

# Riemannian geometric aspects of Penrose-type inequalities

Sumio Yamada <sup>2(a)</sup>

<sup>(a)</sup>*Mathematical Institute, Tohoku University, Sendai 980-8578, Japan*

## Abstract

We investigate a series of inequalities which gives bounds on a set of asymptotically defined invariant associated with blackhole geometries.

## 1 Introduction

We suppose that  $\mathcal{S}^4$  is a smooth manifold with a Lorentz signature metric  $g$ , representing a space-time.

Matter in relativity is represented by tensor fields over  $\mathcal{S}$ , and the spacetime metric  $g$  represents the gravitational field. The matter fields evolve from initial data via their equations of motion, and the gravitational field evolves via the Einstein equation

$$\text{Ric}(g) - \frac{1}{2}R_g = 8\pi T$$

where  $\text{Ric}$  denotes the Ricci curvature and  $R = \text{Tr}_g(\text{Ric}(g))$  is the scalar curvature. When there are no matter fields present the right hand side  $T$  is zero, and the equation reduces to  $\text{Ric}(g) = 0$ . These equations are called the vacuum Einstein equation.

The solution to the Einstein equation is determined by initial data given on a spacelike hypersurface  $M^3$  in  $\mathcal{S}^4$ . The fields at a point  $x_0$  in the space-time  $\mathcal{S}^4$  are determined by initial data in the part of  $M^3$  which lies in the past of  $x_0$ . The initial data for  $g$  are the induced (Riemannian) metric  $g|_M$ , also denoted  $g$  for simplicity, and the second fundamental form  $p$ . These play the role of the initial position and velocity for the gravitational field. An initial data set is a triple  $(M, g, p)$ . With this set of data, solving the Einstein equation amounts to finding the integral curve with the initial position and the initial velocity. That  $p$  describes how  $M^3$  sits inside  $\mathcal{S}$ , and hence it cannot be chosen arbitrarily.

Using the Einstein equations together with the Gauss and Codazzi equations, the constraint equations  $g$  and  $p$  need to satisfy are

$$\mu = \frac{1}{16\pi}(R_M + \text{Tr}_g(P)^2 - \|p\|^2), \quad J_i = 8\pi \sum_{j=1}^3 \nabla^j \pi_{ij}$$

where  $\mu$  is the local energy density,  $J$  is the local current energy with respect to an adapted frame  $\{e_i\}_{i=0}^3$  to  $M^3 \subset \mathcal{S}^4$  with  $e_1, e_2, e_3$  tangential to  $M$ . where  $\pi_{ij} = p_{ij} - \text{Tr}_g(p)g_{ij}$ , and  $\nabla$  is the covariant derivative with respect to  $g$ . In case there is no matter present, the vacuum constraint equations become

$$R_M + \text{Tr}_g(p)^2 - \|p\|^2 = 0, \quad \nabla^j \pi_{ij} = 0$$

for  $i = 1, 2, 3$  where  $R_M$  is the scalar curvature of  $M^3$ . For spacetimes with matter, the stress-energy tensor is normally required to satisfy the dominant energy condition

$$\mu \geq \|J\|^2$$

The most natural boundary condition for the Einstein equations is the condition of asymptotic flatness [1, 21]. This boundary condition describes isolated systems which are the analogues of finite mass distributions in Newtonian gravity. The requirement is that the initial manifold  $M$  outside a compact set be diffeomorphic to the exterior of a ball in  $\mathbb{R}^3$  and that there be coordinates  $x = (x^1, x^2, x^3)$  in which  $g$  and  $p$  have appropriate fall of

$$g_{ij} = \delta_{ij} + O_2(|x|^{-1}), \quad p_{ij} = O_1(|x|^{-2})$$

<sup>2</sup>Email address: yamada@math.tohoku.ac.jp

where  $O_i(|x|^{-j})$  means that the function falls off like  $C/|x|^j$  and its  $i$ -th derivative falls off like  $C'/|x|^{j+i}$ . The following are two basic examples of asymptotically flat spacetimes:

- The Minkowski space-time is  $\mathbb{R}^4$  with the flat metric  $g = -dx_0^2 + \sum_{i=1}^3 dx_i^2$ . It is the space-time of special relativity.
- The Schwarzschild space-time with mass  $m > 0$  is  $\mathbb{R} \times (2m, \infty) \times S^2$  with metric

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2g_0$$

where  $g_0$  is the standard metric on the unit sphere  $S^2$ ,  $r = \sqrt{\sum_{i=1}^3 x_i^2}$ . It is a vacuum solution describing the exterior of a static black hole with mass  $m$ . It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

For general asymptotically flat initial data sets, there is a notion of total energy-momentum which was defined by Arnowitt, Deser, and Misner (ADM) [1]. There is no energy density for the gravitational field so these quantities are computed in terms of the asymptotic behavior of  $g$  and  $p$ . For these definitions we fix asymptotically flat coordinates  $x = (x^1, \dots, x^n)$ . Then the ADM-mass  $M$  is defined as

$$M = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{|x|=r} \sum_{i,j=1}^n (g_{ij,i} - g_{ii,j}) \nu_0^j d\sigma_0$$

where  $\nu_0$  is the unit outer normal vector to the  $(n-1)$ -dimensional standard Euclidean unit sphere  $S^{n-1}$ ,  $\sigma_0$  is the area element of  $S^{n-1}$ , and  $\omega_{n-1}$  is the volume of that sphere.

## 2 Variational Formulations of Mass Inequalities

In the above setting, there have been several important results in Riemannian geometric contexts. The first is the so-called positive mass/energy theorem.

**The Positive Mass Theorem** (Schoen-Yau 1979 [19, 20], Witten 1981 [23])

*Among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations, flat Euclidean  $n$ -space is the unique minimizer of the total ADM-mass. Namely the following inequality holds*

$$m \geq 0.$$

We make the following remarks regarding the history.

- For  $3 \leq n \leq 7$  this theorem was proven by R. Schoen and S.-T. Yau around 1980 [19, 20]. The proof uses variational properties of minimal hypersurfaces.
- For spin manifolds of any dimension the theorem follows from an argument of E. Witten in the early 1980's [23]. The proof uses the Dirac operator.
- For  $n = 3$ , a new proof was given in 2000 by G. Huisken and T. Ilmanen [13]. It uses the inverse mean curvature flow, making rigorous an argument proposed by R. Geroch [10]. We will come back to this argument.

Based on the conjectured picture of gravitational collapse to a black hole, Penrose [17] proposed a sharp strengthening of the positive mass/energy theorem, which is the so-called Riemannian Penrose inequality. In what follows, we restrict our attention to the case  $n = 3$ .

**The Riemannian Penrose Inequality** (Huisken-Ilmanen 2001 [13], Bray 2001 [2])

*Among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations with*

an outermost minimal surface  $\Sigma$  of area  $A$ , the Schwarzschild slice is the unique minimizer of the total mass. Namely we always have the inequality

$$m \geq \frac{1}{2}R$$

where  $R$  is the area radius  $\sqrt{\frac{A}{4\pi}}$ .

Recall that the event horizon and the apparent horizon coincided in time-symmetric settings, and the horizon consist of a collection of minimal spheres. This follows from the stability of the minimal surfaces, namely the second variations of the area functional of the horizon are positive definite. We also make the following remarks concerning the Penrose inequality.

- Around 2000 this was proven by G. Huisken and T. Ilmanen using the inverse mean curvature flow. This made rigorous a proof originally proposed by P. Jang and R. Wald [14].
- Shortly after the Huisken/Ilmanen proof [13], a very different proof was given by H. Bray [2] which extends the result to the case of a possibly disconnected boundary surface. The inverse mean curvature flow approach does not work in that case, namely, in the Huisken/Ilmanen argument, what they showed is the ADM mass is bounded below by the area radius of the largest connected component of the outermost minimal surfaces. Bray's proof also involved a flow, but it was a conformal flow of the metric which deformed the given metric to a Schwarzschild metric.

### 3 Natural Questions

We ask the following set of questions, naturally led by the variational interpretation of the preceding two inequalities. Those inequalities share the property that in each case, the equality is realized by one of the exact solutions of the Einstein equation. Incidentally a good exposition on these subjects, whose scope is much wider than the current article is Marc Mars's article [16].

#### 3.1 The Angular Momentum Case

**Question:** *Is the Kerr slice the unique minimizer of the total mass among all asymptotically flat axisymmetric maximal gauge initial data sets for the Einstein-Vacuum Equations with an outermost minimal surface  $\Sigma$  of area  $A$  and (Komar) angular momentum  $J$ ? Namely does the following inequality always hold?*

$$m \geq \frac{1}{2} \left( R^2 + \frac{4J^2}{R^2} \right)^{1/2}$$

Here the Komar angular momentum  $J(S)$  for a surface  $S$  is defined for axisymmetric data as

$$J(S) = \frac{1}{8\pi} \int_S p_{ij} X^i n_S^j d\sigma_S$$

where  $S$  is a sphere homological to the two-sphere at the infinity,  $X$  is the Killing vector field which generates the axisymmetry,  $n_S$  is the unit normal vector to the surface  $S$ , and  $d\sigma_S$  is the area element of  $S$ . This quantity is known to be a homological quantity, and is equal to the total angular momentum  $J$  in this setting. The equality is assumed for the Kerr slice.

This question remains open. Technically this is due to the fact that the setting is no longer Riemannian, as the effect of Killing vector field  $X$  and the extrinsic curvature  $p$  have to be taken into account.

#### 3.2 The Charged Case I

**Question (cf. Jang 1979 [15]):** *Is the Reissner-Nordstrom slice the unique minimizer of the total mass among all asymptotically flat time-symmetric initial data sets for the Einstein-Maxwell Equations with*

an outermost minimal surface  $\Sigma$  of area  $A$  and charge  $Q$ ? Namely does the following inequality always hold?

$$m \geq \frac{1}{2} \left( R + \frac{Q^2}{R} \right)$$

where the total charge  $Q$  is defined by  $Q(S) = \int_{S^2_\infty} E_i n^i d\sigma_S$  for  $S$  a sphere homological to the sphere at the asymptotically flat end. This represents the sum of the charges  $\{Q_i\}$  each trapped inside the horizons  $\Sigma_i$ .

Recall that the Time-Symmetric Einstein-Maxwell Constraints

$$S_g = 2(|E|_g^2 + |B|_g^2), \quad \operatorname{div}_g E = \operatorname{div}_g B = 0, \quad E \times_g B = 0$$

as well as the nonexistence of magnetic monopole  $\int_S B_i n^i d\sigma_S = 0$  for a sphere  $S$  homological to the sphere at the asymptotically flat end.

### 3.3 The Charged Case II

**Question (Gibbons 1984 [12]):** Is the Majumdar-Papapetrou slice with the horizon consisting of two components of opposite charges the unique minimizer of the total mass among all asymptotically flat time-symmetric initial data sets for the Einstein-Maxwell Equations with an outermost minimal surface  $\Sigma = \cup \Sigma_i$  of area  $A = \sum A_i$  and charges  $\{Q_i\}$ ? Namely does the following inequality always hold?

$$m \geq \sum_i \frac{1}{2} \left( R_i + \frac{Q_i^2}{R_i} \right)$$

## 4 Answers to Questions

In this section, we present answers to each of the two questions concerning the charged cases I and II.

### 4.1 Charged Case I

For the Penrose-type inequality  $m \geq \frac{1}{2} \left( R + \frac{Q^2}{R} \right)$ , the following is known:

- **Yes**, provided  $\Sigma$  is connected (Jang 1979 [15], Huisken-Ilmanen 2001 [13]).
- **No**, in general (Weinstein-Yamada 2004 [22]).

It is interesting to note that the topological conditions affect the Penrose-type inequality in an essential manner. We will outline the arguments for the two cases now.

#### 4.1.1 Jang/Huisken-Ilmanen argument for $\Sigma$ connected

As was done for the proof of the Riemannian Penrose inequality, flow the connected component  $\Sigma$  of the horizon of the biggest area by Inverse Mean Curvature;  $\frac{\partial x}{\partial t} = \frac{1}{H} \nu$ . Then use the Geroch Monotonicity (sharpened by Jang to incorporate the electric field  $E$ ) which says

$$\frac{dm_H(\Sigma(t))}{dt} \geq \frac{R}{32\pi} \int_{\Sigma(t)} S_g \geq 0$$

where  $m_H(\Sigma) = \frac{R}{2} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \right)$  is the Hawking Mass and the scalar curvature of  $(M^3, g)$  has the following expression  $S_g = 2(|E|^2 + |B|^2)$  from the Einstein constraint equation. The Hawking mass converges to ADM mass as  $\Sigma \rightarrow S^2_\infty$ , while the Cauchy-Schwarz inequality and  $Q = \int_{S^2_\infty} E_i n^i d\sigma$  gives the expression for the right hand side of the inequality  $m \geq \frac{1}{2} \left( R + \frac{Q^2}{R} \right)$ . In taking the limit of the evolution  $\Sigma \rightarrow S^2_\infty$  via the inverse mean curvature flow, and jump over discontinuities with Huisken-Ilmanen's weak flow as necessary. By backtracking the argument, one can check that the equality holds if and only if the initial data set is that of Reissner-Nordstrom with the appropriate mass and charge.

### 4.1.2 Weinstein-Yamada argument for $\Sigma$ of two components

We recall the so-called Majumdar-Papapetrou Metric [6], or sometimes better known as the extremal Reissner-Nordstrom metric in the sense that the usual inequality  $M \geq |Q|$  is saturated  $M = |Q|$ . We are here concerned with the case when  $\Sigma$  is of two components, and the charge is split evenly and of the same sign causing the repulsive force. To be precise, the underlying differential manifold is  $\mathbb{R}^3 \setminus \{p_1, p_2\}$  with  $p_1 = (0, 0, 1), p_2 = (0, 0, -1)$  on which we have the following gravitational and electric fields

- $g_{ij} = u^4 \delta_{ij}, u = \left(1 + \frac{\mu}{r_1} + \frac{\mu}{r_2}\right)^{1/2}$
- $E_i = 2\nabla_i \log u, B_i = 0$

where  $r_1, r_2$  are the distances to  $p_1$  and  $p_2$ . Then the scalar curvature is  $S_g = 2\|E\|_g^2$

The total mass of this initial data is  $m = 2\mu$  and the total asymptotic area of  $\Sigma$  is  $A = 8\pi\mu^2$ , while the total charge is  $Q = 2\mu$ . This gives a counterexample to the inequality  $m \geq \frac{1}{2}\left(R + \frac{Q^2}{R}\right)$  as

$$m - \frac{1}{2}\left(R + \frac{Q^2}{R}\right) = \mu\left(2 - \frac{3}{\sqrt{2}}\right) < 0.$$

However, we are not quite done yet, for this manifold  $(\mathbb{R}^3 \setminus \{p_1, p_2\}, g, E, B = 0)$  is not asymptotically flat. It has two ends, one of which is asymptotically flat, but the Majumdar-Papapetrou slice  $(\mathbb{R}^3 \setminus \{p_1, p_2\}, g)$  has no horizon, as there are two infinite cylinders centered at  $p_1$  and  $p_2$ . In [22], two copies of Majumdar-Papapetrou are truncated at their necks, and glued to rectify those features.

We note that the so-called Cosmic Censorship proposed by Penrose [17] is intact. The Cosmic Censorship says that the evolution of space-time driven by the Einstein equation would force singularities to be hidden behind the event horizons which is the surface of the blackhole regions, so that there is no naked singularities to the observer in this part of the space-time. As those blackholes would coalesce together by the gravitational effect, the time-asymptotic behavior of the universe would settle down to the Kerr-Newman solution.

Setting aside the angular momentum, Jang [15] has adapted the Cosmic Censorship statement to the Penrose-type inequality. First he notes that

$$m \geq \frac{1}{2}\left(R + \frac{Q^2}{R}\right)$$

is equivalent to the following pair of inequalities

$$m - \sqrt{m^2 - Q^2} \leq R \leq m + \sqrt{m^2 - Q^2}$$

Only the upper bound on  $R$  follows from Cosmic Censorship using Penrose's heuristic argument, as the coalescing process would cause the increase of the size of the horizon, whose upper bound is set by the asymptotic invariants from the Hamiltonian formulation by Arnowitt, Deser, and Misner [1].

Note that our counter-example violates the lower bound.

## 4.2 Charged Case II

For the Penrose-type inequality

$$m \geq \sum_i \frac{1}{2}\left(R_i + \frac{Q_i^2}{R_i}\right)$$

the following is known: the so-called Brill-Lindquist initial data set [3] satisfies  $m < \frac{1}{2}\sum_{i=1}^2 R_i$  (Dain-Weinstein-Yamada 2010 [9]), which violates the proposed inequality when  $Q_i = 0$ .

By setting quasi-local mass-like quantities  $m_i$ , the same inequality  $m < \frac{1}{2}\sum_{i=1}^2 R_i$  functions as a counterexample for each of the following cases.

Namely by defining

$$m_i = \frac{1}{2}\left(R_i + \frac{Q_i^2}{R_i}\right), \quad \text{or} \quad \sqrt{\frac{1}{4}R_i^2 + \frac{J_i^2}{R_i^2}}$$

the inequalities

$$m \geq \sum m_i.$$

do not hold, as they reduce to  $m \geq \frac{1}{2} \sum R_i$  in vacuum.

We will explain the counterexample to  $m \geq \frac{1}{2} \sum_{i=1}^2 R_i$  now.

#### 4.2.1 Brill-Lindquist initial data

The so-called Brill-Lindquist initial data [3] has been known for many years, which models a time-symmetric initial data set  $(\mathbb{R}^3 \setminus \{p_1, p_2\}, g)$  with  $p_1 = (0, 0, 1), p_2 = (0, 0, -1)$ , which has three asymptotically flat ends, which are linked by two Einstein-Rosen bridges. To be precise,

$$\bullet \quad g_{ij} = u^4 \delta_{ij}, \quad u = \left(1 + \frac{\mu}{2r_1} + \frac{\mu}{2r_2}\right)$$

where  $r_1, r_2$  are the distances to the punctures  $p_1$  and  $p_2$  in  $\mathbb{R}^3$ . The metric has zero scalar curvature, as the function  $u$  is harmonic.

Seen from afar, the metric  $g$  looks like a Schwarzschild metric of mass  $\mu + \mu$ . Hence the total mass is  $m = 2\mu$ . By a simple comparison argument, the area radii are  $R_1 = R_2 > 2\mu$  (Gibbons 1972 [11]). Once we establish that the outermost horizon is not connected for  $\mu$  sufficiently small, we have

$$m - \frac{1}{2} \sum_i R_i < 0.$$

To check that for  $\mu$  sufficiently small, the horizon cannot be a single minimal sphere, we use the following properties of the stable minimal surfaces in three dimensional manifolds of bounded geometry.

If the outermost horizon is connected, then there exists  $x \in \Sigma \cap \{z = 0\}$ . Then an estimate by Schoen [18] says there exists some  $\varepsilon > 0$  such that

$$\sup_{\Sigma_1 \cap B(x, \varepsilon)} |A| \leq C \int_{\Sigma_1 \cap B(x, 2\varepsilon)} |A|^2 dx$$

where  $C$  is independent of  $\mu \ll 1$ . Note that the right hand side is  $o(\mu)$  as  $\mu \rightarrow 0$  as the space becomes increasingly flatter. This means that the piece of the surface  $\Sigma_1 = \Sigma \cap B(x, r)$  with  $r < 1$  becomes a graph over a two-dimensional disc centered at  $x$ , as the pointwise small extrinsic curvature implies the bound of turning of the normal vector to the surface. This says  $|\Sigma_1| > C$ , with  $C$  independent of  $\mu$ .

On the other hand, Bray's Penrose inequality [2]

$$2\mu \geq \sqrt{\frac{|\Sigma|}{16\pi}}$$

says  $|\Sigma_1| \rightarrow 0$  as  $\mu \rightarrow 0$ , a contradiction.

## 5 Concluding Remarks

We have seen the following set of results concerning the Penrose-type inequalities involving the Einstein-Maxwell Constraints:

- $\Sigma$  connected  $\Rightarrow m \geq \frac{1}{2} \left(R + \frac{Q^2}{R}\right)$  always holds by inverse mean curvature flow and the monotonicity of Hawking mass.
- $\Sigma$  not connected  $\Rightarrow m - \frac{1}{2} \left(R + \frac{Q^2}{R}\right) < 0$  is possible, in fact achieved by an asymptotic time-symmetric initial data set obtained by glueing and perturbing a pair of the Majumdar-Papapertou data.

- $\Sigma$  not connected  $\Rightarrow m - \frac{1}{2} \sum_i \left( R_i + \frac{Q_i^2}{R_i} \right) < 0$  is possible, and achieved by the Brill-Lindquist initial data when the two neck are sufficiently apart compared to the assigned masses.

This would lead to the following open problems

**Open Problem 1:** The Charged Case for the asymptotically flat time-symmetric initial data set  $(M^3, g, p, E)$  without any topological assumption.

$$R \leq m + \sqrt{m^2 - Q^2}$$

Note this is the part consistent with the Cosmic Censorship, of the two inequalities proposed by Jang. The variational interpretation is that the Reissner-Nordstrom slice *maximizes* the area of the horizon among all asymptotically flat time-symmetric initial data satisfying the Einstein-Maxwell constraint with ADM mass  $m$  and the total charge  $Q$ .

**Open Problem 2:** For the axisymmetric maximal ( $\text{Tr}_g p = 0$ ) initial data set whose the horizon  $\Sigma$  is connected,

$$m \geq \frac{1}{2} \left( R^2 + \frac{4J^2}{R^2} \right)^{1/2}.$$

Note that the equality here is realized by the Kerr slice. Note the Einstein-Maxwell setting, the analogous result was obtained by inverse mean curvature flow. However, the difficulty here is the fact that the time-symmetry is no longer assumed, and there is a nontrivial Killing vector field around the symmetry axis.

We also remark that S. Dain recently obtained a series of results [8] which address the inequality

$$m \geq \sqrt{|J|}$$

where the equality is achieved by the extremal Kerr initial data set, using the technique of harmonic maps with singularities.

**Open Problem 3:** When one relaxes the topological type of the horizon, we ask the following stronger inequality than Open Problem 3 for axisymmetric maximal ( $\text{Tr}_g p = 0$ ) initial data sets

$$\frac{R^2}{2} \leq m^2 + \sqrt{m^4 - J^2}.$$

The variational interpretation is that the Kerr slice *maximizes* the area of the horizon among all asymptotically flat time-symmetric initial data satisfying the Einstein constraint with ADM mass  $m$  and the total angular momentum  $J$ . The challenge here is that one needs to capture the lack of symmetry of the space, caused by inhomogeneity of the metric or by multiple horizons, in terms of the deviation of the right hand side from the left hand side of the inequality. Such was achieved in Bray's ingenious proof [2] of the Riemannian Penrose inequality by means of a conformal change trick by Bunting-Masood-ul-Alam [4].

## References

- [1] R. Arnowitt, S. Deser, C. Misner. (Republication of:) The dynamics of general relativity. *General Relativity and Gravitation* 40, 1997–2027, 2008.
- [2] H. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geometry*, 59, 177–367, 2001.
- [3] D. Brill and R. Lindquist. Interaction energy in geometrostatics. *Phys. Rev.*, 131, 471–476, 1963.
- [4] G. Bunting and A. Masood-ul-Alam. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time *General Relativity and Gravitation* 19, 147–154, 1987.

- 
- [5] P. T. Chruściel, J. Corvino and J. Isenberg. Construction of  $N$ -body time-symmetric initial data sets in general relativity. *Proceedings of Complex Analysis and Dynamical Systems IV. Intl. Conf. on Complex Analysis and Dynamical Systems*. Ed. by Mark Agranovsky et al. American Mathematical Society, 2011.
- [6] P. Chruściel and N. Nadirashvili. All Majumdar-Papapetrou black holes with a non-singular domain of outer communication. *Classical and Quantum Gravity* 12, L17–L23 (1995).
- [7] S. Dain. Extreme throat initial data set and horizon area-angular momentum inequality for axisymmetric black holes. *Phys. Rev. D*, 82, 2010.
- [8] S. Dain. Proof of the angular momentum-mass inequality for axisymmetric black holes. *J. Differential Geometry*, 79, 33–67 (2008).
- [9] S. Dain, G. Weinstein and S. Yamada. A counterexample to a Penrose inequality conjectured by Gibbons. *Class. Quantum Grav.* 28, (2011).
- [10] R. Geroch. *Ann. N. Y. Acad. Sci.* 224, 108 (1973).
- [11] G. Gibbons. The time symmetric initial value problem for black holes. *Commun. Math. Phys.*, 27, 87–102, 1972.
- [12] G. Gibbons. The isoperimetric and bogomolny inequalities for black holes. In T.J. Willmore and N. Hitchin, editors, *Global Riemannian Geometry*, pages 194–202. John Wiley & Sons, New York, 1984.
- [13] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geometry*, 59, 352–437, 2001.
- [14] P. Jang and R. Wald. *J. Math. Phys.* 18, 41 (1977).
- [15] P. Jang. Note on cosmic censorship. *Phys. Rev. D*, 20, 834–837, 1979.
- [16] M. Mars. Present status of the Penrose inequality. *Class. Quant. Grav.*, 26, 2009.
- [17] R. Penrose. Naked singularities. *Ann. New York Acad. Sci.*, 224:125–134, 1973.
- [18] R. Schoen. Estimates for stable minimal surfaces in three-dimensional manifolds. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 111–126. Princeton Univ. Press, Princeton, NJ, 1983.
- [19] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65, 45–76, 1979.
- [20] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, 79, 231–260, 1981. August 2005.
- [21] R. Wald. *General Relativity* Univ. Chicago Press, 1984.
- [22] G. Weinstein and S. Yamada. On a Penrose inequality with charge. *Commun. Math. Phys.*, 257, 703–723, 2005.
- [23] E. Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80, 381–402, 1981.