{u})^2}D_\mu u D^\mu \bar{u} - \frac{1}{(1+u\bar{u})^4}[(D_\mu u D^\mu \bar{u})^2 - \quad (13.80)$$

$$- (D_\mu u D^\mu u)(D_\nu \bar{u} D^\nu \bar{u})] - (1+u\bar{u})^2 \mathcal{U}_u \mathcal{U}_{\bar{u}} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (13.81)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (13.82)$$

To summarize, we just find the gauged version of the baby Skyrme model, where partial derivatives are replaced by covariant derivatives, and a Maxwell term is included. This model is known to support soliton solutions [144]. We remark that, exactly as in the ungauged case, within this SUSY extension it is not possible to eliminate the (gauged) quadratic, i.e., nonlinear sigma model, term without eliminating, at the same time, the potential. That is to say, we cannot construct the gauged BPS baby Skyrme model [145] within this SUSY extension. More general $N = 1$ extensions which do allow to find the $N = 1$ extension of the gauged BPS baby Skyrme model certainly will exist, like in the ungauged case (see Section 13.4.2). Here we shall consider, instead, directly the $N = 2$ SUSY extension of the gauged BPS baby Skyrme model (Section 13.9), which turns out to exist, exactly as for the ungauged case.

13.6 N=2 Supersymmetry in 2+1 dimensions

In this section we shall introduce our conventions for $N = 2$ supersymmetry in $2 + 1$ dimensions. We have four independent Grassmann variables, θ^α and $\bar{\theta}^{\dot{\alpha}}$, and the corresponding superderivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{13.83}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{13.84}$$

With these definitions it is easy to check the following anticommutation relations,

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{13.85}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \tag{13.86}$$

The supersymmetric generators Q and $\bar{Q}$ have the same structure as the superderivatives, up to a relative sign,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{13.87}$$

$$\bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \tag{13.88}$$

therefore the anticommutation relations are

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{13.89}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \tag{13.90}$$

and the mixed anticommutators all vanish,

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{13.91}$$

Now we introduce the superfields. To construct our model, we will need only chiral and anti-chiral superfields satisfying the following constraints (for chiral and anti-chiral, respectively)

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{13.92}$$

$$D_\alpha\Phi^\dagger = 0. \tag{13.93}$$

It is easy to solve the above constraints by introducing the chiral variables

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{13.94}$$

(we assume dotted indices for variable with bar, and undotted without bar). These new variables satisfy the chiral constraint

$$\bar{D}_{\dot{\alpha}}(x^\mu + i\theta\sigma^\mu\bar{\theta}) = 0, \tag{13.95}$$

therefore, by building superfields with this variable and expanding, the chiral constraint is automatically implemented. Concretely, for the chiral superfield

$$\begin{aligned} \Phi &= u(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu u(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box u(x) + \sqrt{2}\theta\psi(x) - \\ &- \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x) \end{aligned} \tag{13.96}$$

and analogously for the anti-chiral superfield

$$\begin{aligned} \Phi^\dagger &= \bar{u}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{u}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{u}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \\ &+ \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) + \bar{\theta}\bar{\theta}\bar{F}(x). \end{aligned} \tag{13.97}$$

13.7 The baby Skyrme model and N=2 supersymmetry

In a first step, let us try to find an $N = 2$ extension which produces the two kinetic terms L_2 and L_4,

$$L_2 + L_4 = \frac{\partial_\mu u\partial^\mu\bar{u}}{(1+|u|^2)^2} + \frac{(\partial_\mu u)^2(\partial_\nu\bar{u})^2 - (\partial_\mu u\partial^\mu\bar{u})^2}{(1+|u|^2)^4}. \tag{13.98}$$

In order to generate the quadratic term, we need only a D-term involving a Kähler potential (this is just the $N = 2$ CP(1) σ-model), with the lagrangian density

$$L_2 = \frac{1}{16}\int d^2\theta d^2\bar{\theta}\,\ln(1+\Phi\Phi^\dagger) \tag{13.99}$$

where Φ is a N=1 chiral superfield in (2+1) dimensions and $\Phi^\dagger$ the respective antichiral superfield. Taking into account that $K(\Phi,\Phi^\dagger) = \ln(1+\Phi\Phi^\dagger)$ is a

Kähler potential with Kähler metric

$$g(u,\bar{u}) = g_{\bar{u}u} = \partial_u \partial_{\bar{u}} K(u,\bar{u}) = \frac{1}{(1+\bar{u}u)^2} \tag{13.100}$$

the only non-zero Christoffel symbols are

$$\Gamma^u_u = g^{u\bar{u}} \partial_u g(u,\bar{u}) \tag{13.101}$$
$$\Gamma^{\bar{u}}_{\bar{u}\bar{u}} = g^{\bar{u}u} \partial_{\bar{u}} g_{\bar{u}u} \tag{13.102}$$

or, explicitly,

$$\Gamma^u_{uu} = \frac{-2\bar{u}}{1+u\bar{u}} \tag{13.103}$$
$$\Gamma^{\bar{u}}_{\bar{u}\bar{u}} = \frac{-2u}{1+u\bar{u}}. \tag{13.104}$$

The lagrangian can be written in components as

$$\begin{aligned} L_2 &= g(u,\bar{u})[\partial^\mu u \partial_\mu \bar{u} - \frac{i}{2}\psi\sigma^\mu \mathcal{D}^\mu \bar{\psi} + \frac{i}{2}\mathcal{D}^\mu \psi \sigma_\mu \bar{\psi} + F\bar{F}] \\ &+ \frac{1}{4}\mathcal{R}_{u\bar{u}u\bar{u}}(\psi\psi)(\bar{\psi}\bar{\psi}) \end{aligned} \tag{13.105}$$

where

$$g(u,\bar{u}) = \frac{1}{(1+u\bar{u})^2} \tag{13.106}$$
$$\mathcal{R}_{u\bar{u}u\bar{u}} = -\frac{2}{(1+u\bar{u})^4} \tag{13.107}$$
$$\mathcal{D}_\mu \psi_\alpha \equiv \left(\partial_\mu - \frac{2\bar{u}}{1+u\bar{u}}\partial_\mu u\right)\psi_\alpha \tag{13.108}$$
$$\mathcal{D}_\mu \psi^{\dagger\dot{\alpha}} \equiv \left(\partial_\mu - \frac{2u}{1+u\bar{u}}\partial_\mu \bar{u}\right)\psi^{\dagger\dot{\alpha}}. \tag{13.109}$$

In a next step, we have to generate the $N=2$ supersymmetric version of the quartic terms in (13.98). We might choose a supersymmetric lagrangian starting from a superfield quartic in superderivatives and depending on both chiral and anti-chiral superfields. Let $\tilde{\mathcal{L}}_4$ be this quartic superfield,

$$\tilde{\mathcal{L}}_4 = \frac{1}{16} D^\alpha \Phi D_\alpha \Phi \bar{D}^{\dot{\beta}} \Phi^\dagger \bar{D}_{\dot{\beta}} \Phi^\dagger \tag{13.110}$$

then after integration in the Grassmann variables we get for the bosonic sector

$$\tilde{L}_{4,bos} = (\partial^\mu u)^2 (\partial^\nu \bar{u})^2 + 2\bar{F}F\partial^\mu u \cdot \partial_\mu \bar{u} + (\bar{F}F)^2. \qquad (13.111)$$

Right now, this quartic lagragian is still quite different from the quartic part of (13.98). The first observation in that we can multiply this lagrangian by a prefactor depending on the superfields. Let this prefactor be $h(\Phi, \Phi^\dagger)$, then the new superfield has the following form,

$$\mathcal{L}_4 = \frac{1}{16} h(\Phi, \Phi^\dagger) D^\alpha \Phi D_\alpha \Phi \bar{D}^{\dot{\beta}} \Phi^\dagger \bar{D}_{\dot{\beta}} \Phi^\dagger \qquad (13.112)$$

and after the $\theta-$integration the bosonic sector of the corresponding lagrangian is

$$L_{4,bos} = h(u,\bar{u})[(\partial^\mu u)^2 (\partial^\nu \bar{u})^2 + 2\bar{F}F\partial^\mu u \cdot \partial_\mu \bar{u} + (\bar{F}F)^2]. \qquad (13.113)$$

The reason for this result is that each superderivative $D^\alpha \Phi$ is at least linear in θ or $\bar{\theta}$ in the bosonic sector, and the above superfield contains four powers of $D^\alpha\Phi$'s. Therefore, only the θ-independent part of the prefactor contributes to the bosonic sector.

Adding the bosonic sector of the quadratic lagrangian to the above quartic bosonic lagrangian we get

$$\begin{aligned} L_{T,bos} &= g(u,\bar{u})[\partial^\mu u \partial_\mu \bar{u} + F\bar{F}] + \\ &+ h(u,\bar{u})[(\partial^\mu u)^2 (\partial^\nu \bar{u})^2 + 2\bar{F}F\partial^\mu u \cdot \partial_\mu \bar{u} + (\bar{F}F)^2] \end{aligned} \qquad (13.114)$$

Finally solving the algebraic equation of motion for $F\bar{F}$:

$$F\bar{F} = -\partial_\mu u \partial^\mu \bar{u} - \frac{g(u,\bar{u})}{2h(u,\bar{u})} \qquad (13.115)$$

we get

$$L_{T,bos} = h(u,\bar{u})[(\partial^\mu u \partial_\mu u)(\partial^\mu \bar{u} \partial_\mu \bar{u}) - (\partial^\mu u \partial_\mu \bar{u})^2] - \frac{g(u,\bar{u})^2}{4h(u,\bar{u})}. \qquad (13.116)$$

For our special case with $g(u,\bar{u}) = 1/(1+u\bar{u})^2$, $h(u,\bar{u}) = 1/(1+u\bar{u})^4$, this turns into

$$L_{T,bos} = \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u \partial_\mu u)(\partial^\mu \bar{u} \partial_\mu \bar{u}) - (\partial^\mu u \partial_\mu \bar{u})^2] - \frac{1}{4}, \qquad (13.117)$$

so we apparently find a constant "potential" $\mathcal{V} = (1/4)$. The important observation here is that after the substitution of the auxiliary field F by its on-shell value, the quadratic, nonlinear sigma model term has completely disappeared from the above bosonic lagrangian, for arbitrary choices of g and h. There is, therefore, no more reason to restrict the Kähler metric g and the corresponding Kähler potential to their CP(1) form. Then, choosing $h(u,\bar{u}) = 1/(1+u\bar{u})^4$ (which we maintain, because we want the standard quartic term of the baby Skyrme model), and a general Kähler manifold (different from CP(1)) with metric $g(u,\bar{u})$ we have in the bosonic sector

$$\begin{aligned} L_{T,bos} &= \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u\partial_\mu u)(\partial^\mu\bar{u}\partial_\mu\bar{u}) - (\partial^\mu u\partial_\mu\bar{u})^2] - \\ &- \frac{g(u,\bar{u})^2}{4}(1+u\bar{u})^4. \end{aligned} \tag{13.118}$$

This is precisely the lagrangian of the BPS baby Skyrme model with the potential term

$$\mathcal{V}(u,\bar{u}) = \frac{g(u,\bar{u})^2}{4}(1+u\bar{u})^4 \tag{13.119}$$

where g is a Kähler metric. The potential in the $N=2$ extension is, therefore, induced by the Kähler potential of the nonlinear sigma-model type lagrangian (13.99), and *not* by a superpotential term. The addition of a superpotential is, in fact, forbidden in the sense that it would transform the algebraic field equation of the auxiliary field F into a fourth order equation with complicated roots of u and $\partial_\mu u$ as solutions. The resulting lagrangian would then be completely different from the baby Skyrme lagrangian.

To summarize, we found the $N=2$ supersymmetric extension of the restricted baby Skyrme model. Let us give some concrete examples. For the following family of potentials $\mathcal{V}(u,\bar{u})$ depending on the parameter s,

$$\mathcal{V}(u,\bar{u}) = \left(\frac{u\bar{u}}{1+u\bar{u}}\right)^s \tag{13.120}$$

the corresponding Kähler metrics generating these potentials are

$$g(u,\bar{u}) = 2\frac{(u\bar{u})^{\frac{s}{2}}}{(1+u\bar{u})^{\frac{s+4}{2}}}. \tag{13.121}$$

Integrating this metric we obtain the Kähler potential, which at the superfield level is

$$K(\Phi, \Phi^\dagger) = \frac{8(\Phi\Phi^\dagger)^{\frac{s+2}{2}}}{(2+s)^2} {}_2F_1[\frac{2+s}{2}, \frac{2+s}{2}, \frac{4+s}{2}, -\Phi\Phi^\dagger], \tag{13.122}$$

for example,

$$s = 1, \qquad K(\Phi, \Phi^\dagger) = \operatorname{arcsinh}(\sqrt{\Phi\Phi^\dagger}) - \sqrt{\frac{\Phi\Phi^\dagger}{1+\Phi\Phi^\dagger}} \tag{13.123}$$

$$s = 2, \qquad K(\Phi, \Phi^\dagger) = \frac{1}{1+\Phi\Phi^\dagger} + \ln(1+\Phi\Phi^\dagger). \tag{13.124}$$

Reintroducing the coupling constant μ of the potential terms, we get the bosonic lagrangians

$$\begin{aligned} s = 1, \qquad L_T^1 &= \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u \partial_\mu u)(\partial^\mu \bar{u} \partial_\mu \bar{u}) - (\partial^\mu u \partial_\mu \bar{u})^2] \qquad (13.125) \\ &- 2\mu^2 \left(\frac{u\bar{u}}{1+u\bar{u}}\right) \\ s = 2, \qquad L_T^2 &= \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u \partial_\mu u)(\partial^\mu \bar{u} \partial_\mu \bar{u}) - (\partial^\mu u \partial_\mu \bar{u})^2] \qquad (13.126) \\ &- 2\mu^2 \left(\frac{u\bar{u}}{1+u\bar{u}}\right)^2 \\ \ldots \end{aligned}$$

We remark that the parameter μ is introduced in the D-term generated by the Kähler potential, hence it is present in the fermionic sector of this term.

13.8 Bogomol'nyi equation

The BPS baby Skyrme model is well-known to support BPS solitons, that is, solitons which saturate the Bogomol'nyi bound and obey the corresponding first order BPS equation. In addition, we just found that the model admits an $N = 2$ SUSY extension, so the natural question arises whether these BPS solitons may be recovered as one-half BPS states of the supersymmetrically extended theory. SUSY BPS states are characterized by the fact that they are annihilated by some of the SUSY charges or, in the case of classical BPS solutions, that some SUSY charges (SUSY transformations) are zero when evaluated for the BPS states. We, therefore, need the $N = 2$ SUSY transformations in a first step. More concretely, a SUSY BPS solution has the fermionic components of the basic superfield Φ equal to zero, and only the scalar field u and the auxiliary field F are nontrivial. Further, the SUSY transformation of both u and F is proportional to a fermion and therefore trivially zero for a BPS state. The only nontrivial conditions, thus, come from the SUSY transformations of the spinors. The $N = 2$ transformations of the spinors have the following form

$$\delta\psi_\beta = -i\partial_{\beta\dot\alpha}u\bar\epsilon^{\dot\alpha} + F\epsilon_\beta \qquad (13.127)$$

$$\delta\bar\psi^{\dot\beta} = i\partial^{\dot\beta\alpha}\bar u\epsilon_\alpha + \bar F\bar\epsilon^{\dot\beta} \qquad (13.128)$$

where ϵ_α and $\bar\epsilon^{\dot\alpha}$ are the Grassmann-valued SUSY transformation parameters. For static (time-independent) fields we find in components

$$\delta\psi_1|_{static} = \partial_1 u\bar\epsilon^{\dot 1} - \partial_2 u\bar\epsilon^{\dot 2} + F\epsilon_1 \qquad (13.129)$$

$$\delta\psi_2|_{static} = -\partial_1 u\bar\epsilon^{\dot 2} - \partial_2 u\bar\epsilon^{\dot 1} + F\epsilon_2 \qquad (13.130)$$

$$\delta\bar\psi^{\dot 1}|_{static} = \partial_1\bar u\epsilon_1 - \partial_2\bar u\epsilon_2 + \bar F\bar\epsilon^{\dot 1} \qquad (13.131)$$

$$\delta\bar\psi^{\dot 2}|_{static} = -\partial_1\bar u\epsilon_2 - \partial_1\bar u\epsilon_1 + \bar F\bar\epsilon^{\dot 2} \qquad (13.132)$$

or

$$\delta\vec\psi = M\vec\epsilon \qquad (13.133)$$

where $\vec\psi = (\psi_1, \psi_2, \bar\psi^{\dot 1}, \bar\psi^{\dot 2})^t$, $\vec\epsilon = (\epsilon_1, \epsilon_2, \bar\epsilon^{\dot 1}, \bar\epsilon^{\dot 2})^t$, and M is the matrix

$$M = \begin{pmatrix} \partial_1 u & -\partial_2 u & F & 0 \\ -\partial_2 u & -\partial_1 u & 0 & F \\ \bar F & 0 & \partial_1\bar u & -\partial_1\bar u \\ 0 & \bar F & -\partial_2\bar u & -\partial_1\bar u \end{pmatrix}. \qquad (13.134)$$

The condition that some (linear combinations of the) SUSY transformations $\delta\psi$ are zero is equivalent to the condition $\det M = 0$, therefore we now need the eigenvalues of M. These eigenvalue may be calculated to be $(\lambda_+, -\lambda_+, \lambda_-, -\lambda_-)$, where

$$\lambda_\pm^2 = -\partial_i u\partial^i\bar u \pm \sqrt{(\partial_i u\partial^i\bar u)^2 - (\partial_i u)^2(\partial_j\bar u)^2} - F\bar F, \qquad (13.135)$$

and the determinant is

$$\det M = (\partial_i u)^2(\partial_j \bar{u})^2 + 2\partial_i u \partial^i \bar{u} F\bar{F} + (F\bar{F})^2. \tag{13.136}$$

The condition det $M = 0$ therefore leads either to $\lambda_+^2 = 0$, that is,

$$F\bar{F} = -\partial_i u \partial^i \bar{u} + \sqrt{(\partial_i u \partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2} \tag{13.137}$$

or to $\lambda_-^2 = 0$, that is,

$$F\bar{F} = -\partial_i u \partial^i \bar{u} - \sqrt{(\partial_i u \partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2}, \tag{13.138}$$

corresponding to soliton and antisoliton, respectively. As the eigenvalues come in pairs, each condition has multiplicity two, and possible BPS solutions are, therefore, always one-half BPS states (they leave invariant one-half of the supersymmetries). We remark that the discussion up to now has been completely general, and the above equations are therefore the *completely general* one-half BPS equations for any $N = 2$ supersymmetric field theory constructed from a chiral superfield. Specific models are characterized by the specific field equations for the auxiliary field F.

Concretely, for the $N = 2$ BPS baby Skyrme model, the equation of motion for $F\bar{F}$ in the static regime is

$$F\bar{F} = -\partial_i u \partial^i \bar{u} - \frac{g(u,\bar{u})}{2h(u,\bar{u})}, \tag{13.139}$$

and we obtain the BPS equations

$$\mp\sqrt{(\partial_i u \partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2} = \frac{g(u,\bar{u})}{2h(u,\bar{u})}. \tag{13.140}$$

In order to demonstrate that this is, indeed, precisely the BPS equation of the BPS baby Skyrme model, we use

$$(\partial_i u \partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2 = (i\epsilon_{jk} u_j \bar{u}_k)^2 \tag{13.141}$$

and the expression for the topological charge density $q \equiv K^0/2$ where

$$q(x) = \frac{2i\epsilon_{jk} u_j \bar{u}_k}{(1+u\bar{u})^2}, \qquad \int d^2x q(x) = 4\pi k. \tag{13.142}$$

The normalization of q is useful because then q is just the pullback (under the map defined by u) of the area two-form on the target space unit sphere (the area of the unit sphere is 4π). Using this expression, and $h = (1+u\bar{u})^{-4}$, we get for the BPS equation

$$q(x) = \pm g(u,\bar{u})(1+u\bar{u})^2 = \pm 2\sqrt{\mathcal{V}(u,\bar{u})} \tag{13.143}$$

where

$$\mathcal{V}(u,\bar{u}) = \frac{1}{4} g(u,\bar{u})^2 (1+u\bar{u})^4. \tag{13.144}$$

This is precisely the BPS equation of the BPS baby Skyrme model, see e.g. [123], [124] (in those papers the r.h.s. of Eq. (13.143) reads $\pm\sqrt{2\mathcal{V}}$, because there the potential shows up in the lagrangian like $-(\mu^2/2)\mathcal{V}$, whereas it appears without the factor 1/2 in the present paper).

Remark: it might appear that the on-shell value (13.139) for $F\bar{F}$ is negative, which would be contradictory. Here we want to show that, at least for field configurations which are sufficiently close to the BPS bound, this is not the case. Indeed, from (13.139) we easily derive

$$\frac{2F\bar{F}}{(1+u\bar{u})^2} = \frac{2\nabla u \cdot \nabla \bar{u}}{(1+u\bar{u})^2} - 2\sqrt{\mathcal{V}} \tag{13.145}$$

and, using the BPS equation (13.143),

$$\frac{2F\bar{F}}{(1+u\bar{u})^2} = \frac{2\nabla u \cdot \nabla \bar{u}}{(1+u\bar{u})^2} \pm q(x) \geq 0. \tag{13.146}$$

13.9 N=2 SUSY gauged Skyrme model in 2+1 dimensions

Recently it has been found that the gauged BPS baby Skyrme model still has a BPS bound and supports soliton solutions saturating this bound [145], so it is natural to attempt an $N = 2$ SUSY extension for this case, as well. For this purpose, we need the formalism for $N = 2$ supersymmetric gauge fields, concretely for abelian gauge fields (Maxwell electrodynamics). For the gauged version of the Kähler potential term (i.e., the quadratic kinetic term), we use the well-known fact that the combination of superfields $\Phi^\dagger e^V \Phi$

is gauge invariant, where V is the real vector multiplet with components (in the Wess-Zumino gauge)

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda - \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D + \theta\gamma^5\bar{\theta}\sigma. \tag{13.147}$$

Here D and σ are real fields. Again we need the chiral and antichiral superfields (13.97), (13.98), which are $N = 2$ supersymmetric by construction. The gauged quadratic term may now be constructed starting from

$$L_2^g = \int d^2\theta d^2\bar{\theta} K(\Phi^\dagger e^V \Phi) \tag{13.148}$$

where K is a generalized Kähler potential. Integrating we obtain the lagrangian

$$\begin{aligned} L_2^g &= g_{u\bar{u}}\left(D^\mu u D_\mu \bar{u} - \frac{i}{2}\psi\sigma^\mu\mathcal{D}^\mu\bar{\psi} + \frac{i}{2}\mathcal{D}^\mu\psi\sigma_\mu\bar{\psi} + F\bar{F}\right) \\ &+ \frac{1}{4}\mathcal{R}_{u\bar{u}u\bar{u}}(\psi\psi)(\bar{\psi}\bar{\psi}) + \sigma^2\bar{u}u g_{u\bar{u}} + \left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial \bar{u}}\right)D - \\ &- i g_{u\bar{u}}(u\lambda\psi + \bar{u}\bar{\lambda}\bar{\psi}). \end{aligned} \tag{13.149}$$

Here, D^μ is the standard covariant derivative, and $\mathcal{D}^\mu$ is the covariant derivative on spinors,

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + (\partial_\mu u)\Gamma^u_{uu}\psi + ieA_\mu\psi. \tag{13.150}$$

Further, $g_{u\bar{u}}$ is the Kähler metric.

In a next step, we have to covariantize the quartic term. This is easily done by introducing the spinor gauge superfields defined by

$$\Gamma_\alpha = D_\alpha V \tag{13.151}$$

$$\bar{\Gamma}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}} V \tag{13.152}$$

and changing the superderivatives to the covariant superderivatives, $\tilde{D}_\alpha$ and $\tilde{\bar{D}}^{\dot{\alpha}}$,

$$\tilde{D}_\alpha = D_\alpha + \Gamma_\alpha \tag{13.153}$$

$$\tilde{\bar{D}}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}} + \tilde{\bar{\Gamma}}^{\dot{\alpha}}, \tag{13.154}$$

hence L_4^g is the $\theta^2\bar{\theta}^2$ component of the superfield

$$\mathcal{L}_4^g = \frac{1}{16}h(\Phi,\Phi^\dagger)\tilde{D}^\alpha\Phi\tilde{D}_\alpha\Phi\bar{\tilde{D}}^{\dot\beta}\Phi^\dagger\bar{\tilde{D}}_{\dot\beta}\Phi^\dagger. \tag{13.155}$$

The bosonic part of this lagrangian reads in components

$$L_{4,bos}^g = h(u,\bar{u})[(D^\mu u)^2(D^\nu\bar{u})^2 + 2\bar{F}FD^\mu u\cdot D_\mu\bar{u} + F^{\dagger 2}F^2] + o(\sigma^2). \tag{13.156}$$

Finally, we need the $N=2$ extension of the Maxwell lagrangian, which is constructed from the superfields

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \tag{13.157}$$

$$\bar{W}_{\dot\alpha} = -\frac{1}{2}DD\bar{D}_{\dot\alpha}V. \tag{13.158}$$

The corresponding Maxwell lagrangian then is

$$L_M = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}) \tag{13.159}$$

or

$$L_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - i\lambda\gamma^\mu\partial_\mu\bar{\lambda}. \tag{13.160}$$

The complete lagrangian in the bosonic sector, therefore, reads

$$\begin{aligned} L_b^g &= g_{u\bar{u}}\left(D^\mu u D_\mu\bar{u} + F\bar{F} + (u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial\bar{u}})D\right) + \\ &+ h(u,\bar{u})\left((D^\mu u)^2(D^\nu\bar{u})^2 + 2\bar{F}FD^\mu uD_\mu\bar{u} + F^{\dagger 2}F^2\right) - \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 + o(\sigma^2). \end{aligned} \tag{13.161}$$

The real scalar field σ appears at least quadratically, therefore, the trivial vacuum configuration $\sigma = 0$ always is a solution. We eliminate σ using this trivial solution. Further, the (algebraic) field equations for the auxiliary fields F and D are solved by

$$F\bar{F} = -D^\mu uD_\mu\bar{u} - \frac{g(u,\bar{u})}{2h(u,\bar{u})} \tag{13.162}$$

$$D = -\left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial\bar{u}}\right) \tag{13.163}$$

and, using them (and $\sigma = 0$), the complete bosonic lagrangian finally reads

$$\begin{aligned} L_b^g &= h(u,\bar{u})\left((D^\mu u)^2(D^\nu \bar{u})^2 - (D^\mu u D_\mu \bar{u})^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{g(u,\bar{u})^2}{4h(u,\bar{u})} - \frac{1}{2}\left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial \bar{u}}\right)^2 . \end{aligned} \tag{13.164}$$

We, therefore, found a bosonic lagrangian where the quadratic, sigma-model type contribution has disappeared, again, the quartic Skyrme term is covariantized, a Maxwell term has been created, and, finally, a potential has been produced by the two auxiliary fields F and D, which explicitly reads

$$\mathcal{V}(u,\bar{u}) = \frac{g(u,\bar{u})^2}{4h(u,\bar{u})} + \frac{1}{2}\left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial \bar{u}}\right)^2 \tag{13.165}$$

or

$$\mathcal{V}(u,\bar{u}) = \frac{1}{4h(u,\bar{u})}\left(\frac{\partial^2 K}{\partial u \partial \bar{u}}\right)^2 + \frac{1}{2}\left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial \bar{u}}\right)^2 . \tag{13.166}$$

For later use we now assume that $h(u,\bar{u}) = h(u\bar{u})$ and $K(u,\bar{u}) = K(u\bar{u})$, and define $K' \equiv \partial_{u\bar{u}} K$, then

$$\mathcal{V}(u\bar{u}) = \frac{1}{4h(u\bar{u})}\left(K' + u\bar{u}K''\right)^2 + 2\left(u\bar{u}K'\right)^2 \tag{13.167}$$

or

$$\mathcal{V}(u\bar{u}) = \frac{1}{4h(u\bar{u})}\mathcal{W}'^2 + 2\mathcal{W}^2 , \quad \mathcal{W} \equiv u\bar{u}K'. \tag{13.168}$$

13.9.1 Bogomol'nyi equations

In a next step, we want to recover the BPS equations of the gauged BPS baby Skyrme model as one-half BPS states of the $N = 2$ supersymmetrically extended theory, as in the ungauged model. For this purpose, we need the supersymmetric transformations of the multiplet (for fermions)

$$\delta\lambda_\alpha = -\eta_\alpha D - \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}(\gamma_\lambda)_\alpha^\beta \eta_\beta - i(\gamma^\mu)_\alpha^\beta \partial_\mu \sigma \eta_\beta \tag{13.169}$$

$$\delta\bar{\lambda}^{\dot{\alpha}} = -\bar{\eta}^{\dot{\alpha}} D - \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}(\gamma_\lambda)^{\dot{\alpha}}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}} + i(\gamma^\mu)^{\dot{\alpha}}_{\dot{\beta}} \partial_\mu \sigma \bar{\eta}^{\dot{\beta}} \tag{13.170}$$

$$\delta\psi_\beta = -iD_{\beta\dot{\alpha}} u \bar{\epsilon}^{\dot{\alpha}} + F\epsilon_\beta + i\eta_\alpha \sigma u \tag{13.171}$$

$$\delta\bar{\psi}^{\dot{\beta}} = iD^{\dot{\beta}\alpha}\bar{u}\epsilon_\alpha + \bar{F}\bar{\epsilon}^{\dot{\beta}} - i\bar{\eta}_{\dot{\alpha}}\sigma\bar{u}. \tag{13.172}$$

We, again, restrict to the trivial solution $\sigma = 0$ for the σ field to obtain

$$\delta\lambda_\alpha = -\eta_\alpha D - \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}(\gamma_\lambda)_\alpha^\beta\eta_\beta \quad (13.173)$$

$$\delta\bar{\lambda}^{\dot{\alpha}} = -\bar{\eta}^{\dot{\alpha}}D - \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}(\gamma_\lambda)^{\dot{\alpha}}_{\dot{\beta}}\bar{\eta}^{\dot{\beta}} \quad (13.174)$$

$$\delta\psi_\beta = -iD_{\beta\dot{\alpha}}u\bar{\epsilon}^{\dot{\alpha}} + F\epsilon_\beta \quad (13.175)$$

$$\delta\bar{\psi}^{\dot{\beta}} = iD^{\dot{\beta}\alpha}\bar{u}\epsilon_\alpha + \bar{F}\bar{\epsilon}^{\dot{\beta}}. \quad (13.176)$$

Now we are ready to repeat the strategy of the ungauged model. That is to say, we have to calculate the matrices of the susy transformations of both spinors, take the determinants (or their eigenvalues) and extract the Bogomol'nyi equations. In the last step we then have to take into account the on-shell values of the auxiliary fields. The matrices of the SUSY transformations for static fields are

$$M_\psi|_s = \begin{pmatrix} D_1u & -D_2u & F & 0 \\ -D_2u & -D_1u & 0 & F \\ \bar{F} & 0 & D_1\bar{u} & -D_2\bar{u} \\ 0 & \bar{F} & -D_2\bar{u} & -D_1\bar{u} \end{pmatrix}$$

and (where we also assume $A_0 = 0$)

$$M_\lambda|_s = \begin{pmatrix} -D & \frac{i}{2}\epsilon^{ij}F_{ij} & 0 & 0 \\ -\frac{i}{2}\epsilon^{ij}F_{ij} & -D & 0 & 0 \\ 0 & 0 & -D & \frac{i}{2}\epsilon^{ij}F_{ij} \\ 0 & 0 & -\frac{i}{2}\epsilon^{ij}F_{ij} & -D \end{pmatrix},$$

and from $\det(M_\psi|_s) = 0$ and $\det(M_\lambda|_s) = 0$ we obtain the general BPS equations

$$F\bar{F} = -D_iuD_i\bar{u} \pm \sqrt{(D_iuD_i\bar{u})^2 - (D_iu)^2(D_j\bar{u})^2} \quad (13.177)$$

$$D = \pm\epsilon^{ij}F_{ij}. \quad (13.178)$$

We emphasize that, again, these are the completely general BPS equations for a general $N = 2$ chiral superfield coupled to an $N = 2$ extended abelian gauge field. Specific models result from specific solutions for the auxiliary fields F and D.

Concretely, for the gauged BPS baby Skyrme model we get

$$\frac{g}{2h} = \pm\sqrt{(D_i u D_i \bar{u})^2 - (D_i u)^2 (D_j \bar{u})^2} \quad (13.179)$$

$$\left(u\frac{\partial K}{\partial u} + \bar{u}\frac{\partial K}{\partial \bar{u}}\right) = \pm\epsilon^{ij} F_{ij}. \quad (13.180)$$

For a comparison to known results it is useful to simplify the square root in the first equation,

$$(D_i u D_i \bar{u})^2 - (D_i u)^2 (D_j \bar{u})^2 = (i\epsilon_{jk} D_j u D_k \bar{u})^2 \quad (13.181)$$

and

$$i\epsilon_{jk} D_j u D_k \bar{u} = i\epsilon_{jk} u_j \bar{u}_k + e\epsilon_{jk} A_k \partial_j (u\bar{u}), \quad (13.182)$$

then we get

$$\frac{\mathcal{W}'}{2h} = \pm\left(i\epsilon_{jk} u_j \bar{u}_k + e\epsilon_{jk} A_k \partial_j (u\bar{u})\right) \quad (13.183)$$

$$2\mathcal{W} = \pm\epsilon^{ij} F_{ij} \quad (13.184)$$

where we also assumed $K = K(u\bar{u})$, as above. Introducing now the topological charge density q and its "covariant" version Q,

$$Q = \frac{i\epsilon_{jk} D_j u D_k \bar{u}}{(1+u\bar{u})^2} = q + \frac{e}{(1+u\bar{u})^2}\epsilon_{jk} A_k \partial_j (u\bar{u}) \quad (13.185)$$

and using the explicit expression $h = (1+u\bar{u})^{-4}$, we finally get the BPS equations

$$\frac{(1+u\bar{u})^2}{2}\mathcal{W}' = \pm Q \quad (13.186)$$

$$\mathcal{W} = \pm B \quad (13.187)$$

where B is the magnetic field, $B = \epsilon_{ij}\partial_i A_j = F_{12}$. For a direct comparison with the results of [145] we should take into account that in that paper the potential $\mathcal{V}$ and the "superpotential" $\mathcal{W}$ were treated as functions of n_3 instead of $u\bar{u}$, where

$$n_3 = \frac{1-u\bar{u}}{1+u\bar{u}} \quad \Rightarrow \quad \partial_{u\bar{u}} = -\frac{2}{(1+u\bar{u})^2}\partial_{n_3} \quad (13.188)$$

which leads to the BPS equations

$$\mathcal{W}_{n_3} = \mp Q \qquad (13.189)$$

$$\mathcal{W} = \pm B. \qquad (13.190)$$

These are precisely the BPS equations of Ref. [145], after the corresponding coupling constants have been reintroduced. Finally, for the relation between $\mathcal{W}$ and $\mathcal{V}$ we get

$$\mathcal{W}_{n_3}^2 + 2\mathcal{W}^2 = \mathcal{V} \qquad (13.191)$$

which again, coincides with the relation (the "superpotential equation") of Ref. [145]. In the present $N = 2$ SUSY context, this relation may be easily understood from the fact that both $\mathcal{W}$ and $\mathcal{V}$ are derived from the same Kähler potential K.

13.10 Bogomol'nyi solitons in a gauged O(3) sigma model from $N = 2$ SUSY

As emphasized already, our method for the calculation of BPS equations for $N = 2$ SUSY extended theories is completely general for chiral $N = 2$ superfields with or without gauge interaction, therefore we may use it to study further models. Concretely, we want to employ it to obtain the Bogomol'nyi equations of the gauged nonlinear sigma model originally analyzed in [146]. We remark that the $N = 2$ SUSY extension of this model in the O(3) formulation has already been discussed in [147]. The gauged non-linear sigma term results from the generalized Kähler term

$$L_2^g = \int d^2\theta d^2\bar{\theta} \ln(1 + \Phi^\dagger e^V \Phi) \qquad (13.192)$$

where now the target space metric (=the Kähler metric) is the one of the CP(1) model and, therefore, the corresponding Kähler potential is fixed. The resulting lagrangian is like in Eq. (13.149), but for fixed $g_{u\bar{u}} = (1 + u\bar{u})^{-2}$. Further, we need the $N = 2$ extension of the Maxwell lagrangian, Eqs. (13.159) and (13.160). Focusing on the D-dependent terms for the moment we find (for a general Kähler potential $K(u\bar{u})$)

$$(L_2^g + L_M)|_D = Du\bar{u}K' + \frac{1}{2}D^2 \qquad (13.193)$$

(remember $K' \equiv \partial_{|u|^2} K$) with the solution

$$D = -u\bar{u}K' \tag{13.194}$$

and, therefore, the potential term contribution to the lagrangian is

$$\mathcal{V} = \frac{1}{2}(u\bar{u}K')^2. \tag{13.195}$$

For the specific Kähler potential of the CP(1) model, $K = \ln(1 + u\bar{u})$, we get

$$\mathcal{V} = \frac{1}{2}\left(\frac{u\bar{u}}{1+u\bar{u}}\right)^2. \tag{13.196}$$

We emphasize that this potential stems exclusively from the auxiliary field D, and that its form is fixed by the target space geometry (by the Kähler potential). Specifically, there is no superpotential contribution to this potential, and the only solution for the auxiliary fields F for this lagrangian is the trivial solution $F = \bar{F} = 0$. Using these solutions for F and D, and setting the scalar σ from the Maxwell superfield equal to its trivial solution, $\sigma = 0$, we get the lagrangian in the bosonic sector

$$(L_2^g + L_M)|_{bos} = \frac{D^\mu u D_\mu \bar{u}}{(1+u\bar{u})^2} - \frac{1}{2}\left(\frac{u\bar{u}}{1+u\bar{u}}\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{13.197}$$

that is, precisely the Lagrangian of the gauged nonlinear sigma model. Further, inserting the on-shell values for the D and F fields into the general $N = 2$ BPS equations (13.177), (13.178), we find the Bogomol'nyi equations

$$D_1 u = \pm i D_2 u \tag{13.198}$$

$$B \equiv F_{12} = \pm\frac{|u|^2}{(1+|u|^2)}, \tag{13.199}$$

which coincide precisely with the ones of Ref. [146].

We remark that in this case, in principle, we may add a superpotential term

$$L_0 = \int d^2\theta \mathcal{U}(\Phi) + \int d^2\bar{\theta}\mathcal{U}^\dagger(\Phi^\dagger), \tag{13.200}$$

which leads to the F-dependent contribution

$$(1+u\bar{u})^{-2}F\bar{F} + \mathcal{U}_u F + \mathcal{U}^\dagger_{\bar{u}}\bar{F} \tag{13.201}$$

and to the on-shell values

$$\bar{F} = -(1+u\bar{u})^2 \mathcal{U}_u \ , \quad F = -(1+u\bar{u})^2 \mathcal{U}^\dagger_{\bar{u}} \tag{13.202}$$

and, therefore, to the further contribution to the potential

$$\tilde{\mathcal{V}} = (1+u\bar{u})^4 |\mathcal{U}_u|^2. \tag{13.203}$$

The BPS equations in this case read

$$F_{12} = \pm \frac{|u|^2}{(1+|u|^2)} \tag{13.204}$$

$$(1+u\bar{u})^4 |\mathcal{U}_u|^2 = D_i u D_i \bar{u} \pm \sqrt{(D_i u D_i \bar{u})^2 - (D_i u)^2 (D_j \bar{u})^2}. \tag{13.205}$$

The second BPS equation may be rewritten like

$$2(1+u\bar{u})^4 |\mathcal{U}_u|^2 = (D_i u \pm i\epsilon_{ij} D_j u)(D_i \bar{u} \mp i\epsilon_{ik} D_k u)$$

or (after introducing the complex base space variable $z = (1/2)(x+iy)$), depending on the sign, as

$$2(1+u\bar{u})^4 |\mathcal{U}_u|^2 = (D_{\bar{z}} u)(D_z \bar{u})$$

or as

$$2(1+u\bar{u})^4 |\mathcal{U}_u|^2 = (D_z u)(D_{\bar{z}} \bar{u})$$

where $\partial_z = \partial_x - i\partial_y$ and $A_z = A_x + iA_y$.

It might be interesting to investigate whether in this class of field theories some models (i.e., some nontrivial choices of $\mathcal{U}$) can be found which support genuine solitons.

13.11 Conclusions

It was the purpose of the present work to investigate in detail possible supersymmetric extensions of baby Skyrme models. First of all, we found that the complete baby Skyrme model, consisting of three terms (potential, quadratic and quartic term), allows for an $N=1$ SUSY extension where, in addition, the potential derives from a superpotential via the field equation of the auxiliary field, as usual. This finding is related to the fact that for this $N = 1$ extension, the SUSY extension of the quartic term does not depend on the auxiliary field, at least in the bosonic sector. As a consequence, this SUSY extension cannot be used for the so-called BPS baby Skyrme model (a submodel without the quadratic term), because then the equation for the auxiliary field automatically eliminates the potential. Still, there exists another $N = 1$ SUSY extension which automatically eliminates the quadratic term and induces the potential from the Kähler metric (and *not* from a superpotential), leading directly to the BPS baby Skyrme model in the bosonic sector. It turns out that this $N = 1$ extension is, in fact, secretly $N = 2$. We explicitly constructed this $N = 2$ extension and demonstrated that, again, the equation for the auxiliary field eliminates the quadratic term and induces the potential from the Kähler metric. In a next step, we derived the general BPS equations for any $N = 2$ supersymmertic field theory of chiral superfields and used this construction to demonstrate that the BPS solitons of the BPS baby Skyrme model are one-half BPS states of the corresponding $N = 2$ supersymmetric extension. Then we turned to the investigation of SUSY extensions of gauged baby Skyrme models, i.e., of baby Skyrmions coupled to an abelian gauge field. We found that the complete gauged baby Skyrme model, too, has an $N = 1$ extension. Further, the gauged BPS baby Skyrme model (without the quadratic term, but coupled to a gauge field) again has an $N = 2$ extension where the auxiliary field of the chiral multiplet eliminates the quadratic term, whereas both auxiliary fields (from the chiral and the gauge multiplets) induce the potential in terms of the Kähler potential. We derived the completely general BPS equations for any $N = 2$ chiral multiplet coupled to an $N = 2$ gauge multiplet and used this result to re-derive the BPS equations of the gauged BPS baby Skyrme model [145] as one-half BPS equations of the $N = 2$ extension. Finally, we applied our general $N = 2$ BPS equations to the gauged nonlinear sigma model as a further, concrete application.

With these results at hand, the issue of possible applications and generalizations arises naturally. First of all, our BPS equations hold completely generally for *any* $N = 2$ supersymmetric field theory of (gauged or ungauged) chiral superfields, so it can obviously be used to find BPS equations for other models. Baby Skyrmions as such have found some applications in brane cosmology [120], so their supersymmetric extensions may be of interest in this context. Another interesting issue is related to generalizations to higher dimensions. An $N = 2$ supersymmetric theory in $d = 2 + 1$ dimensions leads in a natural way to an $N = 1$ theory in one dimension higher, i.e., in $d = 3 + 1$ dimensions. For the Skyrme-Faddeev-Niemi (SFN) model (same field content and lagrangian as the baby Skyrme model, but in $d = 3 + 1$), we conclude that we cannot find an $N = 1$ extension with our methods, in agreement with the findings of [15], [16]. on the other hand, for the restricted or extreme SFN model consisting of the quartic term and a potential only, we conclude that an $N = 1$ SUSY extension does exist. This model has been investigated recently [148], [149] where it was shown that it supports knotted and linked solitons (Hopfions), like the full SFN model. In the same line of reasoning, we conclude that the gauged nonlinear sigma model in $d = 3 + 1$ dimensions has an $N = 1$ SUSY extension.

This naturally leads to the question of SUSY extensions of the Skyrme model in $d = 3 + 1$ dimensions. Indeed, the Skyrme model, too, has a submodel which supports BPS solitons [125], and the results of the present work make it plausible to conjecture that this submodel might allow for an $N = 2$ extension, as well, but now in $d = 3 + 1$. This then implies that there should exist certain generalizations (i.e., more general submodels of the Skyrme model) which, while not possessing $N = 2$ extensions, still allow for $N = 1$ extensions. It would be very interesting to find these Skyrme submodels amenable to supersymmetry, to determine their SUSY extensions, and to investigate whether these supersymmetrizable Skyrme models are of special relevance in other contexts.

Another interesting class of problems is related to (and requires the determination of) the fermionic sectors of the SUSY extensions of the non-standard kinetic terms. Due to their complexity, these fermionic sectors have remained undetermined in almost all calculations up to now. Their knowledge, however, would allow to determine explicitly the supercharges (not only their evaluation on BPS solutions) and to calculate the resulting SUSY algebra with its possible central extensions. It is well-known that in the presence of topological solitons these central extensions have to be expected [44]. In addition, the inclusion of the fermions would allow to study the presence of fermionic zero modes in the background of topological solitons and, therefore, to investigate the corresponding index theorems relating the topological charges to the number of zero modes. These issues are under current investigation.

To summarize, in the present work we have made some important steps towards a better understanding of SUSY extensions of field theories with non-standard kinetic terms and, specifically, of non-standard field theories which support topological solitons. We found - among other results - that also for these theories the existence of BPS solitons is related to the existence of higher SUSY extensions, such that the BPS solitons are realized as BPS states in the SUSY extended theories, which for this type of theories is a new result.

Chapter 14

Symmetries of the BPS Skyrme model

The ultimate goal of the research was the supersymmetrization of the field theories relevant for strong interactions, providing among other aspects stability under quantum corrections. For this purpose one requires some refinements like the one presented in this chapter. The BPS Skyrme model is our main candidate. It is a specific subclass of Skyrme-type field theories which possesses both a BPS bound and infinitely many soliton solutions (skyrmions) saturating that bound, a property that makes the model a very convenient first approximation to the study of some properties of nuclei and hadrons. A related property, the existence of a large group of symmetry transformations, allows for solutions of rather general shapes, among which some of them will be relevant to the description of physical nuclei. We study here the classical symmetries of the BPS Skyrme model, applying them to the construction of soliton solutions with some prescribed shapes, what constitutes a further important step for the reliable application of the model to strong interaction physics. This chapter consists of a paper published in [101].

Symmetries and exact solutions of the BPS Skyrme model

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Abstract: The BPS Skyrme model is a specific subclass of Skyrme-type field theories which possesses both a BPS bound and infinitely many soliton solutions (skyrmions) saturating that bound, a property that makes the model a very convenient first approximation to the study of some properties of nuclei and hadrons. A related property, the existence of a large group of symmetry transformations, allows for solutions of rather general shapes, among which some of them will be relevant to the description of physical nuclei. We study here the classical symmetries of the BPS Skyrme model, applying them to construct soliton solutions with some prescribed shapes, what constitutes a further important step for the reliable application of the model to strong interaction physics.

14.1 Introduction

The Skyrme model [1] (SkM), a non-linear field theory for an $SU(2)$-valued field, is meant to be a low energy effective theory, describing some interesting aspects of strong interaction physics. In this model, pions play the role of primary fields (excitations around the trivial vacuum), whereas nucleons and nuclei are, on the other hand, represented by topological solitons, collective excitations which are part of the nonperturbative spectrum of the theory.

The application of the SkM to nuclear and hadronic physics has been quite successful at a qualitative level [102], [103], [150], [151], but it encounters some problems once a more detailed, quantitative agreement, is required. The main obstacle for this is the absence of (almost) BPS solutions in the original SkM, as well as in its standard generalizations. Indeed, although there exists a BPS bound already in the original model, as proposed by Skyrme, nontrivial soliton solutions cannot saturate this bound. As a consequence, higher solitons, meant to describe larger nuclei, are strongly bound, in striking contrast to the weak binding energies of physical nuclei.

Some alternatives approaches to improve this situation have been recently advanced. Basically they imply the extension of the symmetries of the Skyrme type theory to conformal transformations [152] or to volume preserving diffeomorphisms [153]. It it the aim of this paper to further elaborate on one of them, namely, the proposal of [153].

The SkM may be generalized in a rather straightforward way, by simply adding some judiciously chosen extra terms to its defining Lagrangian [154]-[158]. Indeed, the addition of extra terms becomes a quite natural step when one recalls the fact that the SkM is an effective theory, supplemented with the condiment of some simplicity and symmetry constraints. In fact, assuming, as one usually does, that we want to maintain the field content of the original model, as well as its Poincaré invariance and the standard Hamiltonian interpretation (Lagrangian quadratic in time derivatives), the number of possible terms is in fact quite restricted. One may then just have: a potential term (no derivatives), a standard kinetic term (the nonlinear sigma model term) quadratic in first derivatives, the 'Skyrme term' originally introduced by Skyrme (quartic in derivatives) and, finally, a term which is the square of the baryon number current (topological current), which is sextic in derivatives.

As it has been demonstrated in [153], there is a submodel, termed 'BPS Skyrme model' (BPSSkM), defined by a Lagrangian consisting of just the potential and sextic terms, which satisfies some quite interesting properties. Indeed, it possesses a BPS bound, and infinitely many BPS solutions saturating this bound. Besides, it has been also shown in [153] that the static energy functional of the model is invariant under an infinite number of symmetry transformations, a fact that is obviously related to the properties enunciated in the previous sentence.

Among the symmetry transformations, an interesting type are the volume preserving diffeomorphisms (VPDs), since they are precisely the symmetries of an incompressible fluid, a fact pointing to a possible relation to the liquid drop model for nuclei. The BPSSkM, therefore, has several appealing features from the point of view of the description of nuclei (see, for example, [153], [159]). The model is, in fact, constructed assuming that the coherent (topological) excitations play an especially important role in strong interaction physics. This assumption is directly related to the suppression of the usual kinetic term in the Lagrangian and, as a consequence, one might expect that the BPSSkM will not lead to reliable results in the weak field regime. To overcome these shortcomings, it may be necessary to augment the lagrangian by further structures for a more consistent description of nuclear or hadron physics. There are, for instance, initial data for which the BPSSkM does not have a well-defined Cauchy problem; thus, a standard kinetic term must either be added explicitly or induced by quantum corrections, to remedy this situation.

We think that the BPSSkM provides an approximation which may be quite reliable for the study of static properties and for the dynamics in a region of relatively high density (i.e., with a not too small baryon charge density) like, e.g., in a soliton background. On the other hand, it will generally not be reliable in near-vacuum regions, and moreover cannot be applied at all to consider perturbative phenomena corresponding to quantum fluctuations of the pion field around the vacuum, since the dominant term would then be non-quadratic.

Because of the above, it would be important to relate the properties of solutions of the BPSSkM to the corresponding solutions of more general Skyrme-type models. A stumbling block which immediately pops up when attempting this task is the different sizes of the respective spaces of solutions, which are in turn due to the different symmetry groups of the models. The solutions of the BPSSkM may have almost any symmetry, due to the huge symmetry group of the field equations. In particular, there are spherically symmetric solutions (i.e., with spherically symmetric energy densities) for all the possible values of the baryon charge, Q_B. This is not the case, on the other hand, for the original SkM and its non BPS generalizations. Typically, the $Q_B = 1$ skyrmion is spherically symmetric, the $Q_B = 2$ skyrmion has cylindrical symmetry, while higher-charge skyrmions have, at most, a set of discrete symmetries. Indeed, their energy densities are invariant under some discrete subgroup of the rotation group SO(3) (see, for example, [106], [107], [162]). A skyrmion of the BPSSkM with the same set of discrete symmetries would, therefore, be a good starting point for the inclusion of physical effects induced by adding other extra terms to the Lagrangian. Because of this, it would be important to find a method for the systematic construction of solutions of the BPS Skyrme model with some prescribed symmetries.

It is the purpose of the present notes to investigate the space of BPS solutions further, making explicit use of its symmetries as a tool to generate new solutions. To that end, we shall take the spherically symmetric ones as a starting point for the construction. Finally, we shall show that all local solutions may, in fact, be constructed in this way.

This article is organized as follows: In section 14.2, we define the model, and introduce our notation and conventions. Then we construct the classical Hamiltonian in section 14.3. The BPS bound is considered in 14.4. In 14.5 we explore the issue of symmetries, within both the Lagrangian and Hamiltonian contexts. In 14.6 we derive and discuss the main properties of the BPS solutions of the model. We also construct several explicit classes of solutions with some prescribed symmetries, including the important case of discrete symmetries. In 14.7 we summarize our results and conclusions.

14.2 The model

The Lagrangian density $\mathcal{L}$, which has an $SU(2)$ valued field U as dynamical variable, may be written as follows:

$$\mathcal{L} \,=\, \mathcal{L}_{06} \,=\, -\lambda^2\pi^2\, B_\mu B^\mu - \mu^2\mathcal{V}(U,U^\dagger)\ , \tag{14.1}$$

where λ is a positive constant, B^μ denotes the topological current:

$$B^\mu \,=\, \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\big(L_\nu L_\rho L_\sigma\big)\ , \quad L_\mu \equiv U^\dagger\partial_\mu U\ , \tag{14.2}$$

and $\mathcal{V}$ is a potential density. The current B^μ is 'topologically conserved', namely, it can be shown to be conserved, regardless of the equations of motion. The resulting conserved charge, Q_B, is therefore given by:

$$
\begin{aligned}
Q_B &= \int d^3x\, B_0 = \frac{1}{24\pi^2}\int d^3x\,\epsilon^{ijk}\mathrm{tr}\big(L_iL_jL_k\big) \\
&= \frac{1}{24\pi^2}\int d^3x\,\epsilon^{ijk}\,\mathrm{tr}\big(U^\dagger\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U\big)\ , \qquad (14.3)
\end{aligned}
$$

the degree of the map $\mathbb{R}^3 \to S^3$, an integer which is invariant under arbitrary globally well-defined coordinate transformations, as well as under global isospin rotations of U. It is, in fact, invariant under the much bigger group of target space transformations leaving invariant a certain target space volume form, see below.

To proceed to the classical equations of motion, it is convenient to introduce a specific parametrization for the three degrees of freedom of U.

Following [153] , we use a real scalar field ξ plus a 3-component unit vector $\hat{\mathbf{n}}$, so that:

$$U(x) = e^{i\xi(x)\hat{\mathbf{n}}(x)\cdot\tau}\ , \qquad (14.4)$$

where τ are the three Pauli matrices. The real scalar ξ runs from 0 to π, while the two independent parameters defining $\hat{\mathbf{n}}$ may be taken as the two components of a complex variable u, by means of a stereographic projection:

$$\hat{\mathbf{n}} = \frac{1}{1+|u|^2}\big(u+\bar{u}, -i(u-\bar{u}), |u|^2-1\big)\ . \qquad (14.5)$$

In this way, one obtains for the Lagrangian density an expression in terms of ξ, u and $\bar{u}$:

$$\mathcal{L} = \frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}\ (\epsilon^{\mu\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2 - \mu^2\mathcal{V}(\xi) \qquad (14.6)$$

where the lower indices in those variables denote partial derivatives with respect to the spatial coordinates, and we have assumed that the potential may only depend on U through $\mathrm{tr}\,U$.

With the notation $\mathcal{V}_\xi \equiv \partial_\xi\mathcal{V}$, the Euler–Lagrange equations read:

$$
\begin{aligned}
\frac{\lambda^2\sin^2\xi}{(1+|u|^2)^4}\partial_\mu(\sin^2\xi\ H^\mu) + \mu^2\mathcal{V}_\xi = 0 \\
\partial_\mu\left(\frac{K^\mu}{(1+|u|^2)^2}\right) = 0\ , \qquad (14.7)
\end{aligned}
$$

where

$$H_\mu = \frac{\partial(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \xi^\mu}, \quad K_\mu = \frac{\partial(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \bar{u}^\mu}.$$

These objects satisfy, by construction, the relations

$$H_\mu u^\mu = H_\mu \bar{u}^\mu = 0,\ K_\mu \xi^\mu = K_\mu u^\mu = 0,\ \ H_\mu \xi^\mu = K_\mu \bar{u}^\mu = 2(\epsilon^{\alpha\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2 \ , \tag{14.8}$$

which are often useful.

14.3 Hamiltonian and static energy

In order to construct the Hamiltonian, we first introduce a more compact notation, in terms of three real fields $\xi^{(a)}$, with $a = 1, 2, 3$, such that $u = \xi^{(1)} + i\xi^{(2)}$ and $\xi^{(3)} \equiv \xi$.

Then $\mathcal{L}$ may be written as follows ($\xi_0^{(a)} \equiv \partial_0 \xi^{(a)}$):

$$\mathcal{L} \ = \ \frac{1}{2}\xi_0^{(a)} G_{(ab)} \xi_0^{(b)} - \frac{4\lambda^2 \sin^4 \xi^{(3)} \left(\epsilon^{ijk}\xi_i^{(1)}\xi_j^{(2)}\xi_k^{(3)}\right)^2}{\left[1+(\xi^{(1)})^2+(\xi^{(2)})^2\right]^4} - \mu^2 \mathcal{V}(\xi) \ . \tag{14.9}$$

where the kinetic term is determined by a metric $G_{(ab)}$, given by:

$$G_{(ab)} \ = \ \frac{2\lambda^2 \sin^4(\xi^{(3)})}{[1+(\xi^{(1)})^2+(\xi^{(2)})^2]^4}\, \mathcal{Q}_i^{(a)} \mathcal{Q}_i^{(b)} \tag{14.10}$$

where:

$$\mathcal{Q}_i^{(a)} \ = \ \epsilon_{ijk}\, \epsilon^{abc} \xi_j^{(b)} \xi_k^{(c)} \ . \tag{14.11}$$

In order to see whether the system defined by $\mathcal{L}$ is regular or not, we note that $\mathbb{Q} \equiv [\mathcal{Q}_i^{(a)}]$ the 3×3 matrix defined by the nine elements $\mathcal{Q}_i^{(a)}$ ($i = 1, 2, 3;$ $a = 1, 2, 3$) is proportional to the cofactor matrix of the matrix $\mathbb{X} \equiv [\xi_i^{(a)}]$:

$$\mathbb{Q} \ = \ 2\,\mathrm{cof}(\mathbb{X}) \ . \tag{14.12}$$

Thus, we see that the metric $[G_{(ab)}]$ (hence, the Lagrangian system) is regular if and only if $\det[\xi_i^{(a)}] \neq 0$. In other words, the regularity of the system is equivalent to the non vanishing of the Jacobian determinant:

$$\mathcal{J} \ \equiv \ \det[\mathbb{X}] \ = \ \det\left[\frac{\partial \xi^{(a)}}{\partial x_i}\right] \neq 0 \ , \tag{14.13}$$

for the mapping between the sphere (i.e., one-point compactified $\mathbb{R}^3$) in coordinate space and the one in $SU(2)$.

Under the assumption that (14.13) holds true, the inverse of $\mathbb{G} = [G_{(ab)}]$ may be found by elementary algebra. Indeed,

$$[\mathbb{G}^{-1}]^{(ab)} = \frac{[1+(\xi^{(1)})^2+(\xi^{(2)})^2]^4}{8\lambda^2\,\mathcal{J}^2\,\sin^4(\xi^{(3)})}\xi_i^{(a)}\xi_i^{(b)}\ . \tag{14.14}$$

Thus, the Hamiltonian density in terms of the variables $\xi^{(a)}$, its spatial derivatives, and their canonical momenta $\Pi^{(a)}$, becomes:

$$\begin{aligned}\mathcal{H} &= \frac{[1+(\xi^{(1)})^2+(\xi^{(2)})^2]^4}{16\lambda^2\,\mathcal{J}^2\,\sin^4(\xi^{(3)})}\Pi^{(a)}\xi_i^{(a)}\,\xi_i^{(b)}\Pi^{(b)} \\ &+ \frac{4\lambda^2\sin^4\xi^{(3)}\left(\epsilon^{ijk}\xi_i^{(1)}\xi_j^{(2)}\xi_k^{(3)}\right)^2}{\left[1+(\xi^{(1)})^2+(\xi^{(2)})^2\right]^4}+\mu^2\mathcal{V}(\xi)\ ,\end{aligned} \tag{14.15}$$

which, for a Lagrangian like the one we are considering, coincides with the energy density of the system. In particular, for the static configuration case to be considered in the forthcoming sections, the total energy E is:

$$E = \int d^3x\Big\{4\lambda^2\frac{\sin^4\xi^{(3)}\left(\epsilon^{ijk}\xi_i^{(1)}\xi_j^{(2)}\xi_k^{(3)}\right)^2}{\left[1+(\xi^{(1)})^2+(\xi^{(2)})^2\right]^4}+\mu^2\mathcal{V}(\xi)\Big\}\ . \tag{14.16}$$

We have shown that the regularity of the system depends on the field configurations considered. Specifically, the system is singular in regions where the fields take their vacuum values ($\xi^{(a)}$ = const. such that $\mathcal{V}(\xi^{(3)}) = 0$). This already demonstrates that, while the system may provide a good approximation to the description of static properties of nucleons and nuclei via solitons (Skyrmions) and for the dynamics in regions with nonzero baryon charge density (where it is regular by construction), its fully consistent application to dynamical nuclear physics requires additional structures like, e.g., quantum corrections, or the inclusion of further terms in the Lagrangian.

14.4 BPS bound

The static energy functional in (14.16), or, in terms of the variables ξ and u introduced previously,

$$E = \int d^3x\left[\frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}(\epsilon^{mnl}i\xi_m u_n\bar{u}_l)^2+\mu^2\mathcal{V}(\xi)\right] \tag{14.17}$$

obeys a Bogomol'nyi bound. Indeed,

$$E = \int d^3x \left(\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l \pm \mu\sqrt{\mathcal{V}} \right)^2 \mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{\mathcal{V}}}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l$$

$$\geq \mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{\mathcal{V}}}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l =$$

$$\pm(2\lambda\mu\pi^2) \left[\frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{\mathcal{V}}}{(1+|u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right] \equiv 2\lambda\mu\pi^2 \langle \sqrt{\mathcal{V}} \rangle |B| \quad (14.18)$$

where $\langle \sqrt{\mathcal{V}} \rangle$ is the average value of $\sqrt{\mathcal{V}}$ on the target space S^3. The corresponding Bogomol'nyi (first order) equation is

$$\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l = \mp\mu\sqrt{\mathcal{V}}. \quad (14.19)$$

The static second order field equations may be derived from the squared Bogomol'nyi equation by applying a gradient ∂_k and by projecting onto $\epsilon_{ijk}\partial_j \xi^{(a)}$ where $\xi^{(a)} \equiv (\xi, u, \bar{u})$. We remark that a completely analogous BPS bound can be found for the BPS baby Skyrme model in one lower dimension [163]-[166].

Another interesting observation is that the BPS equation can be formulated in the language of a non-linear generalization of the static (vacuum) Nambu-Poisson equation. Indeed the left hand side can be recast into the Nambu-Poisson three-bracket [167]

$$\left\{X^A, X^B, X^C\right\} = \epsilon^{mnl} \frac{\partial X^A}{\partial x^m} \frac{\partial X^B}{\partial x^n} \frac{\partial X^C}{\partial x^l} \quad (14.20)$$

where the target space embedding coordinates X^A, $A = 1, 2, 3, 4$ form a three-sphere S^3 (i.e., $(X^A)^2 = 1$) and are related to the previous coordinates like $X^a = n^a \sin \xi$, $a = 1, 2, 3$, and $X^4 = \cos \xi$. Then, the generalized Nambu-Poisson dynamics is given by

$$\frac{dX^A}{dt} = \epsilon^{ABCD} \left\{X^B, X^C, X^D\right\} + X^A \sqrt{\mathcal{V}(X^4)}, \quad (14.21)$$

which differs from the standard case by the additional factor $\sqrt{\mathcal{V}}$ in the last term [167]. obviously, although the dynamics of the BPS Skyrme model is profoundly different, the BPS equation provides static solutions to this generalized Nambu-Poisson equation. Such solutions may be interpreted as vacuum configurations of the underlying hyper-membrane Lagrangian [170]. We remark that if one assumes from the outset that the target space variables X^A span a three-sphere, as we do in this paper, then there is no dynamics in Eq. (14.21), i.e., $\frac{dX^A}{dt} = 0$, as follows from the fact that the r.h.s. of (14.21) is proportional to X^A in this case. This just corresponds to the well-known result that the static vacuum equations for the hyper-membrane imply that the brane embedding coordinates X^A span a three-sphere [170]. So, our model generalizes the static hyper-membrane action, with a correspondence between the BPS solitons and the vacuum membrane configurations, but with completely different dynamics.

It may be instructive to compare the BPS bound arising above with a $1 + 1$ dimensional analogue: the search for (non-trivial) static minimum energy configurations for the Sine-Gordon model. Here, a real scalar field φ is in the presence of a potential density $\mathcal{U}(\varphi)$ which allows for non-trivial topology. The Lagrangian density is:

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu \varphi)^2 - \mathcal{U}(\varphi) \tag{14.22}$$

$$\mathcal{U}(\varphi) \;=\; \frac{m^4}{\lambda}\big[1 - \cos(\frac{\sqrt{\lambda}}{m}\varphi)\big] \;. \tag{14.23}$$

The static energy is then:

$$E_\varphi \;=\; \int_{-\infty}^{+\infty} dx_1 \big[\frac{1}{2}(\partial_1 \varphi)^2 + \mathcal{U}(\varphi)\big] \;. \tag{14.24}$$

The non-negative potential has non-trivial minima for

$$\varphi = \varphi_N = \frac{2\pi m}{\sqrt{\lambda}} N \;, \quad N \in \mathbb{Z} \;, \tag{14.25}$$

all of them having zero energy. Finite energy vacuum configurations must tend to one of the minima when $x_1 \to \pm\infty$.

The topologically conserved current is $j^\mu = \frac{\sqrt{\lambda}}{2\pi m}\epsilon^{\mu\nu}\partial_\nu \varphi$ $(\mu, \nu = 1, 2)$, which obviously satisfies $\partial \cdot j = 0$. Its associated topological charge is quantized:

$$Q_\varphi \;=\; \frac{\sqrt{\lambda}}{2\pi m} \int_{-\infty}^{+\infty} dx_1 \partial_1 \varphi(x) \;=\; N \;, \tag{14.26}$$

it is a constant of motion, and it is akin to a winding number, if one interprets φ as an angular variable.

Note the striking similarity with the BPS Skyrme model, when one writes the energy as follows:

$$E_\varphi = \int_{-\infty}^{+\infty} dx_1 \left[\frac{1}{2}(\frac{2\pi m}{\lambda})^2 (j_0)^2 + \mathcal{U}(\varphi)\right] . \tag{14.27}$$

The static energy then also verifies a Bogomol'nyi-like bound, since:

$$E_\varphi = \int_{-\infty}^{+\infty} dx_1 \left[\frac{1}{\sqrt{2}}\frac{d\varphi(x_1)}{dx_1} \pm \sqrt{\mathcal{U}(\varphi)}\right]^2 \mp \sqrt{2}\int_{-\infty}^{+\infty} dx_1 \frac{d\varphi(x_1)}{dx_1}\sqrt{\mathcal{U}(\varphi)} . \tag{14.28}$$

Thus:

$$\begin{aligned} E_\varphi &\geq \pm\sqrt{2}\int_{-\infty}^{+\infty} dx_1 \frac{d\varphi(x_1)}{dx_1}\sqrt{\mathcal{U}(\varphi)} \\ &= \pm\sqrt{2}\,\frac{2\pi m}{\sqrt{\lambda}}\,|\langle\mathcal{U}\rangle|\,|Q_\varphi| \\ &= 2\sqrt{2}\,\pi\,\frac{m^3}{\lambda}\,|Q_\varphi| . \end{aligned} \tag{14.29}$$

where:

$$|\langle\mathcal{U}\rangle| = \frac{1}{\varphi_1 - \varphi_0}\int_{\varphi_0}^{\varphi_1} d\varphi\sqrt{\mathcal{U}(\varphi)} , \tag{14.30}$$

the average of $\sqrt{\mathcal{U}(\varphi)}$ over the fundamental region.

of course, the first order equations that result from saturating the bound may be found by other methods; they lead to the well-known static solutions by a single quadrature. What we learn from the comparison with this model is that the particular form of the Lagrangian of the BPS Skyrme model involving the square of the topological current, is what makes it produce quite powerful constraints on the solution. It is interesting to note that the kinetic term in this 1+1 dimensional example allows for two different interpretations, either as a standard kinetic term or as the topological current squared, which is no longer true in higher dimensions. In other words, the simple Sine-Gordon type soliton model in 1+1 dimensions allows for two different generalizations to higher dimensions, generalizing either the standard kinetic term or the topological current, and the model studied in the present paper just corresponds to the second case.

14.5 Symmetries

The Lagrangian certainly has the standard Poincaré symmetries. Besides, the sextic term is the square of the pull back of the target space volume form on S^3,

$$dV = -i\frac{\sin^2 \xi}{(1+|u|^2)^2} d\xi du d\bar{u} \tag{14.31}$$

so this sextic term is invariant under target space diffeos which do not change this form (the volume preserving diffeos (VPDs) on S^3). The potential only depends on ξ, so it is still invariant under those diffeomorphisms which do not change ξ, i.e., under the diffeos which obey

$$\xi \to \xi\,, \quad u \to \tilde{u}(u, \bar{u}, \xi)\,, \quad (1+|\tilde{u}|^2)^{-2} d\xi d\tilde{u} d\bar{\tilde{u}} = (1+|u|^2)^{-2} d\xi u d\bar{u}.$$

The symmetries mentioned so far are symmetries of the action, i.e. Noether symmetries.

The static energy functional has some further symmetries. Indeed, it is invariant under volume preserving diffeos on the base space $\mathbb{R}^3$, as can be seen easily. The Bogomol'nyi equation has even more symmetries as we want to demonstrate now. For this purpose we introduce the new target space coordinates

$$u = g e^{i\Phi} = \tan(\chi/2) e^{i\Phi}, \quad H(g) = \frac{1}{1+g^2},$$

(for later convenience we also introduced χ, which together with ξ and Φ provides the standard hyperspherical coordinates on the target S^3), and

$$F(\xi) = \frac{\lambda}{\mu} \int d\xi \frac{\sin^2 \xi}{\sqrt{\mathcal{V}(\xi)}} \tag{14.32}$$

and rewrite the Bogomol'nyi equation as

$$\nabla F(\xi) \cdot \nabla H(g) \times \nabla \Phi = \pm 1 \tag{14.33}$$

or, in terms of differential forms

$$dF dH d\phi = \pm dx^1 dx^2 dx^3 \tag{14.34}$$

from which it is obvious that the Bogomol'nyi equation has as its symmetries all the VPDs both in base space and in a modified target space defined by the volume form $dFdHd\Phi$. The above equation implies, in fact, that all local VPDs on base space produce local solutions of the BPS equation. The problem is that, in general, a local solution cannot be extended to a global one, because of the different geometry and topology of the base space and the modified target space. The modified target space is defined by the volume form

$$dFdHd\Phi = -\frac{\lambda}{\mu}\frac{\sin^2\xi}{\sqrt{\mathcal{V}(\xi)}}\sin\chi d\xi d\chi d\Phi \tag{14.35}$$

and differs from the volume form on S^3 by the additional factor $1/\sqrt{\mathcal{V}}$. There does not exist a unique riemannian metric giving rise to this volume form, but a natural choice which assumes that the S^2 spanned by u (i.e., χ and Φ) remains intact is

$$ds^2 = d\xi^2 + \frac{\sin^2\xi}{\sqrt{\mathcal{V}(\xi)}}(d\chi^2 + \sin^2\chi d\Phi^2). \tag{14.36}$$

For $\mathcal{V} = 1$ this is just the round metric on S^3 in hyperspherical coordinates, but for nontrivial potentials the resulting target space manifold is different. Indeed, potentials which may support finite energy skyrmion solutions must have vacua $\xi = \xi_0$ where $\mathcal{V}(\xi_0) = 0$, and the above metric is singular at the vacuum values ξ_0. These singularities may either be integrable (i.e., the function F defined in (14.32) is well-defined and finite even at vacuum values $\xi = \xi_0$), in which case the total volume of the modified target space is still finite. In the opposite case, the total volume is infinite. one further conclusion may be drawn immediately by integrating Eq. (14.34). If the total volume of the modified target space is finite, then any skyrmion solution of the BPS equation must have compact support (i.e., be a "compacton"). Further, its volume must be equal to $|B|$ times the total volume of the modified target space, where B is the winding number. For equivalent results for the case of the BPS baby Skyrme model in one lower dimension, we refer to [166].
We remark that for $\mathcal{V} = \sin^4\xi$ the metric on the target space describes in fact a 3 dimensional cylinder with a very simple skyrmion solution (see below).

14.6 Solutions

As already said, locally, any VPD on base space will provide a solution of the BPS equation, but this solution will, in general, not be extendible to a global, genuine one (i.e., a skyrmion), because of the nontrivial topology one should have on the modified target space. A more promising strategy is the following: start from a simple known solution which may follow from a simple ansatz. Then one may generate new solutions by composing the given solution with a VPD on base space $\mathbb{R}^3$. If the VPD is well-defined on the whole of $\mathbb{R}^3$, then it will map genuine skyrmions into genuine skyrmions. In the case of compactons, we may even relax this condition, since it is then sufficient for the VPD on base space to be well-defined in the region of the compacton.

To proceed, let us first find some simple solutions with the help of an ansatz in spherical polar coordinates

$$\xi = \xi(r), \quad \chi = \chi(\theta), \quad \Phi = n\varphi \tag{14.37}$$

which inserted into the BPS equation yields:

$$-\frac{\lambda}{\mu}\frac{\sin^2\xi}{\sqrt{\mathcal{V}(\xi)}}\sin\chi d\xi d\chi d\Phi = \mp r^2 \sin\theta dr d\theta d\varphi \ , \tag{14.38}$$

leading to $\chi = \theta$ and

$$-\frac{n\lambda}{\mu}\frac{\sin^2\xi}{\sqrt{\mathcal{V}}}d\xi = \mp r^2 dr$$

or, after the coordinate transformation:

$$y = \frac{\mu}{3\sqrt{2}\lambda n}r^3 \tag{14.39}$$

to the autonomous oDE:

$$\frac{\sin^2\xi}{\sqrt{2\mathcal{V}(\xi)}}\xi_y = -1 \ . \tag{14.40}$$

We have chosen the sign which leads to a negative ξ_y, which is compatible with the boundary conditions $\xi(r=0)=\pi$, $\xi(r=\infty)=0$ for a potential which takes its vacuum at $\xi_0 = 0$.

Let us consider now the symmetries of these solutions. This issue depends on the criterion used to characterize that symmetry. Note that a given solution will not be invariant under any rotation, because it depends on the two angular coordinates θ and φ. The energy density, on the other hand, depends only on the radial coordinate r and is, therefore, spherically symmetric. Note, however, that there exists another symmetry criterion, often used for solitons, whereby there is spherical symmetry when the effect of a base space rotation on a solution can be undone by a corresponding target space rotation. Under this criterion, only the solution with topological charge $n = 1$ is spherically symmetric (i.e., all rotations can be undone). Solutions with higher winding number n only have cylindrical symmetry, i.e., only a rotation about the z axis $\varphi \to \varphi + \alpha$ can be undone by a target space rotation (a phase transformation $u \to e^{-in\alpha}u$).

In any case, we shall call all solutions of the spherically symmetric ansatz "spherically symmetric solutions" in what follows. We shall first review some general properties of these spherically symmetric solutions and, in a next step, construct solutions with lesser symmetries.

14.6.1 Solutions with spherical symmetry

Many qualitative aspects of solutions maybe easily derived from the particular form of the potential, which should be contrasted with the typical situation in general Skyrme models, where similar results usually require a full three-dimensional numerical simulation.

First of all, depending on the form of the potential in the vicinity of the vacuum, one can distinguish three types of solitonic configurations: compactons (where the solution approaches its vacuum value at a strictly finite distance) and exponentially as well as power-like localized solutions. Using the BPS equation and expanding the potential at a vacuum (e.g., at $\xi = 0$), $\mathcal{V} = \mathcal{V}_0 \xi^\alpha + ...$, one easily finds that for $\alpha < 6$ one gets compactons. There is also one exponentially localized solution for $\alpha = 6$, while for $\alpha > 6$ we find power-like localized solitons.

Another important feature of solutions reflects the number of vacua of the potential. It is easy to prove that for one-vacuum potentials the BPS solutions

are of the nucleus type (no empty regions in the interior), while two-vacuum potentials lead to shell-like configurations.
Let us present some particular examples. For the most elaborated family of one vacuum potentials, the so-called old potentials

$$\mathcal{V}_{old} = \left(\mathrm{Tr} \left(\frac{1-U}{2} \right) \right)^a \quad \rightarrow \quad \mathcal{V}(\xi) = (1-\cos\xi)^a \tag{14.41}$$

(where a is a real positive parameter), we find (besides the previously known compacton) a solution with exponential tail ($a = 3$) in implicit form

$$\cos\frac{\xi}{2} + \ln\tan\frac{\xi}{4} = -\frac{y}{2}$$

, and power-like localized solutions. E.g., for $a = 6$ we get

$$\xi = 2 \text{ arc cot} \sqrt[3]{3\sqrt{2}y}.$$

A family of two-vacuum potentials is given by

$$\mathcal{V}_{shell\ I} = \left(\mathrm{Tr} \left(\frac{1-U}{2} \right) \mathrm{Tr} \left(\frac{1+U}{2} \right) \right)^a \quad \rightarrow \quad V(\xi) = (1-\cos^2\xi)^a, \tag{14.42}$$

which is the chiral counterpart of the so-called new baby potential. The vacua exactly coincide with the boundary values for the scalar field i.e., $\xi = 0, \pi$. From the BPS property of the solution one can immediately see that the energy density should have a shell structure with two zeros: one at the center of the soliton, while the second (outer zero) can be located at a finite distance (compact shells) or approached asymptotically at infinity. Without losing generality (the potential is symmetric under the change of the vacua) we assume that $\xi = 0$ is the outer vacuum. of course, the inner vacuum can only be reached at a finite point as $y \geq 0$. This implies that only compact solitonic shells are acceptable. Specific examples of exact solutions are, for $a = 1$

$$\xi = \begin{cases} f \quad \arccos(\sqrt{2}y - 1) & y \in \left[0, \sqrt{2}\right] \\ 0 & y \geq \sqrt{2}, \end{cases}$$

and for $a = 2$

$$\xi = \begin{cases} \pi - \sqrt{2}y & y \in \left[0, \frac{\pi}{\sqrt{2}}\right] \\ 0 & y \geq \frac{\pi}{\sqrt{2}}. \end{cases}$$

The latter solution is, in fact, a solution for the case when the target space is a three-dimensional cylinder, as $\sin^2 \xi / \sqrt{\mathcal{V}} = const.$
In order to deal with non-compact shell skyrmions, we need to modify our potential in such a way that one vacuum (say, the inner vacuum at $\xi = \pi$) is always approached in a compacton manner . A simple choice is

$$\mathcal{V}_{shell\ II} = \mathrm{Tr}\left(\frac{1+U}{2}\right)\left(\mathrm{Tr}\left(\frac{1-U}{2}\right)\right)^a \quad \rightarrow \quad V(\xi) = (1+\cos\xi)(1-\cos\xi)^a \tag{14.43}$$

Again, we find compact shell skyrmions $a < 3$

$$\xi = \begin{cases} \arccos\left[1 - \left(2^{\frac{3-a}{2}} - \frac{3-a}{\sqrt{2}} y\right)^{\frac{2}{3-a}}\right] & y \leq \frac{\sqrt{2}}{3-a} \\ 0 & z \geq \frac{\sqrt{2}}{3-a} \end{cases}$$

an exponentially localized skyrmion for $a = 3$

$$\xi = \xi = \arccos\left[1 - 2e^{-\sqrt{2}y}\right],$$

and shell skyrmions which extend to infinity but are localized in a power-like manner ($a > 3$)

$$\xi = \arccos\left[1 - \left(2^{\frac{3-a}{2}} + \frac{a-3}{\sqrt{2}} y\right)^{\frac{2}{3-a}}\right].$$

14.6.2 Solutions with cylindrical symmetry

Now we assume that a spherically symmetric solution has been found, and we want to use symmetry transformations to map them to new solutions. In a first step we construct solutions with cylindrical symmetry, using the ansatz (in cylindrical coordinates)

$$\xi = \xi(\rho, z), \quad g = g(\rho, z), \quad \Phi = n\varphi \tag{14.44}$$

where $\rho^2 = (x^1)^2 + (x^2)^2$, $z = x^3$. The Bogomol'nyi equation for this ansatz may be written like

$$dF^{(n)} dH = \pm dq dp \tag{14.45}$$

where $F^{(n)} = nF$ and

$$q = \frac{\rho^2}{2}, \quad p = z$$

or like the Poisson bracket

$$\{F^{(n)}, H\} \equiv \frac{\partial F^{(n)}}{\partial q}\frac{\partial H}{\partial q} - \frac{\partial F^{(n)}}{\partial p}\frac{\partial H}{\partial q} = \pm 1. \tag{14.46}$$

Further, we know that it has the spherically symmetric solution

$$g = g_s = \tan(\theta/2) = \frac{\rho}{\sqrt{\rho^2 + z^2} + z} = \frac{\sqrt{2q}}{\sqrt{2q + p^2} + p} \equiv g_s(q, p) \tag{14.47}$$

and (depending on the potential)

$$\xi = \xi_s(r) = \xi_s(\sqrt{2q + p^2}) \equiv \xi_s(q, p). \tag{14.48}$$

As a consequence, a general solution with spherical symmetry may be written like

$$\xi(q, p) = \xi_s(Q(q, p), P(q, p)), \quad g(q, p) = g_s(Q(q, p), P(q, p)) \tag{14.49}$$

where (Q, P) are related to (q, p) via a canonical transformation, i.e., $\{Q, P\} = 1$.

A first class of examples is given by

$$Q = U(q), \quad P = \frac{p}{U'(q)}$$

where $U'(q) \neq 0 \; \forall \; q$ must hold. Further, it should hold that $\lim_{q \to 0} U(q)/q = const.$ to have a well-behaved function near $\rho = 0$. Among these examples the scale transformation $Q = a^2 q, P = a^{-2} p$ can be found, which corresponds to the scale transformation $x^1 \to a x^1$, $x^2 \to a x^2$ and $x^3 \to a^{-2} x^3$. Another class of examples is

$$Q = \frac{q}{U'(p)}, \quad P = U(p).$$

14.6.3 Solutions with discrete symmetries

Here, we want to construct a class of base space VPDs which transform solutions with spherical or cylindrical symmetry into solutions which only preserve symmetries w.r.t. to some discrete rotations about the z axis. Concretely, we want to consider solutions which may be written like

$$\xi = \xi(\rho, z) = \xi_s(\tilde{\rho}, z), \quad g = g(\rho, z) = g_s(\tilde{\rho}, z), \quad \Phi = n\tilde{\varphi} \tag{14.50}$$

where ξ_s, g_s, $\Phi = n\varphi$ constitute a known solution with either spherical or cylindrical symmetry. That is to say, we consider base space VPDs which act nontrivially only on ρ and φ, where for simplicity we restrict ourselves to the following transformations,

$$\tilde{\rho} = \tilde{\rho}(\rho, \varphi), \quad \tilde{\varphi} = \tilde{\varphi}(\varphi). \tag{14.51}$$

Using $q = \rho^2/2$ as before, and $\tilde{q} = \tilde{q}(q, \varphi)$, the condition for the transformation to be a VPD simplifies to

$$d\tilde{q}d\tilde{\varphi} = dqd\varphi. \tag{14.52}$$

A class of formal solutions is given by

$$\begin{aligned} \tilde{q} &= (f')^{-1} q \\ \tilde{\varphi} &= f(\varphi) \end{aligned} \tag{14.53}$$

in close analogy to the results of the last section. In order to define genuine diffeomorphisms, however, the transformations have to obey some further conditions. In particular, for the new coordinates $\tilde{q}$ and $\tilde{\varphi}$ to define polar coordinates on $\mathbb{R}^2$ they must satisfy the boundary conditions

$$\begin{aligned} \tilde{q}(q = 0, \varphi) = 0, &\qquad \tilde{q}(q = \infty, \varphi) = \infty, \\ \tilde{\varphi}(\varphi = 0) = 0, &\qquad \tilde{\varphi}(\varphi = 2\pi) = 2\pi. \end{aligned} \tag{14.54}$$

In addition, the vector field generating the flow induced by the coordinate transformation must be well-defined (nonzero and nonsingular) on the whole of $\mathbb{R}^2$. A class of examples fulfilling all the required conditions is given by $f = \varphi + (c/m)\sin m\varphi$, i.e., by the class of transformations

$$\begin{aligned} \tilde{q} &= (1 + c\cos m\varphi)^{-1} q && m \in \mathbb{N} \\ \tilde{\varphi} &= \varphi + \frac{c}{m}\sin m\varphi && c \in \mathbb{R}, \quad |c| < 1. \end{aligned} \tag{14.55}$$

Clearly, if a solution $\xi_s^{(a)}(\rho, z, \varphi)$ is invariant under rotations about the z axis (in the sense that its energy density is invariant under these rotations), then the new solution $\xi_s^{(a)}(\tilde{\rho}, z, \tilde{\varphi})$ is invariant only under the discrete set of rotations $\varphi \to \varphi + (2\pi/m)$.

14.7 Summary

We explored in detail the symmetries of the static energy functional of the BPSSkM, and of its related BPS equation. Then we applied these symmetries to the systematic construction of new solutions, starting from known ones. This is in the spirit of the dressing methods of classical integrability [171], which is an open problem for higher dimensional generalizations [172], an initial motiviation of this work. Specifically, this allowed us to construct solutions with some prescribed symmetries, what is quite relevant to the physical problem one wants to consider. We gave concrete examples of solutions with cylindrical symmetry and with symmetries w.r.t. some discrete subgroup of the group $SO(2)$ of rotations about the z axis. In this context, it would be interesting to construct solutions with the symmetries of platonic bodies or other discrete subgroups of the full rotation group $SO(3)$ (crystallographic groups), because solitons with these symmetries frequently show up as true minimizers of the energy in the original Skyrme model or some of its generalizations [106], [162]. The corresponding volume-preserving diffeomorphisms producing solutions with these symmetries will be more complicated than the ones constructed in the present paper, and it almost certainly will be more difficult to find them.
This issue is under current investigation.

Chapter 15

A final summary

In this chapter we collect the main results obtained along this Ph.D. thesis: *"Non-perturbative methods in non-linear field theories and their supersymmetric extensions"*.

The first original presentation for K field theories is performed in chapter 7. We proposed a simpler supersymmetric extension of such theories. With this extension it is easy to calculate supersymmetric charges, and therefore all the supersymmetric algebra is found. Nevertheless, in the last section we demonstrate that with this oversimplified scheme of supersymmetrization the model contains ghosts.

In chapter 8 we performed another way of SUSY extension of K field theories (of the form $L = \sum_k \alpha_k(\phi)X^k - V(\phi)$) without ghost fields. In the bosonic sector our supersymmetric action possesses terms with higher derivatives but also a polynomial in F (the auxiliary field) of degree N (being N the highest degree in the derivative terms) plus the contribution of the derivative of the superpotential. Instead of solving the algebraic equation for F we choose to do it in the other way: We solve the superpotential in terms of F, and the field lagrangian depends on a polynomial on F plus derivative terms. At this point we can choose a ϕ-dependence for F, to have an appropriate potential. Once we choose this dependence, the calculation of the original superpotential is trivial. We calculate specific solutions with their respective energy for different potentials finding, for example, compactons (solitons with compact support), $\mathcal{C}^1$-kinks and $\mathcal{C}^\infty$-kinks.

For standard scalar theories the relation between central charges of the SUSY algebra and topological charges for a soliton configuration has been known for a long time [44]. In chapter 9, following our investigation on K field theories, we demonstrated that all domain wall solutions which exist for the class of theories mentioned above , are, in fact, BPS solutions and further, these BPS solitons are invariant under part of SUSY transformations. For kink configurations, despite of the obvious differences between K field theories and theories with standard kinetic term, we found strong indications that here we have the same relation between central charges and topological charges, i.e., for a kink annihilated by one on the supercharges, the central extension of the SUSY algebra coincides with the difference between values of the superpotential evaluated on the asymptotic values of the kink.

In chapter 10 we continue with our characterization of K field theories. If the generalized dynamics of such theories is taken into account, then there exist the possibility of the so-called twin-like models, that is, pairs of models (one standard (with ordinary kinetic term) and one K) sharing the same topological defect solutions with the same energy density. We found the algebraic conditions that such twin theories must satisfy. In fact, this characterization provides a method for studying K field theories in terms of the standard ones. We demonstrated that, given a standard theory it is always possible to find a K field theory twin. For SUSY K field theories we found that, in addition to the equivalence between solutions and energies, the corresponding auxiliary field coincide on-shell. We also gave the conditions which allow the existence of twins coupled to gravity.

The problem of quantization of K field remains unsolved, and, due to the higher degree on non-linearity on these theories, it seems to be a quite difficult issue. In chapter 11 we solve partially this problem. We give a set of algebraic conditions that a pair of twin-like models must satisfy, to have the same linear fluctuation spectra. This implies (under these assumptions) that the semiclassical quantization around the topological defect gives the same results for the standard defect and its K field twin. The framework of SUSY K field theories gives us the possibility of the inclusion of fermions and the corresponding consequences of the coincidence fluctuation spectra, but this questions remains still unsolved.

Then we moved to a particular K field theory of special interest, the baby Skyrme model (bSkM). In chapter 12 we proposed the first $N = 1$ SUSY extension of the bSkM. We analyzed different features related to the supersymmetric structure. In particular, we demonstrate that this scheme of supersymmetrization prohibit the BPS bSkM (consisting on quartic term plus potential) which possesses topological solitons saturating a Bogomol'nyi type energy bound. The possibility of SUSY extension of the BPS BSkM seems to imply a second supersymmetry, and, in fact, in chapter 13 we proposed another $N = 1$ SUSY extension which has a second hidden supersymmetry. This extension allows the existence of the BPS bSkM. Then we constructed explicitly an $N = 2$ extension. Moreover, a completely general BPS equation (depending on the auxiliary field) for scalar models in $2 + 1$ dimensions and extended SUSY bSkM is proposed. We also calculated the $N = 2$ SUSY extension for the gauged model, and again, as a consequence of the SUSY transformations, general BPS equations for scalar coupled to abelian gauge fields are found. The information related to the particular model is encoded in the dependence of such equations on the auxiliary fields. The auxiliary field F coming from the scalar multiplet defines one of the equations, and D, the auxiliary field coming from the vector multiplet, defines the other.

Finally, in chapter 14, we analyzed another relevant example of K field theory, a Skyrme type model in $3 + 1$ dimensions, the BPS Skyrme model, consisting of a sextic term in derivatives plus potential. The existence of infinitely many solutions saturating the BPS bound makes it a good candidate to be supersymmetrized. As a first step we analyzed different solutions of the model. We used the Volume Preserving Diffeomorphisms (VPD) symmetry of the base space in order to generate different genuine skyrmions. We composed a simple known solution with a VPD generating in this ways solutions with different symmetries both continuous and discrete.

Chapter 16

Conclusions

In this Ph.D. thesis, we have studied different important aspects of a large class of generalized field theories which are characterized by the presence of higher (than just second) powers of first derivatives in the Lagrangian. These theories are frequently called K field theories, because of the generalized kinetic term.

Many of these theories are known to support topological soliton solutions, where, in more than two space dimension the presence of higher kinetic terms avoids the Derrick theorem.

Further, in some cases, these solitons saturate a topological bound and verify first order equations.

In this context, one first natural question is whether the corresponding K field theories allow for SUSY extension and whether BPS soliton solutions reappear as SUSY BPS states (which are invariant under some supersymmetries) in the SUSY extension, as is the case for standard field theories.

One of the main results of this thesis is that this is indeed true both for scalar field theories in $d = 1 + 1$ and for planar (baby) Skyrme models in $d = 2 + 1$.

These results are materialized in the following publications: In [95], SUSY extension of general K field theories in lower dimensions are described, whereas in [96] the BPS bounds and equations and the related central charges of the SUSY algebra are investigated. The result is that the BPS energies are related to the central charges (the BPS solutions are the SUSY BPS states), the only difference being that, due to the higher degree of non-linearity, there appear several BPS solutions (several roots of the first order equations) with the corresponding central charges. In [99], the first $N = 1$ SUSY extension of the baby Skyrme Model is presented, and consequences of this supersymmetrization are analyzed. And finally, the N=1 gauged SUSY extension and the N=2 gauged and ungauged SUSY extensions of the Baby Skyrme model are presented in [100]. In this last paper, we obtain that only BPS BSkM may be extended to $N = 2$ SUSY, i.e. the quadratic term ($\sigma-$model) is "eaten" by extended SUSY.

Soliton solutions of K field theories are, in general, quite different from the solitons of standard field theories. In some instances it happens, however, that a K field theory has the same soliton with the same energy density as a related standard field theory (the so-called twin models). A second major result of this thesis is a more profound algebraic analysis of these models. Concretely in [97] we fix the algebraic constraints that mutually twin-models must verify and in [98] these constraints are extended to ensure even the same fluctuation spectra between such models.

In higher-dimensional generalized field theories with their inherent high degree of non-linearity, it is usually difficult to find explicit (especially exact) solutions. One available method is to use symmetries of the theory for the generation of solutions. A third relevant result of this Ph.D. thesis is the use of symmetry transformations for the generation of solutions of arbitrary shapes for a certain Skyrme model which supports BPS solitons and has already found interesting applications to strong interaction physics. Concretely in [101] we determine the BPS bound of the BPS Skyrme model and exploit its symmetry under Volume Preserving Diffeomorphisms to find different solutions.

Briefly I repeat the potential applications of the results of this Ph.D. thesis:

- Effective field theories at low energies (Skyrme model $\leftrightarrows$ QCD at low energies)
- Applications to cosmology, e.g. inflation

- Theories descending from higher dimensions, e.g. brane world scenario
- Supersymmetry as a fundamental tool in the analysis of solutions of non-linear field theories

Chapter 17

Brief outlook

We have obtained several new results related to the supersymmetric extensions of theories with higher derivative terms, in particular the $N = 1$ and $N = 2$ supersymmetric extension of the planar Skyrme model, where BPS equations and bound have been derived from the supersymmetric structure, extension of general K field models and explicit solutions, etc. But there are still a lot of open questions to answer, some of them are:

- In general supersymmetric K field theories, the natural question at this point is the inclusion of fermions. Because of supersymmetry and translational invariance one should expect the existence of fermionic zero modes for each kink background.

- The addition of fermions would allow to study the corresponding index theorems explicitly, equating topological charges and zero modes.

- Despite some specific examples where the explicit calculation of the supersymmetry algebra is possible, the study of the superalgebra structure in general K theories could be interesting, in particular, the determination of central charges implied by the existence of solitons, as a generalization of the results of Witten and Olive.

- If these theories are in fact effective theories of a more fundamental one, one interesting issue is their quantization, even though the nonlinearity of these theories implies further complications.

- The extension of these results to higher dimensions. The supersymmetric extension of Skyrme type model in $d = 3 + 1$ with more general Skyrme terms and its possible applications to others fields.

- In the case of supersymmetric extensions of Skyrme type models the inclusion of fermions could be especially relevant. For example, the explicit calculation of the supercharges in such models allows us to determine explicitly the central extensions of the algebra.

- The issue of twin-like models in higher dimensions (vortices, monopoles, skymions...) and possible applications of the correspondence, e.g. trying to find realistic applications of the duality in such theories.

- Finally, it would be interesting trying to understand more deeply the issue of the Volume Preserving Diffeomorphisms group. For example, usual integrability exists in $(1 + 1)$ or $(2 + 0)$ dimensions and is closely related (but not equal) to conformal invariance with Virasoro algebra. In higher dimensions, the conformal group is finite dimensional, which can be another source for problems with the extension of integrability to higher dimensions. But in higher dimensions there is a natural group wich remains infinite dimensional, namely the VPD group. Such groups also provide an interesting connection to Virasoro as (at least for APD (area preserving diffeomorphisms) in $\mathbb{S}^2$, $\mathbb{R}^2$ and $\mathbb{T}$) the Virasoro is a subalgebra of the algebra of APD for these spaces. One concrete aim is to better understand these problems and the connection between generalized integrability and VPD.

Chapter 18

Resumo

A descrición da natureza pode ser afrontada dende moi diversos puntos de vista. Dende o punto de vista da física teórica podemos dicir, ainda a risco de ser simplistas, que a natureza pode ser clasificada nunha escala de enerxías (ou distancias) onde a simetría xoga un papel fundamental. Así por exemplo a mecánica cuántica revélase máis útil para explicar fenómenos que teñen lugar a moi pequenas distancias (ainda que existen fenómenos cuánticos perceptibles a grandes lonxitudes), ou a relatividade xeral de Einstein para explicar o Universo a distancias interplanetarias. Ata onde coñecemos hoxe en día só existenten catro interacción fundamentais: interaccións forte e débil (que xogan o seu papel máis relevante a moi pequenas distancias dentro dos nucleos atómicos por exemplo) e as interaccións electromágnetica e gravitatoria, éstas últimas sinxelas de observar na nosa vida diaria.

Neste proceso de síntese que sempre caracterizou á física dúas das interaccións nomeadas antes xa foron unificadas nunca única teoría, a teoría electrodébil, mais os físicos teóricos son conscientes hoxe en día de que a coexistencia das teorías electrodébil e forte por unha a banda (denominadas en conxunto teorías cuánticas de campos) e a Gravidade pola outra non é consistente (a cuantización da Gravidade é o impedimento crucial). Así diferente candidatas a teorías unificadoras xurdiron nos últimos corenta anos, a máis prometedora e á vez máis popular de todas elas é a Teoría de Cordas, ainda que podería haber outras opcións, podemos citar por exemplo a Gravidade Cuántica de Bucles.

Así e todo, obter resultados directamente a partir duna teoría fundamental é complicado e en ocasión imposible. Hai casos nos que resulta interesante adoptar un punto de vista diferente, a teoría efectiva. Unha teoría efectiva pode ser vista coma un certo límite duna teoría máis fundamental, por exemplo o límite de baixas enerxías da QCD (teoría que explica as interaccións fortes) pode ser descrito por unha teoría efectiva, o modelo de Skyrme, estudado amplamente neste traballo. Éste resulta máis simple que aquela e permite obter información prácticamente imposible de obter directamente da teoría orixinal. Este é o punto de vista adoptado ó largo desta tese, o estudo de teorías efectivas que poden ter aplicación en teorías máis fundamentais.

Como dixemos ó inicio, a simetría xoga un papel angular na física de hoxe en día, pero de entre todas as posibles simetrías hai unha que destaca especialmente, a supersimetría. Esta simetría liga bosóns e fermións e dota de certas propiedades interesantes ás teorías fundamentais. Por exemplo o modelo estándar (que describe conxuntamente as partículas que coñecemos), e dito sea de paso, a teoría máis precisa que coñecemos, posúe unha grave imperfección, o denominado *problema do axuste fino*, se dotamos a este modelo de supersimetría o problema parece arranxanse. Por citar outro caso, se un engade supersimetría á teoría de cordas consegue reducir as 26 dimensións nas que orixinalmente é formula a teoría a ton so 10. Ademáis, dende un punto de vista matemático, a supersimetría proporciona xeralizacións interesantes a supervariedades ou superalxebras de Lie. Agora ben, mentres ésta sexa unha boa candidata a simetría da natureza, e forme parte de teorías fundamentais, parece lóxico que sexa herdada por teoría efectivas que as describan en certo límite. Neste marco, xustificamos a meirande parte do traballo realizado nesta tese doutoral sobre extensión supersimétricas de teorías de campos.

Contido da tese

Nesta tese tratamos as extensións supersimétricas de teorías de campos K, é decir, teorías de campos con termos cinéticos superiores, prestando unha especial atención ó modelo de Skyrme. o traballo aquí presentado vese reflectido nas publicacións [95], [96], [97], [98], [99], [100] e [101]. Esta formada por dúas partes ben diferenciadas. A primeira parte está constituída por: nos capítulos 2 ó 6 ofrécese unha introducción a supersimetría e o modelo de Skyrme, fixando así mesmo a notación. A segunda céntrase no estudo das teorías K es as súas extensións supersimétricas:

No capitulo 7, presentamos os primeiros problemas que subxacen ós intentos de supersimetrización das teorías de campos K (que poden funcionar coma teorías efectivas en certo réxime).

Nos capítulos 8 e 9 móstranse os primeiros intentos existosos da devandita supersimetrización e análise de diversas propiedades. Continuando coa análise de estas teorías nos capítulos 10 e 11, móstrase una correspondencia entro ditas teorias e teorías estándar que pode facilitar o seu estudo.

Os capítulos 12 e 13 están xa centrados nun exemplo fundamental de teoría K, o modelo baby Skyrme. Preséntase aquí a primeira extensión supersimétrica do mesmo cunha e dúas supersimetrías e diversas propiedades herdadas desta estructura supersimétrica son estudadas amplamente.

No capítlo 14, que constitúe case un anexo, preséntase o que podería ser o seguinte paso no análise de teorías supersimétricas tipo Skyrme, o BPS Skyrme. Son estudadas diferentes simetrías e calculadas explícitamente solucions.

18.1 Teorías de campos K e supersimetría

En xeral as teorías escalares en física están caracterizadas por un Lagranxiano da forma:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{18.1}$$

onde $V(\phi)$ é o potencial e $\frac{1}{2}\partial_\mu\phi\partial\mu\phi$ é o termo cinético que denominaremos estándar, por conter dúas derivadas (supoñemos que traballamos en $2+1$ dimensións). Cómo supersimetrizar este tipo de modelos é coñecido. Utilizando o formalismo supercampos, non temos máis que promocionar os campos escalares a supercampos e as derivadas ordinarias a superdivadas. o lagragiano supersimétrico completo virá dado por:

$$S = \int d^3x d^2\theta \left[-\frac{1}{4}D^\alpha\Phi D_\alpha\Phi + P(\Phi)\right] = \int d^3x D^2 \left[-\frac{1}{4}D^\alpha\Phi D_\alpha\Phi + P(\Phi)\right]\Big| \tag{18.2}$$

se integramos agora nas variables de Grassman:

$$S = \int d^3x\Big[\frac{1}{2}F^2 + \frac{1}{2}i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}P''(\phi)\bar{\psi}\psi + P'(\phi)F\Big]. \tag{18.3}$$

F é o denominado campo auxiliar, e non é dinámico. Podemos eliminalo coa súa ecuación de movemento, resultando:

$$S = \int d^3x\Big[-\frac{1}{2}(P'(\phi))^2 + \frac{1}{2}i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}P''(\phi)\bar{\psi}\psi\Big]. \tag{18.4}$$

se identificamos o potencial en (18.1) con correspondente en (18.4), $V(\phi) = \frac{1}{2}(P'(\phi))^2$ temos obtida a extensión supersimétrica. Ademáis é posible comprobar que a acción fica invariante baixo as transformación de supersimetría seguintes:

$$\delta\phi(x) = \epsilon^\alpha\psi_\alpha(x) \tag{18.5}$$

$$\delta\psi_\alpha(x) = \epsilon^\beta(C_{\alpha\beta}F(x) + i\partial_{\alpha\beta}\phi(x)) \tag{18.6}$$

$$\delta F(x) = -i\epsilon^\alpha\partial_\alpha^\beta\psi_\beta \tag{18.7}$$

Unha teoría de compos K contén termos cinéticos con máis de dúas derivadas. Éstas teorías son relevantes en cosmoloxía e describen adecuadamente propiedades no periodo inflacionario. Ademáis moitos fenómenos relevantes son asociados a elas, por exemplo Galileons, condensados ghost ou inflación DBI son estudados en [135] ou en [136]. Dito isto, se resultan ser teorías efectivas dunha teoría máis fundamental e asumimos que a natureza é supersimétrica a extensión das mesmas é precisa. Sexa $X = \partial_\mu\phi\partial^\mu\phi$ entón centrarémonos agora en modelos da forma:

$$L = \sum_k \alpha_k X^k - V(\phi) \tag{18.8}$$

A supersimetrización destes modelos resulta ser moito máis complexa. Para chegar a ela, precisamos un par de pasos previos. Primeiro definimos:

$$\begin{aligned}(\mathcal{L}^{(k,n)})_{\psi=0} &= -\left(D^2[(\frac{1}{2}D^\alpha\Phi D_\alpha\Phi)(\frac{1}{2}D^\beta D^\alpha\Phi D_\beta D_\alpha\Phi)^{k-1}(D^2\Phi D^2\Phi)^n]|\right)_{\psi=0} \\ &= (F^2 + \partial_\mu\phi\partial^\mu\phi)^k F^{2n} \end{aligned} \tag{18.9}$$

Agora eliximos unha combinación linear particular:

$$\begin{aligned}(\mathcal{L}^{(k)})_{\psi=0} &\equiv (\mathcal{L}^{(k,0)})_{\psi=0} - \binom{k}{1}(\mathcal{L}^{(k-1,1)})_{\psi=0} + \binom{k}{2}(\mathcal{L}^{(k-2,2)})_{\psi=0} + \ldots \\ &\ldots + (-1)^{k-1}\binom{k}{k-1}(\mathcal{L}^{(1,k-1)})_{\psi=0} = (\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}. \end{aligned} \tag{18.10}$$

Finalmente engadindo o potencial e restrinxíndonos ó sector bosónico chegamos a:

$$\begin{aligned}\mathcal{L}_b^{(\alpha,P)} &= \sum_{k=1}^N \alpha_k(\mathcal{L}^{(k)})_{\psi=0} + P'F \\ &= \sum_{k=1}^N \alpha_k[(\partial^\mu\phi\partial_\mu\phi)^k + (-1)^{k-1}F^{2k}] + P'(\phi)F \end{aligned} \tag{18.11}$$

E unha vez eliminado o campo auxiliar recuperamos (18.8). Ainda que a ecuación resultante é altamente non linear é posible encontrar solucións explícitamente e calcular a súa enerxía nalgúns casos, consultar [96]. É máis, para determinadas confuguracións é posible explorar a extensión central da álxebra supersimétrica destes modelos. Escribamos explícitamente a álxebra supersimétrica:

$$\begin{aligned} Q_1^2 &= \Pi_0 + Z \\ Q_2^2 &= \Pi_0 - Z \\ \{Q_1, Q_2\} &= 2\Pi_1 \end{aligned} \tag{18.12}$$

onde as $Q's$ son os xeradores da supersimetría, $\Pi_\nu = (\Pi_0, \Pi_1)$ son os operadores de enerxía e momento e Z é a posible extensión central. Se tomamos unha configuración de kink (que existen nesten modelos, ver [96]), que teña valores asintóticos $\phi_\pm$ pode probarse que a súa enerxía ve determianda por:

$$E_k = P(\phi_+) - P(\phi_-) \tag{18.13}$$

Agora ben, se supoñemos que este kink corresponde coa supercarga Q_2 (ver [96]) sustituindo na álxebra obtemos:

$$Q_2^2 = 0 = E_k - Z = P(\phi_+) - P(\phi_-) - Z \quad \Rightarrow \quad Z = P(\phi_+) - P(\phi_-) \tag{18.14}$$

mentres que para a configuración antikink:

$$Z = P(\phi_-) - P(\phi_+) \tag{18.15}$$

obténdose un resultado análogo ó de Witten e Olive [44] tamén nas teorías K, ligando as configuración solitónicas e as cargas centrais da superálxebra.

18.2 Modelos Xemelgos

Baixo certas condicións é posible atopar teorías K cuxas solucións comparten densidade de enerxía e perfil das sas solucións. As condicións que deben verificar son puramente alxebraicas. Se temos unha teoría con termo cinético esténdar e potencial $V(\phi)$ as condicións necesarias para que unha teoría K definida por $\mathcal{L}_k$ sexa xemelga dela son:

$$\mathcal{L}_k|_{X=-U} \equiv \mathcal{L}| = -2U \tag{18.16}$$
$$\mathcal{L}_{k,X}|_{X=-U} \equiv \mathcal{L}_{,X}| = 1 \tag{18.17}$$

por exemplo se tratamos coa seguinte familia de teorías K:

$$\mathcal{L}_k = \sum_{k=1}^{N} f_k(\phi)X^k - U(\phi) \tag{18.18}$$

tense que:

$$\mathcal{L}_{k,X}|_{X=-V} = \sum_{k=1}^{N} k f_k(\phi)(-V)^{k-1} \equiv 1 \tag{18.19}$$
$$\mathcal{L}_k|_{X=-V} = \sum_{k=1}^{N} f_k(\phi)(-V)^k - U(\phi) = -2V(\phi) \tag{18.20}$$

Podemos por exemplo, fixar as funcións f_k para $k > 1$ e determinar f_1 e U coas ecuacións anteriores. Se a estas condicións engadimos as seguintes:

$$\mathcal{L}_{k,XX}| = 0 \tag{18.21}$$
$$[\mathcal{L}_{k,XU} + 2U(\mathcal{L}_{k,XXX} - \mathcal{L}_{k,XXU})]| = 0 \tag{18.22}$$
$$[\mathcal{L}_{k,UU} + \mathcal{L}_{k,XU} + 2U(\mathcal{L}_{k,XUU} - \mathcal{L}_{k,XXU})]| = 0 \tag{18.23}$$

e

$$(\mathcal{L}_{k,U} + 2U\mathcal{L}_{k,XU})| = -1. \tag{18.24}$$

podemos asegurar a coincidencia do espectro de fluctuacións linear o que implica que unha cuantización semiclásica no entorno dun defecto topolóxico proporciona exactamente os mesmos resultados para unha teoría estándar e a súa correspondente K xemelga. Podemos extender directamente estes resultados ás correspondentes extensións supersimtéricas, logo, esta correspondencia pode ser utilizada para estudar teorías K supersimétricas a través de teorías con termos cinéticos estándar.

18.3 O baby Skyrme supersimétrico

O modelo de Skyrme pode ser visto como o límite de baixas enerxías e alto número de cores da QCD, ademáis de ser un exemplo destacado de teoría tipo K. Unha versión a dimensións baixas do modelo de Skyrme, o denominado modelo baby Skyrme está caracterizado polo seguinte lagraxiano:

$$L = \frac{\lambda_2}{2} L_2 + \frac{\lambda_4}{4} L_4 + \frac{\tilde{\lambda}_4}{4} \tilde{L}_4 + \lambda_0 L_0 \tag{18.25}$$

onde

$$L_2 = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \tag{18.26}$$

$$L_4 = -(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 \tag{18.27}$$

$$\tilde{L}_4 = (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 \tag{18.28}$$

e

$$L_0 = -V(\phi_3) \tag{18.29}$$

Este modelo formulado en $2+1$ dimensións posúe solitóns topolóxicos e ten a $\mathbb{S}^2$ como espazo rango, é similar en moitos sentidos modelo de Skyrme orixinal, e ademáis pode servir coma modelo de xoguete para estudar problemas concernentes s solitóns topolóxicos. Por suposto, ten aplicacións directas, a descrición de ferromagnetos Hall cuánticos [85] ou para estudar texturas de spin [86] e [87]. En calqueira caso a súa supersimetrizacin ($N = 1$) resulta interesante e foi realizada por primeira vez en [99]. Os detalles da supersimetrización poden ser consultados en [99] e o resultado no sector bosónico ten propiedades interesantes, vexamos:

$$\begin{aligned} (\mathcal{L})_{\psi=0} &= \frac{\lambda_2}{2}[(\vec{F})^2 + \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}] - \frac{\lambda_4}{4}(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 + \lambda_0 F_3 P' + \mu_\phi(\vec{\phi}^2 - 1) \\ &= \frac{\lambda_2}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{\lambda_4}{4}(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 - \frac{\lambda_0^2}{2\lambda_2}(1 - \phi_3^2) P'^2 + \mu_\phi(\vec{\phi}^2 - 1) \end{aligned} \tag{18.30}$$

que corresponde co baby Skyrme con potencial

$$V(\phi_3) = \frac{\lambda_0}{2\lambda_2}(1 - \phi_3^2) P'^2(\phi_3). \tag{18.31}$$

O que vemos é que, polo menos con nesta extensión non está permitido o BPS baby Skyrme (que consiste en termo cuártico e potencial), que posúe solucións saturando a cota BPS, pois o límite $\lambda_2 \to 0$ que elimina o termo cuadrático crea un potencial diverxente.

A posible supersimetrización do modelo BPS baby Skyrme ($N = 1$) suxire a existencia dunha segunda supersimetría oculta. O caso é que é posible construir explícitamente este modelo con $N = 2$ como veremos. Tomaremos a versión $\mathbb{C}P^1$:

$$L_2 + L_4 = \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} + \frac{(\partial_\mu u)^2(\partial_\nu \bar{u})^2 - (\partial_\mu u \partial^\mu \bar{u})^2}{(1+|u|^2)^4}. \tag{18.32}$$

A supersimetrización do termo cuadrático é sinxela, basta tomar o potential de kehler correspondente a $\mathbb{C}P^1$:

$$L_2 = \frac{1}{16}\int d^2\theta d^2\bar{\theta}\,\ln(1+\Phi\Phi^\dagger) \tag{18.33}$$

onde $\Phi, \Phi^\dagger$ son respectivamente campos quiral e antiquiral con supersimetría $N = 2$. Para o termo cuártico precisamos:

$$\tilde{\mathcal{L}}_4 = \frac{1}{16} D^\alpha \Phi D_\alpha \Phi \bar{D}^{\dot{\beta}} \Phi^\dagger \bar{D}_{\dot{\beta}} \Phi^\dagger \tag{18.34}$$

O interesante é que o sumar as compoñentes de ambos lagranxianos a contribución cuadrática desaparece, de tal forma que escollemos adecuadamente o potencial de Kähler podemos xerar un termo potencial que depende déste, por exemplo para:

$$s = 1, \qquad K(\Phi, \Phi^\dagger) = \operatorname{arcsinh}(\sqrt{\Phi\Phi^\dagger}) - \sqrt{\frac{\Phi\Phi^\dagger}{1+\Phi\Phi^\dagger}} \tag{18.35}$$

$$s = 2, \qquad K(\Phi, \Phi^\dagger) = \frac{1}{1+\Phi\Phi^\dagger} + \ln(1+\Phi\Phi^\dagger). \tag{18.36}$$

obtemos trala eliminación do campo auxiliar:

$$\begin{aligned} s=1, \qquad L_T^1 &= \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u\partial_\mu u)(\partial^\mu \bar{u}\partial_\mu \bar{u}) - (\partial^\mu u\partial_\mu \bar{u})^2] - && (18.37)\\ &- 2\mu^2\left(\frac{u\bar{u}}{1+u\bar{u}}\right) \\ s=2, \qquad L_T^2 &= \frac{1}{(1+u\bar{u})^4}[(\partial^\mu u\partial_\mu u)(\partial^\mu \bar{u}\partial_\mu \bar{u}) - (\partial^\mu u\partial_\mu \bar{u})^2] - && (18.38)\\ &- 2\mu^2\left(\frac{u\bar{u}}{1+u\bar{u}}\right)^2 \\ &... \end{aligned}$$

é dicir, que anque partiamos dun modelo con termo cuadrático, unha vez eliminado F o que obtemos é o BPS baby Skyrme con supersimetría $N=2$. Das transformación de supersimetría é posible obter unha ecuación BPS xeral que so depende do modelo particular a través da forma específica do campo auxiliar, sendo a seguinte:

$$F\bar{F} = -\partial_i u\partial^i \bar{u} - \sqrt{(\partial_i u\partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2}, \qquad (18.39)$$

e unha vez substituido F o que obtemos é:

$$(\partial_i u\partial^i \bar{u})^2 - (\partial_i u)^2(\partial_j \bar{u})^2 = (i\epsilon_{jk}u_j\bar{u}_k)^2 \qquad (18.40)$$

que constitúe exactamente a ecuación BPS do modelo BPS baby Skyrme que foi deducida diractamente da supersimetría. Resultados análogos son obtidos se se engaden campos gauge, e de novo, somentes das transformación de supersimetría é posible deducir as ecuacións BPS do modelo. No caso gauge:

$$\begin{aligned} F\bar{F} &= -D_i u D_i \bar{u} \pm \sqrt{(D_i u D_i \bar{u})^2 - (D_i u)^2(D_j \bar{u})^2} && (18.41)\\ D &= \pm\epsilon^{ij}F_{ij}. && (18.42) \end{aligned}$$

onde D_i é a derivada covariante, F_{ij} a curvatura da conexión correspondente, F o campo auxiliar do multiplete quiral e D o campo auxiliar do multiplete vectorial. Remarcamos de novo, que estas ecuacións BPS son completamente xerais para supercampos quirais $N=2$ acoplados a campos gauge abelianos $N = 2$. Se temos en conta que podemos reducir dimensionalmente de $3+1$ dimensions e unha supersimetría a $2+1$ e dúas supersimetrías, e plausible extender estes resultados ó espazo ordinario polo que estes modelos con supersimetría extendida e dimensións baixas se fan aplicables néste. En conclusión, este nestraballo realizáronse importantes avances cara unha mellor comprensión das extensión supersimétricas de teorías de campos con termos cinéticos non lineares e en particular das que soportan solitóns topolóxicos.

Chapter 19

Conclusións

Nesta tese estudiamos diferentes aspectos relevantes dunha grande clase de teorías de campos xerais caracterizadas pola presenza de termos con derivadas superiores. Estas teorías son frecuentemente denominadas teorías de campos K debido ó termo cinéticos xeralizado (*kinetic term*).

É coñecido que moitas destas teorías conteñen solucións solitónicas, debido a que en dimensións maiores que un, a presenza de termos cinéticos superiores evita a restricción que impón o teorema de Derrick. Admais en moitos casos, estas solución satisfan ecuación de primeira orde (solucións BPS) e saturan unha cota topolóxica.

Neste contexto, unha posible primeira pregunta natural é determinar en que condicións é posible extender supersimétricamente e en que condición as devanditas solucións reaparecen coma estados BPS supersimétricos (invariantes baixo certa supersimetría) na extensión supersimétrica, como acontece nos casos estándar.

Un dos resultados fundamentais é que isto é certo para teorías K en $1+1$ diemensións e para modelo de Skyrme planar (baby) en $2+1$.

Estes resultados atópanse materializados nas seguintes publicacións: En [95], a extensión supersimétrica xeral para teorías K é descrita en dimensións baixas, mentres que en [96] son investigadas as cotas e ecuacións BPS así como extensión centrais da álxebra supersimétrica. o resultado é que as enerxías BPS están relacionadas coas cargas centrais (as solucións BPS son estados BPS supersimétricos), a única diferencia é que dado o alto grao de non linearidade xurden varias solución BPS (as raíces da ecuación de primeira orde) coas súas correspondentes cargas centrais. En [99] é presentada a primeira extensión supersimétrica $N = 1$ do modelo baby Skyrme e as consecuencias de dita supersimetrización son analizadas. Finalmente, as extensións $N = 2$ con e sen gauge e a $N = 1$ con gauge do modelo baby Skyrme son presentadas, ver [100]. Neste último artigo obtivemos como conclusión que somentes o BPS baby Skyrme (que consiste en termo cuártico máis potencial) pode ser extendido ate $N = 2$, é dicir, o termo correspondente ó modelo σ, o cuadrático, é comido pola supersimetría.

As solucións solitónicas das teorías de campos K, son en xeral, moi diferentes das correspondentes das teorías estándar. Pero baixo determinadas circunstancias acontence sen embargo que unha teoría K posúe os mesmos solitóns coa mesma densidade de enerxía que as de algunha teoría estándar (por estándar facemos referencia a modelos con termos cinéticos ordinarios), neste caso falamos de *teorías xemelgas*. Un segundo resultado relevante desta tese é unha análise alxebraico máis profundos destes modelos. Concretamente en [97] fixamos as condicións alxebraicas que han de verificar dúas teorías mutuamente xemelgas e en [98] estas condición son restrinxidas para asegurar que ademais de compartiren densidade de enerxía e perfil da solución, compartan o espectro de fluctuacións lineais é dicir, que no caso de cuantización semiclásica entorno a un defecto, podemos afirmar que ambalasdúas teorías proporcionarían os mesmos resultados.

No eido das teorías xeralizadas en dimensións altas co seu inherente alto grao de non linearidade, é a miúdo difícil atopar explícitamente solucións. Un posible método para atopalas consiste no uso de simetrías da teoría para xeras solucións. Un terceiro resultado desta tese é este uso de transformacións de simetría para a xeración de solucións de formas arbitrarias para un certo tipo de modelos de Skyrme en $3 + 1$ dimensions que soporta solitóns BPS e que xa atopou certo aplicación na física das interaccións fortes. Concretamente en [101] somos quen de determinar a cota BPS do modelo de Skyrme BPS e explotar a simetría baixo difeomorfismos que preservan o volume para atopar diferentes solucións.

Brevemente repetimos areas de aplicación xerais dos resultados contidos nesta tese:

- Teorías efectivas a baixas enerxías (Modelo de Skyrme $\leftrightarrows$ QCD a baixas enerxías)
- Aplicación á cosmoloxía, por exemplo no estudio do periodo inflacionario.
- Teorías derivadas de dimensións altas, por exemplo no escenario *brane world*.
- Supersimetría coma unha ferramenta fundamental na análise de teorías de campos non lineais.

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