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Thermodynamic stability of ACGL Chern-Simons Black Hole and Optimal Processes

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Abstract

We investigate the thermodynamic stability of the (2+1)-dimensional $\beta^2 = 0$ ACGL Cern-Simons black hole solution of the topologically massive gravitoelectrodynamics. We show that the system is globally unstable against thermal fluctuations, but still able to retain some local regions of stability with respect to certain processes. As a consequence we are able to define an optimal Penrose process and study its properties via the concept of thermodynamic length. We estimate the thermodynamic time and speed of the process with respect to several Hessian metrics on the space of macro states.

1 Introduction

For long time gravity in three dimensions has been unduly neglected. The situation radically changed after the discovery of the famous Banados-Teitelboim-Zanelli (BTZ) black hole solution [1] and the string/holographic revolution [2]. As of today, variety of such toy models have been investigated in great details. Due to their relative simplicity and topological nature an abundance of highly non-trivial structures and phenomena were discovered, many of which find applications in more general theories.

An advantage of three-dimensional gravity is that it can be defined in a multitude of different ways. For example, one can modify the Einstein-Hilbert action by electromagnetic or higher-derivative terms [1, 3, 4]. While another approach adds topological Chern-Simons (CS) terms [5–9]. A third path takes on nontrivial reductions of higher-dimensional models down to $D = 3$. In this way a Gauss-Bonnet-like black hole solutions were recently derived [10–12].

In this paper we focus on the Chern-Simons approach, where a sequence of Chern-Simons black hole solutions were already found [9, 13, 14]. The latter admit highly non-trivial statistical, differential and algebraic structures, which one can use to extract relevant physical data encoded in these systems [15–18]. Most importantly, three-dimensional black holes can be interpreted as the heat bath source for the finite temperature in the holographically dual field theory. For this reason we will be most interested in the thermodynamic aspects of these black holes. Particularly, we conduct our study on the $\beta^2 = 0$ ACGL[1] CS black hole beyond the traditional formalism. Our approach is towards the study of its thermodynamic stability under fluctuations in the framework of thermodynamic information geometry [19–22]. This approach can additionally be utilized to identify most efficient ways to extract energy from the system.

As a first step we implement the classical conditions for thermodynamic equilibrium with respect to the Hessian matrix of the mass-energy potential. If some of these criteria are violated, we say that the black hole is thermodynamically unstable and thus it can radiate. In the second step we utilize the Hessian as a metric on the space of states and use it to define a probabilistic measure of the fluctuations.

Additionally, studying the stability against thermal fluctuations is indicative of the possibility to extract energy from the system. One such process of extraction in black hole physics is the famous Penrose process. One can ask the natural question what is the optimal way or protocol to extract energy with minimal effort. The answer to this question is most conveniently encoded in the properties of the so called thermodynamic length (see [23] and references therein).

The concept of thermodynamic (TD) length is not new, but its rather powerful in thermodynamic optimization and control theory for various systems [24–28]. It quantifies the amount of work or effort required to change the thermodynamic state of a physical system. The definition depends on the notion of metric on the thermodynamic state space. The simplest such metrics are proportional to the Hessian of the given thermodynamic potential and determine the probability for fluctuations between states. Most prominent ones are the Hessian of the energy (Weinhold metric) [19] or the Hessian of the entropy (Ruppeiner metric) [20]. Their importance is evident in fluctuation theory as shown by Ruppiner [20]. The general idea is that the probability of fluctuating from state S to state $S + \Delta S$, where $\Delta S = S - \bar{S}$, is proportional to the thermodynamic distance or length between these two states. It is important to stress, that this is not the traditional geometric distance between these two states on the thermodynamic manifold, but rather the Fisher distance between probability distributions [29, 30].

Given the metric $g_{ab}(x)$ on some metric space with coordinates x one can define the functional of the length in two different forms. The first definition uses the natural coordinates x of the chosen thermodynamic potential, without any explicit parametrization of the path γ in x space:

$$\mathcal{L}[\gamma] = \int\limits_{\gamma} \sqrt{g_{ab}(x) dx^a dx^b}. \tag{1.1}$$

The second definition takes an affine parameter t along the path γ, which yields

$$\mathcal{L}(\tau) = \int_0^{\tau} \sqrt{g_{ab}(x) \dot{x}^a(t) \dot{x}^b(t)}\, dt. \tag{1.2}$$

Here τ is the final value of t. The main difference between the two lengths is that $\mathcal{L}[\gamma]$ does not necessarily give the optimal thermodynamic distance, while $\mathcal{L}(\tau)$, evaluated on

[1] K. Ait Moussa, G. Clement, H. Guennoune and C. Leygnac (ACGL).

a geodesic profile for $x^a(t)$, is optimal. The extremal values of the thermodynamic length represent the minimum amount of work required to take a system from one equilibrium state to another in a reversible manner. Reversible means that the system goes through a sequence of equilibrium states during the transformation, and at each step, it is in thermal equilibrium with its surroundings. This implies that the process is quasi-static and infinitesimally slow, so no entropy is generated along the way. Therefore, the TD length sets a lower bound on dissipation. This is implemented by the simple Cauchy-Schwarz inequality

$$\mathcal{J} = \tau \int_0^\tau g_{ab}(\vec{S})\dot{S}^a\dot{S}^b dt \geq \mathcal{L}^2, \tag{1.3}$$

where the thermodynamic divergence $\mathcal{J}$ measures the efficiency of the quasi-static protocols.

The structure of this work is the following. In Section 2 we briefly present the concepts of global and local classical thermodynamic stability. In Section 3 we introduce the (2+1)-dimensional $\beta^2 = 0$ Cern-Simons black hole solution of TMGE and its properties. In Section 4 we show that the system is globally unstable against thermal fluctuations, but still able to retain local thermodynamic stability with respect to certain processes. In Section 5 we consider optimal processes in view of different thermodynamic representations. Finally, in Section 6 we make a brief discussion on our results.

2 Classical criteria for thermodynamic stability

The study of thermal systems starts by referring to an appropriate thermodynamic potential, which properly reflects the constraints imposed on the system. This is equivalent to determining the suitable thermodynamic representation. In general, one starts with the energy E or the entropy S of the system and derive all other representations via some Legendre transformation. For this purpose one needs to know the fundamental relations $E = E(\vec{E})$ or $S = S(\vec{S})$, where $\vec{E} = (E^1, E^2, ..., E^n)$ and $\vec{S} = (S^1, S^2, ..., S^n)$ are the natural extensive variables of the energy and the entropy, respectively. All other properties, including the intensive ones, follow naturally from these relation. For example, in energy representation, the natural extensive variables $\vec{E}$ and their thermodynamically conjugate intensive ones $\vec{I} = (I_1, I_2, ..., I_n)$, are related by the equations of state:

$$I_a = \left.\frac{\partial E(\vec{E})}{\partial E^a}\right|_{E^1,...,\hat{E}^a,...,E^n}. \tag{2.1}$$

Here the parameters in the subscript are kept fixed except for $\hat{E}^a$. Taking a differential of the energy and using these equations one can write the first law of thermodynamics as a generalized work:

$$dE = I_a dE^a = \vec{I}.d\vec{E}. \tag{2.2}$$

This form of the first law is specifically chosen to represent I_a as generalized thermodynamic forces and E^a as generalized thermodynamic coordinates by analogy of classical mechanics. Similar expressions can be written in entropy or other representations. All of these representations contain full thermodynamic information of the system. The choice of a particular potential only depends on the control parameters and the constraints imposed on the system.

The fundamental relation $E = E(\vec{E})$ is even more important, because it can be used to determine the thermodynamic stability of the system, i.e. for the set of control parameter for which the system is in equilibrium. We say that a thermodynamic system is in equilibrium with its surroundings if the state quantities do not spontaneously change over considerably long period of time. According to the laws of thermodynamics [31–35] the necessary, but not sufficient, conditions for thermodynamic equilibrium between the system and its surroundings can be established by the equalities of the corresponding intensive parameters, $I_a = I_a^*$, of the system I_a and the reservoir I_a^*. These parameters may include temperature, pressure, chemical potentials etc. The conditions can easily be derived by the restriction on the first variation of the internal energy of the system during a virtual process:

$$\delta^{(1)} E(E^a) - I_a^* \delta E^a = \left(\left. \frac{\partial E}{\partial E^a} \right|_{E^1,\ldots,\hat{E}^a,\ldots,E^n} - I_a^* \right) \delta E^a = 0. \tag{2.3}$$

Due to the first law in equilibrium one has (2.1), thus the necessary (but not sufficient) conditions for equilibrium become

$$I_a = I_a^* = const. \tag{2.4}$$

Naturally, the sufficient conditions for global thermodynamic equilibrium, and thus global thermodynamic stability, follow from the second variation of the energy:

$$\delta^{(2)} E = \delta \vec{E}^T . \hat{\mathcal{H}}^{(E)}(\vec{E}) . \delta \vec{E} > 0. \tag{2.5}$$

The positive sign of $\delta^{(2)} E > 0$ reflects the fact that in equilibrium the energy of the system assumes its minimum and should be a strictly convex function. Here $\hat{\mathcal{H}}^{(E)}$ is the symmetric $n \times n$ Hessian matrix of the energy given by

$$\mathcal{H}_{ab}^{(E)}(\vec{E}) = \left. \frac{\partial^2 E(\vec{E})}{\partial E^a \partial E^b} \right|_{E^1,\ldots,\hat{E}^a,\ldots,\hat{E}^b,\ldots,E^n}, \quad a, b = 1, 2, \ldots, n. \tag{2.6}$$

The inequality $\delta^{(2)} E > 0$ defines $\hat{\mathcal{H}}^{(E)}$ as a positive definite quadratic form. This means that for global equilibrium it is sufficient that all eigenvalues $\varepsilon_a > 0$, $a = 1, \ldots, n$, of the Hessian of the energy be strictly positive. The positive definiteness of the Hessian is a consequence of the convexity of the energy potential, which can be expressed in the most general form by the Jensen inequality for convex functions [32]:

$$E\big(E^1 + \Delta E^1, E^2 + \Delta E^2, \ldots\big) + E\big(E^1 - \Delta E^1, E^2 - \Delta E^2, \ldots\big) > 2E(E^1, E^2, \ldots). \tag{2.7}$$

Expanding these strict global conditions in powers of the fluctuations $\Delta E^a = E^a - \bar{E}^a$, one by one or in conjunction to each other, one finds another form of the global criteria for thermodynamic stability, namely the Sylvester criterion for positive definiteness of a quadratic form. In this case, the criterion states that all the principal minors $\Delta_k > 0$ of the Hessian of the energy must be strictly positive[2]. For example, for $n = 2$ system, the Hessian $\hat{\mathcal{H}}$ is 2×2 symmetric matrix and the Sylvester criterion imposes the conditions:

$$\Delta_1 = \mathcal{H}_{11} = \left. \frac{\partial^2 E}{(\partial E^1)^2} \right|_{E^2}, \quad \Delta_2 = \mathcal{H}_{22} = \left. \frac{\partial^2 E}{(\partial E^2)^2} \right|_{E^1}, \quad \Delta_3 = \det \hat{\mathcal{H}} = \begin{vmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{12} & \mathcal{H}_{22} \end{vmatrix} > 0. \tag{2.8}$$

[2]The same criteria were found valid for general black holes by A. K. Sinha [36–38].

Finally, we would like to briefly discuss on the concept of local thermodynamic stability of the system. The system is said to be in a local thermodynamic equilibrium if it can be divided into smaller constituents, which are individually in approximate thermodynamic equilibrium, only small gradients are allowed. The study of local thermodynamic stability is based on the admissible heat capacities. In this case, for a system to be locally stable with respect to a perturbation in a set of parameters, the corresponding heat capacities must be strictly positive. When there is a heat transfer the change in temperature is defined by $đQ = CdT = TdS$, where the extensive quantity C is the total heat capacity of the system. One can also write

$$C = \frac{đQ}{dT} = T\frac{\partial S}{\partial T}, \tag{2.9}$$

and since $đQ$ depends on the nature of the process[3], so does C. In multiparameter systems the heat capacity $C_{x^1,x^2,...,x^{n-1}}$, at fixed set of thermodynamic parameters $(x^1, x^2, ..., x^{n-1})$, is defined by the temperature gradient of the entropy in a certain space of variables $(y^1, y^2, ..., y^n)$, namely [39]:

$$C_{x^1,x^2,...,x^{n-1}}(y^1, y^2, ..., y^n) = T\frac{\partial S}{\partial T}\bigg|_{x^1,x^2,...,x^{n-1}} = T\frac{\{S, x^1, x^2, ..., x^{n-1}\}_{y^1,y^2,...,y^n}}{\{T, x^1, x^2, ..., x^{n-1}\}_{y^1,y^2,...,y^n}}. \tag{2.10}$$

The Nambu brackets { } generalize the Poisson brackets for three or more independent variables (see Appendix C). We say that the parameters $(y^1, ..., y^n)$ define the coordinates on the space of macro states of the system.

Local heat capacities are also important for identifying critical properties and possible phase transitions of the system. For example, the divergences and the zeroes of the heat capacity signal the presence of a phase transition and the breakdown of the equilibrium description of the system. This was first pointed out by Paul Davies for black hole [40], where the energy and the mass of the black hole can be identified as equivalent[4].

3 Chern-Simons black hole solutions of TMGE

The topologically massive gravitoelectrodynamics (TMGE) is a three-dimensional Einstein-Maxwell theory augmented by gravitational and electromagnetic Chern-Simons terms. The action for TMGE consists of several parts:

$$S = S_E + S_M + S_{CSG} + S_{CSE}, \tag{3.1}$$

where S_E is the Einstein action, S_M is the Maxwell part, S_{CSG} and S_{CSE} are the gravitational and the electromagnetic Chern-Simons action terms, respectively. Explicitly, one has

$$S_E = \frac{1}{2\kappa}\int d^3x\sqrt{|g|}(R - 2\Lambda), \tag{3.2}$$

$$S_M = -\frac{1}{2}\int d^3x\sqrt{|g|}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}, \tag{3.3}$$

[3]Hence the inexact differential $đ$.

[4]Unless you consider a pV extended thermodynamics, where the mass becomes the enthalpy of spacetime.

$$S_{CSG} = \frac{1}{4\kappa\mu_G}\int d^3x\, \epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right), \tag{3.4}$$

$$S_{CSE} = \frac{\mu_E}{2}\int d^3x\, \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{3.5}$$

where $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric Levi-Civita symbol, Λ is the cosmological constant, μ_G and $\mu_E > 0$ are the Chern-Simons coupling constants, and $\kappa = 8\pi G$ is the Einstein gravitational constant. Note that in (2+1)-dimensional Einstein gravity, the sign of the gravitational constant κ is not fixed a priori [5, 13, 41]. Therefore, we shall consider both signs to be possible.

Several regular rotating Chern-Simons black hole solutions of TMGE were found in [13]. In this case, the type of the black hole solution is determined by the values of the parameter β^2:

$$\beta^2 = \frac{1}{2(1-\eta)}\left(1 - 2\eta - 4\frac{\Lambda}{\mu_E}\right), \quad \eta \equiv \frac{\mu_E}{2\mu_G}. \tag{3.6}$$

Different, causal and regular Chern-Simons black hole solutions exist for $0 < \beta^2 < 1$, $\beta^2 = 1$ and $\beta^2 = 0$. We focus here on the properties of the $\beta^2 = 0$ Chern-Simons black hole, which thermodynamics admits a natural representation of the mass potential.

The line element of the $\beta^2 = 0$ solution is given by[5] [13]:

$$ds^2 = \left[dt - \left(\rho + \nu + \omega\right)d\varphi\right]^2 - 2\nu\rho d\varphi^2 + \frac{d\rho^2}{2\nu\rho\mu_E^2}, \tag{3.7}$$

where $\nu > 0$ and $\omega \in \mathbb{R}$ are the parameters of the solution. This is a causally regular $(R = \mu_E^2/2)$ black hole, with a single horizon at $\rho = 0$ for $\nu > -2\omega$. Its thermodynamics is represented by

$$M = \frac{2\pi\nu\mu_E}{\kappa}(\lambda - 1), \quad J = \frac{2\pi\nu\mu_E}{\kappa}\big((\lambda-1)\omega - \nu\big), \tag{3.8}$$

$$S = \frac{4\pi^2}{\kappa}\big((1+\lambda)\nu + (1-\lambda)\omega\big), \quad \Omega = \frac{1}{\nu+\omega}, \quad T = \frac{\nu\mu_E}{2\pi(\nu+\omega)}, \tag{3.9}$$

where M is the mass of the black hole, J is the angular momentum, Ω is the angular velocity, and T is the Hawking temperature. In order to avoid extremality one has to assume $M, S, \Omega, T > 0$, which lead to three admissible sectors for ω, ν and the coupling parameter λ. All other sectors of the ACGL CS black hole are presented in Appendix A.

- Sector I: $\kappa > 0$, $\lambda > 1$, $\nu \geq \omega > 0$: nonextremal.
- Sector II: $\kappa < 0$, $\lambda < -\frac{\nu+\omega}{\nu-\omega} < 0$, $0 < \omega < \nu$: nonextremal.
- Sector III: $\kappa < 0$, $\lambda < -\frac{\nu+\omega}{\nu-\omega} < 0$, $\nu > -\omega > 0$: nonextremal.

Finally, solving κ, ω, μ_E and λ in terms of (S, J, Ω, T) and inserting them in M one finds the Smarr relation:

$$M = ST + 2J\Omega. \tag{3.10}$$

In what follows, we will investigate the stability of the ACGL CS black hole against thermal fluctuations with respect to the strict local and global classical criteria for thermodynamic stability.

[5]The authors of [13] consider additional time scale c, which consequently was set to $c = 1$.

4 Thermodynamic stability of the Chern-Simons black hole

This study is entirely within the classical criteria for thermodynamic stability, where quantum fluctuations are neglected. Also extremal cases are discarded due to the third law of thermodynamics[6].

4.1 Thermodynamic stability in Sector I

The strategy is to solve for the free parameters ω and ν in terms of (S, J), keeping the couplings μ_E, λ and κ constant:

$$\nu = \frac{\kappa S + \sqrt{\kappa\left(\frac{32\pi^3 J\lambda}{\mu_E} + \kappa S^2\right)}}{8\pi^2\lambda}, \quad \omega = \frac{(\lambda+1)\sqrt{\kappa\left(\frac{32\pi^3 J\lambda}{\mu_E} + \kappa S^2\right)} - \kappa(\lambda-1)S}{8\pi^2\lambda(\lambda-1)}. \tag{4.1}$$

After inserting these expressions in M, T and Ω one finds the mass-energy representation:

$$M(S,J) = \frac{(\lambda-1)\mu_E}{4\pi\kappa\lambda}\left(\kappa S + \sqrt{\kappa\left(\frac{32\pi^3\lambda}{\mu_E}J + \kappa S^2\right)}\right), \tag{4.2}$$

$$T(S,J) = \left.\frac{\partial M}{\partial S}\right|_J = \frac{(\lambda-1)\mu_E}{4\pi\lambda}\left(1 + \frac{\kappa S}{\sqrt{\kappa\left(\frac{32\pi^3\lambda}{\mu_E}J + \kappa S^2\right)}}\right), \tag{4.3}$$

$$\Omega(S,J) = \left.\frac{\partial M}{\partial J}\right|_S = \frac{4\pi^2(\lambda-1)}{\sqrt{\kappa\left(\frac{32\pi^3\lambda}{\mu_E}J + \kappa S^2\right)}}. \tag{4.4}$$

The fundamental relation $M = M(S,J)$ allows one to test for global thermodynamic stability via the Sylvester criterion for positive definiteness of the Hessian of the mass,

$$\hat{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_{SS} & \mathcal{H}_{SJ} \\ \mathcal{H}_{SJ} & \mathcal{H}_{JJ} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 M}{\partial S^2} & \frac{\partial^2 M}{\partial S \partial J} \\ \frac{\partial^2 M}{\partial J \partial S} & \frac{\partial^2 M}{\partial J^2} \end{pmatrix} = \begin{pmatrix} \frac{8\pi^2\sqrt{\kappa}J(\lambda-1)}{\left(\frac{32\pi^3 J\lambda}{\mu_E}+\kappa S^2\right)^{3/2}} & -\frac{4\pi^2\sqrt{\kappa}(\lambda-1)S}{\left(\frac{32\pi^3 J\lambda}{\mu_E}+\kappa S^2\right)^{3/2}} \\ -\frac{4\pi^2\sqrt{\kappa}(\lambda-1)S}{\left(\frac{32\pi^3 J\lambda}{\mu_E}+\kappa S^2\right)^{3/2}} & -\frac{64\pi^5(\lambda-1)\lambda}{\sqrt{\kappa}\mu_E\left(\frac{32\pi^3 J\lambda}{\mu_E}+\kappa S^2\right)^{3/2}} \end{pmatrix}. \tag{4.5}$$

The stability criteria requires the following inequalities to be satisfied simultaneously:

$$\mathcal{H}_{SS} > 0, \quad \mathcal{H}_{JJ} > 0, \quad \det\hat{\mathcal{H}} > 0. \tag{4.6}$$

However, in this sector, one notes that $\mathcal{H}_{JJ} < 0$ and $\det\hat{\mathcal{H}} < 0$, where

$$\det\hat{\mathcal{H}} = -\frac{16\pi^4(\lambda-1)^2\mu_E^2}{\left(32\pi^3 J\lambda + \kappa\mu_E S^2\right)^2} < 0. \tag{4.7}$$

Therefore, the $\beta^2 = 0$ ACGL CS black hole is globally unstable against thermal fluctuation as far as classical thermodynamics is concerned. This result follows also by the eigenvalue

[6] For a discussion of how the third law can be violated by black holes see [42]. Extremal cases might still be interesting for string theory and other approaches to quantum gravity.

criterion, presented in Appendix B. Under these conditions the system is always radiating through some process.

In the case of local thermodynamic stability with respect to certain processes, one should study the heat capacities of the system in (S, J) space:

$$C_M(S,J) = T\frac{\partial S}{\partial T}\bigg|_M = T\frac{\{S,M\}_{S,J}}{\{T,M\}_{S,J}} = \frac{\sqrt{32\pi^3 J\lambda + \kappa\mu_E S^2}}{\sqrt{\kappa\mu_E}} > 0, \tag{4.8}$$

$$C_\Omega(S,J) = T\frac{\partial S}{\partial T}\bigg|_\Omega = T\frac{\{S,\Omega\}_{S,J}}{\{T,\Omega\}_{S,J}} = C_M + S > 0, \tag{4.9}$$

$$C_J(S,J) = T\frac{\partial S}{\partial T}\bigg|_J = T\frac{\{S,J\}_{S,J}}{\{T,J\}_{S,J}} = \frac{\kappa\mu_E}{32\pi^3 J\lambda} C_\Omega C_M^2 > 0. \tag{4.10}$$

Here, we used the Nambu bracket formalism to calculate the corresponding quantities (see Appendix C). One notes that all of the heat capacities are positive in this sector, hence the CS black hole retains full local stability with respect to either fixed mass, angular velocity or angular momentum, respectively.

One can also look for critical behavior. For example, all heat capacities diverge at $\mu_E \to 0$ or $\lambda \to \infty$. For processes at constant J, the corresponding heat capacity C_J also diverges at $\mu_E \to \infty$ and $J \to 0$. These critical points indicate the break down of the classical thermodynamic description.

4.2 Thermodynamic stability in Sectors II and III

It turns out that Sectors II and III share the same expressions for the state quantities and thus, the same macro properties. Here we set $\kappa = -|\kappa| < 0$ and $\lambda = -|\lambda| < 0$. The solutions for ω and ν in terms of (S, J), are given by

$$\nu = \frac{|\kappa|S + \sqrt{|\kappa|\left(|\kappa|S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)}}{8\pi^2|\lambda|}, \tag{4.11}$$

$$\omega = -\frac{|\kappa|(|\lambda|+1)S - (|\lambda|-1)\sqrt{|\kappa|\left(|\kappa|S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)}}{8\pi^2|\lambda|(|\lambda|+1)}. \tag{4.12}$$

After inserting these expressions in those for M, T and Ω one finds the mass-energy representation:

$$M(S,J) = \frac{(|\lambda|+1)\mu_E}{4\pi|\kappa\lambda|}\left(|\kappa|S + \sqrt{|\kappa|\left(|\kappa|S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)}\right), \tag{4.13}$$

$$T(S,J) = \frac{\partial M}{\partial S}\bigg|_J = \frac{(|\lambda|+1)\mu_E}{4\pi|\lambda|}\left(1 + \frac{|\kappa|S}{\sqrt{|\kappa|\left(|\kappa|S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)}}\right), \tag{4.14}$$

$$\Omega(S,J) = \frac{\partial M}{\partial J}\bigg|_S = \frac{4\pi^2(|\lambda|+1)}{\sqrt{|\kappa|\left(|\kappa|S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)}}. \tag{4.15}$$

The Hessian of the mass is

$$\hat{\mathcal{H}} = \begin{pmatrix} \frac{\partial^2 M}{\partial S^2} & \frac{\partial^2 M}{\partial S \partial J} \\ \frac{\partial^2 M}{\partial J \partial S} & \frac{\partial^2 M}{\partial J^2} \end{pmatrix} = \begin{pmatrix} \frac{8\pi^2 \sqrt{|\kappa|} J(|\lambda|+1)}{\left(|\kappa| S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)^{3/2}} & -\frac{4\pi^2 \sqrt{|\kappa|}(|\lambda|+1) S}{\left(|\kappa| S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)^{3/2}} \\ -\frac{4\pi^2 \sqrt{|\kappa|}(|\lambda|+1) S}{\left(|\kappa| S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)^{3/2}} & -\frac{64\pi^5 (|\lambda|+1)\lambda}{\sqrt{|\kappa|}\mu_E \left(|\kappa| S^2 + \frac{32\pi^3 J|\lambda|}{\mu_E}\right)^{3/2}} \end{pmatrix}. \tag{4.16}$$

One notes that the Sylvester criterion is violated by $\mathcal{H}_{JJ} < 0$ and the determinant

$$\det \hat{\mathcal{H}} = -\frac{16\pi^4 (|\lambda|+1)^2 \mu_E^2}{\left(|\kappa| \mu_E S^2 + 32\pi^3 J |\lambda|\right)^2} < 0, \tag{4.17}$$

hence the ACGL CS black hole is globally unstable against thermal fluctuation in these sectors.

The heat capacities of the system in (S, J) space are correspondingly:

$$C_\Omega(S, J) = C_M + S > 0, \tag{4.18}$$

$$C_M(S, J) = \frac{\sqrt{|\kappa| \mu_E S^2 + 32\pi^3 J|\lambda|}}{\sqrt{|\kappa| \mu_E}} > 0, \tag{4.19}$$

$$C_J(S, J) = \frac{|\kappa| \mu_E}{32\pi^3 J |\lambda|} C_\Omega C_M^2 > 0, \tag{4.20}$$

thus retaining local thermodynamic stability.

5 Optimal processes

The negative definite Hessian of the mass showed that the ACGL CS black hole is globally unstable against thermal fluctuations. Therefore, the system is not in equilibrium and always radiates energy. This leads to certain irreversible changes in the thermodynamic parameters of the black hole. They could be optimized so that to provide an answer to the following question: what kind of processes most efficiently change the state of the system? The answer lies in the extremization of the functional of the thermodynamic length with respect to Hessian metrics.

In what follows we will study optimal processes for the $\beta^2 = 0$ ACGL CS black hole.

5.1 Optimal processes in energy representation

Here we show that even non-positive definite Hessian metrics of the CS black hole may lead to positive definite thermodynamic (TD) lengths. For this purpose, let us identify $x = (S, J)$ and $\hat{g} \equiv \epsilon \hat{\mathcal{H}}$ from (4.5). The choice $\epsilon = \pm 1$ corresponds to elliptic ($R^{(TD)} > 0$) or hyperbolic ($R^{(TD)} < 0$) information geometry with respect to the thermodynamic curvature

$$R^{(TD)} = \frac{4\pi \sqrt{\kappa} \lambda}{\epsilon(\lambda - 1)\sqrt{\mu_E (32\pi^3 J \lambda + \kappa \mu_E S^2)}}. \tag{5.1}$$

We prefer to work in Sector I, but similar analysis can be implemented in the other two sectors. The TD length in energy natural parameters is given by

$$\mathcal{L}[\gamma] = 2\pi i \int\limits_\gamma \frac{\sqrt{2\epsilon \mu_E (\lambda - 1)}}{\kappa \left(\frac{32\pi^3 \lambda}{\mu_E} J + \kappa S^2\right)^{3/4}} \sqrt{\left(\frac{8\pi^3 \lambda}{\mu_E} dJ + \kappa S dS\right) dJ - \kappa J dS^2}. \tag{5.2}$$

Evidently, the properties of the length depends on the process (the path γ).

For example, one can consider an isentropic process $S = const$ from initial state J_0 to final state J, hence

$$\mathcal{L}_S(J_0, J) = \int\limits_{J_0}^{J} \frac{4i\pi^2\sqrt{4\pi\epsilon\lambda(\lambda-1)}}{\kappa\left(\frac{32\pi^3\lambda}{\mu_E}J+\kappa S^2\right)^{3/4}}\, dJ = i\sqrt{\frac{\mu_E(\lambda-1)\epsilon}{\pi\lambda\kappa}}\sqrt[4]{\kappa\left(\frac{32\pi^3\lambda}{\mu_E}J+\kappa S^2\right)}\Bigg|_{J_0}^{J}. \quad (5.3)$$

The TD length $\mathcal{L}_S$ is real and positive for $\epsilon = -1$ and $J_0 > J$ (Penrose process). One notes that $\mathcal{L}_S(J_0, J)$ has only two reference measurements at J_0 and J and may not give the optimal distance between the initial and the final state.

In order to estimate the optimal thermodynamic path of the process, one needs the geodesic profiles of $S(t)$ and $J(t)$. The corresponding geodesic equation,

$$\ddot{x}^\sigma(t) + \Gamma^\sigma_{\mu\nu}(g)\dot{x}^\mu(t)\dot{x}^\nu(t) = 0, \quad (5.4)$$

yields a system of two ordinary nonlinear differential equations:

$$\ddot{S} - \frac{16\pi^3\lambda\dot{J}\dot{S}}{32\pi^3\lambda J+\kappa\mu_E S^2} - \frac{\kappa\mu_E S\dot{S}^2}{32\pi^3\lambda J+\kappa\mu_E S^2} = 0, \quad (5.5)$$

$$\ddot{J} - \frac{2\kappa\mu_E S\dot{J}\dot{S}}{32\pi^3\lambda J+\kappa\mu_E S^2} + \frac{\kappa\mu_E J\dot{S}^2}{32\pi^3\lambda J+\kappa\mu_E S^2} - \frac{24\pi^3\lambda\dot{J}^2}{32\pi^3\lambda J+\kappa\mu_E S^2} = 0. \quad (5.6)$$

For $S = const$ the second equation for $J(t)$ decouples,

$$\ddot{J}(t) - \frac{24\pi^3\lambda}{32\pi^3\lambda J(t)+\kappa\mu_E S^2}\dot{J}^2(t) = 0. \quad (5.7)$$

The general solution is given by

$$J(t) = \frac{4096\pi^{12}c_1^4\lambda^4(t+c_2)^4 - \kappa\mu_E S^2}{32\pi^3\lambda}, \quad (5.8)$$

where $c_{1,2}$ are constants of integration. In this case, one can impose two different initial and boundary conditions. For $J(0) = J_0$ and $\dot{J}(0) = \dot{J}_0$ the solution is given by

$$J(t) = J_0 + \dot{J}_0 t + \frac{12\pi^3\dot{J}_0^2\lambda}{32\pi^3 J_0\lambda+\kappa\mu_E S^2}t^2 + \frac{64\pi^6\dot{J}_0^3\lambda^2}{\left(32\pi^3 J_0\lambda+\kappa\mu_E S^2\right)^2}t^3 + \frac{128\pi^9\dot{J}_0^4\lambda^3}{\left(32\pi^3 J_0\lambda+\kappa\mu_E S^2\right)^3}t^4. \quad (5.9)$$

The TD length becomes

$$\mathcal{L}_S(\tau) = \int_0^\tau \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b}\, dt = \frac{8i\pi^{5/2}\dot{J}_0\sqrt{\epsilon\lambda(\lambda-1)}\sqrt[4]{\mu_E}}{\sqrt[4]{\kappa}\left(32\pi^3 J_0\lambda+\kappa\mu_E S^2\right)^{3/4}}\tau = v\tau, \quad (5.10)$$

where $v = \dot{\mathcal{L}}(\tau)$ is the thermodynamic speed of the process and $\dot{J}_0$ is the initial rate of change of the angular momentum. The length $\mathcal{L}(\tau)$ is positive definite for $\epsilon = -1$ and $\dot{J}_0 = -|\dot{J}_0| < 0$. Equating (5.10) and (5.3) one finds the thermodynamic time of the issentropic Penrose process from J_0 to J:

$$\tau = \frac{32\pi^3 J_0\lambda+\kappa\mu_E S^2 - \sqrt[4]{32\pi^3 J\lambda+\kappa\mu_E S^2}\left(32\pi^3 J_0\lambda+\kappa\mu_E S^2\right)^{3/4}}{8\pi^3|\dot{J}_0|\lambda}. \quad (5.11)$$

Here J is the final value of the angular momentum of the black hole.

Solving for $J(t)$ with boundary conditions $J(0) = J_0$ and $J(\tau) = J = const$, one has implicit equations for c_1 and c_2 as functions of the final time τ:

$$c_1^4 c_2^4 = \frac{32\pi^3 J_0 \lambda + \kappa\mu_E S^2}{4096\pi^{12}\lambda^4}, \quad c_1^4(\tau + c_2)^4 = \frac{32\pi^3 J\lambda + \kappa\mu_E S^2}{4096\pi^{12}\lambda^4}. \tag{5.12}$$

After calculating the TD length for (5.8), and using the relations above, one recovers the original length $\mathcal{L}_S(J_0, J)$ from (5.3).

Let us consider a process from state S_0 to state S at constant angular momentum $J = const$. In this case, Eq. (5.2) becomes

$$\begin{aligned}\mathcal{L}_J(S_0, S) &= -2\pi\sqrt{2J\epsilon(\lambda-1)}\sqrt[4]{\kappa}\int_{S_0}^{S}\left(\frac{\mu_E}{32J\pi^3\lambda + \kappa\mu_E S^2}\right)^{3/4} dS \\ &= -\frac{\mu_E S\sqrt{(\lambda-1)\epsilon}\sqrt[4]{\kappa}}{4\pi\sqrt[4]{2J\pi\lambda^3\mu_E}}\, {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -\frac{\kappa\mu_E S^2}{32J\pi^3\lambda}\right)\Bigg|_{S_0}^{S}.\end{aligned} \tag{5.13}$$

This quantity is real and positive definite for $\epsilon = 1$ and under decreasing entropy $S < S_0$. The latter seems counter intuitive and suggests that this process may not be possible. This is evident from the geodesic equations at $J = const$, where one can show that entropy should also be a constant. Other possibility is for $J = 0$, where the entropy decreases exponentially

$$\ddot{S}(t) - \frac{\dot{S}(t)^2}{S(t)} = 0 \quad \Rightarrow \quad S(t) = S_0 e^{-\frac{|\dot{S}_0|}{S_0}t}, \quad S_0 = S(0), \quad \dot{S}_0 = \dot{S}(0). \tag{5.14}$$

However, in this case, the thermodynamic length becomes zero, $\mathcal{L}_{J\to 0} = 0$, which indicates a phase transition in the system at $J \to 0$. This is evident from $C_J \to \infty$ at $J \to 0$. Therefore, in the energy representation, it is not possible to decrease the entropy of the black hole with an optimal process at $J = const$.

5.2 Optimal processes in entropy representation

In order to study processes with constant mass of the black hole one has to go to entropy representation:

$$S(M, J) = \frac{2\pi\lambda M}{(\lambda-1)\mu_E} - \frac{4\pi^2 J(\lambda-1)}{\kappa M}. \tag{5.15}$$

The Hessian of the entropy is given by

$$\hat{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_{MM} & \mathcal{H}_{MJ} \\ \mathcal{H}_{MJ} & \mathcal{H}_{JJ} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 S}{\partial M^2} & \frac{\partial^2 S}{\partial S \partial J} \\ \frac{\partial^2 S}{\partial J \partial M} & \frac{\partial^2 S}{\partial J^2} \end{pmatrix} = \begin{pmatrix} -\frac{8\pi^2 J(\lambda-1)}{\kappa M^3} & \frac{4\pi^2(\lambda-1)}{\kappa M^2} \\ \frac{4\pi^2(\lambda-1)}{\kappa M^2} & 0 \end{pmatrix}. \tag{5.16}$$

Notably, in entropy representation, the Hessian does not depend on the CS coupling μ_E. The two eigenvalues of the Hessian are given by

$$s_- = \frac{4\pi^2(\lambda-1)\left(J - \sqrt{J^2 + M^2}\right)}{\kappa M^3}, \quad s_+ = \frac{4\pi^2(\lambda-1)\left(J + \sqrt{J^2 + M^2}\right)}{\kappa M^3}. \tag{5.17}$$

They appear with different signs, $s_- < 0$ and $s_+ > 0$, hence CS is globally unstable against thermal fluctuations as expected. This is also confirmed by the Sylvester criterion for negative definite quadratic forms. In this case, entropy is a concave function and the principal minors of the Hessian should satisfy $(-1)^k\Delta_k > 0$, hence $\mathcal{H}_{MM} < 0$, $\mathcal{H}_{JJ} < 0$ and $\det\hat{\mathcal{H}} > 0$. For the CS system this criterion is not satisfied:

$$\mathcal{H}_{MM} = -\frac{8\pi^2 J(\lambda-1)}{\kappa M^3} < 0, \quad \mathcal{H}_{JJ} = 0, \quad \det\hat{\mathcal{H}} = -\frac{16\pi^4(\lambda-1)^2}{\kappa^2 M^4} < 0. \tag{5.18}$$

Considering $\hat{g} = \epsilon\hat{\mathcal{H}}$ (Ruppeiner metric) with $\epsilon = \pm 1$ and $x = (M, J)$, one can calculate the TD length in (M, J) space:

$$\mathcal{L} = 2\pi i \int_\gamma \sqrt{\frac{2\epsilon(\lambda-1)(JdM - MdJ)dM}{\kappa M^3}}. \tag{5.19}$$

It is evident that $\mathcal{L} = 0$ for processes with constant mass $M = const$. A process at constant $J = const$ is defined by

$$\mathcal{L}_J(M_0, M) = -4\pi i \sqrt{\frac{2\epsilon J(\lambda-1)}{\kappa M}}\Bigg|_{M_0}^{M}. \tag{5.20}$$

This quantity is positive definite if $\epsilon = -1$ and the mass of the black hole decreases $M < M_0$.

The geodesic equations for $M(t)$ and $J(t)$ are

$$\ddot{M} = \frac{\dot{M}^2}{M}, \quad \ddot{J} + \frac{J\dot{M}^2}{M^2} - \frac{2\dot{J}\dot{M}}{M} = 0. \tag{5.21}$$

The first equation for the mass decouples and it can be solved to give

$$M(t) = M_0 e^{\frac{\dot{M}_0}{M_0}t}, \quad M_0 = M(0), \quad \dot{M}_0 = \dot{M}(0). \tag{5.22}$$

Consequently the solution for $J(t)$ becomes

$$J(t) = \frac{e^{\frac{\dot{M}_0}{M_0}t}}{M_0}\big((\dot{J}_0 M_0 - J_0\dot{M}_0)t + J_0 M_0\big), \quad J(0) = J_0, \quad \dot{J}(0) = \dot{J}_0. \tag{5.23}$$

One notes that $J = const$ leads to $M = const$. Therefore we cannot decrease the mass of the black hole quasistatically. The process is intrinsically irreversible. Hence, there are no optimal processes in entropy representation along independent directions.

However, one may consider optimal paths between states with both M and J fluctuating. In this case, the total thermodynamic length, between the initial state (M_0, J_0) and state (M, J) at time τ, becomes:

$$\mathcal{L}(\tau) = 2\pi\sqrt{\frac{2\epsilon\dot{M}_0(\lambda-1)(\dot{J}_0 M_0 - J_0\dot{M}_0)}{\kappa M_0^3}}\,\tau = \upsilon\tau. \tag{5.24}$$

It is positive definite assuming several restrictions. For $\epsilon = 1$ one has $\dot{J}_0 M_0 > J_0\dot{M}_0$ and $\dot{M}_0 > 0$, or $\dot{J}_0 M_0 < J_0\dot{M}_0$ and $\dot{M}_0 < 0$. For $\epsilon = -1$ one has $\dot{J}_0 M_0 < J_0\dot{M}_0$ and $\dot{M}_0 > 0$, or $\dot{J}_0 M_0 > J_0\dot{M}_0$ and $\dot{M}_0 < 0$.

In order to extract τ one has to solve (5.21) with boundary conditions:

$$M(t) = M_0\left(\frac{M}{M_0}\right)^{t/\tau}, \quad M(0) = M_0, \quad M(\tau) = M, \tag{5.25}$$

where M_0 is the initial mass and M is the final mass of the black hole. Therefore, the $J(t)$ profile assumes the form:

$$J(t) = \frac{J_0 M(\tau - t) + J M_0 t}{M\tau} \left(\frac{M}{M_0}\right)^{t/\tau}, \quad J(0) = J_0, \quad J(\tau) = J. \tag{5.26}$$

The TD length now evaluates to

$$\mathcal{L}(M_0, J_0; M, J) = 2\pi\sqrt{2\epsilon(\lambda - 1)}\sqrt{\frac{J_0 M - J M_0}{\kappa M_0 M}}\sqrt{\ln\frac{M_0}{M}}. \tag{5.27}$$

It is positive definite for $\epsilon = 1$, together with $M < M_0$ and $JM_0 < J_0 M$. Comparing to (5.24) one finds (together with $M < M_0$, $\dot{M}_0 < 0$, $\dot{J}_0 M_0 < J_0 \dot{M}_0$):

$$\tau = \frac{M_0\sqrt{J_0 M - J M_0}}{\sqrt{M\dot{M}_0}\sqrt{\dot{J}_0 M_0 - J_0 \dot{M}_0}}\sqrt{\ln\frac{M_0}{M}} > 0. \tag{5.28}$$

5.3 Optimal processes in Helmholtz representation

The analysis of optimal processes can be extended to other representations. For example, in (T, J) space, the appropriate potential is the Helmholtz free energy,

$$F(T, J) = M - TS = 4\pi\sqrt{\frac{JT\big((\lambda-1)\mu_E - 2\pi\lambda T\big)}{\kappa\mu_E}}, \quad T < T_c, \tag{5.29}$$

which defines the canonical ensemble. The Helmholtz free energy exists only below the critical temperature defined by

$$T_c = \frac{(\lambda - 1)\mu_E}{2\pi\lambda}. \tag{5.30}$$

The components of the Hessian of the Helmholtz potential yield

$$\hat{\mathcal{H}} = \begin{pmatrix} \frac{\partial^2 F}{\partial T^2} & \frac{\partial^2 F}{\partial T \partial J} \\ \frac{\partial^2 F}{\partial J \partial T} & \frac{\partial^2 F}{\partial J^2} \end{pmatrix} = \begin{pmatrix} \frac{\pi(\lambda-1)^2\sqrt{J\mu_E^3}}{\sqrt{\kappa T^3}((\lambda-1)\mu_E - 2\pi\lambda T)^{3/2}} & -\frac{\pi((\lambda-1)\mu_E - 4\pi\lambda T)}{\sqrt{\kappa J \mu_E T((\lambda-1)\mu_E - 2\pi\lambda T)}} \\ -\frac{\pi((\lambda-1)\mu_E - 4\pi\lambda T)}{\sqrt{\kappa J \mu_E T((\lambda-1)\mu_E - 2\pi\lambda T)}} & -\frac{\pi\sqrt{T((\lambda-1)\mu_E - 2\pi\lambda T)}}{J^{3/2}\sqrt{\kappa\mu_E}} \end{pmatrix}. \tag{5.31}$$

The Hessian metric $\hat{g} = \epsilon\hat{\mathcal{H}}$ is positive definite for $\epsilon = -1$, and negative definite for $\epsilon = 1$.

The analysis of the geodesic equations shows that only an optimal isothermal $T = T_0 = const$ process is possible at a very specific temperature:

$$T_0 = \frac{(\lambda - 1)\mu_E}{4\pi\lambda} = \frac{T_c}{2}. \tag{5.32}$$

In this case, the profile of the angular momentum is

$$\ddot{J}(t) - \frac{3\dot{J}(t)^2}{4J(t)} = 0 \quad \Rightarrow \quad J(t) = \frac{(4J_0 + \dot{J}_0 t)^4}{256 J_0^3}. \tag{5.33}$$

The thermodynamic length at constant temperature is

$$\mathcal{L}_T(\tau) = \frac{|\dot{J}_0|\sqrt[4]{\pi\mu_E}\sqrt{\epsilon(\lambda - 1)}}{\sqrt[4]{2^3 J_0^3 \kappa\lambda}}\tau = v\tau. \tag{5.34}$$

It is positive definite in the elliptic case $\epsilon = 1$. The TD length in terms of Helmholtz natural parameters is

$$\mathcal{L}_T(J_0, J) = -2\sqrt{(\lambda-1)\epsilon}\,\sqrt[4]{\frac{2\pi J \mu_E}{\kappa\lambda}}\Bigg|_{J_0}^{J}. \qquad (5.35)$$

It is positive definite for $J < J_0$ and $\epsilon = 1$. Equating both lengths we find the thermodynamic time and the speed of the optimal Penrose process in Helmholtz representation:

$$\tau = \frac{4\left(J_0 - \sqrt[4]{J J_0^3}\right)}{|\dot{J}_0|}, \quad v = \dot{\mathcal{L}}(\tau) = \frac{|\dot{J}_0|\sqrt[4]{\pi\mu_E}\sqrt{(\lambda-1)}}{\sqrt[4]{2^3 J_0^3 \kappa\lambda}}. \qquad (5.36)$$

It is notable that, in this case, the thermodynamic time τ does not depend on any of the coupling constants μ_E, λ or κ, but only from the initial J_0 and the final value J of the angular momentum.

6 Conclusion

Gravity in three dimensions prove itself as an invaluable source of renormalizable toy models, which provide a rich set of new structures and phenomena. This is especially important in the context of string theory and holography, where many such novel systems turn out to be dual to a lower-dimensional quantum field theories. The presence of black holes in such models provide an external heat bath in the quantum field theory side and thus placing it at finite temperature. This leads to the appearance of criticality and the possibility of thermal phase transitions.

One way a system to be driven to a certain phase transition is through natural fluctuations or external processes conducted within its boundaries. Natural small fluctuations around an equilibrium state are always present, but in many cases they can be neglected, hence the system can be considered stable against thermal fluctuations. If strong fluctuations appear the system is driven out of equilibrium and becomes unstable. The criteria for thermodynamic stability are well established and can be used to identify the parametric regions of stability of any thermodynamic system.

In this context, we investigated the regions of local and global thermodynamic stability of the (2+1)-dimensional $\beta^2 = 0$ ACGL CS black hole solution of TMGE. We found that it is globally unstable against thermal fluctuations in all admissible sectors for its parameters, thus it always radiates energy. However, we showed that local thermodynamic stability with respect to certain processes is still attainable.

The intrinsic global thermodynamic instability of the CS black hole allowed us to investigate the possibility of extracting energy out of the system in an optimal way. For this purpose we utilized the methods of thermodynamic geometry to define a proper Riemannian metrics on the space of macro states of the system. The latter are chosen to be proportional to the Hessians of the mass-energy, the entropy and the Helmholtz free energy of the CS black hole. We were able to define and calculate the corresponding thermodynamic distances (lengths) between macro states in two different ways: one optimal (quasi-equilibrium) along a geodesic curve on the thermodynamic manifold, and another as a non-optimal (non-equilibrium) version. As a consequence we managed to define an optimal Penrose process in every representation and estimate its thermodynamic time and speed. Finally, the natural requirement for positive definiteness of the thermodynamic length also uniquely defines the direction of the Penrose process.

It would be interesting to extend this analysis towards the framework of finite-time thermodynamics [43], where one can consider these systems as black hole heat engines. In this case, one can define a step-wise Carnot-like cycles on the system and calculate their efficiency against an ideal case, hence one can actually find the amount of useful energy, which can be extracted from the system under such processes. A second avenue of investigation is towards geometrothermodynamics [44, 45], where one defines a set of Legendre invariant metrics on the state manifold. The difference from the Hessian approach is that the Legendre invariance preserves the physical properties under the change of representation. In this case, one can compare the efficiency of the Penrose process with respect to different metric measures.

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A Sectors of the ACGL CS black hole

Assuming $M, S, \Omega, T > 0$ and the properties of the metric one can identify several regions for the ω, ν and the coupling parameter λ. For $\kappa > 0$ one finds:

- Sector Ia: $\lambda > 1$, $\nu \geq \omega > 0$: admits positive mass, entropy and temperature.
- Sector Ib: $\lambda > -\frac{\nu+\omega}{\nu-\omega} \geq 1$, $0 < \nu < \omega$: not allowed due to negative entropy.
- Sector Ic: $\lambda < -\frac{\nu+\omega}{\nu-\omega} < \frac{1}{3}$, $0 < \omega < \nu$: not allowed due to negative mass and entropy.
- Sector Id: $\frac{1}{3} \leq \lambda < 1$: not allowed due to negative mass.
- Sector Ie: $\lambda < -\frac{\nu+\omega}{\nu-\omega} < \frac{1}{3}$, $\omega < -\frac{\nu}{2} < 0$: not allowed due to negative mass and entropy.

For $\kappa < 0$ one finds:

- Sector IIa: $\lambda > 1$, $\nu \geq \omega > 0$: not allowed due to negative mass and entropy.
- Sector IIb: $\lambda > -\frac{\nu+\omega}{\nu-\omega} \geq 1$, $0 < \nu < \omega$: not allowed due to negative mass.
- Sector IIc: $\lambda < -\frac{\nu+\omega}{\nu-\omega} < \frac{1}{3}$, $0 < \omega < \nu$: allowed.
- Sector IId: $\frac{1}{3} \leq \lambda < 1$: not allowed due to negative entropy.

- Sector IIe: $\lambda < -\frac{\nu+\omega}{\nu-\omega} < \frac{1}{3}$, $\omega < -\frac{\nu}{2} < 0$: allowed if $\nu > -\omega$ for $T > 0$, thus $\lambda < -\frac{\nu+\omega}{\nu-\omega} < 0$.

One notes that only three sectors are allowed, namely Ia, IIc and IId. The latter we called Sector I, II and III, respectively.

B Eigenvalues of the Hessian

The eigenvalues of the Hessian of the mass (4.5) in Sector I satisfy the following quadratic equation

$$\varepsilon^2 - \varepsilon\left(\mathcal{H}_{JJ} + \mathcal{H}_{SS}\right) - \mathcal{H}_{SJ}^2 + \mathcal{H}_{JJ}\mathcal{H}_{SS} = 0. \tag{B.1}$$

Its explicit solutions are given by

$$\varepsilon_\pm = \frac{4\pi^2(\lambda-1)\left(\kappa J\mu_E - 8\pi^3\lambda \pm \sqrt{\kappa^2\mu_E^2\left(J^2+S^2\right) + 16\pi^3\kappa J\lambda\mu_E + 64\pi^6\lambda^2}\right)}{\mu_E\sqrt{\kappa}\left(\frac{32\pi^3 J\lambda}{\mu_E} + \kappa S^2\right)^{3/2}}. \tag{B.2}$$

A closer study of their signs suggests that $\varepsilon_+ > 0$ and $\varepsilon_- < 0$ in Sector I. Therefore, confirming the result from the Sylvester criterion.

In Sectors II and III one finds:

$$\varepsilon_\pm = \frac{4\pi^2(|\lambda|+1)\left(8\pi^3|\lambda| - |\kappa|J\mu_E \pm \sqrt{\kappa^2\mu_E^2\left(J^2+S^2\right) + 16\pi^3|\kappa\lambda|J\mu_E + 64\pi^6\lambda^2}\right)}{\mu_E\sqrt{|\kappa|}\left(\frac{32\pi^3 J|\lambda|}{\mu_E} + |\kappa|S^2\right)^{3/2}}. \tag{B.3}$$

A closer look at their signs suggests that $\varepsilon_+ < 0$ and $\varepsilon_- > 0$ in these sectors. Therefore, confirming the result from the Sylvester criterion.

C Nambu brackets

The Nambu brackets generalizes the Poisson brackets for three or more variables. For example, for $n = 2$ one has the traditional Poisson brackets:

$$\{f,x\}_{u,v} = \begin{vmatrix} \frac{\partial f}{\partial u}\big|_v & \frac{\partial f}{\partial v}\big|_u \\ \frac{\partial x}{\partial u}\big|_v & \frac{\partial x}{\partial v}\big|_u \end{vmatrix} = \frac{\partial f}{\partial u}\bigg|_v \frac{\partial x}{\partial v}\bigg|_u - \frac{\partial f}{\partial v}\bigg|_u \frac{\partial x}{\partial u}\bigg|_v. \tag{C.1}$$

For $n = 3$ one has:

$$\{f,x,y\}_{u,v,w} = \begin{vmatrix} \frac{\partial f}{\partial u}\big|_{v,w} & \frac{\partial f}{\partial v}\big|_{u,w} & \frac{\partial f}{\partial w}\big|_{u,v} \\ \frac{\partial x}{\partial u}\big|_{v,w} & \frac{\partial x}{\partial v}\big|_{u,w} & \frac{\partial x}{\partial w}\big|_{u,v} \\ \frac{\partial y}{\partial u}\big|_{v,w} & \frac{\partial y}{\partial v}\big|_{u,w} & \frac{\partial y}{\partial w}\big|_{u,v} \end{vmatrix}, \tag{C.2}$$

and so on. In general, Nambu brackets account for the determinant of the Jacobian when working in certain coordinates, i.e.

$$\{f, x^1, ..., x^{n-1}\}_{y^1,y^2,...,y^n} = \begin{vmatrix} \frac{\partial f}{\partial y^1}\Big|_{y^2,y^3,...,y^n} & \frac{\partial f}{\partial y^2}\Big|_{y^1,y^3,...,y^n} & \cdots & \frac{\partial f}{\partial y^n}\Big|_{y^1,y^2,...,\hat{y}^n} \\ \frac{\partial x^1}{\partial y^1}\Big|_{y^2,y^3,...,y^n} & \frac{\partial x^1}{\partial y^2}\Big|_{y^1,y^3,...,y^n} & \cdots & \frac{\partial x^1}{\partial y^n}\Big|_{y^1,y^2,...,\hat{y}^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x^{n-1}}{\partial y^1}\Big|_{y^2,y^3,...,y^n} & \frac{\partial x^{n-1}}{\partial y^2}\Big|_{y^1,y^3,...,y^n} & \cdots & \frac{\partial x^{n-1}}{\partial y^n}\Big|_{y^1,y^2,...,\hat{y}^n} \end{vmatrix}. \tag{C.3}$$

Let us show how this works for C_M from (4.8):

$$\{S, M\}_{S,J} = \begin{vmatrix} \frac{\partial S}{\partial S} & \frac{\partial S}{\partial J} \\ \frac{\partial M}{\partial S} & \frac{\partial M}{\partial J} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{\partial M}{\partial S} & \frac{\partial M}{\partial J} \end{vmatrix} = \frac{\partial M}{\partial J}\Big|_S = \frac{4\pi^2(\lambda - 1)}{\sqrt{\kappa\left(\frac{32\pi^3 J\lambda}{\mu_E} + \kappa S^2\right)}}, \tag{C.4}$$

$$\begin{aligned} \{T, M\}_{S,J} &= \begin{vmatrix} \frac{\partial T}{\partial S} & \frac{\partial T}{\partial J} \\ \frac{\partial M}{\partial S} & \frac{\partial M}{\partial J} \end{vmatrix} = \frac{\partial T}{\partial S}\Big|_J \frac{\partial M}{\partial J}\Big|_S - \frac{\partial T}{\partial J}\Big|_S \frac{\partial M}{\partial S}\Big|_J \\ &= \frac{\pi(\lambda - 1)^2 \mu_E^{5/2}\left(\sqrt{\kappa\left(\frac{32\pi^3 J\lambda}{\mu_E} + \kappa S^2\right)} + \kappa S\right)}{\sqrt{\kappa}\lambda\left(32\pi^3 J\lambda + \kappa\mu_E S^2\right)^{3/2}}. \end{aligned} \tag{C.5}$$

Taking into account the expression (4.3) for the temperature T, one finds

$$C_M(S, J) = T\frac{\{S, M\}_{S,J}}{\{T, M\}_{S,J}} = \frac{\sqrt{32\pi^3 J\lambda + \kappa\mu_E S^2}}{\sqrt{\kappa\mu_E}}. \tag{C.6}$$

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