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Time dependent black holes and gravitational wave in Einstein–Gauss–Bonnet theory with two scalar fields

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Abstract In this paper, we investigate the time dependent black holes in the frame of Einstein–Gauss–Bonnet theory having two scalar fields and investigate the propagation of the gravitational wave (GW). In the reconstructed models, there often appear ghosts, which could be eliminated by imposing some constraints. We investigate the behavior of high-frequency gravitational waves by examining the effects of varying Gauss–Bonnet coupling during their propagation. The speed of propagation changes due to the coupling during the black hole formation process. The propagation speed of gravitational waves differs when they enter the black hole compared to when they exit.

1 Introduction

Black holes reside within the rapidly growing universe. Specifically, in our growing universe, there are no black holes that are asymptotically flat. Therefore, it is essential to study the physics and thermodynamics of black holes within a growing cosmological environment. Thus, in recent years, there has been much research on black holes that are asymptotic to the expanding universe, referred to as "Cosmological Black Holes". These black holes raise various questions as: What effects does the cosmic expansion have on the local physics of black holes in the entire cosmic epochs? What impact does the universe's content have on the black hole? What changes should be made to the physics of black holes in light of the expanding universe? In an expanding universe, how should the definitions of black hole horizon, singularities, mass, and thermodynamics be altered? McVittie's solution is one of the earlier studies that explains black holes within the Friedmann–Robertson–Walker (FRW) universe [1]. Subsequently, solutions such as those developed by Einstein and Strauss [2], Lemaitre–Tolman–Bondi (LTB) [3–5], and Vaidya [6] were introduced. In this study, the significant aspect is the reconsideration of boundaries using indigenous ideas instead of asymptotically flat circumstances, a concept proposed as trapping horizon by Hayward [7] and as dynamical horizon [8]. In addition to the LTB metric's dynamic nature, the FLRW metric can be represented as a background and is a specific instance of the LTB metric. By utilizing the characteristics of the LTB metric, it is possible to create a cosmological black hole [9], with both its singularity and horizon emerging as it collapses [10]. References [11,12] provide useful overviews of different types of horizons such as event, Killing, apparent, trapping, isolated, and dynamical horizons. Within modified theories like $f(R)$, $f(G)$, etc. it is difficult to derive time dependent solution [13,14]. However, for theories coupled with scalar fields it can be applicable to derive time-dependent solution [15].

Einstein–Gauss–Bonnet (EGB) theories are significant because they emerge as ultraviolet corrections to the minimally coupled scalar field theory [16]. The EGB theory is a promising candidate for explaining the inflationary epoch [17–73]. One of its key features is its ability to generate a desirable blue-tilted tensor spectral index. This property is crucial because it enables the stochastic gravitational waves produced during an EGB inflationary era to be potentially detected in future gravitational wave experiments. Recent research [15] demonstrated that an arbitrary spherically symmetric spacetime can be constructed using Einstein's gravity coupled with two scalar fields, even when the spacetime itself is dynamic. Research has shown that ghost degrees of freedom can emerge in certain models. In classical theory, these ghosts have kinetic energy that is unbounded from below, and in quantum theory, they produce negative norm states, a phenomenon also observed in quantum chromodynamics [74]. This is significant because it conflicts with the Copenhagen interpretation of quantum theory. The presence

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of ghost degrees of freedom poses a challenge to physical reality. Recently, an extension of the formulation in Ref. [15] has led to proposed models describing static wormholes [75] and gravastars [76]. Furthermore, these models have been investigated for the propagation of gravitational waves in spherically symmetric spacetimes [77]. Gravastars offer an alternative to conventional black holes and could potentially form through Bose–Einstein condensation [78]. In previous works [15,75], constraints were imposed on the scalar fields to prevent the emergence of ghost degrees of freedom. The constraints used resembled those employed in mimetic gravity [79]. Consequently, the scalar fields lose their dynamical behavior, effectively prohibiting fluctuations and preventing the propagation of these fields. In mimetic gravity, the constraints employed mirror those used, causing the scalar fields to lose their dynamic behavior. This effectively prevents fluctuations and hinders the propagation of these fields. In the context of the original mimetic gravity approach, this scenario closely resembles a situation where effective dark matter emerges. Unlike standard dust, this effective dark matter exhibits vanishing pressure, yet it remains non-dynamical and does not undergo gravitational collapse.

In this study, we examine an extension of the model presented in [15], which involves coupling the Gauss–Bonnet invariant with two scalar fields because of the substantial phenomenological implications of EGB. Furthermore, the Gauss–Bonnet term emerges as α correction from string theory to Einstein's gravity, with the parameter $1/\alpha$ representing the string tension. We designate this model, which encompasses both scalar fields and the Gauss–Bonnet invariant, as EGB gravity. Moreover, we show that within the present study it is impossible to construct a model in which the propagation speed of gravitational waves coincides with that of light in dynamical spherically symmetric spacetimes.

The arrangement of this study is as follows: in Sect. 2, we present in a brief scenario the EGB theory coupled with two scalar fields. In Sect. 3, we apply the field of EGB coupled with two scalar fields to a time dependent spherically symmetric spacetime and obtained a system of differential equations. This system is in a closed form which enabled us to derive explicitly the forms of the coefficients of the scalar field besides the potential. In Sect. 4 we present three time dependent black holes and derive their relevant coefficient of scalar field as well as the potential. In Sect. 5 we discuss the gravitational wave by present the perturbed form of EGB theory coupled with two scalar fields. In the final section we investigate the results contained in this study.

2 Ghost-free Einstein–Gauss–Bonnet gravity coupled with two scalar fields

According to the study presented in [15] it is shown that any spherically symmetric spacetime, whether dynamic or static, can be constructed using Einstein's gravity in conjunction with two scalar fields. The model in [15], however, includes ghost[1] degrees of freedom. In the present study, we first develop models without ghosts by introducing constraints via Lagrange multiplier fields, similar to the constraints found in the mimetic gravity theoretical framework [79]. Moreover, to account for variations in the propagation speed of gravitational waves, we incorporate the coupling of two scalar fields with the Gauss–Bonnet topological invariant, following the approach outlined in the references [26,27].

We start with EGB gravity coupled to two scalar fields, denoted as φ and ψ. The action for this system is given by:

$$
\begin{aligned}
S_{\varphi\psi} = \int d^4x\sqrt{-g}\Big\{&\frac{R}{2\kappa^2} - \frac{1}{2}\beta(\varphi,\psi)\partial_\mu\varphi\partial^\mu\varphi \\
&-\sigma(\varphi,\psi)\partial_\mu\varphi\partial^\mu\psi - \frac{1}{2}\alpha(\varphi,\psi)\partial_\mu\psi\partial^\mu\psi - \Phi(\varphi,\psi) \\
&-\zeta(\varphi,\psi)\mathcal{G} + \mathcal{L}_{\text{matter}}\Big\} \,. \qquad (1)
\end{aligned}
$$

Here, $\Phi(\varphi,\psi)$ represents the potential which is a function of φ and ψ, and $\zeta(\varphi,\psi)$ is coupling function of the Gauss–Bonnet which is a function of φ and ψ. Here $\mathcal{G}$ represents Gauss–Bonnet invariant term given by

$$
\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} \,, \qquad (2)
$$

that is a topological density. In four dimensions Gauss–Bonnet is a total derivative which yields the Euler number.

Varying the action of Eq. (1) regarding metric $g_{\mu\nu}$, gives

$$
\begin{aligned}
0 = &\frac{1}{2\kappa^2}\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) \\
&+ \frac{1}{2}g_{\mu\nu}\Big\{-\frac{1}{2}\beta(\varphi,\psi)\partial_\rho\varphi\partial^\rho\varphi - \sigma(\varphi,\psi)\partial_\rho\varphi\partial^\rho\psi \\
&-\frac{1}{2}\alpha(\varphi,\psi)\partial_\rho\psi\partial^\rho\psi - \Phi(\varphi,\psi)\Big\} \\
&+ \frac{1}{2}\Big\{\beta(\varphi,\psi)\partial_\mu\varphi\partial_\nu\varphi \\
&+\sigma(\varphi,\psi)\left(\partial_\mu\varphi\partial_\nu\psi + \partial_\nu\varphi\partial_\mu\psi\right) + \alpha(\varphi,\psi)\partial_\mu\psi\partial_\nu\psi\Big\} \\
&- 2\left(\nabla_\mu\nabla_\nu\zeta(\varphi,\psi)\right)R + 2g_{\mu\nu}\left(\nabla^2\zeta(\varphi,\psi)\right)R \\
&+ 4\left(\nabla_\rho\nabla_\mu\zeta(\varphi,\psi)\right)R_\nu^{\ \rho} + 4\left(\nabla_\rho\nabla_\nu\zeta(\varphi,\psi)\right)R_\mu^{\ \rho} \\
&- 4\left(\nabla^2\zeta(\varphi,\psi)\right)R_{\mu\nu} - 4g_{\mu\nu}\left(\nabla_\rho\nabla_\sigma\zeta(\varphi,\psi)\right)R^{\rho\sigma}
\end{aligned}
$$

[1] The ghost mode has negative kinetic energy classically and generates negative norm states as a quantum theory. Therefore the existence of the ghost mode tells that the model is physically inconsistent.

$$+4\left(\nabla^\rho\nabla^\sigma\zeta(\varphi,\psi)\right)R_{\mu\rho\nu\sigma}+\frac{1}{2}T_{\text{matter}\,\mu\nu}\,, \tag{3}$$

The field equations of the scalar fields are derived by varying the action with respect to the scalar fields φ and ψ resulting in the following expressions:

$$\begin{aligned}0=&\frac{1}{2}\beta_\varphi\partial_\mu\varphi\partial^\mu\varphi\\&+\beta\nabla^\mu\partial_\mu\varphi+\beta_\psi\partial_\mu\varphi\partial^\mu\psi\\&+\left(\sigma_\psi-\frac{1}{2}\alpha_\varphi\right)\partial_\mu\psi\partial^\mu\psi+\sigma\nabla^\mu\partial_\mu\psi-\Phi_\varphi-\zeta_\varphi\mathcal{G}\,,\\0=&\left(-\frac{1}{2}\beta_\psi+\sigma_\varphi\right)\partial_\mu\varphi\partial^\mu\varphi\\&+\sigma\nabla^\mu\partial_\mu\varphi+\frac{1}{2}\alpha_\psi\partial_\mu\psi\partial^\mu\psi+\alpha\nabla^\mu\partial_\mu\psi\\&+\alpha_\varphi\partial_\mu\varphi\partial^\mu\psi-\Phi_\psi-\zeta_\psi\mathcal{G}\,,\end{aligned} \tag{4}$$

where $\beta_\varphi=\partial\beta(\varphi,\psi)/\partial\varphi$, etc. In Eq. (3), $(T_{\text{matter}})_{\mu\nu}$ represents the energy-momentum tensor of matter.

Considering the general spherically symmetric and time-dependent spacetime, described by the metric,[2]

$$\begin{aligned}ds^2=&-\mathrm{e}^{2\nu(t,r)}dt^2+\mathrm{e}^{2\nu_1(t,r)}dr^2\\&+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\,,\end{aligned} \tag{5}$$

where $\nu(t,r)$ and $\nu_1(t,r)$ are two unknown function of time and radial coordinate. Furthermore, we use following identifications,

$$\varphi=t\,,\qquad \psi=r\,. \tag{6}$$

This assumption does not lead to a loss of generality for the following reason: for the spherically symmetric solutions (5) of the theory (1), in general φ and ψ depend on both coordinates t and r. Given a solution, the t- and r-dependence of φ and ψ can be determined and φ and ψ are then given by specific functions $\varphi(t,r)$, $\psi(t,r)$. On spacetime regions where these relations are one-to-one, and provided that $\partial_\mu\varphi$ is timelike and $\partial_\mu\psi$ is spacelike, one can invert these specific functional forms and redefine the scalar fields to replace t and r with new scalar fields, say, $\bar\varphi$ and $\bar\psi$ with $\varphi(t,r)\to\varphi(\bar\varphi,\bar\psi)$ and $\psi(t,r)\to\psi(\bar\varphi,\bar\psi)$. We can then identify the new fields with t and r as in (6), $\bar\varphi\to\varphi=t$ and $\bar\psi\to\psi=r$. The change of variables $(\varphi,\psi)\to\left(\bar\varphi,\bar\psi\right)$ can then be absorbed into redefinitions of β, σ, α, and Φ in the action (1). Therefore, the assumption (6) does not lead to loss of generality [77].

For the purpose of eliminating any ghosts that could appear we use the Lagrange multiplier fields λ_φ and λ_ψ, that can be added to the action $(S_{\mathrm{GR}})_{\varphi\psi}$ given by Eq. (1) as $(S_{\mathrm{GR}})_{\varphi\psi}\to(S_{\mathrm{GR}})_{\varphi\psi}+S_\lambda$,

$$\begin{aligned}S_\lambda=\int d^4x\sqrt{-g}\Big[&\lambda_\varphi\left(\mathrm{e}^{-2\nu(t=\varphi,r=\psi)}\partial_\mu\varphi\partial^\mu\varphi+1\right)\\&+\lambda_\psi\left(\mathrm{e}^{-2\lambda(t=\varphi,r=\psi)}\partial_\mu\psi\partial^\mu\psi-1\right)\Big]\,.\end{aligned} \tag{7}$$

Variations of Eq. (7) with respect to λ_φ and λ_ψ yield,

$$\begin{aligned}&0=\mathrm{e}^{-2\nu(t=\varphi,r=\psi)}\partial_\mu\varphi\partial^\mu\varphi+1\,,\\&0=\mathrm{e}^{-2\nu_1(t=\varphi,r=\psi)}\partial_\mu\psi\partial^\mu\psi-1\,,\end{aligned} \tag{8}$$

whose solutions are consistently given by Eq. (6).

The constraints in Eq. (8) make the scalar field φ and ψ non-dynamical, that is, the fluctuation of φ and ψ from the background (6) do not propagate. In fact, by considering the perturbation from of Eq. (6) we get:

$$\varphi=t+\delta\varphi\,,\qquad \psi=r+\delta\psi\,, \tag{9}$$

By using Eq. (8), we find

$$\partial_t\left(\mathrm{e}^{2\nu(t=\varphi,r=\psi)}\delta\varphi\right)=\partial_r\left(\mathrm{e}^{2\nu_1(t=\varphi,r=\psi)}\delta\chi\right)=0\,. \tag{10}$$

Eq. (10) tells that by imposing the initial condition $\delta\varphi=0$ (because φ corresponds to the time coordinate) and by imposing the boundary condition $\delta\psi\to0$ when $r\to\infty$ (because ψ corresponds to the radial coordinate), we find that both of $\delta\varphi$ and $\delta\psi$ vanish in the whole spacetime, $\delta\varphi=0$ and $\delta\psi=0$ not only at the initial surface for $\delta\varphi$ and boundary surface for $\delta\psi$. This tells that both φ and ψ are non-dynamical or frozen degrees of freedom. References [75,76,80,81] tell that $\lambda_\varphi=\lambda_\psi=0$ consistently appear as a solution even in the model with the modified action $S+S_\lambda$. This tells any solution of Eqs. (3) and (4) which are based on the original action (1) is a solution even for the modified model with the action $S+S_\lambda$.

Upon modifying $S_{\mathrm{GR}\varphi\psi}\to S_{\mathrm{GR}\varphi\psi}+S_\lambda$ in (7), the equations in (3) and (4) undergo the following modifications:

$$\begin{aligned}0=&\frac{1}{2\kappa^2}\left(-R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}R\right)\\&+\frac{1}{2}g_{\mu\nu}\Big\{-\frac{1}{2}\beta(\varphi,\psi)\partial_\rho\varphi\partial^\rho\varphi\\&-\sigma(\varphi,\psi)\partial_\rho\varphi\partial^\rho\psi\\&-\frac{1}{2}\alpha(\varphi,\psi)\partial_\rho\psi\partial^\rho\psi-\Phi(\varphi,\psi)\Big\}\\&+\frac{1}{2}\big\{\beta(\varphi,\psi)\partial_\mu\varphi\partial_\nu\varphi\\&+\sigma(\varphi,\psi)\left(\partial_\mu\varphi\partial_\nu\psi+\partial_\nu\varphi\partial_\mu\psi\right)+\alpha(\varphi,\psi)\partial_\mu\psi\partial_\nu\psi\big\}\end{aligned}$$

[2] The justifications that metric (5) is a general one are presented in [15].

$$
\begin{aligned}
& -2\left(\nabla_\mu \nabla_\nu \zeta(\varphi, \psi)\right) R+2 g_{\mu \nu}\left(\nabla^2 \zeta(\varphi, \psi)\right) R \\
& +4\left(\nabla_\rho \nabla_\mu \zeta(\varphi, \psi)\right) R_\nu^{\ \rho} \\
& +4\left(\nabla_\rho \nabla_\nu \zeta(\varphi, \psi)\right) R_\mu^{\ \rho} \\
& -4\left(\nabla^2 \zeta(\varphi, \psi)\right) R_{\mu \nu}-4 g_{\mu \nu}\left(\nabla_\rho \nabla_\sigma \zeta(\varphi, \psi)\right) R^{\rho \sigma} \\
& +4\left(\nabla^\rho \nabla^\sigma \zeta(\varphi, \psi)\right) R_{\mu \rho \nu \sigma} \\
& +\frac{1}{2} g_{\mu \nu}\left\{\lambda_{\varphi}\left(\mathrm{e}^{-2 \nu(r=\psi)} \partial_\rho \varphi \partial^\rho \varphi+1\right)\right. \\
& \left.+\lambda_\psi\left(\mathrm{e}^{-2 \lambda(r=\psi)} \partial_\rho \psi \partial^\rho \psi-1\right)\right\} \\
& -\lambda_{\varphi} \mathrm{e}^{-2 \nu(r=\psi)} \partial_\mu \varphi \partial_\nu \varphi-\lambda_\psi \mathrm{e}^{-2 \lambda(r=\psi)} \partial_\mu \psi \partial_\nu \psi \\
& +\frac{1}{2} T_{\text{matter}\, \mu \nu}\,,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
0= & \frac{1}{2} \beta_{\varphi} \partial_\mu \varphi \partial^\mu \varphi+\beta \nabla^\mu \partial_\mu \varphi+\beta_\psi \partial_\mu \varphi \partial^\mu \psi \\
& +\left(\sigma_\psi-\frac{1}{2} \alpha_{\varphi}\right) \partial_\mu \psi \partial^\mu \psi+\sigma \nabla^\mu \partial_\mu \psi-\Phi_{\varphi}-\zeta_{\varphi} \mathcal{G} \\
& -2 \nabla^\mu\left(\lambda_{\varphi} \mathrm{e}^{-2 \nu(r=\psi)} \partial_\mu \varphi\right),
\end{aligned}
$$

$$
\begin{aligned}
0= & \left(-\frac{1}{2} \beta_\psi+\sigma_{\varphi}\right) \partial_\mu \varphi \partial^\mu \varphi \\
& +\sigma \nabla^\mu \partial_\mu \varphi+\frac{1}{2} \alpha_\psi \partial_\mu \psi \partial^\mu \psi+\alpha \nabla^\mu \partial_\mu \psi \\
& +\alpha_{\varphi} \partial_\mu \varphi \partial^\mu \psi-\Phi_\psi-\zeta_\psi \mathcal{G} \\
& -2 \nabla^\mu\left(\lambda_\psi \mathrm{e}^{-2 \lambda(r=\psi)} \partial_\mu \psi\right) .
\end{aligned} \tag{12}
$$

Let's examine the solutions to the equations in (3) and (4) under the assumptions (5) and (6). Subsequently, the (t, t) and (r, r) components in (11) yield:

$$
0=\lambda_{\varphi}=\lambda_\psi\,. \tag{13}
$$

The remaining components in (11) are identically satisfied. Additionally, Eq. (12) yields

$$
0=\nabla^\mu\left(\lambda_{\varphi} \mathrm{e}^{-2 \nu(r=\psi)} \partial_\mu \varphi\right)=\nabla^\mu\left(\lambda_\psi \mathrm{e}^{-2 \lambda(r=\psi)} \partial_\mu \psi\right). \tag{14}
$$

The satisfaction of Eq. (14) implies the satisfaction of Eq. (13). Consequently, the solution to Eqs. (3) and (4) corresponds to a solution of Eqs. (11) and (12) associated with the modified action as $S_{\mathrm{GR}\varphi\psi} \rightarrow S_{\mathrm{GR}\varphi\psi}+S_\lambda$ in (7) if λ_{φ} and λ_ψ are nil (13). However, it's worth noting that for Eqs. (11) and (12), there exists a solution where λ_{φ} and λ_ψ do not equal zero.

The detailed perturbation of the field equations (11) is given in (see for example [77,80]. Therefore in this study we will not repeat these calculation but give the final perturbation form which yields:

$$
\begin{aligned}
0= & {\left[\frac{1}{4 \kappa^2} R+\frac{1}{2}\left\{-\frac{1}{2} \partial_\rho \varphi \partial^\rho \varphi-\Phi\right\}\right.} \\
& \left.-4\left(\nabla_\rho \nabla_\sigma \zeta\right) R^{\rho \sigma}\right] h_{\mu \nu} \\
& +\left[\frac{1}{4} g_{\mu \nu}\left\{-\beta \partial^\tau \varphi \partial^\eta \varphi-\sigma\left(\partial^\tau \varphi \partial^\eta \psi+\partial^\eta \varphi \partial^\tau \psi\right)\right.\right. \\
& \left.-\alpha \partial^\tau \psi \partial^\eta \psi\right\}-2 g_{\mu \nu}\left(\nabla^\tau \nabla^\eta \zeta\right) R \\
& -4\left(\nabla^\tau \nabla_\mu \xi\right) R_\nu^{\ \eta}-4\left(\nabla^\tau \nabla_\nu \xi\right) R_\mu^{\ \eta}+4\left(\nabla^\tau \nabla^\eta \zeta\right) R_{\mu \nu} \\
& +4 g_{\mu \nu}\left(\nabla^\tau \nabla_\sigma \zeta\right) R^{\eta \sigma}+4 g_{\mu \nu}\left(\nabla_\rho \nabla^\tau \zeta\right) R^{\rho \eta} \\
& \left.-4\left(\nabla^\tau \nabla^\sigma \zeta\right) R_{\mu\ \ \nu \sigma}^{\ \eta}-4\left(\nabla^\rho \nabla^\tau \zeta\right) R_{\mu \rho \nu}^{\quad\ \eta}\right\} h_{\tau \eta} \\
& +\frac{1}{2}\left\{2 \delta_\mu^{\ \eta} \delta_\nu^{\ \zeta}\left(\nabla_\kappa \zeta\right) R-4 \delta_\rho^{\ \eta} \delta_\mu^{\ \zeta}\left(\nabla_\kappa \zeta\right) R_\nu^{\ \rho}\right. \\
& -4 \delta_\rho^{\ \eta} \delta_\nu^{\ \zeta}\left(\nabla_\kappa \zeta\right) R_\mu^{\ \rho}+4 g_{\mu \nu} \delta_\rho^{\ \eta} \delta_\sigma^{\ \zeta}\left(\nabla_\kappa \zeta\right) R^{\rho \sigma} \\
& \left.-4 g^{\rho \eta} g^{\sigma \zeta}\left(\nabla_\kappa \zeta\right) R_{\mu \rho \nu \sigma}\right\} g^{\kappa \lambda}\left(\nabla_\eta h_{\zeta \lambda}\right. \\
& \left.+\nabla_\zeta h_{\eta \lambda}-\nabla_\lambda h_{\eta \zeta}\right) \\
& -\left\{\frac{1}{4 \kappa^2} g_{\mu \nu}-2\left(\nabla_\mu \nabla_\nu \zeta\right)+2 g_{\mu \nu}\left(\nabla^2 \zeta\right)\right\} R^{\mu \nu} h_{\mu \nu} \\
& +\frac{1}{2}\left\{\left(-\frac{1}{2 \kappa^2}-4 \nabla^2 \zeta\right) \delta_{\ \mu}^\tau \delta_{\ \nu}^\eta+4\left(\nabla_\rho \nabla_\mu \zeta\right)\right. \\
& \left.\delta_{\ \nu}^\eta g^{\rho \tau}\ +4\left(\nabla_\rho \nabla_\nu \zeta\right) \delta_{\ \mu}^\tau g^{\rho \eta}-4 g_{\mu \nu} \nabla^\tau \nabla^\eta \zeta\right\} \\
& \times\left\{-\nabla^2 h_{\tau \eta}-2 R_{\eta\ \tau}^{\lambda\ \varphi} h_{\lambda \varphi}+R_{\ \tau}^{\varphi} h_{\varphi \eta}+R_{\ \tau}^{\varphi} h_{\varphi \eta}\right\} \\
& +2\left(\nabla^\rho \nabla^\sigma \zeta\right)\left\{\nabla_\nu \nabla_\rho h_{\sigma \mu}-\nabla_\nu \nabla_\mu h_{\sigma \rho}-\nabla_\sigma \nabla_\rho h_{\nu \mu}\right. \\
& \left.+\nabla_\sigma \nabla_\mu h_{\nu \rho}+h_{\mu \varphi} R_{\ \rho \nu \sigma}^{\varphi}-h_{\rho \varphi} R_{\ \mu \nu \sigma}^{\varphi}\right\} \\
& +\frac{1}{2} \frac{\partial T_{\text{matter}\, \mu \nu}}{\partial g_{\tau \eta}} h_{\tau \eta}\,.
\end{aligned} \tag{15}
$$

The observation of GW170817 gives the constraint on the propagating speed c_{GW} of the gravitational wave as follows,

$$
\left|\frac{c_{\mathrm{GW}}{}^2}{c^2}-1\right|<6 \times 10^{-15}, \tag{16}
$$

where c denotes the speed of light. In order to investigate if the propagating speed c_{GW} of the gravitational wave $h_{\mu\nu}$ could be different from that of the light c, we only need to check the parts including the second derivatives of $h_{\mu\nu}$,

$$
\begin{aligned}
I_{\mu \nu} \equiv & \ I_{\mu \nu}^{(1)}+I_{\mu \nu}^{(2)}, \\
I_{\mu \nu}^{(1)} \equiv & \ \frac{1}{2}\left\{\left(-\frac{1}{2 \kappa^2}-4 \nabla^2 \zeta\right) \delta_{\ \mu}^\tau \delta_{\ \nu}^\eta+4\left(\nabla_\rho \nabla_\mu \xi\right) \delta_{\ \nu}^\eta g^{\rho \tau}\right. \\
& \left.+4\left(\nabla_\rho \nabla_\nu \xi\right) \delta_{\ \mu}^\tau g^{\rho \eta}\ -4 g_{\mu \nu} \nabla^\tau \nabla^\eta \zeta\ \right\} \nabla^2 h_{\tau \eta}, \\
I_{\mu \nu}^{(2)} \equiv & \ 2\left(\nabla^\rho \nabla^\sigma \zeta\right)\left\{\nabla_\nu \nabla_\rho h_{\sigma \mu}-\nabla_\nu \nabla_\mu h_{\sigma \rho}\right. \\
& \left.-\nabla_\sigma \nabla_\rho h_{\nu \mu}+\nabla_\sigma \nabla_\mu h_{\nu \rho}\right\}.
\end{aligned} \tag{17}
$$

Now we are ready to derive time dependent black hole in the frame of EGB theory coupled with scalar fields.

3 Time dependent black hole

Now let us apply the line element given by Eq. (5) to the field equations (3) and get:

$$tt-component: \tag{18}$$

$$\frac{1-2\nu_1'(t,r)\,r-e^{2\nu_1(t,r)}}{r^2e^{2\nu_1(t,r)}}=\frac{-\kappa^2}{2r^2e^{2[\nu(t,r)+2\nu_1(t,r)]}}$$
$$\Big[16e^{2\nu(t,r)}\left\{1-e^{2\nu_1(t,r)}\right\}\xi''(t,r)+16\zeta'(t,r)\,e^{2\nu(t,r)}$$
$$\left\{e^{2\nu_1(t,r)}-3\right\}\nu_1'(t,r)$$
$$+\left\{16\left(e^{2\nu_1(t,r)}-1\right)\dot\zeta(t,r)\,\dot\nu_1(t,r)\right.$$
$$+\left[\left\{2e^{2\nu(t,r)}\Phi(t,r)+\beta(t,r)\,\dot\varphi^2(t)\right\}e^{2\nu_1(t,r)}\right.$$
$$\left.\left.+e^{2\nu(t,r)}\psi'^2(r)\,\alpha(t,r)\right]r^2\right\}e^{2\nu_1(t,r)}\Big],$$

$$tr-component: \tag{19}$$

$$\frac{2\dot\nu_1(t,r)}{re^{2\nu_1(t,r)}}=\frac{\kappa^2}{r^2e^{4\nu_1(t,r)}}$$
$$\times\left[\left\{8-8e^{2\nu_1(t,r)}\right\}\dot\zeta'(t,r)+\left\{8e^{2\nu_1(t,r)}-24\right\}\right.$$
$$\zeta'(t,r)\,\dot\nu_1(t,r)+\left\{8e^{2\nu_1(t,r)}-8\right\}\dot\zeta(t,r)\,\nu'(t,r)$$
$$\left.+\gamma(t,r)\,\dot\varphi(t)\,\psi'(r)\,r^2e^{2\nu_1(t,r)}\right],$$

$$rt-component=e^{2\nu_1(t,r)}(tr-component): \tag{20}$$

$$rr-component: \tag{21}$$

$$\frac{1+2\nu'(t,r)\,r-e^{2\nu_1(t,r)}}{r^2e^{2\nu_1(t,r)}}=\frac{\kappa^2}{2r^2e^{2[\nu(t,r)+2\nu_1(t,r)]}}$$
$$\Big[16e^{2\nu_1(t,r)}\left\{1-e^{2\nu_1(t,r)}\right\}\ddot\xi(t,r)$$
$$+16\zeta'(t,r)\,e^{2\nu(t,r)}\left\{e^{2\nu_1(t,r)}-3\right\}\nu_1'(t,r)$$
$$+\left\{16\left(e^{2\nu_1(t,r)}-1\right)\dot\zeta(t,r)\,\dot\nu_1(t,r)\right.$$
$$+\left[\left\{\beta(t,r)\,\dot\varphi^2(t)-2e^{2\nu(t,r)}\Phi(t,r)\right\}e^{2\nu_1(t,r)}\right.$$
$$\left.\left.+e^{2\nu(t,r)}\psi'^2(r)\,\alpha(t,r)\right]r^2\right\}e^{2\nu_1(t,r)}\Big],$$

$$\theta\theta=\phi\phi-component: \tag{22}$$

$$\frac{\left[(1+\nu'(t,r)\,r)(\nu'(t,r)-\nu_1'(t,r))+\nu''(t,r)\,r\right]e^{-2\nu_1(t,r)}+r\left[\dot\nu_1(t,r)\,\dot\nu(t,r)-\ddot\nu_1(t,r)-\dot\nu_1^2(t,r)\right]e^{-2\nu(t,r)}}{r}$$
$$=\frac{\kappa^2}{2r}\Big[16\left\{\ddot\nu_1(t,r)\,\zeta'(t,r)+2\dot\nu_1(t,r)\,\dot\zeta'(t,r)-\nu_1'(t,r)\,\ddot\zeta(t,r)-\dot\nu_1(t,r)\left[\dot\nu_1(t,r)+\dot\nu(t,r)\right]\right\}\zeta'(t,r)$$
$$-16\dot\zeta(t,r)\left\{\dot\nu_1(t,r)\,\nu'(t,r)-\nu_1'(t,r)\,\dot\nu(t,r)\right\}e^{-2[\nu_1(t,r)+\nu(t,r)]}-e^{-4\nu_1(t,r)}$$
$$\left\{16\zeta'(t,r)\,\nu''(t,r)+16\nu'(t,r)\times[\zeta''+\zeta'\{\nu'(t,r)-3\nu_1'(t,r)\}]\right\}$$
$$+r\left\{e^{-2\nu(t,r)}\beta(t,r)\,\dot\varphi^2(r)-e^{-2\nu_1(t,r)}\alpha(t,r)\,\psi'^2(r)-2\Phi(t,r)\right\}\Big] \tag{23}$$

The above system of differential equations can be solved for the coefficients of the scalar fields in addition to the potential to have the form:

$$\beta(t,r)=\frac{e^{-4\nu_1(t,r)}}{\kappa^2r^2}\Big[\Big\{8\kappa^2\zeta''(t,r)+\nu''(t,r)\,r^2$$
$$-8\kappa^2\zeta'(t,r)\,\nu_1'(t,r)$$
$$+\left(r-r^2\nu'(t,r)\right)\nu_1'(t,r)$$
$$+r\nu'(t,r)-1+\nu'^2(t,r)\,r^2\Big\}$$
$$\times e^{2\nu(t,r)+2\nu_1(t,r)}+e^{2\nu(t,r)+4\nu_1(t,r)}$$
$$+\Big[8\kappa^2\ddot\zeta(t,r)\,\nu_1'(t,r)\,r-8\kappa^2\zeta'(t,r)\,\ddot\nu_1(t,r)\,r$$
$$-16\kappa^2\dot\nu_1(t,r)\,\dot\zeta'(t,r)\,r$$
$$+8\kappa^2\dot\nu_1^2(t,r)\,\zeta'(t,r)\,r$$
$$+\Big(8\kappa^2\zeta'(t,r)\,\dot\nu(t,r)\,r+8\kappa^2\dot\zeta(t,r)$$

$$
\begin{aligned}
&+8\kappa^2\dot{\zeta}(t,r)\,\nu'(t,r)\,r\Big)\,\dot{\nu}_1(t,r)\\
&-8\kappa^2\dot{\zeta}(t,r)\,\dot{\nu}(t,r)\,\nu_1'(t,r)\,r\Big]e^{2\nu_1(t,r)}\\
&+\Big[\Big(8\kappa^2\nu'(t,r)\,r\\
&-8\kappa^2\Big)\zeta''(t,r)+8\kappa^2\zeta'(t,r)\,\nu''(t,r)\,r\\
&+\Big(\Big(-24\kappa^2\nu'(t,r)\,r+24\kappa^2\Big)\nu_1'(t,r)\\
&+8\kappa^2\nu'(t,r)^2 r\Big)\zeta'(t,r)\Big]e^{2\nu(t,r)}\\
&+\Big[\Big(\dot{\nu}(t,r)\,r^2-8\kappa^2\dot{\zeta}(t,r)\Big)\dot{\nu}_1(t,r)\\
&-\ddot{\nu}_1^2(t,r)-\dot{\nu}_1^2(t,r)\,r^2\Big]e^{4\nu_1(t,r)}\Big],\\
\gamma(t,r)=&\frac{2e^{-2\nu_1(t,r)}}{\kappa^2r^2}\Big[\Big\{4\kappa^2\dot{\zeta}'(t,r)\\
&+\Big(r-4\kappa^2\zeta'(t,r)\Big)\dot{\nu}_1(t,r)\\
&-4\kappa^2\dot{\zeta}(t,r)\,\nu'(t,r)\Big\}e^{2\nu_1(t,r)}\\
&+4\kappa^2\left(3\zeta'(t,r)\,\dot{\nu}_1(t,r)\right.\\
&\left.-\dot{\zeta}'(t,r)+\dot{\zeta}(t,r)\,\nu'(t,r)\right)\Big],\\
\alpha(t,r)=&-\frac{e^{-2\nu(t,r)-2\nu_1(t,r)}}{\kappa^2r^2}\Big[\Big\{\nu''(t,r)\,r^2+\nu'^2(t,r)\,r^2\\
&+\Big(8\kappa^2\zeta'(t,r)-r-\nu_1'^2(t,r)\Big)\nu'(t,r)\\
&-\nu_1'(t,r)\,r-1\Big\}e^{2\nu(t,r)+2\nu_1(t,r)}\\
&+e^{2\nu(t,r)+4\nu_1(t,r)}+\Big[\Big(8\kappa^2\nu_1'(t,r)\,r+8\kappa^2\Big)\ddot{\zeta}(t,r)\\
&-8\kappa^2\zeta'(t,r)\,\ddot{\nu}_1(t,r)\,r-16\kappa^2\dot{\nu}_1(t,r)\,\dot{\zeta}'(t,r)\,r\\
&+8\kappa^2\dot{\zeta}(t,r)\,\dot{\nu}_1(t,r)\,\nu'(t,r)\,r\\
&+8\Big(\kappa^2\dot{\nu}(t,r)\,\dot{\nu}_1(t,r)\,r+\kappa^2\dot{\nu}_1^2(t,r)\,r\Big)\zeta'(t,r)\\
&-8\kappa^2\dot{\zeta}(t,r)\,\dot{\nu}(t,r)\\
&-8\kappa^2\dot{\zeta}(t,r)\,\dot{\nu}(t,r)\,\nu_1'(t,r)\,r\Big]\\
&\times e^{2\nu_1(t,r)}+\Big[\dot{\nu}(t,r)\,\dot{\nu}_1(t,r)\,r^2\\
&-\ddot{\nu}_1(t,r)\,r^2-\dot{\nu}_1^2(t,r)\,r^2+8\kappa^2\dot{\zeta}(t,r)\,\dot{\nu}(t,r)\\
&-8\kappa^2\ddot{\zeta}(t,r)\Big]e^{4\nu_1(t,r)}\\
&+\Big[8\kappa^2\zeta'(t,r)\,\nu''(t,r)\,r+8\kappa^2\zeta''(t,r)\,\nu'(t,r)\,r\\
&+8\kappa^2\zeta'(t,r)\,\nu'^2(t,r)\,r\\
&-24\Big(\kappa^2\nu_1'(t,r)\,r+\kappa^2\Big)\zeta'(t,r)\,\nu'(t,r)\Big]e^{2\nu(t,r)}\Big],\\
\Phi(t,r)=&-\frac{e^{-2\nu(t,r)-4\nu_1(t,r)}}{\kappa^2r^2}\Big[\Big\{-4\kappa^2\zeta''(t,r)\\
&-4\kappa^2\left[\nu'(t,r)\right.\\
&\left.-\nu_1'(t,r)\right]\zeta'(t,r)-\nu_1'(t,r)\,r+1\\
&+r\nu'(t,r)\Big\}e^{2\nu(t,r)+2\nu_1(t,r)}\\
&\times -e^{2\nu(t,r)+4\nu_1(t,r)}+\Big\{-4\kappa^2\ddot{\zeta}(t,r)\\
&+\Big(-4\kappa^2\dot{\nu}_1(t,r)+4\dot{\nu}(t,r)\,\kappa^2\Big)\dot{\zeta}(t,r)\Big\}e^{2\nu_1(t,r)}\\
&+\Big\{4\kappa^2\ddot{\zeta}(t,r)+\Big(-4\dot{\nu}(t,r)\,\kappa^2\\
&+4\kappa^2\dot{\nu}_1(t,r)\Big)\dot{\zeta}(t,r)\Big\}e^{4\nu_1(t,r)}+\Big\{4\kappa^2\zeta''(t,r)\\
&+\Big\{12\kappa^2\nu'(t,r)-12\kappa^2\nu_1'(t,r)\Big\}\\
&\zeta'(t,r)\Big\}e^{2\nu(t,r)}\Big].
\end{aligned}
\tag{24}
$$

4 Time dependent black hole solutions

In this section, we are going to derive the coefficients of the scalar fields, φ and ψ, that are time dependent black hole.

4.1 First time dependent black hole

In this case we take the ansatzs of the BH to have the from:[3].

$$
e^{2\nu(t,r)}=e^{-2\nu_1(t,r)}=1-\frac{2M(t)}{r}\,, \tag{25}
$$

where the Misner–Sharp–Hernandez mass $M(t)$ is positive and depends only on time. A negative mass, M, indicates a violation of the energy conditions and makes apparent horizons impossible. The Ricci scalar related to ansatz (25) is given by:

$$
R=\left(1-\frac{2M}{r}\right)^{-1}\left[\frac{2\ddot{M}}{r}+\frac{\left(\frac{2\dot{M}}{r}\right)^2}{1-\frac{2M}{r}}\right]. \tag{26}
$$

Equation (26) demonstrates that when M remains constant, R becomes zero, indicating the geometry is the same as Schwarzschild geometry. In this scenario, there is a single visible horizon, with a surface radius of $r(t)=2M(t)$. This horizon, which is constantly changing, represents a black hole horizon as it is a unique solution to the equation $\nabla^\alpha r\nabla_\alpha r=0$. A curvature singularity occurs when $2M=r$ given that $\frac{2\ddot{M}}{r}\left(1-\frac{2M}{r}\right)$. The value of $\left(\frac{2\dot{M}}{r}\right)^2$ decreases significantly as $\mathcal{O}\left(\left(1-\frac{2M}{r}\right)^2\right)$ approaches zero when $2M=r$, which is not possible. Using Eq. (25) in Eq. (24) we present

[3] In this study, we assume $\nu(t,r)$ is equal to $-\nu_1(t,r)$, as seen in the Schwarzschild. The line element in spherical spacetime is expressed using the areal radius r with $g_{tt}g_{rr}=-1$, where r serves as an affine parameter along radial null geodesics [82]. Additionally, this spacetime possesses unique algebraic characteristics: the Ricci tensor's double projection onto radial null vectors results in zero [82–84]. On the other hand, the Ricci tensor's restriction to the (t,r) submanifold is directly related to the restriction of the metric $g_{\mu\nu}$ to this specific subspace [82]. Several solutions in general relativity or alternative gravities are characterized by the condition $g_{tt}g_{rr}=-1$, such as vacuum solutions, electrovacuum solutions with Maxwell or nonlinear Born–Infeld electrodynamics, and the string hedgehog global monopole [82], also in higher dimensions, as shown in [85,86].

the coefficient of the scalar fields and the potential in the Appendix due to its lengthy:

4.2 Second time dependent black hole

The second time dependent ansatz of the BH that we will deal takes the form:

$$e^{2\nu(t,r)} = e^{-2\nu_1(t,r)} = \left(1 - \frac{r_0}{r}\right)\frac{t_0}{t}\,, \tag{27}$$

with r_0 and t_0 being positive constants. The Misner–Sharp–Hernandez mass $M(t)$ of the above line element takes the form:

$$M(t) = \frac{rt - t_0 r + t_0 r_0}{2t}\,. \tag{28}$$

Using Eq. (27) in Eq. (24) we get the coefficient of the scalar fields and the potential in the Appendix. Equation (28) demonstrates that only a single apparent horizon exists at $r = 2M\,(t, r)$, resulting in $r = r_0$. Since this is a single root, we have a black hole apparent horizon. Equation (28) reveals that there is just a single visible horizon situated at $r = 2M\,(t, r)$, where $r = r_0$. This horizon is both a null surface and an event horizon. The Ricci curvature of Eq. (28) has the form:

$$R = \frac{2}{r^2}\left(1 - \frac{t_0}{t}\right). \tag{29}$$

The geometry of Eq. (29) has no spacetime singularities except for the usual one at $r = 0$ and the Big Bang at $t = 0$.

4.3 Third time dependent black hole

The second time dependent ansatz of the BH that we will deal takes the form:

$$e^{2\nu(t,r)} = e^{-2\nu_1(t,r)} = \frac{1 - \frac{r_0}{r}}{1 + \frac{t r_0}{r t_0}}\,, \tag{30}$$

where the Misner–Sharp–Hernandez mass $M(t)$ of the above line element take the form:

$$M(t) = \frac{r r_0 (t_0 + t)}{2(r t_0 + t r_0)}\,. \tag{31}$$

Equation (31) displays that the apparent horizons can be found at $r = 2M$, resulting in the sole root being $r = r_0$. This represents a lone root and the circumference of a stationary black hole event horizon. The mass located at this boundary is equal to half the radius $M(r_0) = r_0/2$ and remains constant over time. As r approaches infinity, the relationship $\mathrm{e}^{2\nu} = \mathrm{e}^{-2\nu_1} \to 1$ indicates that the geometry is flat at infinity. As t approaches infinity while r remains constant, the metric can be approximated by $\mathrm{e}^{2\nu} \simeq \frac{t_0 r}{t r_0}\left(1 - \frac{r_0}{r}\right)$. Next, by defining a new time parameter τ as $d\tau = \frac{dt}{\sqrt{t}}$ (or $\tau(t) = 2\sqrt{t}$), the metric equation can be rewritten as:

$$ds^2 = -\frac{t_0\, r}{r_0}\left(1 - \frac{r_0}{r}\right) d\tau^2 + \frac{r_0 \tau^2}{4 t_0\, r\,(1 - r_0/r)}\, dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{32}$$

as the time τ (or t) increases, the radial direction expands resembling a throat. The horizontal radius of the horizon stays the same. The factor depending on time $\left(\frac{r_0}{4t_0\, r\left(1-\frac{r_0}{r}\right)}\,\tau^2\right)$ affects only dr^2 and not the angular component of the metric, unlike in a FLRW universe with a central object (the metric approaches flat instead of FLRW at infinity) [87]. The Ricci scalar of the geometry (30) is

$$R = \frac{2\left(1 + \frac{t_0}{t}\right)\frac{r_0}{r^3}}{\left(1 + \frac{t r_0}{t_0 r}\right)^3}\left[1 - \frac{t}{t_0} + 2\left(1 - \frac{t}{t_0}\right)\frac{t r_0}{t_0\, r} + \frac{t^2 r_0^2}{t_0^2\, r^2}\right], \tag{33}$$

which is continuous at the horizon $r = r_0$, despite the presence of singularities at $r = 0$, $t = 0$, and $t \to \infty$. Using Eq. (27) in Eq. (24) we we present the coefficient of the scalar fields as well as the potential in the Appendix.

5 Gravitational wave propagation

Now, let us investigate the GW propagation. The EGB theory requires that the propagation speed of gravitational waves with a single scalar fields corresponds to the speed of light in the FLRW metric spacetime has been studied in [52]. The condition of propagation of gravitational waves for spacetime with two scalar field was provided in [68]. Now let us use the condition given by Eq. (17). In our examination of the GW propagation velocity, denoted as c_{GW}, our focus is solely on the terms that encompass the second-order derivatives of the GW tensor $h_{\mu\nu}$. Given the assumption of minimal interaction between matter and gravity, contributions from matter do not interact with the derivatives of the gravitational wave tensor $h_{\mu\nu}$. As a result, they are not present in the tensor $I_{\mu\nu}$. Put simply, the existence of matter has no impact on the GW's propagation velocity. Importantly, the tensor $I^{(1)}_{\mu\nu}$ does not change the GW s speed, which continues to match that of light. In general, the tensor $I^{(2)}_{\mu\nu}$ has the potential to change the GW's speed, deviating it from the speed of light. In this analysis, we have omitted the disturbances of the scalar fields φ and ψ, as these fluctuations can be rendered nonexistent in accordance with the constraints outlined in Eq. (8). When there are no constraints present, or if only a single scalar field interacts with the Gauss–Bonnet invariant, the scalar mode perturbations become intertwined with the scalar modes within the metric's fluctuation. Provided that we focus on the massless spin-two modes, representative of the conventional GW, these modes become independent from the scalar mode at the foremost order. When taking into account the second-order perturbation, it's possible for the quadratic moment to emerge as a result of the scalar mode's perturbation. This quadratic moment acts as the origin of the GW. Hence, the propagation characteristics of the GW may vary between scenarios with the constraints, as specified in Eq. (8), and those without them.

In the context of the metric outlined in Eq. (5), the sole connection coefficients that are not zero are the following:

$$\begin{aligned}
&\left\{{}^{t}_{tt}\right\} = \dot{\mu}\,, \qquad \left\{{}^{r}_{tt}\right\} = \mathrm{e}^{-2(\nu_1-\nu)}\nu'\,, \qquad \left\{{}^{t}_{tr}\right\} = \left\{{}^{t}_{rt}\right\} = \nu'\,,\\
&\quad \left\{{}^{t}_{rr}\right\} = \mathrm{e}^{2\nu_1-2\mu}\dot{\nu}_1\,, \qquad \left\{{}^{r}_{tr}\right\} = \left\{{}^{r}_{rt}\right\} = \dot{\nu}_1\,,\\
&\left\{{}^{r}_{rr}\right\} = \nu_1'\,, \quad \left\{{}^{i}_{jk}\right\} = \overline{\left\{{}^{t}_{tt}\right\}}\,, \qquad \left\{{}^{r}_{ij}\right\} - \mathrm{e}^{-2\nu_1} r\bar{g}_{ij}\,,\\
&\quad \left\{{}^{i}_{rj}\right\} = \overline{\left\{{}^{i}_{jr}\right\}}\frac{1}{r}\delta^{i}{}_{j}\,,
\end{aligned} \tag{34}$$

with $\overline{\left\{{}^{i}_{jk}\right\}}$ represents the metric connection of $\bar{g}_{ij}$, an overdot signifies differentiation with respect to time t, and a prime indicates differentiation with respect to radial distance r. Using Eq. (34) we get the following formula for the Riemann tensor:

$$\begin{aligned}
R_{rtrt} =& -\mathrm{e}^{2\nu_1}\left\{\ddot{\nu}_1 + (\dot{\nu}_1 - \dot{\nu})\,\dot{\nu}_1\right\} + \mathrm{e}^{2\nu}\left\{\nu'' + \left(\nu' - \nu_1'\right)\nu'\right\}\,,\\
R_{titj} =& \, r\nu'\mathrm{e}^{-2\nu_1+2\nu}\bar{g}_{ij}\,,\\
R_{rirj} =& \, \nu_1' r\bar{g}_{ij}\,, \quad R_{tirj} = \dot{\nu}_1 r\bar{g}_{ij}\,,\\
R_{ijkl} =& \left(1 - \mathrm{e}^{-2\nu_1}\right) r^2\left(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}\right)\,,\\
R_{tt} =& -\left\{\ddot{\nu}_1 + (\dot{\nu}_1 - \dot{\nu})\,\dot{\nu}_1\right\}\\
&+ \mathrm{e}^{-2\nu_1+2\nu}\left\{\nu'' + \left(\nu' - \nu_1'\right)\nu' + \frac{2\nu'}{r}\right\}\,,\\
R_{rr} =& \, \mathrm{e}^{2\nu_1-2\nu}\left\{\ddot{\nu}_1 + (\dot{\nu}_1 - \dot{\nu})\,\dot{\nu}_1\right\}\\
&- \left\{\nu'' + \left(\nu' - \nu_1'\right)\nu'\right\} + \frac{2\nu_1'}{r}\,,\\
R_{tr} =& \, R_{rt} = \frac{2\dot{\nu}_1}{r}\,,\\
R_{ij} =& \left\{1 + \left\{-1 - r\left(\nu' - \nu_1'\right)\right\}\mathrm{e}^{-2\nu_1}\right\}\bar{g}_{ij}\,,\\
R =& \, 2\mathrm{e}^{-2\nu}\left\{\ddot{\nu}_1 + (\dot{\nu}_1 - \dot{\nu})\,\dot{\nu}_1\right\}\\
&+ \mathrm{e}^{-2\nu_1}\left\{-2\nu'' - 2\left(\nu' - \nu_1'\right)\nu' - \frac{4\left(\nu' - \nu_1'\right)}{r}\right.\\
&\left. + \frac{2\mathrm{e}^{2\nu_1} - 2}{r^2}\right\}\,,
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\nabla_t\nabla_t\zeta =& \, \ddot{\zeta} - \dot{\nu}\dot{\zeta} - \mathrm{e}^{-2(\nu_1-\nu)}\nu'\zeta'\,,\\
\nabla_r\nabla_r\zeta =& \, \zeta'' - \mathrm{e}^{2\nu_1-2\nu}\dot{\nu}_1\dot{\zeta} - \nu_1'\zeta'\,,\\
\nabla_t\nabla_r\zeta =& \, \nabla_r\nabla_t\zeta = \dot{\zeta}' - \nu'\dot{\zeta} - \dot{\nu}_1\zeta'\,,\\
\nabla_i\nabla_j\zeta =& -\mathrm{e}^{-2\nu_1} r\bar{g}_{ij}\zeta'\,,\\
\nabla^2\zeta =& -\mathrm{e}^{-2\nu}\left(\ddot{\zeta} - (\dot{\nu} - \dot{\nu}_1)\,\dot{\zeta}\right)\\
&+ \mathrm{e}^{-2\nu_1}\left(\zeta'' + \left(\nu' - \nu_1' - \frac{2}{r}\right)\zeta'\right)\,,
\end{aligned} \tag{36}$$

The metric tensor $(\bar{g}_{ij})$ can be expressed as

$$\sum_{i,j=1}^{2}\bar{g}_{ij}dx^i dx^j = d\vartheta^2 + \sin^2\vartheta, d\varphi^2,$$

where $\left(x^1 = \vartheta, x^2 = \varphi\right)$, and $\overline{\left\{{}^{i}_{jk}\right\}}$ are the coordinates, denoted by $x^1 = \vartheta$ and $x^2 = \varphi$, respectively. Additionally, $\overline{\left\{{}^{i}_{jk}\right\}}$ denotes the connection coefficients associated with the metric tensor $\bar{g}_{ij}$. The symbols "dot" and "prime" signify the derivatives with respect to t and r, respectively. We will proceed to analyze the GW that is propagating outwards along the radial axis that has the form:

$$\begin{aligned}
h_{ij} =& \frac{\mathrm{Re}\left(\mathrm{e}^{-i\varpi t+ikr}\right)h^{(0)}_{ij}}{r}\\
&\times\left(i, j = \theta, \phi, \quad \sum_i h^{(0)\,i}_{\ i} = 0\right)\,,\\
&\text{other components} = 0\,.
\end{aligned} \tag{37}$$

We proceed with the assumption that k is sufficiently large, which leads us to retain terms that are quadratic in relation to k and/or ϖ, while disregarding any terms that are linear with respect to k or ϖ. Then Eq. (17) with (36) gives

$$\begin{aligned}
0 =& \left[-\frac{1}{4\kappa^2} - 2\left\{-\mathrm{e}^{-2\nu}\left(2\ddot{\zeta} - (2\dot{\nu} - \dot{\nu}_1)\,\dot{\zeta}\right)\right.\right.\\
&\left.\left. + \mathrm{e}^{-2\nu_1}\left(\zeta'' + \left(2\nu' - \nu_1'\right)\zeta'\right)\right\}\right]\mathrm{e}^{-2\nu}\varpi^2\\
&- \left[-\frac{1}{4\kappa^2} - 2\left\{-\mathrm{e}^{-2\nu}\left(\ddot{\zeta} - (\dot{\nu} - 2\dot{\nu}_1)\,\dot{\zeta}\right)\right.\right.\\
&\left.\left. + \mathrm{e}^{-2\nu_1}\left(2\zeta'' + \left(\nu' - 2\nu_1'\right)\zeta'\right)\right\}\right]\mathrm{e}^{-2\nu_1}k^2\\
&- 4\left(\dot{\zeta}' - \nu'\dot{\zeta} - \dot{\nu}_1\zeta'\right)\mathrm{e}^{-2\nu-2\nu_1}k\varpi\,.
\end{aligned} \tag{38}$$

Upon assuming that ζ is sufficiently small, it is observed that,

$$\begin{aligned}
\frac{\omega}{k} =& \, \mathrm{e}^{-\nu_1+\nu}\left[\pm\left\{1 - 4\kappa^2\left(-\ddot{\zeta} + \dot{\zeta}\,(\dot{\nu}_1 + \dot{\nu})\right)\mathrm{e}^{-2\nu}\right.\right.\\
&\left.\left. - 4\left\{-\zeta'' + \zeta'\left(\nu_1' + \nu'\right)\right\}\kappa^2\mathrm{e}^{-2\nu_1}\right\}\right.
\end{aligned}$$

$$-8\mathrm{e}^{-\nu_1-\nu}\kappa^2\left(\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'\right)\Big] . \tag{39}$$

The speed of light is determined by the equation $c = e^{-\nu_1+\nu}$. Consequently, if

$$\Big\{\left(\ddot{\zeta}-\dot{\zeta}\,(\dot{\nu}_1+\dot{\nu})\right)\mathrm{e}^{-2\nu}+\Big\{\zeta''-\zeta'\,\big(\nu_1' +\nu'\big)\Big\}\,\mathrm{e}^{-2\,\nu_1}\Big\}\pm 2\mathrm{e}^{-\nu_1-\nu}\left(\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'\right).$$

Should the value be positive (negative), it indicates that the GW travels faster/slower than the speed of light. As denoted in Eq. (39), a positive sign is associated with the GW moving away from the black hole, whereas a sign negative indicates the GW moving towards the black hole. Thus, the velocity of the GW as it moves into the black hole is characterized by

$$\begin{aligned}v_{\mathrm{in}} \equiv \mathrm{e}^{-\nu_1+\nu}\Big[1-4\kappa^2\left\{-\ddot{\zeta}+\dot{\zeta}\,(\dot{\nu}_1+\dot{\nu})\right\}\mathrm{e}^{-2\nu}\\ -4\kappa^2\left\{-\zeta''+\zeta'\left(\nu_1'+\nu'\right)\right\}\mathrm{e}^{-2\nu_1}\\ +8\mathrm{e}^{-\nu_1-\nu}\kappa^2\left(\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'\right)\Big] ,\end{aligned} \tag{40}$$

where the velocity of GWs is identical to the speed of light, as they propagate outward from their source

$$\begin{aligned}v_{\mathrm{out}} \equiv \mathrm{e}^{-\nu_1+\nu}\Big[1-4\kappa^2\left\{-\ddot{\zeta}+\dot{\zeta}\,(\dot{\nu}_1+\dot{\nu})\right\}\mathrm{e}^{-2\nu}\\ -4\kappa^2\left\{-\zeta''+\zeta'\left(\nu_1'+\nu'\right)\right\}\mathrm{e}^{-2\nu_1}\\ -8\mathrm{e}^{-\nu_1-\nu}\kappa^2\left(\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'\right)\Big] .\end{aligned} \tag{41}$$

When ζ depends on the radial coordinate r or the scalar field ψ and ν_1 depends on the time t or the scalar field φ, we encounter the following conditions: If, $\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'>0$ then $v_{\mathrm{in}}>v_{\mathrm{out}}$. Conversely, if $\dot{\zeta}\,\nu'+\zeta'\dot{\nu}_1-\dot{\zeta}'<0$, we have $v_{\mathrm{in}}<v_{\mathrm{out}}$. For instance, we can examine the expression for $\zeta(t,r)$, provided in the Appendix given by Eq. (A2), and restate it using the variables (φ,ψ) yields:

$$\zeta(\varphi,\psi)=\frac{\zeta_0\chi^2}{\chi^2+{r_0}^2\mathrm{e}^{-\frac{2\varphi}{t_0}}} . \tag{42}$$

In the scenario where ζ_0 and t_0 are positive constants, and under the condition specified by Eq. (42), we observe that ζ approaches zero as either t tends toward negative infinity or r approaches zero. This observation suggests that the speed at which gravitational waves propagate matches that of light either in the distant past or at the center of a black hole. An important observation is that when $|t|>0$, we have $\frac{\dot{\zeta}}{\zeta_0}>0$, and if $t<\frac{t_0}{2}$, then $\frac{\ddot{\zeta}}{\zeta_0}<0$. Conversely, when $t>\frac{t_0}{2}$, we find $\frac{\ddot{\zeta}}{\zeta_0}<0$. We observe the following properties for the function $\zeta(t)$: i-$\frac{\dot{\zeta}'}{\zeta_0}>0$, is always positive ii-if $t>-\frac{t_0}{2}$, then $\frac{\dot{\zeta}'}{\zeta_0}<0$, iii-if $t>\frac{t_0}{2}$, then $\frac{\dot{\zeta}'}{\zeta_0}>0$, iv-if $t<\frac{t_0}{2}$, then $\frac{\dot{\zeta}'}{\zeta_0}>0$. Conversely, we discover for the first black hole we get:

$$\begin{aligned}\dot{\nu}_1&=\frac{\dot{M}(t)}{r(1-\frac{2M}{r})}>0\,,\\ \nu_1'&=-\frac{M(t)}{r^2(1-\frac{2M}{r})}<0\,,\end{aligned} \tag{43}$$

and for the second black hole we get:

$$\begin{aligned}\dot{\nu}_1&=\frac{1}{2t}>0\,,\\ \nu_1'&=-\frac{r_0}{2r^2(1-\frac{r_0}{r})}<0\,,\end{aligned} \tag{44}$$

and finally for the third black hole we get:

$$\begin{aligned}\dot{\nu}_1&=\frac{-r_0}{2rt_0(1+\frac{tr_0}{rt_0})}>0\,,\\ \nu_1'&=\frac{r_0(t+t_0)}{2(r-r_0)(rt_0+tr_0)}>0\,,\end{aligned} \tag{45}$$

provided that $r_0<0$.

Discussing the general choice of parameters and regions is a complex task and not very beneficial, so we can focus on some specific simplified cases instead.

- Initially, we examine the region near the horizon where $r_0\approx 2M(t)$. Subsequently, in Eqs. (40) and (41), the terms involving ν_1 and ν may dominate. However, due to the conditions $\mathrm{e}^{2\nu}=\mathrm{e}^{-2\nu_1}=0$. Hence, the prevailing terms may be expressed as:

$$v_{\mathrm{in}}\sim v_{\mathrm{out}}\sim c\left[1-4\,\kappa^2\dot{\zeta}\,(\dot{\nu}_1+\dot{\nu})\right] . \tag{46}$$

  Importantly, the distinction in velocities between inward-propagating and outward-propagating gravitational waves diminishes. If $\zeta_0<0$, we find that $\dot{\zeta}\,(\dot{\nu}_1+\dot{\nu})<0$, leading to the GW's propagation speed exceeding that of light. On the other hand, when $\zeta_0>0$, the propagation speed diminishes, eventually falling below the speed of light.
- Additionally, we examine the region where r_0 significantly exceeds $2M(t)$. Subsequently, we make the following observations:

$$v_{\mathrm{in}}\sim v_{\mathrm{out}}\sim c\left[1+4\kappa^2\zeta''\right] . \tag{47}$$

  Once again, the disparity in speeds between inward-propagating and outward-propagating GWs diminishes. When $\zeta_0>0$, we observe $\zeta''<0$, in the region, resulting in a decrease in the GW's propagation speed, which falls below the speed of light once again. Conversely, if $\zeta_0<0$, the propagation speed exceeds the speed of light.

In both of the aforementioned regions, the disparity between the speeds of inward-propagating and outward-propagating GWs can be disregarded. Consequently, a distinction may arise in the intermediate region. Let's highlight that in both of the aforementioned regions, when $\zeta_0 < 0$, the gravitational wave's propagation speed exceeds that of light. Conversely, if $\zeta_0 > 0$, the propagation speed falls below the speed of light.

The discussion above indicates that the propagation speed of gravitational waves in Einstein–Gauss–Bonnet gravity, coupled with two scalar fields, generally differs from the speed of light. This discrepancy suggests that black holes predicted by this theory may be incompatible with the observations from GW170817. In other words, unless a novel scenario is proposed to reconcile these differences, black holes in this theory may not represent realistic astrophysical objects.

Now, let's analyze the implications of a gravitational wave propagation speed that differs from the speed of light in the context of black hole construction. For a fixed frequency, if the propagation speed is higher (lower) than the speed of light, the wavelength becomes longer (shorter). While a longer wavelength could be associated with lower frequencies, which are harder to detect, it may also imply changes in the distribution and detectability of primordial gravitational waves depending on the energy spectrum. Thus, a higher (lower) propagation speed might influence the characteristics of the primordial gravitational wave background, though not necessarily its abundance.

Another consideration is the interaction between different modes of gravitational waves and the black hole horizon. In modified gravity theories, the propagation speed of gravitational waves, if different from that of light or scalar fields, can influence the dynamics of gravity-related fluctuations. This could affect the propagation of tensor modes and potentially leave imprints on cosmological observations, such as B-mode polarization in the Cosmic Microwave Background.

6 Conclusion

Black holes, once considered bizarre solutions to Einstein's equations and nowadays play a central role in astrophysics, cosmology, and foundational physics. Recent radio images of supermassive black holes provide valuable insights into matter dynamics near event horizons. While deviations from GR are possible, identifying the correct alternative theory remains challenging. In this study, we explore the dynamical black holes on frame of Einstein Gauss–Bonnet theory with two scalar fields.

In this study, we provided three ansatzes to create time dependent black holes with visible horizons that increase in time based on the Misner-Sharp-Hernandez mass. This horizon will appear identical to all observers associated with a spherically symmetric foliation [88], but it will differ for observers moving relative to the original frame in a way that breaks this symmetry, such as by a Lorentz boost in a non-radial direction [89,90].

Our models, when constructed, may incorporate ghosts. The presence of ghosts can be mitigated by imposing constraints, akin to the mimetic constraint represented by (8). These constraints have been applied to various spacetime scenarios, as demonstrated in the referenced paper by Nojiri et al. (for example, see [75]). Evidence indicates that the conventional cosmological solutions and self-gravitating entities-such as planets, the Sun, and various types of stars-within Einstein's gravitational framework also qualify as solutions in this particular model. We explored the propagation of high-frequency gravitational waves by selecting the GB coupling described in equation (A2). According to the research in [52,68], the propagation speed undergoes changes due to the coupling effect during the black hole formation process. In general, the speed at which GWs propagate into a black hole differs from the speed of the wave going out, although these effects occur at next-to-leading order. Through our investigation of speed expressions, we have identified conditions under which the propagation speed remains below the speed of light and adheres to causality.

It is essential to emphasize that the current formulation highlights the non-dynamical nature of the scalar fields. The scalar fields act like fluids with energy density and pressure, although they are not actual material fluids in the conventional sense: The scalar fields do not vary, so unlike regular fluids, there is no fluctuation in sound speed. In the standard fluid, breaking energy conditions can result in a sound speed exceeding the speed of light, leading to potential causality violations. However, our study does not experience this breakdown. The lack of variation shows that the solution remains stable despite not following the energy conditions. The restrictions utilized in this study mirror the limitation used in the framework of mimetic gravity [79], where there seems to be a type of effective dark matter that acts like dust matter but lacks pressure. The active dark matter is not tangible matter but non-physical and shows no variations in spatial distribution. In addition, functional dark matter does not undergo collapse because of gravity. It is intriguing that dynamic black holes like those analyzed in this study could offer a different option to inflation for the early Universe era. The specifics of this early Universe black holes period will be talked about in another place.

So far, there have not been many physically plausible solutions to the field equations of different theories that directly describe dynamic black holes; instead, naked singularities and wormholes are more frequently observed [87,91–118]. Another issue is that certain analytical solutions become complicated when described in terms of the surface area radius, and at that times it is difficult to find the apparent horizons analytically or their expressions are not straightforward [87]. In general, it is challenging to design black holes that depend on time, and we have deciphered the connection functions of gravity theories that have specified apparent horizons as their solutions.

It will be useful to do such procedure for regular dynamical black hole to see what is the effect of the GW. This will be done elsewhere in our future study.

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Appendix A: The explicate form of the scalar fields

1. The scalar fields of the dynamical first black hole

$$
\begin{aligned}
\beta(t,r) = \Bigg[& -3{r_0}^4 \Bigg\{ (r-2M(t))\,{t_0}^2 r^4 \Bigg\{ r^3 + \frac{16}{3}\kappa^2\xi_0 r \\
& -\frac{32}{3}\kappa^2\xi_0 M(t) \Bigg\} \ddot{M}(t) \\
& +4{t_0}^2 r^4 \left(r^3 + \frac{8}{3}\kappa^2\xi_0 r - \frac{16}{3}\kappa^2\xi_0 M(t) \right) \dot{M}^2(t) \\
& +\frac{64}{3}\xi_0 (r-2M(t))\, t_0 r^4 \kappa^2 \left(r - \frac{3}{2}M(t) \right) \dot{M}(t) \\
& +\frac{32}{3}\xi_0 (r-2M(t))^2 \kappa^2 \\
& \left(r^4 + \frac{1}{2}{t_0}^2 r^2 - 4M(t)\,{t_0}^2 r + 6M^2(t)\,{t_0}^2 \right) M(t) \Bigg\} \\
& \times e^{2\frac{t}{t_0}} - r^2 \Bigg\{ 3{r_0}^2 \Bigg[(r-2M(t))\,{t_0}^2 r^4 \\
& \left(r^3 + \frac{16}{3}\kappa^2\xi_0 r - \frac{32}{3}\kappa^2\xi_0 M(t) \right) \ddot{M}(t) \\
& +4{t_0}^2 r^4 \left(r^3 + \frac{8}{3}\kappa^2\xi_0 r - \frac{16}{3}\kappa^2\xi_0 M(t) \right) \dot{M}^2(t) \\
& -\frac{64}{3}\left(r - \frac{5}{2}M(t) \right) \xi_0 (r-2M(t))\, t_0 r^4 \kappa^2 \dot{M}(t) \\
& -\frac{32}{3}\xi_0 (r-2M(t))^2 \kappa^2 M(t) \\
& \left(r^4 - \frac{13}{2}{t_0}^2 r^2 + 28M(t)\,{t_0}^2 r \right. \\
& \left. -30M^2(t)\,{t_0}^2 \right) \Bigg] e^{4\frac{t}{t_0}} + \Big((r-2M(t))\,\ddot{M}(t) \\
& +4\dot{M}^2(t) \Big) {t_0}^2 r^3 \left({r_0}^6 + r^6 e^{6\frac{t}{t_0}} \right) \Bigg\} \Bigg] \\
& \left[r^5 \kappa^2 (r-2M(t))^2 {t_0}^2 \left(r^2 e^{2\frac{t}{t_0}} + {r_0}^2 \right)^3 \right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma(t,r) = \Bigg[& 6{r_0}^4 \Bigg\{ \left[r^3 + \frac{16}{3}\kappa^2\xi_0 r - 16\kappa^2\xi_0 M(t) \right] t_0 \dot{M}(t) \\
& +\frac{32}{3}M(t)\,\xi_0 \left(r - \frac{5}{2}M(t) \right) \kappa^2 \Bigg\} e^{2\frac{t}{t_0}} \\
& +2r \Bigg\{ 3r{r_0}^2 \Bigg[\Bigg(r^3 + \frac{16}{3}\kappa^2\xi_0 r \\
& -16\kappa^2\xi_0 M(t) \Bigg) t_0 \dot{M}(t) \\
& -\frac{32}{3}M(t) \left(r - \frac{3}{2}M(t) \right) \xi_0 \kappa^2 \Bigg] e^{4\frac{t}{t_0}} \\
& +t_0 \dot{M}(t) \left({r_0}^6 + r^6 e^{6\frac{t}{t_0}} \right) \Bigg\} \Bigg] \\
& \left[r^2 t_0 \kappa^2 (r-2M(t)) \left(r^2 e^{2\frac{t}{t_0}} + {r_0}^2 \right)^3 \right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\alpha(t,r) = \Bigg[& 3{r_0}^4 \Bigg\{ r^4 \left(r^3 + \frac{16}{3}\kappa^2\xi_0 r - \frac{32}{3}\kappa^2\xi_0 M(t) \right) {t_0}^2 \\
& (r-2M(t))\,\dot{M}(t) \\
& +4r^4 \left(r^3 - \frac{16}{3}\kappa^2\xi_0 M(t) + \frac{8}{3}\kappa^2\xi_0 r \right) {t_0}^2 \dot{M}^2(t) \\
& +\frac{64}{3}r^4 t_0 \left(r - \frac{3}{2}M(t) \right) \xi_0 (r-2M(t))\,\kappa^2 \dot{M}(t) \\
& +32 \Bigg(r^4 + 1/2{t_0}^2 r^2 - \frac{8}{3}M(t)\,{t_0}^2 r \\
& +\frac{10}{3}(M(t))^2 {t_0}^2 \Bigg) M(t)\,\xi_0 \kappa^2 \\
& \times (r-2M(t))^2 \Bigg\} e^{2\frac{t}{t_0}} + r^2 \Bigg\{ 3{r_0}^2 \Bigg[r^4 \Bigg(r^3 \\
& +\frac{16}{3}\kappa^2\xi_0 r - \frac{32}{3}\kappa^2 x_0 M(t) \Bigg) {t_0}^2 (r-2M(t)) \\
& \ddot{M}(t) + 4r^4{t_0}^2 \left(r^3 - \frac{16}{3}\kappa^2 x_0 M(t) + \frac{8}{3}\kappa^2\xi_0 r \right) \\
& \dot{M}^2(t) - \frac{64}{3}r^4 t_0 \xi_0 \left(r - \frac{5}{2}M(t) \right) \\
& (r-2M(t))\,\kappa^2 \dot{M}(t) - 32 \Bigg(r^4 - \frac{7}{6}{t_0}^2 r^2 \\
& +\frac{16}{3}M(t)\,{t_0}^2 r - 6M^2(t)\,{t_0}^2 \Bigg) M(t)\,\xi_0 \\
& \times (r-2M(t))^2 \kappa^2 \Bigg] e^{4\frac{t}{t_0}} + r^3 \left({r_0}^6 + r^6 e^{6\frac{t}{t_0}} \right) {t_0}^2 \\
& \Big\{ (r-2M(t))\,\ddot{M}(t) + 4\dot{M}^2(t) \Big\} \Bigg\} \Bigg] \\
& \left[r^3 {t_0}^2 (r-2M(t))^4 \kappa^2 \left(r^2 e^{2\frac{t}{t_0}} + {r_0}^2 \right)^3 \right]^{-1},
\end{aligned}
$$

$$
\Phi(t,r) = 2{r_0}^2\xi_0 M(t) \Big[- \Big\{ r^4 \dot{M}(t)\, t_0
$$

$$
\begin{aligned}
&+\left(r^4+\frac{1}{2}t_0{}^2r^2-3M(t)\,t_0{}^2r+4(M(t))^2t_0{}^2\right)\\
&(r-2M(t))\}\,r_0{}^2e^{2\frac{t}{t_0}}+r^2e^{4\frac{t}{t_0}}\\
&\times\left\{\left(r^4-\frac{5}{2}t_0{}^2r^2-r^4\dot{M}(t)\,t_0+11M(t)\,t_0{}^2\right.\right.\\
&\left.\left.r-12M^2(t)\,t_0{}^2\right)(r-2M(t))\right\}\Big]\\
&\times\left[r^4t_0{}^2(r-2M(t))^2\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},
\end{aligned}
\tag{A1}
$$

where we have assume the form the Gauss–Bonnet coupling has the form

$$
\xi(t,r)=\frac{\xi_0r^2}{r^2+r_0{}^2\mathrm{e}^{-\frac{2t}{t_0}}}\,. \tag{A2}
$$

2. The scalar fields of the dynamical second black hole

$$
\begin{aligned}
\beta(t,r)=&\left[3\left\{\left[-t_0{}^3t+\left(t^2+\frac{8}{3}\kappa^2\xi_0\right)t_0{}^2\right.\right.\right.\\
&\left.-8\kappa^2t\xi_0t_0-\frac{8}{3}\kappa^2t^2\xi_0\right]r^5-\left(tt_0{}^2\right.\\
&\left.+\frac{16}{3}\kappa^2t\xi_0-\frac{8}{3}t_0\xi_0\kappa^2\right)r_0\,(t-t_0)\,r^4\\
&+\frac{16}{3}t_0{}^3\kappa^2\xi_0\,(t-t_0)\,r^3\\
&-8\left(t-\frac{2}{3}t_0\right)\xi_0t_0{}^3\kappa^2r_0r^2\\
&\left.+\frac{8}{3}r_0{}^2t_0{}^3\kappa^2\xi_0\,(3t_0+t)\,r-8t_0{}^4r_0{}^3\kappa^2\xi_0\right\}r_0{}^4e^{2\frac{t}{t_0}}\\
&+\left\{3\left[\left\{\left(t^2+\frac{8}{3}\kappa^2\xi_0\right)t_0{}^2-t_0{}^3t\right.\right.\right.\\
&\left.+\frac{40}{3}\kappa^2t\xi_0t_0-\frac{8}{3}\kappa^2t^2\xi_0\right\}r^5-\left[\left(t^2+\frac{8}{3}\kappa^2\xi_0\right)t_0{}^2\right.\\
&\left.-t_0{}^3t+\frac{40}{3}\kappa^2t\xi_0t_0-\frac{16}{3}\kappa^2t^2\xi_0\right]r_0r^4\\
&-16t_0{}^3\kappa^2\xi_0\,(-t_0+t)\,r^3\\
&+\frac{104}{3}r_0t_0{}^3\kappa^2\xi_0\,(-2t_0+t)\,r^2\\
&\left.-\frac{56}{3}r_0{}^2t_0{}^3\kappa^2\xi_0\,(t-5t_0)\,r-40t_0{}^4r_0{}^3\kappa^2\xi_0\right]r_0{}^2e^{4\frac{t}{t_0}}\\
&\left.\left.+tt_0{}^2\,(r-r_0)\left(r_0{}^6+r^6e^{6\frac{t}{t_0}}\right)(-t_0+t)\right\}r^2\right]\\
&\times\left[r^5t_0\kappa^2t^3\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\gamma=&\left[3\left\{-\frac{40}{3}\xi_0t_0\left(t-\frac{3}{5}t_0\right)\kappa^2r_0{}^2-\left\{r^2tt_0\right.\right.\right.\\
&\left.+\frac{40}{3}\xi_0\left(t^2+\frac{6}{5}t_0{}^2-2tt_0\right)\kappa^2\right\}rr_0\\
&\left.+\left[r^2tt_0+\frac{32}{3}\xi_0\left(t^2+\frac{3}{4}t_0{}^2-\frac{5}{4}tt_0\right)\kappa^2\right]r^2\right\}\\
&\times r_0{}^4e^{2\frac{t}{t_0}}+\left\{3\left[8t_0\xi_0\kappa^2\,(t_0+t)\,r_0{}^2\right.\right.\\
&-\left(r^2tt_0-8\kappa^2\xi_0\left(-2tt_0+t^2-2t_0{}^2\right)\right)rr_0+r^2\\
&\left.\left(r^2tt_0-\frac{32}{3}\xi_0\left(t^2-\frac{3}{4}tt_0-\frac{3}{4}t_0{}^2\right)\kappa^2\right)\right]\\
&\left.\left.\times\ rr_0{}^2e^{4\frac{t}{t_0}}+tt_0\,(r-r_0)\left(r_0{}^6+r^6e^{6\frac{t}{t_0}}\right)\right\}r\right]\\
&\times\left[r^2t^2\,(r-r_0)\,\kappa^2t_0\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\alpha(t,r)=&\left[3r_0{}^4\left\{r^4\left(r^3+\frac{16}{3}\kappa^2\xi_0r\right.\right.\right.\\
&\left.-\frac{32}{3}\kappa^2\xi_0M(t)\right)t_0{}^2\,(r-2M(t))\,\ddot{M}(t)\\
&+4r^4\left(r^3-\frac{16}{3}\kappa^2\xi_0M(t)+\frac{8}{3}\kappa^2\xi_0r\right)t_0{}^2\dot{M}^2(t)\\
&+\frac{64}{3}r^4t_0\left[r-\frac{3}{2}M(t)\right]\xi_0\,(r-2M(t))\,\kappa^2\dot{M}(t)\\
&+32\left[r^4+\frac{t_0{}^2r^2}{2}-\frac{8}{3}M(t)\,t_0{}^2r\right.\\
&\left.\left.+\frac{10}{3}(M(t))^2t_0{}^2\right]M(t)\,\xi_0\,(r-2M(t))^2\kappa^2\right\}\\
&\times e^{2\frac{t}{t_0}}+r^2\left(3r_0{}^2\left(r^4\left[r^3\right.\right.\right.\\
&\left.+\frac{16}{3}\kappa^2\xi_0r-\frac{32}{3}\kappa^2\xi_0M(t)\right]t_0{}^2\,(r-2M(t))\,\ddot{M}(t)\\
&+4r^4\left(r^3-\frac{16}{3}\kappa^2\xi_0M(t)+\frac{8}{3}\kappa^2\xi_0r\right)t_0{}^2\dot{M}^2(t)\\
&-\frac{64}{3}r^4t_0\xi_0\left(r-\frac{5}{2}M(t)\right)(r-2M(t))\\
&\kappa^2\dot{M}(t)-32\left(r^4-\frac{7}{6}t_0{}^2r^2\right.\\
&\left.+\frac{16}{3}M(t)\,t_0{}^2r-6(M(t))^2t_0{}^2\right)M(t)\,\xi_0e^{4\frac{t}{t_0}}\\
&\left.\times(r-2M(t))^2\kappa^2\right)+r^3\left(r_0{}^6+r^6e^{6\frac{t}{t_0}}\right)t_0{}^2\\
&\left.\left.\left((r-2M(t))\,\ddot{M}(t)+4\dot{M}^2(t)\right)\right)\right]\\
&\left[r^3t_0{}^2\,(r-2M(t))^4\kappa^2\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\Phi=&\,32r_0{}^2\xi_0\left[-\left\{r^4\dot{M}(t)\,t_0+\left(r^4+\frac{1}{2}t_0{}^2r^2\right.\right.\right.\\
&\left.\left.-3M(t)\,t_0{}^2r+4M^2(t)\,t_0{}^2\right)(r-2M(t))\right\}\\
&\times\ r_0{}^2e^{2\frac{t}{t_0}}+r^2e^{4\frac{t}{t_0}}\left(-r^4\dot{M}(t)\,t_0\right.\\
&+\left(r^4-\frac{5}{2}t_0{}^2r^2+11M(t)\,t_0{}^2r-12(M(t))^2t_0{}^2\right)\\
&\left.\left.(r-2M(t))\right)\right]M(t)\\
&\times\left[r^4t_0{}^2\,(r-2M(t))^2\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},
\end{aligned}
\tag{A3}
$$

where we have used the coupling Gauss–Bonnet given by Eq. (A2).

3. The scalar fields of the dynamical third black hole

$$
\beta(t,r)=\left[3r_0{}^4\left\{\frac{1}{8}\left(r_0\left[8\left(\frac{16}{3}\kappa^2\xi_0-r^2\right)t^2r_0^3\right.\right.\right.\right.
$$

$$+\left(16t^2r^3-96r\kappa^2\xi_0t^2-r^5\right)r_0^2+\Big[r^4$$
$$+\left(\frac{32}{3}\kappa^2\xi_0-9t^2\right)r^2+\frac{160}{3}\kappa^2t^2\xi_0\Big]r^2r_0$$
$$-\frac{32}{3}r^5\kappa^2\xi_0\Big]t_0{}^{5/2}-8\left[\frac{5}{2}r_0\left\{\left(-\frac{56}{15}\kappa^2\xi_0\right.\right.\right.$$
$$\left.+\frac{4}{5}r^2\right)r_0{}^2+\left(-\frac{17}{10}r^3+\frac{128}{15}\kappa^2\xi_0r\right)r_0$$
$$\left.-\frac{24}{5}\kappa^2\xi_0r^2+r^4\right\}tt_0{}^{7/2}$$
$$+r\left\{\left[\left(-4\kappa^2\xi_0+5/8r^2\right)r_0{}^2\right.\right.$$
$$\left.+\left(\frac{28}{3}\kappa^2\xi_0r-\frac{3}{2}r^3\right)r_0+r^4-\frac{16}{3}\kappa^2\xi_0r^2\right]t_0{}^{9/2}$$
$$-\frac{1}{8}\left(r-r_0\right)r_0{}^2\left(r\left(r_0r-\frac{64}{3}\kappa^2\xi_0\right)t_0{}^{3/2}\right.$$
$$\left.\left.\left.\left.-\frac{32}{3}t\kappa^2\xi_0\sqrt{t_0}r_0\right)t\right\}\right]r\right)\sqrt{r-r_0}$$
$$\sqrt{t_0r+tr_0}+\left(r-r_0\right)\Big[-t\Big[\Big(\frac{4}{3}t_0\kappa^2\xi_0$$
$$-4t\kappa^2\xi_0+t_0t^2\Big)r^2+\frac{16}{3}t_0{}^2t\kappa^2\xi_0\Big]t_0r_0{}^4$$
$$+\left\{\left(\left(-\frac{8}{3}\kappa^2\xi_0+t_0{}^2\right)t^3-\left(\frac{20}{3}t_0\kappa^2\xi_0+3t_0{}^3\right)t^2\right.\right.$$
$$\left.+\frac{28}{3}t_0{}^2t\kappa^2\xi_0-\frac{4}{3}t_0{}^3\kappa^2\xi_0\right)r^2$$
$$\left.+\frac{40}{3}\left(t-\frac{3}{5}t_0\right)t\kappa^2\xi_0t_0{}^3\right\}rr_0{}^3$$
$$+3r^2t_0\left[\left\{\left(-\frac{16}{9}\kappa^2\xi_0+t_0{}^2\right)t^2\right.\right.$$
$$\left.-\left(\frac{40}{9}t_0\kappa^2\xi_0+t_0{}^3\right)t+\frac{16}{9}t_0{}^2\kappa^2\xi_0\right\}r^2$$
$$\left.-\frac{8}{3}tt_0{}^2\kappa^2\xi_0\left(-3t_0+t\right)\right]r_0{}^2$$
$$+3\left\{\left[\left(t_0{}^2-\frac{8}{9}\kappa^2\xi_0\right)t-\frac{1}{3}t_0{}^3-\frac{20}{9}t_0\kappa^2\xi_0\right]r^2\right.$$
$$\left.-\frac{16}{3}\kappa^2\xi_0\left(t-\frac{1}{3}t_0\right)t_0{}^2\right\}r^3t_0{}^2r_0$$
$$\left.\left.+\left(-\frac{16}{3}\kappa^2\xi_0+r^2\right)r^4t_0{}^5\right]\right\}e^{2\frac{t}{t_0}}$$
$$+3r_0{}^2r^2\left[\frac{1}{8}\sqrt{r-r_0}\right.$$
$$\times\left[\left\{\left(\frac{736}{3}r\kappa^2\xi_0t^2-r^5+16t^2r^3\right)r_0{}^2\right.\right.$$
$$-8t^2\left(16\kappa^2\xi_0+r^2\right)r_0{}^3$$
$$+r^2\left\{r^4+\left(\frac{32}{3}\kappa^2\xi_0-9t^2\right)r^2-\frac{352}{3}\kappa^2t^2\xi_0\right\}r_0$$
$$\left.-\frac{32}{3}r^5\kappa^2\xi_0\right\}r_0t_0{}^{5/2}$$
$$-8\left\{\frac{5}{2}\left[\left(\frac{40}{3}\kappa^2\xi_0+\frac{4}{5}r^2\right)r_0{}^2\right.\right.$$
$$-\left(\frac{17}{10}r^3+\frac{128}{5}\kappa^2\xi_0r\right)r_0$$
$$\left.+\frac{184}{15}\kappa^2\xi_0r^2+r^4\right]r_0tt_0{}^{7/2}$$
$$+\left[\left(\left(\frac{52}{3}\kappa^2\xi_0+\frac{5}{8}r^2\right)r_0{}^2\right.\right.$$
$$\left.-\left\{\frac{100}{3}\kappa^2\xi_0r+\frac{3}{2}r^3\right\}r_0+r^4+16\kappa^2\xi_0r^2\right)t_0{}^{9/2}$$
$$-\frac{1}{8}\left(r-r_0\right)r_0{}^2\left\{r\left(r_0r-\frac{64}{3}\kappa^2\xi_0\right)t_0{}^{3/2}\right.$$
$$\left.\left.\left.-\frac{32}{3}t\kappa^2\xi_0\sqrt{t_0}r_0\right\}t\right]r\right\}r\Big]\sqrt{t_0r+tr_0}$$
$$+\Big[-\Big\{\left(\frac{20}{3}t\kappa^2\xi_0+\frac{4}{3}t_0\kappa^2\xi_0+t_0t^2\right)r^2$$
$$\left.-16t_0{}^2t\kappa^2\xi_0\right\}tt_0r_0{}^4+r\left[\left\{\left(\frac{8}{3}\kappa^2\xi_0+t_0{}^2\right)t^3\right.\right.$$
$$+\left(\frac{28}{3}t_0\kappa^2\xi_0-3t_0{}^3\right)t^2$$
$$\left.-12t_0{}^2t\kappa^2\xi_0-\frac{4}{3}t_0{}^3\kappa^2\xi_0\right\}r^2-24t\kappa^2\xi_0t_0{}^3$$
$$\left.\left(t-\frac{5}{3}t_0\right)\right]r_0{}^3+3\left[\left\{\left(\frac{16}{9}\kappa^2\xi_0+t_0{}^2\right)t^2\right.\right.$$
$$\left.+\left(\frac{56}{9}t_0\kappa^2\xi_0-t_0{}^3\right)t-\frac{16}{9}t_0{}^2\kappa^2\xi_0\right\}r^2$$
$$\left.-\frac{16}{9}t_0{}^2\kappa^2\xi_0\right\}r^2+8/3\kappa^2$$
$$\left.\left\{-\frac{23}{3}tt_0+10/3t_0{}^2+t^2\right\}\xi_0t0^2\right]r^2t_0r_0{}^2$$
$$+3\left[\left\{\left(t_0{}^2+\frac{8}{9}\kappa^2\xi_0\right)t-1/3t_0{}^3+\frac{28}{9}t_0\kappa^2\xi_0\right\}r^2\right.$$
$$\left.+\frac{64}{9}t_0{}^2\kappa^2\xi_0\left(t-2t_0\right)\right]r^3t_0{}^2r_0$$
$$\left.\left.+r^4t_0{}^5\left(16\kappa^2\xi_0+r^2\right)\right]\left(r-r_0\right)\right]e^{4\frac{t}{t_0}}$$
$$+\left(\frac{1}{8}\left[r_0{}^2\left\{r^4-8t^2r_0{}^2\right.\right.\right.$$
$$\left.-\left(r^3-16rt^2\right)r_0-9t^2r^2\right\}t_0{}^{5/2}$$
$$-8r\left\{\frac{5}{2}\left(\frac{4}{5}r_0{}^2+r^2-\frac{17}{10}r_0r\right)r_0tt_0{}^{7/2}\right.$$
$$+r\left(\left(-\frac{3}{2}r_0r+r^2+\frac{5}{8}r_0{}^2\right)t_0{}^{9/2}\right.$$
$$\left.\left.\left.-\frac{1}{8}r_0{}^3t_0{}^{3/2}t\left(r-r_0\right)\right)\right\}\right]\sqrt{r-r_0}\sqrt{t_0r+tr_0}$$
$$\left.+t_0{}^2\left(r-r_0\right)^2\left(t_0r+tr_0\right)^3\right)\left(r_0{}^6+r^6e^{6\frac{t}{t_0}}\right)\Big]$$
$$\times\left[t_0{}^{3/2}\left(r-r_0\right)^{3/2}\left(t_0r+tr_0\right)^{7/2}\right.$$
$$\left.r^{-2}\kappa^2\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},$$

$$\gamma(t,r)=\frac{r_0}{2}\left[3r_0{}^3\left\{-\frac{64}{3}\sqrt{r-r_0}\right.\right.$$
$$\left(\left(r^2-3/2r_0r+3/8r_0{}^2\right)t_0{}^{3/2}\right.$$
$$\left.+\frac{7}{8}\left(r-\frac{8}{7}r0\right)\sqrt{t_0}r_0t\right)\xi_0\kappa^2$$
$$\sqrt{t_0r+tr_0}+\left(t_0r+tr_0\right)$$
$$\left[\left\{\frac{64}{3}\xi_0\left(\frac{t_0}{8}-t\right)\kappa^2-t_0r^2\right\}r_0{}^2+\left(t_0r^2+\frac{56}{3}\right.\right.$$
$$\left.\left.\left.\left(t-\frac{10}{7}t_0\right)\xi_0\kappa^2\right)rr_0+\frac{64}{3}\kappa^2\xi_0t_0r^2\right]\right\}e^{2\frac{t}{t_0}}$$
$$+3r^2r_0\left(\frac{64}{3}\sqrt{r-r_0}\xi_0\kappa^2\right.$$

$$\times\left(\left(r^2-\frac{r_0}{2}r-3/8r_0{}^2\right)t_0{}^{3/2}\right.$$
$$\left.+\frac{9}{8}\left(r-\frac{8}{9}r_0\right)\sqrt{t_0}r_0t\right)\sqrt{t_0r+tr_0}+(t_0r+tr_0)$$
$$\times\left[\left(\frac{64}{3}\xi_0\left(\frac{1}{8}t_0+t\right)\kappa^2-t_0r^2\right)r_0{}^2\right.$$
$$+r\left[t_0r^2-24\xi_0\left(t-\frac{2}{3}t_0\right)\kappa^2\right]r_0$$
$$\left.\left.-\frac{64}{3}\kappa^2\xi_0t_0r^2\right]\right)e^{4\frac{t}{t_0}}$$
$$\left.+t_0(r-r_0)\left(r_0{}^6+r^6e^{6\frac{t}{t_0}}\right)(t_0r+tr_0)\right]$$
$$\left[r(t_0r+tr_0)^2t_0\kappa^2(r-r_0)\right.$$
$$\left.\times\left(r^2e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]^{-1},$$

$$\alpha(t,r)=-\left[3rr_0{}^4\left\{1/8\sqrt{r-r_0}\left[\left\{\left(r_0{}^2-\frac{256}{3}\kappa^2\xi_0\right)r^5\right.\right.\right.\right.$$
$$+\left(\frac{224}{3}r_0\kappa^2\xi_0-r_0{}^3\right)r^4$$
$$-9\left(t^2-\frac{32}{27}\kappa^2\xi_0\right)r_0{}^2r^3+16r^2t^2r_0{}^3\Big)$$
$$\times\ 8t^2\left(\frac{4}{3}\kappa^2\xi_0-r_0{}^2\right)r_0{}^2r-\frac{32}{3}r_0{}^3\kappa^2\xi_0t^2\Big\}t_0{}^{7/2}$$
$$+\left(\frac{32}{3}r_0r^2\kappa^2\xi_0-\frac{32}{3}r_0{}^2r\kappa^2\xi_0\right.$$
$$\left.-8r^5-5r_0{}^2r^3+12r_0r^4\right)t_0{}^{11/2}$$
$$+t\left\{\left(-20r^4+34r_0r^3+\left(-16r_0{}^2+\frac{32}{3}\kappa^2\xi_0\right)r^2\right.\right.$$
$$\left.-\frac{32}{3}r_0{}^2\kappa^2\xi_0\right)t_0{}^{9/2}$$
$$+\left[\left\{\left(r_0{}^2-256\kappa^2\xi_0\right)r-\frac{64}{3}r_0\kappa^2\xi_0\right\}rt_0{}^{5/2}\right.$$
$$-256t\xi_0\Big((r+1/24r_0)t_0{}^{3/2}$$
$$\left.\left.\left.+1/3r_0t\sqrt{t_0}\right)\kappa^2r_0\right]r(r-r_0)\right\}r_0\Big]\sqrt{rt_0+tr_0}$$
$$+t_0\left[\left(t_0{}^5+\frac{32}{3}t_0{}^3\kappa^2\xi_0\right)r^5\right.$$
$$+3t_0{}^2r_0\left\{\left(t_0{}^2+\frac{88}{9}\kappa^2\xi_0\right)t\right.$$
$$\left.-\frac{1}{3}t_0{}^3-\frac{52}{9}\kappa^2\xi_0t_0\right\}r^4$$
$$+3t_0\left\{\left(t_0{}^2+\frac{80}{9}\kappa^2\xi_0\right)t^2-\left(t_0{}^3+\frac{136}{9}\kappa^2\xi_0t_0\right)t\right.$$
$$\left.+\frac{16}{9}t_0{}^2\kappa^2\xi_0\right\}r_0{}^2r^3+\left[\left\{\left(8\kappa^2\xi_0+t_0{}^2\right)t^3\right.\right.$$
$$+\left(-\frac{116}{3}\kappa^2\xi_0t_0-3t_0{}^3\right)t^2+\frac{28}{3}t_0{}^2\kappa^2\xi_0t$$
$$\left.\left.-\frac{4}{3}t_0{}^3\kappa^2\xi_0\right\}r_0{}^2-16/3\kappa^2\xi_0t_0{}^4(t+t_0)\right]r_0r^2$$
$$-\left[t\left\{\left(t_0{}^2+\frac{32}{3}\kappa^2\xi_0\right)t^2\right.\right.$$
$$\left.-4t_0\kappa^2\xi_0t+4/3t_0{}^2\kappa^2\xi_0\right\}r_0{}^2$$
$$\left.+8/3t_0{}^3\kappa^2\xi_0(t+t_0)(t-2t_0)\right]r_0{}^2r$$
$$\left.+8/3tt_0{}^3r_0{}^3\kappa^2\xi_0(t+t_0)\right]$$
$$\times\ (r-r_0)\}\,e^{2\frac{t}{t_0}}+3r^3\left[1/8\sqrt{r-r_0}\right.$$
$$\left\{\left[\left(r_0{}^2+\frac{256}{3}\kappa^2\xi_0\right)r^5+\left(-96r_0\kappa^2\xi_0-r_0{}^3\right)r^4\right.\right.$$
$$-9\left(t^2-\frac{32}{27}\kappa^2\xi_0\right)r_0{}^2r^3+16r^2t^2r_0{}^3$$
$$\left.-8t^2\left(-\frac{4}{3}\kappa^2\xi_0+r_0{}^2\right)r_0{}^2r-\frac{32}{3}r_0{}^3\kappa^2\xi_0t^2\right]t_0{}^{7/2}$$
$$+\left(-\frac{32}{3}r_0{}^2r\kappa^2\xi_0+\frac{32}{3}r_0r^2\kappa^2\xi_0\right.$$
$$\left.-8r^5-5r_0{}^2r^3+12r_0r^4\right)t_0{}^{11/2}$$
$$+t\left[\left\{-20r^4+34r_0r^3+\left(-16r_0{}^2+\frac{32}{3}\kappa^2\xi_0\right)r^2\right.\right.$$
$$\left.-\frac{32}{3}r_0{}^2\kappa^2\xi_0\right\}t_0{}^{9/2}$$
$$+\left(r\left(\left(r_0{}^2+256\kappa^2\xi_0\right)r-\frac{64}{3}r_0\kappa^2\xi_0\right)t_0{}^{5/2}\right.$$
$$+256t\left\{\left(r-\frac{r_0}{24}\right)t_0{}^{3/2}\right.$$
$$\left.\left.\left.\left.+\frac{r_0t}{3}\sqrt{t_0}\right\}\xi_0\kappa^2r_0\right)r(r-r_0)\right]r_0\right\}\sqrt{rt_0+tr_0}$$
$$+\left[\left(t_0{}^5-\frac{32}{3}t_0{}^3\kappa^2\xi_0\right)r^5\right.$$
$$+3t_0{}^2\left\{\left(t_0{}^2-\frac{88}{9}\kappa^2\xi_0\right)t\right.$$
$$\left.-\frac{1}{3}\left(-20\kappa^2\xi_0+t_0{}^2\right)t_0\right\}r_0r^4$$
$$+3t_0\left[\left(t_0{}^2-\frac{80}{9}\kappa^2\xi_0\right)t^2\right.$$
$$\left.+\left(\frac{152}{9}\kappa^2\xi_0t_0-t_0{}^3\right)t-\frac{16}{9}t_0{}^2\kappa^2\xi_0\right]r_0{}^2r^3$$
$$+r_0\left[\left\{\left(t_0{}^2\right)t^3-8\kappa^2\xi_0+\left(\frac{124}{3}\kappa^2\xi_0t_0-3t_0{}^3\right)t^2\right.\right.$$
$$\left.-12t_0{}^2\kappa^2\xi_0t-4/3t_0{}^3\kappa^2\xi_0\right\}r_0{}^2$$
$$\left.-\frac{32}{3}\kappa^2\xi_0t_0{}^4(t+t_0)\right]r^2$$
$$-\left[t\left\{\left(t_0{}^2-\frac{32}{3}\kappa^2\xi_0\right)t^2+\frac{20}{3}t_0\kappa^2\xi_0t\right.\right.$$
$$\left.+4/3t_0{}^2\kappa^2\xi_0\right\}r_0{}^2+8(t+t_0)$$
$$\left.\xi_0t_0{}^3(t-4/3t_0)\kappa^2\right]$$
$$\left.\times\ r_0{}^2r+8tt_0{}^3r_0{}^3\kappa^2\xi_0(t+t_0)\right]$$
$$\times t_0(r-r_0)]\,r_0{}^2e^{4\frac{t}{t_0}}+\left(r^6e^{6\frac{t}{t_0}}+r_0{}^6\right)$$
$$\times\left\{\frac{1}{8}\sqrt{r-r_0}\left[r_0{}^2\left(r^4-9r^2t^2-8t^2r_0{}^2\right.\right.\right.$$
$$\left.-r_0r^3+16r_0rt^2\right)t_0{}^{7/2}-8r\left\{\left(\frac{5}{8}r_0{}^2\right.\right.$$
$$\left.-\frac{3}{2}r_0r+r^2\right)rt_0{}^{11/2}$$
$$-\frac{t}{8}\left[\left(34r_0r-16r_0{}^2-20r^2\right)\right.$$
$$\left.\left.\left.\left.\times\ t_0{}^{9/2}+(r-r_0)rr_0{}^2t_0{}^{5/2}\right]r_0\right\}\right]$$
$$\left.\left.\sqrt{rt_0+tr_0}+t_0{}^3(r-r_0)^2(rt_0+tr_0)^3\right\}\right]$$
$$\times\left[(rt_0+tr_0)^{5/2}(r-r_0)^{5/2}t_0{}^{7/2}r^2\kappa^2\right.$$

$$\left(r^2 e^{2\frac{t}{t_0}} + r_0{}^2\right)^3\Bigg]^{-1},$$

$$\begin{aligned}
\Phi(t,r) = \frac{1}{2}\Bigg[& 6r_0{}^4\Bigg\{-\frac{16}{3}\Bigg[\frac{9}{32}\left(-\frac{2}{3}r_0+r\right)r_0 t \\
& \left(r^2-\frac{8}{3}\kappa^2\xi_0\right)\left(\frac{3}{16}\left(r-\frac{1}{2}r_0\right)\right. \\
& \left(r^2-\frac{8}{3}\kappa^2\xi_0\right)t_0{}^{9/2}+\left(2r_0\left(\frac{1}{8}r_0+r\right)tt_0{}^{3/2}\right. \\
& \left.\left.+\left(r+\frac{1}{4}r_0\right)rt_0{}^{5/2}+r_0{}^2\sqrt{t_0}t^2\right)\xi_0 r\kappa^2\right)r\Bigg] \\
& \sqrt{r-r_0}\sqrt{rt_0+tr_0}+t_0\left(t\left(\left(\frac{16}{3}t\kappa^2\xi_0\right.\right.\right. \\
& \left.\left.+\frac{4}{3}\kappa^2\xi_0 t_0+tt_0{}^2\right)r^2+\frac{8}{3}t_0{}^3\kappa^2\xi_0\right)r_0{}^2 \\
& +2\Bigg[\left(tt_0{}^2+\frac{2}{3}\kappa^2\xi_0 t_0+\frac{16}{3}t\kappa^2\xi_0\right)r^2 \\
& -\frac{10}{3}\xi_0 t_0{}^2\kappa^2\left(t+\frac{t_0}{5}\right)rt_0 r_0+r^2 t_0{}^2\Bigg\} \\
& \left(t_0{}^2+\frac{16}{3}\kappa^2\xi_0\right)r^2-\frac{8}{3}t_0{}^2\kappa^2\xi_0\Bigg\}\Bigg](r-r_0)\Bigg)e^{2\frac{t}{t_0}} \\
& +6r^2 r_0{}^2\Bigg[\frac{16}{3}\Bigg\{\frac{r_0}{32}\left(3\left(2r^2+16\kappa^2\xi_0{}^2\right)r_0-3r^3\right. \\
& \left.-40r\kappa^2\xi_0\right)tt_0{}^{7/2}+\Bigg[\Bigg\{\left(\frac{3}{32}r^2+\frac{7}{4}\kappa^2\xi_0\right)r_0 \\
& -\frac{3}{1}6r\left(8\kappa^2\xi_0+r^2\right)\Bigg\}t_0{}^{9/2}+\Bigg\{2\left(r-\frac{r_0}{8}\right)r_0 tt_0{}^{3/2} \\
& +\left(r-\frac{r_0}{4}\right)rt_0{}^{5/2}+r_0{}^2\sqrt{t_0}t^2\Bigg\}\xi_0 r\kappa^2\Bigg]r\Bigg\} \\
& \times\sqrt{r-r_0}\sqrt{rt_0+tr_0} \\
& +t_0\Bigg\{\left(\frac{4}{3}\kappa^2\xi_0 t_0-\frac{16}{3}t\kappa^2\xi_0+tt_0{}^2\right)r^2 \\
& -8t_0{}^3\kappa^2\xi_0\Bigg\}r_0{}^2+2r\Bigg\{\left(tt_0{}^2+\frac{2}{3}\kappa^2\xi_0 t_0\right. \\
& \left.-\frac{16}{3}t\kappa^2\xi_0\right)r^2+2t_0{}^2\kappa^2\xi_0\left(t-3t_0\right)\Bigg\}t_0 r_0+r^2 \\
& \times\ \Bigg\{\left(t_0{}^2-\frac{16}{3}\kappa^2\xi_0\right)r^2+8t_0{}^2\kappa^2\xi_0\Bigg\}t_0{}^2\Bigg] \\
& \times\ (r-r_0)\Big\}e^{4\frac{t}{t_0}}+2\left(r^6 e^{6\frac{t}{t_0}}+r_0{}^6\right) \\
& \times\Bigg\{\Bigg[\frac{3}{2}\left(\frac{r_0}{3}r_0-r\right)r_0 tt_0{}^{7/2}-t_0{}^{9/2}\left(r-\frac{r_0}{2}\right)r\Bigg] \\
& \sqrt{r-r_0}\sqrt{rt_0+tr_0}+t_0^3(r-r_0)(rt_0+tr_0)^2\Bigg\}\Bigg] \\
& \times\left[t_0{}^3\kappa^2 r^2(rt_0+tr_0)^2(r-r_0)\left(r^2 e^{2\frac{t}{t_0}}+r_0{}^2\right)^3\right]. \qquad \text{(A4)}
\end{aligned}$$

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