MATTER AND GEOMETRY IN A UNIFIED THEORY

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The conditions which are imposed by mathematical axioms can in general only within limits be fulfilled by physical objects. The integers which occur in arithmetics may still rather well be in harmony with atomistic physics. Points, lines, planes, etc., defined by the continuum in geometry obey however definite relations which can at best in crude approximation be identified with measurable physical systems. This is apparent from the one to one mapping of sets of the continuum on subsets. One can expect from the foregoing that any good and therefore clear physical theory involving a continuum will lead eventually to extreme results where physics can no longer do justice to the axioms so that no reasonable person can believe in the absurdity of its predictions. Riemann had already recognized the problem of the continuum in the complementarity of geometry and physics for the description of nature. He devoted a section of his habilitation work to a discrete description. One can not expect that in the sophisticated spacetime continuum of the general theory of relativity the consequences of Riemann's critique of such sharply contoured geometric constructions as points, lines – and even the light cone, will not come to the light when describing extreme physical situations. This is already known in microscopic physics where the uncertainty relation rules out the identification of points with physical objects. The persistence of the curse of the thirteenth fairy – (with which Schrödinger poetically compares the continuum because it proceeded the birth of our science) – results strangely enough from macroscopic physics. The Einstein-Hilbert equations of general relativity predict inevitably the gravitational collapse of a sufficiently large cloud of dust to a point, irrespective of the nature of the short range interaction between the dust particles. The point of view that this extreme result is a manifestation of the predicted absurdity and has not the character of a physical law, is not shared today by many physicists. Einstein himself and also Schrödinger did however not advocate the last mentioned trend. This is witnessed by the article [1] which introduces a modified interpretation of the field equations to abandon the domain beyond the horizon. One sees in the apparent inevitability of gravitational collapse rather one of the greatest revolutions in our physical world picture. The argument for this collapse is based on the fact that the curvature near the formation of a horizon can remain small. The principle of equivalence seems then to rule out any reason why physics should be different in this domain than in others so that a given solution of the Einstein-Hilbert field equations does apply everywhere.

The author's counter-argument is based on macroscopic quantum effects induced by the curvature. The earliest discovered of these effects is Schrödinger's alarming phenomenon of elementary particle pairs created in the time dependent metric of an expanding universe [2]. Associated with it are the contributions of virtual elementary particle pair effects, which became known as the gravitational analogues of the Uehling term of the Lamb shift [3] and of the Casimir effect [4]. There exist other more complicated contributions of quantized fields in classical gravitational fields, even in low order approximations. The terms due to virtual particle contributions are in general divergent and non-renormalizable. Every quantum field contributes to an additional source term of the gravitational field equations. The gravitational field itself has also to be considered – but we can hardly do more than speculate about the microscopic manifestations of the gravitational field. Solutions of the classical Einstein-Hilbert equations can not account for the appearance of such source terms.

We summarize the conclusions we draw from the foregoing considerations:

1. The gravitational collapse of dust to a geometrical point predicted by classical general relativity is tentatively considered as an absurdity of the kind discussed.
2. Our knowledge about quantum effects in classical gravitational fields excludes a *rigorous* macroscopic description of extreme situations in terms of the Einstein-Hilbert equations alone. These cannot produce the Schrödinger phenomenon and thus also not its virtual manifestations which ought to be considered *before* conclusions about horizon formation are drawn.
3. We lack empirical knowledge about the structure of the gravitational interaction in microscopic regions and lack adequate knowledge of particle – and field theory to even estimate the magnitude of the mentioned macroscopic quantum effects in classical gravitational fields.
4. We have no direct observational criterium to distinguish a highly collapsed system from a true black hole with a horizon.
5. To avoid the fallacies cited under point 3 we search for modified macroscopic equations which are hoped to include the macroscopic quantum effects in average and tend to eliminate the absurdity. These equations must be of higher order than the second and must give a good approximation to general relativity in less extreme situations. The coupling of matter with gravitation should contain nonminimal terms to produce

the Schrödinger phenomenon.

What about the principle of equivalence, Einstein's ingenious bridge between physics and geometry? Einstein and Rosen [1] conclude with its help that something different than the vacuum must be found (latest) at the horizon – if the rest of physics is to remain valid. They postulate there a source in accordance with Einsteins geometrization program. On the other hand the ultrarelativists – those who for one or the other reason follow an orthodox course without much consideration about results of quantum and particle physics – they conclude from the principle of equivalence that space-time has to be extended unaltered beyond the horizon until it ends in a singularity. Their use of the prescribed mathematics is certainly correct – but they risk to fall just because of this onto the mentioned absurdity. The Schrödinger phenomenon demands modifications of the classical equations already before and outside the horizon.

Equations with an admixture of fourth order terms, derivable from a Lagrangian of the form:

$$\mathcal{L} = \sqrt{g}\,(R + aR^2 + bR_{hijk}\,R^{hijk}) \qquad (1)$$

a,b constants of dimension (length) have early been considered [2]. Their vacuum solutions include all of those of general relativity. Other physically significant vacuum solutions are not known. The presence of matter requires here solutions different from general relativity but none are known either. Other lower order effects of quantum field theory are even more difficult to incorporate into classical equations.

The search for modified equations need not to be restricted to the perturbation formalism of quantum field theory. The approach from a gauge principle and in particular from Kałuża-Klein models appears promising. The latter achieve a quasi-unification of general relativity in interaction with a gauge field of vanishing rest mass in a (somewhat mutilated) metrical space of $4+n$ dimensions. The Schrödinger phenomenon, of particular interest for a massless gauge field, does not appear in the classical theory. A nonmininal interaction is required to obtain it classically. There are too many possibilities to arrive at such equations. We shall follow one way led by an early attempt of the author to describe the inner quantum number of spin by a higher dimensional Kałuża-Klein generalization. The gauge group is in the simplest case that of the tetrade rotations. The theory has the unique feature of convertability of the inner quantum number (spin) into a dynamical variable (angular momentum) [5,6].

The theory is formulated on the ten-dimensional manifold of the anti-De Sitter group $G = SO(3,2)$. The subgroup $H = SO(3,1)$ is the gauge group. The principal fibre bundle $P(G,\ H,\ G/H,\ \pi)$ has the anti-De Sitter universe with the topology of G/H as base manifold and the natural projection π : $G \to G/H$. The Cartan-Killing metric γ of every semi-simple Lie group G_r,

$$\gamma_{uv} = \mathrm{Tr}\{(\mathrm{Ad}A_u) \circ (\mathrm{Ad}A_v)\}, \tag{2}$$

with A_u, A_v left invariant vectors of G_r, fulfills Einstein equations,

$$R_{uv} - \tfrac{1}{2}\gamma_{uv}\left(R - \tfrac{r-2}{4}\right) = 0. \tag{3}$$

A metric $g = \pi'\,\gamma$ is then defined on the base. It is in this case the anti-De Sitter metric which fulfills:

$$B_{ik} - \tfrac{1}{2}g_{ik}\,(B+1) = 0 \tag{4}$$

with B_{ik} the Ricci tensor of the base space. The left invariant vectors A_R ($R = 1\dots 10$) are Killing vectors of γ. We shall label henceforth indices pertaining to the base space by letters $A\dots L$ running from $1\dots 4$ and those pertaining to the fibre by letters $M\dots Q$ running from $5\dots 10$. General indices $R\dots Z$ run from $1\dots 10$. This rule will be applied without further warning also to the Einstein summation convention.

We consider more general metrics γ which are solutions of the Einstein equations (3) and keep the six Killing's vectors with unaltered commutation relations on each fibre,

$$[A_P,\, A_Q] = c^M_{PQ}\, A_M. \tag{5}$$

The structure of the principal fibre bundle P and of the subgroup H on the fibres thus still exists. In the space perpendicular to the A_M there exist four orthonormal vector fields A_E with the unaltered commutation relations of the group G:

$$[A_E,\, A_M] = c^H_{EM}\, A_H \tag{6}$$

only the commutation relations:

$$[A_E,\, A_F] = C^R_{EF}(x)\, A_R \tag{7}$$

are modified to base point dependent general structure constants.

The metric γ defines a connection on P with horizontal vectors A_E perpendicular to the fibre. The generalized structure constants C^M_{EH}, C^J_{EH} determine respectively curvature and torsion two forms over the base. The topology of the base remains that of the anti- De Sitter universe, but the metric $g = \pi'\gamma$ is now generalized.

The construction constitutes a generalized classical Kałuża-Klein theory with a gauge field F^M which is determined by the C^M_{EH}. The geometry on the base is non-Riemannian. The torsion two-form is in general not vanishing. The gauge group H is a pseudo orthogonal subgroup of $GL(4,\mathbb{R})$ which allows the decomposition of the connection into a Riemannian part and contortion,

$$\Gamma^{H}{}_{EF} = \left\{ {H \atop EF} \right\} - K^{H}_{EF}, \tag{8}$$

$$K^{H}_{EF} = \frac{1}{2}\left(T^{H}_{EF} + T_{FE}{}^{H} + T_{EF}{}^{H}\right) \tag{9}$$

with T the torsion tensor.

The components of the curvature tensor F of the two form F^M can likewise be decomposed:

$$F_{AEIJ} = B_{AEIJ} + Q_{AEIJ}, \tag{10}$$

$$Q_{AEIJ} = K_{AEI;J} - K_{AEJ;I} + K_{AHI}\, K^{H}_{EJ} - K_{AHJ}\, K^{H}_{EI} \tag{11}$$

with the Riemann tensor B and contortion K. The semicolon denoting the Riemannian covariant derivative.

Such a decomposition cannot be achieved with the full $GL(4,\mathbb{R})$ as gauge group. The assumptions about Riemannian curvature found in the literature [7] in connection with this gauge group can thus in general not be right. See ref. 6.

The purely vertical component of the ten-dimensional equation (3) is eliminated with Lagrange multipliers to restrict only to such solutions for which the natural metric on the fibres is preserved and the Planck length (in units with $\hbar = c = 1$ the square root of the gravitational constant G) is introduced on the base manifold as physical unit of length instead of that of the radius of the universe. The theory cannot yield a relationship between these two lengths without altering the topology of the manifolds.

The mixed horizontal-vertical components of equation (3) are

$$F^{A}{}_{HI}{}^{J}{}_{;J} = 0 \tag{12}$$

this becomes if torsion vanishes

$$B^{A}{}_{HI}{}^{J}{}_{;J} = 0 \tag{13}$$

and due to the Bianchi identities:

$$B^{A}{}_{H;I} - B^{A}{}_{I;H} = 0 \tag{14}$$

related by Yang to a gauge theory of $GL(4,\mathbb{R})$ [7]. The absence of torsion which can in this case not be separated, is not accounted for in Yang's paper and equation (14) alone also admit unphysical solutions. Yet the term (12) is the Riemannian analog of Maxwell's equations. It is supplemented in equation (11) by a source formed out of torsion and by the purely horizontal components of equation (3),

$$B^I_E - \tfrac{3}{2} G F_{AHDE}\ F^{AHDI} - \tfrac{1}{2}\delta^I_E \left(B - \tfrac{3}{4} G F_{AHDJ}\ F^{AHDJ} + 1\right) = 0 \quad (15)$$

We are inclined to relate the torsion term of equation (12) to a nonminimal interaction of torsion with elementary particle spin. Equation (13) admits all vacuum solutions of general relativity. Equation (14) consists of the Einstein's term with cosmological member and the energy-momentum tensor of the Yang-Mills field, which can be decomposed again into metric curvature and torsion; it is of vanishing trace. Vanishing torsion leaves this term bilinear in the metric curvature – apparently an additional vacuum energy of virtual matter fields which remains small with the curvature. The real field part is bilinear in Q and the term linear in B and Q constitutes the nonminimal interaction which can give rise to particle creation by gravitation, the Schrödinger phenomenon, of which even the virtual part appears. Einstein's request for the geometric expression of the matter tensor is fulfilled – yet as its vanishing trace shows, the model describes only very special matter. The spherically symmetric vacuum solution of general realtivity satisfies also equations (12,13,14) but other solutions of Einstein's theory in general do not, due to the nonlinear term.

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