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## Teaching renormalization, scaling, and universality with an example from quantum mechanics

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E-mail: [paik\\_steve@smc.edu](mailto:paik_steve@smc.edu)**Keywords:** renormalization, scaling, inverse square potential, universality, critical exponents, graduate level pedagogy**Abstract**

We discuss the quantum mechanics of a particle restricted to the half-line  $x > 0$  with potential energy  $V = \alpha/x^2$  for  $-1/4 < \alpha < 0$ . It is known that two scale-invariant theories may be defined. By regularizing the near-origin behavior of the potential by a finite square well with variable width  $b$  and depth  $g$ , it is shown how these two scale-invariant theories occupy fixed points in the resulting  $(b, g)$ -space of Hamiltonians. A renormalization group (RG) flow exists in this space and scaling variables are shown to exist in a neighborhood of the fixed points. Consequently, the propagator of the regulated theory enjoys homogeneous scaling laws close to the fixed points. Using RG arguments it is possible to discern the functional form of the propagator for long distances and long imaginary times, thus demonstrating the extent to which fixed points control the behavior of the cut-off theory. By keeping the width fixed and varying only the well depth, we show how the mean position of a bound state diverges as  $g$  approaches a critical value. It is proven that the exponent characterizing the divergence is universal in the sense that its value is independent of the choice of regulator. Two classical interpretations of the results are discussed: standard Brownian motion on the real line, and the free energy of a certain one-dimensional chain of particles with prescribed boundary conditions. In the former example,  $V$  appears as part of an expectation value in the Feynman–Kac formula. In the latter example,  $V$  appears as the background potential for the chain, and the loss of extensivity is dictated by a universal power law.

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**1. Introduction**

The inverse-square potential in quantum mechanics has a rich history that continues to be updated in light of new connections to diverse physical phenomena including electron capture by neutral polar molecules [1], the Efimov effect in a system of three identical bosons [2], the transition between asymptotically free and conformal phases in QCD-like theories as a function of the ratio of the number of quark flavors to colors, and the AdS/CFT correspondence [3]<sup>1</sup>. A common thread running through these applications, and the reason for our interest, is that the attractive  $1/x^2$  potential is a fascinating case study that naturally calls upon the framework of the renormalization group (RG). We will see that the RG approach mirrors, in many ways, the modern treatment of quantum effective field theories whereby one demands that long-distance observables remain insensitive to the adjustment of fine details at short distances.

The purpose of this article is twofold. It is primarily a pedagogical treatment of renormalization in the context of single-particle quantum mechanics and it is intended for teachers of quantum field theory. We feel that the example presented herein can provide an instructive introduction to basic RG ideas and terminology. Because our RG analysis involves a quantum theory where one maintains full nonperturbative control, teachers may find that it serves as a useful aid for the beginning graduate student who is learning field theoretic renormalization, but having difficulty separating the core principles from the technology required to do

<sup>1</sup> Many topics beyond the scope of our article are discussed in this paper; the regularization and renormalization of the inverse-square potential occupies section 3.

perturbative renormalization with many degrees of freedom. We should mention that other pedagogical presentations of the renormalization of the inverse-square potential exist. Readers may wish to consult [4, 5] for additional background. However, in these works they study the case  $\alpha < -1/4$ .

The other purpose of the article is to continue exploring the implications of the RG results uncovered in the work of Kaplan *et al* [3] (see footnote 1). Although their work is certainly not the first instance in which the inverse-square potential is discussed, it is notable for, among other things, discussions of the beta function, operator anomalous dimensions at the fixed points, and the general phenomenon of conformal to non-conformal phase transitions. In particular, we follow up their understanding of the fixed-point structure with a natural extension to the quantum mechanical propagator, and we use well-known quantum–classical equivalences to extract statistical lessons for specific one-dimensional classical systems.

Consider a particle in one spatial dimension subject to the potential

$$V(x) = \begin{cases} \infty & x \leq 0 \\ \alpha/x^2 & x > 0 \end{cases} \quad (1)$$

for  $-1/4 < \alpha < 0$ . We work in units where  $\hbar = 1$  so that energy is the reciprocal of time, and  $\hbar^2/2m = 1$  so that energy is also the reciprocal of length-squared. This means that time and length-squared have equivalent dimensions. In these units,  $\alpha$  is a dimensionless number whose value we do not imagine changing in any of our analyses.

The paper is organized as follows. In section 2 we explain why the choice  $-1/4 < \alpha < 0$ , although giving perfectly consistent dynamics, is still peculiar. We then explain that two distinct Hilbert spaces may be defined. In section 3 the Hamiltonian is modified at short-distances at the expense of a dimensionful length scale and a coupling constant so that these two Hilbert spaces meld into one. In section 4 we discuss how the process of renormalization allows one to continuously vary the short-distance modification without affecting a long-distance observable. This naturally leads to the construction of the propagator in section 5 and analysis of its properties in section 6 when there is a continuous spectrum. In section 7 we analyze the low-energy discrete spectrum when it exists. Classical applications of quantum mechanics are given in section 8. Finding these applications were, in fact, the original source of motivation for this work. In section 9 remarks that generalize the inverse-square potential to three dimensions and  $\alpha < -1/4$  are given.

Since this paper is part pedagogical guide, part research, a remark about which portions are novel is in order. Sections 2–4 follow the path set forth [3] (set forth in footnote 1), although we provide a different interpretation of the fixed points than in that work. To the best of our knowledge, sections 5–8 are new. Lastly, some of the remarks made in section 9 briefly summarize what is already well-established in the literature regarding the renormalization of the  $\alpha < -1/4$  case.

## 2. Pure inverse-square potential

Does the Hamiltonian given by  $H = P^2 + V(X)$  define a sensible quantum theory? To answer this question one may proceed to construct the physical Hilbert space of states  $\mathcal{H}$  over which the operators  $X, P$ , and  $H$  are self-adjoint. It is important to remember that the self-adjointness property is not inherent to a differential expression for an operator—one must also consider the vector space of functions on which it acts and the boundary conditions satisfied by those functions. And so it should be stressed that the physical Hilbert space is not necessarily the space of square-integrable complex-valued functions  $L^2(0, \infty)$  equipped with the standard inner product  $(\cdot, \cdot)$ . While this is a bonafide Hilbert space in the functional analysis sense and is consistent with the degrees of freedom available to a single spinless particle moving along a line, it is not necessarily the physical Hilbert space because certain functions exist whose behavior at  $x = 0$  or  $x = \infty$  ruin the hermiticity of observables. Nevertheless, it is certainly true that  $\mathcal{H} \subset L^2(0, \infty)$ . We shall construct  $\mathcal{H}$  by forming linear combinations of the eigenfunctions of  $H$ . It is crucial that  $\mathcal{H}$  be complete with respect to the distance function inherited from the inner product so that any Cauchy sequence built from elements of  $\mathcal{H}$  converge in  $L^2$ -norm to a limit also in  $\mathcal{H}$ . This is guaranteed, according to Sturm–Liouville theory, for a self-adjoint Hamiltonian [6].

The position operator  $X$  is already self-adjoint in the space  $L^2(0, \infty)$ . The momentum operator  $P$  is self-adjoint if and only if  $[f^*g]_0^\infty = 0$  for any  $f, g \in \mathcal{H}$ . The energy operator  $H$  is self-adjoint if and only if  $[f^*g' - f'^*g]_0^\infty = 0$ . Both of these conditions arise as boundary terms resulting from an integration by parts in trying to establish the equality of  $(f, \mathcal{O}g)$  and  $(\mathcal{O}f, g)$ .

The general solution to the eigenvalue equation

$$\psi'' = (\alpha/x^2 - E)\psi \quad (2)$$

may be constructed as a linear combination of two Frobenius series. For  $E \neq 0$  and  $x \ll |E|^{-1/2}$ , the solution is  $\psi \sim C_+x^{1/4} + C_-x^{3/4}$ , where

$$\nu_{\pm} = \frac{1}{2} \pm \omega, \quad \omega = \sqrt{\frac{1}{4} + \alpha},$$

are the roots of the characteristic equation  $\nu(\nu - 1) = \alpha$ . We note that the range  $-1/4 < \alpha < 0$  implies  $0 < \omega < 1/2$ , so both solutions satisfy  $\psi(0) = 0$ . We will see that the regularity of both solutions is what makes the quantum mechanics of the inverse-square potential so interesting.

It is straightforward to construct eigenfunctions of  $H$  using series, and it turns out that they define ordinary or modified Bessel functions. The simplest way to see this is to define a dimensionless variable  $x = |E|^{-1/2}\xi$  and let  $\psi = \xi^{1/2}\varphi(\xi)$ . The resulting ode for  $\varphi$  is a variant of Bessel's equation. For  $E < 0$  there is a unique linear combination of the independent solutions (with  $0 < |C_+/C_-| < \infty$ ) that exhibits asymptotic exponential decay and is therefore square-integrable, namely  $\psi_E = x^{1/2}K_{\omega}(|E|^{1/2}x)$ . Here  $K$  is a modified Bessel function of the second kind. Such an eigenfunction is inadmissible as an element of  $\mathcal{H}$  for several related reasons. The scale invariance of the eigenvalue equation implies that if  $\psi_E$  is a normalized state, then so is  $\psi_{\lambda^2 E} = \lambda^{1/2}\psi_E(\lambda x)$  for  $\lambda > 0$ . But that would indicate a continuum of square-integrable states, a situation which directly contradicts the Feynman–Hellmann theorem as  $\lambda$  is not an explicit parameter of  $H$  [7]. Furthermore,  $\psi_E$  and  $\psi_{\lambda^2 E}$  are not orthogonal when  $\lambda \neq 1$ . Most damning of all, by scaling we can make  $E$  arbitrarily negative so there is no ground state!

The root of these problems lie in the fact that  $\psi_E$  fails to satisfy the required boundary conditions. Generally, a solution to equation (2) that is also square-integrable vanishes at spatial infinity and has vanishing first derivative. In addition,  $\psi_E$  also vanishes at the origin. Therefore,  $P$  is self-adjoint with regard to such eigenfunctions. However, self-adjointness of  $H$  requires

$$f^*(0)g'(0) - f^{*'}(0)g(0) = 0. \quad (3)$$

But

$$\lim_{x \downarrow 0} \left[ \psi_E(x) \frac{d}{dx} \psi_{\lambda^2 E}(x) - \psi_{\lambda^2 E}(x) \frac{d}{dx} \psi_E(x) \right] \propto \frac{C_+}{C_-} \omega (\lambda^{\nu_+} - \lambda^{\nu_-}) \neq 0.$$

Essentially, hermiticity fails because both near-origin solutions  $x^{\nu_{\pm}}$  are acceptable. One can force the self-adjointness of  $H$  by artificially introducing another boundary condition to select one solution or the other, or by modifying the near-origin behavior of the potential.

For  $E > 0$  a continuum of eigenfunctions exist,

$$e_k(x) = (kx)^{1/2} J_{\pm\omega}(kx), \quad k = E^{1/2}.$$

Both signs satisfy the important closure relation  $\int_0^{\infty} e_k(x)e_k(y)dk = \delta(x - y)$  since the order of the Bessel function is greater than  $-1/2$  [8]. Since both positive and negative orders give independent representations of the identity operator, we have the freedom to use either sign in forming physical states. That is, consider the subspace of  $L^2(0, \infty)$  whose elements may be expressed as

$$f(x) = \int_0^{\infty} \tilde{f}(k)e_k(x)dk.$$

Here we assume that  $\|f\| = \|\tilde{f}\| = 1$  so that the integral exists by Plancherel's theorem. This describes an inverse Hankel transform. A function constructed by this kind of superposition inherits the same near-origin behavior as that of the Bessel  $J$  function. To wit, near the origin,  $e_k(x) \sim (E^{1/2}x)^{\nu_{\pm}}$ . Notice that the dependence on  $E$  and  $x$  is separable. Therefore, in an integral of the form  $\int_0^{\infty} g(E)e_k(x)dE$ , where the  $x$ -dependence is parametric, the result will be proportional to  $x^{\nu_{\pm}}$ . This ensures that  $H$  is self-adjoint and that one obtains two possible Hilbert spaces  $\mathcal{H}_{\pm}$  depending on the sign chosen in the order of the Bessel function. A final remark:  $E = 0$  is the infimum of the eigenvalues, but strictly speaking there is no zero-energy eigenstate.

There is a simple physical argument that explains why the Hilbert space is not unique [9]. Imagine scattering a de Broglie wave of fixed energy  $E > 0$  coming from  $x = \infty$ . This is somewhat artificial since we should really speak in terms of a normalizable wave packet, but such extra rigor does not change the essential conclusion. For  $x \gg k^{-1}$  the wavefunction takes the form  $e^{-ikx} + re^{ikx}$ . It should be possible to express the reflection amplitude  $r$  in terms of  $C_+/C_-$ , however there is no condition that determines the value of  $C_+/C_-$  itself. Ordinarily, a boundary condition like  $\psi(0) = 0$  suffices to fix the ratio of coefficients of linearly independent solutions. In this situation,  $\psi(0) = 0$  rules out neither  $x^{\nu_+}$  nor  $x^{\nu_-}$ .

### 3. Breaking scale invariance

Our goal is to connect the two scale-invariant theories  $\mathcal{H}_{\pm}$  in the Wilsonian sense by linking them in a continuous space of Hamiltonians. However, the kind of explicit symmetry breaking needed in  $V$  must take place at  $x = 0$  and must be sufficient to resolve the singularity. When framed as a two-body problem in three

dimensions, it is obvious that ‘near-origin’ is synonymous with ‘short-distance’ (i.e., nearly coincident particles).

Before moving on it is worth mentioning what happens if  $V$  is modified at long distances instead of short distances. For example, one analytically attractive method is to add a harmonic trap,  $\Omega^2 x^2$ . Here  $\Omega^{-1/2}$  serves as an explicit length scale. Miraculously, there exist raising and lowering operators that create two entirely independent ladders of states with equally spaced rungs [7]. Or, imagine a hard wall at some position  $x = L$ . This imposes a quantization of energies related to the zeros of the Bessel  $J$  function. The zeros may be those of  $J_{+\omega}$  or  $J_{-\omega}$ , which are, in general, distinct.

### 3.1. Regularizing with a square well

Modify the potential in equation (1) so that it reads

$$V(x) = \begin{cases} -g/(bx_0)^2 & 0 < x < bx_0 \\ \alpha/x^2 & x > bx_0 \end{cases}, \quad (4)$$

where  $g > 0$  and  $b > 0$  are dimensionless parameters, and we regard  $x_0$  as a fixed length scale. As usual, hermiticity of the momentum and kinetic energy operators  $-id/dx$  and  $-d^2/dx^2$  constrain a wavefunction and its derivative to exist and be everywhere continuous, even at the jump discontinuity in  $V$  [10].

Our expressions are naturally stated in terms of a dimensionless wavenumber,

$$\xi = bx_0|E|^{1/2}.$$

### 3.2. Discrete eigenfunctions

Let  $E < 0$ . The exact bound state eigenfunction is

$$\psi_{E<0} = \begin{cases} A \sin(x\sqrt{E + g/b^2x_0^2}) & 0 \leq x < bx_0 \\ C\sqrt{|E|^{1/2}x}K_\omega(|E|^{1/2}x) & x > bx_0 \end{cases}.$$

The allowed energies follow from

$$\sqrt{g - \xi^2} \cot \sqrt{g - \xi^2} = \frac{1}{2} + \xi \frac{K'_\omega(\xi)}{K_\omega(\xi)}. \quad (5)$$

Constants  $A$  and  $C$  are fixed by continuity and normalization. Imagine graphing each side of equation (5) with respect to  $\xi$ : the left side is monotone increasing but the right side is monotone decreasing. Thus, a root occurs only if the left side starts somewhere below, or at, the starting point of the right side. By making use of the identity  $\lim_{\xi \downarrow 0} \xi K'_\omega(\xi)/K_\omega(\xi) = -\omega$ , the root  $\xi = 0$  obtains when  $\sqrt{g} \cot \sqrt{g} = \nu_-$ . Let us denote the solution to this equation by  $g_-$ . Therefore, a valid bound state exists when  $g > g_-$ .

### 3.3. Continuum eigenfunctions

Let  $E > 0$ . The exact eigenfunction is

$$\psi_{E>0} = \begin{cases} A \sin(x\sqrt{E + g/b^2x_0^2}) & 0 \leq x < bx_0 \\ C_+\sqrt{E^{1/2}x}J_\omega(E^{1/2}x) \\ + C_-\sqrt{E^{1/2}x}J_{-\omega}(E^{1/2}x) & x > bx_0 \end{cases}.$$

The implicit energy equation is

$$\sqrt{g + \xi^2} \cot \sqrt{g + \xi^2} = \frac{1}{2} + \xi \frac{C_+J'_\omega(\xi) + C_-J'_{-\omega}(\xi)}{C_+J_\omega(\xi) + C_-J_{-\omega}(\xi)}, \quad (6)$$

where primes denote derivatives with respect to the whole argument. It is important to note that although we have chosen to write these expressions using the notation of Bessel functions, the analyses in this article rely on little more than the first couple terms in their power series for small argument.

Suppose we wish either  $C_+$  or  $C_-$  to vanish. Using the identity  $\lim_{\xi \downarrow 0} \xi J'_\omega(\xi)/J_\omega(\xi) = \omega$ , equation (6) becomes, as  $b \downarrow 0$ ,  $\sqrt{g} \cot \sqrt{g} = \nu_\pm$ . Denote the roots of this equation as  $g_\pm$ . That is,

$$\begin{aligned} C_- = 0: & \quad \sqrt{g_+} \cot \sqrt{g_+} = \nu_+ \\ C_+ = 0: & \quad \sqrt{g_-} \cot \sqrt{g_-} = \nu_-. \end{aligned}$$

Since  $\sqrt{g} \cot \sqrt{g}$  is monotonically decreasing for  $0 < g < \pi^2$  it follows that  $g_+ < g_-$ . For instance, if  $\alpha = -3/16$ , then  $g_+ \approx 0.7136$  and  $g_- \approx 1.9411$ .

The explicit formula for the ratio  $C_+/C_-$  is

$$\frac{C_+}{C_-}(g, \xi) = \frac{-\xi J'_{-\omega}(\xi) + J_{-\omega}(\xi) \left[ \sqrt{g + \xi^2} \cot \sqrt{g + \xi^2} - \frac{1}{2} \right]}{\xi J'_{\omega}(\xi) - J_{\omega}(\xi) \left[ \sqrt{g + \xi^2} \cot \sqrt{g + \xi^2} - \frac{1}{2} \right]}.$$

Assume that  $E$  is fixed. For any desired positive value of  $C_+/C_-$  there is a corresponding value of  $g \in (g_+, g_-)$ . We will primarily be interested in the analytical form of the ratio for small  $b$ , or equivalently,  $\xi \ll 1$ . Then

$$\frac{C_+}{C_-} = \frac{-2^{2\omega} \Gamma(1 + \omega)}{\Gamma(1 - \omega)} \xi^{-2\omega} \frac{\sqrt{g} \cot \sqrt{g} - \nu_-}{\sqrt{g} \cot \sqrt{g} - \nu_+} [1 + o(b)]. \quad (7)$$

We have established that for  $g_+ < g < g_-$  every eigenfunction  $\psi_{E>0}$  will involve a well-defined linear combination of the fundamental solutions characterized by a finite and nonzero value for  $C_+/C_-$ . In particular,  $C_+/C_-$  scales as  $E^{-\omega}$  for fixed  $b$  and  $g$ .

## 4. Renormalizing the coupling

### 4.1. $C_+/C_-$

Consider a point in the space of Hamiltonians parametrized by the  $(b, g)$ -plane. We would like to preserve the quantity  $C_+/C_-$  as  $b$  is taken to zero. In order to do this,  $g$  must vary too. The physical motivation for preserving  $C_+/C_-$  is, as will be discussed later, its direct relation to the scattering phase shift and indirect relation to the binding energy. These are low-energy observables that may be measured at spatial infinity.

Take equation (7) for fixed  $E$  and small nominal  $b$ . As  $b$  decreases further, the factor  $\xi^{-2\omega}$  increases, and so  $C_+/C_-$  remains constant only if  $(\sqrt{g} \cot \sqrt{g} - \nu_-)/(\sqrt{g} \cot \sqrt{g} - \nu_+)$  approaches zero from below.

Shrinking  $b$  ought to be understood as a flow to the infrared. Why? The position of a particle at  $x$  can be said to be neither close to the origin nor far unless a comparison is made to some length scale.  $V$  as given by equation (1) lacks such a scale, but introducing the cutoff  $bx_0$  in equation (4) makes it possible to judge whether a physical distance is small or large. For instance, for a nominal value of  $b = 1$ ,  $x/x_0 \ll 1$  indicates that the particle is ‘close’ to the origin, while  $x/x_0 \gg 1$  indicates that it is ‘far.’ In three dimensions, for a fixed spatial separation  $x$  between two particles, rescaling the cutoff from  $x_0$  to  $bx_0$  makes the distance between the particles larger in units of the cutoff. That is,  $x/(bx_0)$  increases as  $b$  decreases. This is precisely the regime one must examine to understand the long-distance behavior of the interaction.

The infrared flow described by  $b \downarrow 0$  takes a coupling  $g$  within the interval  $(g_+, g_-)$  and makes it tend toward the root of  $\sqrt{g} \cot \sqrt{g} = \nu_-$ . In other words,  $g \uparrow g_-$ . The precise manner by which  $g$  needs adjustment is called *renormalization*. This indicates that  $(b, g) = (0, g_-)$  is the *infrared-attractive fixed point* (IRFP) of the flow. This is depicted in figure 1.

We have learned that the IRFP corresponds to the eigenfunction when  $C_+ = 0^2$ . Why does this make sense? Recall that both of the fundamental solutions to the Schrödinger equation with  $V$  given by equation (1) satisfy the boundary condition  $\psi(0) = 0$ . Thus, any linear combination of the fundamental set is acceptable near the origin. Once a regulator is introduced (take  $b = 1$ , say) and a unique linear combination is chosen (up to scalar multiplication), we may ask, in generic terms: how should the coefficients of these solutions be adjusted so that the appropriate  $C^1$ -matching can be done at the cutoff scale  $x_0$ ? Assuming that  $x_0 \ll E^{-1/2}$ , the solution  $x^{\nu_-}$  dominates over  $x^{\nu_+}$  near the cutoff because  $\nu_- < \nu_+$ . In other words,  $x^{\nu_-}$  changes more rapidly than  $x^{\nu_+}$  does for small  $x$ . As such, if one desires to have a scattering phase shift dictated by the  $x^{\nu_+}$  solution, then one must *finely tune* the coefficient  $C_-$  to be zero. Therefore, from the point of view of  $x \gg x_0$ , the phase shift is generically dictated by the  $x^{\nu_-}$  solution, whereas the  $x^{\nu_+}$  solution is rather special.

Let us be precise about how  $g$  must change as  $b$  does. We work in the limit of small  $b$ . A certain change of variables makes the analysis elegant [3] (see footnote 1):

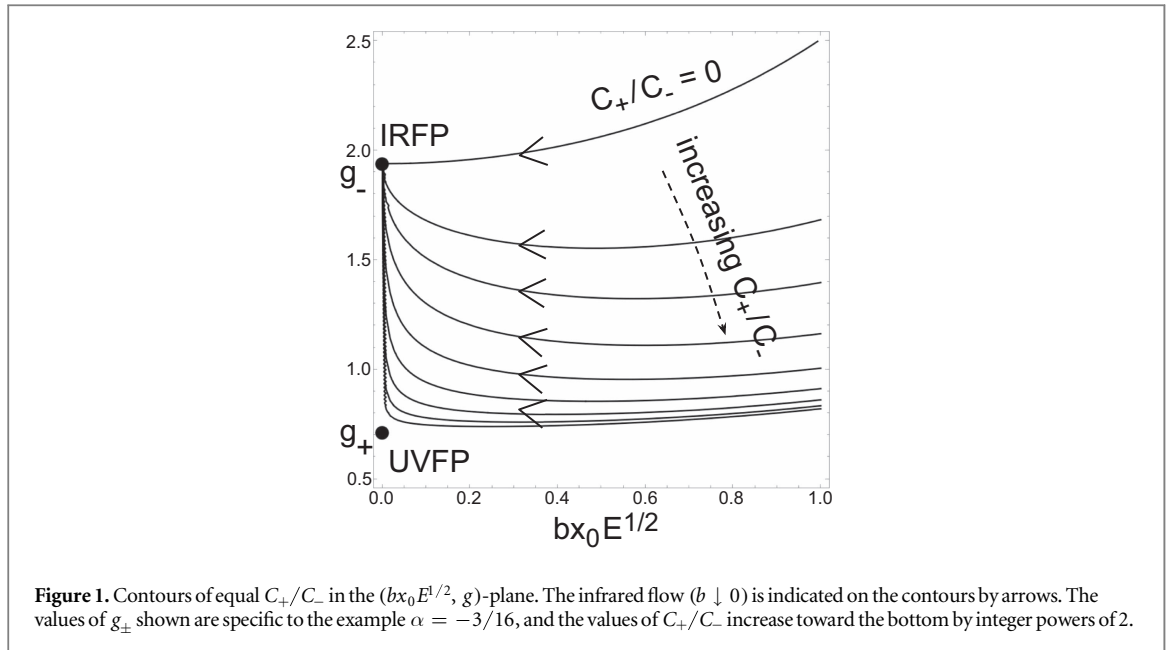
$$\gamma(g) = \sqrt{g} \cot \sqrt{g}.$$

Then for a given  $E$  and  $b \ll 1$ ,

$$\frac{C_+}{C_-} \sim b^{-2\omega} \frac{\gamma - \nu_-}{\gamma - \nu_+}$$

up to a multiplicative factor independent of  $b$ . From the invariance condition  $d(C_+/C_-)/db = 0$  we obtain a *RG equation*:

<sup>2</sup> Strangely, in [3] (see footnote 1) it is claimed that the  $C_- = 0$  solution should be identified with the IRFP. See p 6 in the paragraph below equation (13) and also p 8 in the paragraph below equation (19).



$$b \frac{d\gamma}{db} = -(\gamma - \nu_-)(\gamma - \nu_+), \quad (8)$$

which is exact in the  $b = 0$  limit. Consider  $db < 0$ . For  $\nu_- < \gamma < \nu_+$ , the right-hand side of equation (8), denoted  $\beta(\gamma)$  and called the *beta function*, is positive and so  $d\gamma < 0$ . Thus,  $\gamma = \nu_-$  is the IRFP, while  $\gamma = \nu_+$  is an *infrared-repulsive fixed point*, or, equivalently, an *ultraviolet-attractive fixed point* (UVFP). This is consistent with what we discovered in terms of the parameter  $g$ . There are two fixed points<sup>3</sup>.

It is both interesting and particularly simple to study the behavior of  $\gamma$  close to a zero of the beta function. Here a scaling behavior emerges. Let  $\gamma = \gamma(b)$  indicate the coupling associated to some choice of  $b$ . Let  $b' = b(1 - \epsilon)$  for some infinitesimal  $\epsilon > 0$ . It follows that  $b' < b$ . Let  $\gamma' = \gamma(b')$ . So  $\gamma' \approx \gamma(b) - b\epsilon \frac{d\gamma}{db}|_b = \gamma - \epsilon\beta(\gamma)$ . Differentiate with respect to  $\gamma$  to obtain  $d\gamma'/d\gamma \approx 1 - \beta'(\gamma)\epsilon$ . At a zero  $\gamma_*$  of the beta function, the coupling does not change. One hypothesizes that the reduced coupling  $\gamma - \gamma_*$  obeys a simple scaling in the vicinity of the zero:  $\gamma' - \gamma_* = (\gamma - \gamma_*)(1 - \epsilon)^y$ . In the language of the RG, the difference  $\gamma - \gamma_*$  is called a *scaling variable* and the exponent  $y$  is called an *RG eigenvalue* [11]. But this implies  $d\gamma'/d\gamma \approx 1 - y\epsilon$ . Equating both expressions for  $d\gamma'/d\gamma$  implies that  $y = \beta'(\gamma_*)$ —the RG eigenvalue is equal to the *slope* of the beta function at the fixed point. Specifically,  $\beta'(\gamma) = 1 - 2\gamma$  so  $y = \mp 2\omega$  for  $\gamma = \nu_{\pm}$ .

A nice way to summarize what we have found is to define a *reduced coupling* in the vicinity of each fixed point. Note that  $g_+ < g < g_-$  maps to  $\nu_+ > \gamma > \nu_-$ . Near the IRFP define  $u = \gamma - \nu_-$ . Then

$$u' = u(1 - \epsilon)^{2\omega}. \quad (9)$$

Since  $u' < u$ , we learn that  $u$  is an *irrelevant* variable that tends to shrink as one enlarges the system. Near the UVFP define  $u = \nu_+ - \gamma$ . Then

$$u' = u(1 - \epsilon)^{-2\omega}. \quad (10)$$

Since  $u' > u$ , we learn that  $u$  is a *relevant* variable that tends to grow.

The assignment of the descriptions ‘irrelevant’ and ‘relevant’ to equations (9) and (10) might sound backwards to the reader familiar with real-space renormalization, but are, in fact, consistent. For instance, the real-space approach assigns irrelevant scaling variables negative RG eigenvalues, not positive ones like ours. This is because the sense by which one progresses to long distances in, say, a discrete lattice model, is a bit different. On the lattice there is a spacing  $a$  which cannot be adjusted. Instead, one coarse-grains over successively larger chunks of the (infinite) lattice, each step producing an intermediate lattice with larger effective size  $a' = (1 + \epsilon)a$  for  $\epsilon > 0$ . Such procedure reduces the measure of the dimensionless correlation length and is a way of probing successively longer physical distances. A coupling obeying the relation  $u' = u(1 + \epsilon)^{\gamma_{\text{LAT}}}$  for  $\gamma_{\text{LAT}} < 0$  is then said to be irrelevant because it shrinks. This is equivalent to the reduction  $b' = (1 - \epsilon)b$  used in our analysis of the inverse-square potential since any physical distance  $x$  grows bigger in units of the cutoff scale  $bx_0$ .

<sup>3</sup> Note that  $\gamma(g)$  is monotonically decreasing for  $0 < g < \pi^2$ . However, it is quasiperiodic in the sense that the shape of its graph repeats for intervals  $(\pi^2, (2\pi)^2)$ ,  $((2\pi)^2, (3\pi)^2)$ , etc. Since  $0 < \nu_{\pm} < 1$ , there are a pair of fixed points in each of these intervals.

Lastly, it is worth reframing the scaling in terms of the original coupling  $g$ . By the same logic as above it should be that, near a fixed point,  $g' - g_* = (g - g_*)(1 - \epsilon)^{\tilde{y}}$  where  $\tilde{y} = \beta'(g_*)$ . One should not confuse  $\beta(g)$  with  $\beta(\gamma(g))$ . Rather,  $\beta(g) = b \frac{dg}{db} = b \frac{dg}{d\gamma} \frac{d\gamma}{db} = \beta(\gamma)/\gamma'(g)$  by the inverse function theorem as long as  $\gamma'(g) \neq 0$ . Differentiating with respect to  $g$  yields  $\beta'(g) = \beta'(\gamma) - \beta(\gamma)\gamma''(g)/[\gamma'(g)]^2$ . However, at a fixed point  $\beta(\gamma_*) = 0$  so the extra term vanishes and we have  $\beta'(g_*) = \beta'(\gamma_*)$ , or  $\tilde{y} = y$ . Hence, the reduced coupling  $g_- - g$  is irrelevant, and  $g - g_+$  is relevant with exactly the same RG eigenvalues as in equations (9) and (10).

#### 4.2. Scattering phase shift

One may also frame the renormalization condition as the requirement that the relative phase between incoming and outgoing plane waves remain invariant as one varies the wavelength. Let  $\mu = kbx_0$ . For small  $\mu$  we find that  $\mu \frac{d\gamma}{d\mu} = \beta(\gamma)$  with exactly the same beta function as in equation (8). The salient details are presented in an [appendix](#).

There are two ways to interpret the findings: (i) one could regard the cutoff  $b$  as held fixed at some nominal value and imagine varying  $k$ . Observe the phase shift experimentally for some  $k$ . The value so obtained locates a unique point in the  $(\mu, g)$ -plane. Taking the long-wavelength limit  $k \downarrow 0$  (and hence  $\mu \downarrow 0$ ) requires adjusting  $g$  so that one remains on a certain integral curve in this plane. (ii) Another approach is to regard the wavenumber  $k$  of the incoming plane wave as held fixed, but allow the freedom to adjust  $b$ . A choice of  $(b, g)$  uniquely specifies a Hamiltonian. For this Hamiltonian, there will be a definite phase shift. Now as we take  $b \downarrow 0$ , it is possible to adjust  $g$  so that the phase shift does not change if, once again, we follow an integral curve in the  $(\mu, g)$ -plane. In this interpretation there is a flow between Hamiltonians that preserves a long-distance observable. Thus, taking  $b$  to zero is an equivalent way of reaching a long-wavelength approximation.

#### 4.3. Binding energy

Although our discussion of renormalization has been limited to  $g \in (g_+, g_-)$ , we may also consider  $g > g_-$ . At least one bound state will be present with ground state energy  $E$ . One might suspect that  $dE/db = 0$  leads to the same RG equation for  $g$ , namely equation (8), as the previously studied conditions  $d(C_+/C_-)/db = 0$  and  $dr/db = 0$  in sections 4.1, 4.2, and the [appendix](#). Our expectation is that this should be true at least in the limit  $b \downarrow 0$ .

On general grounds, the phase shift is essentially the phase angle of the reflection amplitude  $r$ , whereas the ground state energy  $E$  is the pole of  $r$  (more generally, the  $S$ -matrix) in the complex  $k$ -plane. Since  $r$  has modulus one, it is possible to express it as  $r = (s - ik)/(s + ik)$ , where  $s$  is some real constant and  $k$  the real wavenumber. However, there is a simple pole at  $k = is$ , and so  $E = k^2 = -s^2$ . Requiring that  $r$  remain constant as  $b$  changes implies that the value of  $k/s$  remains constant as  $b$  is adjusted. Thus, the RG equations are identical.

The bound state energy for  $g \gtrsim g_-$  is given by solving equation (5). For  $0 < \omega < 1/2$  and  $\xi \ll 1$ ,  $\xi K_\omega'(\xi)/K_\omega(\xi) = -\omega - 2 \frac{\Gamma(1-\omega)}{\Gamma(\omega)} (\xi/2)^{2\omega} + O(\xi^2)$ . Expanding to lowest order in both  $\xi$  and  $g - g_-$ ,

$$E \sim -C \frac{(g - g_-)^{1/\omega}}{(bx_0)^2}, \quad (11)$$

where  $C = [2^{2\omega-2}(1 + \alpha/g_-)\Gamma(\omega)/\Gamma(1 - \omega)]^{1/\omega}$ , a positive constant that depends only on the value of  $\alpha$ . Equation (11) shows once more that  $E$  remains invariant as long as the reduced coupling  $g - g_-$  scales as  $b^{2\omega}$ , the same scaling found when  $g$  was just below  $g_-$ . Note that as  $b \downarrow 0$ , as long as  $g - g_-$  follows this scaling rule, a single bound state (of arbitrary energy) remains! This is an example of *dimensional transmutation*.

### 5. Propagator

A choice of  $(b, g)$  with  $b > 0$  and  $g_+ < g < g_-$  selects a particular Hamiltonian  $H_{b,g}$ . From this we will now construct the Green's function—the position-space realization of unitary time evolution. However, we shall work with pure negative imaginary times,  $G_{b,g}(x, -it; y) = \langle x|e^{-tH}|y\rangle$ ,  $t \geq 0$ . This imaginary-time propagator is the solution to  $\frac{\partial}{\partial t}G = \left(\frac{\partial^2}{\partial x^2} - V(x)\right)G$  with boundary condition  $G(x, -it \downarrow 0; y) = \delta(x - y)$ . The completeness property of eigenfunctions of  $H_{b,g}$  means that

$$G_{b,g}(x, -it; y) = \int_0^\infty dE e^{-tE} \psi_E(x) \psi_E(y).$$

We are motivated to demonstrate the following: two Hamiltonians, one at scale  $b$  and the other at scale  $b'$ , but both with couplings close to  $g_\pm$ , will give equivalent long-distance behavior (as measured by  $G$  at fixed  $x, y > bx_0$  and fixed  $t$ ) provided that the coupling  $g$  is renormalized from  $b$  to  $b'$  according to the scaling laws found in equations (9) and (10).

At this point we remember that  $C_+/C_-$  is not the only ratio needed in order to fully specify eigenfunctions. We also need

$$\frac{C_-}{A}(g, \xi) = \frac{\sin \sqrt{g + \xi^2}}{\frac{C_+}{C_-}(g, \xi) \sqrt{\xi} J_\omega(\xi) + \sqrt{\xi} J_{-\omega}(\xi)}$$

and the normalization factor  $A^2$ . This can be fixed by remembering that the set of eigenfunctions  $\{\psi_{E>0}\}$  must satisfy orthonormality and closure relations. Let us focus on the closure property which must hold for any choice of  $x, y$  and  $g$ . Therefore, take  $x, y < bx_0$  and  $g = 0$  so that  $\int_0^\infty A^2 \sin(x\sqrt{E}) \sin(y\sqrt{E}) dE = \delta(x - y)$ . We recognize here an identity of Fourier sine transforms so it is clear that  $A^2 = 1/\pi\sqrt{E}$ . This is consistent with dimensional analysis since  $\psi_E$  should have dimensions of length<sup>1/2</sup>. For  $g > 0$  we may write

$$A^2 = \frac{B(g, \xi)}{\pi\sqrt{E}}$$

for some dimensionless function  $B$  which is strictly positive. The propagator may be written as

$$G_{b,g}(x, -it; y) = \int_0^\infty \frac{BdE}{\pi\sqrt{E}} e^{-tE} \left[ \frac{C_-}{A} \frac{C_+}{C_-} \sqrt{E^{1/2}x} J_\omega(E^{1/2}x) + \frac{C_-}{A} \sqrt{E^{1/2}x} J_{-\omega}(E^{1/2}x) \right] \times [x \leftrightarrow y], \quad x, y > bx_0. \tag{12}$$

Without explicitly evaluating the integral, we are interested in proving that  $G_{b,g}$  is simply related to  $G_{b',g'}$  for  $b' < b$ . The simple relation we seek is an equivalence of the two propagators up to an overall scale factor with physical lengths and time held fixed. This is a homogeneous transformation with respect to the parameters  $b$  and  $g$ . The existence of such a relation would place a severe constraint on the form of the ratio  $C_+/C_-$ . The reason is that the contours in figure 1 show exactly how to preserve the propagator as  $b \downarrow 0$ . However, individual points in figure 1 are not representative of a choice of  $(b, g)$ . Rather, a given  $(b, g)$  corresponds to a horizontal line in the plot—all of the uncountably many eigenfunctions of  $H_{b,g}$  correspond to points on this line. Under a shrinking of  $b$ , one can preserve the propagator by following the contour passing through each point to its left. We manage, on an individual basis, to keep all eigenfunctions unchanged if we can adjust  $g$  so that each eigenfunction's  $C_+/C_-$  ratio remains the same. However, there is no guarantee that the new value of the coupling for one energy will coincide with the new coupling required for a different energy! More precisely, recall that  $C_+/C_-$  is a function of two variables  $g$  and  $\xi$ . Write  $\xi = bE^{1/2}$  (suppressing  $x_0$ ) so that  $C_+/C_- = f(g, bE^{1/2})$ . Consider any two distinct energies  $E_1$  and  $E_2$ . By changing  $b$  to  $b'$  it is possible to find some  $g'$  such that

$$f(g, bE_1^{1/2}) = f(g', b'E_1^{1/2}). \tag{13}$$

And it must be possible to find some  $g''$  such that

$$f(g, bE_2^{1/2}) = f(g'', b'E_2^{1/2}). \tag{14}$$

These statements are illustrated schematically in figure 2. However,  $G_{b',g'} \propto G_{b,g}$  requires  $g' = g''$ , a highly nontrivial condition! There is no obvious reason why, starting from generic  $g$ , the renormalized couplings  $g'$  and  $g''$  ought to be the same. Therefore, this appears to be an obstruction to finding a simple scaling law for the propagator.

However, it is easy to see that in the vicinity of a fixed point  $(0, g_*)$  the nontrivial condition requiring the equivalence of  $g'$  and  $g''$  can be satisfied. The crux of our argument is this: it is only close to a fixed point that  $f$  takes the asymptotic form

$$f(g, bE^{1/2}) \sim \alpha(g - g_*)^\beta (bE^{1/2})^\gamma$$

for some real numbers  $\alpha, \beta$ , and  $\gamma$  which may be extracted by studying equation (7). Then equations (13) and (14) become  $g' - g_* = (b/b')^{\gamma/\beta} (g - g_*)$  and  $g'' - g_* = (b/b'')^{\gamma/\beta} (g - g_*)$ , respectively. Together they imply  $g' = g''$  as desired.

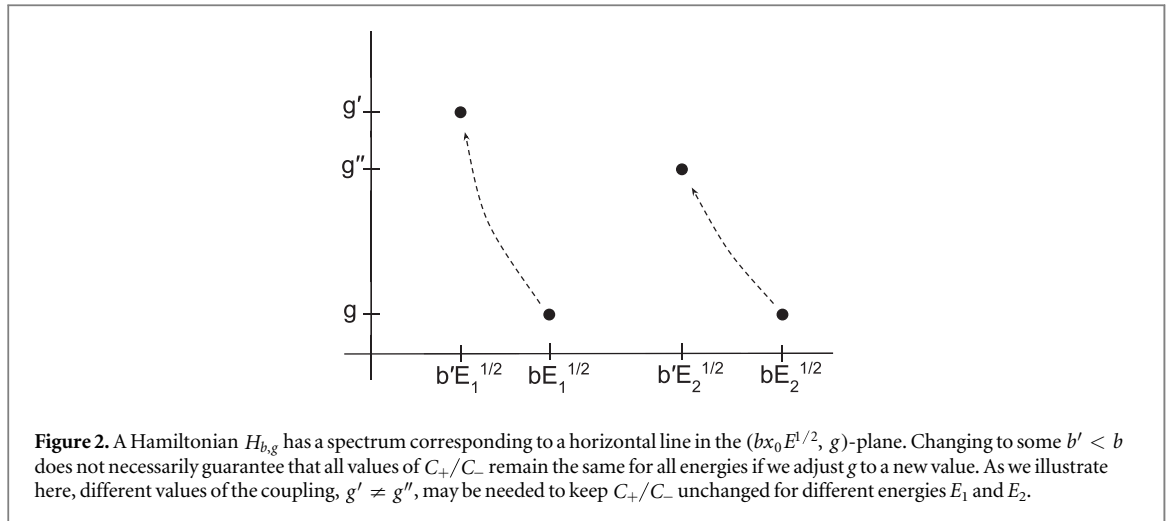
### 5.1. Homogeneous transformation laws

The scaling relation we seek for the propagator may be obtained from the following two transformation laws.

#### 5.1.1. An exact one

The following equation is exact. Suppose  $x, y > bx_0$  and  $b' = b/\lambda$  for some  $\lambda > 1$ . Then

$$G_{b,g}(\lambda x, -i\lambda^2 t; \lambda y) = \lambda^{-1} G_{b',g}(x, -it; y). \tag{15}$$



**Proof.** In equation (12) make a change of variables  $\tilde{E} = \lambda^2 E$ . Within  $B$ ,  $C_+/C_-$ , and  $C_-/A$  this can be perfectly compensated by a redefinition of the scale factor  $b$  since  $\xi = bx_0E^{1/2} = (b/\lambda)x_0\tilde{E}^{1/2} = b'x_0\tilde{E}^{1/2}$ . Since  $b' < b$ , the appropriate eigenfunctions are still Bessel J.

### 5.1.2. One valid only close to a fixed point

The following equation is correct only asymptotically close to a fixed point. Suppose  $x, y > bx_0$ ,  $b \ll 1$ , and  $\lambda > 1$ . Then

$$G_{b,u}(\lambda x, -i\lambda^2 t; \lambda y) \sim \lambda^{2\nu_{\pm}-1} G_{b,u'}(x, -it; y), \tag{16}$$

for (upper sign)  $u = g - g_+ \ll 1$  and  $u' = \lambda^{-2\omega} u$ , or (lower sign)  $u = g_- - g \ll 1$  and  $u' = \lambda^{2\omega} u$ .

*Proof of upper sign case:* Once again, in equation (12), make a change of variables to  $\tilde{E} = \lambda^2 E$ . Near a fixed point it is possible to absorb factors of  $\lambda$  into a redefinition of the reduced coupling rather than a redefinition of  $b$ . Before doing this, replace  $C_{\pm}$  by their asymptotic forms for  $\xi \ll 1$ . (This is justified by taking the upper limit of integration to be some large, but finite, value  $M/x_0^2$  for  $M \gg 1$ . We shall take  $M \rightarrow \infty$  later. This means that  $E^{1/2}$  is bounded above by  $\sqrt{M}/x_0$ , and hence,  $\xi < b\sqrt{M}$ . Let us choose  $b \ll 1/\sqrt{M}$  so that  $\xi$  is small over the entire integration region.) So in the small- $b$  regime,

$$\frac{C_{\pm}}{A}(g, \xi) \sim \frac{\Gamma(\pm\omega)}{2^{1\mp\omega}} \frac{\sin \sqrt{g}(\sqrt{g} \cot \sqrt{g} - \nu_{\mp})}{\xi^{\nu_{\pm}}}. \tag{17}$$

Also, we replace  $B(g, \xi)$  by its limiting value  $B(g_+, 0)$  assuming it exists. Thus,

$$G_{b,g}(\lambda x, -i\lambda^2 t; \lambda y) = \frac{1}{\lambda} \int_0^{\infty} \frac{B(g_+, 0) d\tilde{E}}{\pi \sqrt{\tilde{E}}} e^{-t\tilde{E}} \left[ \frac{\text{cst.} \lambda^{\nu_+}}{(bx_0 \tilde{E}^{1/2})^{\nu_+}} \sqrt{\tilde{E}^{1/2} x} J_{\omega}(\tilde{E}^{1/2} x) + \frac{\text{cst.} u \lambda^{\nu_-}}{(bx_0 \tilde{E}^{1/2})^{\nu_-}} \sqrt{\tilde{E}^{1/2} x} J_{-\omega}(\tilde{E}^{1/2} x) \right] \times [x \leftrightarrow y].$$

The constants ‘cst.’ appearing above result from expanding Expr. (17) around  $g = g_+$ . They are nonzero. By writing  $u = \lambda^{\nu_+ - \nu_-} u'$ , it is seen that two factors of  $\lambda^{\nu_+}$  may be pulled out of the integral. Although this concludes the proof, it is worth explaining it heuristically. For  $g$  close to  $g_+$ , we can set  $C_- = 0$  so that the eigenfunctions are, for small  $x$ , power laws of the form  $x^{\nu_+}$ . Therefore, in the propagator we have two wavefunctions of the form  $(\lambda x)^{\nu_+}$  and  $(\lambda y)^{\nu_+}$ . This is how a factor of  $\lambda^{2\nu_+}$  is obtained. A proof for the lower sign case is similar.

## 6. Scaling

By combining equation (15) with (16) we are able to derive the scaling relation for the propagators evaluated at the same values of  $x, y, t$ , but corresponding to different Hamiltonians  $H_{b,g}$  and  $H_{b',g'}$ . We shall suppress dependence on the variables  $x, y, t$  which are imagined to be held fixed, and instead highlight the dependence of the propagator on the parameters  $b$  and  $g$ . Then

$$G(b, u) \sim \lambda^{-2\nu_{\pm}} G(b/\lambda, u \lambda^{\pm 2\omega}). \tag{18}$$

The upper sign is for  $g \gtrsim g_+$  and the lower sign for  $g \lesssim g_-$ . Equation (18) is a central result of this article. One can recognize the RG eigenvalues of  $u$  found earlier in equations (9) and (10).

An equivalent way to state equation (18) is that the propagator satisfies a certain first-order pde which is obtained by imagining an infinitesimal change in parameters. Let  $\lambda = 1 + \delta\lambda$  for infinitesimal positive  $\delta\lambda$ . Expanding to first order in  $\delta\lambda$ , we obtain

$$\left[ b \frac{\partial}{\partial b} \mp 2\omega u \frac{\partial}{\partial u} + 2\nu_{\pm} \right] G = 0. \tag{19}$$

The physical interpretation of equation (19) is that a change of  $b$  in  $G$  can be absorbed through a compensating change in  $u$ . In other words, scale dependency may be exchanged for coupling dependency, a fact already clear from our derivation of equation (16). Notice that the coefficients  $\mp 2\omega$  of the operator  $u \frac{\partial}{\partial u}$  are precisely the slope of the beta function at the fixed points  $\gamma = \nu_{\pm}$ . Equation (19) is reminiscent of the Callan–Symanzik equation in quantum field theory which describes how correlation functions change with the renormalization scale.

**6.1. At the fixed points**

We wish to discuss the propagator at the fixed points. We have seen that  $g = g_{\pm}$  corresponds to  $C_{\mp} = 0$  only in the limit that  $b$  equals 0. Taking this limit squeezes the sinusoidal part of the eigenfunction into an infinitesimally narrow region around the origin. Adapting an integral from [12], we obtain

$$G_{0,g_{\pm}}(x, -it; y) = \frac{\sqrt{xy}}{2t} e^{-(x^2+y^2)/4t} I_{\pm\omega} \left( \frac{xy}{2t} \right),$$

where  $I$  is a modified Bessel function of the first kind. Using the well-known properties  $I_{\nu}(z) \approx e^z / \sqrt{2\pi z}$  for  $|z| \gg 1$ , and  $I_{\nu}(z) \approx (z/2)^{\nu} / \Gamma(\nu + 1)$  for  $|z| \ll 1$ , the asymptotic behavior is given by

$$G_{0,g_{\pm}}(x, -it; y) \sim \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} & t \downarrow 0, \\ \frac{1}{2^{1\pm 2\omega} \Gamma(1 \pm \omega)} \frac{1}{\sqrt{t}} \left( \frac{xy}{t} \right)^{\nu_{\pm}} & t \rightarrow \infty \end{cases}.$$

The fixed-point theories both reproduce free-particle behavior at short times, but have differing power-laws at long times.

**6.2. Close to the fixed points**

For  $(b, g)$  in the vicinity of  $(0, g_{\pm})$ , but not strictly at those points, the functional form of the propagator may be uncovered by RG scaling analysis [11]. We illustrate these standard techniques near the UVFP.

**6.2.1. RG for  $u > 0$**

For  $u = g - g_{+} \ll 1$  but strictly greater than zero,  $G(b, u) \sim \lambda^{-2\nu_{+}} G(b/\lambda, \lambda^{2\omega}u)$ . Iterating  $n$  times gives  $G(b, u) \sim \lambda^{-2\nu_{+}n} G(b/\lambda^n, \lambda^{2\omega n}u)$ . Since  $u$  is relevant, the effective reduced coupling grows under iteration so  $n$  cannot be taken arbitrarily large or else the asymptotic approximation breaks down. So we stop the iteration at the point where  $\lambda^{2\omega n}u = u_0$ , where  $u_0$  is an arbitrary but fixed constant that is sufficiently small. Solving yields  $\lambda^n = (u_0/u)^{1/2\omega}$ . Plugging this back in gives  $G(b, u) \sim (u/u_0)^{\nu_{+}/\omega} G(b(u/u_0)^{1/2\omega}, u_0)$ . Observe that the parametric dependence of  $G$  on  $b$  and  $u$  has simplified to the point where we may now express it as

$$G(b, u) \sim (u/u_0)^{1+1/2\omega} \Phi(b(u/u_0)^{1/2\omega}), \tag{20}$$

for some function  $\Phi$ . At first sight it might appear that the functional form of  $\Phi$  depends on the specific value of  $u_0$ . Indeed, we could have imagined the iteration stopping at some  $u'_0 \neq u_0$ . Along the lines of equation (20) we could then write  $G(b, u) \sim (u/u'_0)^{1+1/2\omega} \Psi(b(u/u'_0)^{1/2\omega})$  for some apparently unrelated function  $\Psi$ . But the important fact is that the left-hand side of equation (20) is insensitive to whether the scaling variable halts at  $u_0$  or  $u'_0$ . This implies an equality:  $\Psi(z) = (u'_0/u_0)^{1+1/2\omega} \Phi((u'_0/u_0)^{1/2\omega}z)$ , and means that the ratio  $u'_0/u_0$ , and any powers thereof, may be absorbed into a redefinition of the scaling variable  $u$ . Thus, the form of  $\Phi$  must be independent of  $u_0$ . In other words,  $\Phi$  is truly a function with a single argument. In such instance,  $\Phi$  is referred to as a *scaling function*.

**6.2.2. RG for  $u = 0$**

For  $u = 0$  our  $n$ -times-iterated homogeneous transformation law for the propagator may be written  $G\left(\frac{b}{\sqrt{xy}/x_0}, 0\right) \sim (\lambda^{-n})^{2\nu_{+}} G\left(\frac{b\lambda^{-n}}{\sqrt{xy}/x_0}, 0\right)$ . We have reintroduced the spatial variables  $x$  and  $y$  using dimensional analysis. This may be thoroughly justified by redefining the integration variable in equation (12) from  $E$  to  $E/t$ . Without explicitly evaluating the integral it is easy to see that  $G = t^{-1/2} f(x/bx_0, x^2/t, y/bx_0, y^2/t)$  for some function  $f$ . Our recursion relation is not sensitive to terms like  $x^2/t$  or  $y^2/t$ ; it must be correct only for  $t \gg x^2, y^2$ . Furthermore, the propagator must be symmetric in  $x$  and  $y$ . Since  $\lambda > 1$  one may imagine  $n$  being so large that  $b\lambda^{-n}/(\sqrt{xy}/x_0) = \epsilon$  is an exceedingly small fixed number. It could be so small that  $\epsilon \sqrt{xy}/x_0 \ll 1$  (recall that  $x$  and  $y$  are fixed). This requires that we choose  $n \gg \log b / \log \lambda$ , which is always possible for a given

$b$  and  $\lambda$ . So

$$G\left(\frac{b}{\sqrt{xy}/x_0}, 0\right) \sim \epsilon^{2\nu_+} \left(\frac{\sqrt{xy}}{bx_0}\right)^{2\nu_+} G(\epsilon, 0).$$

Note that  $\epsilon^{2\nu_+}G(\epsilon, 0)$  is a constant. Including an overall  $t^{-1/2}$  factor from dimensional analysis, we obtain

$$G\left(\frac{b}{\sqrt{xy}/x_0}, 0\right) \sim \frac{1}{\sqrt{t}} \left(\frac{xy}{b^2x_0^2}\right)^{\nu_+}. \tag{21}$$

### 6.2.3. Verification

Fortunately, analytical evaluation of  $G$  is rather simple for asymptotically large  $t$ . This allows us to check equations (20) and (21). For large  $t$  the integral in equation (12) is dominated by values of  $E$  near zero which allows us to replace  $\sqrt{E^{1/2}x} J_{\pm\nu}(E^{1/2}x)$  by  $(E^{1/2}x)^{\nu_{\pm}}$ . After rescaling the dummy integration variable to extract  $t$ , we obtain, up to an overall constant factor,

$$G_{b,g \sim g_+} \sim \frac{1}{\sqrt{t}} \left[ \left(\frac{x}{bx_0}\right)^{\nu_+} + cu \left(\frac{x}{bx_0}\right)^{\nu_-} \right] \times [x \leftrightarrow y], \tag{22}$$

where  $u = g - g_+ \ll 1$  and  $c$  is some constant.

Setting  $u = 0$  in equation (22) gives precisely equation (21). It takes a little more effort to show that equation (22) is equivalent to equation (20) when  $u > 0$ . Consider

$$\begin{aligned} & b^{-\nu_+} + cub^{-\nu_-} \\ &= (bu^{1/2\omega}u^{-1/2\omega})^{-\nu_+} + cu(bu^{1/2\omega}u^{-1/2\omega})^{-\nu_-} \\ &= (bu^{1/2\omega})^{-\nu_+}u^{1/4\omega+1/2} + c(bu^{1/2\omega})^{-\nu_-}u^{1+1/4\omega-1/2} \\ &= u^{1/2+1/4\omega}\Phi(bu^{1/2\omega}), \end{aligned}$$

where  $\Phi(z) = z^{-\nu_+} + cz^{-\nu_-}$ . Of course, there is also a similar factor involving  $y$ . Thus, the factor  $u^{1/2+1/4\omega}$  gets squared and becomes  $u^{1+1/2\omega}$  as desired.

## 7. Disappearing bound state

In this section we set  $b = 1$  in the regulator equation (4) and do not allow the well width to vary. For clarity we also take  $x_0 = 1$  (it can always be restored by looking at dimensions). For  $g \gtrsim g_-$  the ground state energy scales as  $E \propto (g - g_-)^{1/\omega}$ . As  $g \downarrow g_-$ , this binding energy vanishes at a rate described by the exponent  $1/\omega$ . It must also be the case that the probability to find the particle within any finite interval tends to zero—the mean position of the particle (and all higher moments) should run off to infinity in a continuous fashion. The rate at which this occurs is given by computing the divergent part of  $\int x|\psi|^2 dx$ . In fact, we need only focus on the exponentially-falling tail of the wavefunction. We find that  $\int_1^\infty x|\psi|^2 dx / \int_1^\infty |\psi|^2 dx \sim \frac{1}{2}|E|^{-1/2}$ . Up to a constant factor,

$$\langle x \rangle \sim (g - g_-)^{-1/2\omega}.$$

The rate of vanishing of  $E$ , or the degree of divergence of  $\langle x \rangle$ , is controlled by the long-distance part of  $V$ . The specific scheme chosen for the short-distance part (e.g., square well) has no effect other than to modify the location of the critical value of  $g$ . This independence of regulator scheme is an example of *universality*: physical systems with very different ultraviolet behavior (in this case, Hamiltonians with very different excited spectra) may have similar low-energy behavior.

A nice exercise is to try a regulator like the linear well,  $V = -gx$  for  $0 < x < 1$ . Like the square well, it has a single dimensionless parameter  $g > 0$  responsible for controlling the well slope. The same scaling exponent  $1/\omega$  emerges for the ground state energy as in equation (11), but the coefficient  $C$  and the location of the fixed point  $g_-$  are not the same as in the square well case. One says that  $1/\omega$  is universal, but  $C$  and  $g_-$  are not.

It is readily proven that the same exponent results for a generic regulator. Consider a scheme given by

$$V(x) = \begin{cases} -gf(x) & 0 < x < 1 \\ \alpha/x^2 & x > 1 \end{cases},$$

where  $f$  is a function defined on  $[0, 1]$  with additional properties to be imposed below. First we prove that for sufficiently large  $g$  the potential binds. It is sufficient to show that there exists some  $\psi \in \mathcal{H}$  with  $\langle \psi, H\psi \rangle < 0$  since this implies that at least one of the discrete eigenvalues of  $H$  is negative. Note that  $\psi$  need not be a state of definite energy. A convenient choice is  $\psi = x \exp(-x/2)$ . So

$$\int_0^\infty \psi(-\psi'' + V\psi)dx = \frac{1}{2} - g \int_0^1 f(x)x^2e^{-x}dx + \frac{\alpha}{e}.$$

Assume  $f$  is integrable. A negative expectation value is guaranteed for  $g > \left(\frac{1}{2} + \frac{\alpha}{e}\right) / \int_0^1 f(x)x^2e^{-x}dx$ .

Next we prove that for sufficiently small  $g$  there is no bound state. Recall the fact that the square well regulator, equation (4), does not produce binding if  $g < g_-$ , where  $g_-$  is entirely determined by  $\alpha$ . So take  $g < g_-/f(x)$  for all  $x < 1$ . We assume  $f$  is bounded. This means that our potential  $V$  is everywhere bounded below by the square well.

The solution to the eigenvalue equation is

$$\psi_{E<0} = \begin{cases} A\varphi_{g,\epsilon}^{(1)}(x) + B\varphi_{g,\epsilon}^{(2)}(x) & 0 \leq x < 1 \\ C\sqrt{\epsilon^{1/2}x}K_\omega(\epsilon^{1/2}x) & x > 1 \end{cases},$$

where  $\varphi_{g,\epsilon}^{(i)}$  are the linearly independent solutions to  $\varphi'' + g f(x)\varphi - \epsilon\varphi = 0$ , with  $\epsilon = -E$ . Suppose that  $g$  is large enough to admit at least one bound state so that  $\epsilon$  is a real positive number. Call the critical value of  $g$  needed for this to be true some  $g_*$ . By continuity,

$$\frac{\varphi_{g,\epsilon}^{(1)'}(1)\varphi_{g,\epsilon}^{(2)}(0) - \varphi_{g,\epsilon}^{(1)}(0)\varphi_{g,\epsilon}^{(2)'}(1)}{\varphi_{g,\epsilon}^{(1)}(1)\varphi_{g,\epsilon}^{(2)}(0) - \varphi_{g,\epsilon}^{(1)}(0)\varphi_{g,\epsilon}^{(2)}(1)} = \frac{1}{2} + \epsilon^{1/2} \frac{K_\omega'(\epsilon^{1/2})}{K_\omega(\epsilon^{1/2})}. \tag{23}$$

Call the left side of equation (23),  $L(g, \epsilon)$ . Since  $x = 0, 1$  are ordinary points of the ode,  $\varphi_{g,\epsilon}^{(i)}$  and their derivatives are finite when evaluated at  $x = 0, 1$ . Now let us consider how  $L$  depends on the parameters. Expanding in energy  $\epsilon \ll 1$ , we claim that  $L(g, \epsilon)$  equals  $L(g, 0)$  plus a term whose order is no larger than  $\epsilon^{1/2}$ . (This is because nonanalyticity in the  $\epsilon$ -dependence of the solutions to the ode develops when  $f$  vanishes. For example, if  $f = 0$  near  $x = 0$ , then  $\varphi \sim e^{\pm\sqrt{\epsilon}x} \approx 1 \pm \sqrt{\epsilon}x$ .) Call the right side of equation (23),  $R(\epsilon)$ . Expanding in  $\epsilon \ll 1$ , we obtain  $R(0) + O(\epsilon^\omega)$ . Since  $\omega < 1/2$ , this  $O(\epsilon^\omega)$  term dominates over the  $O(\epsilon^{1/2})$  term obtained by expanding the left side. Expanding  $L$  once more, but this time in  $g$  around  $g_*$ , we get  $L(g_*, 0) + O(g - g_*)$ . Finally, if  $g_*$  is such that  $L(g_*, 0) = R(0)$ , then we have  $O(g - g_*) = O(\epsilon^\omega)$ . It follows that  $\epsilon \sim (g - g_*)^{1/\omega}$ .

## 8. Some classical applications

### 8.1. Brownian motion

Brownian motion is a stochastic process which may be regarded as the symmetric random walk in the limit of infinitesimally small time increments. In a one-dimensional process, a real random variable  $x(t)$  varies with a time parameter  $t$ . The map  $x(t)$  is continuous, satisfies an initial condition and may satisfy a final condition as well. Crucially, the process is such that increments in  $x$  are independent and identically distributed (i.e., the probability distribution for  $x(t_2) - x(t_1)$  is independent of  $x(t_1)$  for  $t_2 > t_1 > 0$ , but they are identical in the sense that  $\mathbb{P}(x(t_2) - x(t_1) < a) = \mathbb{P}(x(t_1) < a)$ ). These distributions are normal with mean zero and variance equal to twice the time interval. It is well known how to rigorously assign a probabilistic measure to such a set of real-valued continuous functions.

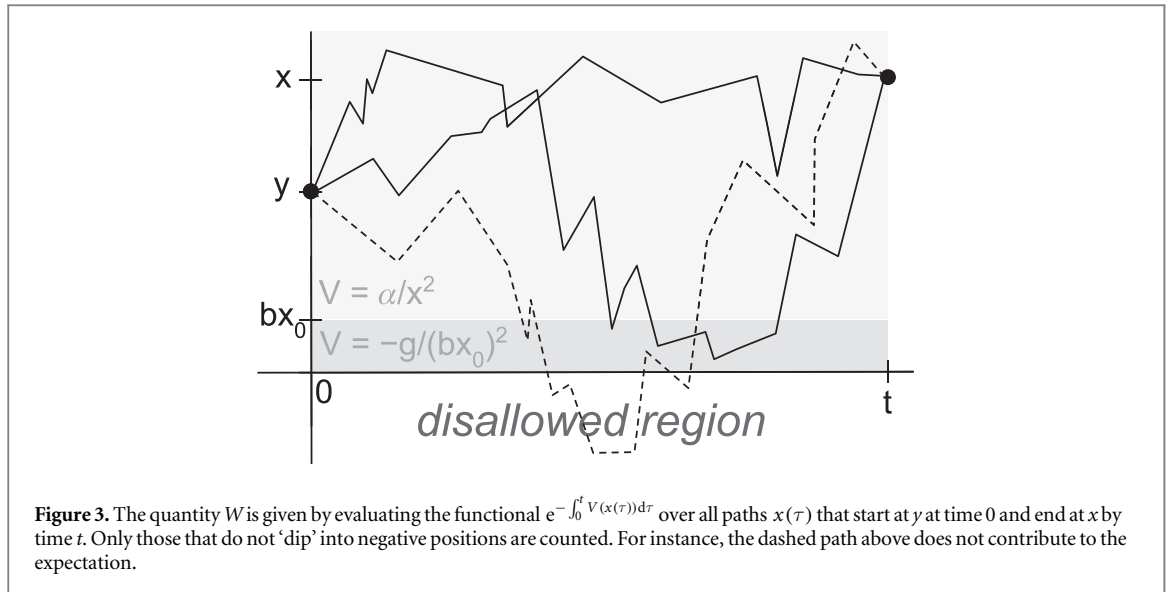
We are interested in the following conditional expectation value,<sup>4</sup>

$$\begin{aligned} W(x, t; y) &= \mathbb{E}'_{x,t;y,0} \left[ e^{-\int_0^t V(x(\tau))d\tau} \right] \\ &= \lim_{N \rightarrow \infty} \int_{-\infty}^\infty dx_1 \cdots dx_{N-1} (4\pi\epsilon)^{-N/2} \\ &\quad \times \exp\left(-\sum_{j=0}^{N-1} \frac{(x_{j+1} - x_j)^2}{4\epsilon} + \epsilon V(x_j)\right), \end{aligned}$$

with  $\epsilon = t/N$ ,  $x_0 = y$ , and  $x_N = x$ . Since  $V$  is infinite for all  $x \leq 0$ , any path that ‘dips’ into  $x \leq 0$  at any time will be weighted by  $e^{-\infty} = 0$ . This effectively eliminates from consideration all paths that travel left of the origin. Thus, we are considering Brownian motion on the half-line  $x > 0$ . See figure 3.

Kac proved that  $W$  satisfies the pde  $\frac{\partial}{\partial t}W = \left(\frac{\partial^2}{\partial x^2} - V\right)W$  with boundary condition  $\lim_{t \downarrow 0} W(x, t; y) = \delta(x - y)$ . This is mathematically identical to the problem of finding the quantum mechanical Green’s function in imaginary time [13].

<sup>4</sup> It should be noted that  $W$  is not quite a true expectation value because it is not properly normalized. The correctly normalized expression turns out to be  $W(x, t; y)/\rho(x, t)$ , where  $\rho(x, t) = e^{-x^2/4t}/\sqrt{4\pi t}$ . See [13].



Consider  $V$  given by equation (4) and let  $g = g_{\pm}$ . According to the quantum result equation (21), for  $x, y > bx_0, b \ll 1$ , and asymptotically large  $t, W(x, t; y) \sim t^{-1/2}(xy/b^2x_0^2)^{\nu_{\pm}}$ . Consider two Brownian expectations: one for all continuous paths from  $y$  to  $x$ , and another from  $\lambda'y$  to  $\lambda x$ , where all positions are strictly greater than  $bx_0$ . Then in the large- $t$  limit,

$$W(\lambda x, t; \lambda'y) \sim (\lambda\lambda')^{\nu_{\pm}}W(x, t; y).$$

In particular, observe that  $\lambda' = 1/\lambda$  leads to asymptotic equivalence. This is rather surprising from the stochastic point of view because the interval where the potential depends on  $g$  can be made arbitrarily narrow, nevertheless it exerts an outsized influence on the sum over paths! More precisely, consider the set of continuous functions that begin at  $(y, 0)$  and end at  $(x, t)$  while always satisfying  $x(\tau) > 0$  for all  $\tau \in (0, t)$ . Sampling only from this space, what is the conditional probability that a particle passes through the portal  $0 < x < bx_0$  at some intermediate time  $t_1$ ? There is a finite answer to this question<sup>5</sup> and it can be made arbitrarily small by taking  $b \rightarrow 0$ . A naive conclusion might be then that the value of  $g$  has little effect on  $W$ . However, the analysis shows the opposite:  $W$  exhibits very different scaling at two special values of  $g$ .

### 8.2. Statistical mechanics of a chain

Yet another way to interpret the imaginary-time Green's function is as a classical partition function. Consider a discrete and finite one-dimensional system with  $N$  real degrees of freedom  $x_j$ . Impose fixed boundary conditions  $x_0 = y$  and  $x_{N+1} = x$ . On the set of  $\{x_j\}$  define a probability measure by the Boltzmann factor,  $\mu = \exp(-S)/Z$ , where the energy of a configuration is

$$S = \sum_{j=0}^N \left[ \frac{1}{4\epsilon}(x_{j+1} - x_j)^2 + \frac{\epsilon}{2}V(x_j) + \frac{\epsilon}{2}V(x_{j+1}) \right].$$

$\epsilon$  is a nearest-neighbor coupling and  $V$  is an external potential. We take  $\epsilon > 0$  so that the site-to-site coupling is attractive—it is energetically favorable for  $x_j$  to be similar in value to  $x_{j\pm 1}$ . Lastly,  $Z$  is a constant chosen to satisfy the normalization condition. Explicitly,

$$Z_{xy}(N, \epsilon) = \int dx_1 \cdots dx_N e^{-S}.$$

*A priori* there is no relation between  $N$  and  $\epsilon$ . One is typically interested in evaluating expectations in the measure  $\mu$  in the infinite volume limit.

It is well known that, in the double scaling limit  $N \rightarrow \infty, \epsilon \rightarrow 0$  such that  $N\epsilon = t$  fixed, there is a correspondence between the classical statistical mechanics problem we have just defined and quantum mechanics of a particle propagating in negative imaginary time<sup>6</sup>. In fact,

<sup>5</sup> For some  $0 < t_1 < t$ , the probability is given by  $\int_0^{bx_0} \rho(x_1, t_1)\rho(x - x_1, t - t_1)dx_1 / \int_0^{\infty} \rho(x_1, t_1)\rho(x - x_1, t - t_1)dx_1$ , where  $\rho$  is given in footnote 4.

<sup>6</sup> Here is a reminder of how that correspondence works. Consider the operator mechanics given by the hermitian transfer matrix  $T = \exp[-\epsilon V(X)/2]\exp[-\epsilon P^2]\exp[-\epsilon V(X)/2]$ , where  $[P, X] = -i$ . Then  $\langle x|T^{N+1}|y \rangle$  equals, using the completeness property for  $P$  and  $X, (2\pi\epsilon)^{-(N+1)/2} \int dx_1 \cdots dx_N e^{-S}$ . But  $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \log T = P^2 + V(X)$ . Thus,  $\langle x|e^{-t(P^2+V(X))}|y \rangle = (2\pi\epsilon)^{-(N+1)/2}Z_{xy}(t)$ . We recognize the left-hand side as  $G(x, -it; y)$ . A trace over positions is not taken.

$$W(x, t; y) \propto Z_{xy}(t),$$

where the proportionality constant is independent of  $g$ . The limit we have taken is a continuum limit.

Now we wish to take a thermodynamic limit by making the ‘volume’  $t$  arbitrarily large. In this limit we are interested in extracting the free energy density defined by  $f_{xy} = \lim_{t \rightarrow \infty} \frac{-1}{t} \log Z_{xy}(t)$ . According to the discussion in section 7, for  $g > g_*$  a bound state exists so the dominant behavior of  $W(x, t; y)$  is exponential in  $t$  for large  $t$ . That is,  $W(x, t; y) \sim e^{-E_0 t} \psi_0(x) \psi_0(y)$  plus exponentially suppressed corrections. Here  $E_0$  is the ground state energy. Therefore,  $f_{xy} \sim E_0$ . Terms like  $(\log \psi_0(x))/t$  vanish in the large- $t$  limit so the specific choice of  $x$  and  $y$  do not affect the free energy density in the thermodynamic limit. Thus,

$$f_{xy} \propto (g - g_*)^{1/\omega}, \quad g \gtrsim g_*$$

This is finite and so an extensive phase exists if  $g > g_*$ . However, if  $g < g_*$ , then  $W(x, t; y)$  is asymptotically a power law in  $t$  so the system is no longer extensive. In other words,

$$f_{xy} \sim 0, \quad g < g_*$$

We see that the loss of extensivity as the parameter  $g \downarrow g_*$  is characterized by a universal critical exponent  $1/\omega$ .

The phase transition described by the crossing of  $g$  across  $g_*$  is different from the kind of finite-temperature phase transition familiar from lattice models. One prominent difference is that in short-range lattice spin systems, the free energy is extensive on both sides of the critical point. Nevertheless, similar extensive-to-nonextensive phase transitions are described in [14] for the case  $\alpha < -1/4$ . These authors treat  $\alpha$  as a variable thermal parameter. In one experimentally realizable system it is explained how the approach  $\alpha \uparrow -1/4$  reproduces Berezinskii–Kosterlitz–Thouless scaling related to a topological transition between winding states of a floppy polymer circling a defect.

## 9. Other aspects of the inverse-square potential

While there are other features and applications of the inverse-square potential, here we make two brief remarks that extend our analysis and bridge our work with other pedagogical discussions of the inverse-square potential [4, 5].

### 9.1. Three dimensions

Our results apply to two particles interacting in the  $s$ -wave via a central potential that is  $\alpha/r^2$  at large  $r$  [5]. Since the potential is time-independent the nontrivial part of the two-particle wavefunction is obtained by solving  $-\frac{\hbar^2}{2m} \nabla_r^2 \psi + V\psi = E\psi$ , where  $m$  is the reduced mass. In a zero-orbital-angular-momentum state,  $\psi \propto U(r)/r$ . It is well known that the radial function  $U$  obeys an equation that looks just like the Schrödinger equation on the half-line,  $-\frac{\hbar^2}{2m} d^2U/dr^2 + V(r)U = EU$ ,  $r \geq 0$ .

It turns out that the appropriate boundary condition is that  $U(0) = 0$ , precisely analogous to that of our one-dimensional example. In [15] a proof is given by demanding that  $-d^2/dr^2$  be hermitian with respect to the space obtained from all square-integrable linear combinations of functions  $U$  satisfying the eigenvalue equation. This argument is repeated below.

Let  $U_1$  and  $U_2$  be two such functions. The combination of square integrability and satisfaction of the ode implies that  $U_i$  and  $U_i'$  limit to 0 as  $r$  goes to  $\infty$ . Then operator hermiticity necessitates that  $(U_1'(0)/U_1(0))^* = U_2'(0)/U_2(0)$ , which is precisely equation (3) again. From this we see that  $U_1(0)$  and  $U_2(0)$  are equal up to a multiplicative constant for which the only self-consistent choice is 1. Thus, all  $U_i(0)$  must equal the same constant  $c$ . However, if  $c \neq 0$ , then  $\psi \sim 1/r$  which is problematic because  $-\nabla_r^2(1/r) \propto \delta^3(\mathbf{r})$  yet there is no corresponding delta function in  $V$ . The only way to avoid this situation is to have  $c = 0$ .

### 9.2. $\alpha < -1/4$

While the case  $\alpha < -1/4$  has been well-studied in the literature (see, for example, [4, 9]) it is worth highlighting its RG analysis for its qualitatively different long-distance behavior.

The pure  $\alpha/x^2$  potential with  $\alpha < -1/4$  has a spectrum unbounded from below so one really does need to regulate the Hamiltonian with a cutoff in order to obtain a healthy theory. Suppose this is done with the finite square well, equation (4). Let  $\epsilon = -Ex_0^2$ . Then for the shallowest bound states satisfying  $\epsilon \ll 1$ , the implicit energy equation is

$$\sqrt{g} \cot \sqrt{g} \sim \frac{1}{2} - |\omega| \tan(|\omega| \log b + \frac{1}{2} |\omega| \log \epsilon + \phi), \quad (24)$$

where the phase  $\phi$  is determined by the normalizable solution and is not a free parameter. This regulator breaks the continuous scale symmetry to a discrete subgroup (notice that  $\epsilon \rightarrow e^{-2\pi/|\omega|} \epsilon$  leaves equation (24) invariant). Fix  $\epsilon$  and consider the locus of points in the  $(\log b, g)$ -plane obeying equation (24). If  $(\log b_0, g_0)$  is any such

point, then a continuous curve of points exist in its vicinity. By following the curve to more negative  $\log b$ ,  $g$  tends toward zero. Eventually, there will come a time when  $g$  attains zero, but at this point we may jump to another curve and start the flow all over again with a finite value of  $g$ . This is possible because there are an infinite number of solutions  $g$  for a given  $b$ . The periodicity of tangent means that the coupling  $g$  is renormalized, but never approaches a limit, as  $b$  decreases. Instead, each time  $\log b$  is shifted by  $-\pi/|\omega|$ ,  $g$  may be identified with its starting value. Such identification is natural because it keeps  $g$  positive. The process may be replicated indefinitely;  $b$  remains strictly positive because it is reduced by a multiplicative factor. This describes a RG *limit cycle*—the coupling  $g$  cycles through an interval of values.

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### Appendix. Integral curves of constant phase shift

For  $k = E^{1/2} > 0$ , the time-independent part of the solution to the Schrödinger equation with  $V$  given by equation (4) may be written

$$\psi = \sqrt{\frac{\pi}{2}} e^{-i\omega\pi/2 - i\pi/4} \times \begin{cases} A \sin(x\sqrt{k^2 + g/b^2x_0^2}) & 0 \leq x < bx_0 \\ \sqrt{kx} H_\omega^{(2)}(kx) + re^{i\omega\pi + i\pi/2} \sqrt{kx} H_\omega^{(1)}(kx) & x > bx_0 \end{cases}$$

$H_\omega^{(i)}$  are Hankel functions. The asymptotic form of this solution is  $\psi \sim e^{-ikx} + re^{ikx}$ . Doing the matching at  $x = bx_0$  yields the following explicit formula for the reflection amplitude,

$$r = ie^{-i\omega\pi} \frac{1 - ic}{1 + ic},$$

where

$$c = \frac{Y'_\omega(\mu) - dY_\omega(\mu)}{J'_\omega(\mu) - dJ_\omega(\mu)},$$

$$d = \frac{2\sqrt{\mu^2 + g} \cot \sqrt{\mu^2 + g} - 1}{2\mu}.$$

It is obvious that  $|r| = 1$  so this is pure phase.

Demand that  $dr/d\mu = 0$ . This is only possible if  $g$  is allowed to ‘run’ with  $\mu$ . Therefore, the  $\mu$  dependence of the phase  $r$  is present both explicitly and via implicit dependence through  $g$ . Write  $r = R(\mu, g(\mu))$ . Then

$$0 = \frac{dr}{d\mu} = \frac{\partial R}{\partial \mu} + \frac{\partial R}{\partial g} \frac{\partial g}{\partial \mu}.$$

This is a linear first-order pde of the form  $R_\mu + f(\mu, g)R_g = 0$ , with

$$f = \frac{-\alpha\zeta \sin^2 \zeta + g\zeta \cos^2 \zeta - g \sin \zeta \cos \zeta}{(\mu/2)(\zeta - \sin \zeta \cos \zeta)}, \quad \zeta = \sqrt{\mu^2 + g}.$$

The directional derivative of  $R$  in the direction of the vector field  $(1, f(\mu, g))$  is zero. Therefore, in the  $(\mu, g)$ -plane there is a one-parameter family of solutions  $\phi(\mu, g) = c$  for constant  $c$ . Since  $R$  is constant along each integral curve, this means that  $R$  depends only on the value of  $c$ , or, equivalently, all the  $\mu$  and  $g$  dependence in  $R$  appears as  $r = R(c) = R(\phi(\mu, g))$ .

Define the phase shift  $\delta$  by  $r = -e^{2i\delta}$ . The rationale for this definition is that for scattering off an attractive potential,  $\delta$  will be positive and the wave function is ‘drawn into’ the well by a distance given by  $\delta/k$  relative to the case of scattering off a hard wall. When the de Broglie wavelength of the incident particle is much longer than the width of the square well,  $k^{-1} \gg bx_0$  (i.e.,  $\mu \ll 1$ ),

$$\delta = \frac{\pi}{4}(1 - 2\omega) - \frac{\pi}{\omega(2^\omega \Gamma(\omega))^2} \frac{\sqrt{g} \cot \sqrt{g} - \nu_+}{\sqrt{g} \cot \sqrt{g} - \nu_-} \mu^{2\omega} + O(\mu^{4\omega}).$$

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