GRAND UNIFICATION AND SUPERSYMMETRIC GRAND UNIFICATION

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1. INTRODUCTION

Over the last decade remarkable progress has been made in understanding the strong, weak and electromagnetic interactions. The most successful theory we have is quantum electrodynamics, a theory for electromagnetism. It has been tested to great accuracy. For example the prediction for the magnetic moment of the electron is 1.0011596553 in natural units and experiment gives a value of 1.0011596524, both with uncertainties $\pm$ 0.0000000030. Quantum electrodynamics is a gauge field theory and may be derived from the requirement that electric charge is locally conserved.

It now seems likely that the theories for the strong and the weak forces may also be gauge field theories derived from the requirement of local invariances under new symmetries. For the strong interactions the symmetry group is SU(3) acting in the colour quantum number of quarks. For the weak and electromagnetic interactions the symmetry group is that introduced by Glashow, Salam and Weinberg namely SU(2)xU(1) which contains charge conservation. Together the gauge field theory built on the symmetry SU(3)xSU(2)xU(1) has come to be known as the "standard" model and its predictions are entirely consistent with all experimental data.

If the strong, weak and electromagnetic interactions are separately described by gauge field theories, it is natural to ask whether they are all related. In grand unified theories (GUT) this idea is (realised by) embedding SU(3)xSU(2)xU(1) in a semisimple group G (e.g., SU(5)) with a single coupling constant g_{GU}. The strong, weak and electromagnetic interactions are then seen to be different facets of the same fundamental interaction based on a field theory with local gauge invariance under G. The group G is spontaneously broken at a scale M_X to SU(3)xSU(2)xU(1) and the strong weak and electromagnetic couplings are related to g_{GU} by radiative corrections. Remarkable the predictions for these couplings agree with experiment provided the scale M_X is very large

($\simeq 10^{15}$ GeV in SU(5)); the reason such a large scale arises is that the radiative corrections needed to get agreement depend only logarithmically on M_X. However a large mass scale turns out to be essential for in SU(5), and in most GUTs, the new gauge bosons mediate novel processes, in particular proton decay. To avoid violating the current lower bound on the proton lifetime of $O(10^{30}$ years) the mass M_X of these new gauge bosons must be $\gtrsim O(10^{15}$ GeV).

The appearance of such a large scale gives rise to a serious problem for GUTs, the "hierarchy" problem. In these theories the natural size for the weak interaction breaking scale M_W is $O(M_X)$ yet the actual value needed for M_W/M_X is $O(10^{-13})$. To achieve this parameters in the theory must be tuned to an accuracy of one part in 10^{13}, and no one has given a reason why this should be so within the framework of SU(5). Recently there has been much interest in a generalisation of GUTs which avoids this problem through the introduction of a new symmetry, supersymmetry,which guarantees that M_W should be small. Grand unified theories with supersymmetry, SUSY-GUTs, have been constructed and provide the first self-consistent GUTs. A particularly interesting feature of these models is that they require a new set of states, supersymmetric partners of the observed states, which must be relatively light ($O(M_W)$) and should be observable with the new machines such as LEP.

In these lectures I will discuss the status of GUTs and SUSY-GUTs. In section 2 the successes and failures of the standard model are reviewed. Section 3 introduces GUTs and discusses the minimal SU(5) version. Section 4 discusses the classic predictions of SU(5) and gives a critique of its achievements. Finally section 5 and 6 discuss supersymmetric grand unification in the globally supersymmetric case (SUSY GUTs) and in the locally supersymmetric case (SUGRA GUTs).

2. SUCCESSES AND FAILURES OF THE STANDARD MODEL

2.1. The standard model

Building on the local gauge principle, gauge theories for the weak, electromagnetic and strong interactions have been constructed. QCD, the theory for the strong interactions is based on the gauge group SU(3), which transforms the colour quantum number carried by all strongly interacting particles. The Glashow, Salam and Weinberg model of the weak and electromagnetic interactions is based on the gauge group SU(2)xU(1) which transform weak isospin and hypercharge. Together the $SU(3)_c xSU(2)xU(1)$ model, or (3,2,1) model, provides a potentially complete description of the strong, weak and electromagnetic interactions. It already has much experimental evidence in favour of it, as discussed in Professor Okun's lectures.

The structure of the model is given by the Lagrangian density

$$L_{(3,2,1)} = L_{kin} + L_{Yuk} + L_{scalar},$$

where the kinetic term L_{kin}, describes the kinetic energy of the gauge and matter fields and through the local gauge principle the coupling of the gauge bosons.

$$L_{kin} = \sum_{\text{fermions } j} i\bar{\psi}_j(\partial_\mu - ig_3A^a_\mu \frac{\lambda^a}{2} - ig_2 V^b_\mu \frac{\sigma^b}{2} - ig_1B_\mu Y)\gamma^\mu\psi_j$$

$$+ \sum_{\text{scalars } k} |(\partial_\mu - ig_2W^b_\mu \frac{\sigma^b}{2} - ig_1B_\mu Y)\phi_k|^2$$

$$- \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4} W^j_{\mu\nu} W^{\mu\nu}_j - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \qquad (2.1)$$

where $\frac{\lambda^a}{2}$ and $\frac{\sigma^b}{2}$ represent the generators of SU(3) and SU(2) respectively, and the kinetic terms for the gauge bosons involve the gluon field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_3 f^{abc} A^b_\mu A^c_\nu, \qquad a = 1\ldots 8$$

and the W and B field strengths

$$W^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu + g_2 \varepsilon^{jkm} W^k_\mu W^m_\nu,$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu . \qquad (2.2)$$

Y is chosen to satisfy

$$Q = \frac{\sigma_3}{2} + Y . \qquad (2.3)$$

Fermions appear to be grouped in three families (the e, μ and τ families) with SU(3)xSU(2) content

$$(3,2) + 2(\bar{3},1) + (1,2) + (1,1) \qquad (2.4)$$

as shown in Fig.2.1.

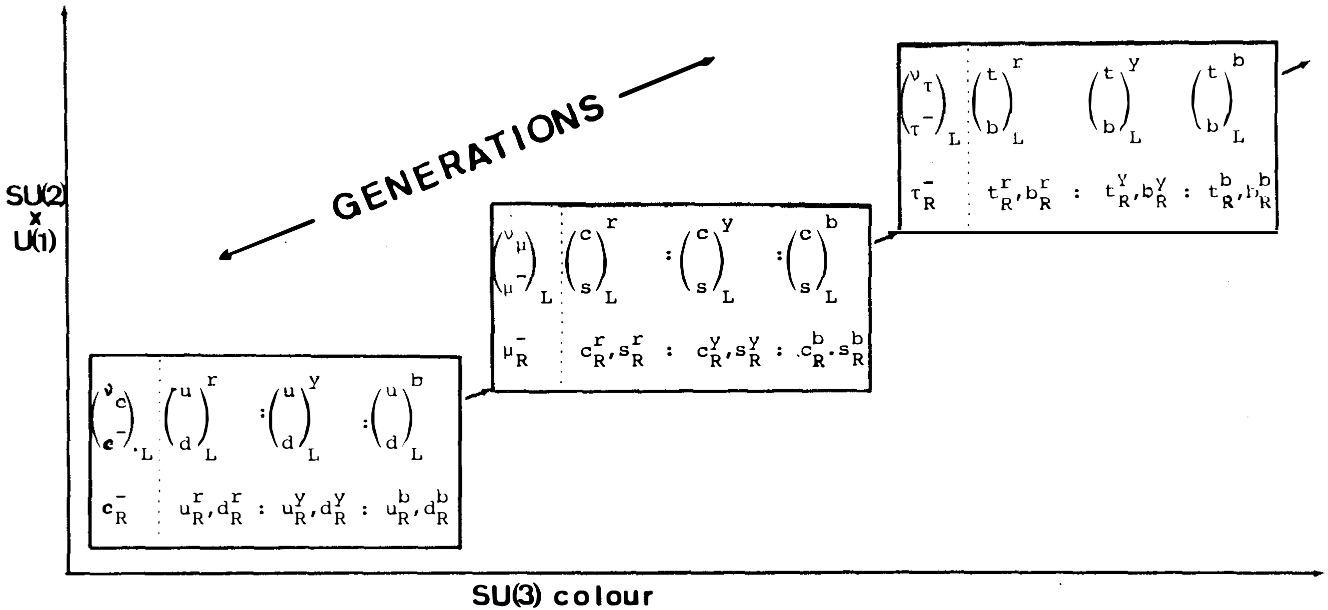

GENERATIONS
SU(2)
x
U(1)
SU(3) colour
ν_e, e⁻ L : u, d L r : u, d L y : u, d L b
e⁻_R : u^r_R, d^r_R : u^y_R, d^y_R : u^b_R, d^b_R
ν_μ, μ⁻ L : c, s L r : c, s L y : c, s L b
μ⁻_R : c^r_R, s^r_R : c^y_R, s^y_R : c^b_R, s^b_R
ν_τ, τ⁻ L : t, b L r : t, b L y : t, b L b
τ⁻_R : t^r_R, b^r_R : t^y_R, b^y_R : t^b_R, b^b_R

Fig.2.1.

L_{Yuk} describes the coupling of matter fermions to scalars in the theory. It is needed to introduce masses to the quarks and leptons, for the gauge interactions of L_{kin} preserve the chirality of quarks and leptons, while mass terms mix chirality. For the minimal (3,2,1) model it is possible to give quarks and leptons mass with a single doublet ϕ.

$$L_{Yuk} = \sum_{\substack{\text{quarks} \\ \text{generations } j,k}} c_{jk}(\bar{u}'_j, \bar{d}'_j)_L \begin{pmatrix} \phi^{o*} \\ -\phi^{-*} \end{pmatrix} u_{k'R} + \tilde{c}_{jk} (\bar{u}'_j, \bar{d}'_j)_L \begin{pmatrix} \phi^{+} \\ \phi^{o} \end{pmatrix} d'_{kR}$$

$$+ \sum_{\substack{\text{leptons} \\ \text{generations } i}} \tilde{d}_{ij}(\bar{\nu}_i, \bar{\ell}_i)_L \begin{pmatrix} \phi^{+} \\ \phi^{o} \end{pmatrix} \ell_{jR} + \text{h.c.} \qquad (2.5)$$

When ϕ^o develops a vacuum expectation value (vev), $\langle\phi\rangle$, the terms in L_{Yuk} generate quark and lepton masses. u'_i, d'_i and ℓ'_i are mixtures of mass eigenstates u_i, d_i and ℓ_i. In terms of these eigenstates the neutral currents all remain diagonal but the charged currents coupled to the W bosons, in eq. (2.1), may be written

$$L_{c.c.} = \frac{g}{2\sqrt{2}} (\bar{u}, \bar{c}, \bar{t})\gamma^\lambda (1-\gamma_5) U \begin{pmatrix} d \\ s \\ b \end{pmatrix} W^-_\lambda + \text{h.c.} \qquad (2.6)$$

where the Kobayashi-Maskawa mixing matrix U may be written in the form

$$U = \begin{bmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & -c_1s_2c_3-c_2s_3e^{i\delta} & -c_1s_2s_3+c_2c_3e^{i\delta} \end{bmatrix}, \qquad (2.7)$$

where $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ and θ_i and δ are arbitrary.

Finally there is the lagrangian density describing the interaction of scalar fields to trigger spontaneous symmetry breakdown

$$L_{scalar} = -V(\phi) = -\frac{1}{2}\lambda^2 |\phi|^4 + \frac{1}{2}\mu^2|\phi|^2 \qquad (2.8)$$

giving

$$|\langle\phi\rangle|^2 = \frac{\mu^2}{2\lambda^2} . \qquad (2.9)$$

After spontaneous symmetry breakdown, $W^{\pm}_{\mu}$ and Z_{μ} acquire masses while the photon field A^{γ}_{μ} remains massless

$$A^{\gamma}_{\mu} = B_{\mu}\cos\theta_W + W^3_{\mu}\sin\theta_W ,$$

$$Z_{\mu} = -B_{\mu}\sin\theta_W + W^3_{\mu}\cos\theta_W , \qquad (2.10)$$

where

$$\tan\theta_W = \frac{g_1}{g_2} ,$$

$$e = \frac{g_1 g_2}{(g_1^2+g_2^2)^{1/2}} ,$$

$$M_W = \frac{37.3}{\sin\theta_W} \text{ GeV} ,$$

$$M_Z = M_W/\cos\theta_W ,$$

$$M_{\phi}^2 = 2\mu^2 . \qquad (2.11)$$

The neutral current coupling is given by

$$J^Z_{\mu} = \sum_i \bar{\psi}^i_L \gamma_{\mu}(T_3 - Q\sin^2\theta_W)\psi^i_L$$

$$+ \sum_j \bar{\psi}^j_R \gamma_{\mu}(-Q\sin^2\theta_W)\psi^j_R \qquad (2.12)$$

and T_3 is the third component of weak isospin ($T_{3_L} = \pm\frac{1}{2}$).

2.2. Successes of the standard model

The theory described above with the multiplet structure summarized in fig.2.1. has an impressive list of successes.

(1) It is renormalisable and perturbatively unitary. As a result the theory may be used beyond the tree level to predict to arbitrary accuracy (limited by the endurance of the calculator and the convergence of the perturbation series) all but a finite number of quantities - those quantities being the fundamental parameters of the theory which require renormalisation. The perturbative unitarity of the theory means that an amplitude calculated at a given order in perturbation theory has good high energy behaviour and does not violate unitarity bounds. That this is so is highly nontrivial for it comes about as a result of cancellation of graphs which separately are much larger than the final amplitude. For example in $\nu\bar{\nu} \to W^+W^-$ good high energy behaviour is arranged by a cancellation with s channel Z exchange, see Fig.2.2(a).

(a)

(b)

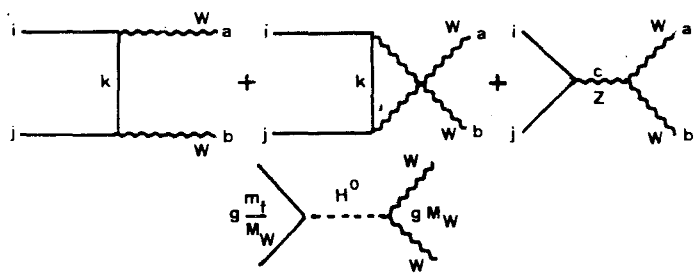


Fig.2.2: Graphs contributing to $f\bar{f} \to W^+W^-$.

Each term separately grows like s in cross section but there is a cancellation because their contributions are proportional to $(L^aL^b)_{ij}$ - $(L^bL^a)_{ij}$ - $if_{abc}L^c{}_{ij}$, and this term vanishes in gauge theories because the L^s form a Lie algebra with structure functions f. However there is still a residue violation of unitarity like $\sqrt{s}$ which is only cancelled by the scalar (Higgs) exchange of Fig.2.2(b).

Obviously this cancellation will not occur until energies above the Higgs mass, and if unitarity is not to be violated this imposes the condition on the Higgs mass m_H

$$m_H \lesssim 1 \text{ TeV} . \qquad (2.13)$$

Of course if this condition is not satisfied unitarity does not break down, but it is achieved only through higher order perturbative terms becoming large, i.e., the alternative is a breakdown of perturbation theory.

(2) The strong interactions (QCD plus quark mass terms) automatically conserve P, C, and strangeness. There is an approximate chiral symmetry which may be realised nonlinearly. This can only occur in a theory with strong forces mediated by vector gluons, an interaction which preserves the fermion chirality. The theory is asymptotically free allowing for explanation of the observed near scaling in large momentum reactions.

(3) The low-energy weak interactions are well described by the currents following from eqs. (2.1) and (2.6) with a single parameter θ_W relating charged and neutral current phenomena

$$\sin^2\theta(M_W)_{\overline{MS}} = 0.215 \pm 0.015 \tag{2.14}$$

(Here $\sin^2\theta$ is corrected for the predicted radiative corrections of the model).

The recent discovery of the W is consistent with the (radiatively corrected) predictions

$$M_W = 83.1^{+3.1}_{-3.8} \text{ GeV} ,$$

$$M_Z = 93.9^{+2.5}_{-2.2} \text{ GeV} . \tag{2.15}$$

2.3. Limitations of the standard model

Although the standard model has many impressive successes it falls short of a complete theory of the strong electromagnetic and weak interactions for several reasons.

(1) There are too many parameters (mainly connected with the Higgs sector) needed to describe the standard model. The model of eqs. (2.1)-(2.7) has seventeen, six quarks and three lepton masses, 3 mixing angles and a phase parameterising CP violation, three gauge couplings and two boson mass scales M_W and M_ϕ. There is a further parameter θ which describes potential strong violation of CP which, it has been realised, must be included due to the anomaly in the axial vector current. There must be added to the (3,2,1) Lagrangian a term

$$L_\theta = \frac{1}{32\pi^2}\,\theta_{QCD} F^a_{\mu\nu}\,\tilde{F}^{\mu\nu}_a , \tag{2.16}$$

where

$$\tilde{F}^{\mu\nu}_a = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\tau} F^a_{\rho\tau} . \tag{2.17}$$

This term violates CP and in order to be consistent with experiment θ_{QCD} must be less than 10^{-9}. It is possible to modify the standard model by adding a further Higgs doublet to replace the $\bar{\phi}^*$ term in eq. (2.5) which generates up quark masses. Then one may show θ_{QCD} is zero automatically, but the model then predicts a light pseudogoldstone state, the axion, which has not been found experimentally.

(2) There is no reason why the matter multiplet structure chosen for the standard model in Fig. 2.1 should be as it is. Also there is no understanding of the family replication.

(3) Charge quantisation is not explained as Y in eq. (2.3) is arbitrary. The relation of quark to lepton charges is also not understood. Also we do not understand why the charged weak interactions should be left handed for both quarks and leptons.

(4) There is no explanation of even the gross features of the mass spectrum. Why are quarks and leptons much lighter than the W and Z? Why are families different in mass, and what relates quark and lepton masses? Neutrinos are massless because one excludes right handed neutrinos, but why are neutrinos different in this respect?

For many people these limitations suggest that the (3,2,1) model is only a step towards a more fundamental theory and that at best it is an effective theory valid up to a scale M_X at which the underlying theory that will answer the above questions appears.

There are two main possibilities for this underlying theory, if it exists. The first is that some or all of the fields of the standard model may be composite and there is some more fundamental level of structure. The second is that the fields of the standard model are themselves fundamental, but they are related by further symmetries, broken at the scale M_X. The latter approach leads to grand unified theories (GUTs) and to supersymmetric theories (SUSY-GUTs), and are the subject of these lectures. In GUTs the additional symmetries are gauge symmetries based on larger Lie algebra than SU(3)xSU(2)xU(1) which may relate particles of the same spin. In the ideal GUT all the fundamental fields of a given spin will belong to a single irreducible representation of a gauge group G and hence their interactions will also be related by the (gauge) transformations of G. In SUSY-GUTs the additional symmetry is based on graded Lie algebra which may relate particles of different spin and ideally may relate all matter particles and all interactions to the fundamental gauge bosons and gauge interactions. How far along this road it is possible to proceed we will discuss in the following sections.

3. SU(5) - THE PROTOTYPE GUT

We turn now to the construction of a complete Grand Unified theory (GUT for short). Our approach is to apply the rules used in the building of the standard SU(3)xSU(2)xU(1) gauge model, but this time for a simple group G with a unique coupling constant g. The steps in this program are

(i) Choose a suitable gauge group G. This, through the requirement of local gauge invariance, specifies the spin one gauge bosons in the model.

(ii) Choose the fermion representations so that the standard SU(3)xSU(2)xU(1) low energy structure emerges. The coupling of these fermions to the gauge bosons is now specified by local gauge invariance.

(iii) Choose scalar representations and scalar couplings to give a phenomenologically acceptable pattern of symmetry breaking of G down to SU(3)xSU(2)xU(1).

(iv) Specify the Yukawa couplings in the theory, ensuring that an acceptable pattern of fermion masses results after spontaneous breakdown.

Applying these rules to the simplest possible model gives the minimal SU(5) theory originally proposed by Georgi and Glashow. In this chapter we will study this model in detail because it contains much of the structure found in a general GUT, and because its phenomenological implications have been extensively studied. Indeed it is only in the minimal model that such predictions may be made precise and though there is no à priori reason why the simplest model should be realistic the success of the very simple SU(3)xSU(2)xU(1) structure in describing low energy phenomena is an encouragement to study its minimal Grand Unified extensions.

3.1. The choice of the gauge group G

The standard model SU(3)xSU(2)xU(1) has four diagonal generators corresponding to T_3 and Y_c of colour, T_3 of weak isospin and Y, and the observed states carry definite values of these quantum numbers. Any group G SU(3)xSU(2)xU(1) must be large enough to contain these four diagonal generators, i.e., it must be at least rank 4. We know that there are only a finite number of groups with the minimal rank 4 which is either simple or is the product of identical simple factors and SU(5) is the only rank 4 group which contains SU(3)xSU(2)xU(1) and can accommodate the spectrum of Fig.2.1.

3.2. The generators and gauge bosons of SU(5)

SU(5) is defined by its fundamental representation, which is the group of 5x5 complex unitary matrices with determinant one. There are 25 independent real 5x5 matrices, i.e., 50 independent complex matrices U. The unitary condition $UU^+=1$ and the unimodular condition det U = 1 give 25 + 1 constraints leaving the 24 independent matrices defining SU(5). U may be written in the form

$$U = \exp(-i \sum_{j=1}^{24} \beta^i L^i), \qquad (3.1)$$

where the 24 generators L^i are Hermitian (ensuring $UU^+=1$) and traceless (ensuring det U=1). The transformation of an arbitrary representation of SU(5) may be described in terms of L^i and it is useful at this stage to choose a convenient basis for L allowing us to identify the 24 associated vector bosons, $V_\mu^{a=1...24}$, of SU(5). We choose the 5x5 matrices L such that the colour group SU(3) acts on the first three rows and columns, while the SU(2) group operates on the last two rows and columns. This gives the SU(3)xSU(2) subgroup structure of SU(5). Thus, for generators normalised such that

$$\mathrm{Tr}\left[L^a, L^b\right] = 2\delta^{ab} \tag{3.2}$$

we have

$$L^a = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \lambda^a & & 0 & 0 \\ & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]; \quad \begin{array}{l} a = 1...8 \\ \text{(generators} \\ \text{of SU(3))} \end{array}, \tag{3.3}$$

where λ^a are the usual Gell-Mann Zweig matrices acting on the colour indices. $V_\mu^{a=1..8}$ are the gauge bosons of SU(5) which are to be identified with the gluons.

$$L^{9,10} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 0 & \multicolumn{2}{c}{\sigma_{1,2}} \end{array}\right] \quad \begin{array}{l} \text{(charged} \\ \text{generators} \\ \text{of SU(2)).} \end{array} \tag{3.4}$$

Here $\sigma_{1,2}$ are the non diagonal Pauli spin matrices. Then $\frac{1}{\sqrt{2}}(V^9 \pm iV^{10})$ are the $W^\pm$ of the standard model.

In addition to the two diagonal generators L^3 and L^8 of the colour group SU(3) there are two further diagonal generators of SU(5) which may conveniently be chosen as proportional to the third component of weak isospin and the weak hypercharge

$$L^{11} = \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ 0 & & & & -1 \end{bmatrix}, \qquad L^{12} = \frac{1}{\sqrt{15}} \begin{bmatrix} -2 & & & & 0 \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ 0 & & & & 3 \end{bmatrix} \tag{3.5}$$

V^{11}_μ and V^{12}_μ are the weak gauge bosons W^3_μ and B_μ respectively. Finally there are the twelve additional Hermitian generators of SU(5) which do not correspond to any of the generators of SU(3)xSU(2)xU(1). These are represented by the matrices $L^{13} \ldots L^{18}$ and $L^{18} \ldots L^{24}$, where

$$L^{13} = \left[\begin{array}{ccc|cc} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right], \quad L^{14} = \left[\begin{array}{ccc|cc} & & & -i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline i & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{array}\right] \tag{3.6}$$

$$L^{19} = \left[\begin{array}{ccc|cc} & & & 0 & 1 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline 0 & 0 & 0 & & \\ 1 & 0 & 0 & & 0 \end{array}\right] \quad L^{20} = \left[\begin{array}{ccc|cc} & & & 0 & -i \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline 0 & 0 & 0 & & \\ i & 0 & 0 & & 0 \end{array}\right]$$

and the others are obtained by putting 1 and $\pm i$ in the same pattern. The associated vector bosons $V^{a=13\ldots18}_\mu$ and $V^{a=19\ldots24}_\mu$ are the twelve new gauge bosons needed in SU(5). They are called X and Y bosons.

In order to calculate the SU(5) gauge invariant interaction we will find it useful to define the 5x5 matrix V_μ by

$$\frac{1}{\sqrt{2}} V_\mu = \frac{1}{2} \sum_{a=1}^{n^2-1} V^a_\mu L^a . \tag{3.7}$$

In terms of the vector bosons introduced above we have

$$V_\mu = \left[\begin{array}{ccc|cc} G^1_1 - \frac{2B}{\sqrt{30}} & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\ G^2_1 & G^2_2 - \frac{2B}{\sqrt{30}} & G^2_3 & X^2 & Y^2 \\ G^3_1 & G^3_2 & G^3_3 - \frac{2B}{\sqrt{30}} & X^3 & Y^3 \\ \hline X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{30}} + \frac{3B}{\sqrt{30}} \end{array}\right] \tag{3.8}$$

we have suppressed the vector indices for clarity.

The G^i_1 are the gluons of eq. (2.1) $\left[G^1_{\mu 2}(G^2_{\mu 1}) = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu)\right]$ etc., $W^\pm_\mu, W^3_\mu$ and B are the gauge bosons of SU(2)xU(1), again appearing in eq. (2.1). The new gauge bosons X^i_μ; i = 1,2,3 transform as colour antitriplets and carry charge $+\frac{4}{3}$.

The new gauge bosons Y^i_μ; i = 1, 2, 3 also transform as colour antitriplets and have charge $+\frac{1}{3}$. They, together with their antiparticles X^i_μ and Y^i_μ make up the 12 new gauge bosons of SU(5).

Now the gauge invariant kinetic energy term generalising eq. (2.1) is

$$L_{Kin} = -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu}{}_a \qquad (3.9)$$

$$= -\frac{1}{4} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

where

$$(F_{\mu\nu})^i_j = \partial_\mu(V_\nu)^i_j - \frac{ig}{\sqrt{2}}(V_\mu)^i_k(V_\nu)^k_j - (\mu \rightleftarrows \nu). \qquad (3.10)$$

3.3. The choice of fermion representations

As discussed in the appendix,we may choose to write the fermions f in terms of either their left and right handed components f_L and f_R, or in terms of states of a definite helicity f_L and f^c_L (or f_R and f^c_R). Here f^c is the charge conjugate spinor $f^c = c\bar{f}^{T}_{L}$. In constructing a GUT the fermions in a representation of the grand unified group G are all of a single helicity and for this reason it is useful to work in the basis with fermions of definite helicity.

The simplest representation of SU(5) is the five dimensional fundamental one ψ_5, which may be represented by the column matrix

$$\psi_5 = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \\ a^4 \\ a^5 \end{pmatrix}. \qquad (3.11)$$

The covariant derivative of ψ_5 is easily written down using the matrix representations derived above

$$(D_\mu\psi_5)^i = \left[\partial_\mu\delta^i_j - \frac{ig}{2}\sum_{a=1}^{24} V^a_\mu(L^a)^i_j\right]\psi_5{}^j \qquad (3.12)$$

$$= \left[\partial_\mu\delta^i_j - i\frac{g}{\sqrt{2}}(V_\mu)^i_j\right]\psi_5{}^j.$$

Remembering that the SU(2) generators operate only on rows 4 and 5 we see that a_1, a_2 and a_3 are unaffected by the operation of the generators of SU(2) and are thus singlets under SU(2). However from the definition of the generators L^a given in eq.(3.3) we see that a_1, a_2 and a_3 form an SU(3) triplet. Similarly we easily see that the last two entries a_4 and a_5 are SU(3) singlets and SU(2) doublets. Putting this together we have the result.

$$5 = (3,1) + (1,2) \text{ under } SU(3)xSU(2). \tag{3.13}$$

If we now refer to the Fig.(2.1), we see that, for the first family, the (3.1) can only be identified with quarks q^i, either d^i_R or u^i_R while the (1,2) must be identified with the $\begin{pmatrix}\nu\\ e\end{pmatrix}_L$ doublet. Using our freedom to rewrite left handed states in terms of their right handed charge conjugate states we have

$$\psi_5 = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \\ e^c \\ -\nu^c \end{pmatrix}_R . \tag{3.14}$$

The final identification of the quark states follows from the fact that if SU(2)xU(1) is embedded in SU(5) the photon, which is a combination of the gauge bosons of SU(2)xU(1), is a gauge boson of SU(5) and thus the charge operator Q must be identified with one of the traceless generators of SU(5). For the multiplet ψ_5 the traceless condition requires

$$Tr(Q) = 3Q_q + Q_{e^c} + Q_{\nu^c} = 0. \tag{3.15}$$

Here we have written the charge of the quarks q_i as Q_q. It is the same for each q_i since they form a colour triplet and the charge operates commutes with $SU(3)_c$. Then

$$Q_q = \frac{1}{3}\, e, \tag{3.16}$$

where e is the charge on the electron. Thus the quarks q_i must be identified with the down quarks d_{i_R}, giving finally

$$\psi_5 = \begin{pmatrix} d^1 \\ d^2 \\ d^3 \\ e^c \\ -\nu^c \end{pmatrix}_R = (3,1) + (1,2) \text{ of } SU(3)xSU(2). \tag{3.17}$$

We may now uniquely identify the charge operator

$$Q = \begin{bmatrix} -\frac{1}{3} & & & & 0 \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ 0 & & & & 0 \end{bmatrix} = \frac{1}{2} L^{11} + \sqrt{\frac{5}{3}} L^{12} \qquad (3.18)$$

this being the only combination of generators of SU(5) which can give the charges for the particle assignments in ψ_5.

The coupling of the gauge bosons to the 5 of fermions is given by the gauge invariant kinetic energy term

$$L^5_{Kin} = i\overline{\psi}_{5_i}\gamma^\mu(D_\mu\psi_5)^i$$
$$= i\overline{\psi}_{5_i}\gamma^\mu(\partial_\mu\delta^i_j - i\tfrac{g}{\sqrt{2}}(V_\mu)^i_j)\psi^j_5 . \qquad (3.19)$$

Already some of the beautiful features of SU(5) are evident. The charge is quantised as expected but as a bonus we find the prediction that quarks must carry third integral charges because quarks come in 3 colours (giving the factor 3 on evaluating the trace in eq. (3.15)). Moreover we found that the right handed SU(2) doublets of leptons must be partnered by a right handed SU(2) singlet of quarks (if we had tried to build ψ_5 using a left handed SU(2) doublet of leptons the charge condition eq.(3.15) would have been impossible to satisfy with triplets of quarks). This means SU(5) predicts that right handed quarks are SU(2) singlets in accord with experiment.

What about the assignment of the remaining quarks and leptons of the standard model shown in Fig.(2.1). We may build further representations of SU(5) by taking products of the fundamental 5. The simplest possibility is to take the product of two 5s. This gives

$$5 \times 5 = 10 + 15. \qquad (3.20)$$

The 10 is the antisymmetric product ψ^{ij}, where $\chi^{ij} = \frac{1}{\sqrt{2}}\ a^i a^j - a^j a^i$; $i,j = 1...5$ (3.21)

and a^i a^j are components of the two 5s in eq.(3.14). We have just seen for $i,j = 1,2,3$, a^i and a^j transform as (3,1) under SU(3)xSU(2) and thus this product contains the product of two (3,1) representations. The antisymmetric product of two SU(3) triplets is a $\bar{3}$($3 \times 3 = 6 + \bar{3}$). Thus χ^{ij}, $i,j = 1,2,3$ transforms as ($\bar{3}$,1) and must be identified with the left handed u^c_L quarks.

This may be written as

$$u^c_{k_L} = \frac{1}{\sqrt{2}} \varepsilon_{kij}\, a^i a^j;\; i,\, j,\, k = 1,2,3 \tag{3.22}$$

and so multiplying by $\varepsilon^{ki'j'}$ and using the properties of the totally antisymmetric matrix ε^{ijk} we find

$$\chi^{ij} = \varepsilon^{ijk} u^c_{k_L}. \tag{3.23}$$

χ^{j4}(χ^{j5}) represents a colour triplet which has the third component of weak isospin $\frac{1}{2}$($-\frac{1}{2}$), being the product of (3,1) x (1,2) representations. Thus we identify

$$\chi^{j4} = (u^j)_L;\quad j = 1,\, 2,\, 3, \tag{3.24}$$

$$\chi^{j5} = (d^j)_L;\quad j = 1,\, 2,\, 3.$$

Finally ψ^{45} is clearly a singlet under SU(3)xSU(2) because it is the antisymmetric product of two SU(3) singlets, SU(2) doublets. It thus neatly accommodates the last remaining state of the lowest family namely the e^+_L.

Putting all this together in matrix notation by the ten dimensional representation is $(\chi_{10})^i_j = \frac{1}{\sqrt{2}} \chi^{ij}$

$$\chi_{10} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u^c_3 & -u^c_2 & u^1 & d^1 \\ -u^c_3 & 0 & u^c_1 & u^2 & d^2 \\ u^c_2 & -u^c_1 & 0 & u^3 & d^3 \\ u^1 & u^2 & u^3 & 0 & e^+ \\ d^1 & d^2 & d^3 & -e^+ & 0 \end{bmatrix}_L \tag{3.25}$$

= (3,2) + (3,1) + (1,1) of SU(3)xSU(2).

It is now straightforward to write the gauge invariant kinetic energy term for the 10 dimensional representation

$$L^{10}_{Kin} = i\, \bar{\chi}_{10ki}\, \gamma^\mu (\partial_\mu \delta^i_j - i\sqrt{2}\, g (V_\mu)^i_j)\, \chi^{jk}_{10}. \tag{3.26}$$

By construction of the SU(3)xSU(2)xU(1) properties of the fermions in ψ_5 and χ_{10} are in accord with the standard assignments of Fig.2.1. It is a remarkable fact that a single family fits neatly into the fifteen states available in the $\bar{5}$ + 10 representations.

Finally the two remaining families of Fig.2.1 are assigned to two copies of $\bar{5}$ + 10 representations.

3. 4. Fermion interactions in SU(5)

Having assigned the fermions to multiplets of SU(5) the couplings of the gauge bosons are defined as in eqs. (3.19 and 3.22). By construction the gauge couplings of the SU(3)xSU(2)xU(1) bosons are as given in eq.(2.1).

The coupling of the neutral fields W^3_μ and B_μ to matter is determined in general by the covariant derivative

$$D_\mu = \partial_\mu - i\frac{g}{2}(W^3_\mu T^{11} + B_\mu T^{12}), \qquad (3.27)$$

where T^{11} and T^{12} are the representations of the generators L^{11} and L^{12} for the matter representation. (As in eq.(3.19) for ψ_5 or eq.(3.26) for χ_{10}). We may, using eq.(2.10), rewrite this in terms of the photon A_μ and the neutral weak boson Z_μ.

$$\begin{aligned} D_\mu &= \partial_\mu - i\frac{g}{2}\Big[(\sin\theta_W T^{11} + \cos\theta_W T^{12})A_\mu \\ &\quad + (\cos\theta_W T^{11} - \sin\theta_W T^{12})Z_\mu\Big] \\ &= \partial_\mu - i\Big[eQA_\mu + gQ_Z Z_\mu\Big] \end{aligned} \qquad (3.28)$$

the charge operator is defined by eq.(3.18)

$$Q = \frac{1}{2}(T^{11} + \sqrt{\frac{5}{3}}\, L^{12}). \qquad (3.29)$$

Using this in eq.(3.24) gives

$$\tan\theta_W = \sqrt{\frac{3}{5}}\ ; \quad \sin\theta_W = \sqrt{\frac{3}{8}} \qquad (3.30)$$

and

$$\frac{g}{2} = \sqrt{\frac{2}{3}}\, e\,. \qquad (3.31)$$

Such a prediction for $\sin^2\theta_W$ is expected since it is (cf.eq.(2.11)) related to g_2 and g_1 and in SU(5) these are both related to the single coupling g. However the value is clearly inconsistent with the experimental result. This appeared disastrous for the SU(5) model until Georgi, Quinn and Weinberg realised that this prediction applies at a scale M_X at which SU(5) is a good symmetry and that it must be corrected when comparing with ($t_p \gtrsim 10^{30}$yrs) imposes a stringent limit on the mass of the baryon number violating X and Y bosons. As we will discuss a proton lifetime of 10^{30} years corresponds to an X or Y boson mass of $O(10^{15}$ GeV). Thus SU(5) Grand Unification requires the appearance of a scale for new physics 10^{13} times that of the W and Z bosons responsible for the weak interactions. In the next section we will discuss how this may come about as a result of spontaneous symmetry breakdown.

3.5. Spontaneous symmetry breakdown of SU(5)

In order to achieve a phenomenologically acceptable model it is necessary to break SU(5) in two stages

$$SU(5) \xrightarrow{M_X} SU(3) \times SU(2) \times U(1) \xrightarrow{M_X} SU(3) \times U(1)_{em} . \qquad (3.32)$$

Other possible breaking sequences are possible, for example SU(5) could first break to SU(4)xU(1), but this does not happen for the simplest choices of scalar potentials.

With this pattern of symmetry breaking at the first stage the X and Y bosons receive a mass of order M_X leaving the other twelve gauge fields of SU(3)xSU(2)x U(1) massless. The second stage gives mass of order M_W to the $W^\pm$ and Z bosons.

The first stage of breaking is achieved through an adjoint (24) of scalars Σ_a. The coupling of gauge fields to Σ is given via the kinetic Lagrangian

$$L_{Kin} = \frac{1}{2} \sum_{a=1}^{28} \left(D_\mu \Sigma\right)^+_a \left(D_\mu \Sigma\right)_a , \qquad (3.33)$$

where the covariant derivative is

$$D_\mu \Sigma = \partial_\mu \Sigma + ig \left[V_\mu^a \frac{L^a}{2}, \Sigma\right] \quad ; \quad \Sigma \equiv \Sigma_a L^a/2 , \qquad (3.34)$$

and $$L^\Sigma_{Kin} = Tr \left[(D_\mu \Sigma)^+ D^\mu \Sigma\right] . \qquad (3.35)$$

We may now easily discuss what happens when Σ develops a vacuum expectation value. Through the kinetic energy term L^Σ_{Kin} and using eq.(3.11) we find a mass matrix for the vector fields of the form

$$\frac{1}{2} g^2 Tr \left[V_\mu, \langle\Sigma\rangle\right]^2 \qquad (3.36)$$

$$\equiv m^2_{ab} V^a_\mu V^{\mu, b} ,$$

where $\langle\Sigma\rangle$ denotes the vacuum expectation value of Σ.

If, as a result of Σ acquiring a vev, the X and Y bosons are to receive a mass while the remaining bosons remain massless the form of $\langle\Sigma\rangle$ is restricted to be diagonal

$$\langle\Sigma\rangle = v\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} = -\frac{v}{2}\, L^{12}. \tag{3.37}$$

Because $\langle\Sigma\rangle$ is chosen to be proportional to the unit matrix in the subspace of SU(3) (row and columns 1 to 3) and in the subspace of SU(2) (row and columns 4 and 5), it is clear that the only non-zero commutators of L^a with $\langle\Sigma\rangle$ are for $L^{a=13...24}$, and hence only X and Y get masses.

Using eq.(3.36) we immediately find

$$m_X^2 = m_X^2 = \frac{25}{8} g^2 v^2. \tag{3.38}$$

The second step in building the scalar sector is to construct a potential for Σ which leads to the desired vev. This should be the most general form of scalar couplings of dimension <4 consistent with gauge invariance and, possibly, invariance under additional discrete symmetries. All such terms are needed to build a renormalisable theory. The possible gauge invariant quantities involving the Σ field are easily formed by taking the trace of powers of Σ. Hence

$$-L^{\Sigma}_{int} = + V(\Sigma) = -\mu^2 Tr(\Sigma^2) + \frac{1}{4} a\left[Tr(\Sigma^2)\right]^2 + \frac{1}{2} bTr(\Sigma^4). \tag{3.39}$$

In this equation we have dropped a possible cubic term by imposing a discrete symmetry under $\Sigma \to -\Sigma$. This simplifies the potential and the subsequent analysis for minima, but is not essential in constructing a suitable potential. $V(\Sigma)$ has a unique minimum with $\Sigma = \langle\Sigma\rangle$ of eq.(3.37) provided $b > 0$, $\mu^2 > 0$ and $a > (-\frac{7}{5})b$. Assuming this to be true v is found solving the equation

$$\mu^2 = \frac{15}{2} av^2 + \frac{7}{2} bv^2. \tag{3.40}$$

Thus the introduction of an adjoint representation Σ allows us to break SU(5) to SU(3)xSU(2)xU(1) as desired. The second stage of breaking, the electroweak breaking of the standard model requires the introduction of a Higgs field with components transforming as a doublet under weak SU(2). The simplest possibility is to introduce a 5 of Higgs H. From our discussion of section (3.3) we know H has the form

$$H = \begin{pmatrix} h^1 \\ h^2 \\ h^3 \\ h^+ \\ -h^o \end{pmatrix} = (3,1) + (1,2) \text{ of } SU(3)xSU(2). \tag{3.41}$$

We now, introduce the potential

$$V(H) = -\frac{1}{2}\nu^2 H^2 + \frac{1}{4}\lambda(H^2)^2 \;;\; \nu^2, \lambda > 0 . \tag{3.42}$$

This drives a vacuum expectation value for H_5 which, for the moment, we assume to be along the neutral direction

$$<(H)^5> = <-h^\circ> = v_0 \;, \tag{3.43}$$

where

$$\nu^2 = \frac{\lambda}{2} v_0^2 . \tag{3.44}$$

This will induce the desired pattern of SU(2)xU(1) breaking with

$$M_W^2 = m_Z^2 \cos^2\theta_W = \frac{1}{4} g_2^2 v_0^2 . \tag{3.45}$$

There are, however, two problems with this potential. The first is that, to $O(\frac{v_0}{v})$, the colour triplet of Higgs fields h^i remains massless. It is obvious that $V(H)$ can only give masses $O(M_W)$ to h^i and to $O(\frac{v_0}{v})$, these fields are not "eaten" by the $\overline{Y}^i_\mu$ gauge bosons. However, these triplet Higgs fields have baryon number violations couplings to quarks and leptons and will mediate proton decay far too fast.

The second problem is that we have not allowed cross terms coupling the H and Σ fields. Even if we try to omit such terms they will be induced by radiative corrections, which, moreover, are divergent. Thus without the cross terms in the original Lagrangian to act as counter terms the theory has unregulated divergences and is nonrenormalisable. If we add all gauge invariant cross terms of dimensions < 4 to avoid this we find a potential

$$V(\Sigma,H) = \alpha H^2 \, Tr(\Sigma^2) + \beta \overline{H}\Sigma^2 H . \tag{3.46}$$

Actually these terms give a mass to the triplet Higgs fields

$$m^2_{h^i} = -\frac{5}{2}\beta v^2 \tag{3.47}$$

which is $O(M_X^2)$ and consequently reduces Higgs mediated proton decay to an acceptable level. Thus the terms of eq.(3.46) solve both of the problems discussed above. Now eqs.(3.40) and (3.44) which determine v and v_0 are modified by the new terms of eq.(3.46). They become

$$\mu^2 = \frac{15}{2} a v^2 + \frac{7}{2} b v^2 + \alpha v_0^2 + \frac{9}{30}\beta v_0^2 , \tag{3.48}$$

$$\nu^2 = \frac{1}{2}\lambda v_0^2 + 15\alpha v^2 + (\frac{9}{2} - 3\epsilon)\beta V^2 .$$

Note that for β-ve (as required by eq.(3.47)) the potential now favours a vacuum expectation value for H along the 4,5 directions rather than along the 1, 2, 3 direction. This is as desired and the resulting vacuum expectation value may be rotated into the 5 direction as in eq.(3.43).

However a new problem arises because of the cross terms in eq.(3.46). When Σ acquires its large vacuum expectation value of order V these terms generate contributions to the mass of the doublet fields in H_5 of order V. In order that the net doublet mass is of order v_0, with $\frac{v_0}{v} \approx \frac{M_W}{M_X} \approx 10^{-12}$, so that the vacuum expectation value of $h^{\bullet}$ can also be of order v_0, there must be a delicate cancellation between the various mass terms. Thus in eq (3.48) we need

$$\nu^2 - (15\alpha + \frac{9}{2}\beta)\, v^2 = \frac{1}{2}\lambda v_0^2 = 0(10^{-24}) v^2. \qquad (3.49)$$

Even if this cancellation could be arranged at tree level, radiative corrections discussed above will re-introduce the problem generating a mass for H_5 of order $\frac{g^2}{4\pi}$ v; the relation eq.(3.49) must be valid with the renormalised (running) parameters ν, α, β, and v evaluated at a scale corresponding to the minimum of the potential. The difficulty in explaining the origin of this delicate and unnatural cancellation is known as the hierarchy problem. We will return to a fuller discussion of it later.

If we accept this cancellation the terms of eqs(3.40), (3.42) and (3.46) provide a completely acceptable potential which generates the desired pattern of symmetry breakdown of eq.(3.32).

3.6. Fermion masses in SU(5)

The left handed fermions in SU(5) transform as $\bar{5}$ + 10. As discussed in the appendix , fermion masses involve the product of two left handed fermion fields, so the representation content of these masses is obtained from the product ($\bar{5}$ + 10) x ($\bar{5}$ x 10). We have

$$\bar{5} \times 10 = 5 + \overline{45}$$
$$10 \times 10 = \bar{5} + 45 + 50$$
and
$$\bar{5} \times \bar{5} = \overline{10} + \overline{15}. \qquad (3.50)$$

A mass term in the Lagrangian must be gauge invariant so clearly, as in the standard model, there can be no bare fermion masses for there is no singlet component in these products. Masses will only arise via spontaneous breaking through gauge invariant couplings of these fermion products to Higgs scalars, and the form of these masses will depend on which scalars are present. Let us first discuss the pattern of masses in the minimal model with only the 5 and 24 of Higgs scalars needed in section 5.5 to break SU(5) to $SU(3)_c \times U(1)_{em}$. None of the products include a 24 so the adjoint Σ^a_b does not couple to fermions. As a result the scale for fermion masses is $O(M_W)$ through their coupling to H and not $O(M_X)$ which would be quite unacceptable. The possible Yukawa couplings to H are

$$L \doteq (\bar{\psi}_{R_i})_\alpha M^D_{ij} \chi^{\alpha\beta}_{Lj} \bar{H}^+_\beta - \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta\epsilon} \chi^{T\alpha\beta}_{Li} C M^U_{ij} \chi^{\gamma\delta}_{Lj} H^\epsilon + h.c. \qquad (3.51)$$

Here i and j are generation indices and $\alpha,\beta,\gamma,\delta$, are SU(5) indices. When H develops a vacuum expectation value as in eq.(3.41) it will generate down quark and charged masses through the term proportional to M^D and up quark masses through the term proportional to M^U.

We may assume M^D has been diagonalised by unitary rotations in flavour space of the fermion fields

$$\psi_{R_i} \to \left[U_1\right]_{ij} \psi_{R_j} \quad ; \quad \chi_{L_i} \to \left[U_2\right]_{ij} \chi_{L_j} . \qquad (3.52)$$

Then we obtain the masses

$$\begin{aligned} m_d &= m_e = M^D_{11} v_0 , \\ m_s &= m_\mu = M^D_{22} v_0 , \\ m_b &= m_\tau = M^D_{33} v_0 , \end{aligned} \qquad (3.53)$$

These predictions must be corrected by radiative corrections and we discuss these in the next section. Up quark masses come from the term involving M^U, but because there are right handed neutrinos, and hence no neutrino masses, there are no further relations between quark and lepton masses.

3.7. Mixing Angles in SU(5)

The general structure of the fermion mass terms found above is

$$L_F = -\bar{u}_R M^u u_L - \bar{d}_R M^d d_L - \bar{e}^+_R M^e e^+_L + h.c. , \qquad (3.54)$$

where for the general Higgs structure the mass matrices M^u, M^d and M^e are arbitrary $N_g x N_g$ matrices. For the minimal Higgs structure with just a 5 and 24 (or indeed for an arbitrary number of 5) we saw above that $M^d = M^e$ and also, since χ occurs twice in eq.(3.51) coupled by the totally antisymmetric tensor, M^u is symmetric in generation space $M^u = M^{uT}$.

As in the standard SU(3)xSU(2)xU(1) model, when these mass matrices are diagonalised and the theory is expressed in terms of mass eigenstates mixing angles and phases appear in the interactions. It is clearly important to establish whether the mixing angles relevant to proton decay are related to the Kobayashi-Maskawa angles in the standard model.

We first form diagonal masses $M_D{}^d$, $M_D{}^e$ and $M_D{}^U$ by unitary matrices $U^d{}_L$, $U^d{}_R$, $U^e{}_L$, $U^e{}_R$ and $U^u{}_L$ respectively where

$$M^d{}_D = U^{d+}{}_R M^d U^d{}_L \text{ etc.} \tag{3.55}$$

The mass eigenstates $d^m_{L,R}$, $e^{+m}_{L,R}$ and $u^m_{L,R}$ are given in terms of the current eigenstates by

$$d_{L,R} = U^d_{L,R} d^m_{L,R} \text{ etc.} \tag{3.56}$$

To exhibit the effects of these rotations on the interactions in the theory it is convenient to introduce a shorthand notation for the fermion representations in which the colour index is suppressed. Thus for the first family

$$5 : \begin{pmatrix} d \\ \hline \bar{e} \\ \nu_e \end{pmatrix}_R \qquad 10 : \left(\begin{array}{c|cc} u^c & u & d \\ \hline u & & \\ & & e \\ d & & \end{array}\right)_L . \tag{3.57}$$

Here the SU(2) doublet partners are grouped in a submatrix while the singlets occur on their own. The X and Y bosons couple fermions in different submatrices.

The rotation of eq.(3.56) now, for example, transform ψ_{5_1}, the 5 representation of the first family, as follows

$$\psi_{5_1} = \begin{pmatrix} d \\ \hline \bar{e} \\ \nu_e \end{pmatrix}_R = \begin{pmatrix} U_R^d & d^m \\ \hline U_R^{\bar{e}} & \bar{e}^m \\ \nu_e & \end{pmatrix}_R , \tag{3.58}$$

where now $\bar{e}^m$ and d^m are column vectors in generation space with the charged antileptons and down quarks as elements respectively. For massless neutrinos we may redefine the neutrino fields $\bar{\nu}_e = U_R^{\bar{e}}\bar{\nu}_e^m$ etc.so that

$$\psi_5 = U_R^{\bar{e}} \begin{pmatrix} U_R^{\bar{e}+} U_R^d d^m \\ \bar{e}^m \\ -\bar{\nu}_e^m \end{pmatrix}_R , \tag{3.59}$$

We are free to pick a basis $U_R^{\bar{e}\dagger}\psi_5$ in which $\bar{e}_i^m$ are diagonal and define new unitary matrices $A_R^d = U_R^e U_R^d$ so that the representations involving mass eigenstates may be written

$$5: \quad \begin{pmatrix} A^d_R & d^m \\ & \bar{e}^m \\ & -\bar{\nu}^m_e \end{pmatrix}_R \qquad (3.60)$$

and similarly in a basis in which d_{L_i} are diagonal

$$10: \quad \left(\begin{array}{c|c} u^{m^c} \; A^{u+}_R & A^u_L u^m \qquad d^m \\ \hline A^u_L \; u^m & A^{e-}_L e \\ d^m & \end{array}\right)_L . \qquad (3.61)$$

Now for the low energy SU(3)xSU(2)xU(1) structure only the entries involving doublet states are relevant since the neutral currents are unaffected by this rotation and the charged currents involve only the doublet states. Thus from eqs.(3.60) and (3.61) we see that only A^a_L is involved for the weak currents connecting the u_L, d_L doublet fields.

In diagonalising the mass matrix eq.(3.48) there is an arbitrariness corresponding to $\psi_L \to K\psi_L$, $\psi_R \to K^* \psi_R$, where K is a diagonal matrix

$$K = \begin{pmatrix} e^{i\alpha_1} & & 0 \\ & e^{i\alpha_2} & \\ 0 & & e^{i\alpha_{N_g}} \end{pmatrix} . \qquad (3.62)$$

Thus by using this freedom for the u_L and d_L quarks we can always rewrite A^u_L in the form

$$A^u_L \to K^*_{d_L} A^u_L K_{u_L} . \qquad (3.63)$$

We now choose K_{d_L} and K_{u_L} such that we remove the phases from the first row and column of A^u_L

$$K^*_{d_L} = \begin{pmatrix} e^{i\phi_{11}} & & 0 \\ & e^{i\phi_{12}} & \\ 0 & & e^{i\phi_{1N_g}} \end{pmatrix} . \qquad (3.64)$$

$$K_{u_L} = \begin{pmatrix} e^{i\phi_{11}} & & 0 \\ & e^{i\phi_{21}} & \\ 0 & & e^{i\phi_{N_g a}} \end{pmatrix} .$$

The resulting matrix is in the form of the standard Kobayashi Maskawa matrix of eq(2.7)

$$K^{*}_{d_L} A^{u}_{L} K_{u_L} \equiv U_{Km} \qquad (3.65)$$

and so finally the representations have the form

$$5: \begin{pmatrix} A^{d}_{R} \quad d^{m} \\ \bar{e}^{m} \\ -\bar{\nu}^{m}_{e} \end{pmatrix}_R \qquad 10: \left(\begin{array}{c|c} U^{m^c} \quad A^{u+}_{R} & U_{KM} U^{m} \quad d^{m} \\ \hline U_{KM} u^{m} & A_L{}^{\bar{e}} \bar{e} \\ d^{m} & \end{array}\right) \qquad (3.66)$$

with new unitary matrices A^{c+}_{L} and A^{u+}_{C} and A^{d}_{R}. Clearly now the standard model SU(3)xSU(2)xU(1) charged and neutral gauge interaction of section (2) are recovered. However the new interactions of the X and Y bosons involve the new matrices A^{e+}_{L}, A^{u+}_{R} and A^{d}_{e}. One could even choose them so that, for example, the u and d quarks occur in multiplets only containing the τ lepton. In this case proton decay could not occur at tree level since the proton is too light to decay to a τ.

In the case of the minimal model, however, we have the constraints on the mass matrix $M^d = M^e$, which imply $U^{d}_{L,R} = U^{e+}_{L,R} = I$ (in the basis of eq(3.60)). Also $M^u = M^{UT}$ implies it can be diagonalised by a unitary transformation. Thus $U^{u}_{L} = U^{u+}_{R}$ up to the phase ambiguity discussed above, i.e., $A^{u^*}{}_{R} = A^{u}_{L} K^{*}_{u_L}$.

So in this case the SU(5) multiplets are

$$5: \begin{pmatrix} d^{m} \\ \hline \bar{e}^{m} \\ -\bar{\nu}^{m}_{e} \end{pmatrix}_R \qquad 10: \left(\begin{array}{c|c} U_{KM} K^{*}_{u_L} u^{m^c} & U_{KM} U^{m} d_{m} \\ \hline U_{KM} u^{m} & \bar{e} \\ d^{m} & \end{array}\right)_L . \qquad (3.67)$$

In this case the interactions of the X and Y bosons with fermions are described in terms of the Kobayashi-Maskawa matrix plus the (N_g-1) phases, extra sources of CP violation observable only in nucleon decay processes. Note that in this case we do not have the freedom to rotate away proton decay.

4. THE CLASSIC PREDICTIONS OF GRAND UNIFIED THEORIES

In this chapter we discuss the prediction of grand unified theories for gauge couplings, quarks and lepton (including neutrino) masses and proton decay. We will concentrate on the predictions of the minimal SU(5) model since these have been most fully worked out. Many of the qualitative features remain the same in most Grand Unified theories.

4.1. Gauge couplings

In the previous section we derived the relations

$$\sin^2{}_W = \frac{3}{8} = \frac{g_1^2}{g_1^2+g_2^2} \cdot \qquad (4.1)$$

In addition we have a relation for the strong and weak couplings g_3 and g_2

$$g_3 = g_2 = g \cdot \qquad (4.2)$$

These results are phenomenologically unacceptable, the strong coupling g_3 being much bigger than the weak coupling constant g_2 and $\sin^2\theta_W = \frac{3}{8}$. It was the realisation however of Georgi, Quinn and Weinberg that the above predictions apply at a scale $O(M_X)$ at which SU(5) is a good symmetry, and that in comparing with experiment it is necessary to include radiative corrections to continue the coupling and masses to a scale O(1 GeV) at which laboratory measurements are made. That this is a possibility follows from the calculation of these radiative corrections which gives for the evaluation of the effect couplings of the SU(3)xSU(2)xU(1) model below M_X. We know that the renormalised couplings depend on the energy scale E at which they are measured through calculable radiative corrections

$$\frac{1}{\alpha_3(E)} = \frac{1}{\alpha_{GU}} + \frac{1}{6}\,(4N_G-33)\,\ln(\frac{M_X}{E}) + \ldots$$

$$\frac{1}{\alpha_2(E)} = \frac{\sin^2(E)}{\alpha_{1m}(E)} = \frac{1}{\alpha_{GU}} + \frac{1}{6}\,(4N_G-22+\frac{1}{2})\,\ln(\frac{M_X}{E}) + \ldots, \qquad (4.3)$$

$$\frac{1}{\alpha_1(E)} = \frac{3}{5}\,\frac{\cos^2(E)}{\alpha_{1m}(E)} = \frac{1}{\alpha_{GU}} + \frac{1}{6}\,(4N_G+\frac{3}{10})\,\ln(\frac{M_X}{E}) + \ldots,$$

where N_G is the number of generations, $\alpha_i = \frac{g_i^2}{4\pi}$, and α_{GU} is related to the single SU(5) coupling $\alpha_{GU} = \frac{g^2}{4\pi}$.

We see that the couplings of SU(3) and SU(2) decrease with increasing energy (corresponding to the asymptotic freedom of nonabelian gauge groups) while the U(1) coupling increases. Moreover the negative coefficients of the log term of eq.(4.3) is larger for SU(3) than SU(2) so that α_3 falls faster than α_2. The result is sketched in Fig. 4.1 and shows that even though α_3 is initially larger than α_2 and α_1 it will eventually become equal to them at some large scale M_X. Eqs.(4.3) contain one unknown parameter M_X which may be determined in terms of α_3 and α_{em} via a combination or these three equations. Then it may be used to make a prediction for the third coupling (or more conveniently $\sin^2\theta_W$),

$$\frac{3}{5\alpha_{em}(\mu)} - \frac{3}{5\alpha_3(\mu)} = \frac{201}{30}\ln\frac{M_X}{\mu}\,, \qquad \sin^2\theta_w(\mu) = \frac{3}{8}\,(1 - \frac{109}{18}\ln\frac{M_X}{\mu}). \quad (4.4)$$

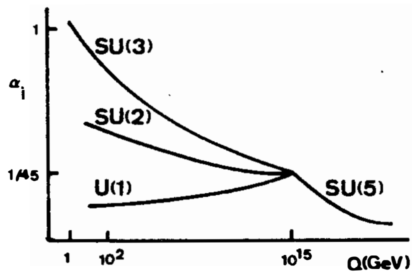


Fig 4.1. Behaviour of the (3,2,1) couplings with energy.

These lead to

$$M_X = \mu \exp \frac{30}{201}\left[\frac{3}{5\alpha_{em}(\mu)} - \frac{3}{5\alpha_3(\mu)}\right]$$

$$\simeq 2 \times 10^{15} \text{ GeV}$$

and $\sin^2 \theta(\mu) = 0.21$.

The reason such a large scale emerges follows from the fact that the evolution of eq.(4.3) is only logarithmic so that $\frac{M_X}{\mu}$ is exponentially related to the coupling constant differences. This is a remarkable result for, as we saw in section (3.4) it is also crucial that this scale should be large to inhibit proton decay mediated by X and Y bosons. The agreement of $\sin^2\theta_W(\mu)$ with experiment is also impressive, particularly as the initial value of $\frac{3}{8}$ was in clear disagreement. These two facts, more than any others, encourage us to believe that GUTs have something to do with reality.

A more careful analysis, including threshold effects and higher order corrections gives for $50\text{MeV} < \Lambda_{\overline{MS}} < 500$ MeV (4.5)

$$M_X = 3.6\left\{{}^{+3.4}_{-3.2} \times 10^{14} \text{ GeV for } M_t = 20 \text{ GeV}\right.$$

and $\sin^2\theta_W = 0.206^{+0.016}_{-0.004}$. (4.6)

4.2. Quark and lepton masses

In minimal SU(5) we derived the relations

$m_d = m_e$; $m_s = m_\mu$; $m_b = m_\tau$.

As was the case for gauge couplings the masses require renormalisation and the mass relations apply for masses defined at scales > M_X where SU(5) is a good symmetry and SU(5) symmetry breaking effects can be neglected. As was the case for couplings radiative corrections cause the masses to "run" as the scale at which they are measured changes. Graphs giving these corrections are shown in Fig.4.2. Computing these

Fig.4.2. Radiative corrections to quark and lepton masses. The cross denotes a quark or lepton mass and the curly line denotes a gluon.

gives

$$\left[\frac{m_b(\mu^2)}{m_t(\mu^2)}\right] = \left[\frac{m_b(M_X^2)}{m_t(M_X^2)}\right]\left[\frac{\alpha_3(\mu^2)}{\alpha_3(M_X^2)}\right]^{\frac{4}{11-\frac{2}{3}n_q}}\left[\frac{\alpha_1(\mu^2)}{\alpha_1(M_X^2)}\right]^{\frac{3}{2n_q}}, \qquad (4.7)$$

where n_q is the number of quarks with mass $< \mu^2$. Using this formula, and the value of m_t we can predict the b quark mass. This is "measured" in e^+e^- annihilation as one half the Υ mass ($b\bar{b}$ bound state) and it is reasonable to interpret this value for m_b as $m_b(E_0)$ where $E_0 = 2m_b(E_0) = m_\Upsilon$. From eq(4.7) with six quark flavours

$$m_b = 5\text{-}5.5\,\text{Gev for } 50 \leqslant \Lambda_{\overline{MS}} \leqslant 500\text{ MeV}. \qquad (4.8)$$

Including threshold and higher order effects this estimate can be refined and it is found

$$\begin{aligned} m_b &= 5.3\text{ Gev} \quad \text{for } n_q = 6 \text{ for } \Lambda_{\overline{MS}} = 300\text{ MeV} \\ &\ 5.8\text{ Gev} \quad n_q = 8 \\ &\ 6.9\text{ Gev} \quad n_q = 10. \end{aligned} \qquad (4.9)$$

The experimental value is 5.5Gev remarkably close to the prediction for six quark flavours. Indeed the flavour dependence is so strong that we may argue that minimal SU(5) is consistent with the b quark mass for not more than three families.

The prediction for the strange quark mass is

$$m_s = 500\text{ MeV for } n_q = 6.$$

The experimental determination of m_s varies between 150 MeV (from sum rule estimates) to 300 MeV (from bag model estimates). These are substantially lower than the SU(5) prediction.

Even worse are the predictions for m_d. For convenience we quote this as the ratio $\frac{m_d}{m_s}$ which is not renormalised substantially relative to $\frac{m_e}{m_\mu}$. Thus the minimal SU(5) prediction is

$$\frac{m_d}{m_s} = \frac{m_e}{m_\mu} = \frac{1}{207}. \qquad (4.10)$$

This is to be compared with the value obtained from current algebra $\frac{m_d}{m_s} = \frac{1}{24}$.

While the discrepancy for the strange quark might be ascribed to uncertainties in its measurement, the discrepancy with $\frac{m_d}{m_s}$ seems impossible to accept.

One possible explanation of this discrepancy that grand unified structure at a scale above M_X may give small additional contributions to fermion masses. Since the d quark is very light ($\simeq$ 10 Mev) it will be the most significantly affected by small corrections and this could account for the bad prediction eq(4.10). A second possibility, is to extend the minimal Higgs structure to include a 45 dimensional representation. In this case, however, the mixing angles relevant to proton decay do not reduce to the Kobayashi-Maskawa angles as in eq.(3.67).

4.3. Nucleon decay in SU(5)

One of the features we noted above was the fact that the X and Y gauge bosons did not conserve B and L separately but only the combination (B-L). Guts seek to combine quarks and leptons in a single multiplet and so it is a general property of them that there will be gauge interactions coupling leptons and quarks. This does not mean that baryon number is violated. For example in eq(4.11) the terms coupling X and Y to the $\bar{5}$ all have $B = -\frac{1}{3}$, $L = -1$, so if this was the only fermion representation one could ascribe these quantum numbers to the $\bar{X}$ and $\bar{Y}$ fields and B and L would separately be conserved. It is the combination of this plus the coupling to the 10, with quarks and antiquarks in the same representation, that means B and L are separately violated for the terms involving $\overline{u_L{}^c}\gamma^\mu u_\lambda$ have $B = \frac{2}{3}$, L = 0 and no choice of B and L for X and Y will conserve B or L.

In the "age of the gauge". where all (continuous) symmetries are expected to be gauge symmetries, one expects the only conserved numbers to correspond to massless gauge fields. Apart from the photon, gluons and the graviton we know of no massless fields so it seems reasonable to expect there are no new absolutely conserved quantities such as B or L. In SU(5), B and L are violated, but at a very slow rate because of the large scale of breaking. What about the residual (B-L) symmetry? In SU(5) this is an accidental global symmetry (it is easy to check that it is preserved by the Yukawa couplings of eq(3.51). However further Yukawa couplings involving new Higgs representation will violate it at a rate depending on the mass of these new scalar states. Alternatively, following our principle that all symmetries should be gauge symmetries one may enlarge the gauge group to include the (B-L) generator. Then (B-L) will be violated at a rate depending on the mass of new gauge boson.

In the minimal SU(5) scheme the X and Y interactions violate baryon number. In terms of mass eigenstates of eq(3.61) in the minimal model these are

$$\frac{g}{\sqrt{2}}\bar{X}^{\alpha}\left[\bar{d}_{R_\alpha}\gamma^{\mu} e_R^{+} + \bar{d}_{L_\alpha}\gamma^{\mu}e_L^{+} + \varepsilon_{\alpha\beta\gamma}\bar{u}_L^{C\gamma} K\gamma^{\mu}u_L^{\beta}\right]$$
$$+\frac{g}{\sqrt{2}}\bar{Y}^{\alpha}_{\mu}\left[-\bar{d}_{R\alpha}\gamma^{\mu}\nu_R^{C} - \bar{u}_{L\alpha}U^{+}_{KM}\gamma^{\mu}e_L^{+} + \epsilon_{\alpha\beta\gamma}\bar{u}_L^{C\gamma}KU^{+}_{KM}\gamma^{\mu}d_L^{\beta}\right] + \text{h.c.} \qquad (4.11)$$

At low energies ($<< M_X$) exchange of these bosons gives an effective 4-Fermi interaction.

As we observed in section (3.7) in the minimally coupled theory there is no freedom to inhibit proton decay by choosing mixing angles so that it couples only to heavy leptons. Keeping only the large Cabibbo mixing between the first two generators eq(4.11) becomes

$$\frac{1}{4} L = e^{i\theta} \frac{g^2}{8M_X^2}\Big[\epsilon_{ijk}\bar{u}^c_{kL}\gamma_\mu u_{jL}\Big)\Big\{ \Big[(1+\cos^2\theta_c)\bar{e}^+_L$$
$$+ \sin\theta_c \cos\theta_c \bar{\mu}^+_c\Big] \gamma^\mu d_{iL} + \Big[(1 + \sin^2\theta_c)\bar{\mu}^+_L + \sin\theta_c \cos\theta_c \bar{e}^+_L\Big]\gamma^\mu s_{iL} \tag{4.12}$$
$$+\bar{e}^+_R\gamma^\mu d_{iR} + \bar{\mu}^+_R\gamma^\mu s_{iR}\Big\} -\Big[\epsilon_{ijk}\bar{u}^c_{kL}\gamma_\mu(d_{jL}\cos\theta_c+s_{jL}\sin\theta_c)\Big]$$
$$\Big[\bar{\nu}^c_{eR}\gamma^\mu d_{iR}+\bar{\nu}^c_{\mu R}\gamma^\mu s_{iR}\Big]\Big] + h.c.$$

This gives quantitative predictions for relative decay rates

$$\frac{\Gamma(N\to\mu^+ + \text{non-strange})}{\Gamma(N\to e^+ + \text{non-strange})} = \frac{\sin^2\theta_c \cos^2\theta_c}{(1+\cos^2\theta_c)^2+1} \tag{4.13}$$

$$\frac{\Gamma(N\to e^+ + \text{strange})}{\Gamma(N\to\mu^+ + \text{strange})} = \frac{\sin^2\theta_c\cos^2\theta_c}{(1+\sin^2\theta_c)^2+1} .$$

These predictions are specific to the minimal SU(5) scheme. If, we include the 45 couplings the mixing angles involved in proton decay are no longer just those of the Kobayashi-Maskawa matrix and the pattern of proton decay is dependent on unknown parameters.

4.4. Proton lifetime

We now try to estimate the proton lifetime following from the Lagrangian of eq(4.12). Gluon corrections to the Born diagram give rise to terms involving $\left[\alpha_s \log(\frac{M_X}{M_p})\right]^n$. These potentially large terms may conveniently be summed by the usual operator renormalisation group techniques. The terms of eq(4.12) give rise to the combination $2\,O_1 + O_2$ where the operators $O_{1,2}$ are (neglecting mixing angles)

$$O_1 = (\epsilon_{ijk}\bar{u}^c_{RL}\gamma_\mu u_{jL})\ \bar{e}^+_L\ \gamma^\mu d_{iL}) ,$$
$$O_2 = (\epsilon_{ijk}\bar{u}^c_{kL}\gamma_\mu)\ (u_{jL}\bar{e}^+_R + d_{jL}\bar{\nu}^c_{eR})\gamma^\mu\ d_{iR} . \tag{4.14}$$

The advantage in defining 0_1 and 0_2 lies in the fact that they do not mix through the logarithmic corrections we are seeking to include. The anomalous dimensions in leading order of 0_1 and 0_2 coming from gluon exchange give rise to the enhancement factors

$$A_3 = \left[\frac{\alpha_3(1\ \text{Gev})}{\alpha_3(M_X)}\right]^{\frac{2}{11-4/3N_g}} \tag{4.15}$$

Similarly one may compute the enhancement due to $W^{\pm}$, Z and γ give

$$A_{21} = \left[\frac{\alpha_2(M_W)}{\alpha_2(M_X)}\right]^{\frac{27}{86-4N_g}} \times \begin{cases} \left[\frac{\alpha_1(M_W)}{\alpha_1(M_X)}\right]^{-\frac{69}{6+20N_g}} & \text{for } 0_1 \\ \left[\frac{\alpha_1(M_W)}{\alpha_1(M_X)}\right]^{-\frac{33}{6+20N_g}} & \text{for } 0_2 \,. \end{cases} \tag{4.16}$$

These gauge boson corrections sum the large logs coming from perturbative corrections to the fundamental processes. In addition one must estimate non-perturbative effects in going from the quark fields of eq(4.14) to the proton, i.e., we must compute the hadronic matric elements of the operators 0_1 and 0_2. There are two contributions that have been estimated in several ways to be of comparable magnitude (cf.Fig4.3).

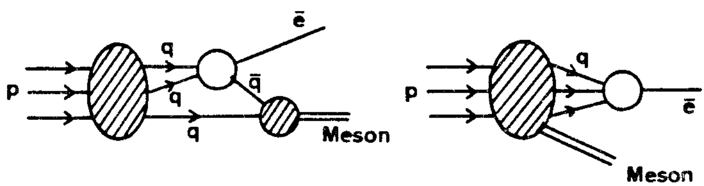


Fig.4.3. Operator matrix elements.

The first involves a spectator quark. The second involves the meson emission first followed by the 3 quark overlap probability at the origin. A variety of estimate have been made giving the range

$$t_{p,n} = (0.1 \text{ to } 5) \times 10^{30} \text{yrs} \left(\frac{M_X}{5 \times 10^{14} \text{GeV}}\right)^4 , \tag{4.17}$$

where a factor of $3\frac{1}{2}$ to 4 has been included from eqs(4.16).

M_X as we discussed in the previous section depends almost linearly on $\Lambda_{\overline{MS}}$ giving for a range (150 to 500) MeV for $\Lambda_{\overline{MS}}$

$$m_X = (1 \text{ to } 6) \times 10^{14} \text{ GeV} \tag{4.18}$$

and

$$t_{p,n} = \frac{1}{5}\, 10^{27} \text{ to } 10^{31} \text{ yrs} \,.$$

The estimates of operator matrix elements also predict the decay modes expected. The favoured mode is $p\to e^+\pi^0$ and, with the range of $\Lambda_{\overline{MS}}$ used above

$$\frac{1}{\Gamma}(p\to e^+\pi^0) = 4.5 \times 10^{29\pm 1.7} yrs . \qquad (4.19)$$

Baryon number violating neutron decay is also generated and using isospin, reactions gives

$$\frac{1}{\Gamma}(n\to e^+\pi^-) = (\tfrac{1}{2})\frac{1}{\Gamma}(p\to e^+\pi^0) = 2.2 \times 10^{29\pm 1.7} yrs . \qquad (4.20)$$

For comparison the most recent results on experimental limits on proton and neutron τ p, n $\to e\pi > 2.10^{31}$ yrs. It can be seen that these limits are already inconsistent with the minimal SU(5) values quoted above.

4.5. Neutrino masses

In the Appendix we discuss the possible Lorentz invariant forms for fermion masses. The most familiar one is the Dirac mass term involving the terms $m(\bar{\psi}_L\psi_R+\bar{\psi}_R\psi_L)$ and in the SU(5) model, as constructed up to now, it is absent for neutrino simply because there is no right handed neutrino state ν_R. However there is another possible mass term, the Majorana mass which is of the form $m(\psi_L{}^T C\psi_L)$. This term violates any symmetry under which ψ_L has non-trivial transformation properties and cannot, for example, give mass to a charged particle if charge conservation is a good symmetry. However neutrinos occupy a unique place in Grand Unified theories for they are the only fermions in the theory which carry neither charge nor colour. It is therefore quite possible that they should have a Majorana mass for while this will violate lepton number, we expect this to be violated at some level.

In the standard SU(3)xSU(2)xU(1) model Majorana masses did not arise because the combination $\nu_L^T C_L \nu_L$ transforms under SU(2) as an I = 1, I_3 = 1 object and there are no I = 1 Higgs fields which could, on spontaneous symmetry breakdown, generate such a mass term. Equivalently no neutrino mass terms arise because we know lepton number is conserved in the standard model (essentially through the absence of such terms). In minimal SU(5) there are I = 1 Higgs fields in the adjoint representation but these are not coupled to fermions so again there is no Majorana mass for the neutrinos. It is also forbidden by the exact (B-L) symmetry of SU(5). Thus both in the standard model and in minimal SU(5) there is no neutrino mass term either of the Dirac or Majorana type.

This is not an entirely convincing result for the absence of ν_R is put in by hand (nothing prevents us from adding an SU(5) singlet). Indeed in most generalisations of SU(5) such fields appear. If it is included SU(5) does not prevent a Dirac mass term $\bar{\nu}_L\nu_R$ arising from a gauge invariant coupling of the 5 of fermions to the $\bar{5}$ of Higgs, $\bar{H}$ and the singlet ν_R field. The natural scale for such a term would be of the order of a quark or lepton mass term quite unacceptably large compared with the experimental limits on neutrino masses.

Moreover, as we have already stressed, it seems likely that the (B-L) will be broken at some level so we should expect higher order corrections to generate a term of the form $\nu_L^T C \nu_L H^0 H^0$ in which the $I = \frac{1}{2}$ Higgs fields combine to supply the $I = 1$ Higgs component needed to generate a Majorana mass. One of the most elegant results of Grand Unification is the observation of Gell-Mann, Ramond and Slansky that even if we include these terms we can understand why neutrinos are light and different from other fermions. Let us see how this works.

If we allow both ν_L and ν_R fields the mass matrix involving both Dirac and Majorana masses is of the form

$$(\nu_L, \bar{\nu}_R) \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} \begin{pmatrix} \nu_L \\ \bar{\nu}_R \end{pmatrix}, \tag{4.21}$$

where, for notational convenience, we have denoted a Majorana mass term by $m\nu_L\nu_L$. As we have already discussed it is reasonable to choose $m_2 = O(m_q)$ on purely dimensional grounds. m_1 coming for example from a $(H^0\nu_L)^2$ term will be of order $v_0^2/M_{X'}$ where $M_{X'}$ is the scale associated with (B-L) violation $(M_{X'} > M_X)$.

Finally what is the expected size of m_3? Since ν_R is an SU(5) singlet there is no symmetry reason forbidding a $\nu_R\nu_R$ mass term. It is likely therefore that this term will be of magnitude $O(M_X)$. Putting all this together we need to diagonalise a mass matrix of the form

$$(\nu_L, \bar{\nu}_R) \begin{pmatrix} \sim \frac{m_q^2}{M_X} & m_q \\ m_q & M_X \end{pmatrix} \begin{pmatrix} \nu_L \\ \bar{\nu}_R \end{pmatrix}. \tag{4.22}$$

The mass eigenstates are $\nu_L + \varepsilon\nu_R$ and $\nu_R + \varepsilon\nu_L$ where $\varepsilon = m_q^2/M_X^2$ with masses $\sim \frac{m^2}{M_X}$ respectively. Due to the enormous mass M_X particular to GUTs the neutrino mass is expected to be very small $O(10^{-6} - 10^{-3}$ eV).

4.6. SU(5) - A critique

In section 1 we motivated the need for GUTs by listing many unsatisfactory features of the standard model. Let us take a look at how well SU(5) answers these criticisms.

(1) By fitting a family into a $\bar{5} + 10$ the multiplet structure has simplified and the LH structure of quarks and leptons is better. However there is no explanation of the family structure and there is no connection between the vector, fermion and scalar representations. Although anomalies cancel between the $\bar{5}$ and 10 it is unclear why this happens.

(2) Charge is quantised and the third integral nature of quark charges is explained - a remarkable result. The gauge couplings are all related and

the prediction for $\sin^2\theta_W$ is in good agreement with experiment. The value for M_X is such that proton decay should be seen soon.

(3) The situation with Yukawa couplings is not so good. There are three mass predictions which have varying success. There is no prediction for the Kobayashi Maskawa angles and there are (N_g-1) additional phases which contribute in proton decay. Similarly there is no improvement in the scalar potential where more couplings are needed in SU(5) than in the standard model.

(4) The appearance of a large mass scale is a mixed blessing. It explains why the proton should decay slowly and why neutrinos should have a low mass, but there is no explanation of why this mass should be so much greater than the weak interaction breaking scale. Indeed to accommodate these two scales it is necessary to fine tune parameters in the scale potential to one part in 10^{13}. Also even though M_X is approaching the Planck mass gravitational corrections have been ignored.

It is clear that SU(5) is not the final theory. However many of its properties are so pretty that many of us would be very unhappy to give them up. The GUT addict sees SU(5) as the first approximation and hopes that the missing elements will be supplied by more complete theory. What could this theory be?

5. SUPERSYMMETRY

5.1. Why supersymmetry?

Grand Unified theories are based on Lie groups which assign particles of a given spin to representations of the group. However they do not connect particles of different spin and so the unification achieved is not complete. In particular there is no understanding of the large number of Yukawa and scalar interactions, only the vector interactions are unique following from the local gauge principle.

An obvious generalisation of Grand Unified theories is to build a symmetry relating different spins. If this is done it may be possible to connect the representations and interactions of scalars and fermions to that of the gauge bosons. In this chapter we will introduce a symmetry called supersymmetry that can do this, and discuss how supersymmetry may be used for unification.

Early attempts to combine local symmetries with internal symmetries in a non-trivial way suggested this was impossible. The reason is easy to understand at the heuristic level and we will briefly sketch the problem following the approach of Coleman and Mandula. They point out that to construct a scattering amplitude it is necessary to satisfy the constraints of the local symmetry group. In, say, a two to two scattering of spinless particles there are four momenta (see Fig. 5.1). Energy momentum

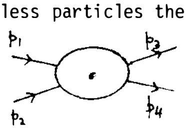


Fig 5.1. Kinematics of 2 - 2 scattering.

conservation reduces theseto three independent momentum vectors $p_{i\mu}$. The scattering amplitude must be a Lorentz scalar and so may depend only on the independent Lorentz scalar quantities constructed from $p_{i\mu}$. Apart from the particle masses $p_{i\mu}p_i^{\mu}$ there are three invariants that may be formed from three vectors, usually chosen as

$$\begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1-p_2)^2 \\ u &= (p_2-p_3)^2. \end{aligned} \tag{5.1}$$

However the condition $m_4{}^2 = p_{4\mu}p_4{}^{\mu}$ relates these invariants

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2. \tag{5.2}$$

Thus the scattering amplitude may be described by two invariants, the total centre of mass energy $\sqrt{s}$, and the centre of mass scattering angle θ. The requirement of Lorentz invariance has limited the number of independent variables. If one imposes additional conservation laws they will further limit the variables allowing only a discrete set of scattering angles. Since the amplitude is analytic in the scattering angle the only solution is the trivial one with vanishing amplitude everywhere. For example if there is a conserved tensor $Q_{\alpha\beta}$, traceless and symmetric then for a one particle state

$$\langle p|Q_{\alpha\beta}|p\rangle = p_\alpha p_\beta - \frac{1}{4} g_{\alpha\beta}m^2. \tag{5.3}$$

Applied to the two to two process with equal masses gives

$$p_{1\beta}p_{1\gamma} + p_{2\beta}p_{2\gamma} = p_{3\beta}p_{3\gamma} + p_{4\beta}p_{4\gamma}. \tag{5.4}$$

This is satisfied only when $\theta = 0$.

Coleman and Mandula proved in general that in a theory with nonzero scattering amplitudes in 4 dimensions the only possible conserved quantities that transform as tensors under the Lorentz group are the generators of the Poincaré group P_μ and $M_{\mu\nu}$ and Lorentz invariant quantum numbers Q_i, the charges of the internal symmetries.

However the Coleman-Mandula theorem does not forbid conserved charges transforming as spinors under the Lorentz group and it is this possibility that is exploited in supersymmetry. It will prove to be most convenient to work with two-component Weyl spinors as introduced in the Appendix. Let us start with the simplest such spinor charges Q_α, $\overline{Q}_\alpha$ transforming as $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ respectively, under the spinor group SL(2,C). These Qs satisfy anticommutation relations and, being spin $\frac{1}{2}$, their anticommutators can be a spin 1 vector. However if the Qs are conserved charges, as we would like, their anticommutator must be too. The Coleman-Mandula theorem permits a vector conserved quantity provided it is formed from the generators of the Lorentz group. Consider the vector operator P_μ. To appear on the right hand side of the anticommutator the Lorentz index must be contracted with γ^μ (it has both vector and spinor indices as required). In the Weyl basis γ^μ is off diagonal

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{5.5}$$

and so the allowed anticommutator relations are

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \\ \{Q_\alpha, Q_\beta\} &= 2\sigma^\mu_{\alpha\beta} P_\mu \\ \left[Q_\alpha, P_\mu\right] &= \left[\bar{Q}_{\dot{\alpha}}, P_\mu\right] = 0. \end{aligned} \tag{5.6}$$

This is the simplest supersymmetry algebra. In writing eq(5.6) we set the commutator of Q_α with P_μ to zero because we have introduced no conserved spinor charge $Q_{\alpha\mu}$, $\bar{Q}_{\alpha\mu}$ with vector and spinor indices and components of spin up to $\frac{3}{2}$. In fact the Coleman-Mandula theorem suggests it is not possible to introduce such a charge for their anticommutators would have spins up to e and the Coleman-Mandula theorem does not permit conservation of a spin 3 operator in an interacting theory. One may try to generalise the algebra of eq(5.6) by adding terms proportional to $\sigma^{\mu\nu}M_{\mu\nu}$ but this too is not possible for one can show that such terms do not satisfy the Jacobi identity and the algebra does not close. As we will discuss in the next section it is possible to build a finite set of different supersymmetries involving N spinoral charges ($N \leq 8$), the so-called N extended supersymmetries. The remarkable thing is that these, together with the usual space-time and internal symmetries, are the only possible symmetries of a physically reasonable theory. For this reason alone it is worthwhile exploring the implications of supersymmetry.

5.2. Representations of supersymmetry

In the last section we introduced supersymmetry, generated by the charges Q_α and $\bar{Q}_{\dot{\alpha}}$ satisfying the anticommutation, and commutation relations of eq(5.6). Such a structure generalises the concept of a Lie Algebra and is known as a Graded Lie Algebra. In this section we will construct its representations. Since the Q_α, $\bar{Q}_{\dot{\alpha}}$ are spinors we expect that operating on boson fields they will produce a fermion and vice versa.

$$Q_\alpha |B\rangle = |F\rangle$$

$$Q_\alpha |F\rangle = |B\rangle \cdot$$

As a first step in constructing representations let us consider the massless case. In this case P_m is light-like ($P_m P^m = 0$) and we can choose P_μ of the form (1, 0, 0,1). The supersymmetry algebra in this case is

$$
\begin{aligned}
\{Q_1, \bar{Q}_1\} &= 1 \\
\{Q_2, \bar{Q}_2\} &= 0 \\
\{Q_1, \bar{Q}_2\} &= \{Q_2, \bar{Q}_1\} = 0 \\
\{Q_i, Q_i\} &= \{\bar{Q}_i, \bar{Q}_i\} = 0 .
\end{aligned}
\tag{5.7}
$$

Let us start with a massless state $|\lambda>$ of a given helicity λ satisfying the constraint that

$$\bar{Q}_1|\lambda> = \bar{Q}_2|\lambda> = 0 . \tag{5.8}$$

Note that it is always possible to construct such a state since if, for example, $\bar{Q}_1|\lambda>$ is non-null then choose it as the initial state. By the anticommutation relations eq(5.6), $\bar{Q}_1|\lambda'> = 0$. Starting with $|\lambda>$ there is only one non-null state that can be formed, namely $Q_1|\lambda>$. The other possible states such as $Q_2|\lambda>$, $Q_1Q_2|\lambda>$ etc. all have zero norm, e.g., by eq.(5.6) we have

$$<\lambda|\bar{Q}_2Q_2|\lambda> = <\lambda|Q_2\bar{Q}_2|\lambda> = 0 . \tag{5.9}$$

Thus the massless representations of the supersymmetry algebra consists of two states of helicity λ and $\lambda+\frac{1}{2}$. A parity operator can be defined by the transformation properties of the spinorial charge Q_α under space reflection

$$(Q_\alpha)_P = (\sigma^0\bar{Q})_\alpha . \tag{5.10}$$

If parity is included the representations consist of helicities $\pm\lambda$, $\pm(\lambda+\frac{1}{2})$.

Simple examples of massless supersymmetric multiplets are "chiral" supermultiplets with $\lambda = 0$, and describing particles of helicities $(0, \pm\frac{1}{2})$, "vector" supermultiplets with $\lambda = \frac{1}{2}$ and helicity content $(\pm\frac{1}{2}, \pm 1)$ which may be used to partner a vector boson with a fermion, and the "graviton" supermultiplet with $\lambda = 1$ and helicities $(\pm\frac{3}{2}, \pm 2)$ which can contain the graviton responsible for the gravitational force.

It is possible to generalise the supersymmetry algebra by adding a flavour i to the supersymmetry generators $Q_\alpha \to Q_\alpha^{i=1\ldots N}$.

Now the algebra is

$$
\begin{aligned}
\{Q^i_\alpha, Q^j_\beta\} &= \{\bar{Q}^i_\alpha, \bar{Q}^j_\beta\} = 0 \\
\{Q^i_\alpha, \bar{Q}^j_\beta\} &= 2\delta^{ij}\sigma^\mu_{\alpha\beta} P_\mu \\
\left[Q^i_\alpha, P_\mu\right] &= \left[\bar{Q}^i_\alpha, P_\mu\right] = 0.
\end{aligned}
\tag{5.11}
$$

The massless representations of this algebra may be constructed in an analogous way to that obtained above. Starting with the state $|\lambda>$ annihilated by $Q_\alpha^{i=1...N}$ the non-null states are easily constructed. For example with N = 2 we find the states

$$\begin{aligned} &|\lambda> \\ &Q_1^1\,|\lambda>,\quad Q_1^2|\lambda> \\ &Q_1^2Q_1^1|\lambda>. \end{aligned} \tag{5.12}$$

All other states vanish by the anticommutation relations eq(5.6). Thus a N = 2 supermultiplet may be built, starting with $\lambda = 0$, from one (complex) scalar state, two fermion states $\lambda = \frac{1}{2}$, and one vector state $\lambda = 1$.

This could be used to accommodate the vector bosons. N = 2 matter supermultiplets may similarly be constructed starting with a right handed fermion, $\lambda = -\frac{1}{2}$, two (complex) scalars and a left handed fermion, $\lambda = +\frac{1}{2}$. The appearance of mirror fermion states giving left and right handed fermion partners is characteristic of all N > 1 supersymmetries and is the main reason such theories have been little used for phenomenology for, as we saw in Fig.2.1, the known fermions do not have mirror partners. However it is conceivable that such partners exist and current experiment only requires that they be heavier than about 20 GeV.

For higher N the multiplets contain more states formed by building from the lowest non-null state. In Table 5.1 we list the number of these states of a given spin, including the charge conjugate states.

Table 5.1. Particle content of extended supergravity theories

	Particle content				
Theory	Spin=0	Spin=½	Spin=1	Spin=3/2	Spin=2
N=1				1	1
N=2			1	2	1
N=3		1	3	3	1
N=4	2	4	6	4	1
N=5	10	11	10	5	1
N=6	30	26	16	6	1
N=7	70	56	28	7	1
N=8	70	56	28	8	1

We see that there is an intimate relationship between the helicity and the multiplicity of the states, giving rise to the hope that these theories may provide the basis for the ultimate unification of local and internal symmetries. Moreover the class of these N extended supergravity theories, is very small for they terminate at N = 8 if we require that spins no greater than two should be present. It appears that this is a necessary condition as theories with N > 8 suffer from defects such as the appearance of ghost states (negative norm states) which make them unacceptable.

Unfortunately, even though the theories for large N appear to possess a large number of states, they are insufficient to accommodate the states needed for the standard model. To see this note from Table 5.1 that there are $\frac{1}{2}N(N-1)$ vector fields in a supermultiplet. These vectors form an adjoint representation of O(N) and if the symmetry is made local they will be the gauge bosons of the theory. Consider the largest N possible, N = 8. Its internal symmetry group O(8) contains SU(3)xU(1)xU(1) but not SU(3)xSU(2)xU(1). Thus even the largest theory cannot accommodate the $W\pm$ bosons. A check of the fermion spectrum shows there are also missing fermions, for example, there is no room for the τ of the μ.

It seems therefore that these theories cannot be used to generalise the standard model in a simple way. Two possible ways out of this impasse have been explored. The first is that the extended supersymmetry is a symmetry relevant at a more fundamental level, and that the W's and the μ,τ, etc.,are composite states. This idea has received some support from the fact that the N = 8 theory has global symmetries larger than the O(8) symmetry of the fundamental vector fields. If these global symmetries should through the dynamics of the theory be realised locally, there will be additional gauge bosons which arise as bound states of the fundamental fields in the theory. At present we are unable to solve the theory in the non-perturbative way necessary to show if this idea works, but simpler models with global symmetries have been shown to behave in this manner.

The second possibility is to give up the hope that the extended supersymmetry models should contain the spectrum of observed states in a single representation and to build models based on the direct product structure G x [N-extended supersymmetry]. At first sight this appears to lose many of the potential benefits of the supersymmetry, and indeed in this type of theory there still remain many arbitrary parameters. However such models can solve the hierarchy problem common to all Grand Unified Theories, and for this reason they have been extensively studied. If such a model proves realistic it does not mean we have to give up the idea of ultimately unifying interactions in a unique extended supergravity theory; it is possible such models result as the low energy effective Lagrangians of the underlying extended supergravity theory.

6. SUPERSYMMETRIC MODELS

6.1. A supersymmetric version of the standard model-multiplet structure

The simplest supersymmetric model which can be constructed is the direct product of the internal symmetry group with a (N=1) supersymmetry group. The basic building blocks for such a model are the massless supersymmetry multiplets either chiral or vector supermultiplets as shown in Table 6.1 .

We will first construct a supersymmetric version of the standard SU(3)xSU(2) xU(1) model with the particle content of Fig.2.1. One of the reasons for studying supersymmetry is the hope that the complicated multiplet structure shown there will be simplified by assigning fermion and boson states to the same supermultiplet.

Table 6.1 . Fundamental supermultiplets in N=1 supersymmetry

Name		Particle Content
Chiral, ϕ	ψ	2 component (left handed) Weyl fermion
	ϕ	2 real scalar fields A,B(ϕ = A+B)
Vector, v	v_μ	2 component vector
	λ	2 component (left handed Weyl fermion

However this proves to be impossible for the simple N=1 supersymmetric models. We know vector bosons belong to the adjoint representation of the gauge group while, as can be seen from Fig.2.1, the known fermions belong to fundamental representations of SU(3)xSU(2)xU(1) (cf eq(2.4)). Therefore when we assign the gauge bosons of the standard model to a vector multiplet v_μ^a transforming as the adjoint we cannot identify the fermions λ^a in that supermultiplet with the known fermions because the λ^a are in the same SU(3)xSU(2)xU(1) representation as their vector boson partners. These new states are called "gauginos", see Table 6.2.

We still have the problem of assigning the fermions to supermultiplets. If we want to build a renormalisable theory all vector fields in our theory must be gauge vector fields. Thus, without enlarging the gauge groups, the fermions cannot be assigned to vector supermultiplets and must belong to chiral supermultiplets. Once again we are forced to double the number of states introducing (complex) scalar fields to partner the known fermions in chiral supermultiplets. These fields are in the same SU(3)xSU(2)xU(1) representations as their fermion brothers. They are known as squarks (for scalar quarks) and sleptons, see Table 6.2.

Finally we must assign the Higgs scalars of Fig.2.1 to chiral supermultiplets. Our original hope in constructing a supersymmetric theory was that the Higgs sector would be simplified by assigning the Higgs scalars to the same supermultiplets as the known fermions. An obvious possibility is to assign the Higgs SU(2) doublet to partner a lepton doublet. However this is not possible for such an assignment if supersymmetry does not give an acceptable pattern of fermion masses. The reason is that supersymmetry restricts the possible forms of Yukawa couplings and (with the above assignment) the couplings necessary to give down quarks and charge leptons a mass (eq(2.5)) are not present. To see this we need to know the allowed form of the Yukawa couplings and the scalar couplings which, through supersymmetry, are related to them. This is most simply expressed by describing these couplings in terms of a superpotential P. P is a gauge invariant function of dimension $\leq$ 3 constructed from the chiral superfields of the model (but not their complex conjugates). Then the Yukawa and scalar couplings are given in terms of P by

$$\text{Yukawa} = \sum_{i,j} \frac{\partial^2 P}{\partial\phi_i \partial\phi_j} \psi_i \psi_j \quad , \tag{6.1}$$

$$\text{scalar} = \sum_i \left|\frac{\partial P}{\partial\phi_i}\right|^2 \equiv \sum_i F_i^* F_i . \tag{6.2}$$

Here ψ,ϕ refer to the (LH) fermion and scalar components of the chiral supermultiplets respectively and the sums over i,j run over all the chiral supermultiplets. The F_i are the auxiliary fields.

To reproduce eq(2.5) for the Yukawa couplings needed to give all charged fermions a mass we need a superpotential of the form

$$P = m_{ij}^{(d)\alpha\beta} \psi_{i,\alpha}^{(q)} H_{2\beta} \psi_j^{(d)} + m_{ij}^{(u)\alpha\beta} \psi_{i,\alpha}^{(q)} H_{1\beta} \psi_j^{(u)} + \sum_{i=e,\mu,\tau} m_i \psi_{i,\alpha} H_{2\beta} \psi_j^{(1)}, \tag{6.3}$$

where $H_{1\alpha}$ and $H_{2\alpha}$ are chiral supermultiplets transforming as doublets under SU(2) but with U(1) charge $\pm\frac{1}{2}$ respectively so that their charge states are

$\begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix}$ and $\begin{pmatrix} H_2^0 \\ H_2^- \end{pmatrix}$ respectively. i and j are family indices and $m_{ij}^{(d)}$, $m_{ij}^{(u)}$

are the mass matrices for the up and down quark masses. The lepton doublets, $\psi_{i,\beta}^{(1)}$, have the correct charges to be identified with H_β. In the non-supersymmetric standard model $H_{2\beta} = \varepsilon_{\beta\gamma} H_1^{+\gamma}$ but in the supersymmetric case the rules for writing P state that P can only be formed using products of (left handed) chiral supermultiplets and not their (right handed chiral) conjugates. Thus in eq(6.3) H_2 must be identified with a completely new chiral supermultiplet. In addition more states are needed for H_2 contains new, charged, Weyl fermions and we must add further charged fermions allowing the construction of Dirac masses for them to ensure the final theory has no massless charged states (remember that we cannot give charged fermions a Majorana mass without violating charge conjugation - see Appendix . The simplest solution is to introduce another new SU(2) doublet chiral superfield which is usually identified with H_β.

In constructing simple grand unified generalisations of the standard model this, in fact, is the only possibility for if H_β is identified with a lepton doublet proton decay proceeds too fast.

Thus the final multiplet structure for a supersymmetric version of the standard model includes two new chiral supermultiplets whose scalar partners are to be identified with the Higgs scalars need to break the SU(3)xSU(2)xU(1) to $SU(3)xU(1)_{em}$ and to give all charged fermions a mass. The full multiplet structure is given in Table 6.2.

Although the SU(3)xSU(2)xU(1) x [N=1 supersymmetry] structure fails to simplify the multiplet structure of the original model (indeed it more than doubles the spectrum!) it does have a redeeming property that has caused it to be studied intensively recently as a possible theory for the strong, weak and electromagnetic interactions relevant at relatively low energy scales - it solves the hierarchy problem. In the next section we will construct the Lagrangian for this theory and show how this solution is achieved.

6.2. The SU(3)xSU(2)xU(1) supersymmetric Lagrangian

Once the transformation properties of the supermultiplets under the gauge group are specified and the superpotential is given the Lagrangian density may be immediately constructed using the results of section (6.1). As usual we write the Lagrangian as the sum of the two parts

$$L = L_{kin} + L_{Int} . \qquad (6.4)$$

L_{kin} is the supersymmetric form of the locally gauge invariant kinetic energy. The requirements of local gauge invariance uniquely specify how the gauge fields couple and the requirements of supersymmetry relate these gauge field couplings to gaugino couplings

$$\begin{aligned} L_{kin} = & -\frac{1}{4}\mathrm{Tr}\{W^{\mu\nu}W_{\mu\nu}\} - i\mathrm{Tr}\{\lambda\sigma^m D_m\bar{\lambda}\} \\ & + \sum_j A_j D_m D^m A_j^* + i\sum_j D_m\bar{\psi}_j\bar{\sigma}^m\psi_j \\ & - \frac{i}{\sqrt{2}}\sum_{j,a}(A_j\tau^a\bar{\psi}_j - A_j^*\tau^a\psi_j)\lambda^a \\ & + \sum_a g_a^2 \sum_j A_j^+\tau^a A_j{}^2 . \end{aligned} \qquad (6.5)$$

In this the trace implies a sum over all the gauge indices a of SU(3)xSU(2)x U(1) and the sum over j is over all the chiral superfields of Table 6.2. D_m are the usual gauge covariant derivatives. For convenience we split L_{kin} up into the usual (non-supersymmetric) kinetic energy term L'_{kin}, a term L^a_{Yuk}

Table 6.2. Multiplet structure for the minimal supersymmetric SU(3)xSU(2)xU(1) theory

Vector Supermultiplets			Spin J
V_G	$g^{a=1...8}$	Gluons	1
	$\tilde{g}^{a=1...8}$	Gluinos	½
V_W	$W^{\pm}$, Z	W,Z bosons	1
	$\tilde{W}^{\pm}$, $\tilde{Z}$	Winos, Zino	½
V_γ	A_μ	Photon	1
	$\tilde{A}$	Photino	½

Chiral Supermultiplets			Spin J
$S_q, S_{\bar{q}}$	q_L, q_R	Quarks	½
	$\tilde{q}_L, \tilde{q}_R$	Scalar quarks	0
$S_\ell, S_{\bar{\ell}}$	ℓ_L, ℓ_R	Leptons	½
	$\tilde{\ell}_L, \tilde{\ell}_R$	Scalar leptons	0
S,T	$\tilde{H}_1, \tilde{H}_2$	Fermionic Higgs	½
	H_1, H_2	Higgs doublets	0

describing the new Yukawa interactions induced by the supersymmetric form of the gauge interactions and L^D the scalar interactions obtained by eliminating the D auxiliary fields and commonly called the "D" term.

$$L'_{kin} = -\frac{1}{4}\,\mathrm{Tr}\{W^{\mu\nu}W_{\mu\nu}\} - i\mathrm{Tr}\{\lambda\sigma^m D_m \bar{\lambda}\} + \sum_j A_j D_m D^m A_j^* + i\sum_j D_m\bar{\psi}_j\bar{\sigma}^m\psi_j \,. \qquad (6.6)$$

Apart from the terms of the standard model this gives the following Higgsino leptonic couplings (quark couplings are omitted for their SU(2)xU(1) couplings are identical to the corresponding leptonic couplings and the gluino couplings are easily generated).

$$L'_{kin} = (e/\cos\phi_W)\tfrac{1}{2} B_\mu(\bar{\tilde{H}}_{1+}\bar{\sigma}^\mu\tilde{H}_{1+} + \bar{\tilde{H}}_{10}\bar{\sigma}^\mu\tilde{H}_{10}$$

$$- \bar{\tilde{H}}_{2-}\bar{\sigma}^\mu\tilde{H}_{2-} - \bar{\tilde{H}}_{2o}\bar{\sigma}^\mu\tilde{H}_{2o}$$

$$- \bar{\nu}\,\bar{\sigma}^\mu\nu - \bar{e}_-\,\bar{\sigma}^\mu e_- + 2\,\bar{e}_+\bar{\sigma}^\mu e_+)$$

$$+(e/\sin\theta_W)W^+_\mu\,\{\tfrac{1}{\sqrt{2}}\cdot\;(\bar{\tilde{H}}_{1+}\bar{\sigma}^\mu\tilde{H}_{1o}+\bar{\tilde{H}}_{2o}\bar{\sigma}^\mu\tilde{H}_{2-})$$

$$+ (-\bar{\tilde{W}}_+\bar{\sigma}^\mu\tilde{W}_o + \bar{\tilde{W}}_o\bar{\sigma}^\mu\tilde{W}_-)$$

$$+ \tfrac{1}{\sqrt{2}}\,\bar{\nu}\,\bar{\sigma}^\mu e_- +\}\text{ h.c.} \qquad (6.7)$$

$$+ (e/\sin\theta_W)W^o_\mu\;\tfrac{1}{2}\cdot(\bar{\tilde{H}}_{1+}\bar{\sigma}^\mu\tilde{H}_{1+} - \bar{\tilde{H}}_{1o}\bar{\sigma}^\mu\tilde{H}_{1o}$$

$$+ \bar{\tilde{H}}_{2o}\bar{\sigma}^\mu\tilde{H}_{2o} - \bar{\tilde{H}}_{2-}\bar{\sigma}^\mu\tilde{H}_{2-})$$

$$+(\bar{\tilde{W}}_+\bar{\sigma}^\mu\tilde{W}_+ - \bar{\tilde{W}}_-\bar{\sigma}^\mu\tilde{W}_-)$$

$$+ \tfrac{1}{2}\,(\bar{\nu}\bar{\sigma}^\mu\nu - \bar{e}_-\bar{\sigma}^\mu e_-)\}.$$

$$L^G_{Yuk} = (e/\cos\theta_W)\quad i\cdot\tfrac{\sqrt{2}}{2}\cdot\{\;H^*_{1+}\tilde{B}_o\tilde{H}_1 + H^*_{1o}\tilde{B}\,\tilde{H}_{1o}$$

$$-H^*_{2o}\tilde{B}_o\tilde{H}_{2o} - H^*_{2-}\,\tilde{B}_o\tilde{H}_{2-}$$

$$-\tilde{\nu}^*\;\tilde{B}_o\nu - \tilde{e}^*_-\,\tilde{B}_o e_- + 2\,\tilde{e}^*_+\,\tilde{B}_o e_+\} + \text{h.c.}$$

$$+ (e/\sin\theta_W\;(i\{\,H^*_{1+}\tilde{W}_+\tilde{H}_{1o} + H_{1o}\tilde{W}_-\,\tilde{H}_{1+}$$

$$+ H^*_{2o}\tilde{W}_+\tilde{H}_{2-} + H^*_{2-}\tilde{W}_-\tilde{H}_{2o} \qquad (6.8)$$

$$+ \tilde{\nu}^*\;\tilde{W}_+e_- + \tilde{e}^*_-\,\tilde{W}_-\nu\} + \text{h.c.}$$

$$+ (e/\sin\theta_W)\;\tfrac{i}{\sqrt{2}}\{H^*_{1+}\tilde{W}_o\tilde{H}_{1+} - H^*_{1o}\tilde{W}_o\tilde{H}_{1o}$$

$$+ H^*_{2o}\tilde{W}_o\tilde{H}_{2o} - H^*_{2-}\tilde{W}_o\tilde{H}_2$$

$$+ \tilde{\nu}^*\tilde{W}_o\nu - \tilde{e}^*_-\,\tilde{W}_o e_-\} + \text{h.c.}$$

The Yukawa couplings may be read immediately from the superpotential using the form of eq(6.1). They are of the form given in the standard model, but, with H_1 coupling to the right handed up quarks and H_2 coupling to the right handed down quarks and leptons. The scalar interactions of H_1 and H_2 follow from P using eq.(6.2).

The two-component interactions used here are simply related to the more familiar four component form by noting, in the Weyl basis, that

$$\lambda\sigma^{\mu}\psi = \Lambda\gamma^{\mu}\left(\frac{1-\gamma^5}{2}\right)\psi,$$

where λ and ψ are 2 component Weyl spinors related (see Appendix) to the usual four component Dirac spinors Λ and ψ by

$$\psi = \begin{pmatrix}\psi \\ \text{any}\end{pmatrix} \quad \Lambda = \begin{pmatrix}\lambda \\ \text{any}\end{pmatrix}.$$

We may also write this vertex in the form

$$\overline{\lambda\sigma}^{\mu}\psi = -\overline{\Lambda}'\gamma^{\mu}\left(\frac{1+\gamma_5}{2}\right)\psi' \tag{6.9}$$

where

$$\psi' = \begin{pmatrix}\text{any} \\ \psi\end{pmatrix}, \quad \Lambda' = \begin{pmatrix}\text{any} \\ \lambda\end{pmatrix}. \tag{6.10}$$

For scalar couplings $\lambda\psi = \psi\boldsymbol{\lambda}$ so

$$\lambda\psi = \bar{\Lambda}_R'\psi_L = \bar{\psi}_R'\Lambda_L. \tag{6.11}$$

6.3. A SUSY-GUT Supersymmetric SU(5)

We are now in a position to construct the minimal (N=1) supersymmetric extension of SU(5). The first step is to assign SU(5) multiplets to supersymmetric multiplets. The gauge fields must be assigned to an adjoint (24) vector supermultiplet with gaugino partners also in the adjoint representation. The quarks and leptons are assigned to (left handed) chiral supermultiplets transforming as $N_G x(\bar{\mathbf{5}} + 10)$ under SU(3). In order to give them mass it is necessary to choose a supermultiplet of the form

$$P_{5_M} = -\sqrt{2}M_{ij}^{(d)}\psi_i^{\alpha}\chi_{\alpha\beta}H_{\mathbf{2}}^{\beta} - \frac{1}{4}M_{ij}^{(u)}\varepsilon^{\alpha\beta\gamma\delta\rho}\chi_{i\alpha\beta}\chi_{j\gamma\delta}H_{1\rho}, \tag{6.12}$$

where H_1 and H_2 are (left handed) chiral superfields transforming as 5 and $\bar{\mathbf{5}}$ respectively under SU(5). The normalisation is chosen so that the χ fermion kinetic term is correctly normalised, i.e., $\chi_{15} = \sqrt{\frac{1}{2}}\, d_{1L}$, etc., as in section (3.3). As we discussed in section (6.1) it is necessary that H_1 and H_2 be distinct chiral supermultiplets, and not hermitian conjugates for the rules of section (6.1) do not allow us to build the superpotential with a chiral superfield and its conjugate. It is also impossible to identify $\bar{H}$ transforming as a $\bar{5}$ with one of the $\bar{5}^s$ introduced to describe the quark and lepton sector. The reason is that the colour triplet components of H_2 mediate proton decay (see section (7.2)) and must have very large mass ($\gtrsim O(10^{10}$ Gev) if the proton is not to decay too quickly. If we identify the doublet components of H_2 with the sneutrinos, selectron members of a $\bar{5}$ then the triplet components will be partners of the down antiquarks. Since, as discussed below, supermultiplets splitting in a gauge nonsinglet representative is < O(1 Tev^2 $\frac{\pi}{\alpha}$) the triplet components will mediate proton decay far too fast (cf of section (7.2)). Consequently we must choose new chiral supermultiplets H and $\bar{H}$ transforming as a 5 and $\bar{5}$ to accommodate the Higgs scalars. We must also ensure that the H and $\bar{H}$ split so that their triplet components are heavy ($\gtrsim 10^{10}$Gev) while leaving the doublets light < O(1TeV).

It is necessary to add an adjoint chiral supermultiplet, Σ, to accommodate the adjoint of Higgs scalars necessary to break SU(5) to SU(3)xSU(2)xU(1).

The final multiplet choice for our SU(5)x N=1 supersymmetry model is given in Table 6.3.

Table 6.3. Chiral supermultiplets used in SU(5) SUSY GUT

Role	Notation	SU(5) Representation Content
Matter	$\psi^{\alpha}_{a}, \chi_{\alpha\beta}$	$N_G \times (\underline{\bar{5}} + \underline{10})$
	$H_{1\alpha}$, H_2^{α}	$(\underline{\bar{5}} + \underline{5})$
Higgs	Σ^{β}_{α}	$\underline{24}$

The interaction Lagrangian is given by eqs (6.1) and (6.2) where the superpotential is as chosen in eq (6.12). To this must be added a superpotential which will give the scalar components of the adjoint chiral supermultiplet a vacuum expectation value to break SU(5) to SU(3)xSU(2)xU(1), and also the underlying supersymmetry to give the new supersymmetric states a mass. Obviously supersymmetry must be broken for we have not seen, for example, a scalar electron (selectron) degenerate with the electron. If the selectron has a mass $\gtrsim$ 15 GeV it would have escaped detection so far).

There is considerable ambiguity in these symmetry breaking mechanisms and we will not describe them here. Instead we will treat the masses of the new states as arbitrary up to the constraints imposed by the requirement that supersymmetry should solve the hierarchy problem. This requires the new gauge nonsinglet supersymmetric partners should have a mass $\lesssim O$ (1 TeV). We also assume H_1^0 and H_2^0 acquire vevs v_1 and v_2 breaking SU(2)xU(1) to $U(1)_{em}$.

6.4 The hierarchy problem

In SU(5) we met the problem that the Higgs scalar mass is naturally close to M_X, but it must be < O(1Tev) in order to generate electroweak breaking. Although this can be arranged at tree level through the unnatural cancellation of eq(3.49) a large mass for the doublets reappears in higher order through radiative corrections involving superheavy virtual states as in Fig.6.1(a). Calculation of these graphs gives for the effective Higgs doublet mass at laboratory energies, μ

$$M^2(\mu) = M^2(M_X) + \sum_i c_i \alpha_i M_{X_i}^2, \qquad (6.13)$$

where c_i and α_i are the coefficients and couplings resulting from the graphs in Fig.6.1(a). Thus the natural scale for $M^2(\mu)$

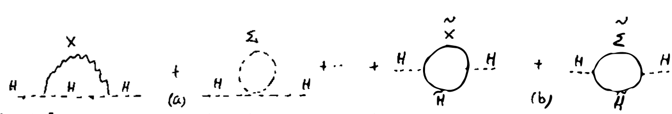


Fig 6.1. Graphs contributing to the Higgs scalar mass.

is $O(\alpha_i M_X^2)$. In supersymmetry however there are additional contributions coming from the gauginos and Higginos as in Fig.6.1(b) with couplings related to those in Fig. 6.1a . Due to the - sign coming from a fermion loop these graphs cancel exactly the contribution of eq(6.13). Of course supersymmetry must be broken leaving a residual contribution $M^2(\mu) = M^2(M_X) + \sum_i c_i \alpha_i \Delta M_L^2$ (6.14)

where Δm_L^2 is the mass splitting between supersymmetric partners. Since $M^2(\mu) \lesssim O(1\text{Tev})$, this mass splitting should be bounded by

$$\Delta M_L^2 < \frac{O((1\,\text{Tev})^2)}{\alpha_i}. \qquad (6.15)$$

This constraint has important phenomenological implications for it means the new supersymmetric states cannot have mass much different from their conventional partners and the light states of Table 6.2 should be produced at energies accessible in the laboratory.

It is for this reason that the hierarchy problem has been stressed so much. The standard SU(3)xSU(2)xU(1) model is unnatural at scales much larger than 1 Tev and any resolution of this requires new physics at these scales; in this case the new supersymmetric states.

7. PHENOMENOLOGY OF SUPERSYMMETRY

7.1. Mass and coupling constant renormalisation

Because a SUSY-GUT involves new light supersymmetric particles it is necessary to compute the values for the grand unification mass and $\sin^2\theta_W$. The initial ratios of the couplings and hence $\sin^2\theta_W$ are unchanged in going to a SUSY-GUT because we still have the same assignment of states to a GUT representation. However the radiative corrections will differ for now we must include loops containing the new states, the gauginos, the squarks, etc. It is easy to compute the expected size of the corrections at one loop assuming all heavy particles have a mass $\approx M_X$. The SU(3)xSU(2)xU(1) effective couplings depend on M_X via

$$\alpha_i(\mu) = C_i + \frac{1}{6\pi} b_i \ln\left(\frac{M_X}{\mu}\right) \; ; \quad i = 1, 2, 3. \tag{7.1}$$

Then

$$M_X = \mu \exp\left[6\pi\left(b_2 + \frac{5}{3}b_0 - \frac{8}{3}b_3\right)^{-1}\left(\frac{1}{\alpha_{em}(\mu)} - \frac{8}{3}\frac{1}{\alpha_3(\mu)}\right)\right]$$

$$\sin^2\theta_W = \frac{3}{8} + \frac{\frac{5}{8}(b_2 - b_0)}{\left(b_2 + \frac{5}{3}b_0 - \frac{8}{3}b_3\right)}\left(1 - \frac{8}{3}\frac{\alpha_{em}(\mu)}{\alpha_3(\mu)}\right). \tag{7.2}$$

The values of b_i are easily calculated using their definitions in eq.(7.1). The results for the standard model of section (6) are $b_3 = 27-4N_G$, $b_2 = 18-4N_G$, $b_0 = -4N_G$ giving, for a value of $\Lambda = 150$ Mev, $\sin^2\theta_W = 0.23$ and $M_X = 4\times10^{16}$ Gev. (For comparison the SU(5) b_i were $b_3 = 33-4N_G$, $b_2 = 22-4N_G$, $b_0 = -4N_G$). The result for M_X is about 20 times the SU(5) value. The reason for this is principally due to the decrease in b_3 compared to the SU(5) value which comes about mainly because the supersymmetric theory requires an SU(3) adjoint of fermions, the gluinos. Due to their large charge they contribute significantly to the β function and because they are fermions they contribute a negative amount to β_3 making the theory less asymptotically free. The value of M_X is, very roughly, the scale at which α_3 equals α_2 (cf.Fig.4.1). Since β_3 is reduced, α_3 varies more slowly; and thus it takes longer for this equality to be achieved. Hence M_X is increased. This can be compensated by speeding up the rate of evolution of α_2 so the value of M_X depends on the number of light Higgs particles included. For example adding the light Higgs SU(2) singlets to mimic a lepton family brings M_X back to the usual SU(5) value and lowers $\sin^2\theta_W$ by 0.015.

Including the two loop corrections gives the best values for $\sin^2\theta_W$ and M_X in the minimal supersymmetric extension of SU(5) for $\Lambda_{\overline{MS}}$ = 160 Mev.

$$M_X = 7.7 \times 10^{15} \text{ Gev},$$
$$\sin^2\theta_W = 0.236. \tag{7.3}$$

One might think that such a large value for M_X will render proton decay unobservable but, due to the presence of the new diagram mediating proton decay in Fig. 7.1 , this is not necessarily so (see next section).

The renormalisation of quark and lepton masses is also sensitive to the change in the unification scale. It also requires evaluation of an additional contribution to the anomalous dimension coming from gaugino contributions to the wave function renormalisation. These give

$$m_b(\mu) = m_b(M_X)\left[\frac{\alpha_3(\mu)}{\alpha_3(M_X)}\right]^{8/9} \quad \tfrac{8}{9} \tag{7.4}$$

which when compared to the SU(5) predictions of eq.(4.7) gives (keeping the dominant SU(3) corrections only)

$$\left(\frac{m_b(\mu)}{m_\tau(\mu)}\right) = \left(\frac{m_b(\mu)}{m_\tau(\mu)}\right)_{\substack{\text{minimal}\\ SU(5)}} \times \left[\frac{\left[\frac{\alpha_3(\mu)}{\alpha_3(M_X)}\right]^{\frac{8}{9}}_{SUSY}}{\left[\frac{\alpha_3(\mu)}{\alpha_3(M_X)}\right]_{\text{minimal } SU(5)}}\right]^{\frac{4}{3}} \tag{7.5}$$

In evaluating this equation there are two competing corrections. Since $\alpha_3(\mu)$(SUSY) varies more slowly than $\alpha_3(\mu)$(minimal SU(5)) we expect a smaller enhancement of m_b. On the other hand the additional contribution to the anomalous dimension in supersymmetry enhances the change in m_b. Together, remarkably, the change in the SU (5) prediction is very small, less than 10%.

7.2. Nucleon decay in the minimal SU(5) SUSY-GUT

The fact that M_X is approximately 20 times the value obtained in conventional SU(5) at first sight suggests proton decay will be so slow as to be invisible. However this is not the case for the minimal supersymmetric SU(5) theory for there are new contributions to proton decay coming from the graph of Fig.7.1(a) involving squarks and sleptons. The reason these contributions dominate is because the dominant contribution comes from a Born graph with a superheavy fermion exchanged and not, as was the case in ordinary SU(5), through a boson exchange. The fermion propagator has the form $(p+m)/(p^2+m^2)$ and the dominant term comes from the piece proportional to m in the numerator. As a result this contribution is proportional to $\frac{1}{M_X}$ and not $\frac{1}{M_X{}^2}$ in amplitude. In Fig.7.1 we show the relevant graph in the minimal model where the term proportional to m is shown as a mass insertion. In Fig.7.1 we show that this contribution comes from the exchange of the coloured fermions in the Higgs multiplets, with the mass insertion connecting H_1 and H_2. Because the Higgs couple preferentially to heavy states the dominant decay modes are (depending on the mixing angle of the top quark) either to ($\bar{\nu}_\mu$ + strange) or ($\bar{\nu}_\mu$ + strange). Charged lepton decay modes are suppressed

$$\frac{B(\bar{\nu}_\mu \text{ strange})}{B(\mu^+ + \text{strange})} = \left(\frac{m_c \sin^2\theta}{m_n}\right)^2 \simeq 100 \quad (7.6)$$

Non-strange modes are suppressed too, but by a smaller factor

$$\frac{B(\bar{\nu}_\mu \text{ or } \tau \text{ non-strange})}{B(\bar{\nu}_\mu \text{ or } \tau \text{ strange})} = 0(\sin^2\theta) \quad 0 \quad 0(\frac{1}{10}) \quad (7.7)$$

The expected rate for $p \to \bar{\nu}_\mu K^+$ is (using SU(6) for the spin-flavour wave functions),

$$\Gamma \approx 4\left(\frac{m_c m_s M_p^2}{2\sqrt{2} v_1 v_2 M_{H_X}}\right)^2 b^{o2} A^2 \sin^4\theta_c (1.39 \times 10^{27})\ y^{-1} \quad (7.8)$$

where b^o is determined by the loop integral in Fig.7.1(b) which thro' wino exchange turns squarks and sleptons into quarks and leptons

$$b^o = \left(\frac{g_2^2\, m_{\tilde{W}}^2}{256\pi^2}\right) \left[f(m_{\tilde{q}}, m_{\tilde{q}}, m_{\tilde{W}}) + f(m_{\tilde{q}}, m_{\tilde{e}}, m_{\tilde{W}})\right], \quad (7.9)$$

where

$$f(m_1, m_2, m_3) = \frac{1}{(m_2^2 - m_3^2)} \left[\frac{m_2^2}{(m_1^2 - m_2^2)} \ln\left(\frac{m_1^2}{m_2^2}\right) - \frac{m_3^2}{(m_1^2 - m_3^2)} \ln\left(\frac{m_1^2}{m_3^2}\right)\right]. \quad (7.10)$$

Here $m_{\tilde{W}}$ is the Wino Majorana mass (it also has a Dirac mass M_W), and M_{H_X} is the Higgs triplet mass. $m_{\tilde{W}}$ is unknown. In some models there is a symmetry which ensures $m_{\tilde{W}} = 0$, but in this case the gluinos and the photino are massless which may already be ruled out. In supergravity models the gauginos can acquire a mass at tree level and for them the natural magnitude for m_W is $>0(M_W)$. In eq(7.8), A is the enhancement factor coming from summing the large logarithmic corrections from gauge loop corrections

$$A = \left(\frac{\alpha_3(1\ \text{Gev})}{\alpha_3(m_c)}\right)^{2/9} \left(\frac{\alpha_3(m_c)}{\alpha_3(mb)}\right)^{6/25} \left(\frac{\alpha_3(m_b)}{\alpha_3(m_t)}\right)^{6/23} \cdot$$

$$\left(\frac{\alpha_3(m_t)}{\alpha_3(M_W)}\right)^{2/7} \left(\frac{\alpha_3(M_W)}{\alpha_3(M_X)}\right)^{4/3} \left(\frac{\alpha_2(M_W)}{\alpha_2(M_X)}\right)^{-3} \left(\frac{\alpha_1(M_W)}{\alpha_1(M_X)}\right)^{\frac{1}{66}} \cdot \quad (7.11)$$

A is ≈ 15 with the parameters chosen as in section (4.1). For $M_{H_X} = M_X$ and $\tau_p \gtrsim 10^{31}$ years consistency with eq(7.8) requires that

$$b^o \gtrsim \frac{1}{3} \times 10^{-8} \text{Gev}^{-1} \quad (7.12)$$

b^o is given in eq(7.9) and for $m_{\tilde{q}}$, $m_{\tilde{l}} >> m_{\tilde{W}}$ means

$$\frac{m_{\tilde{W}}}{m_q^{\,2}} < 10^{-5} \text{Gev}^{-1} \quad (7.13)$$

with the natural choice for $m_{\tilde{W}} = O(M_W)$, $\tau_p > 10^{31}$ years implies the limit

$$m_q > 3 \text{ Tev} \quad (7.14)$$

The current experimental limit on $p \to \bar{\nu} K^+$ or $n \to \bar{\nu} K^o$ is $\tau_p > 0.6 \times 10^{31}$ years.

7.3. The spectrum of new states

The most direct test of supersymmetry is to observe directly one of the many new states predicted by the theory. In section (6.1) we introduced the multiplets needed to build the basic (SU(3)xSU(2)xU(1))x(N=1 supersymmetry) model. The various Grand Unified versions of the theory all have this low energy structure, with the possible addition of a light singlet field. In fact most of the supersymmetric models that have been considered, whether global or local, have the same low energy spectrum. What differs between various models is the pattern of masses. In most models there is a multiplicative R parity conserved, where R is + 1 for conventional hadrons and -1 for the new supersymmetric states. Thus these states may only be produced in pairs and once produced a new supersymmetric state will ultimately decay into the lightest such state. Their decay patterns will thus depend sensitively on the identity of the lightest state. In this section we will try to discuss the possible mass spectra and characteristic signals for the new states for a general class of possible model.

The states of Table 6.2 in addition to the usual ones of the standard model, have squarks and sleptons, the scalar supersymmetric partners of quarks and leptons, gauginos, the fermion partners of the gauge bosons and Higgsinos,the fermion partners of the Higgs scalars.Also there is non-minimal

Higgs structure so there will be, after spontaneous symmetry breakdown, two residual charged Higgs and three neutral Higgs scalars. In addition models of spontaneously broken global supersymmetry necessarily have a massless fermion, the goldstino, arising from the analogous mechanism to the Goldstone theorem for breaking ordinary continuous symmetries.

Its coupling e_q to other fields is determined by the breaking scale of supersymmetry Λ^2_{SUSY}

$$e_q = \frac{I \Delta m^2}{\Lambda^2_{SUSY}}, \qquad (7.15)$$

where

$$\Delta m^2 = (\text{scalar mass})^2 - (\text{fermion mass})^2$$

and the - and + refers to LH or RH fermions respectively.

If the global supersymmetry is made local then the golstino becomes the $\pm \frac{1}{2}$ components of the spin $\frac{3}{2}$ gravitino, which develops a mass;

$$m_{3/2} = \frac{\Lambda^2_{SUSY}}{M_{Planck}}. \qquad (7.16)$$

The squarks and sleptons

Squarks and sleptons have the same quantum numbers as their fermion partners. Their SU(2)xU(1) interactions both with the gauge bosons and gauginos are given explicitly in eqs(6.7) and (6.8), and the $SU(3)_c$ interactions are straightforward generalisations of this.

The most direct way to look for these new scalar states is to pair produce them via their electromagnetic interaction. They have been looked for in the process $e^+e^- \to \tilde{q}\bar{\tilde{q}}$ or $\tilde{l}\bar{\tilde{l}}$. Each real scalar has one degree of freedom compared to the four for a Dirac spinor so the result of this calculation is that a charged (complex) scalar state contributes ½ that of its fermion partner. The sleptons are expected to decay via

$$\tilde{l} \to \tilde{X} + l, \qquad (7.17)$$

where $\tilde{X}$ may be the photino, gravitino or higgsino. Thus the general process is

$$e^+e^- \to \tilde{l}^+\tilde{l}^- \qquad (7.18)$$

$$\to l^+l^- + 2\tilde{X}.$$

The current experimental bound on this process places a lower limit on the slepton mass of about 13 Gev if the state $\tilde{X}$ is massless or nearly so.

The squarks decay via

$$\tilde{q} \to \tilde{X} + q \qquad (7.19)$$

where $\tilde{X}$ will be predominantly a gluino if it is light, otherwise a photino, gravitino or higgsino. In the former case the gluino will subsequently decay to a gluon plus a photino, gravitino or higgsino. Because this process is not so clear the bounds obtained for the squark mass are not so stringent, $m_{\tilde{q}} > 10$ Gev.

There are many other processes sensitive to squarks that have been investigated for any process involving quarks will have its analogue involving squarks. For example in deep inelastic scattering squarks and gluinos will be produced by a virtual photon or W boson if kinematically possible. Estimates of this process show that the squark contribution to electroproduction will be 16% of the total cross section at very high momentum transfers. Another potentially interesting process is $W \to \tilde{e}\tilde{\nu}$, $Z \to \tilde{e}\tilde{e}$. We will not attempt a review of these various predictions here but refer the interested reader to the reviews in the Bibliography.

What is the expectation for squark and slepton masses? One principal motivation for supersymmetry was as a solution to the hierarchy problem which basically is that the natural mass scale for scalar states is that of the heaviest state in the theory. Supersymmetry provides a reason why there should be light scalar states but once supersymmetry is broken the natural scale for the mass of the scalars is as large as possible, i.e., of the order of the supersymmetry breaking scale. ($\lesssim 1$ Tev) In models with a large scale of supersymmetry breaking gravity couples universally to all scalars and gives them a tree level mass of $O(m_{3/2} \lesssim 1$ Tev). Radiative corrections from gauge couplings increase this mass so squarks are heavier than sleptons. Models with a lower scale of supersymmetry breaking rely on gauge and Yukawa couplings to transfer the symmetry breaking to the light sector. In this case to the squarks are usually heavier as they couple via the strong interactions. The sleptons can be quite light. Models with a low scale of supersymmetry breaking may induce only small radiative masses for squarks and sleptons, but in the minimal model the mass spectrum is unacceptable and requires additional Abelian factors in the gauge group.

Gauginos and Higgsinos

The new $J=\frac{1}{2}$ fermion states in the theory are the gauge bosons partners, the gauginos and the fermion partners of the Higgs scalars, the Higgsinos. These are expected to contain the lightest new supersymmetric states, but different models have different states as the lightest ones.

The charged states are $\tilde{W}^+_L$, $\tilde{H}^+_{1L}$, $\tilde{W}^-_L$, $\tilde{H}^-_{2L}$. They will mix with the most general form of the mass matrix

$$(\tilde{W}^+ \quad \tilde{H}^+_1)_L \begin{pmatrix} M_2 & g_2 \nu_2 \\ g_2 \nu_1 & \epsilon \end{pmatrix} \begin{pmatrix} \tilde{W}^- \\ \tilde{H}^-_2 \end{pmatrix}_L \qquad (7.20)$$

M_2 is a Majorana gaugino mass which may be anything in the range $< O(1\ \text{Tev})$. ε is a Higgsino mass which in many models is expected to be very small. The mass matrix is diagonalised with mass eigenstates $\chi^{\pm}_{1,2}$

$$
\begin{aligned}
\tilde{\chi}_1^+ &= \tilde{W}^+ \cos\theta_+ - \tilde{H}_1^+ \sin\theta_+ \\
\tilde{\chi}_2^+ &= +\tilde{W}^+ \sin\theta_+ + \tilde{H}_1^+ \cos\theta_+ \\
\tilde{\chi}_1^- &= \tilde{W}^- \cos\theta_- - \tilde{H}_2^- \sin\theta_- \\
\tilde{\chi}_2^- &= +\tilde{W}^- \sin\theta_- + \tilde{H}_2^- \cos\theta_-
\end{aligned}
\tag{7.21}
$$

In the limit $M_2, \varepsilon \to 0$, then $\theta_+ = 0, \theta_- = \frac{\pi}{2}$ and the mass eigenstates are Dirac fermions ie they are the states $(\tilde{W}^+, \tilde{H}_2^-)$ and $(\tilde{W}^-, \tilde{H}_1^+)$ with masses $g_2 v_2$ and $g_2 v_1$ respectively.

In the limit M_2 large, ε_2 0, the mass eigenstates are $(\tilde{W}^+, \tilde{W}^-)$ and $(\tilde{H}_2^-, \tilde{H}_1^+)$ with masses M_2, $\frac{g_2^2 v_1 v_2}{M_2}$ respectively.

In both cases and for most ranges of parameters we have a new charged fermion $\tilde{\chi}^{\pm}$ with mass $m_{\chi^{\pm}} < m_W$.

The neutral sector is even more complicated for there are four neutral supersymmetric fermions which can mix. This mass matrix is

$$
(\tilde{W}^3, \tilde{B}^o, \tilde{H}_1^o, \tilde{H}_2^o)
\begin{bmatrix}
M_2 & 0 & \frac{-g_2 v_1}{2} & \frac{g_2 v_2}{2} \\
0 & \frac{5}{3}\frac{\alpha_1}{\alpha_2} M_2 & \frac{g_1 v_1}{2} & -\frac{g_1 v_2}{2} \\
\frac{-g_2 v_1}{2} & \frac{g_1 v_1}{2} & 0 & \varepsilon \\
\frac{g_2 v_2}{2} & -\frac{g_1 v_1}{2} & & 0
\end{bmatrix}
\begin{pmatrix} \tilde{W}^3 \\ \tilde{B}^o \\ \tilde{H}_1^o \\ \tilde{H}_2^o \end{pmatrix} .
\tag{7.22}
$$

Again this must be diagonalized. In the limit $M_2, \varepsilon \to 0$ the mass eigenstates are

$$
\tilde{\gamma} = \frac{g_1 \tilde{W}^3 + g_2 \tilde{B}^o}{g_1^2 + g_2^2} \quad ; \quad m_{\tilde{\gamma}} = \frac{8}{3} \frac{g_1^2}{g_1^2 + g_2^2} M_2
$$

$$
\tilde{Z}_1^o \equiv \frac{g_1 \tilde{B}^o - g_2 \tilde{W}^3 \pm (g_1^2 + g_2^2) \tilde{A}^o}{2(g_1^2 + g_2^2)} \quad ; \quad m_{\tilde{Z}_\pm^o} = m_{Z^o} = \frac{(g_1^2 + g_2^2)}{2} v
$$

$$\tilde{S}^o = \frac{v_2\tilde{H}_1^o + v_1\tilde{H}_2^o}{v} \quad ; \qquad m_{\tilde{S}^o} = \frac{2v_1v_2}{v^2}\varepsilon ,$$

where

$$\tilde{A}^o \equiv \frac{v_1\tilde{H}_1^o - v_2\tilde{H}_2^o}{v} \qquad (7.23)$$

and $v = v_1^2+v_2^2$.

In the limit M_2 large, ε small $\tilde{A}^o$ and $\tilde{S}^o$ are mass eigenstates together with $\tilde{W}^3$ and $\tilde{B}^o$.

In general the two lightest neutral fermions are lighter than the lightest charged fermion.

Thus the expectation is that there should be new fermion states $\tilde{X}^{\pm}, \tilde{X}^o, \tilde{X}'^o$ with mass less, and often much less than the W boson. This leads to interesting new decay possibilities for the W boson which may be used to look for these states

$$W^{\pm} \to \tilde{X}^{\pm} + \tilde{X}^o$$
$$\hookrightarrow \tilde{X}^o + (e\nu, \mu\nu, \tau\nu \text{ or } q\bar{q}) \qquad (7.24)$$

$$W^{\pm} \to X^{\pm} + X'^o$$
$$X^o + (\nu\bar{\nu}, \ell^+\ell^- \text{ or } \bar{q}q)$$
$$X^o + (e\nu, \mu\nu, \tau\nu \text{ or } q\bar{q}) .$$

The rate for these events depends on the unknown parameters in the mass matrices eq.(7.20) and (7.22). For a large range of these parameters the rate is close to that for $W^{\pm} \to e^{\pm}_{\nu}$ decays.

These events have characteristic patterns and can be distinguished from heavy lepton production by forward backward asymmetry measurement.

Similarly there are new decay modes for the Z, eg

$$Z \to \tilde{X}^{o\prime}\tilde{X}^{o\prime}$$
$$\hookrightarrow \tilde{X}^o + (\nu\bar{\nu}, \ell^+\ell^- \text{ or } \bar{q}q) \qquad (7.25)$$
$$\hookrightarrow \tilde{X}^o + (\nu\bar{\nu}, \ell^+\ell^- \text{ or } q\bar{q}) .$$

We do not have time to discuss in full these and other possible signals for these new fermonic states. They should be readily produced and observed once the threshold for their production is passed and the liklihood is that this threshold should be less than M_W.

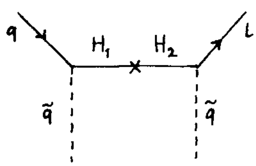
Higgs scalars

Although Higgs scalars are also expected in non-supersymmetric models, as we saw in section (6.1), supersymmetry requires at least two doublets of

Higgs scalars. As a result there should be, after spontaneous symmetry breakdown, two residual charged Higgs and three neutral Higgs scalars, a much richer structure than in the standard model. These are given in Table 7.1 with their expected masses and characteristic couplings. Note there is expected a very light neutral pseudoscalar Higgs, a, coupling more to charge $-\frac{1}{3}$ quarks. As shown in Table 7.1 a is made up of the "uneaten" pseudoscalar components η_1 and η_2 of H_1 and H_2.

Table 7.1. Higgs scalar states. λ is a Yukawa coupling expected to be much smaller than g_2.

scalar state	Mass	Characteristics
$(\nu_1 H_2^0 + \nu_2 H_1^0)/\nu$ +..	$4\lambda^2\nu_2^2$ +...	I=1/2, mainly coupled to charge - 1/3 quarks
$(\nu_1 H_1^0 - \nu_2 H_2^0)/\nu$	$(1/2)(g_2^2+g_1^2)\nu^2$+...	I=1/2, mainly coupled to charge + 2/3 quarks
$a \equiv (\eta_1\nu_2 - \eta_2\nu_1)/\nu$	few Gev?	pseudoscalar coupling, larger for charge-1/3 quarks
$(\nu_1 H_2^{\pm} - \nu_2 H_1)/\nu$	$(1/2)g_2^2\nu^2$+...	conventional charged Higgs boson



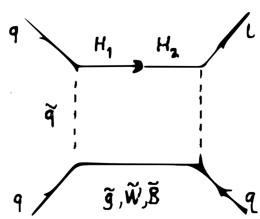


Fig.7.1.

(a) Leading baryon number graph in minimal supersymmetric SU(5)

(b) Gaugino exchange leading to baryon number four fermion operator from the Born graph of Fig.7.1(a).

Appendix

Fermions in Grand Unified Theories are most conveniently described in terms of helicity states. Under the Lorentz group ψ transform as

$$\psi \rightarrow S\psi = e^{\omega\sigma}\psi, \qquad (A1)$$

where

$$\omega\sigma = \omega^{\mu\nu}\sigma_{\mu\nu}, \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \qquad (A2)$$

Since $\sigma_{\mu\nu}$ commutes with γ_5 this transformation does not mix states of different helicity. Define left (right) handed helicity state $\psi_{L(R)}$

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi \qquad \text{(A3)}$$
$$\psi_R = \tfrac{1}{2}(1+\gamma_5)\psi .$$

Then $\psi_{L(R)}$ transform independently under a Lorentz transformation.

It is most convenient when dealing with helicity states to use the chiral (Weyl) representation for γ matrices

$$\gamma_0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \text{(A4)}$$

where are the 2x2 Pauli matrices. They obey

$$\{\gamma_\mu, \gamma_\mu\} = 2g_{\mu\nu} . \qquad \text{(A5)}$$

Now ψ_L and ψ_R are two component spinors corresponding to the upper two and lower two indices of the Dirac spinor ψ ($\psi = \binom{\psi_L}{\psi_R}$). These are Weyl spinors and we will find them useful in building unified theories. For rotations in this representation eq (A1) gives

$$S_{L(R)} = e^{i\frac{\sigma}{2}\cdot\omega} . \qquad \text{(A6)}$$

Boosts in this basis are

$$S_{L(R)} = e^{\pm\frac{\sigma}{2}\cdot\nu} . \qquad \text{(A7)}$$

<u>Charge conjugation</u>

Under a Lorentz transformation the combination $\sigma^2\psi_L^*$ transforms as

$$\sigma^2\psi_L^* \to \sigma^2 S_L^* \psi_L^*$$
$$= \sigma^2 S_L^* \sigma^2 \sigma^2 \psi_L^*$$
$$= S_R \sigma^2 \psi_L^* , \qquad \text{(A8)}$$

where the star denotes complex conjugation and we have used the property of Pauli matrices

$$\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*} \qquad \text{(A9)}$$

to show $$\sigma^2 S_L \sigma^2 = e^{-\frac{i}{2}\vec{\sigma}^*\cdot(\vec{\omega}-i\vec{\nu})} = S_R^* . \qquad \text{(A10)}$$

Eq. (A8) shows that $\sigma^2\psi_L^*$ transforms as $(0,\frac{1}{2})$, i.e., as a right handed spinor. Similarly $\sigma^2\psi_R^*$ transforms as $(\frac{1}{2},0)$ as a left handed spinor under the Lorentz group SL(2,6).

Thus when writing the states in our theory we may use the left and right handed components of a massive fermion or we may use the above transformations to express all components as left handed or right handed. This we will find useful in constructing Grand Unified Theories. We will adopt the usual notation

$$\psi_L^{\,c} = \sigma^2 \psi_R^{*}$$
$$\psi_R^{\,c} = \sigma^2 \psi_L^{*}, \qquad (A.11)$$

where ψ^c is the charge conjugate spinor and eq.(A.11) defines the operation of charge conjugation.

Fermion Masses

It is now easy to build invariants using these representations of the Lorentz group. For example if χ_L and ψ_L are two spinors transforming as $(\frac{1}{2},0)$ then under a Lorentz transformation the quantity $\chi_L^T\sigma^2\psi_L$ is invariant

$$\chi_L^T\sigma^2\psi_L \rightarrow \chi_L^T S_L^T\sigma^2 S_L\psi_L$$
$$=\chi_L^T \sigma^2 \sigma^2 S_L^T \sigma^2 S_L \psi_L \qquad (A.12)$$
$$=\chi_L^T \sigma^2 \psi_2 ,$$

where we have used eq.(A.10) and the hermiticity of the Pauli matrices. With $\chi_L = \sigma^2 \psi_R^*$ this invariant is $i(\sigma_2 \psi_R^*)^T\sigma^2 \psi_L = -i\psi_R^T\psi_L$. (A.13)

In 4 component notation this is the familiar Dirac mass term

$$\psi_R^+\psi_L + \psi_L^+ \psi_R = \psi^+ \gamma^0\psi \equiv \bar{\psi}\psi . \qquad (A.14)$$

However eq.(A.12) shows us there is a further possibility for a mass term for if $\chi_L = \psi_L$ we have the Lorentz invariant quantity. $\psi_L^T\sigma^2\psi_L = \psi_L^c \psi_L$. (A.15)

This is known as a Majorana mass term. It is not invariant under an U(1) transformation $\psi_L \rightarrow e^{i\alpha}\psi_L$ and so any quantum number carried by ψ_L, such as charge, lepton number, is broken if ψ_L has a Majorana mass.

References

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