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Null and timelike circular orbits from equivalent 2D metrics

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Abstract

The motion of particles on spherical $1+3$ dimensional spacetimes can, under some assumptions, be described by the curves on a two-dimensional manifold, the optical and Jacobi manifolds for null and timelike curves, respectively. In this paper we resort to auxiliary two-dimensional metrics to study circular geodesics of generic static, spherically symmetric, and asymptotically flat $1+3$ dimensional spacetimes, whose functions are at least C^2 smooth. This is done by studying the Gaussian curvature of the bidimensional equivalent manifold as well as the geodesic curvature of circular paths on these. This study considers both null and timelike circular geodesics. The study of null geodesics through the optical manifold retrieves the known result of the number of light rings on the spacetime outside a black hole and on spacetimes with horizonless compact objects. With an equivalent procedure we can formulate a similar theorem on the number of marginally stable timelike circular orbits of a given spacetime satisfying the previously mentioned assumptions.

Keywords: black holes, light rings, marginally stable circular orbits

(Some figures may appear in colour only in the online journal)

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1. Introduction

Gibbons and Werner [1] showed how one could use the optical metric of a spherically symmetric spacetime, which is the two dimensional Riemannian metric seen by a massless particle, to compute the deflection angle of light by means of the Gauss–Bonnet theorem on the optical manifold. Later, the procedure was generalised to consider the more general axisymmetric spacetimes in two ways, one developed by Werner [2] which makes use of Nazim's construction of the osculating manifold, and the other developed by Ono *et al* [3] which computes explicitly the contribution of the rotation one-form to the geodesic curvature of curves and even takes into account finite distance corrections [4]. The optical metric has been recently widely used to compute the deflection angle of light passing near a compact object both within and beyond general relativity (GR), e.g. [5–12].

In a procedure similar to the derivation of the optical one can derive the Jacobi metric [13, 14], which is also a two dimensional metric but yields the timelike geodesics of the spacetime. Therefore one can use the Jacobi metric to study the paths of massive (and possibly charged) particles in spacetimes [15]. Applying the same reasoning used to compute the deflection angle of light, the Jacobi metric has been used to compute the deflection angle of massive particles passing near compact objects [16, 17]. Since the Jacobi metric yields the timelike geodesics of the spacetime it can be used, in particular, to study those that are circular, the timelike circular orbits (TCOs).

Unlike light rings (LRs), which are null circular orbits, and exist only for some distinct radial coordinates, TCOs can exist for continuous ranges of the radial coordinate. This means that TCOs form a continuum of connected orbits characterised by the energy and angular momentum of the particle, hence unlike LRs the TCOs are not solely characterised by the properties of the spacetime. However, there are special TCOs which depend only on the underlying spacetime, the ones that separate regions of different stability of TCOs. These are known as marginally stable circular orbits (MSCOs), the innermost of which is the innermost stable circular orbit (ISCO). The ISCO, as the name suggests, separates a region where TCOs are stable (radially above in the Schwarzschild case) from another where they are unstable (radially below). For the Kerr metric, regardless of the spin parameter, there is only one such orbit separating regions of stability; however, for other compact objects one can have much richer structure, namely several disconnected regions where stable TCOs are possible [18]. ISCOs are astrophysically relevant due to their impact on the accretion disk dynamics as well as on the study of extreme mass ratio inspirals where the motion of the lighter body can be modelled as following stable TCOs, moving gradually inwards, until it reaches the ISCO, after which it plunges towards the central object. Therefore the structure of TCOs and the location of the ISCO greatly affects the emitted gravitational waves (GWs). The GWs generated by such events are expected to lie in the sensitivity range of LISA [19]; hence, the study of such matters is quite timely.

Recently the optical metric has also been used in a discussion of LRs on Schwarzschild-like spherically symmetric black holes (BHs) [20]. In the latter it is argued that the Hadamard–Cartan theorem can be used to determine the stability of the LRs of the spacetime. In the present paper a similar study will be made concerning the TCOs of static, spherically symmetric, asymptotically flat $1+3$ dimensional spacetimes, whose metric functions are at least C^2 smooth, using for this purpose the Jacobi metric. We will also extend the analysis to the number of LRs and their stability, similar to what was done in [21] for BHs. In addition, we also recover the results of recent theorems for stationary spacetimes in the spherically symmetric case [22, 23]. However the study of LRs performed here serves to present a novel approach and as a proof of concept before considering TCOs. The equivalent analysis

concerning the Jacobi metric will provide similar theorems concerning the number of MSCO on the considered spacetimes.

2. The spacetime

In this paper a general spherically symmetric, static, asymptotically flat, $1+3$ dimensional metric is considered. It can be described by the following line element:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\Omega^2\,, \tag{1}$$

where $\mathrm{d}\Omega^2$ is the usual metric on the unit 2-sphere and f, g are at least C^2 smooth. Asymptotic flatness is imposed by requiring the metric to be Minkowski at infinity, this is achieved by:

$$\lim_{r\to\infty} f(r) = 1\,, \quad \lim_{r\to\infty} h(r) = 1\,. \tag{2}$$

We will be concerned with the motion of massless as well as massive particles on this spacetime. But we will consider the motion on equivalent bidimensional manifolds, namely the optical and Jacobi manifolds. This allows the usage of several results from differential geometry which can give novel insights into the spacetime geodesics.

Two contrasting spacetime types will be considered: one containing a BH and another describing a horizonless compact object. BHs are characterised by the presence of an event horizon, which for metrics of the form of equation (1) occurs at $r = r_H$ such that (check appendix B)

$$h(r_H) = 0\,. \tag{3}$$

Carter proved that for static BHs the event horizon must always be also a Killing horizon of the time translation Killing vector field ∂_t [24, 25]. In the present case, this implies that $f(r_H) = 0$. This is a purely geometric result that does not invoke Einstein's field equations. This Killing vector field must remain timelike everywhere outside the horizon, such that $f(r) > 0\,, r > r_H$ (see section 12.3 of [26]).

Only non-extremal BHs will be considered. Extremal BHs are defined by the vanishing of their surface gravity, $\kappa_{\rm surf}$. For the spacetimes considered the surface gravity is given by:

$$\kappa_{\rm surf} = \lim_{r\to r_H} \sqrt{\frac{h(r)}{f(r)}}\frac{f'(r)}{2}\,. \tag{4}$$

For spherically symmetric BHs this corresponds to the acceleration of a static observer at the horizon as measured at spatial infinity. The surface gravity of non-extremal BHs satisfies $\kappa_{\rm surf} > 0$, the derivation and a small discussion is found in appendix C.

Like the name suggest horizonless compact objects have no horizon therefore $f(r) > 0$ and $h(r) > 0$ everywhere, for line elements of the form (1). The spacetime should be regular, namely at $r = 0$; then, the functions f and h can be Taylor expanded around $r = 0$. Inserting the resulting expansions on the expressions for the Ricci and Kretschmann scalars one finds that for them to be regular at $r = 0$ one must have as $r \to 0$ [27]:

$$f(r) = f_0 + f_2 r^2 + \mathcal{O}\left(r^4\right)\,, \quad h(r) = 1 + h_2 r^2 + \mathcal{O}\left(r^4\right)\,. \tag{5}$$

In this paper topologically non-trivial spacetimes will not be considered, such as wormhole spacetimes.

3. Light rings

This section will focus on the circular null curves of the spacetime, which when these are geodesics correspond to the LRs. This study will be carried using the optical metric, to be defined below. This formalism will be applied to two distinct compact objects spacetimes, those with horizon (BHs) and those without horizons. Asymptotic flatness of the spacetime implies the same behaviour of the metric at infinity, however in the origin/horizon limit the behaviour will be distinct for the two cases.

3.1. The optical metric

The optical metric is the metric as seen by a massless particle. It is obtained by inserting the null condition, $\mathrm{d}s^2 = 0$, into the line element (1) and solving for $\mathrm{d}t$. This yields

$$\mathrm{d}t^2 = \frac{1}{f(r)}\left(\frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\Omega^2\right) = \frac{1}{f(r)}\left(\frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\phi^2\right). \tag{6}$$

For the last equality we took advantage of the symmetry of the problem and restricted the analysis to the equatorial plane of the optical manifold, $\theta = \pi/2$. The geodesics of this metric are the light rays, which are defined as the spatial projections of the null geodesics of the original spacetime, equation (1). This formulation can be used to compute the deflection angle of light for several spacetimes by means of the Gauss–Bonnet theorem. However for this paper we will consider only the Gaussian curvature of the manifold as well as the geodesic curvature of circular orbits. These are given by, respectively,

$$K = -h'(r)\frac{2f(r) - rf'(r)}{4r} + \frac{h(r)}{2}\left[f'(r)\left(\frac{1}{r} - \frac{f'(r)}{f(r)}\right) + f''(r)\right] \tag{7}$$

$$\kappa_g = \sqrt{\frac{h(r)}{f(r)}}\frac{2f(r) - rf'(r)}{2r}. \tag{8}$$

The full computation can be found in appendix A. By definition, the geodesic curvature vanishes for geodesics, hence the roots of $\kappa_g(r) = 0$ correspond to null circular geodesics of the spacetime, i.e. the LRs. We can make use of this feature to confirm several theorems concerning the number and stability of LRs on a spacetime of the form (1), by studying the asymptotic behaviour of this function. For now only the simplest case will be considered: the zeros of κ_g are assumed simple, i.e. points where $\kappa_g = 0$ and $\kappa_g' \neq 0$. In doing so degenerate LRs, corresponding to the coalescence of two LRs, are avoided. A brief comment on the degenerate case will be made at the end of this section.

3.2. Asymptotic limit

Inserting the conditions for asymptotic flatness, equation (2), into equation (8) one obtains

$$\lim_{r\to\infty} \kappa_g = \frac{1}{r}. \tag{9}$$

This means that κ_g tends asymptotically to zero from positive values. Such result is easily understood. Spacetime asymptotic flatness implies that the optical metric, equation (6), is asymptotically Euclidean and $1/r$ is precisely the curvature of a circumference in Euclidean space.

3.3. BHs—horizon limit

Having considered the asymptotic limit of the geodesic curvature we have to compute the opposite limit, which, for spacetimes containing BHs, corresponds to the event horizon.

Considering the horizon limit in equation (8) one obtains

$$\lim_{r\to r_H}\kappa_g = \lim_{r\to r_H}\sqrt{\frac{h(r)}{f(r)}}\frac{2f(r)-rf'(r)}{2r} \tag{10}$$

$$= \lim_{r\to r_H} -\sqrt{\frac{h(r)}{f(r)}}\frac{f'(r)}{2} = -\kappa_{\rm surf} < 0\,. \tag{11}$$

This, together with $\kappa_g(+\infty) = 0^+$, implies that for non-extremal BH spacetimes κ_g must vanish at least once between the horizon and infinity. Moreover, the number of these zeros, which correspond to LRs of the spacetime, must always be odd. This is in agreement with the general theorem in [23].

3.4. Horizonless compact objects—origin limit

For spacetimes harbouring horizonless compact objects, we must consider the limit $r\to 0$. Following the expansions presented in equation (5) one obtains for the equatorial curvature of circular null geodesics

$$\lim_{r\to 0}\kappa_g = \lim_{r\to 0}\frac{1}{r} = +\infty\,. \tag{12}$$

This result together with the asymptotic behaviour of κ_g indicates that a horizonless compact object does not need to have LRs, and ifit does they will always come in pairs. This is consistent with the general theorem in [22].

An illustration of the different behaviour of κ_g is given in figure 1 for spacetimes harbouring either horizonless compact objects or black holes.

3.5. Stability of the LRs

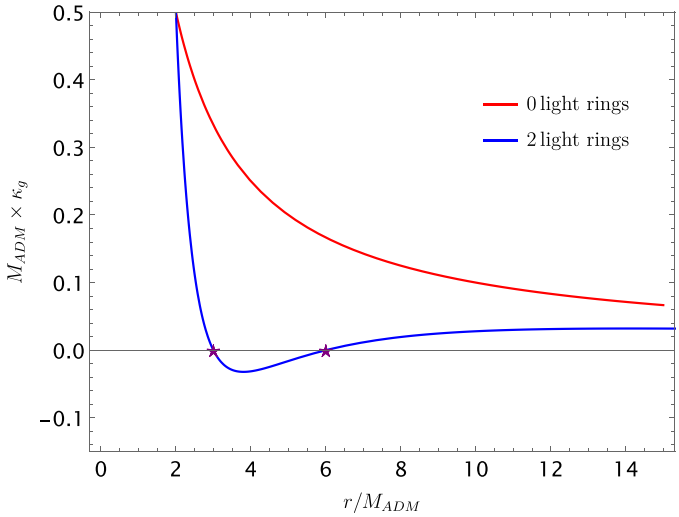
So far we have discussed only the existence and number of LRs but using the Gaussian curvature of the manifold we can compute their stability. The stability of the LRs are determined by the sign of K at that radius, stable (unstable) orbits have a positive (negative) Gaussian curvature. This is discussed in appendix A. Considering the Gaussian curvature, equation (7), at a LR, i.e. at a radial coordinate $r = r_{\rm LR}$ such that $\kappa_g(r = r_{\rm LR}) = 0$, one obtains

$$\begin{aligned} K|_{r=r_{\rm LR}} &= \frac{h(r)}{2}\left(f''(r) - \frac{f'(r)}{r}\right)\Big|_{r=r_{LR}} \\ &= -\sqrt{h(r)f(r)}\,\kappa_g'\big|_{r=r_{\rm LR}}\,. \end{aligned} \tag{13}$$

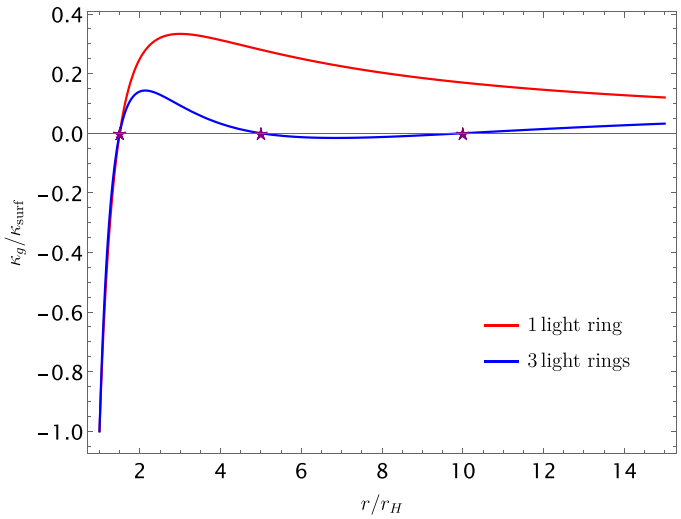
From this relation it follows that the stability of a LR is determined by the way κ_g crosses the horizontal axis at $r = r_{\rm LR}$, if it crosses with positive (negative) slope the associated LR is unstable (stable).

3.6. Some remarks

This discussion shows that asymptotically flat spacetimes containing a non-extremal BH have $2n+1$, $n\in\mathbb{N}_0$ LRs, the first and last of which will be unstable, and they will always alternate.



(a) Horizonless compact objects



(b) Black holes

Figure 1. Two illustrative plots of the behaviour of κ_g for spacetimes containing horizonless compact objects, (a), or BHs, (b). In each plot are represented examples with the lowest and second lowest number of LRs. The LRs are indicated with purple stars.

Spacetimes describing horizonless compact objects, on the other hand, have $2n$, $n \in \mathbb{N}_0$ LRs, where the first (innermost) LR will be stable and the last (outermost) unstable, and they will also always alternate.

It also follows that spacetimes with more than one LR will always have at least one stable LR. The presence of the latter has been argued to imply the spacetime is unstable [28, 29]; in fact this was recently shown in specific examples [30]. Moreover, it has been conjectured that the appearance of more than one LR in BH spacetimes requires the violation of the strong energy condition [21].

In this analysis we did not considered the hypothesis of a degenerate root, this would correspond to the case where a zero of κ_g corresponds to a maximum or minimum. Such points will have $\kappa' = 0 \Rightarrow K = 0$, hence they will be marginally stable, i.e. stable against perturbations in a given direction and unstable against perturbations in the opposite one. Therefore one should compute the derivative of the Gaussian curvature at these points, $r = \bar{r}$, where $\kappa_g(\bar{r}) = 0 = \kappa_g'(\bar{r})$. This is given by

$$K'|_{r=\bar{r}} = -\sqrt{\frac{f(r)}{h(r)}}\kappa_g''\Big|_{r=\bar{r}} . \tag{14}$$

Hence, if $\kappa_g'' > 0$ the LR will be stable against perturbations that increase the radius and unstable otherwise. If $\kappa_g'' < 0$ the converse happens. This further analysis is merely the first level of additional complexity and does not exhaust all the possibilities; it may occur that several of the derivatives of κ_g vanish at some point, and this point is an inflection point if the first non-vanishing derivative is even and an extremum if it is odd. Therefore a full analysis should take this into account. This is however beyond the scope of our discussion; moreover it requires very specific conditions, rather than generic cases.

4. Marginally stable circular orbits

In this section a similar analysis to the one performed in the optical manifold will be performed in the Jacobi manifold. The goal of the analysis will be to obtain the MSCO of the spacetime, and thus the structure of TCOs on it.

4.1. The Jacobi metric

Much like for null particles, the motion of massive ones is also given by the geodesics on a two dimensional Riemannian manifold, the Jacobi metric. This metric depends on the energy of the particle and in the massless limit reduces to the optical metric. To obtain the Jacobi metric we recall that test particles of mass m will move along the timelike geodesics of the spacetime. These can be obtained by extremising the following action

$$S = \int \mathcal{L}\mathrm{d}\lambda = -m\int \sqrt{-g_{\mu\nu}\dot{x}^\nu \dot{x}^\mu}\mathrm{d}\lambda\,, \tag{15}$$

where λ is an arbitrary parameter along the geodesic and the dot denotes differentiation with respect to it. From now on it will be assumed that trajectories are parametrised by the coordinate time t, such that:

$$\mathcal{L} = -m\sqrt{f(r) - g_{ij}\dot{x}^i\dot{x}^j}\,. \tag{16}$$

The associate canonical momentum, p_i, is defined in the usual way

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m\frac{g_{ij}\dot{x}^j}{\sqrt{f(r) - g_{mn}\dot{x}^m\dot{x}^n}}\,. \tag{17}$$

With this it is possible to obtain corresponding Hamiltonian, H, as

$$\begin{aligned} H &= p_i\dot{x}^i - \mathcal{L} = \frac{mf(r)}{\sqrt{f(r) - g_{mn}\dot{x}^m\dot{x}^n}} \\ &= \sqrt{m^2 f(r)^2 + f(r)^2\, g^{ij}p_i p_j}\,. \end{aligned} \tag{18}$$

In the Hamilton–Jacobi formalism the momenta are the gradient of the action function, S,

$$p_i = \partial_i S\,, \tag{19}$$

such that the Hamilton–Jacobi equation reads

$$H\left(x^i, \partial_i S, t\right) = -\frac{\partial S}{\partial t}\,. \tag{20}$$

Since the Hamiltonian does not depend on the parameter t, the right hand side cannot depend on t. Therefore we choose $-\partial_t S = E$, where E is a constant and can be interpreted as the energy of the system. This choice is consistent with the classical result that for conservative systems the Hamiltonian is time independent and corresponds to the total energy of the system. Rearranging then the Hamilton–Jacobi equation one obtains

$$\alpha^{ij} p_i p_j = 1\,, \quad \alpha^{ij} \equiv \frac{f(r)}{E^2 - m^2 f(r)} g^{ij}\,. \tag{21}$$

This is precisely the equation for the geodesics of the Jacobi metric, α_{ij}, defined as

$$\mathrm{d}s_J^2 = \alpha_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{E^2 - m^2 f(r)}{f(r)}\left[\frac{\mathrm{d}r^2}{h\,(r)} + r^2\mathrm{d}\phi^2\right]. \tag{22}$$

This expression makes clear that in the massless limit we recover the optical metric, equation (6), apart from a conformal factor E^2, which does not affect the geodesics.

If a given circular orbit is a geodesic its geodesic curvature vanishes, which is given by

$$\kappa_g = \frac{2f(r)\left(\varepsilon^2 - f(r)\right) - r\varepsilon^2 f'(r)}{2mr\sqrt{f(r)h(r)}\left(\varepsilon^2 - f(r)\right)^{3/2}}\,. \tag{23}$$

where $\varepsilon = E/m$. Much like with the optical metric the radial stability of such orbits is determined by the sign of the Gaussian curvature along it. The Gaussian curvature of this manifold is

$$\begin{aligned} K = &-\frac{1}{4m^2 rf(r)\left(f(r)+\varepsilon^2\right)^3} \\ &\times\left[\varepsilon^2 f(r)^2\left(h'(r)\left(rf'(r)+2\varepsilon^2\right)+2h(r)\left(f'(r)+rf''(r)\right)\right)+2r\varepsilon^4 h(r)f'(r)^2\right. \\ &-4\varepsilon^2 f(r)^3 h'(r)+2f(r)^4 h'(r)+\varepsilon^2 f(r)\left(-f'(r)\left(2h(r)\left(2rf'(r)+\varepsilon^2\right)\right.\right. \\ &\left.\left.\left.+r\varepsilon^2 g'(r)\right)-2r\varepsilon^2 h(r)f''(r)\right)\right]. \end{aligned} \tag{24}$$

It should be noted that since these formulas depend only on the square of the energy these equations are also valid for circular geodesics with purely imaginary energies, $\varepsilon^2 < 0$. These will correspond to circular spacelike orbits.

It will be useful later on to introduce the energy of a circular geodesic at a radius r, obtained from solving $\kappa_g = 0$ for ε; this yields:

$$\varepsilon^2 = -\frac{2f(r)}{2f(r) - rf'\,(r)}\,. \tag{25}$$

As discussed previously the numerator of this expression is positive everywhere outside the horizon. However the denominator can take on either sign and even be zero, possible zeros would correspond to LRs. Since the left hand side is the square of the energy the right hand side must be positive for real energies. However the right hand side can become negative yielding purely imaginary energies, which correspond to spacelike geodesics. Therefore TCOs can only occur in regions where $rf'\,(r) - 2f(r) > 0$. This corresponds to the region between an unstable LR (at $r = r_1$) and a stable one (at $r = r_2$) with $r_2 > r_1$, the region from the outermost LR

(unstable) to infinity, or in the case of horizonless compact objects the region from $r=0$ up to the first LR (stable). This is in agreement with the results found in [18].

Inserting equation (25) into equation (24) one obtains the Gaussian curvature along circular geodesics of a given r, this is denoted by $K^{\rm circ}$ and takes the following form

$$K^{\rm circ}=\frac{h(r)}{m^2r^3f(r)f'(r)^2}\underbrace{\left[2f(r)-rf'(r)\right]}_{\text{Light Rings}}\underbrace{\left[f(r)\left(3f'(r)+rf''(r)\right)-2rf'(r)^2\right]}_{\text{MSCOs}}\,. \quad (26)$$

Since the stability is determined by the sign of this expression the MSCOs occur at radius where $K^{\rm circ}$ vanishes. The numerator of this expression is a product of three different expressions, the first from left to right is $h\,(r)$ which is always positive outside the horizon, the zeros of second one correspond to the LRs, equation (8), and orbits at these radii have infinite energy per unit mass, this means that the MSCOs will be determined by the final term

$$f(r)\left(3f'(r)+rf''(r)\right)-2rf'(r)^2=0\,. \quad (27)$$

One should also take into account possible divergences arising from the vanishing of the denominator. From the previous discussion the only points at which this could happen are the extrema of f, *i.e* some $\tilde{r}$ such that $f'(\tilde{r})=0$. However since the denominator depends on the square of f' it will always be non-negative, hence $K^{\rm circ}$ will have the same sign at $\tilde{r}\pm\delta\,,\delta\ll 1$. Since we are only interested in the sign of $K^{\rm circ}$ such poles do not affect the present analysis. The equivalence between this approach, and the usual one with the effective potential is shown in appendix D.

The fact that equation (26) also yields the LRs justifies the attention given to our previous analysis of the optical metric; indeed, that analysis allows us to clearly identify that one of the factors in equation (26) determines the LRs as its zeros. This is however somewhat undesirable, so one should study more about the behaviour of $K^{\rm circ}$ at the LRs, namely its slope at these points:

$$\left.\frac{\partial K^{\rm circ}}{\partial r}\right|_{r=r_{\rm LR}}=-\left.\frac{h(r)\left[r^2f''(r)-2f(r)\right]^2}{4m^2r^3f(r)^2}\right|_{r=r_{\rm LR}}<0\,. \quad (28)$$

Therefore $K^{\rm circ}$ crosses the horizontal axis at the LRs with negative slope, this means that it must always cross with positive slope between the LRs. If TCOs are possible in that region those crossing points will be the MSCOs and if TCOs are not possible these are marginally stable spacelike orbits.

4.2. Asymptotic limit

As before we will be concerned with the number of zeros of equation (26), so we must study its behaviour in the limits of the region where it is defined. The first limit we consider is spatial infinity. Since the spacetime is asymptotically flat the associated Jacobian metric is asymptotically Euclidean, therefore

$$\lim_{r\to\infty}K^{\rm circ}=0\,. \quad (29)$$

The derivative of f in the denominator of equation (26) means that to obtain the behaviour at infinity one needs also to specify how this function decays. It is expected that near infinity the orbits of the spacetime are essentially Keplerian (e.g. the orbits around the supermassive BH at the centre of the galaxy). These orbits are stable, hence $K^{\rm circ}$ must go to zero from above.

This is equivalent to requiring that the considered spacetimes reduce to Newtonian gravity near infinity, corresponding to

$$f(r) = 1 - \frac{2M}{r} + \mathcal{O}\left(r^{-2}\right), \quad h(r) = 1 + \mathcal{O}(r). \tag{30}$$

Under this assumption one obtains

$$\lim_{r\to\infty} K^{\rm circ} = \frac{2}{m^2 M r}. \tag{31}$$

which yields the expected behaviour.

4.3. BHs—horizon limit

We are interested in the behaviour of equation (26) outside the horizon. So now we will consider the horizon limit. As before the horizon is defined by $h(r_H) = 0 = f(r_H)$, then

$$\begin{aligned}\lim_{r\to r_H} K^{\rm circ} &= \lim_{r\to r_H} \frac{2h(r) r^2 f'(r)^3}{m^2 r^3 f(r) f'(r)^2} \\ &= \frac{2}{m^2 r_H} \kappa^2_{\rm surf} f'(r_H) > 0. \end{aligned} \tag{32}$$

Once again, non-extremality of the BHs is being assumed.

4.4. Horizonless compact objects—origin limit

For horizonless compact objects, we require regularity of the spacetime at the origin, this implies the same behaviour near the origin expressed in equation (5). This leads to

$$\lim_{r\to 0} K^{\rm circ} = \frac{4 f_0}{m^2 r^4 f_2}. \tag{33}$$

From the condition that $f > 0$ for every r it comes that $f_0 > 0$, however the sign of f_2 is not constrained, and the divergence will depend on it. For $f_2 \gtrless 0$ one has $\lim_{r\to 0} K^{\rm circ} = \pm\infty$. To explore the physical meaning of f_2 one studies the Ricci tensor at the origin, to find

$$\lim_{r\to 0} R_{00} = 3 f_2. \tag{34}$$

Raychaudhuri's equation states that in order for gravity to be attractive one must have $R_{00} \geqslant 0$. This is the strong energy condition:

$$R_{\mu\nu} t^\mu t^\nu \geqslant 0, \tag{35}$$

where t^μ is any timelike vector field [31].

4.5. A theorem on the number and location of MSCOs

Considering first BHs, it was seen that $K^{\rm circ} > 0$ both at the horizon and at spatial infinity. At first glance this seems to indicate that BHs do not necessarily possess ISCOs, contradicting previous results. However as seen before BHs always possess at least one unstable LR, at which $K^{\rm circ}$ vanishes with negative slope. The same behaviour occurs at every LR. As discussed above TCOs may occur in a region from an unstable to a stable LR, which means that in that region the spacetime will have $2n+1$ MSCOs with $n \in \mathbb{N}_0$. The same result applies from the outermost LR (which is unstable) up to spatial infinity.

In the case of horizonless compact objects, the behaviour of $K^{\rm circ} > 0$ will depend on the sign of $f''(0) = f_2$. The study of these spacetimes can be divided into three distinct regions:

(a) between $r = 0$ and the innermost LR (which is stable), or spatial infinity if no LRs are present;
(b) between a stable and an unstable LR;
(c) from the outermost LR (which is unstable) to spatial infinity.

The sign of f_2 affects the behaviour near the origin, hence it will only affect region (a), if $f_2 > 0$ one has $2n$ MSCOs and if $f_2 < 0$ one has $2n + 1$ MSCOs with $n \in \mathbb{N}_0$. In regions (b) and (c) there must be at least one MSCO, and if more exist they must come in pairs, meaning $2n - 1$ MSCOs. Recall that this discussion assumes non degenerate MSCOs.

In the regions where TCOs are not possible, i.e. between a stable and an unstable LR the zeros of $K^{\rm circ}$ correspond to marginally stable circular spacelike orbits, of which there is always an odd number.

For a recent and complementary discussion on the number of TCOs using a topological approach, in a stationary BH spacetime, see [32].

5. Discussion and final remarks

This paper considers equatorial circular geodesics in arbitrary static, spherically symmetric, $1 + 3$ dimensional spacetimes that are C^2 smooth and asymptotically flat, with a main focus on the auxiliary 2D optical/Jacobian metrics used in the analysis. The study of null geodesics and LRs using the optical metric formalism was already well known in the literature. Using this method we recovered previous theorems concerning the number and stability of LRs in this class of spacetimes, albeit in a more restrictive case and with a different approach. For instance, one of these results states that BHs satisfying the above spacetime assumptions always have at least one unstable LR, with further LRs always coming in pairs with opposite stability. The analysis was then trivially extended to horizonless spacetimes, for which LRs can only come in pairs with opposing stability, with the innermost (outermost) being stable (unstable).

The LR analysis achieved through the optical metric approach serves additionally as an introduction to the similar study concerning TCOs, but now using the Jacobi rather than the optical metric. To the best of our knowledge, a detailed study of TCOs obtained using the Jacobi metric formalism (and MSCOs/ISCO in particular) constitutes a new contribution to the literature. The latter recovers some recent results concerning the existence of TCOs and LRs within a spacetime. For instance, it is shown that no TCOs are possible radially above (bellow) a stable (unstable) LR. In addition, a novel theorem was proposed concerning the possible number of MSCOs, as well as their location and stability, in regions where TCOs are possible.

Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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Appendix A. Geometry of surfaces

In this appendix the relevant quantities relative to the geometry of two dimensional surface will be introduced and computed. Namely the Gaussian curvature of the manifold and the geodesic curvatures of circular orbits.

To do so a generic two dimensional manifold, $\mathcal{S}$, equipped with a Riemannian metric, α. This can represent both the optical or Jacobi manifold. The line element on such surface is therefore given by:

$$\mathrm{d}\lambda^2 = \alpha_{rr}(r)\,\mathrm{d}r^2 + \alpha_{\phi\phi}(r)\,\mathrm{d}\phi^2\,. \tag{A1}$$

Here λ will be the coordinate time t when considering the optical metric, and the Jacobi parameter s_{J} when discussing the Jacobi metric. It was assumed that the metric components depend only on the radial coordinates, as this will be the case for the metrics considered in this paper.

A.1. Gaussian curvature

Considering the form of the metric, equation (A1), it is useful to introduce the Regge–Wheeler tortoise coordinate, r^*, defined as

$$\mathrm{d}r^* = \sqrt{\alpha_{rr}}\mathrm{d}r\,. \tag{A2}$$

In these coordinates the line element becomes

$$\mathrm{d}\lambda^2 = \mathrm{d}r^{*2} + \alpha_{\phi\phi}(r(r^*))\,\mathrm{d}\phi^2\,. \tag{A3}$$

This makes evident that $\mathcal{S}$ is a surface of revolution, thus its Gaussian curvature is given by

$$K = -\frac{1}{\alpha_{\phi\phi}(r(r^*))}\frac{\mathrm{d}^2\alpha_{\phi\phi}(r(r^*))}{\mathrm{d}r^{*2}}\,. \tag{A4}$$

This set of coordinates is also usually called *geodesic polar coordinates* [33], which are useful to understand the Gaussian curvature. These correspond to polar coordinates in the tangent plane $T_{\mathrm{p}}(S)$ at a given point $p \in S$. This system is usually defined by the polar radius,

r^*, and a polar angle, $\phi \in]0, 2\pi[$, this coordinate system has a pole at the origin of $T_{\rm p}\,(S)$. In these coordinates the metric components obey:

$$\alpha_{r^*r^*} = 1\,,\ \alpha_{r^*\phi} = 0\,,\ \lim_{r^*\to 0}\alpha_{\phi\phi} = 0\,,\ \lim_{r^*\to 0}\frac{\partial\sqrt{\alpha_{\phi\phi}}}{\partial r^*} = 1\,. \tag{A5}$$

This allows for an intuitive interpretation of the Gaussian curvature. The arc-length of the curve with $r^* = $ const. joining two close geodesics with $\vartheta = \vartheta_1$ and $\vartheta = \vartheta_2$, $L(\rho)$, is given by

$$L(r^*) = \int_{\vartheta_1}^{\vartheta_2}\sqrt{\alpha_{\vartheta\vartheta}}\,{\rm d}\vartheta\,. \tag{A6}$$

Then, since

$$\lim_{r^*\to 0}\frac{\partial\sqrt{\alpha_{\phi\phi}}}{\partial r^*} = 1\,,\quad \frac{\partial^2\sqrt{\alpha_{\phi\phi}}}{\partial r^{*2}} = -K\sqrt{\alpha_{\phi\phi}}\,, \tag{A7}$$

we have that if $K<0$ the distance between two geodesics starting close together will continue to grow. On the other hand, if $K>0$ the geodesics can begin to approach after some time.

A.2. Geodesic curvature of circular orbits

Consider a curve γ on $\mathcal{S}$, with arc-length parameter λ its geodesic curvature, κ_g, is given by

$$\kappa_g = \frac{{\rm d}\mathbf{T}}{{\rm d}\lambda}\cdot\mathbf{n}\,, \tag{A8}$$

where $\mathbf{T} = {\rm d}\mathbf{r}\,(\lambda)\,/{\rm d}\lambda$ is the unit tangent vector to γ and $\mathbf{n}$ the unit normal vector to γ. Therefore the geodesic curvature is the component of the proper acceleration along the normal direction to the curve. For circular geodesics, $r = R$, of the metric (A1) one has

$$T^i = \delta^i_\phi\frac{{\rm d}\phi}{{\rm d}\lambda} = \delta^i_\phi\frac{1}{\sqrt{\alpha_{\phi\phi}}}\,. \tag{A9}$$

Since we are considering circular geodesics the unit normal vector to the curve can be $n_i = \sqrt{\alpha_{rr}}\delta^r_i$. Hence the geodesic curvature of circular curves is given by

$$\begin{aligned}\kappa_g &= n_i\frac{{\rm d}T^i}{{\rm d}s}\bigg|_{r=R} = n_iT^j\nabla_jT^i\big|_{r=R}\\ &= n_iT^j\left(\partial_jT^i + \Gamma^i_{jl}T^l\right)\big|_{r=R}\\ &= \delta^r_i\delta^j_\phi\delta^l_\phi\Gamma^i_{jl}\left(\frac{{\rm d}\phi}{{\rm d}\lambda}\right)^2\sqrt{\alpha_{rr}}\bigg|_{r=R}\\ &= -\frac{1}{2\sqrt{\alpha_{rr}}}\frac{\partial\ln\left(\alpha_{\phi\phi}\right)}{\partial r}\bigg|_{r=R}\,.\end{aligned} \tag{A10}$$

It should be stressed that the sign of the geodesic curvature changes when either the orientation of the curve or of the surface changes. This corresponds to choosing the inwards pointing normal vector in our computations, $n_i = -\sqrt{\alpha_{rr}}\delta^r_i$. Therefore, the sign of κ_g simply indicates if the acceleration has the same or opposite sense as the chosen normal vector. For our choice it is negative, since circular orbits around compact objects have inwards pointing accelerations (particles are being pulled to the central object) and we chose the outwards pointing normal vector. Therefore only the absolute value of the geodesic curvature is intrinsic to the curve. Our

choice was made in order to recover the asymptotic $1/r$ behaviour, characteristic of Euclidean geometry, for the optical and Jacobi manifolds.

Appendix B. Horizons

BH spacetimes are characterised by the existence of an event horizon. This horizon corresponds to a hypersurface enclosing all points which are not connected to infinity by a timelike path. The notion of infinity may be very subtle when considering generic spacetimes, however for the present study of asymptotically flat spacetimes infinity corresponds precisely to this region where the spacetime is Minkowski. One can equivalently define the event horizon as the boundary of the closure of the causal past of future null infinity.

This definition makes clear that the horizon is generated by null geodesics, hence it is a null hypersurface, Σ. Such hypersurface can be defined as a level curve of a given scalar function $f(x)$, where x denotes the spacetime coordinates. The gradient of this function is normal to Σ, however, since null vectors are orthogonal to themselves the normal vector $\partial_\mu f$ is also tangent to Σ. This means that a hypersurface is generated by null geodesics.

This local definition relies on having an appropriate set of coordinates which is not always guaranteed. However, in this work we will be concerned only with static, asymptotically flat spacetimes which contain event horizons with spherical topology, such that it is possible to find a suitable coordinate system [31, 34]. These metrics possess a Killing vector field, ∂_t, which is asymptotically timelike, and the metric components can be adapted such that $\partial_t g_{\mu\nu} = 0$. On hypersurfaces where $t =$ const. it is possible to choose coordinates (r, θ, ϕ) in which the metric at infinity looks like Minkowski space in spherical polar coordinates. Hypersurfaces with $r =$ const. will then be timelike cylinders with topology $S^2 \times \mathbb{R}$ at $r \to \infty$. For a clever choice of such coordinates we can have the hypersurfaces $r =$ const. remain timelike from infinity all the way up to some r_H, for which the hypersurface is everywhere null. This will represent the event horizon, as timelike paths that cross into $r < r_H$ will never be able to escape back to infinity. Determining the radius at which the hypersurfaces with $r =$ const. become null is simple, the normal one form normal to such hypersurfaces is $\partial_\mu r$, with norm

$$g^{\mu\nu}\partial_\mu r \partial_\nu r = g^{rr}\,. \tag{B1}$$

The null quality of such hypersurface implies that

$$g^{rr}(r_H) = 0\,. \tag{B2}$$

This is the condition that determines the location of the horizon for static, spherically symmetric, and asymptotically flat spacetimes [31, 37].

Another relevant kind of horizons are the Killing horizons, these correspond to hypersurfaces where a given Killing vector field is null. In the previous discussion about event horizons we made no mention of Killing horizons, however the two definitions are intimately connected by several theorems, known as rigidity theorems which state that under which conditions the event horizon is simultaneously a Killing horizon [24, 36]. In fact for *static* spacetimes the event horizon is simultaneously the Killing horizon of the Killing vector field responsible for time translations at infinity [24].

Appendix C. Surface gravity

A null hypersurface $\mathcal{N}$ is a Killing horizon of a Killing vector field ξ if, on $\mathcal{N}$, ξ null. This Killing vector field will also be normal to $\mathcal{N}$, since a null hypersurface cannot have two linearly independent null tangent vectors.

Since at the horizon ξ is null the hypersurface $\mathcal{N}$ is generated by some $k = f\xi$ also, where f is some function and $k^\mu \nabla_\mu k^\nu = 0$.

The surface gravity $\kappa_{\rm surf}$ at such an horizon is defined as

$$\xi^\alpha \nabla_\alpha \xi^\beta = \kappa_{\rm surf} \xi^\beta \,. \tag{C1}$$

Therefore it is a measure on how orbits of ξ^α fail to be geodesics. This is equivalent to:

$$\kappa^2_{\rm surf} = \frac{1}{2} \nabla_\alpha \xi_\beta \nabla^\alpha \xi^\beta \Big|_{\mathcal{N}} \,. \tag{C2}$$

For the present case we will consider the Killing vector field associated with time translations. Since our metric does not explicitly depend on the time coordinate we have that in our coordinate system $\xi^\alpha = \delta^\alpha_t$. Firs we compute $\nabla_\alpha \xi^\beta$

$$\begin{aligned} \nabla_\alpha \xi^\beta &= \partial_\alpha \xi^\beta + \Gamma^\beta_{\alpha\lambda} \xi^\lambda \\ &= \Gamma^\beta_{\alpha t} \\ &= \left(-\frac{\delta^t_\alpha}{2} g^{\beta\sigma} \partial_\sigma g_{tt} + \delta^\beta_t \partial_\alpha \ln \sqrt{g_{tt}} \right) \\ &= -\frac{\delta^t_\alpha \delta^\beta_r}{2} \partial_r g_{tt} + \delta^\beta_t \delta^r_\alpha \partial_r \ln \sqrt{g_{tt}} \,. \end{aligned} \tag{C3}$$

Expanding the expression defining the surface gravity one obtains

$$\kappa^2_{\rm surf} = \frac{1}{4} g^{tt} g_{rr} \left(g^{rr}\right)^2 \left(\partial_r g_{tt}\right)^2 + \frac{g^{rr}}{2} \partial_r g_{tt} \Big|_{\mathcal{N}} \,. \tag{C4}$$

The event horizon is defined by the condition $g^{rr}(r_H) = 0$, hence the surface gravity at the horizon is given by

$$\kappa_{\rm surf} = \lim_{r \to r_H} \frac{1}{2} \sqrt{\frac{h(r)}{f(r)}} f'(r) \,. \tag{C5}$$

It should be noted that if $\mathcal{N}$ is a Killing horizon of ξ it will also be a Killing horizon of some $c\xi$, where c is some constant, with surface gravity $c^2 \kappa_{\rm surf}$. This means that the surface gravity is not an intrinsic property of $\mathcal{N}$, but also depends on the normalisation of ξ. At the horizon ξ is null, hence it does not admit any natural normalisation there. However, for asymptotically flat spacetimes it can be normalised such that:

$$\lim_{r \to \infty} \xi_\beta \xi^\beta = -1 \,. \tag{C6}$$

This fixes $\kappa_{\rm surf}$ up to a sign, which is further fixed by requiring ξ to be future directed.

Expression (C5) reveals a maybe unexpected relation between the surface gravity and the geodesic curvature, this is by no means fortuitous. For spherically symmetric BHs the surface gravity, with the above normalisation, is the acceleration of a static observer at the horizon, as measured by a static observer at infinity. Therefore it is not surprising that the surface gravity corresponds to the geodesic curvature of the circular null generators of the event horizon, apart from a sign coming from a choice of orientation of the curve.

Appendix D. Potential approach

The motion of test particles in general relativity can be obtained from the following effective Lagrangian:

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \sigma\,, \tag{D1}$$

where $\sigma = -1, 0, +1$ for timelike, null and spacelike orbits, respectively. Using the line element (1) this can be cast in the following form

$$g_{rr}\dot{r}^2 = \sigma - \frac{f(r)L^2 - \varepsilon^2}{r^2 f(r)} \equiv V_{\rm eff}\,, \tag{D2}$$

where the effective potential $V_{\rm eff}$ was introduced. Circular geodesics must have

$$\dot{r} = 0 \wedge \ddot{r} = 0 \Rightarrow V_{\rm eff} = 0 \wedge V'_{\rm eff} = 0\,. \tag{D3}$$

For null geodesics ($\sigma = 0$) the circular null geodesics are obtained then by

$$V'_{\rm eff} = \frac{\varepsilon}{f(r)^2}\frac{2f(r) - rf'(r)}{r} = 0\,. \tag{D4}$$

The roots of this equation are same of equation (8), attesting the equivalence between the two approaches.

For timelike geodesics ($\xi = -1$) we will be concerned with the MSCOs. At a linear level the stability of the TCOs is determined by the sign of $V''_{\rm eff}$, hence one must add the condition $V''_{\rm eff}$ to the ones on equation (D3). This yields:

$$V''_{\rm eff} = 2\frac{f(r)\left(3f'(r) + rf''(r)\right) - 2rf'(r)^2}{rf(r)\left(rf'(r) - 2f(r)\right)}\,. \tag{D5}$$

Once more the roots of this expression coincide with the ones of equation (26), meaning that our approach yields the same results as the usual one. A curious note is that this expression diverges at the LRs while equation (26) vanishes.

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