Charged black holes and the AdS/CFT correspondence

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Abstract

The AdS/CFT duality is an equivalence between string theory and gauge theory. The duality allows to use calculations done in classical gravity to derive results in strongly-coupled field theories. In this thesis, I explore several applications of the duality that have some relevance to condensed matter physics.

In the first of these applications, it is shown that a large class of strongly-coupled $(3+1)$-dimensional conformal field theories undergo a superfluid phase transition in which a certain chiral primary operator develops a non-zero expectation value at low temperatures. A suggestion is made for the identity of the condensing operator in the field theory.

In a different application, the conifold theory, an $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory, is studied at nonzero chemical potential for baryon number density. In the low-temperature limit, the near-horizon geometry of the dual supergravity solution becomes a warped product $\mathrm{AdS}_2 \times \mathbb{R}^3 \times T^{1,1}$, with logarithmic warp factors. This encodes a type of emergent quantum near-criticality in the field theory.

A similar construction is analyzed in the context of M theory. This construction is based on branes wrapped around topologically nontrivial cycles of the geometry. Several non-supersymmetric solutions are found, which pass a number of stability checks. Reducing one of the solutions to type IIA string theory, and T-dualizing to type IIB yields a product of a squashed Sasaki-Einstein manifold with an extremal BTZ black hole. Possible field theory interpretations are discussed.

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Contents

Chapter 1

Introduction

The anti-de Sitter space/conformal field theory (AdS/CFT) duality is a conjectured exact equivalence between string theory or M theory on certain backgrounds, and specific quantum field theories that do not include gravity.

String theory started off in the 60s as an effort to describe the strong nuclear force, but veered off from that goal as it was realized that it might provide a consistent quantum theory of gravity. Around the same time, quantum chromodynamics was developed as a theory of the strong force. The AdS/CFT conjecture, proposed in the late 90s, is a surprising connection back from string theory to gauge theories. In this case, string theory can be seen as a computational tool, useful for understanding strongly-coupled quantum field theories, independent of whether or not it is a fundamental theory of nature.

In the first section of this chapter, I will start with a brief explanation of why AdS/CFT is useful; then I will go into a more detailed description of the duality, and the 'dictionary' that allows mapping results from one side to the other. Along the way, I will mention some of the reasons for which the duality, which as of yet does not have a complete proof, is believed to be true.

The second section of the introduction will focus on a short description of some of the condensed matter topics relevant to the thesis: superconductivity, superfluidity, and quantum criticality.

The subsequent chapters will show three different applications of the duality. Chapter 2 studies an instability reminiscent of superconductivity or superfluidity that occurs in a large class of theories with gravitational duals; chapter 3 studies the conifold gauge theory at nonzero baryonic chemical potential; and chapter 4 looks at a class of theories involving M-theory branes with topological charges. Finally, chapter 5 concludes the thesis.

1.1 AdS/CFT

1.1.1 Motivation

Understanding the strong-coupling regime of quantum field theories (QFTs) is a long-standing problem in theoretical physics. The perturbative expansions employed at weak-coupling break down as the coupling constants become large. One way in which this difficulty can be alleviated is by using a duality transformation, mapping a strongly-coupled theory to a weakly-coupled one. An example of such a duality is the AdS/CFT duality, also known by the more generic names gauge/gravity or gauge/string duality.

In its most basic form, as first conjectured by Maldacena [1] and clarified in [2,3], the AdS/CFT duality states that certain conformal field theories[1] are dual to M or string theories compactified on appropriate backgrounds. Maldacena's conjecture was motivated by a number of previous results connecting the two theories [5–8]. Duality in this context means an exact equivalence between string theory and the conformal field theory, through which results in one of the theories can be mapped exactly

[1]The best known examples are in $2+1$ and $3+1$ dimensions, but the duality has been generalized to field theories from $0+1$ to $5+1$ dimensions (see, for example, [4]).

to corresponding results in the other. This is one of many examples of dualities in physics, ranging from the observation that Maxwell's equations are invariant under the exchange of the electric and magnetic fields, to the dualities relating different string theories.

The power of the gauge/gravity duality is twofold. First, it has the property of mapping weak coupling on one side of the duality to strong coupling on the other. This means that calculations in strongly-coupled QFTs can be performed by doing calculations in weakly-coupled string (or M) theory, which reduce to calculations in supergravity. Such calculations are often tractable, so the AdS/CFT duality can help in deriving results in strongly-coupled field theories.

One caveat here is that as of yet no gravitational duals are known for empirically relevant field theories, like quantum chromodynamics or the effective field theories describing condensed matter systems. Instead, the duality has been most effectively used in theories with some amount of supersymmetry. While these have not been directly observed in nature, qualitative results from the strong-coupling limit of these theories can still be applied in experimental conditions; see, for example, [9, 10].

Secondly, the gauge/gravity duality can provide a non-perturbative definition of string or M theory. String theory is currently only defined as an asymptotic series expansion in the string coupling constant [11], while for M theory only the low-energy limit is known. The AdS/CFT duality might allow a formulation of string and M theory as the duals of some quantum field theories, which can be defined non-perturbatively.

Despite more than a decade of very active work in this field, the AdS/CFT duality is at this point still a mathematical conjecture. The high level of trust that it enjoys is motivated by the fact that it passed many non-trivial checks. See for example [12–18].

There are thus different directions of research in the field:

- using the duality to infer results on one side (usually, the gauge theory side) by doing calculations on the other (usually, the string theory/supergravity side);

- finding calculations that are doable on both sides, and comparing the results, thus providing another check of the duality;
- searching for a proof that the duality is exact, at least for some class of theories.

This thesis focuses on the first idea, of using the AdS/CFT conjecture to infer information about strongly-coupled quantum field theories.

1.1.2 The duality

As was mentioned before, the idea of the gauge/gravity duality is that certain quantum field theories secretly have an equivalent description in terms of string theory. More precisely, the duality maps string theory in ten dimensions, or M theory in eleven dimensions, to a QFT in a lower number of dimensions. This is done in two steps. First, a Kaluza-Klein reduction of string theory is performed, to obtain a lower-dimensional theory. This means reducing the range of some of the dimensions from infinite to a finite size, and introducing appropriate boundary conditions, *e.g.,* periodic boundary conditions. Then the theory can be rewritten in a smaller number of dimensions, by turning the various excitations in the compact directions into Kaluza-Klein modes.

Secondly, the resulting theory, which includes gravity on some manifold with a conformal boundary, is conjectured to be dual to a quantum field theory living on this conformal boundary. In this sense, the duality is reminiscent of the holography proposed by 't Hooft and Susskind [19, 20], which suggests that the physical reality in our $(3 + 1)$-dimensional universe can be described by degrees of freedom living on the boundary of the universe (in 2+1 dimensions).

In order to use the duality quantitatively, we need a precise recipe for mapping field theory quantities to their string theory equivalents. Such a prescription was given in [2, 3]. It essentially states that the partition function of the QFT coincides with the string theory partition function.

More precisely, let ψ^i be fields in string (or M) theory, and let $\mathcal{O}_i$ be their dual operators in the gauge theory. Because of the identification of the gauge theory as the theory living on the boundary of the string theory spacetime, ψ^i will be called *bulk* fields, and $\mathcal{O}_i$ will be called *boundary* operators. The statement of the duality, in Euclidean signature, is that

$$\left\langle \exp \int_{\text{boundary}} \psi_0^i \mathcal{O}_i \right\rangle_{\text{QFT}} = Z_S(\psi_0^i)\,, \tag{1.1}$$

where $Z_S(\psi_0^i)$ is the string theory partition function, with boundary conditions that ψ^i go to ψ_0^i on the boundary. I used Euclidean signature here because it avoids some complications related to boundary conditions, and it makes the process of considering thermal ensembles in the field theory more transparent. This is the formalism that will be used throughout the thesis.

One subtlety here is that, in general, the solutions to the equations of motion for ψ in the bulk might not allow ψ to go to a finite non-zero function on the boundary. In such a case, ψ_0 is defined as the coefficient of the leading term in an expansion of ψ around the boundary. See section 1.1.5 for an example.

In eq. (1.1), the product between ψ_0^i and $\mathcal{O}_i$ is an inner product respecting the symmetries of the field. Consider, for example, the case in which ψ is a gauge field A_a^M, where M is a space-time index, and a is a gauge index. Let A_a^μ be the components of A_a parallel to the boundary. Then the dual boundary fields are conserved currents J_μ^a, and the product in eq. (1.1) is $J_\mu^a A_a^\mu$.[2]

Equation (1.1) can be used to calculate arbitrary correlation functions in the field theory. Indeed, the left-hand-side of that equation is simply the generating functional

[2]The operator dual to a gauge field in the bulk is always a conserved current on the boundary [3]. This can be shown by using the gauge invariance of Z_S, and thus of the left hand side in eq. (1.1).

for correlators, and we have[3]

$$\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_k} \rangle = \frac{1}{Z_S} \frac{\delta}{\delta \psi_0^{i_1}} \cdots \frac{\delta}{\delta \psi_0^{i_k}} Z_S(\psi_0^i) \bigg|_{\psi_0^i = 0} . \tag{1.2}$$

In the low-energy limit, the string partition function can be evaluated in a saddle-point approximation, which depends on the on-shell supergravity action $S_{\rm sugra}(\psi_0^i)$. In this limit, the AdS/CFT duality can be written as

$$\left\langle \exp \int_{\rm boundary} \psi_0^i \mathcal{O}_i \right\rangle_{\rm QFT} \approx \exp\left[-S_{\rm sugra}(\psi_0^i) \right] . \tag{1.3}$$

Using this and eq. (1.2), we can calculate connected correlators by

$$\langle \mathcal{O} i_1 \cdots \mathcal{O}_{i_k} \rangle_{\rm conn} = - \frac{\delta}{\delta \psi_0^{i_1}} \cdots \frac{\delta}{\delta \psi_0^{i_k}} S_{\rm sugra}(\psi_0^i) \bigg|_{\psi_0^i = 0} . \tag{1.4}$$

In the model first proposed in [1], string theory on $\mathrm{AdS}_5 \times S^5$ is dual to the $\mathcal{N} = 4$ super-Yang Mills gauge theory in 3+1 dimensions. A physical picture of the duality can be obtained by thinking of a stack of N parallel D3 branes.[4] There are N^2 types of open strings stretched between the D3 branes, which is proportional to the number of degrees of freedom of a $\mathrm{U}(N)$ gauge field. In fact, as the distance between the branes decreases, the masses of the stretched strings go to zero, and the strings do indeed behave as the components of a gauge field living on the world volume of the branes [21]. The coordinates normal to the D3 branes become non-commuting $N \times N$ matrices—scalars in the adjoint representation of $\mathrm{U}(N)$ in the field theory. This is how the CFT side of the duality is realized.

Closed strings contain gravitons in the spectrum of their excitations. In the low-energy limit this can be modeled by a supergravity theory, in which the metric on

[3]Note that eq. (1.1) implies that the string partition function was normalized such that $Z_S = 1$ when $\psi_0^i = 0$.

[4]D-branes are solitons in string theory on which open strings can end. A Dp brane is a $(p+1)$-dimensional hyperplane.

spacetime is affected by massive objects, like D branes. Thus the interaction between the stack of D3 branes and closed strings curves space, generating an extremal 3-brane, a higher-dimensional analog of an extremal charged black hole. The space near the event horizon turns out to be warped into anti-de Sitter space. Thus the AdS part of the duality is realized as the near-horizon limit of a stack of D3 branes.

Some general features of the generic gauge/gravity duality are as follows. The region close to the conformal boundary of the near-horizon geometry corresponds to the UV regime of the dual field theory. Moving away from the boundary has the field theory interpretation of a renormalization group (RG) flow towards the IR. Of course, for the case of $\mathcal{N} = 4$ Yang-Mills, the RG flow is trivial, since the theory is conformal at the quantum level.

A thermal ensemble in the field theory can be considered by adding a black hole to the dual geometry. The field theory temperature and entropy coincide with the Hawking temperature and the entropy of the black hole [3].

There remains the problem of finding pairs of a quantum field theory and a background space which are dual in the AdS/CFT sense. This is in general highly non-trivial. One uses symmetry arguments (including supersymmetry) and brane constructions to formulate a hypothesis, as in [1, 22, 23], and then finds supporting evidence for this hypothesis by comparing results that can be derived on both sides of the duality. In general it is hard to derive results in strongly-coupled field theories, but this becomes possible when calculating correlation functions of operators protected by supersymmetry.

Finding the boundary operator dual to a certain bulk field or vice-versa is also a non-trivial task. Symmetry arguments can in general be used for such an identification, as in [24]. Similar methods are applied in section 2.3.2 to suggest a field theory interpretation for the superfluid instability found in chapter 2.

1.1.3 Supergravity

In the low-energy limit, the various string theories reduce to the corresponding ten-dimensional supergravity approximation. M theory is defined as the theory that has eleven-dimensional supergravity as a low-energy limit. Since this is the regime in which the gauge/gravity duality is used in this thesis, I will briefly describe the supergravity theories corresponding to M theory, and type IIA/B string theory.

The classical limit of M theory is the unique eleven-dimensional field theory having maximal supersymmetry and Poincaré invariance [11]. This eleven-dimensional supergravity theory has two bosonic fields, the metric G, and a threeform potential A_3 with field strength $F_4 = dA_3$, and the fermionic fields required by supersymmetry. The equations of motion can be derived from the action

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4 \, , \tag{1.5}$$

where κ_{11} is the eleven-dimensional gravitational constant. See appendix A for the conventions used for differential forms. Only the bosonic terms in the action, which will be the most important in what follows, were kept in eq. (1.5). The fermionic terms can be inferred from supersymmetry.

The ten-dimensional type IIA supergravity, which is the low-energy limit of type IIA string theory, can be obtained by dimensional reduction of the eleven-dimensional supergravity. The bosonic sector of the resulting theory has the metric g, a scalar field Φ called the *dilaton*, the NS-NS twoform B_2 with its field strength H_3, and the R-R one- and three-form potentials C_1 and C_3, with field strengths $F_p = dC_{p-1}$. The equations of motion can again be derived from an action, the bosonic part of which is

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g}\, R - \frac{1}{4\kappa_{10}^2} \int \Big[& d\Phi \wedge *d\Phi + e^{-\Phi} H_3 \wedge *H_3 \\ & + e^{\frac{3}{2}\Phi} F_2 \wedge *F_2 + e^{\frac{1}{2}\Phi} F_4 \wedge *F_4 + B_2 \wedge F_4 \wedge F_4 \Big] \, . \end{aligned} \tag{1.6}$$

This is written in the so-called *Einstein frame*, where the gravitational term appears without any dilaton factors.

Finally, type IIB supergravity contains again the metric, the dilaton, and the NS-NS field, but in the R-R sector it features zero-, two-, and four-form potentials C_0, C_2, and C_4. The equations of motion cannot in this case be obtained from an action alone. Instead, the equations of motion obtained from the action need to be supplemented by the self-duality condition for $\tilde{F}_5$:

$$\tilde{F}_5 = *\tilde{F}_5\,. \tag{1.7}$$

The action is

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-g}\,R - \frac{1}{4\kappa^2}\int \Big[& d\Phi\wedge *d\Phi + e^{-\Phi}H_3\wedge *H_3 \\ & + e^{2\Phi}F_1\wedge *F_1 + e^{\Phi}\tilde{F}_3\wedge *\tilde{F}_3 + \frac{1}{2}\tilde{F}_5\wedge *\tilde{F}_5 + C_4\wedge H_3\wedge F_3\Big]\,. \end{aligned} \tag{1.8}$$

Here we have defined

$$\begin{aligned} \tilde{F}_3 &= F_3 - C_0\wedge H_3\,, \\ \tilde{F}_5 &= F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\,, \\ F_{p+1} &= dC_p\,, \\ H_3 &= dB_2\,. \end{aligned} \tag{1.9}$$

Note that imposing the self-duality condition directly in the action (1.8) would give the wrong equations of motion. Self-duality must instead be seen as a constraint *added* to the equations of motion that are derived by varying the action.

1.1.4 Anti-de Sitter space

Anti-de Sitter (AdS) space is a Lorentzian manifold of constant negative curvature. The $(p+1)$-dimensional space AdS_{p+1} is maximally symmetric, so it is the negatively-curved analog of the sphere. It can be embedded into flat $(p+2)$-dimensional space with signature $(-,-,+,\ldots,+)$ through the equation

$$-t_1^2 - t_2^2 + \sum_{\mu=1}^{p} x_\mu^2 = -L^2 \,. \tag{1.10}$$

Note that this is a sphere in the sense that the metric distance to the center is constant. The space obtained by solving eq. (1.10) contains closed time-like curves, because of the rotational symmetry in the (t_1, t_2) plane. Because of this, physicists usually use instead the *universal cover* of anti-de Sitter space, which essentially 'unrolls' the space by breaking the (t_1, t_2) rotational symmetry [25].

The metric on AdS space can be written as

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + \sum_{i=1}^{p-1} dx_i^2\right) . \tag{1.11}$$

This actually only covers half of AdS space, also known as the *Poincaré patch*. Here $z > 0$, so the Poincaré patch is conformally equivalent to the upper half-space of $\mathbb{R}^{p+1}$ with Minkowski signature. As $z \to 0$, we find the conformal boundary of AdS_{p+1}, which is just p-dimensional Minkowski space.

From eq. (1.10), we can see that AdS_{p+1} has $\mathrm{SO}(2,p)$ symmetry. This symmetry group is the same as the conformal group in p dimensions. Translation along the radial z direction is related to scaling transformations in the field theory. This again hints at the relation between AdS space and conformal field theories. See [3] for more details.

The radial z direction can be interpreted as an energy scale in the field theory.

Positions close to the conformal boundary, $z = 0$, correspond to the UV limit of the theory, while positions deep inside the bulk correspond to the IR limit.

Five-dimensional anti-de Sitter space can be obtained as the near-horizon limit of the spacetime around an extremal 3-brane. Indeed, the metric generated by a charged black $(p-1)$-brane is [26, 27]

$$ds^2 = H^{-1/2}(r)\left[-f(r)dt^2 + \sum_{i=1}^{p-1}\left(dx^i\right)^2\right] + H^{1/2}(r)\left[f^{-1}(r)dr^2 + r^2 d\Omega_{9-p}^2\right], \quad (1.12)$$

where $H(r)$ is a harmonic function,

$$H(r) = 1 + \frac{L^{8-p}}{r^{8-p}}, \quad (1.13)$$

and

$$f(r) = 1 - \frac{r_H^{8-p}}{r^{8-p}}. \quad (1.14)$$

Here the horizon of the black brane is at $r = r_H$. The notation $d\Omega_{9-p}^2$ stands for the metric on the $(9-p)$-dimensional sphere.

Let us focus on the limit in which the black brane is *extremal*, when $r_H \to 0$. This implies $f \equiv 1$. In the near-horizon limit, $r \ll L$, we can ignore the constant term in H, and we get

$$ds^2 \approx \left(\frac{r}{L}\right)^{\frac{8-p}{2}} \left[-dt^2 + \sum_{i=1}^{p-1}\left(dx^i\right)^2\right] + \left(\frac{L}{r}\right)^{\frac{8-p}{2}} \left[dr^2 + r^2 d\Omega_{9-p}^2\right]. \quad (1.15)$$

For a 3-brane we have $p = 4$, and defining $z = L^2/r$ puts the metric in the form of $\mathrm{AdS}_5 \times S^5$ (see eq. (1.11)). If $p \neq 4$, the near-horizon metric is not exactly AdS $\times$ sphere, but it is conformally equivalent to that.

If the brane is not extremal, $f \neq 1$, the near-horizon limit is that of a black hole inside AdS space. Black holes emit Hawking radiation at a temperature proportional to their surface gravity. While in Minkowski space this means black holes eventually evaporate (so they are unstable), large enough black holes in AdS space are stable. The AdS/CFT duality maps a geometry with a black hole onto a quantum field theory in a thermal ensemble. The temperature of the QFT is equal to the Hawking temperature of the black hole [3].

In cases where a black hole exists, boundary conditions must be specified not only on the conformal boundary, but also on the black hole horizon. In Euclidean signature, the boundary conditions just require regularity of all the fields on the horizon. In Minkowski signature, several prescriptions for boundary conditions are possible [28]. These correspond to calculating different kinds of correlators in the field theory.

1.1.5 Holographic renormalization

Equation (1.3) cannot usually be applied naively, because the right-hand side diverges.[5] This shouldn't be too surprising, since the left-hand side also diverges without appropriate renormalization. The on-shell supergravity action diverges because of the infinite volume of AdS space. The divergence can be regularized by imposing a cutoff $z > \epsilon$, which gives an on-shell action that depends on ϵ. The divergent terms in ϵ can be canceled by adding local boundary counterterms to the action [3]; this procedure is called *holographic renormalization*.

As an example, let us briefly look at the holographic renormalization for a massive scalar field in the bulk, and how this can be used to calculate correlation functions for the dual field theory operator. The Euclidean action for a massive scalar field is proportional to

$$S = \int d^{p+1}x \, \sqrt{g} \left[g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + m^2 \phi^2 \right] . \tag{1.16}$$

[5]Here by "naive" application I mean only including a Gibbons-Hawking boundary term in the supergravity action. It turns out that, in general, the variational problem in the bulk is not well-posed without adding more boundary terms. Adding these other terms renders the right-hand-side of eq. (1.3) finite [29].

Here we will work, for simplicity, in the approximation in which the scalar field does not backreact on the metric. The metric g is simply that given by eq. (1.11).

Integrating by parts in eq. (1.16) shows that the bulk part of the action vanishes on-shell, since the integrand is proportional to the equations of motion. What is left is a boundary term. Introducing a cut-off at $z = \epsilon$, this boundary term (and thus the on-shell action) is given by[6]

$$S_\epsilon = \int_{z>\epsilon} d^{p+1}x \, \sqrt{g} \, \nabla_\mu \big(g^{\mu\nu} \phi \partial_\nu \phi \big) = - \left(\frac{L}{\epsilon} \right)^{p-1} \int_{z=\epsilon} d^p x \, \phi \partial_z \phi \, . \tag{1.17}$$

Let us Fourier expand ϕ in the boundary directions

$$\phi(x) = \int \frac{d^p k}{(2\pi)^p} \, \phi_k(z) \, e^{ik^a x_a} \, . \tag{1.18}$$

The equation of motion becomes

$$z^{p-1} \partial_z \big(z^{1-p} \partial_z \phi_k \big) - \left(\frac{m^2 L^2}{z^2} + k^2 \right) \phi_k = 0 \, , \tag{1.19}$$

which must hold for all k. The solutions are

$$\phi_k(z) = c_+ \phi_k^{(+)}(z) + c_- \phi_k^{(-)}(z) \, , \tag{1.20}$$

with the following small z expansion:

$$\begin{aligned} \phi_k^{(+)}(z) &= z^{\Delta_+} (1 + \alpha_{1+} z^2 + \cdots) \, , \\ \phi_k^{(-)}(z) &= z^{\Delta_-} (1 + \alpha_{1-} z^2 + \cdots) \, . \end{aligned} \tag{1.21}$$

[6] The minus sign appears because the outward normal at $z = \epsilon$ points towards decreasing z.

The powers $\Delta_\pm$ are the roots of the equation

$$\Delta(\Delta - p) = m^2 L^2 \,. \tag{1.22}$$

Note that $\Delta_+ + \Delta_- = p$, so that one of the roots (call it Δ_-) is smaller than $p/2$, while the other one is larger than $p/2$.

Note also that ϕ_k does not go to a constant as $z \to 0$, unless $m = 0$. In general, the role of ψ_0 from eq. (1.1) is played here by the coefficient of the leading term in ϕ_k, which is c_-.

Let us now focus on the cases in which $\Delta_+ - \Delta_- < 2$. This means that the two leading terms in the expansion of ϕ_k are $z^{\Delta_\pm}$, and we can ignore the $\alpha_{i\pm}$ terms. The on-shell action is obtained from eq. (1.17)

$$S_\epsilon = -L^{p-1} \int \frac{d^p k}{(2\pi)^p} \left[\Delta_+ \left|c_+\right|^2 \, \epsilon^{2\Delta_+ - p} + \Delta_- \left|c_-\right|^2 \, \epsilon^{2\Delta_- - p} + p \, \mathrm{Re}(c_- c_+^*) \right] + \ldots \,, \tag{1.23}$$

where the real-valuedness of ϕ was used. The terms that were ignored vanish in the $\epsilon \to 0$ limit faster than the terms that were kept.

The term $\epsilon^{2\Delta_- - p}$ is divergent at small ϵ, so we need to add a counterterm. The counterterm is given by

$$S_{\mathrm{ct}} = \Delta_- \int_{z=\epsilon} d^p x \, \sqrt{h} \, \phi^2 \,, \tag{1.24}$$

where h is the induced metric on the boundary $z = \epsilon$. In general more counterterms will be needed if $\Delta_+ - \Delta_- \geq 2$, but I will not treat that case here.

The regularized on-shell action is

$$S_{\mathrm{reg}} = L^{p-1} \, (2\Delta_- - p) \int \frac{d^p k}{(2\pi)^p} \; \mathrm{Re}(c_-(k) c_+^*(k)) \,, \tag{1.25}$$

where the limit $\epsilon \to 0$ was already taken. This can be rewritten in position space as

$$S_{\rm reg} = L^{p-1}\,(2\Delta_- - p)\int d^p x\, c_-(x) c_+(x)\,. \tag{1.26}$$

The vacuum expectation value of the boundary operator $\mathcal{O}$ dual to ϕ can be obtained from $S_{\rm reg}$ using eq. (1.4)

$$\langle \mathcal{O} \rangle = -\frac{\delta S_{\rm reg}}{\delta c_-}\Big|_{c_-=0} = (2\Delta_+ - p)\, L^{p-1}\, c_+\,, \tag{1.27}$$

where I used the fact that $\Delta_+ + \Delta_- = p$. Note that the normalization here doesn't mean much (including the dimensions), since I haven't tried to normalize eq. (1.16) correctly. Since $c_+ \propto \langle \mathcal{O} \rangle$, it is common to refer to c_+ itself as the vacuum expectation value ('vev') of the operator $\mathcal{O}$. The value of c_+ can be determined in terms of c_- by imposing boundary conditions at $z \to \infty$. This means that $\langle \mathcal{O} \rangle$ depends on c_-, which earns c_- the name 'source'.

We can also calculate two-point correlation functions, by solving eq. (1.19) to all orders in z. This equation is just the Bessel's differential equation, as can be seen by setting $\phi_k = z^{p/2}\tilde{\phi}_k$ and $z = \tilde{z}/ik$, where $k = \left|\vec{k}\right|$. The solution that is well-behaved at $z \to \infty$ is

$$\phi_k = c_-\,\frac{k^\nu}{2^{\nu-1}\Gamma(\nu)}\, z^{p/2} K_\nu(kz)\,. \tag{1.28}$$

Here I defined $\nu = \sqrt{m^2L^2 + p^2/4}$, as in [2], such that $\Delta_\pm = (p/2) \pm \nu$. We can read off c_+ from the Taylor expansion for the Bessel's K function

$$c_+ = \left(\frac{k}{2}\right)^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)}\, c_-\,. \tag{1.29}$$

Now the (connected) two-point function can be calculated by differentiating $-S_{\rm reg}$

twice, with respect to $c_-(k)$ and $c_-(q)$, and evaluating the result at $c_- = 0$. We get

$$\langle \mathcal{O}(k)\mathcal{O}(q)\rangle_{\text{conn}} = -\frac{2L^{p-1}}{(2\pi)^p}\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}\,\delta^p(k+q)\,. \tag{1.30}$$

Again, note that the overall normalization here is arbitrary.

Since $\langle \mathcal{O}(k)\mathcal{O}(q)\rangle$ is proportional to the $\delta(k+q)$ multiplied by $k^{2\nu}$, the position-space two-point function behaves like $(x-x')^{-(p+2\nu)} = (x-x')^{-2\Delta_+}$, and thus this corresponds to an operator $\mathcal{O}$ of dimension Δ_+. See [30] for more details, including an interpretation for Δ_-. For a more systematic treatment of holographic renormalization in general, see for example [27].

1.1.6 Charged particles in Reissner-Nordström AdS

Over the past few years, considerable effort has been devoted to using the AdS/CFT correspondence for studying strongly coupled field theories at non-vanishing chemical potential for some conserved global charge (see [31,32] for reviews). A global symmetry of a $(p+1)$-dimensional conformal field theory is mapped to a gauge symmetry in $(p+2)$-dimensional anti-de Sitter space. Therefore, properties of a conformal field theory at finite chemical potential μ and temperature T are encoded in charged p-brane solutions that are asymptotic to $\mathrm{AdS}_{p+2}\times Y$, where Y is an Einstein space.

An interesting class of AdS/CFT dualities involves Sasaki-Einstein spaces Y which lead to backgrounds preserving eight supercharges. Type IIB backgrounds of the form $\mathrm{AdS}_5\times Y^5$ are therefore dual to $\mathcal{N}=1$ superconformal gauge theories in four space-time dimensions, while M-theory backgrounds $\mathrm{AdS}_4\times Y^7$ are dual to $\mathcal{N}=2$ superconformal gauge theories in three space-time dimensions [22, 33, 34]. These theories possess $\mathrm{U}(1)_R$ symmetry that in supergravity is realized as an isometry of Y. In an effective $(p+2)$-dimensional description, the charged black branes are described by Reissner-Nordström AdS (RNAdS) backgrounds. One typically finds that as the temperature divided by the chemical potential is reduced, such a charged p-brane solution is not the thermodynamically-preferred phase of the theory [35–37]; it becomes unstable towards developing charged "hair" [38]. Typically, when an R-charged p-brane is embedded into string or M-theory, there are R-charged fields that condense close to the black hole horizon, thus breaking the U(1) gauge symmetry spontaneously [39–42]. The corresponding symmetry breaking in the field theory has been used to model superconductivity [43] or superfluidity [44] in a strongly coupled CFT.

In analyzing the stability of charged black branes, it is useful to start with a simple model, looking at charged particles in the background given by the black brane. This kind of analysis can be used in general for charged branes, whether the charge is the $U(1)_R$ charge described above, or the topological charges studied in chapters 3 and 4.

Let us consider a particle of mass m and charge q in a curved background given by

$$ds_M^2 = -ge^{-w}dt^2 + \frac{dr^2}{g} + \frac{r^2}{L^2}\sum_{i=1}^{3}(dx^i)^2 \,, \tag{1.31}$$

where g and w are functions of r.

The action for such a particle is

$$S_p = -m\int ds \left|\frac{dx^\mu}{ds}\right| - q\int ds\, A_\mu \frac{dx^\mu}{ds} \,. \tag{1.32}$$

We restrict to a radial electric potential $\Phi(r) = A_t(r)$. For a particle sitting at some fixed r and $\vec{x}$, the action becomes

$$S_p = -\int dt\, \left[me^{-w(r)/2}\sqrt{g(r)} + q\Phi(r)\right] \tag{1.33}$$

in the gauge $s = t$. Equation (1.33) shows that the potential for this particle as a function of r is given by

$$V(r) = m e^{-w(r)/2} \sqrt{g(r)} + q\Phi(r)\,. \tag{1.34}$$

The first term is the gravitational attraction of the black hole, and the second term the electrostatic repulsion.

To proceed further, we need explicit results for w, g, and Φ. Let us consider the simple case of a Reissner-Nordström black hole with negative cosmological constant. Given the Einstein-Maxwell action

$$S_{EM} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left[R + \frac{d(d-1)}{L^2} \right] - \frac{1}{4e^2} \int d^{d+1}x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}\,, \tag{1.35}$$

there is a black-brane solution with a horizon at $r = r_h$ and with $w = 0$,

$$g = \frac{r^2}{L^2} \left(1 + Q^2 \left(\frac{r_h}{r} \right)^{2d-2} - (1 + Q^2) \left(\frac{r_h}{r} \right)^{d} \right)\,, \tag{1.36}$$

$$\Phi(r) = Q \frac{e}{\kappa} \frac{r_h}{L} \sqrt{\frac{d-1}{d-2}} \left[\left(\frac{r_h}{r} \right)^{d-2} - 1 \right]. \tag{1.37}$$

The corresponding Hawking temperature is

$$T_H = \frac{d - (d-2)Q^2}{4\pi} \frac{r_h}{L^2}\,. \tag{1.38}$$

Without loss of generality, let's assume $Q > 0$. The RNAdS black hole becomes extremal at $Q = Q_{\max} \equiv \sqrt{\frac{d}{d-2}}$ where T_H vanishes.

Using these formulae for g and Φ, one can check that for values of $q/m < q_{\rm crit}/m$, $V(r)$ is increasing monotonically for all values of Q smaller than $Q_{\max}$, while for large values $q/m > q_{\rm crit}/m$, $V(r)$ has a minimum at some $r = r_*(Q)$ provided that the charge Q of the black hole is larger than some critical value Q_c that depends on q/m. An expression for the critical charge $q_{\rm crit}$ can be found by requiring that the minimum of $V(r)$ occurs right at $r = r_h$ when $Q = Q_{\max}$. The equation $V'(r_h) = 0$ is easily solvable in this case, yielding

$$q_{\text{crit}} = m\frac{\kappa}{e}\,, \tag{1.39}$$

independent of the number of dimensions of the RNAdS space. Actually, this ratio has some broader significance, as emphasized by [45] in the context of a weak gravity bound. If extremal Reissner-Nordström black holes are to be able to decay, there must always exist at least one particle whose charge-to-mass ratio is greater than κ/e.

This picture of an instability is essentially the classical limit of the superfluid or superconducting instability studied in [38, 43]. In these papers, the charged particle is replaced with a charged scalar field ψ. One solution to the equations of motion is $\psi = 0$, but for $Q > Q_c$, there exists a second, more stable branch of solutions with $\psi \neq 0$. The classical analog of this second branch is a cloud of charged particles sitting at the minimum in $V(r)$ described above. Moreover, for large m and q, the classical and field theory results for Q_c agree.

1.2 Condensed matter

1.2.1 Superconductivity and superfluidity

At low temperatures, a large number of materials exhibit a phase transition to a *superconducting* phase, in which the resistivity drops to zero. Apart from the infinite conductivity that gives this phase its name, other important characteristics are the existence of a gap in the energy levels of the conduction electrons, and the Meissner effect—the observation that superconductors expel magnetic fields from their volume.

Superconductivity was first observed experimentally in 1911. A full microscopic theory for the superconducting materials known at the time was worked out by

Bardeen, Cooper, and Schrieffer (BCS) in 1957. The basic idea is that the weak attraction between electrons that is generated by interactions with the phonons is sufficient to produce pairs of electrons, called *Cooper pairs.* The energy gap observed in superconductors is given by the energy required to break a Cooper pair. These pairs obey Bose statistics, and the Bose-Einstein condensate that forms below the critical temperature T_c is responsible for the peculiar characteristics of superconductors.

An empirical theory describing superconductivity was developed in 1935 by the London brothers [46]. In 1950 Ginzburg and Landau worked out a thermodynamically-motivated theory [47] that included the London theory and extended it. I will focus on the Ginzburg-Landau theory in this thesis, since it is more readily related to results from string theory. The description given here follows [48].

In the Ginzburg-Landau (GL) theory, it is assumed that there exists a complex order parameter ψ which is zero in the normal (non-superconducting) phase, and non-zero in the superconducting phase. The function ψ can be viewed as the collective wavefunction for the macroscopically occupied ground state of the condensate. The dynamics of ψ are governed by a variational principle, based on the free energy functional

$$F = \int dx \left[\frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{e^*}{c}\vec{A} \right) \psi \right|^2 + \alpha \left|\psi\right|^2 + \frac{1}{2}\beta \left|\psi\right|^4 \right] , \qquad (1.40)$$

where I kept only the terms that depend on ψ. The parameters $m^* = 2m$ and $e^* = 2e$ are the effective mass and charge of the Cooper pairs.[7] This form can be seen simply as a truncated expansion of F in ψ and gradients of ψ. The expansion (1.40) takes into account the U(1) symmetry under changes of the phase of ψ.

A physical interpretation can be given to ψ by defining the number density of

[7]The mass m^* is different in real metals due to the interaction with the crystal lattice [48]. However, m^* is hard to measure experimentally, so it's conventional to fix its value at $m^* = 2m$. The potential missing factor is absorbed into the normalization of ψ.

electrons in the superconducting phase,

$$n_s = |\psi|^2 \ . \tag{1.41}$$

The superconductor can then be analyzed in a *two-fluid* model, where it is assumed that the electrons, with density N, are split into two parts, a superconducting component with density n_s, and a "normal" component, with density $N - n_s$.

Equation (1.40) is just the action for a complex scalar field ψ in a Mexican-hat potential (provided $\beta > 0$ and $\alpha < 0$), coupled to an external potential $\vec{A}$. If we consider a homogeneous system, such that ψ is constant, and no external field $\vec{A} = 0$, the free energy is given by

$$F = V\,|\psi|^2 \left(\alpha + \frac{1}{2}\,|\psi|^2\right), \tag{1.42}$$

which is minimized by $\psi \equiv 0$ if both α and β are positive, and by

$$|\psi_c|^2 = -\frac{\alpha}{\beta}\,, \tag{1.43}$$

if $\alpha < 0$, $\beta > 0$. We do not take into consideration the case $\beta < 0$, since then the system would find it favorable to increase $|\psi|$ to values where the higher-order terms in eq. (1.40), which we ignored, become important.

To explain superconductivity, we must assume that α is positive for temperatures above T_c, and negative for $T < T_c$. The simplest dependence of α on temperature that obeys these conditions is

$$\alpha = \alpha_0(T - T_c)\,. \tag{1.44}$$

We assume β to be independent of temperature.

Including the backreaction from the superconducting electrons on the electromagnetic field, the free energy becomes

$$F = \int dx \left[\frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{e^*}{c}\vec{A} \right) \psi \right|^2 + \alpha \left|\psi\right|^2 + \frac{1}{2}\beta \left|\psi\right|^4 + \frac{1}{8\pi}h^2 \right] , \tag{1.45}$$

where $\vec{h} = \nabla \times \vec{A}$ is the magnetic field. This is simply the action for the Abelian Higgs model. Below the critical temperature, the nonzero ψ breaks the U(1) symmetry. The phase ϕ of $\psi = |\psi| \exp(i\phi)$ is not an independent degree of freedom, since it can be absorbed into the vector potential by a gauge transformation. Choosing a gauge in which $\phi = 0$, the kinetic term for ψ in eq. (1.45) becomes

$$\frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{e^*}{c}\vec{A} \right) \psi \right|^2 = \frac{\hbar^2}{2m^*} \left(\nabla \left|\psi\right| \right)^2 + \frac{e^{*2}}{2m^* c^2} A^2 \left|\psi^2\right| , \tag{1.46}$$

which generates a mass term for the electromagnetic field. This is the Anderson-Higgs mechanism: the electromagnetic gauge field 'ate' the massless Goldstone boson ϕ which resulted from the spontaneous breaking of the U(1) symmetry that rotates the phase of ψ. The mass term for A is responsible for the Meissner effect: magnetic fields decay exponentially at the surface of a superconductor as a result of turning electromagnetism into a short-ranged interaction.

The differential equations obtained from varying the free energy (1.45) are

$$\alpha\psi + \beta \left|\psi\right|^2 \psi + \frac{1}{2m^*} \left(-i\hbar - \frac{e^*}{c}\vec{A} \right)^2 \psi = 0 \, , \tag{1.47a}$$

$$\vec{J} \equiv \frac{c}{4\pi} \nabla \times \vec{h} = e^* \left|\psi\right|^2 \vec{v}_s \, , \tag{1.47b}$$

where $\vec{v}_s$ is the supercurrent velocity given by

$$m^* \vec{v}_s = \hbar\nabla\phi - \frac{e^*}{c}\vec{A} \, . \tag{1.47c}$$

In a gauge in which $\phi = 0$, which is the same as the *London gauge* $\nabla \cdot \vec{A} = 0$ for slowly varying $n_s = |\psi|^2$, eqns. (1.47b) and (1.47c) imply

$$\vec{J} = -\frac{n_s e^{*2}}{m^* c} \vec{A} . \tag{1.48}$$

Taking the time derivative leads to one of the London equations,

$$\frac{d\vec{J}}{dt} = \frac{n_s e^{*2}}{m^* c} \vec{E} , \tag{1.49}$$

if we ignore the time variation of $|\psi|^2$. This shows that the superconducting electrons experience zero resistivity, since the current is increasing linearly under the effect of a constant electric field. A more careful check for this claim can be done by calculating the DC conductivity with the help of Kubo formulae [49].

Phenomenologically, we thus see that superconductivity comes about when a scalar field ψ charged under a gauge field develops a non-zero expectation value. The Higgs mechanism then implies the expulsion of magnetic fields from the bulk of the superconductor. The current corresponding to ψ encounters no resistance, resulting in zero resistivity. If the order parameter ψ is not coupled to a gauge field, the same mechanism described above can be used as a model for superfluidity [50].

Typical critical temperatures for superconductivity in materials known before 1980 varied across several orders of magnitude, but were no larger than 20-30 Kelvin [51]. Starting in the mid 80s, a series of compounds based on copper were discovered that achieve superconductivity at much higher critical temperatures. There are many examples of so-called *high-T_c superconductors* known today, with critical temperatures going as high as 133 K. Most are based on copper compounds, which earned high-T_c superconductors the nickname *cuprates*, though in 2008 some high-T_c superconductors based on iron were found as well.

The Ginzburg-Landau theory seems to still be applicable to high-T_c superconductors, but the exact pairing mechanism is still under debate [48]. Part of the difficulties encountered in the study of high-T_c superconductivity are related to working in a strong-coupling regime. This is where a description in terms of AdS/CFT could prove fruitful.

1.2.2 Quantum criticality

A quantum phase transition is a phase transition occurring at zero temperature. Since the temperature can no longer be the control parameter, a quantum phase transition occurs at a critical value g_c of some other parameter g [52]. The point where the phase transition occurs is called a *quantum critical point*.

Because of the vanishing temperature, the system is in its ground state, so the transition marks a change in this ground state. The quantum-mechanical ground state of a system is unique, but depends on parameters such as g. As g is varied, one can encounter a level-crossing event, in which the ground state and the first excited state exchange roles. This would lead to a non-analyticity in the ground state as a function of g (see Figure 1.1a). The more typical case for systems of finite size is an "avoided" level-crossing, in which the energy of the excited state approaches that of the ground state, but does not reach it (as in Figure 1.1b). In the infinite-size limit, however, these events can get progressively sharper, leading again to non-analyticities of the ground state [52].

At the point of a second-order quantum phase transition, the length scale of quantum fluctuations diverges, according to a scaling law of the form

$$\xi^{-1} \sim |g - g_c|^{\nu} \,, \tag{1.50}$$

where ν is a *critical exponent*. This exponent exhibits *universality*, meaning it is largely independent of microscopic details. At the same time, the energy scale of the

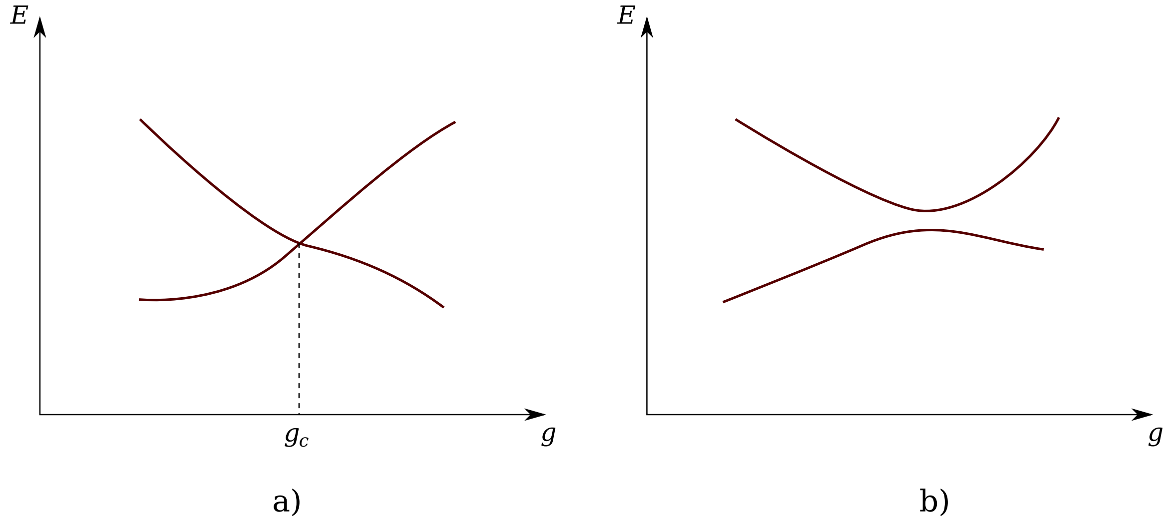

Figure 1.1: Possible dependence of the lowest energy states of a system of finite-size on a parameter g. In a), there is a level crossing, and the ground state is non-analytic as a function of g; in b), the level crossing is avoided, but the ground state's dependence on g might become non-analytic in the limit of infinite size.

fluctuations above the ground state (for example, the energy gap, if it exists) vanishes at the critical point,

$$\Delta \sim |g - g_c|^{z\nu}\,, \tag{1.51}$$

where the ratio z between the critical exponents is called the *dynamic critical exponent*.

The divergence of the correlation length for quantum fluctuations around the critical point leads to long-range quantum entanglement, and the system can be described by a conformal field theory. The conformal symmetry in systems at a quantum critical point is often emergent, in the sense that it is a feature of the IR physics that was not there in their microscopic description.

Since quantum critical points occur at exactly zero temperature, they can't be directly accessed in experiments. It turns out, however, that the existence of quantum critical points has important consequences even at nonzero temperatures [52, 53]. There is in general a wide area in parameter space where the influence of the quantum critical point is felt, as in Figure 1.2. This area represents the region of *quantum criticality* in the phase diagram of a system. The rather surprising fact that the range of g where quantum criticality is observed is growing with temperature is related to the fact that this phase is delimited by the lines $\Delta \sim T$, and thus $|g - g_c| \sim T^{1/z\nu}$.

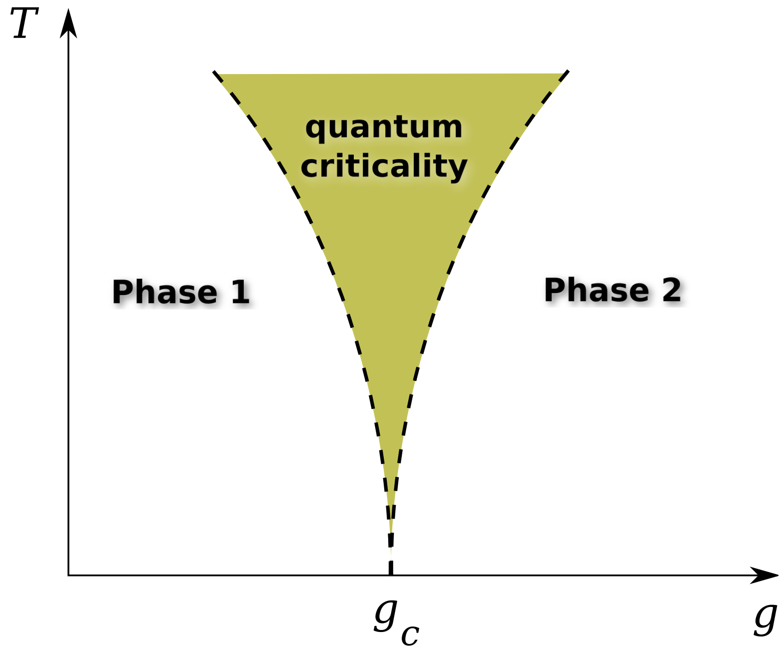

Figure 1.2: A typical phase diagram for a system exhibiting quantum criticality [53]. There is a phase transition at zero temperature when $g = g_c$, but the system exhibits non-classical behavior for a wide area of phase space even at $T \neq 0$.

Quantum criticality has been observed experimentally in various systems. For example [52, 53], the insulators $LiHoF_4$ and $CoNb_2O_6$ are normally ferromagnets at zero temperature, due to the magnetic interactions between the holmium or cobalt ions. However, when subjected to a large enough external magnetic field transverse to the magnetic axis of the system, each of these insulators loses its magnetic order, becoming a *quantum paramagnet*. Signs of quantum criticality were also observed in high-T_c superconductors (see, for example, [54]).

In the AdS/CFT formalism, the existence of quantum criticality is signaled by the appearance of an AdS throat in the IR at zero temperature. Just as on the field theory side of the duality, the conformal symmetry at a critical point need not be present in the microscopic theory (or it can be realized differently), on the gravity side, the IR AdS need not be matched by the UV geometry. Indeed, as an example, several “domain wall” solutions have been constructed that interpolate between AdS behavior in the UV and a different AdS space in the IR, as in [41, 55, 56].

One possibility is that at nonzero temperatures, the throat contains an RNAdS black hole, making the geometry deep in the IR go to $AdS_2 \times \mathbb{R}^d$ as $T \to 0$ [57]. Analysis of the IR behavior of fermions in the AdS_2 throat suggests the existence of a Fermi surface, which would provide another nice connection with condensed matter systems [57–59].

There is a problem with the AdS_2 constructions: the horizon area of the black hole does not go to zero at zero temperature, which on the field theory side implies non-vanishing entropy at $T = 0$. This is indicative of a large ground state degeneracy, whose origin is not yet clear. See [57–59] for more details, and some reviews of the ways in which AdS/CFT was applied to gain insights into strongly-coupled quantum criticality.

Chapter 2

Superconductors from superstrings

The following work was done in collaboration with Steven S. Gubser, Christopher P. Herzog, and Silviu S. Pufu, and is published in [40]. This chapter is a lightly edited version of the published paper.

We establish that in a large class of strongly-coupled $(3+1)$-dimensional $\mathcal{N}=1$ quiver conformal field theories with gravity duals, adding a chemical potential for the R-charge leads to the existence of superfluid states in which a chiral primary operator of the schematic form $\mathcal{O} = \lambda\lambda + \mathcal{W}$ condenses. Here λ is a gluino and $\mathcal{W}$ is the superpotential. Our argument is based on the construction of a consistent truncation of type IIB supergravity that includes a U(1) gauge field and a complex scalar.

2.1 Introduction

Using the AdS/CFT correspondence, refs. [38, 43] argued that a classical scalar-gravity model describes a superconducting phase transition in a dual strongly interacting field theory. Superfluid phase transition is perhaps a more accurate description [44] as there is no Higgs mechanism in the field theory, but for many physical questions, the distinction is irrelevant [60]. The proposal is interesting because it suggests that string theory techniques provide good theoretical control over superfluid transitions in certain strongly-coupled theories, raising the hope that one might extend lessons learned from such theories to real condensed-matter systems. In this chapter, we embed the scalar-gravity model in type IIB string theory. The embedding clarifies the microscopic nature of the $(3+1)$-dimensional field theory dual.

The AdS/CFT correspondence provides a recipe for constructing a large class of $\mathcal{N}=1$ supersymmetric, $(3+1)$-dimensional conformal field theories (SCFTs) by placing a stack of N D3-branes at the tip of a three complex dimensional Calabi-Yau cone X in type IIB string theory [22, 33, 34, 61]. The field theory can be thought of as the open string degrees of freedom propagating on the D-branes at the Calabi-Yau singularity, and is a quiver gauge theory with SU(N) gauge groups and superpotential $\mathcal{W}$.[1]

The AdS/CFT correspondence provides a dual closed string description of the field theory as type IIB string theory in the background curved by the energy density of the stack of D3-branes. In the near horizon limit, *i.e.,* close to the D3-branes, the ten-dimensional space factorizes into a product of anti-de Sitter space and a Sasaki-Einstein manifold, $AdS_5 \times Y$, where Y is a level surface of the cone X. The R-symmetry of the SCFT is realized geometrically as an isometry of Y.

In section 2.2, given a Sasaki-Einstein manifold Y expressible as a U(1) fibration over a compact Kähler-Einstein base, we write down a consistent truncation of type IIB supergravity to five dimensions that includes a complex scalar field Ψ and the gauge field A_μ dual to the R-symmetry current. The field Ψ is dual to a chiral primary operator $\mathcal{O}$ with scaling dimension $\Delta = 3$ in the field theory. In the presence of a chemical potential μ, realized geometrically as the boundary value of A_t, the chiral primary develops an expectation value below a critical temperature T_0. In the dual gravity language, an electrically charged black hole develops scalar hair. By calculating

[1] A quiver gauge theory is a field theory defined in terms of a *quiver diagram*, in which nodes stand for U(N) gauge fields, and edges are bifundamental matter fields between the two gauge groups they connect.

the free energy as a function of T, we demonstrate that this phase transition that spontaneously breaks the U(1) R-symmetry is second order.

In section 2.3.1, we calculate the critical temperature T_p below which a more general complex scalar with conformal dimension Δ and R-charge R will develop a perturbative instability. In some cases, for example for Ψ of section 2.2, this T_p corresponds to the critical temperature of a second order phase transition. However, if the phase transition is first order, $T_c > T_p$ and T_p is instead the temperature below which the symmetry-restored phase of the field theory becomes perturbatively unstable. We show that of all scalar chiral primary operators, $\mathcal{O}$ has the largest T_p if it has lowest conformal dimension. If the latter condition is satisfied, then for reasons presented in section 2.3.2, it is likely that the condensation of $\mathcal{O}$ is responsible for a U(1) symmetry breaking phase transition in the field theory. $\mathcal{O}$ is at least tied for lowest conformal dimension in some quiver theories: in section 2.3.2 we give a particular example based on a $\mathbb{Z}_7$ orbifold of S^5.

Embeddings in M-theory of $(2+1)$-dimensional versions of these field theory models have been discussed in [39]. While ref. [39] treats explicitly a broader range of examples than we do, their analysis of the scalar instability is perturbative. Our consistent truncation allows us to establish the phase transition is second order and to follow the broken phase to arbitrarily low temperatures. Because we are working with a $(3+1)$-dimensional field theory where the AdS/CFT duality is better understood, we are able to say more about the microscopic nature of the field theory dual than has been possible thus far in the $(2+1)$-dimensional case.

2.2 A consistent truncation

Consider the following five-dimensional action, involving a metric, a U(1) gauge field, and a complex scalar:

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left(\mathcal{L}_{\rm EM} + \mathcal{L}_{\rm scalar} \right) , \tag{2.1}$$

where

$$\mathcal{L}_{\rm EM} = R - \frac{L^2}{3} F_{\mu\nu} F^{\mu\nu} + \left(\frac{2L}{3}\right)^3 \frac{1}{4} \epsilon^{\lambda\mu\nu\sigma\rho} F_{\lambda\mu} F_{\nu\sigma} A_\rho \,, \tag{2.2}$$

and[2]

$$\mathcal{L}_{\rm scalar} = -\frac{1}{2} \Big[(\partial_\mu \eta)^2 + \sinh^2 \eta (\partial_\mu \theta - 2A_\mu)^2 - \frac{6}{L^2} \cosh^2 \frac{\eta}{2} \left(5 - \cosh \eta\right) \Big] \,. \tag{2.3}$$

We define $\epsilon^{01234} \equiv 1/\sqrt{-g}$. Note that this has the opposite sign compared to the conventions in appendix A. The real fields η and θ are the modulus and phase of the complex scalar Ψ. Note that for small η, the potential term expands to yield

$$V(\eta) = -\frac{12}{L^2} - \frac{3}{2L^2} \eta^2 + \dots \,. \tag{2.4}$$

The leading order term comes from a negative cosmological constant, $\Lambda = -6/L^2$. The second order term is a mass for the scalar. We have

$$m^2 L^2 = \Delta(\Delta - 4) = -3 \,, \tag{2.5}$$

and so $\Delta = 3$.

The U(1) gauge field has been normalized such that $\mathcal{W}$ has R-charge 2 and chiral primary operators satisfy the relation $\Delta = 3|R|/2$ [63]. Our operator $\mathcal{O}$ has R-charge

[2] S. Franco et al. [62] considered a general class of scalar kinetic terms, to which this example belongs.

$R = 2$ and is indeed chiral primary.

We claim that any solution to the classical equations of motion following from this action lifts to a solution of type IIB supergravity. The lift resembles the consistent truncations of ref. [64], and it also generalizes the Pope-Warner type compactifications of type IIB SUGRA [65]. The ten-dimensional metric for the lift has the form

$$ds^2 = \cosh\frac{\eta}{2}\, ds_M^2 + \frac{L^2}{\cosh\frac{\eta}{2}}\left[ds_V^2 + \cosh^2\frac{\eta}{2}(\zeta^A)^2\right] . \tag{2.6}$$

The metric on the manifold M is a solution to the five-dimensional equations of motion, while V is a two-complex-dimensional manifold with a Kähler-Einstein metric $g_{a\bar{b}}$ such that $R_{a\bar{b}} = 6g_{a\bar{b}}$. Let ω be the Kähler form on V. We construct a U(1) fiber bundle over V with the one form $\zeta^A = \zeta + 2A/3$ and $\zeta = d\psi + \sigma$ such that $d\zeta = 2\omega$.

In the case $A = \eta = 0$, the line element $ds_Y^2 = ds_V^2 + \zeta^2$ on the five-dimensional space Y is a Sasaki-Einstein metric. Moreover, introducing a radial coordinate $r > 0$, the line element $dr^2 + r^2 ds_Y^2$ is a Ricci flat metric on a Kähler manifold X with a conformal scaling symmetry $r \to \lambda r$. In other words, X is a Calabi-Yau cone.

Denoting $F = dA$ and

$$J = \sinh^2\eta\,(d\theta - 2A)\,, \tag{2.7}$$

we can write the self-dual five-form as

$$F_5 = \frac{1}{g_s}\left(\mathcal{F} + *\mathcal{F}\right)\,, \tag{2.8}$$

where

$$\mathcal{F} \equiv -\frac{1}{L}\cosh^2\frac{\eta}{2}\,(\cosh\eta - 5)\,\mathrm{vol}_M - \frac{2L^3}{3}(*_M F)\wedge\omega + \frac{L^4}{4\cosh^4\frac{\eta}{2}} J\wedge\omega^2\,, \tag{2.9}$$

$$*\mathcal{F} = L^4\frac{(\cosh\eta - 5)}{2\cosh^2\frac{\eta}{2}}\zeta^A\wedge\omega^2 + \frac{2L^4}{3}F\wedge\zeta^A\wedge\omega + \frac{L}{2}(*_M J)\wedge\zeta^A\,. \tag{2.10}$$

To specify the two-form gauge potentials we first consider the holomorphic three-form $\hat{\Omega}_3$ on the Calabi-Yau three-fold X normalized so that $\hat{\Omega}_3 \wedge \bar{\hat{\Omega}}_3 = 8\text{vol}_X$. This form decomposes as

$$\hat{\Omega}_3 = r^3 \left(\frac{dr}{r} \wedge \Omega_2 + \Omega_3 \right) , \tag{2.11}$$

where r is the radial coordinate of the cone. The form $e^{-3i\psi}\Omega_2$ is a primitive $(2,0)$ form on V satisfying [14]

$$\begin{aligned} d(e^{-3i\psi}\Omega_2) &= 3ie^{-3i\psi}\sigma \wedge \Omega_2 \,, \\ \Omega_2 \wedge \bar{\Omega}_2 &= 2\omega^2 \,. \end{aligned} \tag{2.12}$$

The two-form potentials are

$$\begin{aligned} F_2 &= L^2 \tanh \frac{\eta}{2} e^{i\theta} \Omega_2 \,, \\ F_2 &\equiv B_2 + i g_s C_2 \,. \end{aligned} \tag{2.13}$$

One can check that the ansatz given by (2.6), (2.8), and (2.13) leads to a consistent reduction with the effective five-dimensional lagrangian given by (2.2) and (2.3).

2.3 The phase transition

We are interested in studying the response of an SCFT to an R-charge chemical potential μ and a temperature T. One expects that for low enough T/μ, R-charged operators develop expectation values that spontaneously break the U(1) R-symmetry. At high temperatures, the field theory is dual to an electrically charged black hole in anti-de Sitter space, while at a sufficiently low temperature, the black-hole acquires scalar hair [38, 43]. The electrically charged black hole which solves the equations of motion following from (2.2), along with a negative cosmological constant $\Lambda = -6/L^2$, takes the form

$$
\begin{aligned}
ds^2 &= \frac{L^2}{z^2}\left[-f(z)\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right], \\
A &= \mu\left[1 - \left(\frac{z}{z_h}\right)^2\right] dt\,,
\end{aligned}
\tag{2.14}
$$

where

$$
f(z) = 1 + Q^2\left(\frac{z}{z_h}\right)^6 - (1+Q^2)\left(\frac{z}{z_h}\right)^4, \tag{2.15}
$$

with the charge $Q = 2z_h\mu/3$. The Hawking temperature of the black hole is

$$
T_H = \frac{2 - Q^2}{2\pi z_h}\,. \tag{2.16}
$$

At low temperatures, a hairy black hole solution with $\eta \neq 0$ becomes available. We find this solution numerically, using the techniques described, *e.g.,* in [60]. By a gauge choice, we can set $\theta = 0$. We require no deformation of the conformal field theory by the symmetry breaking operator $\mathcal{O}$ dual to η. So for small z,

$$
\eta = -z^3\left(\langle\mathcal{O}\rangle\,\frac{\kappa_5^2}{L^3} + \ldots\right), \tag{2.17}
$$

and the expectation value $\langle\mathcal{O}\rangle$ is the order parameter for breaking the $U(1)$ symmetry. With a black hole horizon at $z = z_h$, the other boundary conditions we impose are that

$$
\begin{aligned}
A_t(z{=}0) &= \mu\,, \\
A_t(z{=}z_h) &= 0\,.
\end{aligned}
\tag{2.18}
$$

Figure 2.1 gives $\langle\mathcal{O}\rangle$ as a function of T.

We also plot the difference in pressure between the electrically-charged black hole phase and the hairy black hole phase in figure 2.1. The pressure is a coefficient in the near boundary expansion of g_{tt}:

$$g_{tt} = -\frac{L^2}{z^2} + 2\kappa_5^2 P \frac{z^2}{L} + \dots . \tag{2.19}$$

It is related to the free energy via $\Omega = -P\,\mathrm{Vol}$. To within our numerical precision, $\partial \Delta\Omega / \partial T = 0$ at $T = T_0$, indicating a second order phase transition.

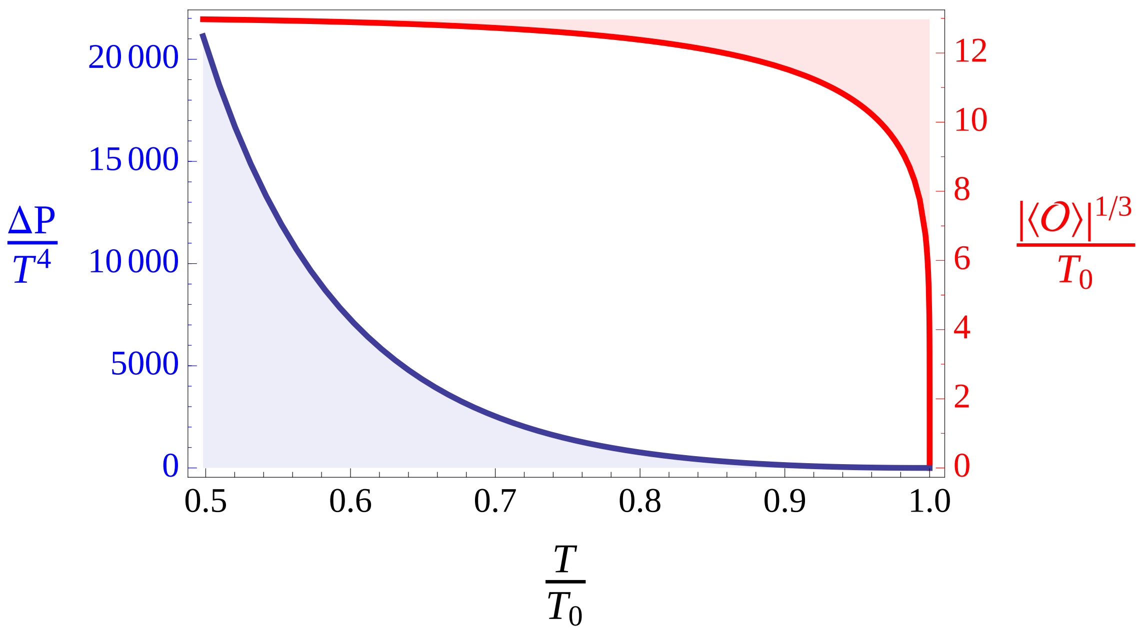

Figure 2.1: Upper-right plot: $|\langle \mathcal{O} \rangle|^{1/3}/T_0$ vs. T/T_0, where $\langle \mathcal{O} \rangle$ is expressed as multiples of L^3/κ_5^2. The critical temperature is $T_0 \approx 0.0607\mu$. Near T_0, $\langle \mathcal{O} \rangle \sim |T - T_0|^{1/2}$, indicating a mean-field critical exponent. Lower-left plot: $\Delta P/T^4$ vs. T/T_0, where ΔP is the difference in pressure between the broken and unbroken phases, calculated in the grand canonical ensemble. Near T_0, one has $\Delta P \sim (T - T_0)^2$, so the phase transition is second order.

We have also calculated the conductivity for this model as a function of frequency and T, using the techniques of ref. [43]. The results are qualitatively similar to those of a free $\Delta = 3$ scalar in the probe limit [66].

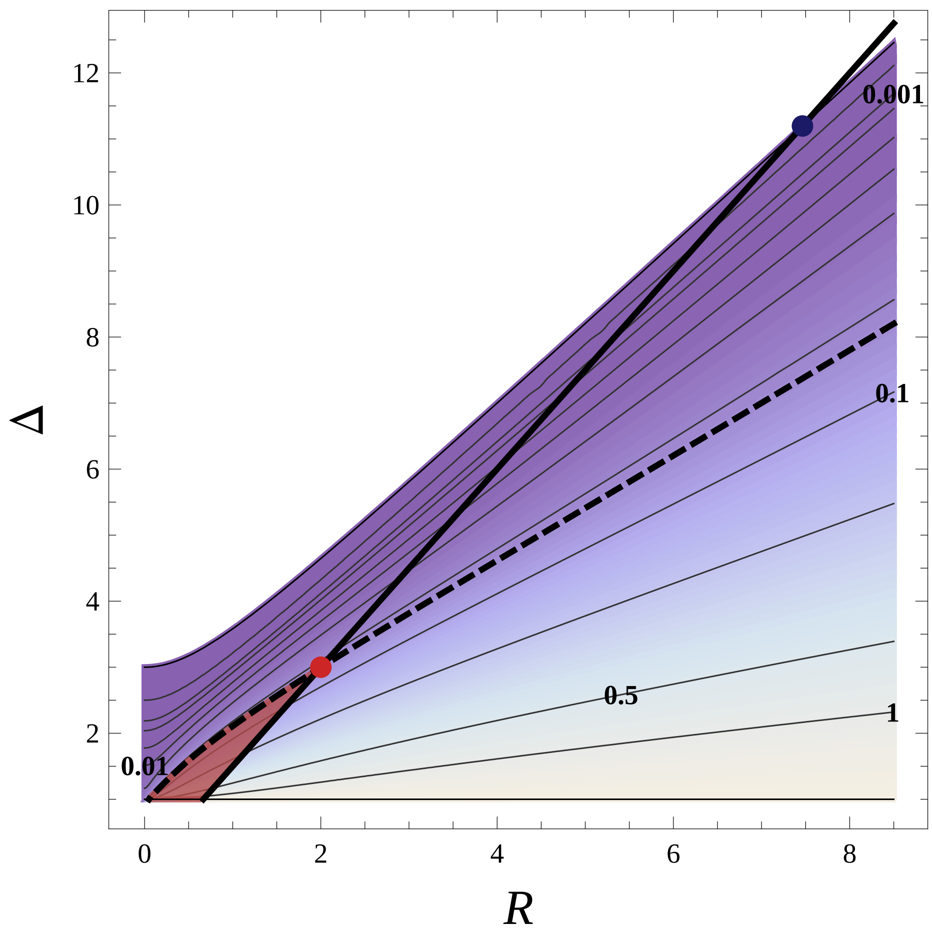

Figure 2.2: A contour plot of T_p/μ as a function of Δ and R. The numbers next to the contour lines represent T_p/μ. We need only consider scalars above the unitarity bound, $\Delta \geq 1$ [67]. The dark solid line is the BPS bound $\Delta = 3R/2$ [68]. Scalars which are less stable than the operator $\mathcal{O}$ are restricted to the triangular, shaded region near the lower-left corner.

2.3.1 Perturbative instabilities

Although our chiral primary leads to spontaneous breaking of the U(1) R-symmetry at low temperature if $\mu \neq 0$, these SCFTs typically have many operators with R-charge. It may be that there exists another operator which produces a phase transition at a higher temperature. Such an operator need not be a scalar. We focus on the case that this "less stable" operator is a scalar, and replace $\mathcal{L}_{\text{scalar}}$ with

$$-\frac{1}{2}\left[(\partial_\mu\eta)^2+\eta^2(\partial_\mu\theta-RA_\mu)^2+m^2\eta^2\right]+\frac{12}{L^2}\,, \tag{2.20}$$

representing the leading quadratic terms for a complex scalar of charge R and conformal dimension satisfying $m^2L^2=\Delta(\Delta-4)$.

These leading quadratic terms are enough to calculate the temperature T_p below which the electrically charged black hole becomes perturbatively unstable with respect to an exponentially growing mode of the scalar. We calculate T_p by looking for a zero mode solution of η in the electrically charged black hole background (2.14). Such a solution will have the leading behavior $\eta\sim z^\Delta$ near the boundary and should be finite at the horizon. Depending on higher order terms in the scalar action, which in general we don't know, T_p may be the point of an actual second order phase transition, as it was in section 2.2, or it may label a spinodal point in a first order phase transition, beyond which the electrically charged black hole ceases to be perturbatively stable.

We solved for this zero mode numerically, and the resulting T_p is plotted in figure 2.2 as a function of Δ and R. As described in [39], there is a critical curve in the R-Δ plane where $T_p=0$. This curve can be determined analytically by considering the behavior of the scalar field in an extremal electrically charged black hole solution:

$$R^2=\frac{2}{3}\Delta(\Delta-4)+2\,. \tag{2.21}$$

Note the BPS line $\Delta=3R/2$ intersects this curve, leaving a region of finite area in the R-Δ plane with $T_p>0$.

Note that T_p is a monotonically increasing function of R and a monotonically decreasing function of Δ. Moreover, along the BPS bound $\Delta=3R/2$, T_p is a decreasing function of Δ. These results suggest that the superfluid phase transition will be caused by an operator at or close to the BPS bound and of small Δ.

We found in section 2.2 that for SCFTs dual to D3-branes at Calabi-Yau singularities, there always exists an operator with $\Delta = 3$ that saturates the BPS bound. Given the existence of such an operator and corresponding T_0, scalars which are perturbatively less stable, should they exist, are restricted to a tiny corner of the R-Δ plane; see figure 2.2. We can in general check if there are any chiral primary operators with $\Delta < 3$. It is less straightforward to rule out unprotected operators with a $T_c > T_0$. Nevertheless, the expectation is that such operators do not exist, at least for strongly interacting theories with AdS/CFT duals. Large couplings are associated with large anomalous dimensions. In these AdS/CFT constructions, we expect generic operators to be dual to string states with masses of order the string scale, $m\ell_s \sim 1$, and thus $\Delta \sim (g_s N)^{1/4}$. It seems unlikely we will find any unprotected operator with a $T_c > T_0$.

2.3.2 A universal chiral primary operator

What chiral primary operators in an SCFT have $R = 2$ and $\Delta = 3$? Our supergravity solutions are dual to superconformal quiver gauge theories via the AdS/CFT correspondence. The lowest component $\mathcal{O}$ of the F-term in the Lagrangian describing these field theories takes the form

$$\mathcal{O} = \mathcal{W}(\phi_i) + \frac{32\pi i}{\sum_j} \tau_j \text{ tr } \lambda_{j\alpha}^2 \,, \tag{2.22}$$

where $\mathcal{W}$ is the superpotential, the gluino field λ_α is the lowest component of the superfield W_α, and the complex scalar fields ϕ_i are the lowest components of the chiral matter superfields Φ_i. The $\tau_j = \theta_j/2\pi + 4\pi i/g_j^2$ are the complexified gauge couplings, and the sum j runs over the gauge groups in the quiver. Because of conformal invariance and holomorphy arguments, we expect $\mathcal{O}$ to be a protected operator (up to non-perturbative corrections, wave-function rescaling, and possible mixing with a descendant of the Konishi operator). It is true that $\mathcal{F}$ has $R = 2$ and $\Delta = 3$. We claim that $\mathcal{O}$ is dual to the complexified scalar supergravity field (η, θ).

Consider the cases where the Sasaki-Einstein manifold is the sphere $Y = S^5$ or the level surface of the conifold $Y = T^{1,1}$. In the first case, the dual field theory is $\mathcal{N} = 4$ SU(N) SYM. In $\mathcal{N} = 1$ notation, the field theory has three chiral superfields X, Y, and Z transforming in the adjoint representation of SU(N), and a superpotential $\mathcal{W} \sim \text{tr}\, X[Y, Z]$. In the second case, the $\text{SU}(N) \times \text{SU}(N)$ field theory has bifundamental fields A_i and B_i, $i = 1, 2$ transforming under the $(N, \bar{N})$ and $(\bar{N}, N)$ representations of the gauge groups and a superpotential $\mathcal{W} \sim \epsilon_{ij}\epsilon_{kl}\, \text{tr}(A_i B_k A_j B_l)$. In both these cases, $\mathcal{O}$ is indeed dual to the complexified scalar (η, θ) [24].

Note that for S^5 and $T^{1,1}$, $\mathcal{O}$ will not cause the phase transition that breaks a U(1) R-symmetry. The reason is that there exist chiral primary operators for these SCFTs with lower conformal dimension. For S^5, $\text{tr}(X^2)$ has $\Delta = 2$ while for $T^{1,1}$, $\text{tr}(A_i B_j)$ has $\Delta = 3/2$, and both of these operators condense at a higher T_c. Thus, we need to look for SCFTs where $\mathcal{O}$ has the lowest conformal dimension among the chiral primaries.

One such theory is $S^5/\mathbb{Z}_7$ where the orbifold acts with weights $(1, 2, 4)$ on the $\mathbb{C}^3 \supset S^5$. The quiver field theory has $G = \text{SU}(N)^7$. The three chiral superfields X, Y, and Z of $\mathcal{N} = 4$ SYM become 21 fields X^i_{i+1}, Y^i_{i+2}, and Z^i_{i+4}. Here X^i_j indicates a field that transforms under the anti-fundamental of the i^{th} gauge group and the fundamental of the j^{th} and $X^{i+7}_j = X^i_j$. A chiral primary tied for smallest conformal dimension in this field theory is $\mathcal{O}$. (The other chiral primary is related to the beta deformation [69].) More generally, we expect an orbifold $S^5/\mathbb{Z}_n$ with weights (w_1, w_2, w_3) such that $n = w_1 + w_2 + w_3$ to be a candidate provided that the w_i are distinct and that $w_i \neq -w_j \mod n$ for all i and j.

Chapter 3

Emergent quantum near-criticality from baryonic black branes

This is work done in collaboration with Christopher P. Herzog, Igor R. Klebanov, and Silviu S. Pufu, and is published in [70]. This chapter is a lightly edited version of the published paper.

We find new black three-brane solutions describing the "conifold gauge theory" at nonzero temperature and baryonic chemical potential. Of particular interest is the low-temperature limit where we find a new kind of weakly curved near-horizon geometry; it is a warped product $AdS_2 \times \mathbb{R}^3 \times T^{1,1}$ with warp factors that are powers of the logarithm of the AdS radius. Thus, our solution encodes a new type of emergent quantum near-criticality. We carry out some stability checks for our solutions. We also set up a consistent ansatz for baryonic black two-branes of M-theory that are asymptotic to $AdS_4 \times Q^{1,1,1}$.

3.1 Introduction

Via the AdS/CFT correspondence [1–3], electrically charged black holes in space-times with negative cosmological constant yield insights into the physics of strongly-interacting systems at nonzero density. Recall that the correspondence relates semi-classical gravity in $d+1$ space-time dimensions to a strongly-interacting field theory in one fewer dimensions. There have been two kinds of approaches to these problems. In the bottom-up approach, a simple and phenomenological gravity model is constructed to which the AdS/CFT dictionary is then applied. It is assumed that there exists some field theory, of which we may have only a qualitative understanding, dual to this space-time, but using the correspondence, we can quickly and efficiently determine the phase structure, the equation of state, and transport coefficients. In the top-down approach, one considers a well-established AdS/CFT duality where the field theory is well-known but may have exotic symmetries and field content. Given the complexity of the known dual pairs, calculations are often more difficult. However, they are worth the extra effort, because they allow us to make precise and reliable statements about actual strongly interacting field theories.

Here we take the second approach and construct a novel type of charged black hole (or, more precisely, black 3-brane) using the baryonic symmetry of the conifold gauge theory [22] dual to the $AdS_5 \times T^{1,1}$ background of type IIB string theory. Our solution is similar to the Reissner-Nordström AdS_5 black hole but is more complicated because the compact space $T^{1,1}$ gets squashed by functions that depend on the radius. In the zero-temperature limit, we find that the near-horizon region becomes similar to $AdS_2 \times \mathbb{R}^3 \times T^{1,1}$ up to slowly varying logarithmic functions. The presence of the logarithms makes our IR solution a new kind of nearly conformal behavior. Thus, very interestingly, our solution exhibits "emergent quantum near-criticality," which could make it useful for exploring connections with condensed matter phenomena.

Recently, there has been much interest in bottom-up approaches to study field theories that undergo superfluid or superconducting phase transitions (see [31, 32] for reviews). Strong electron-electron interactions are believed to play an important role in the physics of high-temperature superconductors. References [57, 71–73] use a bottom-up approach to model a strongly-interacting system of fermions at nonzero density, and find evidence for the existence of a Fermi surface.

While the bottom-up approaches allow one to scan quickly through a number of simple gravity models and search for new phenomena, they have some disadvantages. A major issue is that the precise nature of the dual field theory is typically unclear, and one cannot be certain that it exists. Another disadvantage is related to the notion of a consistent truncation and its stability. The AdS/CFT correspondence in its original incarnation is a mapping between type IIB string theory in an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ super Yang-Mills theory in 3+1 dimensions. In order to reduce a ten-dimensional string theory to a manageable five-dimensional gravity theory, a consistent truncation is made that eliminates all but a small number of fields. The consistency of the truncation guarantees that a solution to the five-dimensional equations of motion for the remaining fields is also a solution to the full ten-dimensional system. However, nothing guarantees that this solution is a global or local minimum of the action in the ten-dimensional setting; indeed, often it is not. The simple gravity models in these bottom-up approaches would be consistent truncations if fit into the larger AdS/CFT framework, and as such they may have instabilities.

In top-down approaches, one of the easiest ways to charge a black hole is through the R-symmetry. The dual field theory often has an R-symmetry which maps to a gauge field in the gravity system. In the grand canonical ensemble, R-charge on the black hole translates into nonzero R-charge chemical potential μ in the field theory. In the most studied case of $\mathcal{N} = 4$ super Yang-Mills in $3 + 1$ dimensions, it is strongly suspected that at any nonzero μ, the theory is only metastable [74–76]. Moreover, if μ becomes sufficiently large compared to the temperature, the theory becomes thermodynamically and perturbatively unstable as well [35, 36, 77].

More generically, charged scalars, if their charge-to-mass ratio is sufficiently large, are a source of instabilities. Starting with refs. [38, 43], these charged scalars have been studied intensively in the context of modeling a superfluid or superconducting phase transition. Truncating such a charged scalar out of the gravity model also spuriously eliminates the phase transition. While it is true that the physical systems of interest often have a superconducting phase transition at very low temperatures, from a theoretical point of view, it is of value to have a model where one may reliably go to low temperatures without worrying about such instabilities.

In this chapter we construct a black three-brane charged under a baryonic $U(1)_B$ symmetry. This solution passes a number of stability checks. We consider the well-known conifold gauge theory and its dual pair, type IIB string theory in an $AdS_5 \times T^{1,1}$ background [22]. The $SU(N) \times SU(N)$ field theory with bifundamental fields A_i and B_j, $i, j = 1, 2$, has a global baryonic $U(1)_B$ symmetry. The corresponding $U(1)_B$ gauge field in AdS_5 comes from the R-R four-form with three indices along a topologically non-trivial three-cycle. This realization of the $U(1)$ symmetry makes our approach different from the previous attempts to embed charged AdS black holes into string theory. In particular, the nature of the charged objects is quite different. The gauge invariant operators with baryonic charge in the conifold gauge theory have conformal dimensions of order N. The smallest such operator in the conifold theory involves determinants of the bifundamental matter fields. In the string dual, such an operator maps to a wrapped D3-brane which may be studied semi-classically. We are able to show explicitly that this wrapped D3-brane has a charge-to-mass ratio that is too small to produce an instability. This check, however, is insufficient to demonstrate the stability of our solution because one of the neutral fields may condense as the temperature is decreased. We demonstrate stability with respect to one seemingly dangerous neutral mode, but leave investigation of other modes for the future.

In [78], another potential instability of charged brane backgrounds was suggested. Such an instability, called the "Fermi seasickness" in [78], is caused by nucleation of a spacetime filling D-brane towards the AdS boundary (for earlier discussions of similar D-brane instabilities see [75, 79]).[1] In the dual gauge theory, this corresponds to an instability with respect to the Coulomb branch, where certain mesonic operators develop vacuum expectation values.

In the background we are studying, this instability can be seen from computing the potential for probe D3-branes filling the $(t, \vec{x})$ directions. Numerical computations show that for temperatures greater than about 0.2μ, the charged black branes constructed here are stable with respect to such D3-brane nucleation, while for lower temperatures they become metastable. For any nonzero temperature, the D3-brane is attracted near the horizon, which means that there exists a potential barrier that for large N prevents brane nucleation to AdS infinity.

The rest of this chapter is structured as follows. In section 3.2, we review the details of the conifold gauge theory and its gravity dual. We also demonstrate a consistent truncation of type IIB supergravity (SUGRA) for the conifold background to a baryonic gauge field and two neutral scalars in five dimensions. Given the effective 5d Lagrangian, in section 3.3 we construct a metric, scalar, and gauge field ansatz for a baryonically charged black 3-brane that depends only on a single radial coordinate. The ansatz is invariant under a certain $\mathbb{Z}_2$ symmetry and leads to a system of non-linear ordinary differential equations. In section 3.4, we find their numerical solutions with $AdS_5 \times T^{1,1}$ boundary conditions at large r. Although we encounter difficulties at very low temperatures, the numerical work provides us with useful intuition concerning the $T = 0$ solution. In section 3.5 we find the near-horizon series expansion in the $T = 0$ limit and show that it has a near-AdS_2 structure. In section 3.6, we study the behavior of the smallest operator with baryonic charge in the conifold theory, the dibaryon. We show that its charge-to-mass ratio is too small to lead to an instability. In section 3.7 we carry out another stability check of our $T = 0$ solution, this time with respect to a neutral $\mathbb{Z}_2$-odd perturbation. Because of nontrivial mixing with

[1] We thank Eva Silverstein for suggesting to us that our background may suffer from this instability.

the $U(1)_R$ gauge field, the analysis requires us to generalize the ansatz of Section 3.2. Section 3.8 contains some final remarks and discussion. In Appendix 3.A, we find an analytic expression for our black hole in the small charge limit.

3.2 Conifold gauge theory and consistent truncation

The principal aim of this chapter is to study the AdS/CFT duality in presence of a chemical potential for baryon number. In a general large N gauge theory, the operators carrying baryon number have dimensions of order N, which distinguishes them from the "mesonic" operators whose dimensions are of order one in the large N limit. In AdS/CFT, the objects dual to baryonic operators are D-branes or M-branes wrapped over non-trivial cycles of the internal manifold. In the maximally supersymmetric version, which relates the $\mathcal{N} = 4$ SYM theory to $AdS_5 \times S^5$, there are no baryonic operators. Their absence is related to the fact that S^5 has no topologically non-trivial three-cycles that could be wrapped by D3-branes. However, there are many known examples where the AdS/CFT duality relates $AdS_5 \times Y$, where Y is a Sasaki-Einstein manifold, to $\mathcal{N} = 1$ superconformal gauge theories that have baryonic operators. The compact space Y typically has non-trivial three-cycles, so that the wrapped D3-branes are topologically stable and the corresponding gauge theory possesses baryonic $U(1)_B$ symmetries. We would like to turn on a chemical potential for the baryon number in the gauge theory; in the string dual this translates into turning on the R-R four-form gauge field that couples to the wrapped D3-branes.

We will present perhaps the simplest example of such a construction, in the context of the duality relating the $AdS_5 \times T^{1,1}$ background of type IIB string theory to the $\mathcal{N} = 1$ superconformal "conifold gauge theory" [22]. This duality is motivated by studying a stack of N D3-branes placed at the tip of the conifold, the Calabi-Yau cone $zw - uv = 0$. The explicit metric of its Sasaki-Einstein base, the $T^{1,1}$, is

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) + \frac{1}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \ . \tag{3.1}$$

The conifold gauge theory with gauge group $SU(N) \times SU(N)$ is coupled to bi-fundamental chiral superfields A_i, $i = 1, 2$ transforming as $(\mathbf{N}, \bar{\mathbf{N}})$, and B_j, $j = 1, 2$ transforming as $(\bar{\mathbf{N}}, \mathbf{N})$. These superfields form doublets under the global symmetries $SU(2)_A$ and $SU(2)_B$, respectively, and all of them carry R-charge 1/2. In addition to the $U(1)_R \times SU(2)_A \times SU(2)_B$ global symmetry, which on the gravity side is realized through isometries of $T^{1,1}$, the gauge theory has a baryonic $U(1)_B$ symmetry under which

$$A_k \to e^{i\theta} A_k \ , \qquad B_l \to e^{-i\theta} B_l \ . \tag{3.2}$$

The spectrum of gauge invariant operators splits into two sectors. The mesonic operators, which are not charged under the $U(1)_B$, have dimensions of order 1; the baryonic operators, which are charged under the $U(1)_B$, have dimensions of order N. The lowest dimension examples of mesonic operators are $\mathrm{Tr} A_i B_j$ of dimension 3/2, and $\mathrm{Tr} A_i \bar{A}_j$, $\mathrm{Tr} B_i \bar{B}_j$, $\mathrm{Tr}(A_i \bar{A}_i - B_j \bar{B}_j)$ of dimension 2. In general, the mesonic operators transform under the $U(1)_R \times SU(2) \times SU(2)$ geometric symmetry of $T^{1,1}$, but are neutral under the $U(1)_B$. The lowest dimension operators carrying the $U(1)_B$ charge are, for example, $\det A_1$ or $\det A_2$. These are the $m = \pm N/2$ states of the spin $N/2$ representation of $SU(2)_A$. The general form of these dimension $3N/4$ operators, $\mathcal{A}_m$, $m = -N, -N+1, \ldots, N-1, N$, may be found in [80]. They carry R-charge $N/2$, and we will normalize their baryon number to 1. The string theory objects dual to these operators are the D3-branes wrapping the (θ_1, ϕ_1, ψ) directions. Quantization of the (θ_2, ϕ_2) collective coordinate gives rise to the $N + 1$ degenerate ground states corresponding to the chiral operators $\mathcal{A}_m$.

Similarly, there exist chiral operators $\mathcal{B}_m$, $m = -N, -N+1, \ldots, N-1, N$, such that $\mathcal{B}_N = \det B_1$ and $\mathcal{B}_{-N} = \det B_2$. These operators have baryon number -1 and R-charge $N/2$; they are dual to D3-branes wrapping the (θ_2, ϕ_2, ψ) directions. Replacing these D3-branes by anti-D3 branes we find objects of baryon number 1 and R-charge $-N/2$ that are dual to the antichiral operators $\bar{\mathcal{B}}_m$ that include $\det\bar{B}_1$. Since the $U(1)_R$ charge couples to mesonic operators and may lead to instabilities mentioned above, we will be interested in objects charged under the $U(1)_B$ symmetry only. The simplest such vertex operators are the $(N+1)^2$ products $\mathcal{A}_{m_1}\bar{\mathcal{B}}_{m_2}$ which are dual to combinations of D3-branes wrapping both the (θ_1, ϕ_1, ψ) and (θ_2, ϕ_2, ψ) directions. Turning on a chemical potential for $U(1)_B$ is expected to create a nonzero spatial density of such wrapped D3-branes. Our goal is to determine the background produced by them. We will use the simplifying assumption that the wrapped D3-branes are appropriately smeared over the $T^{1,1}$ coordinates, so that our solution will have the full $SU(2) \times SU(2)$ symmetry.

The $U(1)_B$ gauge field in $AdS_5 \times T^{1,1}$ is contained in the components of the 4-form R-R-gauge field [30], $C_4 \sim A \wedge \omega_3$, where

$$
\begin{aligned}
\omega_2 &\equiv \frac{1}{2}\left(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2\right) \ , \\
\omega_3 &\equiv g_5 \wedge \omega_2 \ , \\
g_5 &\equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \ .
\end{aligned} \tag{3.3}
$$

Our ansatz for the self-dual 5-form field strength will therefore be[2]

$$
\begin{aligned}
F_5 &= \frac{1}{g_s}\left(\mathcal{F} + *\mathcal{F}\right) \ , \\
\mathcal{F} &= \frac{2L^4}{27}\omega_2 \wedge \omega_3 + \frac{L^3}{9\sqrt{2}} F \wedge \omega_3 \ , \\
*\mathcal{F} &= \frac{4}{L} e^{-20\chi/3} \mathrm{vol}_M + \frac{L^2}{3\sqrt{2}} e^{2\eta - \frac{4}{3}\chi} (*_M F) \wedge \omega_2 \ .
\end{aligned} \tag{3.4}
$$

The normalization of the terms involving F has been chosen so that the kinetic term for

[2] We use the conventions of appendix A for the Hodge dual.

F in the effective five-dimensional action is normalized canonically in the ultraviolet. Quantization of the 5-form flux requires that [81]

$$L^4 = 4\pi g_s N(\alpha')^2 \frac{27}{16} \ . \tag{3.5}$$

At non-linear order, we expect the additional components of F_5 to cause a violation of the Poincaré invariance of the 5d metric, and also to produce a squashing of the internal space $T^{1,1}$. A minimal consistent truncation of type IIB supergravity that contains these effects turns out to be

$$ds_{10}^2 = e^{-\frac{5}{3}\chi} ds_M^2 + L^2 e^{\chi} \left[\frac{e^{\eta}}{6} \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2 \right) + \frac{e^{-4\eta}}{9} g_5^2 \right] \ . \tag{3.6}$$

If $\chi = \eta = 0$, this metric reduces to the direct product between the non-compact space M and $T^{1,1}$. The scalar χ controls the overall size of $T^{1,1}$, while η introduces a stretching of the $U(1)$ fiber relative to the two two-spheres. These scalar fields in AdS_5 are dual to operators of conformal dimension 8 and 6, respectively.

The above ansatz yields a consistent truncation of type IIB SUGRA with the effective five-dimensional lagrangian

$$\mathcal{L}_{\text{eff}} = R - \frac{1}{4} e^{-\frac{4}{3}\chi + 2\eta} F_{\mu\nu}^2 - 5(\partial_\mu \eta)^2 - \frac{10}{3}(\partial_\mu \chi)^2 - V(\eta, \chi) \ , \tag{3.7}$$

where the potential for the two neutral scalars is given by

$$V(\eta, \chi) \equiv \frac{8}{L^2} e^{-\frac{20}{3}\chi} + \frac{4}{L^2} e^{-\frac{8}{3}\chi} \left(e^{-6\eta} - 6e^{-\eta} \right) \ . \tag{3.8}$$

The scalar kinetic terms and potential had been previously determined in [82]. However, the scalar coupling to the $U(1)_B$ gauge fields was not considered there. Indeed, (3.7) shows that the gauge kinetic term depends on the scalars; to study the baryonic black holes we need to include the squashing of $T^{1,1}$.

3.3 Equations of motion

Using (3.7), we will look for time-independent charged black 3-brane solutions. We impose rotation and translation symmetry in the 3 spatial directions, but the Poincaré symmetry is obviously broken. Therefore we will use the ansatz

$$ds_M^2 = -ge^{-w}dt^2 + \frac{dr^2}{g} + \frac{r^2}{L^2}\sum_{i=1}^{3}(dx^i)^2 \ , \tag{3.9}$$

where g and w are functions of r. This choice of parametrization (see, for example, [83]) will prove useful in simplifying the form of the equations of motion. To turn on the baryonic charge density and chemical potential, we need to consider only the time component of the $U(1)_B$ gauge field, $A = \Phi(r)dt$, so that the field strength is $F = dA = \Phi' dr \wedge dt$. We will also assume that the scalars χ and η depend on the radial coordinate r only.

We should note that our full ten-dimensional ansatz (3.4)–(3.6) preserves the $\mathbb{Z}_2$ space-time inversion symmetry where $(t, \vec{x}) \to (-t, -\vec{x})$ is accompanied by the interchange of the two 2-spheres, $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$. The forms ω_3 and ω_2 change sign under this transformation, but the terms $dt \wedge dr \wedge \omega_3$ and $dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega_2$ present in F_5 are invariant. In the gauge theory, this $\mathbb{Z}_2$ symmetry appears to correspond to $(t, \vec{x}) \to (-t, -\vec{x})$ accompanied by $A_i \leftrightarrow \bar{B}_i$.

With this ansatz, the effective one-dimensional lagrangian is

$$L_{\text{eff}} = -5r^3 g e^{-\frac{1}{2}w}\Big(\eta'^2 + \frac{2}{3}\chi'^2\Big) + \frac{1}{2}r^3 e^{\frac{1}{2}w} e^{2\eta - \frac{4}{3}\chi}\Phi'^2 - 3(gr^2)' e^{-\frac{1}{2}w} - r^3 e^{-\frac{1}{2}w} V \ . \tag{3.10}$$

The equations of motion following from this lagrangian are

$$\chi'' + \chi'\left(\frac{3}{r} + \frac{g'}{g} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) - \frac{\Phi'^2}{10g}e^{w+2\eta-\frac{4}{3}\chi} - \frac{3}{20g}\frac{\partial V}{\partial \chi} = 0\ , \tag{3.11a}$$

$$\eta'' + \eta'\left(\frac{3}{r} + \frac{g'}{g} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) + \frac{\Phi'^2}{10g}e^{w+2\eta-\frac{4}{3}\chi} - \frac{1}{10g}\frac{\partial V}{\partial \eta} = 0\ , \tag{3.11b}$$

$$g' + g\left(\frac{2}{r} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) + \frac{r\Phi'^2}{6}e^{w+2\eta-\frac{4}{3}\chi} + \frac{r}{3}V = 0\ , \tag{3.11c}$$

$$\Phi'' + \Phi'\left(\frac{3}{r} + \frac{1}{2}w' + 2\eta' - \frac{4}{3}\chi'\right) = 0\ , \tag{3.11d}$$

$$w' + \frac{10r}{9}\left(3\eta'^2 + 2\chi'^2\right) = 0\ , \tag{3.11e}$$

where primes denote derivatives with respect to r, and V is the scalar potential defined in (3.8).

Equation (3.11d) can be integrated once yielding

$$\Phi' = \frac{Q}{r^3}e^{-\frac{1}{2}w-2\eta+\frac{4}{3}\chi}\ , \tag{3.11f}$$

where Q is an integration constant related to the charge of the black hole. This is the conservation equation for baryonic charge. From now on, we will use (3.11f) instead of (3.11d). Plugging (3.11f) into equations (3.11a)–(3.11c), we get the added bonus of eliminating the dependence on w, so the remaining equations are

$$\chi'' + \chi'\left(\frac{3}{r} + \frac{g'}{g} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) - \frac{Q^2}{10r^6 g}e^{-2\eta+\frac{4}{3}\chi} - \frac{3}{20g}\frac{\partial V}{\partial \chi} = 0\ , \tag{3.12a}$$

$$\eta'' + \eta'\left(\frac{3}{r} + \frac{g'}{g} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) + \frac{Q^2}{10r^6 g}e^{-2\eta+\frac{4}{3}\chi} - \frac{1}{10g}\frac{\partial V}{\partial \eta} = 0\ , \tag{3.12b}$$

$$g' + g\left(\frac{2}{r} + \frac{5r}{3}\eta'^2 + \frac{10r}{9}\chi'^2\right) + \frac{Q^2}{6r^5}e^{-2\eta+\frac{4}{3}\chi} + \frac{r}{3}V = 0\ . \tag{3.12c}$$

To reduce the system from five to three coupled differential equations was the main motivation for using the ansatz (3.9).

3.4 Numerical solutions

Solving the coupled non-linear equations (3.11a)–(3.11f) is in general a difficult task. We will mostly rely on numerical work, but in some limits will be able to present analytical formulae. The simplest situation is when $Q = 0$ and we find the well-known black 3-brane solution in AdS_5. In this solution, the shape of $T^{1,1}$ does not depend on r, i.e. the scalars χ and η vanish. For small values of Q we can use perturbation theory in this small parameter to obtain an analytic expansion of the solution. This exercise is carried out in Appendix 3.A where we find that the scalars χ and η are now of order Q^2 and acquire a dependence on r. In the next section, we present some analytical results in the extremal limit. However, for intermediate values of Q, we know of no good analytical methods and resort to numerical ones.

3.4.1 Setup for numerics

Finite-temperature solutions are found numerically by a standard shooting technique. The numerical solver is seeded close to the boundary, which is located at $r \to \infty$, by using a series expansion that imposes the correct boundary conditions.

The first task is to determine the boundary conditions. All our ten-dimensional metrics must asymptote to $AdS_5 \times T^{1,1}$ at large r. This means that the asymptotic boundary conditions that we require as $r \to \infty$ are

$$\begin{aligned} w &\to 0 \,, \qquad g = \frac{r^2}{L^2} + \mathcal{O}(L^2/r^2) \,, \\ \eta &\to 0 \,, \qquad \chi \to 0 \,. \end{aligned} \tag{3.13}$$

In order to describe states in the dual field theory at nonzero baryon chemical potential, we also require $\Phi \to \Phi_0$, for a constant Φ_0 that will be related to the chemical potential shortly. Generically, solutions satisfying these boundary conditions will have an event horizon at some value $r = r_h$ where $g(r)$ vanishes. The standard boundary conditions required at the horizon are that w, η, and χ should be finite.

From (3.11a)–(3.11f) one can work out a series solution at large r that satisfies the boundary conditions (3.13):

$$\begin{aligned}
w &= \mathcal{O}\left((L/r)^{12}\log^2(r/L)\right) \ , \\
g &= \frac{r^2}{L^2} + \frac{g_2 L^2}{r^2} + \frac{Q^2 L^4}{12 r^4} + \mathcal{O}\left((L/r)^{10}\log^2(r/L)\right) \ , \\
\Phi &= \Phi_0 - \frac{Q}{2r^2} + \mathcal{O}\left((L/r)^{8}\log(r/L)\right) \ , \\
\chi &= -\frac{Q^2 L^2}{200 r^6} + \frac{\chi_8 L^8}{r^8} + \mathcal{O}\left((L/r)^{10}\right) \ , \\
\eta &= \frac{Q^2 L^2 \log(r/L)}{80 r^6} + \frac{\eta_6 L^6}{r^6} + \mathcal{O}\left((L/r)^{10}\log(r/L)\right) \ .
\end{aligned} \tag{3.14}$$

All higher order terms in the series are determined in terms of g_2, Φ_0, χ_8, η_6, and Q.

To proceed further, it is useful to review some of the symmetries of our ansatz. The equations of motion (3.11) and the boundary conditions are invariant under some scaling symmetries which act with the charges summarized in table 3.1. We say that a quantity X has charge q under a scaling symmetry if

$$X \to \lambda^q X \ . \tag{3.15}$$

The first symmetry in table 3.1 is a formal way of expressing the arbitrariness of a

Symmetry	e^{-w}	g	Φ	η	χ	t	$\vec{x}$	r	L
type A	0	0	0	0	0	1	1	1	1
type B	0	2	1	0	0	-1	-1	1	0

Table 3.1: Charges under the scaling symmetries of the equations of motion (3.11) and of the boundary conditions (3.13). These charges are defined as in (3.15).

choice of units in the bulk. It can be used to set $L = 1$. The second symmetry can be used to put $r_h = 1$, but when shooting from the boundary it is more useful to employ the same symmetry to set $g_2 = -1$ instead. We can also set $\Phi_0 = 0$, by using the fact that the potential Φ appears only through its derivatives in the equations of motion, and does not appear in the metric. Having eliminated Φ_0 and fixed Q, we are left with two parameters, χ_8 and η_6, that need to be tuned in order to match to regular solutions at the horizon. Trial values for these parameters are determined from the small Q (large T) expansion of Appendix A, and the results are used to go to progressively smaller temperatures.

Thermodynamic quantities such as the energy density ϵ, entropy density s, temperature T, charge density ρ, and chemical potential μ, can be computed from the following formulae:

$$\begin{aligned} &\epsilon = -\frac{3}{2}\frac{g_2}{\kappa_5^2 L} , \qquad s = \frac{2\pi r_h^3}{\kappa_5^2 L^3} , \qquad T = \frac{g'(r_h)e^{-w_h/2}}{4\pi} , \\ &\rho = \frac{Q}{2\kappa_5^2 L^2} , \qquad \mu = \frac{\Phi_0 - \Phi_h}{L} , \end{aligned} \tag{3.16}$$

where κ_5 is the five-dimensional gravitational constant, $w_h \equiv w(r_h)$, and $\Phi_h \equiv \Phi(r_h)$. Using the parameters from [81] we find

$$\frac{1}{\kappa_5^2} = \frac{27N^2}{64\pi^2 L^3} . \tag{3.17}$$

The energy density can also be computed from

$$\epsilon = \frac{3}{4}\left(Ts + \mu\rho\right) , \tag{3.18}$$

which follows from $\epsilon = Ts - p + \mu\rho$ and the tracelessness of the stress tensor, $\epsilon = 3p$.

3.4.2 Results

Using the shooting method described above, we were able to find numerical black hole solutions for a fairly large range of temperatures. Here are the main features we observe:

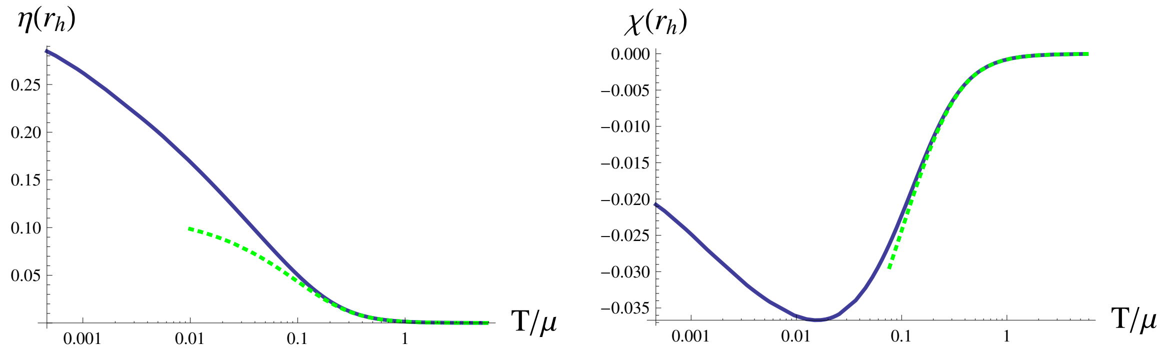

Figure 3.1: The behavior of the horizon values of the scalars as a function of T/μ. The dotted line is the high-temperature behavior obtained from the low Q expansion in appendix 3.A. According to the zero-temperature expansion developed in section 3.5, η_h and χ_h should diverge at $T = 0$. We believe we are far from this regime.

- Our numerical results agree with the analytical computations presented in Appendix 3.A in the limit of high temperatures. See figure 3.1 for a comparison of the values of the scalars at the horizon found numerically to those predicted by the analytic formula (3.55).

- We were able to construct numerical black hole solutions only for temperatures higher than $T \approx 0.0005\mu$ because of loss of numerical precision at lower temperatures. We believe the lowest temperatures we attained are not low enough to provide a thorough check of the analytical zero-temperature horizon expansion constructed in the next section. In fact, we will argue towards the end of the next section that we expect the zero-temperature expansion to become a good approximation to the near-extremal solutions when $\log\log \mu/T \gg 1$, which is beyond the range of temperatures where our numerics are reliable.

- As one decreases the temperature, the Bekenstein-Hawking entropy seems to approach a non-zero value (see figure 3.2). This fact will be confirmed analytically in the next section, where we find that as T approaches zero, $s/\rho \approx 2.09$. However, as will be discussed later, there are stringy effects that make our solution trustworthy only down to an exponentially low temperature of order $\mu e^{-\text{const.}\times(g_s N)^{2/3}}$.

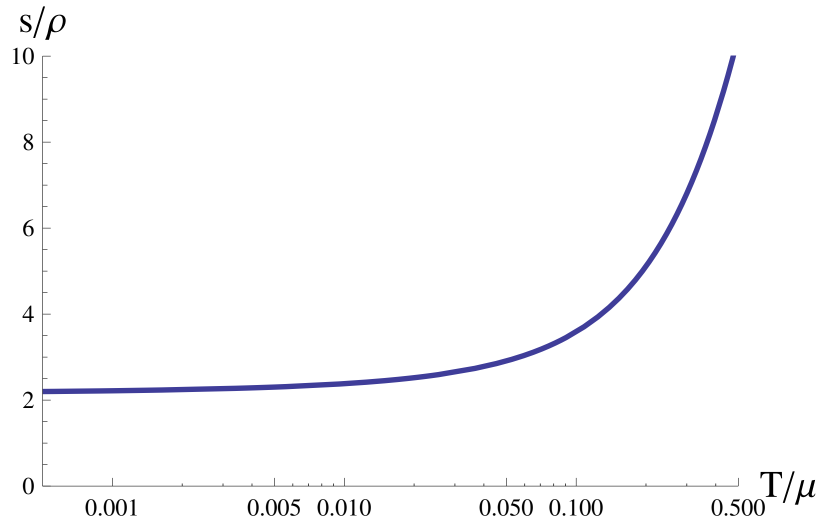

Figure 3.2: Entropy density normalized by charge density, as a function of temperature over chemical potential. The entropy is seen to go to a nonzero constant as $T \to 0$, in agreement with the zero-temperature expansion discussed in section 3.5.

- Lastly, one may worry that at low enough temperatures the curvature of these backgrounds might get large and the supergravity approximation might break down. As can be seen from figure 3.3, the horizon value of the ten-dimensional Riemann tensor squared is uniformly bounded from above. In fact, in the following section we will show that all curvature invariants should vanish at the extremal horizon.

3.5 Near-AdS_2 near horizon

We now find a horizon series expansion at zero temperature. We will restrict ourselves to the set of equations (3.12), since one can always use (3.11e) and (3.11f) to find $w(r)$ and $\Phi(r)$ afterwards. In the following, we use the symmetries in Table 3.1 to set $L = r_h = 1$.

A guess for the zero-temperature value of the charge Q that appears explicitly in equations (3.12) can be found from the following line of reasoning based on properties

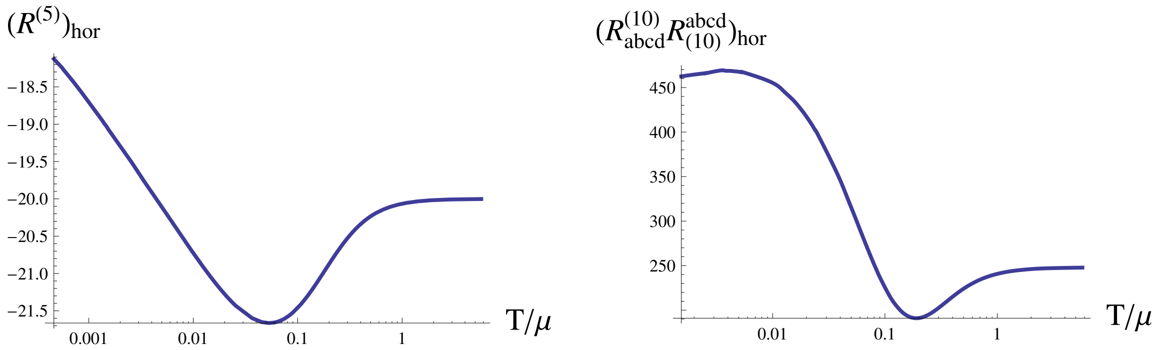

Figure 3.3: The five-dimensional Ricci scalar (left), and the square of the ten-dimensional curvature (right) at the horizon, in units where $L = 1$, as a function of T/μ. As discussed in section 3.5, all curvature invariants evaluated at the extremal horizon vanish.

of the nonzero temperature solutions. At the horizon, $g(r_h = 1) = 0$; since g must be positive outside the horizon, we need $g'(1) \geq 0$. Evaluating eq. (3.11c) at the horizon, we get

$$g'(1) = -\frac{1}{6}\, e^{-6\eta_h - \frac{20}{3}\chi_h} \left(16 e^{6\eta_h} + 8 e^{4\chi_h} - 48 e^{5\eta_h + 4\chi_h} + Q^2 e^{4\eta_h + 8\chi_h}\right) \geq 0 \ , \qquad (3.19)$$

which implies

$$b_1 \leq e^{4\chi_h} \leq b_2 \ , \qquad (3.20)$$

where

$$b_{1,2} = \frac{4}{Q^2 e^{4\eta_h}} \left(-1 + 6 e^{5\eta_h} \pm \sqrt{1 - 12 e^{5\eta_h} - (Q^2 - 36) e^{10\eta_h}}\right) . \qquad (3.21)$$

For $Q > 6$, both b_1 and b_2 are negative or complex for any η_h, resulting in no possible range of χ_h. For $Q < 6$, the smallest positive value of η_h for which $b_{1,2}$ are real is the one for which $b_1 = b_2$. The fact that there are no solutions for $Q > 6$ implies from

(3.16) that

$$\frac{s}{\rho} = \frac{4\pi}{Q} \geq \frac{2\pi}{3} \approx 2.09 \,, \tag{3.22}$$

an inequality that should hold at all T within the supergravity approximation.

When $b_1 = b_2$, we have

$$\eta_c = -\frac{1}{5}\log(6-Q) \,, \qquad \chi_c = -\frac{1}{20}\log(6-Q) - \frac{1}{4}\log\frac{Q}{4} \,, \tag{3.23}$$

which implies that η_c and χ_c go to infinity as $Q \to 6$, while

$$\eta_c - 4\chi_c = \log\frac{Q}{4} \to \log\frac{3}{2} \qquad \text{as } Q \to 6. \tag{3.24}$$

A reasonable guess is that at zero temperature $Q = 6$ and the scalars χ and η diverge at the extremal horizon, while $\eta - 4\chi$ approaches $\log\frac{3}{2}$.

With $Q = 6$, we use the method of dominant balance to find the asymptotic behavior of zero-temperature solutions at the horizon. By adding arbitrary power series to these dominant terms, we find we can satisfy the equations of motion. We obtain a solution of the form

$$\begin{aligned}
\chi &= -\frac{1}{20}\log\left(\frac{2187}{16}\tilde{r}\right) + \frac{1063}{1000}\tilde{r} + \cdots \,, \\
\eta &= -\frac{1}{5}\log(18\tilde{r}) + \frac{463}{250}\tilde{r} + \cdots \,, \\
g &= \tilde{r}^{13/3}\left(\frac{93312 \cdot (12)^{1/3}}{25} + \cdots\right) \,, \\
w &= \frac{5}{36\tilde{r}} + \frac{77}{30}\log\tilde{r} + \tilde{w} + \cdots \,,
\end{aligned} \tag{3.25}$$

where

$$\tilde{r} \equiv r - 1 \,, \tag{3.26}$$

and the dots stand for regular Taylor series.

As $\tilde{r} \to 0$, $g_{00} = -g\, e^{-w}$ has an essential singularity. This prompts us to introduce a better coordinate,

$$y = 3^{-193/120} 2^{-107/120} 5^{3/40} e^{\tilde{w}/2}\, e^{5/(72\tilde{r})} \;, \tag{3.27}$$

which becomes large near the horizon. In terms of this coordinate, and reinstating the factors of L^2, the near-horizon metric becomes

$$ds_{10}^2 = \frac{L^2}{y^2}\Big(-(\log y)^{-37/20} dt^2 + \frac{1}{6\,(30)^{1/4}}(\log y)^{1/4} dy^2\Big) + L^2 \left(\frac{1215}{128}\right)^{1/12} (\log y)^{-1/12} d\vec{x}^2 \\ + \left[\frac{L^2}{6}\left(\frac{15}{8}\right)^{-1/4} (\log y)^{1/4} \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) + \frac{L^2}{9}\left(\frac{125}{96}\right)^{1/4} (\log y)^{-3/4} g_5^2\right] . \tag{3.28}$$

This is a warped product $AdS_2 \times \mathbb{R}^3 \times T^{1,1}$ with warp factors that are powers of the logarithm of the AdS radius y. The appearance of the logarithmic warp factors makes it a novel type of "nearly conformal" IR behavior. Note that the 3d spatial components of the metric tend to zero in the IR as $g_{ij} \sim (\log y)^{-1/12}\delta_{ij}$; this is an important difference from the RNAdS metric where they approach a constant. Remarkably, the extra logarithms actually reduce the curvature, so that we can trust the supergravity approximation everywhere: we have checked that the 10d Kretschmann invariant and the 5d Ricci scalar scale as $R_{abcd}R^{abcd} \sim (\log y)^{-1/2}$ and $R \sim (\log y)^{-1/3}$ respectively. Such an asymptotic reduction of curvature due to the appearance of logarithms also occurs in the UV region of the warped deformed conifold [82, 84].

One can use the asymptotic solution to estimate what happens at temperatures so low that the horizon is located deep inside the near-AdS_2 region, at $\tilde{r}_h \ll 1$. If we assume that (3.25) approximately holds down to the horizon, we can estimate using (3.16) that $T/\mu \sim e^{-\frac{5}{72\tilde{r}_h}}$ up to power law corrections in $\tilde{r}_h$ and numerical factors.

From (3.25), one can then compute the values of the scalars at the near-extremal horizon:

$$\eta_h = 4\chi_h = -\frac{1}{5}\log\tilde{r}_h + \mathcal{O}(\tilde{r}_h^0) = \frac{1}{5}\log\log\frac{\mu}{T} + \mathcal{O}((T/\mu)^0)\ . \tag{3.29}$$

We therefore expect the approach to the extremal limit to be very slow and hard to investigate numerically because one has to go to exponentially small temperatures.

This numerical suggestion of the existence of an exponentially small scale compared to the chemical potential μ in the conifold gauge theory is corroborated by a stringy argument. In the type IIB string theory context, our solution cannot be trusted for arbitrarily low temperatures.[3] In the $T = 0$ solution the ψ-circle shrinks to zero size at the horizon (see the last term of (3.28)), and the standard approach is to T-dualize along this direction for $\log y >$ const. $\times\ (g_s N)^{2/3}$ where the size of the circle becomes of the order of the string length $\sqrt{\alpha'}$. Estimating the temperature at which the size of the ψ-circle at the horizon reaches the string scale, we find

$$\log\frac{\mu}{T} \sim (g_s N)^{2/3}\ . \tag{3.30}$$

It is remarkable that application of string theoretic arguments to our black brane suggests that in the conifold gauge theory at baryon chemical potential μ there exists an energy scale of order $\mu e^{-\text{const.}\times(g_s N)^{2/3}}$. From the point of view of the type IIB σ-model with coupling $\sim (g_s N)^{-1/4}$, such a scale can arise only non-perturbatively. Below this exponentially small scale, the gauge theory presumably exhibits some new effects that can be studied by T-dualizing the solution and lifting it to M-theory. While the background (3.25) has a non-vanishing Bekenstein-Hawking entropy and is smooth, stringy effects become important when the ψ-circle becomes small, and these effects could remove the conflict with the conventional statement of the Third Law

[3] We are very grateful to J. Maldacena and E. Witten for pointing this out to us.

of Thermodynamics [85]. Since the type IIB backgrounds can be trusted when the ψ-circle is above the string scale, we expect the entropy to be approximately constant for $\mu e^{-\text{const.}\times(g_s N)^{2/3}} < T \ll \mu$, i.e. when the horizon is inside the near-AdS_2 throat. It would be very interesting to provide a microscopic origin of this entropy of order N^2 by studying the underlying D3-brane system, which involves two intersecting stacks of D3-branes wrapped over the $T^{1,1}$.

In order to match the asymptotic solution (3.25) to the appropriate behavior at the boundary, we generically need a two-parameter family of solutions, in addition to the integration constant $\tilde{w}$, which is related to a choice of units for time in the boundary theory. Following [83], we linearize around this leading behavior, to find non-analytic pieces that we might have missed. There are two solutions to the linearized equations, of the form

$$\begin{aligned} \delta\chi &= e^{\alpha/\tilde{r}}\tilde{r}^{c_1}(1+\cdots) \ , \\ \delta\eta &= 4\delta\chi + e^{\alpha/\tilde{r}}\tilde{r}^{c_2}(1+\cdots) \ , \\ \delta g &= e^{\alpha/\tilde{r}}\tilde{r}^{c_3}(1+\cdots) \ , \end{aligned} \tag{3.31}$$

where the corrections are just series in positive powers of $\tilde{r}$. The values of α and c_i for the two solutions are

$$\alpha = -\frac{5}{72} \ , \qquad c_1 = \frac{1}{20} \ , \qquad c_2 = c_1 + 1 \ , \qquad c_3 = c_1 + \frac{10}{3} \ , \tag{3.32}$$

and

$$\alpha = -\frac{5}{144}\big(\sqrt{21}-1\big) \ , \qquad c_1 = \frac{179\sqrt{21}-189}{360} \ , \qquad c_2 = c_1 \ , \qquad c_3 = c_1 + \frac{13}{3} \ . \tag{3.33}$$

For a scalar in AdS_2, the behavior of linearized modes as a function of the radius

y is

$$\phi(y,t) \sim \phi_0(t) y^{1-\Delta_{\rm IR}} + A(t) y^{\Delta_{\rm IR}} \ , \tag{3.34}$$

where $\phi_0(t)$ is the source for an operator of dimension $\Delta_{\rm IR}$, and $A(t)$ is its expectation value [30]. In the case at hand, since the background is AdS_2 up to powers of $\log y$, we expect (3.34) to hold up to similar powers. The perturbations (3.31) are indeed of this form. The first of these solutions corresponds to a source for an operator of $\Delta_{\rm IR} = 2$; the second to a source for $\Delta_{\rm IR} = (1+\sqrt{21})/2$. Both of these operators are irrelevant in the nearly conformal quantum mechanics found in the IR; hence, they produce perturbations that decay at large y.

Another type of perturbation that we can study easily is a minimally coupled massless scalar, for example the dilaton. In the near-AdS_2 region, the minimal scalar solutions behave as $a + b\, y (\log y)^{4/5} + \cdots$. We identify the dual operator as a marginal operator with $\Delta_{\rm IR} = 1$. In the case of the dilaton, the operator is exactly marginal because the string coupling is a parameter in our solution.

The extremal solution described above is reminiscent of the "run-away" attractor flows described in [86]. Typically, an attractor flow is a set of solutions to the supergravity equations of motion where the ultraviolet behavior is not universal, but in the infrared the scalars approach fixed values. In the run-away case at least one of these fixed values is at infinity. Our extremal solution would be of run-away type because the scalars η and χ diverge at the extremal horizon, even though the combination $\eta - 4\chi$ stays finite. An important difference between the extremal solutions from [86] and the one we found is that in [86] the entropy density vanishes at extremality while in our case it does not.

3.6 Baryonic operators and D3-brane probes

In this section we investigate whether the baryonic black branes constructed above are stable with respect to condensation of baryonic operators in the conifold gauge theory. As was already described, these operators are dual to wrapped D3-branes in the dual supergravity. These wrapped D3-branes act like charged particles; there is a competition between the gravitational attraction and electrostatic repulsion between the particle and the charged black brane. To test stability, we propose a simple thought experiment. If we place such a wrapped D3-brane into the geometry and the brane falls into the black hole, we conclude the black hole is stable. However, if the brane finds some meta-stable minimum outside the horizon, we conclude the black hole is unstable and more such wrapped D3-branes can bubble off the horizon, find their way to the minimum in the potential, and reduce the baryonic charge on the black hole. It is possible that more exotic bound states of D-branes may lead to instabilities. We leave an investigation of such issues to future work.

We follow an analysis similar to that in section (1.1.6). The probe D3-brane action takes the form

$$S_{D3} = -\mu_3 \int d^4x\, e^{-\phi} \sqrt{-\det G_{ab}} + \epsilon\, \mu_3 \int C_4 \ , \tag{3.35}$$

where $\mu_3 = (2\pi)^{-3}(\alpha')^{-2}$, α' is related to the string tension, and ϕ is the dilaton. For our supergravity solution, the dilaton is a constant we take to be related to the string coupling, $g_s = e^{\phi}$. The parameter ϵ is equal to one, but we leave it arbitrary so that we can tune the charge of the D3-brane. We assume the metric ansatz (3.6) and (3.9).

We assume the probe brane sits at constant θ_2, ϕ_2, x_i, and r and wraps the remaining four directions, including time. From S_{D3} and the ansatz for F_5 (3.4) we deduce a potential for the D3-brane:

$$V(r) = \frac{3N}{4L} e^{-\frac{1}{2}w(r) - \eta(r) + \frac{2}{3}\chi(r)} \sqrt{g(r)} - \epsilon \frac{3N\sqrt{2}}{8L} \Phi(r) \ . \tag{3.36}$$

Comparing (3.36) with (1.34), we see that in the UV where the scalars η and χ are negligible, $e^{w(r)} \approx 1$ and $g \approx r^2/L^2$, the wrapped D3-brane corresponds to a particle in RNAdS with $m/q = \sqrt{2}$. As shown at the end of section 1.1.6, this ratio corresponds to the critical value m/q_{crit} at which in RNAdS the potential $V(r)$ has a local minimum at $r = r_h$ in the extremal case. There is no reason to expect that the critical value of m/q_{crit} should be the same in the presence of scalars, but we will now show that this is nevertheless true.

The simplest demonstration is to plot $V(r)$ numerically. At the lowest temperatures we can access, there is no minimum in the potential when $\epsilon = 1$. However, for $\epsilon > 1$ and T sufficiently low, there is such a minimum, suggesting the D3-brane does indeed have this critical ratio of charge to mass. Indeed, the minimum is observed to occur for $\epsilon > \epsilon_0$, where ϵ_0 is some critical value larger than 1. In figure 3.4, we plotted the dependence of this critical value on T/μ. We see that as the temperature goes to zero, the critical value goes to 1. The position of the minimum moves towards the horizon, showing that wrapped D3-branes have the marginal ratio of charge to mass that means they barely escape condensation.

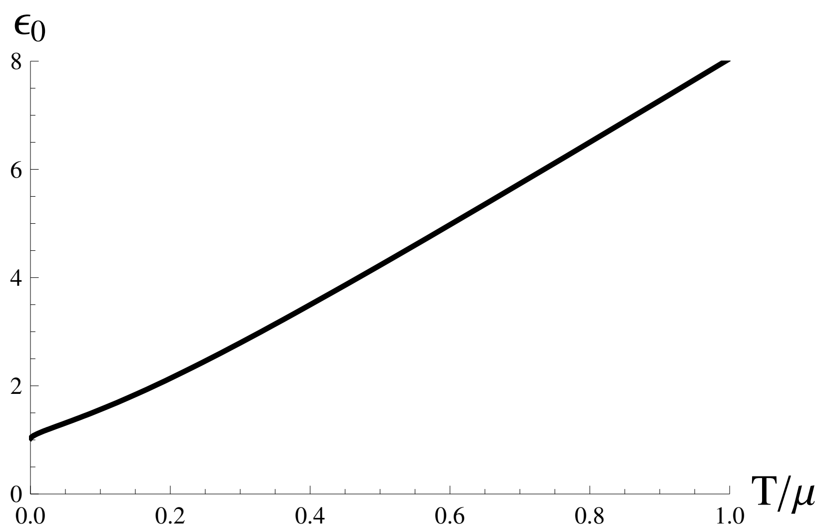

Figure 3.4: Critical value of the parameter ϵ above which condensation occurs, as a function of T/μ. Just above ϵ_0, the position of the potential minimum goes towards the horizon as $T \to 0$.

Some analytic support for this claim comes from studying the conditions that the minimum in the potential disappear, $V'(r_*) = V''(r_*) = 0$. Restoring $\epsilon = 1$, these two vanishing conditions imply the relation

$$e^{\eta(r_*)} = \frac{3}{2} e^{4\chi(r_*)} \ . \tag{3.37}$$

However, we know this condition is satisfied in the $T \to 0$ limit at the horizon from (3.24). Thus the system just begins to become unstable at $T = 0$, and we expect no phase transition.

More explicitly, we examine the potential (3.36) in the $T \to 0$ limit. From the zero-temperature series expansion (3.25), we find that

$$V(r) = N e^{-\frac{5}{72(r-1)} - \frac{1}{2}\tilde{w}} (r-1)^{1/20} \left((1-\epsilon) \frac{324 \cdot 18^{1/6}}{5} (r-1) + O(r-1)^2 \right) \ , \tag{3.38}$$

where we set $L = r_h = 1$. Thus, the leading term in this series expansion vanishes when $\epsilon = 1$.

3.7 Another stability check

The results of the previous section show that the black holes with baryonic charge we have constructed are stable against the simplest condensing operators that carry nonzero baryonic charge. Recall that all such operators have large dimensions of order N. One may worry about a different kind of instability where at low enough temperatures there is a phase transition driven by operators that are uncharged under the baryonic symmetry. That such a phase transition is in principle possible was noted in [39, 40, 60] for the case of the RNAdS black hole. In that case, all uncharged operators with UV conformal dimension smaller than 3 (when the gauge theory is $3+1$-dimensional) could trigger such an instability.

In this section we will study the stability of certain modes in the IR near-AdS_2 region of the $T = 0$ solution. For a minimally coupled scalar,

$$\Delta_{\mathrm{IR}\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m^2 L^2_{AdS_2}} \,. \tag{3.39}$$

Hence, the Breitenlohner-Freedman (BF) stability bound in AdS_2 is

$$m^2 L^2_{AdS_2} \geq -\frac{1}{4} \,, \tag{3.40}$$

and the dimension $\Delta_{\mathrm{IR}-}$ is allowed only in a narrow range of m^2 above this bound [30]. When the BF bound is violated the dimensions become complex, and the modes exhibit the oscillatory behavior as a function of the radius that is characteristic of "bad tachyons" (see for example [87]). This simple analysis is not directly applicable to the case of interest to us because the background is not exactly AdS_2, and the scalars are not minimally coupled. However, we will adopt a similar stability criterion. In the linearized approximation, an instability will be associated with the presence of complex dimensions and the ensuing oscillatory behavior of modes.

The baryonic black 3-branes studied in this chapter come from a ten-dimensional type IIB construction, so one can study fluctuations of various (uncharged) supergravity fields and check their stability. While a full analysis of all possible modes was not performed, we will demonstrate stability with respect to perturbations associated with a field theory operator of UV conformal dimension $\Delta = 2$. This operator is $\mathrm{Tr}(A_i \bar{A}_i - B_j \bar{B}_j)$, and its dual supergravity field is the "resolution mode" of the conifold, λ, that allows the two S^2's to have different sizes. This is the most relevant mode that is odd under the $\mathbb{Z}_2$ space-time inversion symmetry accompanied by the interchange of the two 2-spheres. We thought that this mode was the most likely to cause an instability because it saturates the BF stability bound in $AdS_5 \times T^{1,1}$; luckily, as we show, it does not destroy the stability of the near-AdS_2 solution.

The challenge here is that λ mixes with the time components of certain gauge fields, already at the linearized level. We were nevertheless able to find a consistent set of supergravity fields that include λ and that decouple from all other fluctuations, providing a more general (non-linear) consistent truncation of type IIB supergravity than the one considered in section 3.2. Indeed, the consistent truncation we find reduces to the one in section 3.2 in a particular limit. The full ten-dimensional ansatz and the effective five-dimensional action are given in section 3.7.1. In order to examine the stability of the near-AdS_2 geometry, in section 3.7.2 we linearize the equations of motion around the baryonic black brane background and develop a horizon series expansion at zero temperature using the explicit solution discussed in section 3.5. Remarkably, the linearized equations contain mixings between the modes and effective mass terms that stabilize all potentially unstable modes within this ansatz.

3.7.1 A more general consistent truncation

A consistent truncation that extends (3.4)–(3.6) to include the resolution mode of $T^{1,1}$ can be constructed as follows. Compared to (3.4)–(3.6), this truncation has three additional fields: the scalar field λ, which is the resolution mode of the conifold, and the spin-one fields A_R and $\tilde{A}_R$, which mix to give a gauge field corresponding to the R-symmetry of the gauge theory as well as a massive spin-one field. The metric ansatz is

$$ds_{10}^2 = e^{-5\chi/3} ds_M^2 + L^2 e^{\chi} \left[\frac{e^{\eta+\lambda}}{6} (d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{e^{\eta-\lambda}}{6} (d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{e^{-4\eta}}{9} \left(g_5 + \frac{3}{\sqrt{2}L} A_R \right)^2 \right] . \tag{3.41}$$

Defining

$$g_5^A \equiv g_5 + \frac{3}{\sqrt{2}L} A_R \ , \tag{3.42}$$

the self-dual five-form can be written as

$$
\begin{aligned}
F_5 &= \frac{1}{g_s}\left(\mathcal{F} + *\mathcal{F}\right)\ , \\
\mathcal{F} &= \frac{2L^4}{27}\omega_2\wedge\omega_2\wedge g_5 + \frac{L^3}{9\sqrt{2}}F\wedge\omega_2\wedge g_5^A \\
&\quad - \frac{L^3}{18\sqrt{2}}\tilde{F}_R\wedge dg_5\wedge g_5^A + \frac{L^3}{18\sqrt{2}}\tilde{A}_R\wedge dg_5\wedge dg_5\ , \\
*\mathcal{F} &= \frac{4}{L}e^{-\frac{20}{3}\chi}\mathrm{vol}_M + \frac{L^2}{3\sqrt{2}}e^{-\frac{4}{3}\chi+2\eta}\left[\cosh(2\lambda)*_M F - \sinh(2\lambda)*_M\tilde{F}_R\right]\wedge\omega_2 \\
&\quad + \frac{L^2}{6\sqrt{2}}e^{-\frac{4}{3}\chi+2\eta}\left[\cosh(2\lambda)*_M\tilde{F}_R - \sinh(2\lambda)*_M F\right]\wedge dg_5 \\
&\quad + \frac{2\sqrt{2}}{3}e^{-4\chi-4\eta}*_M(A_R+\tilde{A}_R)\wedge g_5^A\ ,
\end{aligned}
\tag{3.43}
$$

where we have defined the field strengths $F = dA$, $F_R = dA_R$, and $\tilde{F}_R = d\tilde{A}_R$.

The effective five-dimensional action for this consistent truncation can be written as a sum of a bulk piece and a Chern-Simons term:

$$
S = \int d^5x\,\sqrt{-g}\mathcal{L} + S_{\mathrm{CS}}\ . \tag{3.44}
$$

The bulk lagrangian $\mathcal{L}$ is given by

$$
\begin{aligned}
\mathcal{L} &= R - \frac{10}{3}(\partial_\mu\chi)^2 - 5(\partial_\mu\eta)^2 - (\partial_\mu\lambda)^2 - V(\eta,\chi,\lambda) \\
&\quad - \frac{1}{4}e^{2\eta-\frac{4}{3}\chi}\left[\cosh(2\lambda)\left(F_{\mu\nu}F^{\mu\nu} + \tilde{F}^R_{\mu\nu}\tilde{F}^{\mu\nu}_R\right) - 2\sinh(2\lambda)F_{\mu\nu}\tilde{F}^{\mu\nu}_R\right] \\
&\quad - \frac{1}{8}e^{-4\eta+\frac{8}{3}\chi}F^R_{\mu\nu}F^{\mu\nu}_R - \frac{4}{L^2}e^{-4\eta-4\chi}(A^R_\mu + \tilde{A}^R_\mu)^2\ ,
\end{aligned}
\tag{3.45}
$$

where

$$
V(\eta,\chi,\lambda) = \frac{8}{L^2}e^{-\frac{20}{3}\chi} + \frac{4}{L^2}e^{-\frac{8}{3}\chi}\left(e^{-6\eta}\cosh(2\lambda) - 6e^{-\eta}\cosh\lambda\right)\ . \tag{3.46}
$$

The Chern-Simons part of the action is

$$S_{\rm CS} = \frac{1}{2\sqrt{2}} \int \tilde{A}_R \wedge \tilde{F}_R \wedge F_R - \frac{1}{2\sqrt{2}} \int A \wedge F \wedge F_R \ . \tag{3.47}$$

From now on, we restrict to a time-independent background where we have rotation and translation symmetry in the three non-compact directions x^i. The most general ansatz with these symmetries is where the scalars χ, η, and λ depend only on r, and where

$$\begin{aligned} ds_M^2 &= -g e^{-w} dt^2 + \frac{dr^2}{g} + \frac{r^2}{L^2} \sum_{i=1}^{3} (dx^i)^2 \ , \\ A = \Phi(r) dt \ , &\qquad A_R = \Phi_R(r) dt \ , \qquad \tilde{A}_R = \tilde{\Phi}_R(r) dt \ , \end{aligned} \tag{3.48}$$

generalizing the ansatz used in section 3.3. In this case, the equations of motion following from the effective action (3.44) admit two conserved charges Q_B and Q_R associated to the baryonic symmetry and to the R-symmetry of the gauge theory, respectively. The conservation equations take the form

$$\Phi' \cosh(2\lambda) - \tilde{\Phi}_R' \sinh(2\lambda) = \frac{Q_B}{r^3} e^{-\frac{1}{2}w - 2\eta + \frac{4}{3}\chi} \ , \tag{3.49a}$$

$$\tilde{\Phi}_R' \cosh(2\lambda) - \Phi' \sinh(2\lambda) - \frac{1}{2} e^{-6\eta + 4\chi} \ \Phi_R' = \frac{Q_R}{r^3} e^{-\frac{1}{2}w - 2\eta + \frac{4}{3}\chi} \ , \tag{3.49b}$$

where, as usual, primes denote derivatives with respect to r. The first of these two equations is a generalization of (3.11f). In the UV, the scalars are negligible and the above two equations reduce to $\Phi' \approx \frac{1}{r^3} Q_B$ and $\tilde{\Phi}_R' - \frac{1}{2}\Phi_R' \approx \frac{1}{r^3} Q_R$, justifying the interpretation of Q_B as the baryonic charge and of Q_R as the R-charge.

3.7.2 Horizon expansion of linearized fluctuations

As mentioned in section 3.3, our background is invariant under a $\mathbb{Z}_2$ symmetry that acts by flipping the sign of the non-compact coordinates, $(t, \vec{x}) \to (-t, -\vec{x})$, and interchanging the two spheres in the compact space, $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$. Recall that in the background solution only χ, η, and Φ are nonzero. The fluctuations around this background are distinguished by their parity properties under the $\mathbb{Z}_2$ symmetry: $\delta\Phi$, $\delta\chi$, and $\delta\eta$ are even, while $\delta\Phi_R$, $\delta\tilde{\Phi}_R$, and $\delta\lambda$ are odd. Due to the $\mathbb{Z}_2$ symmetry of the background, the even and odd linearized fluctuations cannot mix. Here we are interested in the resolution mode $\delta\lambda$, so we will focus on the mixing among the odd fluctuations.

The linearized equations are

$$\begin{aligned}
\frac{e^{\frac{1}{2}w}}{r^3}\left(\frac{r^3 g}{e^{\frac{1}{2}w}}\delta\lambda'\right)' + e^{2\eta-\frac{4}{3}\chi+w}\Phi'\left[\Phi'\delta\lambda - \delta\tilde{\Phi}'_R\right] + \frac{4}{L^2}e^{-6\eta-\frac{8}{3}\chi}(3e^{5\eta}-2)\delta\lambda = 0\ , \\
\frac{g}{r^3 e^{\frac{1}{2}w}}(r^3 e^{-4\eta+\frac{8}{3}\chi+\frac{1}{2}w}\delta\Phi'_R)' - \frac{16}{L^2}e^{-4\eta-4\chi}(\delta\Phi_R + \delta\tilde{\Phi}_R) = 0\ , \\
\frac{g}{r^3 e^{\frac{1}{2}w}}(r^3 e^{2\eta-\frac{4}{3}\chi+\frac{1}{2}w}\delta\tilde{\Phi}'_R)' - 2g e^{2\eta-\frac{4}{3}\chi}\delta\lambda'\Phi' - \frac{8}{L^2}e^{-4\eta-4\chi}(\delta\Phi_R + \delta\tilde{\Phi}_R) = 0\,,
\end{aligned} \tag{3.50}$$

where χ, η, Φ, g, and w are evaluated at their background values given in (3.11f) and (3.25) for the zero-temperature extremal solution. Let's focus on this extremal solution and find a series expansion in $r-1$ (we set $L = r_h = 1$). This calculation is similar to that of the non-analytic contributions to the background given at the end of section 3.5. Since (3.50) is a system of three second order differential equations, there are six linearly independent solutions whose leading behaviors are of the form

$$\begin{aligned}
\delta\Phi_R &= e^{\frac{\alpha}{\tilde{r}}}\tilde{r}^{b_1}\left(\delta\Phi_R^{(0)} + \delta\Phi_R^{(1)}\tilde{r} + \cdots\right)\ , \\
\delta\tilde{\Phi}_R &= e^{\frac{\alpha}{\tilde{r}}}\tilde{r}^{b_2}\left(\delta\tilde{\Phi}_R^{(0)} + \delta\tilde{\Phi}_R^{(1)}\tilde{r} + \cdots\right)\ , \\
\delta\lambda &= e^{\frac{\alpha}{\tilde{r}}}e^{\frac{5}{72\tilde{r}}}\tilde{r}^{b_3-\frac{21}{20}}\left(\delta\lambda^{(0)} + \delta\lambda^{(1)}\tilde{r} + \cdots\right)\ ,
\end{aligned} \tag{3.51}$$

with $\tilde{r} \equiv r - 1$ as in (3.26). The coefficients α and b_i are given in table 3.2. All six solutions satisfy the $U(1)_R$ charge conservation condition (3.49b) at the linearized level. Of the modes that do not grow with y (I, III, IV, and V), only mode III requires a non-vanishing Q_R, while the others satisfy the conservation equation with $Q_R = 0$. The crucial fact is that α is real for all six solutions, so there are no oscillatory solutions

Solution	α	b_1	b_2	b_3	Δ_{IR}
I	$-\frac{5}{36}$	$\frac{83}{30}$	b_1+1	b_1+1	2, source
II	$\frac{5}{72}$	$-\frac{163}{60}$	b_1+1	b_1+1	2, VEV
III	$-\frac{5}{72}$	$\frac{21}{20}$	b_1	b_1	0, VEV
IV	0	0	b_1	b_1	0, source
V	$-\frac{5(1+\sqrt{5})}{144}$	$\frac{63-7\sqrt{5}}{120}$	b_1	b_1	$\frac{1+\sqrt{5}}{2}$, source
VI	$-\frac{5(1-\sqrt{5})}{144}$	$\frac{63+7\sqrt{5}}{120}$	b_1	b_1	$\frac{1+\sqrt{5}}{2}$, VEV

Table 3.2: The coefficients of the perturbative expansion (3.51) and the IR dimensions of the corresponding operators. The solution IV is in fact an exact pure gauge mode for which $\Phi_R = -\tilde{\Phi}_R = \text{const}$.

in the near-AdS_2 region. The absence of oscillatory solutions means that the black 3-branes with baryonic charge constructed in the previous sections are likely to be stable with respect to the perturbations (3.50).

We discussed in section 3.5 how near the extremal horizon, the geometry is $AdS_2 \times \mathbb{R}^3 \times T^{1,1}$ up to slowly varying logarithmic factors. We can thus ask what the effective dimensions of the operators dual to the modes given in table 3.2 are. Changing variables to the AdS_2 coordinate y defined in (3.27), we see that $\delta\lambda$ behaves for the six solutions as y^{-1}, y^2, y^0, y, $y^{\frac{1-\sqrt{5}}{2}}$, and $y^{\frac{1+\sqrt{5}}{2}}$, respectively.

Using (3.34), the dimensions Δ_{IR} corresponding to various perturbations are given in the last column of table 3.2. Since solution IV is exact, has $\lambda \equiv 0$, and is pure gauge, we suspect that the $\Delta_{\mathrm{IR}} = 0$ modes we are seeing correspond to the conserved charge operator in the effective quantum mechanics. This is consistent with the fact that mode III, which produces a VEV of this operator, is seen to correspond to non-vanishing Q_R. In this chapter we only study solutions with vanishing R-charge, so the charged modes with $\Delta_{\rm IR} = 0$ are not allowed. The remaining dimensions we find, 2 and $1 + \frac{\sqrt{5}}{2}$, correspond to irrelevant operators from the point of view of the IR near-AdS_2 theory. The sources for such operators correspond to modes that fall off near the horizon as y^{-1} or $y^{\frac{1-\sqrt{5}}{2}}$. Since the operators are irrelevant, we expect that inducing them in the IR theory will not destroy the near-conformal IR solution we find.

3.8 Discussion

This work initiates studies of black hole solutions charged under baryonic symmetries. Such solutions are asymptotic to $AdS \times Y$, and the baryonic $U(1)_B$ symmetries appear due to the non-trivial topology of the Einstein space Y. We discussed the type IIB example $AdS_5 \times T^{1,1}$ in some detail. Perhaps our most surprising finding is that the type IIB charged 3-brane solution develops, in the zero-temperature limit, a novel kind of near-horizon region, which is a warped product $AdS_2 \times \mathbb{R}^3 \times T^{1,1}$ with warp factors that are logarithmic in the AdS radius. This supergravity solution is smooth because the logarithms decrease the curvature of the solution; in fact, all curvatures approach zero at the horizon. In this sense this solution is reminiscent of the UV region of another solution based on the conifold, with a topologically non-trivial 3-form flux turned on [82, 84]. That warped deformed conifold solution was supersymmetric and automatically stable. In the present case, where the only non-vanishing supergravity fields are the metric and the self-dual 5-form, the solution does not seem to preserve any supersymmetry, and its stability is a serious issue. We carried out some highly non-trivial stability checks for our solution.

We have shown that the simplest objects charged under the baryonic $U(1)_B$, namely the wrapped D3-branes, do not condense. This still leaves the possibility that one of the neutral fields might cause an instability. Our solution preserves a certain $\mathbb{Z}_2$

symmetry and we have checked stability with respect to one of the modes odd under the $\mathbb{Z}_2$. This well-known mode, dual to the operator $\mathrm{Tr}(A_i \bar{A}_i - B_j \bar{B}_j)$, turns on the difference of the sizes of the two 2-spheres that is present in a small resolution of the conifold, and also mixes with the $U(1)_R$ gauge field. We leave further studies of stability for future work. We also note that we have encountered difficulties in extending our numerical solution all the way to zero temperature. At the lowest temperature we have been able to reach numerically, $w_h/2 \approx 0.85$, which means that the near-AdS_2 throat is only beginning to develop. It would be interesting to construct the full numerical $T = 0$ solution that matches onto the near-horizon form that we found analytically.

The zero-temperature solution is however threatened by another potential instability of our construction. This is the "Fermi seasickness" suggested in [78], which is caused by the nucleation of spacetime-filling D branes outside the black hole horizon. The analysis of this instability can be carried out similar to section 3.6, by calculating the potential for probe spacetime-filling D3 branes. The results are shown in Figure 3.5. It turns out that our background is stable with respect to this kind of nucleation for $T \gtrsim 0.19\mu$. For smaller temperatures, the charged black branes become metastable: spacetime filling D3-branes are attracted to the horizon when they are close to it, and this can be shown analytically to hold even for the zero-temperature solutions of section 3.5; however, the branes can tunnel out to the AdS boundary. The tunneling rate goes to zero for large N, so the metastability might not be a problem in the limit we're considering.

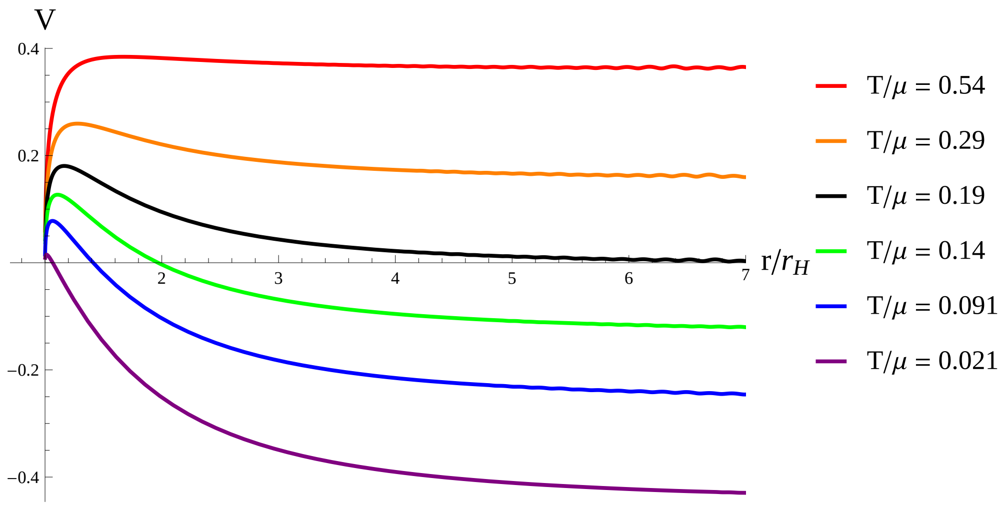

Figure 3.5: The potential for a probe spacetime-filling D3 brane as a function of the distance from the horizon. The branes can tunnel out to the AdS boundary for temperatures smaller than about 0.19μ.

3.A Small-charge limit

Approximate solutions can be found in the small Q limit. Defining the dimensionless parameter $\tilde{Q} \equiv QL/r_h^3$, the small Q expansion takes the form

$$\begin{aligned} w &= \tilde{Q}^2 \delta w^{(2)} + \tilde{Q}^4 \delta w^{(4)} + \cdots \ , \\ g &= \frac{r^2}{L^2}\left(1 - \frac{r_h^4}{r^4}\right) + Q^2 \delta g^{(2)} + Q^4 \delta g^{(4)} + \cdots \ , \\ \Phi &= \frac{Q}{2}\left(\frac{1}{r_h^2} - \frac{1}{r^2}\right) + \tilde{Q}^3 \delta\Phi^{(3)} + \cdots \ , \\ \eta &= \tilde{Q}^2 \delta\eta^{(2)} + \tilde{Q}^4 \delta\eta^{(4)} + \cdots \ , \\ \chi &= \tilde{Q}^2 \delta\chi^{(2)} + \tilde{Q}^4 \delta\chi^{(4)} + \cdots \ , \end{aligned} \tag{3.52}$$

where the starting point of the expansion obtained by setting $Q = 0$ corresponds to the AdS-Schwarzschild solution. The small Q approximation (3.52) can be alternatively thought of as a large temperature expansion.

To second order in Q, defining $\rho \equiv r/r_h$, one finds the solution

$$\begin{aligned}
\delta w^{(2)} &= 0 \ , \qquad \delta g^{(2)} = \frac{1}{12r^4}\left(1-\rho^2\right) \ , \\
\delta\eta^{(2)} &= \frac{1}{20}\left(\frac{1-4\rho^4}{2\rho^2} - Q_{-\frac{3}{2}}\left(1-2\rho^4\right)\right) \ , \\
\delta\chi^{(2)} &= -\frac{1}{40}\left(\frac{1+3\rho^2-6\rho^4}{\rho^2} + 3(1-2\rho^4)\log\frac{\rho^2}{1+\rho^2}\right) \ .
\end{aligned} \tag{3.53}$$

This solution obeys the boundary conditions (3.14) at the conformal boundary and is also regular at the black hole horizon.

The thermodynamic quantities (3.16) become in this limit

$$\begin{aligned}
\rho &= \frac{Q}{2\kappa_5^2 L^2} \ , \qquad s = \frac{2\pi r_h^3}{\kappa_5^2 L^3} \ , \qquad \epsilon = \frac{r_h^4}{8\kappa_5^2 L^5}\left(12+\tilde{Q}^2\right) \ , \\
\mu &= \frac{Q}{2Lr_h^2} \ , \qquad T = \frac{r_h}{24\pi L^2}\left(24-\tilde{Q}^2\right) \ .
\end{aligned} \tag{3.54}$$

One can check explicitly that the relation (3.18) is satisfied. It is also useful to note that the values η_h and χ_h of η and χ at the horizon are

$$\eta_h = \frac{\pi-3}{40}\tilde{Q}^2 \ , \qquad \chi_h = -\frac{\log 8 - 2}{40}\tilde{Q}^2 \ . \tag{3.55}$$

Chapter 4

Membranes with topological charge

This is a lightly edited version of work done with Igor R. Klebanov and Silviu S. Pufu, and published in [88].

If the second Betti number b_2 of a Sasaki-Einstein manifold Y^7 does not vanish, then M-theory on $AdS_4 \times Y^7$ possesses "topological" $U(1)^{b_2}$ gauge symmetry. The corresponding Abelian gauge fields come from three-form fluctuations with one index in AdS_4 and the other two in Y^7. We find black membrane solutions carrying one of these $U(1)$ charges. In the zero temperature limit, our solutions interpolate between $AdS_4 \times Y^7$ in the UV and $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 in the IR. In fact, the $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 background is by itself a solution of the supergravity equations of motion. These solutions do not appear to preserve any supersymmetry. We search for their possible instabilities and do not find any. We also discuss the meaning of our charged membrane backgrounds in a dual quiver Chern-Simons gauge theory with a global $U(1)$ charge density. Finally, we present a simple analytic solution which has the same IR but different UV behavior. We reduce this solution to type IIA string theory, and perform T-duality to type IIB. The type IIB metric turns out to be a product of the squashed Y^7 and the extremal BTZ black hole. We discuss an interpretation of this type IIB background in terms of the $(1+1)$-dimensional CFT on D3-branes partially wrapped over the squashed Y^7.

4.1 Introduction

As was emphasized in sections 1.1.6 and 3.1, charged black hole solutions embedded in string or M theory are often thermodynamically disfavored at small temperatures. While a low-temperature phase transition can be very interesting for applications to superfluidity or superconductivity, for other classes of applications it is desirable that the symmetric phase be stable down to very low (or even vanishing) temperature [57, 71–73, 89]. If so, then there exists a quantum critical phase described by the $AdS_2 \times \mathbb{R}^p$ extremal near-horizon region of the RNAdS background.

In chapter 3, a black 3-brane charged under a topological "baryonic" charge was studied. This is a solution asymptotic to $AdS_5 \times T^{1,1}$, with the topological charge carried by D3 branes wrapped around the nontrivial three-cycle of $T^{1,1}$. In general, for backgrounds of the form $AdS_{p+2} \times Y$, in addition to the isometries of the Sasaki-Einstein space Y, there may exist some non-R $U(1)$ symmetries. The corresponding gauge fields in AdS_{p+2} arise due to the non-trivial topology of Y. The number of such topological $U(1)$ symmetries is given by the second Betti number, b_2, of the internal space Y. In general, the nth Betti number b_n equals the number of linearly independent harmonic n-forms on the manifold Y, each of these forms representing a generator of the de Rham cohomology $H^n(Y)$. In the 10d examples from type IIB supergravity, the space Y is five-dimensional and by the Poincaré duality $b_2 = b_3$; in the 11d examples, the space Y is seven-dimensional, and the Poincaré duality implies $b_2 = b_5$.

While the string theory solution of chapter 3 passes many stability checks, it suffers from "Fermi seasickness", an instability due to nucleation of spacetime filling D3 branes outside the horizon. This chapter will look at a similar construction in M theory that avoids this instability.

Analogously to the solution studied in chapter 3, the connection between topology and supergravity fluctuations comes about as follows [24, 30, 90]. In $AdS_{p+2} \times Y$ compactifications, harmonic forms on Y are all that is needed to construct a consistent linearized set of fluctuations that includes massless gauge fields in AdS_{p+2}, one gauge field for each of the linearly independent harmonic forms. Consider the case of M-theory Freund-Rubin compactifications of the form $AdS_4 \times Y^7$, where Y^7 is a seven-dimensional Sasaki-Einstein manifold with $b_2 > 0$. Denoting by $\omega_2^{(i)}$ the $b_2 (= b_5)$ linearly independent harmonic two-forms on Y^7 and by $\omega_5^{(i)}$ their seven-dimensional Hodge duals (which in this case are harmonic five-forms on Y^7), one can consider the following consistent set of linearized fluctuations of eleven-dimensional supergravity:

$$\delta A_3 = \sum_{i=1}^{b_2} A^{(i)} \wedge \omega_2^{(i)} \ , \qquad \delta A_6 = \sum_{i=1}^{b_2} \tilde{A}^{(i)} \wedge \omega_5^{(i)} \ , \qquad d\tilde{A}^{(i)} = *_4 dA^{(i)} \ , \tag{4.1}$$

where $A^{(i)}$ and $\tilde{A}^{(i)}$ are one-forms in AdS_4. The duality relation $dA_6 = *dA_3$ requires that the fields $A^{(i)}$ and $\tilde{A}^{(i)}$ should be related to each other through $d\tilde{A}^{(i)} = *_4 dA^{(i)}$, and that $\omega_2^{(i)}$ and $\omega_5^{(i)}$ should be harmonic forms. The relation $d\tilde{A}^{(i)} = *_4 dA^{(i)}$ implies that both $A^{(i)}$ and $\tilde{A}^{(i)}$ satisfy the equation of motion for a gauge field, $d *_4 dA^{(i)} = d *_4 d\tilde{A}^{(i)} = 0$. For each i, there are two different boundary conditions in AdS_4 which correspond to treating either $A^{(i)}$ or $\tilde{A}^{(i)}$ as the fundamental variable [91,92]. The two possible conserved charges, electric and magnetic, map in the dual gauge theory to global charge density and magnetic field, respectively [58, 93]. For our purposes, this choice corresponds to allowing either the wrapped M2-branes or the wrapped M5-branes. We will comment on the dual field theory interpretation of the $AdS_4 \times Y^7$ backgrounds, and the meaning of this choice, in section 4.5. The above discussion shows that in the M-theory case the supergravity fluctuation spectrum around $AdS_4 \times Y^7$ contains b_2 independent gauge fields whose existence relies on the existence of harmonic two- and five-forms on Y^7.

In this chapter we will consider Sasaki-Einstein spaces Y^7 which are principal $U(1)$ bundles over a direct product of two Kähler-Einstein spaces, V_1 and V_2. In this case, there exists a universal harmonic two-form ω_2 (or, equivalently, a universal harmonic five-form ω_5) that we exhibit in the next section. We will construct two-brane solutions electrically charged under the corresponding gauge field A coming from δA_3.[1] As in the solutions of [70], several warp factor functions enter our consistent non-linear ansatz. We derive a system of coupled ODEs for these functions and solve them numerically to find the backgrounds for various values of T/μ. The warp factors turn out to stabilize to finite nonzero values at the horizon in the zero-temperature limit, producing an $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 throat region that is also a solution to 11-d supergravity. We find numerically the extremal background interpolating between this throat region in the IR and $AdS_4 \times Y^7$ in the UV. We also find an analytic solution with the same IR but different UV behavior. A possible instability associated with condensation of charged fields would manifest itself in wrapped probe M2-branes being repelled from the horizon. However, using the M2-brane world volume action, we show quite generally that such an instability does not occur. We make some simple checks of stability against condensation of neutral scalar fields, and we again find no instabilities. We also study the potential for a probe space-time filling M2-brane and prove that it vanishes at $T = 0$. Hence, there is no brane nucleation instability, and our solution seems to be a good candidate for embedding the $AdS_2 \times \mathbb{R}^2$ IR behavior into M-theory.

The rest of the chapter is organized as follows. In section 4.2 we describe the eleven-dimensional ansatz and construct the charged black membranes numerically at nonzero temperature and chemical potential. In section 4.3 we find the zero-temperature limit of our backgrounds and show that the $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 throat by itself

[1] An ansatz for magnetically charged solutions was set up in [70], but seems to lead to backgrounds singular in the IR.

satisfies the 11-d supergravity equations of motion. We also present a similar analytic solution with different large r behavior. In section 4.4 we compute the potential for the charged objects in the theory—the M2-branes. In section 4.5 we discuss an interpretation of our results in the dual quiver Chern-Simons gauge theories. The wrapped M2-branes are dual to operators containing non-diagonal magnetic fluxes, and we comment on their fractional statistics. In section 4.6 we use string dualities to map our analytic solution to one in type IIB theory, and find that the type IIB metric is a product of the squashed Y^7 and the extremal BTZ black hole (the one that has the minimum mass for a given angular momentum in AdS_3) [94, 95]. The Appendices contain some further stability checks and constructions of the two-cycles in Y^7.

4.2 A universal consistent truncation

Let us consider a seven-dimensional Einstein space Y^7 that can be written as a $U(1)$ fiber bundle over a direct product of two Kähler-Einstein spaces, V_1 and V_2. The spaces Y^7, V_1, and V_2 could be manifolds or, more generally, orbifolds. The product $V_1 \times V_2$ must describe a space of real dimension six, or complex dimension three, so without loss of generality we assume that V_1 and V_2 have complex dimensions two and one, respectively. In section 4.2.1 we first show explicitly that all the spaces Y^7 with the property mentioned above admit a universal harmonic two-form which can be used to construct a massless gauge field in AdS_4 (4.1), and then give a non-linear consistent truncation of eleven-dimensional supergravity that allows us to construct black membrane solutions with topological charge. In section 4.2.2 we give examples of spaces Y^7. Section 4.2.3 is concerned with examining the thermodynamic properties of the charged black branes at nonzero temperature and charge density. Lastly, in section 4.2.4 we construct these black branes numerically.

4.2.1 The eleven-dimensional background

Quite generally, the Einstein metric on the space Y^7 can be written as

$$ds_Y^2 = ds_{V_1}^2 + ds_{V_2}^2 + (d\psi + \sigma_1 + \sigma_2)^2 \ , \tag{4.2}$$

where each of the connection one-forms σ_i is a pull-back of a locally-defined one-form on V_i. It is convenient to normalize this metric so that in a vielbein basis $R_{ab} = 6\delta_{ab}$. The Einstein condition for Y^7 implies both that

$$d\sigma_i = 2\omega_i \ , \tag{4.3}$$

where ω_i is the Kähler form on V_i, and that the Einstein metric on V_i should be normalized so that the curvature two-form R_i satisfies $R_i = 8\omega_i$. In this normalization, the range of ψ depends on the first Chern class of the fibration; see Appendix 4.A for more details.

The spaces Y^7 admit a universal harmonic two-form given by

$$\omega \equiv \omega_1 - 2\omega_2 \ . \tag{4.4}$$

To see that this form is harmonic, it is helpful to pass to a vielbein basis where, in a small enough coordinate patch, $\omega_1 = e_1 \wedge e_2 + e_3 \wedge e_4$, $\omega_2 = e_5 \wedge e_6$, and $d\psi + \sigma_1 + \sigma_2 = e_7$. In this basis, the volume form on Y^7 is just $\mathrm{vol}_Y = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7$. The Hodge dual of ω can then be computed to be

$$*_Y\, \omega = \omega_1 \wedge (\omega_2 - \omega_1) \wedge (d\psi + \sigma_1 + \sigma_2) \ . \tag{4.5}$$

Using (4.3), (4.5), and the fact that both ω_1 and ω_2 are closed, one can show that

$d\omega = d *_Y \omega = 0$, so ω is indeed harmonic. Note also that $J \wedge *_Y \omega = 0$, where

$$J \equiv \omega_1 + \omega_2 \tag{4.6}$$

is the Kähler form on $V_1 \times V_2$.

One can use the space Y^7 and ω to construct a charged black hole solution to the eleven-dimensional supergravity equations of motion as follows. The eleven-dimensional metric is a warped product of a non-compact four-dimensional space M and a squashed version of (4.2):

$$ds^2 = e^{-7\chi/2} ds_M^2 + 4L^2 e^{\chi} \left[e^{\eta_1} ds_{V_1}^2 + e^{\eta_2} ds_{V_2}^2 + e^{-4\eta_1 - 2\eta_2} (d\psi + \sigma_1 + \sigma_2)^2 \right] , \tag{4.7}$$

where the scalar fields χ, η_1, and η_2 are functions only of the coordinates on M. In fact, we will only look for static solutions that are rotationally symmetric in two of the four non-compact directions, and we write the metric on M in the form

$$ds_M^2 = -g e^{-w} dt^2 + \frac{r^2}{L^2} \left[(dx^1)^2 + (dx^2)^2 \right] + \frac{dr^2}{g} , \tag{4.8}$$

where g, w, χ, η_1, and η_2 depend only on r.

In addition to the metric, we need to specify the four-form F_4:

$$F_4 = -\frac{3}{L} e^{-\frac{21}{2}\chi} \mathrm{vol}_M - 8QL^3 \frac{e^{-\frac{w}{2} - \frac{3}{2}\chi}}{r^2} dt \wedge dr \wedge \left(e^{2\eta_1} \omega_1 - 2 e^{2\eta_2} \omega_2 \right) , \tag{4.9}$$

where Q is a constant related to the charge of the black hole, and the orientation of M is given by

$$\mathrm{vol}_M \equiv \frac{r^2}{L^2} e^{-\frac{1}{2}w} dt \wedge dx^1 \wedge dx^2 \wedge dr . \tag{4.10}$$

Its Hodge dual, F_7, has the form

$$F_7 = 384\, L^6 \mathrm{vol}_Y + 64\, Q L^4 dx^1 \wedge dx^2 \wedge (*_Y \omega)\,, \tag{4.11}$$

with $*_Y\omega$ defined as in (4.5). When Q is small, the 11-d equations of motion imply that η_1, η_2, and χ are of order $\mathcal{O}(Q^2)$, so to linear order in Q, equations (4.9)–(4.11) take the form (4.1) with the gauge fields A and $\tilde{A}$ having only electric and only magnetic components, respectively.

The effective one-dimensional Lagrangian describing the consistent truncation (4.8)–(4.11) is

$$\mathcal{L} = \frac{r^2}{L^2} e^{-\frac{w}{2}} \left[\frac{63g}{8} \chi'^2 + \frac{g}{2}(2\eta_1'^2 + \eta_2'^2) + g(2\eta_1' + \eta_2')^2 + \frac{2g}{r} w' - \frac{2}{r} g' - \frac{2g}{r^2} + V_Q + V_s \right], \tag{4.12}$$

where

$$\begin{aligned} V_Q &= \frac{4L^2}{r^4} e^{-\frac{3}{2}\chi} \left(e^{2\eta_1} + 2e^{2\eta_2} \right) Q^2\,, \\ V_s &= \frac{9}{2L^2} e^{-\frac{21}{2}\chi} - \frac{4}{L^2} e^{-\frac{9}{2}\chi} \left(2e^{-\eta_1} + e^{-\eta_2} \right) + \frac{1}{2L^2} e^{-2(2\eta_1 + \eta_2) - \frac{9}{2}\chi} \left[2e^{-2\eta_1} + e^{-2\eta_2} \right]\,. \end{aligned} \tag{4.13}$$

This Lagrangian needs to be supplemented by the zero-energy constraint

$$\frac{2}{r} g' - g \left[\frac{63}{8} \chi'^2 + \frac{1}{2}(2\eta_1'^2 + \eta_2'^2) + (2\eta_1' + \eta_2')^2 + \frac{2}{r} w' - \frac{2}{r^2} \right] + V_Q + V_s = 0\,. \tag{4.14}$$

The scalar potential V_s agrees with the one derived in [96] for the particular case where the Sasaki-Einstein manifold Y^7 is $Q^{1,1,1}$.

4.2.2 Examples

Examples of spaces Y^7 satisfying the requirements of the previous section are some regular Sasaki-Einstein manifolds and orbifolds thereof. A Sasaki-Einstein manifold Y is a compact Riemannian manifold whose metric cone is Calabi-Yau. Such a manifold is called regular if the fibers all close and have the same length. A regular Sasaki-Einstein manifold can be described as a principal $U(1)$ bundle over a Kähler-Einstein base V, which in general cannot be written as a product $V_1 \times V_2$ as in the previous section [97].

The best known regular Sasaki-Einstein manifolds in seven dimensions are [97]:

I. Regular SE_7 where the base V cannot be written as a product $V_1 \times V_2$:

- S^7, which is a $U(1)$ fibration over $\mathbb{CP}^3$.
- $N^{0,1,0}$, which is a $U(1)$ fibration over the flag manifold $F(1,2)$.
- $V_{5,2}$, which is a $U(1)$ fibration over the Grassmanian manifold $G_{5,2}$.

II. Regular SE_7 where $V = V_1 \times V_2$:

- $Q^{1,1,1}$, which is a $U(1)$ fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.
- $Q^{2,2,2}$, which is a $\mathbb{Z}_2$ orbifold of $Q^{1,1,1}$ and a $U(1)$ fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ also. It differs from $Q^{1,1,1}$ in that the length of the fiber is shorter by a factor of two.
- $M^{1,1,1}$, which is a $U(1)$ fibration over $\mathbb{CP}^2 \times \mathbb{CP}^1$.
- Spaces which we will call $\mathcal{P}_n$ that are appropriate $U(1)$ fibrations over $dP_n \times \mathbb{CP}^1$, $3 \leq n \leq 8$, where dP_n is the nth del Pezzo surface constructed by blowing up $\mathbb{CP}^2$ at n generic points.

From now on we will only be interested in the second group of examples listed above for which the base of the $U(1)$ fibration can be written as a direct product of two

Kähler-Einstein manifolds. Indeed, $V = V_1 \times V_2$ was a necessary ingredient for constructing the consistent truncation of eleven-dimensional supergravity presented in section 4.2.1.

In addition to the regular Sasaki-Einstein spaces we just described, one can also consider their orbifolds. While the regular Sasaki-Einstein spaces under (II) all preserve eight supercharges, their orbifolds generically break all SUSY.

4.2.3 Thermodynamics

Boundary conditions

Before we calculate thermodynamic quantities, we need to discuss the boundary conditions one should impose on the solutions to the equations of motion following from (4.12)–(4.14). At large r, these solutions should asymptote to $AdS_4 \times Y^7$, so

$$\begin{gathered} w \to 0\,, \qquad \chi \to 0\,, \qquad \eta_1 \to 0\,, \qquad \eta_2 \to 0\,, \\ g = \frac{r^2}{L^2} + \mathcal{O}(L/r)\,. \end{gathered} \tag{4.15}$$

Generically, there will be an event horizon at some $r = r_h$ where g vanishes. The remaining boundary conditions come from requiring regularity of all the fields at $r = r_h$.

Let us examine the boundary conditions (4.15) more carefully. From the asymptotic form of the equations at large r we find that there is only one possible behavior for χ consistent with (4.15): $\chi \sim 1/r^6$. The gauge theory operator dual to χ has conformal dimension $\Delta_\chi = 6$, because in general the bulk field dual to a scalar operator of dimension Δ behaves at large r as $r^{-\Delta}$ if no sources for that operator are turned on. A similar asymptotic analysis shows that the fields $\eta_{1,2}$ break up into the combinations

$$\tilde{\eta} = \frac{2\eta_1 + \eta_2}{3}\,, \qquad \tilde{\lambda} = \frac{\eta_1 - \eta_2}{3} \tag{4.16}$$

that have definite scaling dimensions at large r. While for $\tilde{\eta}$ there is only one possible large r behavior consistent with (4.15), $\tilde{\eta} \sim 1/r^4$, corresponding to $\Delta_{\tilde{\eta}} = 4$, $\tilde{\lambda}$ generically behaves as a linear combination of $1/r$ and $1/r^2$ with arbitrary coefficients, both of these behaviors being consistent with $AdS_4 \times Y^7$ asymptotics. One then has a choice of boundary conditions where either $\Delta_{\tilde{\lambda}} = 1$ and the coefficient of $1/r^2$ is required to vanish, or $\Delta_{\tilde{\lambda}} = 2$ and the coefficient of $1/r$ is required to vanish [30]. Here we choose the latter boundary condition on $\tilde{\lambda}$. With this choice, the equations obtained from the Lagrangian (4.12) subject to the zero-energy constraint (4.14) and the other boundary conditions described above can be solved by a power series expansion at large r. The first few terms in the expansion are given below:

$$\begin{aligned}
w &= \mathcal{O}(L^4/r^4)\,, \\
g &= \frac{r^2}{L^2} + \frac{g_1 L}{r} + \mathcal{O}(L^2/r^2)\,, \\
\chi &= \frac{2L^4}{49r^4}\left(2Q^2 - 3\lambda_2^2\right) + \frac{\chi_6 L^6}{r^6} + \mathcal{O}(L^7/r^7)\,, \\
\eta_1 &= \frac{\lambda_2 L^2}{r^2} - \frac{4L^4}{35r^4}\left(2Q^2 - 3\lambda_2^2\right)\log\frac{r}{L} - \frac{L^4}{r^4}\left(\frac{2}{3}Q^2 + \eta_4\right) + \mathcal{O}(L^6/r^6)\,, \\
\eta_2 &= -\frac{2\lambda_2 L^2}{r^2} - \frac{4L^4}{35r^4}\left(2Q^2 - 3\lambda_2^2\right)\log\frac{r}{L} + \frac{L^4}{r^4}\left(\frac{4}{3}Q^2 + \eta_4\right) + \mathcal{O}(L^5/r^5)\,.
\end{aligned} \tag{4.17}$$

All higher order terms are determined in terms of g_1, χ_6, η_4, λ_2, and Q.

The potential conjugate to Q

Quite generally, a global $U(1)$ symmetry in the boundary field theory corresponds to an Abelian gauge symmetry in the bulk. The charge density and its conjugate chemical potential in the boundary theory can be computed from the corresponding bulk gauge field. However, in (4.9) we did not write down a more general formula in terms of a bulk gauge field as in (4.1), but instead we "solved" for the electric component of this gauge field in terms of an integration constant Q from the very beginning. The reason why we did this lies in the intricacies of non-linear consistent truncations: equation (4.9) can probably be generalized to a gauge field with arbitrary components, but one would need to include several other supergravity fields that were consistently set to zero in (4.7)–(4.9). The reason why in the discussion around equation (4.1) this was not an issue is that at the linearized level in the gauge field, it is consistent to set these additional supergravity fields to zero.

A generalization of (4.9) is still possible without having to turn on other supergravity fields: one can find the nonlinear generalization of the time-component of the gauge field appearing in (4.1). To find it, one promotes Q to a canonical momentum in the Hamiltonian associated with the 1-d Lagrangian (4.12). Call the canonically conjugate variable Φ. The equation of motion satisfied by Φ can be found from Hamilton's equation, $\Phi' = \frac{\partial H}{\partial Q}$, which gives

$$\Phi' - \frac{8Q}{r^2} e^{-\frac{1}{2}w-\frac{3}{2}\chi}\left(e^{2\eta_1} + 2e^{2\eta_2}\right) = 0\,. \tag{4.18}$$

Plugging Q from eq. (4.18) in eq. (4.9), we get

$$F_4 = -\frac{3}{L} e^{-\frac{21}{2}\chi}\mathrm{vol}_M - \Phi' \frac{L^3}{e^{2\eta_1} + 2e^{2\eta_2}} dt \wedge dr \wedge \left(e^{2\eta_1}\omega_1 - 2e^{2\eta_2}\omega_2\right). \tag{4.19}$$

One can explicitly check that this still leads to a consistent truncation. The equation of motion for Φ is imposed by the equation of motion for F_4.

It is instructive to decompose the form appearing in (4.19) in terms of ω and J,

$$e^{2\eta_1}\omega_1 - 2e^{2\eta_2}\omega_2 = \frac{e^{2\eta_1} + 2e^{2\eta_2}}{3}\omega + \frac{2(e^{2\eta_1} - e^{2\eta_2})}{3} J\,, \tag{4.20}$$

and rewrite eq. (4.19) as

$$F_4 = -\frac{3}{L} e^{-21\chi/2}\mathrm{vol}_M - \Phi' \frac{L^3}{3} dt \wedge dr \wedge \omega - \Phi' \frac{2L^3}{3} \frac{e^{2\eta_1} - e^{2\eta_2}}{e^{2\eta_1} + 2e^{2\eta_2}} dt \wedge dr \wedge J\,. \tag{4.21}$$

This shows that Φ is the Coulomb potential for the topological charge density; for large r it behaves as $-24Q/r$. With the boundary conditions described in eq. (4.17), the last term in F_4, which contains J, falls off faster than $1/r^2$ and therefore does not correspond to a charge density.

Thermodynamic quantities

Thermodynamic quantities in the boundary theory such as the energy density ϵ, entropy density s, temperature T, $U(1)$ charge density ρ, and chemical potential μ can be calculated from the following formulae

$$\begin{aligned} &\epsilon = -\frac{g_1 e^{-\frac{1}{2}w_0}}{\kappa_4^2 L}\,, \qquad s = \frac{2\pi r_h^2}{\kappa_4^2 L^2}\,, \qquad T = \frac{g'(r_h)e^{-\frac{1}{2}w_h}}{4\pi}\,, \\ &\rho = \frac{Q}{2\kappa_4^2}\,, \qquad \mu = \Phi_0 - \Phi_h\,, \end{aligned} \tag{4.22}$$

where the subscript "h" represents the value of the corresponding field at the horizon, while the subscript "0" represents the value at the conformal boundary.

There is a simple relation between these quantities,

$$\epsilon = \frac{2}{3}(Ts + \mu\rho)\,, \tag{4.23}$$

which holds in any $(2+1)$-dimensional CFT and can be proven from combining the extensivity relation $\epsilon = Ts - p + \mu\rho$ with the tracelessness of the stress-energy tensor $\epsilon = 2p$. One can also prove (4.23) solely from the gravity side by noticing that the "current"

$$j \equiv \frac{r^4}{L^4} e^{\frac{1}{2}w} \left(\frac{L^2}{r^2} e^{-w} g \right)' - \frac{1}{8} r^2 \, \frac{e^{\frac{1}{2}w + \frac{3}{2}\chi}}{e^{2\eta_1} + 2e^{2\eta_2}} \, \Phi\Phi' \tag{4.24}$$

is conserved in the sense that it satisfies $\partial j/\partial r = 0$. One can check that this current is conserved using the equations of motion following from the effective one-dimensional

Lagrangian (4.12). Evaluated at the horizon, eq. (4.24) yields

$$j_h = -2\kappa_4^2\,\Phi_h\rho + 2\kappa_4^2\,Ts\,. \tag{4.25}$$

Evaluated at the conformal boundary, it gives

$$j_0 = -2\kappa_4^2\,\Phi_0\rho + 3\kappa_4^2\,\epsilon\,. \tag{4.26}$$

The equality of the above two relations enforced by the conservation equation yields precisely (4.23).

4.2.4 Numerics at nonzero temperature

For general values of the parameters, it is unlikely that there are analytic solutions to the equations of motion resulting from the Lagrangian (4.12). We thus resort to numerical work. We employ a standard shooting technique where we seed the numerical integrator at large r, and integrate towards the horizon. The initial conditions are then tuned until a solution that is regular at the horizon is found.

The boundary conditions for solving the equations of motion were described in section 4.2.3. A series expansion around $r = \infty$ is used to determine the initial conditions for the numerical integration. The first terms in this expansion are given in eq. (4.17). At fixed Q, there are four free parameters, g_1, χ_6, η_4, and λ_2. One of these parameters can be eliminated by observing that the equations of motion are invariant under the following symmetry transformation:

$$g \to \alpha^2 g\,, \qquad r \to \alpha\, r\,, \qquad t \to \alpha^{-1} t\,, \qquad \vec{x} \to \alpha^{-1}\vec{x}\,, \qquad Q \to \alpha^2 Q\,, \tag{4.27}$$

which can be used to set $g_1 = -1$. The parameters χ_6, η_4, and λ_2 can be fixed by imposing the regularity conditions at the horizon, resulting in one solution for every value of Q. By varying the dimensionless parameter Q we can probe the boundary field theory at various temperatures, or more precisely, at various values of the dimensionless parameter T/μ. The 1-d Lagrangian (4.12) is invariant under $Q \to -Q$, so for each solution with a given value of Q one can find another solution by replacing Q by $-Q$. Without loss of generality, we restrict to the case $Q > 0$.

Our numerical results suggest that nothing drastic happens as the temperature approaches zero. In fact, the scalars χ, η_1, and η_2 seem to approach fairly small values at low temperatures: see figure 4.1. These values will be computed analytically in the next section. The bottom right plot in this figure shows that the horizon value of the eleven-dimensional Riemann tensor squared also stays bounded from above as the temperature is decreased. The lack of divergences means that one can trust the supergravity approximation all the way down to zero temperature.

The thermodynamics of our solutions is similar to that of four-dimensional RNAdS black holes. For example, for both RNAdS and our backgrounds the entropy density approaches a nonzero value at zero temperature (see figure 4.2). Similarly, the specific heat at constant chemical potential grows linearly with temperature at low T, as can be seen from figure 4.3. In the next section, we will in fact prove that at $T = 0$ the near-horizon four-dimensional geometry is $AdS_2 \times \mathbb{R}^2$, as is also the case for RNAdS.

4.3 Extremal solutions

In general, the equations of motion following from (4.12) admit black hole solutions with an event horizon at $r = r_h$. We expect there to exist solutions where the horizon is extremal, which corresponds to having vanishing temperature in the dual field theory. One of the simplest scenarios is that at extremality $r_h > 0$, the functions χ, η_1, η_2, and w approach finite values at $r = r_h$, and g behaves as $(r - r_h)^2$ and thus $g'(r_h) = 0$, giving zero temperature by eq. (4.22). This scenario describes an

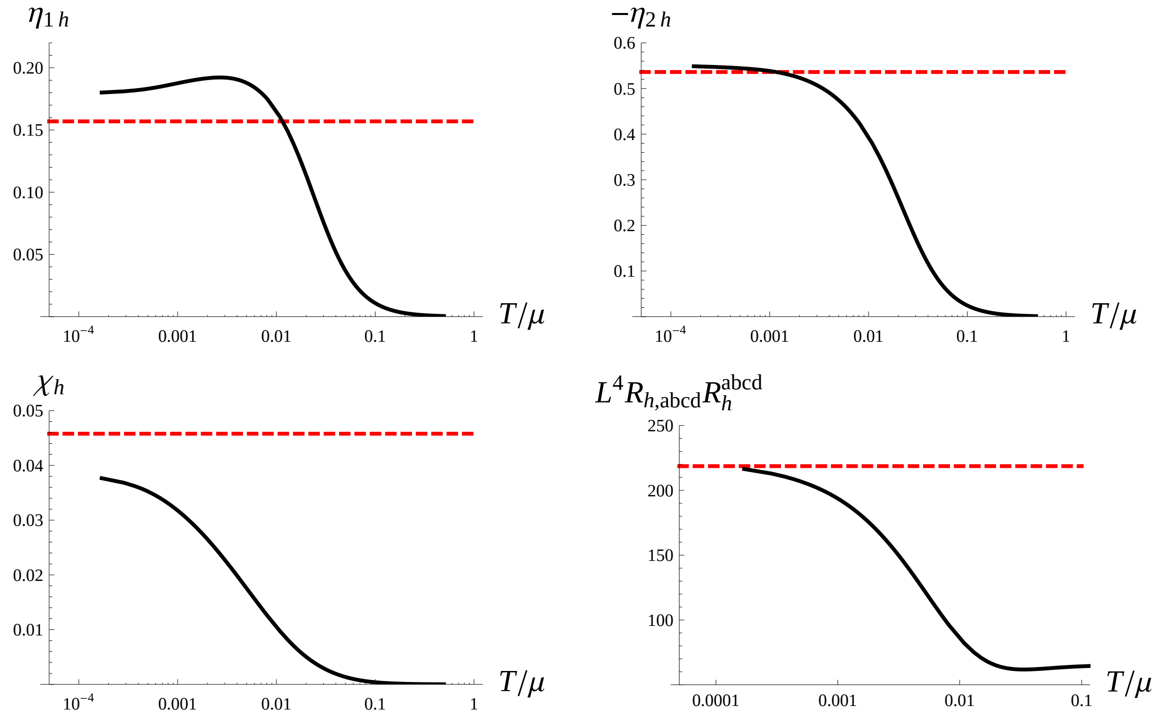

Figure 4.1: The horizon values of the scalars η_1, η_2, and χ and of the squared Riemann tensor as a function of T/μ. The expected zero-temperature values that follow from (4.32) are indicated by red dashed lines. The fact that none of these quantities diverge as $T \to 0$ shows that the supergravity approximation continues to hold down to arbitrarily small temperatures.

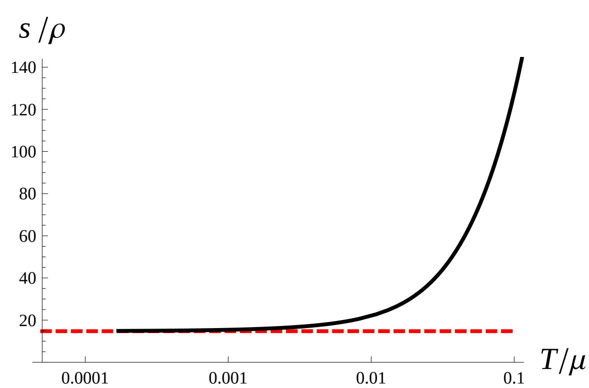

Figure 4.2: The dependence of the ratio of entropy density to charge density on T/μ. The dashed line indicates the value $s/\rho = 4\pi/Q \approx 14.75$ expected from the extremal solution of section 4.3.

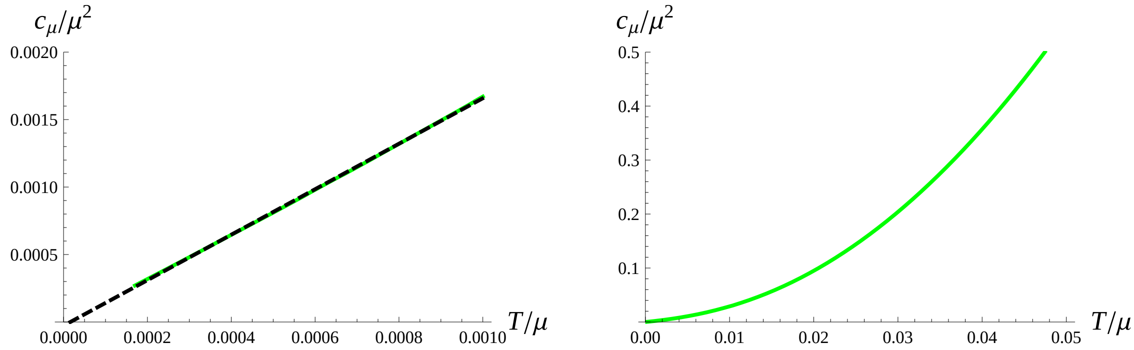

Figure 4.3: The dependence of the specific heat at constant chemical potential on T/μ. The dashed line in the plot on the left is a best fit line, showing the linear behavior of the specific heat at low temperatures.

extremal horizon which is $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7, the amount of squashing in the internal space Y^7 depending on the values of the scalars at the horizon. At extremality, $g(r_h) = g'(r_h) = 0$, and the equations of motion following from the Lagrangian (4.12) together with the zero-energy constraint (4.14) imply that an $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 horizon is possible only if the total potential $V = V_Q + V_s$ (see eq. (4.13)) satisfies

$$V = \frac{\partial V}{\partial \eta_i} = \frac{\partial V}{\partial \chi} = 0 \qquad \text{at } r = r_h. \tag{4.28}$$

These equations are solved by

$$\eta_1 = \frac{1}{7}\log 3\,, \qquad \eta_2 = \frac{1}{7}\log 3 - \log 2\,, \qquad \chi = \frac{5}{14}\log 3 - \frac{1}{2}\log 2\,, \tag{4.29}$$

as well as

$$Q = \pm\frac{2^{\frac{7}{4}}}{3^{\frac{5}{4}}}\frac{r_h^2}{L^2} \approx \pm 0.852\,\frac{r_h^2}{L^2}\,. \tag{4.30}$$

We will see shortly that the $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 space is in fact an exact solution to the 11-d supergravity equations of motion for an appropriate choice of the four-form flux F_4.

For simplicity in the rest of this section we set $L = r_h = 1$. This can be achieved by using an appropriate choice of units in the bulk to set $L = 1$, and then employing the symmetry (4.27) to move r_h to 1. In this section we will describe three solutions to 11-d SUGRA: In section 4.3.1 we start by describing an analytical solution with seemingly unconventional UV behavior and $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 IR asymptotics; in section 4.3.2 we recover the analytical solution $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 mentioned above as a scaling limit of the solution from section 4.3.1; lastly, in section 4.3.3 we present a numerical solution with $AdS_4 \times Y^7$ UV asymptotics and $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 behavior in the IR.

4.3.1 A zero-temperature analytical solution

The 11-d SUGRA equations of motion admit the following analytical solution with extremal $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 horizon:

$$\begin{aligned} ds^2 &= -\frac{2^3}{3^3} e^{-w_0} \frac{(r^4-1)^2}{r^{\frac{8}{3}}} dt^2 + \frac{\sqrt{6} r^{\frac{22}{3}}}{(r^4-1)^2} dr^2 + \frac{2^{\frac{7}{4}}}{3^{\frac{5}{4}} r^{\frac{8}{3}}} d\vec{x}^2 \\ &\quad + 2^{\frac{3}{2}} 3^{\frac{1}{2}} r^{\frac{4}{3}} \left[ds^2_{V_1} + \frac{1}{2} ds^2_{V_2} + \frac{4}{3} (d\psi + \sigma_1 + \sigma_2)^2 \right], \\ F_4 &= -\frac{2^{\frac{21}{4}}}{3^{\frac{11}{4}}} \mathrm{vol}_M - \frac{16}{3} \sqrt{\frac{2}{3}} e^{-\frac{1}{2} w_0} r^3 dt \wedge dr \wedge (2\omega_1 - \omega_2) . \end{aligned} \tag{4.31}$$

This solution is of the form (4.7)–(4.9) given in section 4.2.1 with

$$\begin{aligned} & g = \frac{2^{\frac{5}{4}} (r^4-1)^2}{3^{\frac{7}{4}} r^{12}} , \qquad w = w_0 - 14 \log r , \qquad \Phi = \Phi_h + 4 \sqrt{\frac{2}{3}} e^{-\frac{1}{2} w_0} (r^4 - 1) , \\ & \eta_1 = \frac{1}{7} \log 3 , \qquad \eta_2 = \frac{1}{7} \log 3 - \log 2 , \qquad \chi = \frac{1}{3} \log \frac{3^{\frac{15}{14}} r^4}{2^{\frac{3}{2}}} , \qquad Q = \frac{2^{\frac{7}{4}}}{3^{\frac{5}{4}}} , \end{aligned} \tag{4.32}$$

which shows quite explicitly that in the IR the scalars stabilize to the values calculated above in eq. (4.29), and the charge Q is the same as in (4.30). There is, of course, another solution to the equations of motion that differs from the one above in the sign of Q.

From a field theory perspective, the presence of the AdS_2 factor in the IR geometry means that the effective IR field theory can be thought of as a $(0+1)$-dimensional quantum mechanics, which can perhaps arise from a chiral sector of a $(1+1)$-dimensional CFT. The effective dimensions of various operators are related to the IR behavior of supergravity fluctuations around the background (4.32): a supergravity field dual to an operator $\mathcal{O}$ of dimension $\Delta_{\rm IR}$ has two linearly independent solutions, one behaving as $(r-1)^{\Delta_{\rm IR}}$ and one as $(r-1)^{1-\Delta_{\rm IR}}$ as $r \to 1$. The coefficient of the first of these two solutions corresponds to a source for $\mathcal{O}$, while the coefficient of the second one corresponds to an expectation value.

Some of the simplest operators one can study correspond to fluctuations of the fields already present in the consistent truncation (4.12). It turns out that the linearized equations for the perturbations $(\delta\chi, \delta\eta_1, \delta\eta_2, \delta g, \delta w)$ can be solved exactly. The solution is

$$\begin{gathered} \delta\eta_1 = c_{\eta_1}\,(r^4-1)^\alpha\,, \qquad \delta\eta_2 = c_{\eta_2}\,(r^4-1)^\alpha\,, \qquad \delta\chi = c_\chi\,(r^4-1)^\alpha\,, \\ \delta w = -21\,\delta\chi\,, \qquad \delta g = -\frac{7\cdot 2^{1/4}}{3^{3/4} r^{12}}\,c_\chi (r^4-1)^{\alpha+2}\,, \end{gathered} \tag{4.33}$$

where there are six possible choices for α,

$$\alpha_1 = -\frac{1}{2} \pm \frac{\sqrt{69}}{6}\,, \tag{4.34a}$$

$$\alpha_2 = -\frac{1}{2} \pm \frac{1}{6}\sqrt{66-3\sqrt{73}}\,, \tag{4.34b}$$

$$\alpha_3 = -\frac{1}{2} \pm \frac{1}{6}\sqrt{66+3\sqrt{73}}\,. \tag{4.34c}$$

The coefficients c_i are not independent, but are related by the following equations

$$c_{\eta_1} = -\frac{3c_\chi}{4}\frac{15\alpha^2+15\alpha-28}{6\alpha^2+6\alpha-7}\,, \qquad c_{\eta_2} = \frac{3c_\chi}{8}\frac{126\alpha^4+252\alpha^3-177\alpha^2-303\alpha+140}{6\alpha^2+6\alpha-7}\,. \tag{4.35}$$

These perturbations correspond to three irrelevant operators in the dual quantum mechanics of dimensions

$$\Delta_{\mathrm{IR},1} = \frac{1}{2} + \frac{\sqrt{69}}{6}\,, \qquad \Delta_{\mathrm{IR},2} = \frac{1}{2} + \frac{1}{6}\sqrt{66 - 3\sqrt{73}}\,, \qquad \Delta_{\mathrm{IR},3} = \frac{1}{2} + \frac{1}{6}\sqrt{66 + 3\sqrt{73}}\,. \tag{4.36}$$

Solutions with different UV behavior for the functions appearing in the 11-d metric (4.7) (in particular the one with $AdS_4 \times Y^7$ UV asymptotics we will discuss) generate in the IR sources for these operators. Some fluctuations of 11-d supergravity not included in the consistent ansatz (4.12) are given in Appendix 4.B.

4.3.2 The IR "attractor" as a scaling limit

The $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 IR asymptotics of the exact solution described in the previous section represent in fact another exact solution to the 11-d SUGRA equations of motion. Indeed, the $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 "attractor" arises as a scaling limit of (4.31) where one sends $r \to 1 + y\epsilon$ and $t \to t/\epsilon$ and then takes the limit $\epsilon \to 0$. The background obtained in this limit is $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7 supported by four-form flux:

$$\begin{aligned} ds^2 &= -\frac{2^7}{3^3} e^{-w_0} y^2 dt^2 + \frac{3^{\frac{1}{2}}}{2^{\frac{7}{2}}} \frac{1}{y^2} dy^2 + \frac{2^{\frac{7}{4}}}{3^{\frac{5}{4}}} d\vec{x}^2 \\ &\quad + 2^{\frac{3}{2}} 3^{\frac{1}{2}} \left[ds^2_{V_1} + \frac{1}{2} ds^2_{V_2} + \frac{4}{3}(d\psi + \sigma_1 + \sigma_2)^2 \right]\,, \\ F_4 &= -\frac{2^{\frac{21}{4}}}{3^{\frac{11}{4}}} e^{-\frac{1}{2}w_0} dt \wedge dx^1 \wedge dx^2 \wedge dy - \frac{16}{3}\sqrt{\frac{2}{3}} e^{-\frac{1}{2}w_0} dt \wedge dy \wedge (2\omega_1 - \omega_2)\,. \end{aligned} \tag{4.37}$$

Note that this solution is not of the form (4.7)–(4.9) because the coefficient of $d\vec{x}^2$ in (4.8) cannot be set to a constant. Perturbations around this solution can be computed directly from perturbing the 11-d background (4.37), or can be obtained by taking the scaling limit of perturbations around the background (4.31) such as (4.33).

4.3.3 A numerical solution with $AdS_4 \times Y^7$ asymptotics

Three of the six linearly independent perturbations described in section 4.3.1, namely the ones corresponding to sources for the operators of dimensions (4.36), are well-behaved at the horizon. These three integration constants allow us, at least at the linearized level, to adjust to zero the asymptotic values of the scalars at large r so that our solutions asymptote to $AdS_4 \times Y^7$. Of course, there is no guarantee that the same holds true for the exact equations, but we can check numerically that this is indeed the case. As before, we use a standard shooting technique, this time seeding the numerical integrator very close to the horizon. We use the linearized perturbations as a seed, and tweak the coefficients of the three linearly independent perturbations until we find a solution that obeys the desired boundary conditions at large r.

Plots showing the behavior of the scalars as a function of radial coordinate are given in figure 4.4. We thus see that there exists an extremal black hole solution that interpolates between the attractor solution of the previous section in the IR and $AdS_4 \times Y^7$ in the UV. As a consistency check, we verified that our zero-temperature numerics are consistent with $s/\rho = 4\pi/Q \approx 14.75$ and $R_{h,abcd}R_h^{abcd} = 656/3 \approx 218.67$, which can be calculated directly from the attractor solution (4.32), as these quantities are insensitive to the UV asymptotics. These values are also consistent with the finite-temperature numerics that we discussed in section 4.2.4; see figures 4.1 and 4.2.

4.4 The potential for probe M2-branes

There are two types of M2-branes present in our construction: the M2-branes filling the (t, x^1, x^2) directions, which are responsible for generating the asymptotic $AdS_4 \times Y^7$ space, and M2-branes wrapped over a two-cycle in the internal space which are responsible for the topological charge of the membrane solution. We will henceforth refer to the former type of branes as space-time filling, and to the latter as wrapped.

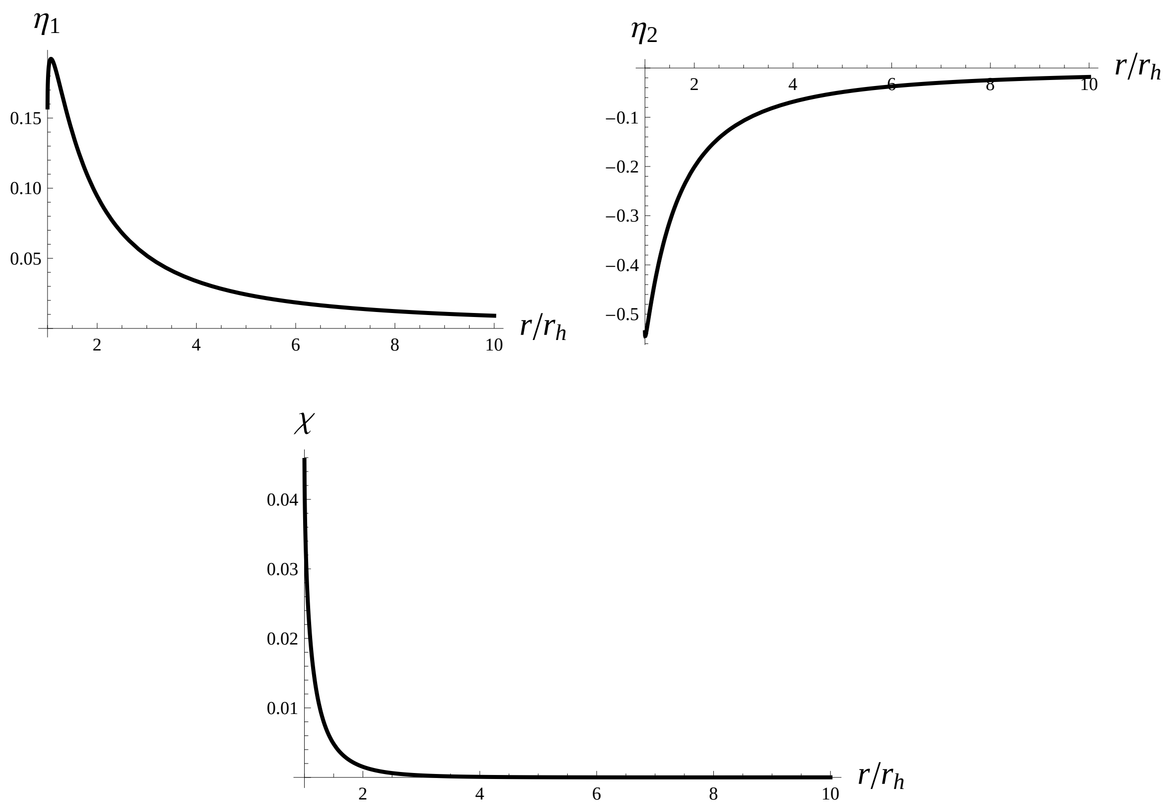

Figure 4.4: The dependence of the scalars η_1, η_2, and χ on the radial variable r at zero temperature. We see that the scalars tend to zero at the boundary since our solution asymptotes to $AdS_4 \times Y^7$ in the UV.

One might wonder whether there is an instability where any of these branes tunnel out to infinity [75, 76, 78]. We investigate this question by computing the potential for a probe brane as a function of the AdS radial variable r. The action for a probe brane is

$$S_{M2} = -\tau_{M2} \int d^3x \sqrt{-G} \pm \tau_{M2} \int A_3 \,, \tag{4.38}$$

where τ_{M2} is the M2-brane tension,

$$\tau_{M2} = \frac{2\pi}{(2\pi \ell_p)^3} \,, \tag{4.39}$$

and A_3 is the three-form gauge potential for $F_4 = dA_3$. We are primarily interested in the sign such that the interaction with A_3 is repulsive, i.e. when the M2-brane has the same charge as the stack that creates our background. Then the force on the brane vanishes in $AdS_4 \times Y^7$. The opposite sign corresponds to a probe anti M2-brane, for which the force is attractive at infinity.

For static embeddings, one can define a potential V for the probe branes through

$$S_{M2} = -\int V \, dt \,. \tag{4.40}$$

Our backgrounds are metastable if the potential is smaller at some $r > r_h$ than at the horizon.

4.4.1 Probe space-time filling M2-branes

Since the volume of these branes is infinite, we will look at their potential energy per unit area. We thus write

$$S_{M2} = -\int dt\, d^2x \left[v_g(r) + v_e(r)\right] , \tag{4.41}$$

where $v_g(r)$ and $v_e(r)$ come from the first and second terms in (4.38), respectively. It is straightforward to calculate these contributions using eq. (4.7). We have

$$v_g(r) = \tau_{M2} r^2 \sqrt{g} e^{-\frac{1}{2}w - \frac{21}{4}\chi} , \qquad v_e'(r) = \mp 3\tau_{M2} r^2 e^{-\frac{1}{2}w - \frac{21}{2}\chi} , \tag{4.42}$$

and we can choose, for example, $v_e(r_h) = 0$. Here and in the rest of this section we set $L = 1$. The minus sign in $v_e(r)$ corresponds to probe branes, while the plus sign corresponds to probe anti-branes. The probe anti-branes are always attracted towards the horizon, so we will only focus on the probe branes. In figure 4.5, we have plotted the potential $v_{\rm tot}(r) \equiv v_g(r) + v_e(r)$ at various temperatures, as a function of r. We see that the potential never dips below the horizon value, so the background is stable with respect to tunneling of space-time filling M2-branes.

One can also evaluate the potential $v_{\rm tot}(r)$ for the space-time filling branes on the analytical extremal solution (4.32). In this case, the potential vanishes identically. From the plots in figure 4.5, it looks like the space-time filling M2-brane potential also vanishes identically in the extremal limit of the solution of section 4.3.3 that asymptotes to $AdS_4 \times Y^7$, so one might wonder whether this result is insensitive to the UV asymptotics of the solution. Indeed, one can prove this result starting with the observation that the force per unit area, $f_{\rm tot} \equiv -v_{\rm tot}'$, satisfies the following first

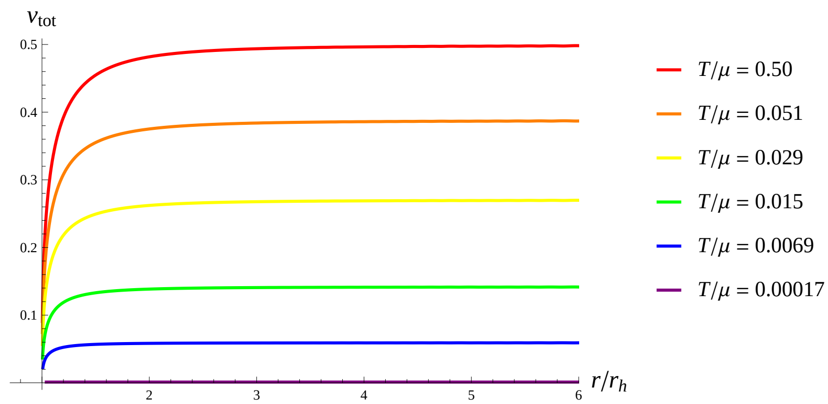

Figure 4.5: The potential energy per unit area of a probe space-time filling M2-brane as a function of the AdS radial coordinate r, at various temperatures. We worked in a gauge where the horizon value of the potential vanishes.

order differential equation:

$$f'_{\text{tot}} + \left[\frac{3e^{-\frac{21}{4}\chi}}{\sqrt{g}} + \frac{g'}{2g} + \frac{21\chi'}{4} \right] f_{\text{tot}} = 0\,. \tag{4.43}$$

Since this equation is linear, its solutions depend on one integration constant that acts as a multiplicative factor. Near the extremal horizon, (4.32) and (4.33) give

$$\frac{3e^{-\frac{21}{4}\chi}}{\sqrt{g}} + \frac{g'}{2g} + \frac{21\chi'}{4} = \frac{2}{r-1} + \text{subleading}\,, \tag{4.44}$$

so

$$f_{\text{tot}} = c\left[(r-1)^2 + \text{subleading}\right]\,. \tag{4.45}$$

The subleading terms in the above two equations are sensitive to the UV asymptotics, but the leading term is not. Since v_{tot} vanishes identically when evaluated on the leading behavior (4.32), it must be that $c = 0$. Therefore, the potential for the space-time filling M2-branes is exactly flat for any solution that connects to the solution (4.32) in the IR. It is worth noting that even though the exact $AdS_2 \times \mathbb{R}^2 \times$squashed Y^7

solution (4.37) cannot be written in the gauge (4.7)–(4.8), one can also show that the potential for space-time filling branes is exactly flat in this case too. The flatness of the potential follows from taking a scaling limit of the exact solution (4.31) as explained at the beginning of section 4.3.2.

The existence of a flat potential for the space-time filling branes is reminiscent of supersymmetric solutions, so one might wonder whether our background preserves any supersymmetry. In a supersymmetric background, the gravitino variation

$$\delta_\mu \epsilon \equiv \nabla_\mu \epsilon + \frac{1}{12}\left(\frac{1}{4!}F_{\nu\rho\lambda\sigma}\Gamma_\mu\Gamma^{\nu\rho\lambda\sigma} - \frac{1}{2}F_{\mu\nu\rho\lambda}\Gamma^{\nu\rho\lambda}\right)\epsilon \tag{4.46}$$

vanishes identically. A necessary condition for this to happen is that

$$[\delta_\mu, \delta_\nu]\epsilon = 0\,, \tag{4.47}$$

which is a linear system of algebraic equations. One can check that this system has no non-trivial solutions for both the backgrounds (4.31) and (4.37).

4.4.2 Probe wrapped M2-branes

Let us consider a static M2-brane embedding where the brane wraps a topologically non-trivial two-dimensional cycle $\mathcal{C}$ in the internal space and sits at some fixed values of r and $\vec{x}$. By the internal space we mean the squashed version $\tilde{Y}^7$ of Y^7 appearing in (4.7) with the metric

$$ds^2_{\tilde{Y}} = e^\chi \left[e^{\eta_1} ds^2_{V_1} + e^{\eta_2} ds^2_{V_2} + e^{-4\eta_1 - 2\eta_2}(d\psi + \sigma_1 + \sigma_2)^2\right]\,. \tag{4.48}$$

For a fixed value of r at which the scalars χ, η_1, and η_2 don't diverge, the topology of $\tilde{Y}^7$ is the same as that of Y^7, and there is a one-to-one correspondence between surfaces in Y^7 and surfaces in $\tilde{Y}^7$. So when we say that a brane wraps a cycle $\mathcal{C}$ in

$\tilde{Y}^7$, we might as well be thinking about the corresponding cycle in Y^7, and indeed we will not be careful about this distinction in the rest of this section unless there is potential for confusion.

The probe-brane action (4.38) takes the form

$$S_{M2} = -\int dt \left[V_g^{\mathcal{C}}(r) + V_e^{\mathcal{C}}(r)\right] , \tag{4.49}$$

where $V_g^{\mathcal{C}}(r)$ and $V_e^{\mathcal{C}}(r)$ come from the first and second terms in (4.38), respectively. We will call $V_g^{\mathcal{C}}(r)$ the gravitational potential and $V_e^{\mathcal{C}}(r)$ the electrostatic potential for such a brane. Stable brane wrappings are of course those that minimize the total potential $V_{\text{tot}}^{\mathcal{C}}(r) \equiv V_g^{\mathcal{C}}(r) + V_e^{\mathcal{C}}(r)$.

A simple way to construct non-trivial two-cycles in $\tilde{Y}^7$ is to start with a two-cycle in the base $V = V_1 \times V_2$, and lift it to $\tilde{Y}^7$. However, not every two-cycle in the base can be lifted to a two-cycle in the total space. The reason for this restriction is that when lifting a two-cycle, one needs to specify what the fiber angle should be at all points on that cycle, and such an assignment may not be consistent because of topological reasons. We include a more technical discussion of these issues in appendix 4.C.1. The upshot is that any (well-defined) two-cycle $\mathcal{C}$ in $\tilde{Y}^7$ satisfies

$$\int_{\mathcal{C}} J = 0 \, , \tag{4.50}$$

where $J = \omega_1 + \omega_2$, as in (4.6). One way to understand this condition is to note that J is a closed form in Y^7 because it obeys $de_\psi = 2J$, where $e_\psi \equiv d\psi + \sigma_1 + \sigma_2$ is a globally-defined one-form on Y^7.[2]

Using (4.21) and the fact that $\int_{\mathcal{C}} J = 0$, one can write the electrostatic potential

[2]Note that $\sigma_1 + \sigma_2$ by itself is not a globally defined one-form, so the condition (4.50) does not hold for two-cycles in $V_1 \times V_2$.

$V_e^{\mathcal{C}}(r)$ as

$$V_e^{\mathcal{C}}(r) = \mp\frac{1}{3}\tau_{M2}\Phi\int_{\mathcal{C}}(\omega_1 - 2\omega_2) = \pm\tau_{M2}\Phi\int_{\mathcal{C}}\omega_2\,. \tag{4.51}$$

This term depends only on the homology class of $\mathcal{C}$ in $H_2(Y^7;\mathbb{Z})$, so in order to find the stable wrappings for a given homology class $H_2(Y^7;\mathbb{Z})$ one has to minimize only the gravitational potential $V_g^{\mathcal{C}}(r)$. Two questions arise:

(I) For a static M2-brane embedding at fixed r, what cycles $\mathcal{C}$ are stable in the sense that they minimize $V_{\text{tot}}^{\mathcal{C}}(r)$, at least compared to neighboring cycles? Since $V_e^{\mathcal{C}}(r)$ is topological and $V_g^{\mathcal{C}}(r)$ is proportional to the volume (or more correctly, area) of $\mathcal{C}$ computed using the induced metric from $\tilde{Y}^7$, this problem reduces to finding the minimal volume cycles of $\tilde{Y}^7$.

(II) How does the minimal value of $V_{\text{tot}}^{\mathcal{C}}(r)$ from (I) depend on r? Are the branes repelled from the black hole horizon, or do they tend to fall into the black hole?

The first question is interesting in its own right, but may be hard to answer in general, especially since there are Sasaki-Einstein manifolds such as the spaces $\mathcal{P}_n$ described in section 4.2.2 for which the metric is not known explicitly. We will therefore content ourselves with finding a lower bound on the volumes of the cycles $\mathcal{C}$ of $\tilde{Y}^7$ in the cases where Y^7 is a regular Sasaki-Einstein manifold. Such a bound can be found by using calibrations, as we discuss in appendix 4.C.2. This bound is

$$\text{Vol}(\mathcal{C}) \geq e^{\chi}\left(e^{\eta_1} + e^{\eta_2}\right)\left|\int_{\mathcal{C}}\omega_2\right|\,. \tag{4.52}$$

For an arbitrary homology class in $H_2(\tilde{Y}^7;\mathbb{Z})$, this inequality may not be saturated by any embedded surfaces in that class. However, as we now explain, the bound (4.52) is restrictive enough to show that wrapped M2-branes do not condense.

Equation (4.52) can be used to find a lower bound on the gravitational potential

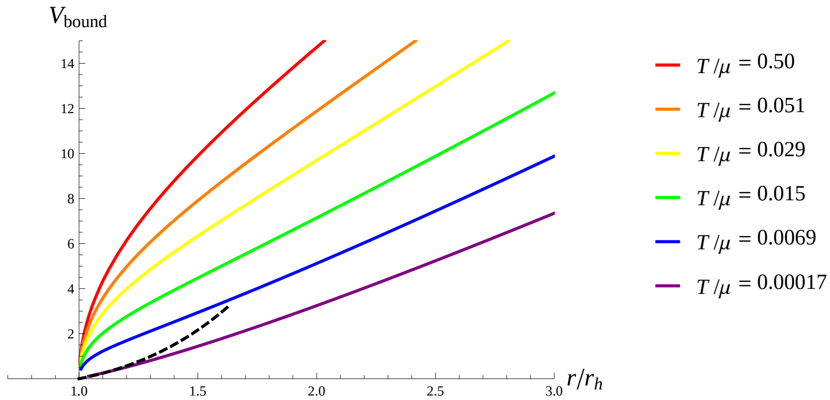

Figure 4.6: The bound (4.54) for the potential for probe M2-branes wrapping a two-cycle $\mathcal{C}$, expressed as multiples of $\tau_{M2}\left|\int_{\mathcal{C}}\omega_2\right|$ and normalized so that it vanishes at the horizon. Each solid curve corresponds to a different temperature. The dashed line represents the analytic approximation (4.56), valid close to the extremal horizon. This bound is saturated for the cycles (4.57) and (4.61) in $M^{1,1,1}$ and $Q^{1,1,1}$, respectively.

for a wrapped M2-brane:

$$V_g^{\mathcal{C}}(r) = 4\tau_{M2}e^{-\frac{7}{4}\chi}\sqrt{g}e^{-\frac{w}{2}}\mathrm{Vol}(\mathcal{C}) \geq 4\tau_{M2}e^{-\frac{3}{4}\chi}\sqrt{g}e^{-\frac{w}{2}}\left(e^{\eta_1}+e^{\eta_2}\right)\left|\int_{\mathcal{C}}\omega_2\right| . \tag{4.53}$$

Combining this equation with the expression for the electrostatic potential (4.51), we find that the total potential satisfies

$$V_{\rm tot}^{\mathcal{C}} \geq V_{\rm bound}^{\mathcal{C}}\,, \qquad V_{\rm bound}^{\mathcal{C}} \equiv \tau_{M2}\left[4e^{-\frac{3}{4}\chi}\sqrt{g}e^{-\frac{w}{2}}\left(e^{\eta_1}+e^{\eta_2}\right)-\Phi\right]\left|\int_{\mathcal{C}}\omega_2\right| . \tag{4.54}$$

From eq. (4.53) it also follows that at the horizon $V_{\rm tot}^{\mathcal{C}}(r_h) = V_{\rm bound}^{\mathcal{C}}(r_h)$, because $V_g^{\mathcal{C}}(r_h) = 0$. From figure 4.6, we see that in a gauge where Φ vanishes at the horizon, $V_{\rm bound}^{\mathcal{C}}(r) > 0$ for all $r > r_h$, implying that

$$V_{\rm tot}^{\mathcal{C}}(r) > V_{\rm tot}^{\mathcal{C}}(r_h)\,, \qquad \text{for } r > r_h. \tag{4.55}$$

This inequality means that the wrapped M2-branes do not condense.

One can check analytically that the wrapped M2-branes are attracted by the horizon at extremality by evaluating the lower bound in (4.53) and the electrostatic potential (4.51) on the exact solution (4.32). The result is

$$\begin{aligned} V_{g,\text{bound}}^{\mathcal{C},\text{extremal}}(r) &= 4\tau_{M2}e^{-\frac{w_0}{2}}(r^4-1)\left|\int_{\mathcal{C}}\omega_2\right| , \\ V_{e}^{\mathcal{C},\text{extremal}}(r) &= \pm 4\sqrt{\frac{2}{3}}\tau_{M2}e^{-\frac{w_0}{2}}(r^4-1)\left(\int_{\mathcal{C}}\omega_2\right) . \end{aligned} \tag{4.56}$$

One can see that the gravitational force (which is always inwards) is larger in magnitude than the electrostatic one by at least a factor of $\sqrt{\frac{3}{2}}$, so all these branes tend to fall into the black hole horizon at extremality. (See figure 4.6 for a comparison between the analytic formulae (4.56) and the numerical results.) By taking a scaling limit of the exact solution (4.32) one can show that these wrapped branes are also always attracted by the extremal horizon in the case of the $AdS_2\times\mathbb{R}^2\times$squashed Y^7 solution (4.37).

Example 1: Probe branes wrapping a two-cycle in $M^{1,1,1}$

The manifold $M^{1,1,1}$ is a $U(1)$ fiber bundle over $\mathbb{CP}^2\times S^2$. It can be parameterized by seven angles: μ, θ_1, ϕ_1, and ψ_1 parameterizing $\mathbb{CP}^2$, θ_2 and ϕ_2 parameterizing S^2, and ψ parameterizing the fiber. In another description, $M^{1,1,1}$ is a $U(1)$ quotient of $S^5\times S^3$. One can parameterize S^5 by three complex coordinates u^i, $i=1,2,3$, with $|u^1|^2+|u^2|^2+|u^3|^2=\text{const}$ and S^3 by two complex coordinates v^j, $j=1,2$, satisfying $|v^1|^2+|v^2|^2=\text{const}$. The $U(1)$ quotient acts by identifying $u^i\sim e^{2i\delta}u^i$ and $v^j\sim e^{-3i\delta}v^j$. An explicit Einstein metric on $M^{1,1,1}$ as well as more details on this space such as topological properties or the relation between the (u^i,v^j) coordinates and the angular ones can be found in Appendix 4.A.1.

As mentioned above, we want to find the two-cycles of $M^{1,1,1}$ (or of the squashed variant thereof $\tilde{M}^{1,1,1}$ as in (4.48)) that are local volume minimizers in their homology class. The second homology of $M^{1,1,1}$ is $H_2(M^{1,1,1};\mathbb{Z})\cong\mathbb{Z}$, so there is only one generator class for it. A minimal volume cycle representing the generator of the second homology of $M^{1,1,1}$ is

$$\mathcal{C}:\begin{cases}\theta_1 = 2\arctan t^2 & \mu = \dfrac{\pi}{2}\\ \theta_2 = 2\arctan t^3 & \psi_1 = \text{const.}\\ \phi_1 = 2\phi & \psi = \text{const.}\\ \phi_2 = -3\phi\,. & \end{cases} \quad\Longleftrightarrow\quad \begin{cases}\left(\dfrac{u^1}{u^2}\right)^2 = \left(\dfrac{\bar{v}_1}{\bar{v}_2}\right)^3\\ u^3 = 0\,.\end{cases} \tag{4.57}$$

In order to cover $\mathcal{C}$ only once, the ranges of t and ϕ should be taken to be $t \geq 0$ and $0 \leq \phi \leq 2\pi$.[3] This cycle is well-defined as it satisfies eq. (4.50) (see also the discussion at the end of Appendix 4.C.1), and has minimal volume since it saturates the bound (4.52), as can be checked by direct computation.

Using the explicit metric on $M^{1,1,1}$ given in Appendix 4.A.1 and the explicit parameterization of the cycle (4.57), one obtains the following gravitational potential:

$$V_g^{\mathcal{C}}(r) = 6\pi\tau_{M2}e^{-\frac{3}{4}\chi}\sqrt{g}e^{-\frac{w}{2}}(e^{\eta_1}+e^{\eta_2})\,. \tag{4.58}$$

Similarly, one can use (4.38) to find the electrostatic potential

$$V_e^{\mathcal{C}}(r) = \mp\frac{3\pi}{2}\tau_{M2}\Phi\,. \tag{4.59}$$

The potential for these branes saturates the bound (4.54), as a consequence of the fact that the cycle they wrap saturates (4.52).

Example 2: Probe branes wrapping a two-cycle in $Q^{1,1,1}$

The manifold $Q^{1,1,1}$ can be described as a $U(1)$ fibration over $S^2\times S^2\times S^2$, so it can be parameterized in terms of three sets of angles (θ_a,ϕ_a), $a = 1,2,3$, each

[3] Similar cycles have been considered in five-dimensional Sasaki-Einstein manifolds. See for example [98].

set parameterizing one of the spheres, and a fiber angle ψ. Another description of $Q^{1,1,1}$ is as a $U(1)^2$ quotient of $S^3 \times S^3 \times S^3$: One can parameterize the S^3's by three sets of two complex coordinates, a^i, b^j, c^k, with $i, j, k = 1, 2$, satisfying $|a^1|^2 + |a^2|^2 = |b^1|^2 + |b^2|^2 = |c^1|^2 + |c^2|^2 = \text{const}$, and take a quotient by a $U(1)$ that acts by $a^i \sim e^{i\delta} a^i$, $b^j \sim e^{-i\delta} b^j$, $c^k \sim c^k$, and by another $U(1)$ that acts by $a^i \sim e^{i\delta} a^i$, $b^j \sim b^j$, $c^k \sim e^{-i\delta} c^k$. More details about $Q^{1,1,1}$ including an explicit Einstein metric, the relation between the complex coordinates and the angles, and some information about its topology are given in Appendix 4.A.2.

The second homology of $Q^{1,1,1}$ is $H_2(Q^{1,1,1}; \mathbb{Z}) \cong \mathbb{Z}^2$. Its generators can be represented by the following minimal volume cycles

$$\mathcal{C}_1 : \begin{cases} \theta_1 = \theta_2 \\ \theta_3 = \text{const.} \\ \phi_1 = -\phi_2 \\ \phi_3 = \text{const.} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \dfrac{a^1}{a^2} = \dfrac{\bar{b}_1}{\bar{b}_2} \\ c^1 = \text{const.} \\ c^2 = \text{const.} \end{cases} \tag{4.60}$$

$$\mathcal{C}_2 : \begin{cases} \theta_1 = \theta_3 \\ \theta_2 = \text{const.} \\ \phi_1 = -\phi_3 \\ \phi_2 = \text{const.} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \dfrac{a^1}{a^2} = \dfrac{\bar{c}_1}{\bar{c}_2} \\ b^1 = \text{const.} \\ b^2 = \text{const.} \end{cases} \tag{4.61}$$

It is straightforward to compute the gravitational and electrostatic potentials for probe M2-branes wrapping these cycles:

$$\begin{aligned} V_g^{\mathcal{C}_1}(r) &= 4\pi\tau_{M2} e^{-\frac{3}{4}\chi}\sqrt{g}e^{-\frac{w}{2}} e^{\eta_1}\,, \qquad V_e^{\mathcal{C}_1}(r) = 0\,, \\ V_g^{\mathcal{C}_2}(r) &= 2\pi\tau_{M2} e^{-\frac{3}{4}\chi}\sqrt{g}e^{-\frac{w}{2}} (e^{\eta_1} + e^{\eta_2})\,, \qquad V_e^{\mathcal{C}_2}(r) = \mp\frac{\pi}{2}\tau_{M2}\Phi\,. \end{aligned} \tag{4.62}$$

The cycle $\mathcal{C}_2$ saturates the bounds (4.52) and (4.54), while $\mathcal{C}_1$ does not.

4.5 Field theory interpretation

Let us discuss a dual 3-d gauge theory interpretation of our brane solutions carrying topological charges. The solutions are asymptotic to $AdS_4 \times Y^7$, with the Sasaki-Einstein space Y^7 having $b_2 > 0$. The classic examples of such backgrounds known since the 80's are $AdS_4 \times M^{1,1,1}$, $AdS_4 \times Q^{1,1,1}$, and $AdS_4 \times Q^{2,2,2}$. The search for the 3-d $\mathcal{N} = 2$ superconformal field theories dual to them began in the late 90's; see, for example, [99]. Following the major progress on formulating the world volume theories of coincident M2-branes [23, 100–103], a recent wave of research has produced compelling proposals for the Chern-Simons (C-S) quiver gauge theories dual to these M-theory backgrounds [104–108]. Interestingly, all these proposals involve $U(N)^{2+b_2}$ gauge theories with a certain set of Chern-Simons levels $k_1, k_2, \ldots, k_{2+b_2}$ that add up to zero.

The discussion of the Abelian $U(1)^{2+b_2}$ subgroup of the gauge group requires special care. None of the matter fields are charged under the diagonal $U(1)$ corresponding to the gauge field $\mathcal{A}_+ \sim \sum_{j=1}^{2+b_2} \mathcal{A}_j$. The existence of magnetic monopole configurations for this diagonal $U(1)$ means that another combination of the $U(1)$'s $\mathcal{A}_b \sim \sum_{j=1}^{2+b_2} k_j \mathcal{A}_j$ gets gauge fixed to a discrete subgroup. The remaining b_2 gauge fields

$$\mathcal{A}_{\vec{m}} \sim \sum_{j=1}^{2+b_2} m_j \mathcal{A}_j \tag{4.63}$$

may be chosen to be orthogonal to each other; they are orthogonal to $\mathcal{A}_+$ and $\mathcal{A}_b$ due to the conditions $\vec{m} \cdot \vec{k} = \vec{m} \cdot \vec{I} = 0$, where $\vec{I} = (1, 1, \ldots, 1)$. The gauge fields $\mathcal{A}_{\vec{m}}$ have C-S terms and are coupled to massless charged matter. For each of them one can define a conserved global current $\mathcal{J}_{\vec{m}} \sim *d\mathcal{A}_{\vec{m}}$. Thus, the C-S gauge theory possesses $U(1)^{b_2}$ global symmetry. Using the equation of motion for $\mathcal{A}_{\vec{m}}$, one can write $\mathcal{J}_{\vec{m}}$ in terms of the bi-fundamental superfields in the quiver gauge theory.

The gauge fields $\mathcal{A}_{\vec{m}}$ are reminiscent of the "statistics gauge fields" for quasi-particles in the effective description of the fractional quantum Hall effect (FQHE) [109] (for a review, see [50]). If $\mathcal{A}$ is one of these $U(1)^{b_2}$ gauge fields for which the Chern-Simons term in the action is

$$\frac{k}{4\pi}\int \mathcal{A}\wedge d\mathcal{A}\,, \tag{4.64}$$

the equation of motion for $\mathcal{A}$ implies that an excitation with charge q under $\mathcal{A}$ is also a vortex with $2\pi q/k$ units of magnetic flux. Interchanging two such vortices results in an additional phase

$$\Delta\phi = \pi\frac{q^2}{k}\,, \tag{4.65}$$

showing that the coupling to $\mathcal{A}$ may change the statistics of the excitations that couple to this gauge field. This situation is reminiscent of the effective description of the FQHE at filling fraction $1/k$ where quasi-particles have non-trivial statistics due to coupling to a Chern-Simons gauge field.

However, our construction differs in an important way from standard FQHE systems because we are studying conformal Chern-Simons gauge theories coupled to massless scalars and fermions. Instead of massive quasi-particles we can only talk about quasi-particle creation operators (a term recently coined for this situation is "quasi-unparticles" [110]). Such operators create vortices that contain the C-S magnetic fluxes and are therefore known as monopole operators. Instead of the diagonal magnetic flux $d\mathcal{A}_+$, which is known to correspond to the Kaluza-Klein charge in M-theory [23], these operators excite the b_2 non-diagonal monopole fields $d\mathcal{A}_{\vec{m}}$. Thus, the non-diagonal monopole operators are the only objects that are charged under the $U(1)^{b_2}$ global symmetry of the C-S gauge theory. It has been argued quite convincingly that the M-theory objects dual to such non-diagonal monopole operators are the M2-branes wrapping some of the b_2 topologically non-trivial cycles [92, 111].

The dimensions of the monopole operators in the non-interacting diagonal $U(1)$ have been studied in [112–114] following [115], but the dimensions of the "non-diagonal" monopole operators appear to be harder to calculate on the gauge theory side. The AdS/CFT correspondence predicts that the dimensions of the operators dual to the wrapped M2-branes scale as $\sqrt{N}$ for large N, but presumably, this is difficult to test. Nevertheless, if we simply accept the proposal of [92,111], we find an interesting picture where the b_2 topological wrapped M2-brane charges in $AdS_4 \times Y^7$ are mapped to the b_2 $U(1)$ global charges in the dual quiver Chern-Simons gauge theory. In particular, a uniform density of such a topological charge corresponds to a uniform $U(1)$ magnetic field in the C-S gauge theory. The magnetic field here is not quite the same as in the duals to the dyonic black holes of [93], where the magnetic field was added as an external background. We may nevertheless speculate that the zero-temperature entropy of our topologically charged brane solution is due to the degeneracy of Landau levels on the gauge theory side.

4.5.1 Boundary conditions in AdS_4 and wrapped branes

In the AdS/CFT correspondence, a conserved current of a field theory is mapped to a massless gauge field in the bulk. The gauge fields corresponding to the conserved currents $\mathcal{J}_{\vec{m}}$ are then the $A^{(i)}$, $i = 1, \ldots, b_2$, that enter the fluctuation δA_3 in (4.1). An additional phenomenon special to AdS_4 is that the dual gauge fields $\tilde{A}^{(i)}$ in (4.1) correspond to the C-S gauge fields $\mathcal{A}_{\vec{m}}$ in the gauge theory. Indeed, as shown in [91], any two gauge fields $\tilde{A}$ and A in AdS_4 that satisfy $d\tilde{A} = *_4 dA$ should be quantized so that one corresponds to a gauge field $\mathcal{A}$ in the dual field theory and the other one to the dual conserved current $\mathcal{J} = *_3 d\mathcal{A}$.

Let us write the AdS_4 metric in the form

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) , \tag{4.66}$$

and pass to a gauge where $\tilde{A}_z = A_z = 0$. Near $z = 0$, the fields $\tilde{A}$ and A have the following expansion

$$\begin{aligned} \tilde{A} &= \tilde{a}^{(0)}_m dx^m + z\tilde{a}^{(1)}_m dx^m + \mathcal{O}(z^2 \log z) \, , \\ A &= a^{(0)}_m dx^m + z a^{(1)}_m dx^m + \mathcal{O}(z^2 \log z) \, . \end{aligned} \tag{4.67}$$

The duality relation between A and $\tilde{A}$ implies that

$$da^{(0)} = *_3 \tilde{a}^{(1)} \, , \qquad d\tilde{a}^{(0)} = *_3 a^{(1)} \, . \tag{4.68}$$

Without loss of generality, let us assume that A is dual to the conserved current $\mathcal{J}$. This means that $a^{(0)}$ should be interpreted as an external source for $\mathcal{J}$, while $a^{(1)}$ as the expectation value of $\mathcal{J}$ (up to normalization). Adding an external source $a^{(0)}$ for $\mathcal{J}$ means that the action changes by

$$\begin{aligned} \delta S &= \int d^3x \sqrt{-g}\, a^{(0)}_m \mathcal{J}^m = \int a^{(0)} \wedge *_3 \mathcal{J} = \int da^{(0)} \wedge \mathcal{A} \\ &= \int *_3 \tilde{a}^{(1)} \wedge \mathcal{A} = \int d^3x \sqrt{-g}\, \tilde{a}^{(1)}_m \mathcal{A}^m \, , \end{aligned} \tag{4.69}$$

where we integrated by parts and used (4.68). Equation (4.69) shows that if $a^{(0)}$ is an external source for $\mathcal{J} = *_3\mathcal{A}$, then $\tilde{a}^{(1)}$, which is related to $a^{(0)}$ through (4.68), is an external source for $\mathcal{A}$. So indeed, if A is dual to $\mathcal{J}$ then $\tilde{A}$ is dual to $\mathcal{A}$, provided $d\tilde{A} = *_4 dA$ and $\mathcal{J} = *_3 d\mathcal{A}$. Similarly, if we assumed that A was dual to $\mathcal{A}$ we would conclude that $\tilde{A}$ should be dual to $\mathcal{J} = *_3\mathcal{A}$.

There are thus two possible boundary conditions for the Abelian gauge fields

A and $\tilde{A}$ in AdS_4. From now on we will assume that A is one of the topological gauge fields $A^{(i)}$ appearing in the expression for δA_3, while $\tilde{A}$ is its dual, as in (4.1). The first (and the more conventional) choice of boundary conditions corresponds to fixing the boundary value of A but allowing the boundary value of $\tilde{A}$ to fluctuate. With this choice, the M2-branes wrapping a certain two-cycle are gauge invariant because they couple electrically to the gauge field A that vanishes at the conformal boundary, but the M5-branes wrapping the dual cycle are not. This statement may seem puzzling, but it agrees with the gauge non-invariance of the baryonic operators in the dual C-S gauge theory [92]. Indeed, operators of the form $\det X$ where X is one of the bi-fundamental fields are not invariant under the $U(1)$ subgroups of the $U(N)^{2+b_2}$ gauge group. Another choice of boundary conditions corresponds to fixing the boundary value of $\tilde{A}$ but allowing the boundary value of A to fluctuate. Now the wrapped M5-branes are gauge invariant, while the wrapped M2-branes are not. This choice should correspond not to the $U(N)^{2+b_2}$ Chern-Simons gauge theories, but to their appropriate Legendre transforms [30, 91] that turn the $U(N)$'s into $SU(N)$'s.[4] In the Legendre transformed theories, baryonic operators like $\det X$ are fully gauge invariant, while it is no longer possible to write down non-diagonal monopole operators that correspond to wrapped M2-branes.

4.5.2 An example: $AdS_4 \times M^{1,1,1}/\mathbb{Z}_k$

The theory conjectured to be dual to M-theory on $AdS_4 \times M^{1,1,1}/\mathbb{Z}_k$ [104, 105] is the $\mathcal{N} = 2$ superconformal $U(N)_1 \times U(N)_2 \times U(N)_3$ C-S gauge theory with levels $(-2k, k, k)$ coupled to three sets of bifundamental chiral superfields $X_{12}^i, X_{23}^i, X_{31}^i$, $i = 1, 2, 3$ (see figure 4.7). The $SU(3) \times U(1)_R$ invariant superpotential is

$$W \sim \epsilon_{ijk} \mathrm{Tr}(X_{12}^i X_{23}^j X_{31}^k)\,. \tag{4.70}$$

[4] We are grateful to D. Jafferis for discussions on this issue.

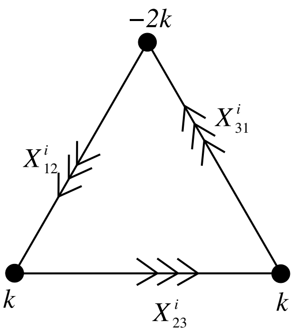

Figure 4.7: The quiver diagram for the C-S gauge theory dual to $AdS_4 \times M^{1,1,1}/\mathbb{Z}_k$ as conjectured in [104, 105]. The numbers next to the gauge nodes represent the C-S levels.

The level assignments break the $\mathbb{Z}_3$ symmetry of the quiver diagram, and the R-charges of the chiral superfields are taken to be [116] $R(X_{12}) = R(X_{31}) = 7/9$, $R(X_{23}) = 4/9$. The natural way to combine the three $U(1)$ gauge fields is

$$\mathcal{A}_+ = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \,, \qquad \mathcal{A}_b = -2\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \,, \qquad \mathcal{A} = \sqrt{2}(\mathcal{A}_2 - \mathcal{A}_3) \;. \quad (4.71)$$

The gauge field $\mathcal{A}$ has the standard Chern-Simons term (4.64), and it also enters the covariant derivatives for bi-fundamental fields. Therefore, the $\mathcal{A}$ equation of motion is

$$\frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda = \mathcal{J}^\mu \,, \quad (4.72)$$

where $\mathcal{J}^\mu$ is the $U(1)$ current

$$\mathcal{J}_\mu \sim \frac{i}{2}\,\mathrm{tr}\left[\bar{X}^i_{12} D_\mu X^i_{12} + \bar{X}^i_{31} D_\mu X^i_{31} - 2\bar{X}^i_{23} D_\mu X^i_{23}\right] + \text{c.c.} + \text{fermionic terms} \,, \quad (4.73)$$

and D_μ is the gauge covariant derivative acting on the bi-fundamental fields X^i_{ab} in the fundamental of $U(N)_a$ and anti-fundamental of $U(N)_b$. The manifold $M^{1,1,1}/\mathbb{Z}_k$ has $b_2 = 1$, and there is one topological $U(1)$ gauge field in AdS_4. In the C-S gauge theory, the current dual to it is $\frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda$.

Some results on matching of the chiral operators in this gauge theory with supergravity fluctuations are available [107], but none of these operators carry the topological $U(1)$ charge. To construct the operators corresponding to the wrapped M2-branes one has to include the monopole operators with the magnetic flux for the field $\mathcal{A}$. If we place a unit charge at the origin, $J^0 = \delta^2(x)$, then (4.72) requires that $\mathcal{A}_\phi = \frac{1}{kr}$. This azimuthal gauge field produces phase $2\pi/k$ when another unit charge circles the one at the origin. This simple field theory argument thus predicts the existence of fractional statistics. It would be interesting to study how this effect arises for wrapped M2-branes in $AdS_4 \times M^{1,1,1}/\mathbb{Z}_k$, but we leave this for future work. We further note that a brane carrying a uniform topological charge density corresponds in the $U(N)^3$ gauge theory described above to the presence of a constant magnetic field $d\mathcal{A}$. The ground state of the charged fields in this background is expected to exhibit the Landau level degeneracy. It would be interesting to investigate if this degeneracy may help explain the large $T = 0$ entropy found on the gravity side.

As reviewed above, the standard boundary conditions in AdS_4 allow the wrapped M2-branes but make the wrapped M5-branes transform under the corresponding $U(1)$ gauge transformations [92, 111]. This agrees with the fact that operators like $\det X_{23}$ transform under the $\mathcal{A}$ gauge transformations in the $U(N)^3$ gauge theory. One can, however, change the AdS_4 boundary conditions to make the wrapped M5-branes allowed and M2-branes forbidden. The corresponding operation in the gauge theory is a Legendre transform [30, 91], which turns the $U(1)$ into a global symmetry. Since the gauge field $\mathcal{A}$ becomes non-dynamical, we can no longer use monopole operators involving this gauge field; this agrees with the fact that the wrapped M2-branes are not allowed. In the Legendre transformed theory we can, however, write down baryonic operators like $\det X_{23}$ of dimension $4N/9$. This dimension agrees with the volume of one of the five-cycles in $M^{1,1,1}$ [99]. This discussion of baryonic operators is rather sketchy, and a number of issues remain to be elucidated. In particular, it would be interesting to study the Legendre transformed theory in more detail.

4.6 A BTZ black hole in type IIB theory

It is interesting to study a reduction of our M-theory membrane solutions to string theory. Since all fields are independent of the two spatial directions x^1 and x^2, we may consider the following strategy. First, we compactify these directions on circles of radii R_1 and R_2, respectively. Then we reduce to type IIA string theory along the x^2 direction and perform T-duality along the x^1 direction to obtain a type IIB background with eight compact dimensions consisting of S^1 times a warped Y^7, and with warp factors depending on the radial coordinate r. What makes these transformations particularly interesting is that our analytic solution (4.32), which seems to have unacceptable large r behavior in M-theory, acquires conventional AdS_3 asymptotics in the type IIB theory. Furthermore, the type IIB background, supported by F_5 flux only, turns out to be the product of a squashed Y^7 space and an extremal BTZ black hole [94, 95].

Some of the reasons for this simplicity can be traced back to our original M-brane construction. We start with a stack of N M2-branes spanning the (t, x^1, x^2) directions placed at the tip of the cone over Y^7, and then add a density of M2-branes wrapping two-cycles inside Y^7. Upon reduction to IIA, the N M2-branes wrapping T^2 turn into N fundamental strings winding around the x^1 circle, while the other wrapped M2-branes turn into wrapped D2-branes. Upon T-duality, the winding modes turn into momentum modes which affect the metric only and do not source the NS-NS two-form B_2, while the wrapped D2-branes turn into wrapped D3-branes. The type IIB background therefore describes D3-branes wrapping a two-cycle in Y^7 and a circle, with N units of momentum flowing along the circle. This setup is very similar to the original D-brane constructions of supersymmetric black holes with non-vanishing Bekenstein-Hawking entropy [117–119]. For example, one such construction involves two stacks of D3-branes wrapping two-tori embedded inside T^6 and intersecting over a circle, while we instead have D3-branes wrapping more complicated cycles inside a squashed Y^7. As a result, our background does not appear to be supersymmetric.

We give the reduction of our general background (4.7)–(4.9) from M-theory to type IIA and the T-duality to type IIB in Appendix 4.D. In this section we will restrict our attention to a slightly generalized version of the exact solution from section 4.3.1, which we will connect through dimensional reduction and T-duality to a locally $AdS_3 \times$ squashed Y^7 type IIB background. We do this starting from the type IIB solution in section 4.6.1. In section 4.6.2 we discuss the corresponding M-theory background.

4.6.1 The type IIB background

Let us start with the following ten-dimensional string frame background describing a product of a locally AdS_3 space and a squashed Y^7:

$$\begin{aligned} ds_{10}^2 &= \left[\frac{r^2}{L_3^2}\left(-dt^2+dx^2\right)+\frac{L_3^2}{r^2}dr^2+\alpha\left(dt+dx\right)^2\right] \\ &\quad + 8L_3^2\left[ds_{V_1}^2+\frac{1}{2}ds_{V_2}^2+\frac{4}{3}(d\psi+\sigma_1+\sigma_2)^2\right], \\ F_5 &= 8\sqrt{\frac{2}{3}}r\,dt\wedge dx\wedge dr\wedge(2\omega_1-\omega_2)-\frac{512}{3}L_3^4\omega_1\wedge(\omega_2-\omega_1)\wedge(d\psi+\sigma_1+\sigma_2)\,, \end{aligned} \tag{4.74}$$

where L_3 is the radius of the asymptotically AdS_3 space and α is an arbitrary constant. The Lorentz boosts $x-t \to \lambda^{-1}(x-t),\ x+t \to \lambda(x+t)$ act as $\alpha \to \lambda^2\alpha$; therefore, there are only three distinct cases: $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The locally AdS_3 space with positive α describes an extremal BTZ black hole [94, 95], which has the smallest mass for a given angular momentum.

This IIB background describes a state in the (1+1)-dimensional CFT on D3-branes wrapped around the x-circle as well as a two-cycle in the internal space. Not much is known about this gauge theory, but using the gauge/string correspondence one can extract the central charge from the Weyl anomaly [120]:

$$c = \frac{3}{2}\frac{L_3}{G_3} = 12\pi\frac{L_3}{\kappa_3^2}\,, \tag{4.75}$$

where κ_3 is the effective gravitational constant in three dimensions. The 3-d gravitational constant can be expressed in terms of the gravitational constant of the type IIB theory, κ_{10}, through

$$\frac{1}{\kappa_3^2} = 2^{10}\sqrt{\frac{2}{3}}L_3^7\mathrm{Vol}(V_1)\mathrm{Vol}(V_2)\Delta\psi\frac{1}{\kappa_{10}^2}\,, \tag{4.76}$$

the factor multiplying $1/\kappa_{10}^2$ in this equation being just the volume of the internal space.

To estimate the number of D3-branes we compute the number of F_5 flux units through a non-trivial five-cycle in the internal space. One of the simplest such five-cycles spans V_1 and the fiber direction. The number of units of D3-brane flux through it can be computed from the standard formulae

$$N_{D3} = \frac{1}{2\kappa_{10}^2\tau_{D3}}\int F_5\,, \qquad \tau_{D3} = \frac{2\pi}{g_s(2\pi\ell_s)^4}\,, \qquad \frac{1}{2\kappa_{10}^2} = \frac{2\pi}{g_s^2(2\pi\ell_s)^8}\,, \tag{4.77}$$

which give

$$N_{D3} = \frac{2^9}{3}\frac{L_3^4}{\sqrt{\pi}\kappa_{10}}\mathrm{Vol}(V_1)\Delta\psi\,. \tag{4.78}$$

Comparing this expression with the one for the central charge above, we notice that $c \sim N_{D3}^2$, suggesting an interpretation of the central charge in terms of intersecting D3-branes.

The gravity background above does not correspond to the vacuum state of the gauge theory—the vacuum has $\alpha = 0$. Nonzero α translates into a nonzero expectation value of the stress-energy tensor. The AdS/CFT dictionary gives

$$\langle T_{tt} \rangle = \langle T_{tx} \rangle = \langle T_{xx} \rangle = \frac{\alpha}{\kappa_3^2 L_3} \,, \tag{4.79}$$

so in the field theory there is conformal matter moving at the speed of light in the negative x direction. If we compactify the x direction on a circle of radius R_x, the entropy of this state can be computed in gravity from the area of the horizon at $r = 0$:

$$S = \frac{(2\pi)^2 \alpha^{\frac{1}{2}} R_x}{\kappa_3^2} \,. \tag{4.80}$$

There is a way of understanding this entropy from field theory considerations, which provides a consistency check on the above formulae. Since the x direction is a circle of radius R_x, the momentum along it needs to be quantized in units of $1/R_x$. The number of momentum units is

$$N = R_x \, |p_x| = 2\pi R_x^2 \, |\langle T_{tx} \rangle| = 2\pi \frac{\alpha R_x^2}{\kappa_3^2 L_3} \,. \tag{4.81}$$

Combining this relation with (4.75) and (4.80) we verify the Cardy formula

$$S = 2\pi \sqrt{\frac{Nc}{6}} \,, \tag{4.82}$$

which can be derived by assuming that the entropy comes from the number of ways of partitioning the N units of momentum into smaller momentum quanta.

4.6.2 The dual M-theory background

T-dualizing (4.74) along the compact direction x and lifting to M-theory by introducing a new coordinate y, one obtains the metric

$$\begin{aligned} ds_{11}^2 &= h^{-\frac{2}{3}}\left[-\frac{r^4}{L_3^4}dt^2 + dx^2 + dy^2\right] + h^{\frac{1}{3}}\frac{L_3^2}{r^2}dr^2 \\ &\quad + 8h^{\frac{1}{3}}L_3^2\left[ds_{V_1}^2 + \frac{1}{2}ds_{V_2}^2 + \frac{4}{3}(d\psi + \sigma_1 + \sigma_2)^2\right], \qquad h \equiv \frac{r^2}{L_3^2} + \alpha\,. \end{aligned} \tag{4.83}$$

The four-form F_4 is

$$F_4 = -2\frac{r\alpha}{h^2 L_3^2}dt \wedge dx \wedge dy \wedge dr - 8\sqrt{\frac{2}{3}}r dt \wedge dr \wedge (2\omega_1 - \omega_2)\,. \tag{4.84}$$

For $\alpha < 0$ the metric contains a naked singularity at finite r, while the $\alpha = 0$ case also appears to be singular. We are therefore primarily interested in the $\alpha > 0$ where the M-theory metric is equivalent to (4.31).

In going from type IIB to type IIA string theory, the circle of radius R_x gets replaced by a circle of radius $\tilde{R}_x = \ell_s^2/R_x$, $\ell_s \equiv \sqrt{\alpha'}$ being the string length. In addition, the string coupling constant g_s of the type IIB theory becomes $\tilde{g}_s = g_s \ell_s / R_x$ in type IIA. The lift to M-theory introduces the new compact direction y of radius $\tilde{R}_y = \tilde{g}_s \ell_s$ and sets the Planck length in eleven dimensions equal to $\ell_p = \tilde{g}_s^{\frac{1}{3}} \ell_s$. The 11-d gravitational constant κ_{11} is related to the gravitational constant κ_{10} in the IIB theory by $\kappa_{11}^2 = 2\pi\kappa_{10}^2 \tilde{R}_x \tilde{R}_y / R_x$ as follows from the relations $2\kappa_{11}^2 = (2\pi\ell_p)^9/(2\pi)$, $2\kappa_{10}^2 = (2\pi\ell_s)^8 g_s^2/(2\pi)$ and the duality transformations described above.

Using the relations between the various constants in M-theory and type IIB mentioned in the previous paragraph, one can easily check that the Bekenstein-Hawking entropy of the 11-d black hole in (4.83) with an event horizon at $r = 0$ agrees precisely with the expression (4.80) that we found in ten dimensions. One can also check that the number of M2-branes filling the (t,x,y) directions,

$$N = \frac{1}{2\kappa_{11}^2 \tau_{M2}} \int F_7 \,, \tag{4.85}$$

agrees with the number of units of momentum in the x direction in the 10-d background that was computed in eq. (4.81).

It is not hard to check that for $L_3 = 2^{\frac{3}{16}} 3^{-\frac{9}{16}}$ and $\alpha = 2^{-\frac{45}{8}} 3^{\frac{39}{8}}$, the change of coordinates $r \to 2^{-\frac{21}{8}} 3^{\frac{15}{8}} \sqrt{r^4-1}$, $t \to 2^{\frac{21}{4}} 3^{-\frac{19}{4}} e^{-\frac{1}{2}w_0} t$, $x \to \frac{3}{2} x^1$, and $y \to \frac{3}{2} x^2$ brings the eleven-dimensional metric (4.83) into the form of the exact solution (4.31). When the size of the torus parameterized by x and y in eleven dimensions is small in Planck units, one can thus view the effective IR theory described by the attractor solution (4.37)—which is the IR limit of (4.31)—as defined through the asymptotically AdS_3 background in type IIB theory that we discussed above. In this limit, one can argue that at nonzero charge density the effective IR description of the $(2+1)$-dimensional C-S gauge theory dual to $AdS_4 \times Y^7$ is the same as that of a chiral sector of a $(1+1)$-dimensional CFT dual to $AdS_3 \times$ squashed Y^7.

4.7 Discussion

We have constructed new charged membrane backgrounds of M-theory that are asymptotic to $AdS_4 \times Y^7$ where Y^7 is a Sasaki-Einstein manifold with non-vanishing b_2. In particular, we considered Y^7 that is a circle bundle over a product of two Kähler-Einstein manifolds, $V_1 \times V_2$. Instead of the $U(1)_R$ charge corresponding to translations of the circle, that was used in previous M-theory constructions [39, 41, 42], we turned on a "topological" charge corresponding to a component of δA_3 along the universal harmonic form $\omega = \omega_1 - 2\omega_2$. As the Hawking temperature of the black membrane horizon is decreased, a $U(1)_R$-charged solution typically undergoes a phase transition due to condensation of charged fields. We showed that such a phase transition does not occur for our topologically charged solutions. At $T = 0$ the near-horizon region becomes $AdS_2 \times \mathbb{R}^2 \times$ squashed Y^7, which signals emergent quantum criticality. This throat region is by itself a solution of the 11-d supergravity equations.

If we compactify the brane coordinates x^1 and x^2 on a two-torus, then the resulting black hole has two kinds of charge. One of them is proportional to the number of M2-branes wrapping the T^2, the other to the number of M2-branes wrapping the two-cycle inside Y^7. To study whether charge condensation occurs, we calculated the potential as a function of r for the different types of wrapped M2-branes. We found that the M2-branes wrapping the internal cycles experience attractive forces at any temperature; the M2-branes wrapped over T^2 experience an attractive force that tends to zero as $T \to 0$ for all r. Thus, unlike the R-charged brane solutions or the type IIB 3-brane solution with a baryonic charge from chapter 3, the new M-theory solution does not suffer from an instability with respect to expulsion of toroidal branes to large r [75, 76, 78].

The fact that there is a moduli space for the M2-branes wrapped over T^2 is consistent with the conjecture that gravity is the weakest force, which implies that there should be a charged object *not* attracted to an extremal charged black hole horizon [45]. Nevertheless, it is very surprising to find the vanishing of the potential for a probe space-time filling M2-brane in a background which apparently does not preserve any supersymmetry. It would be interesting to investigate if this moduli space is lifted by higher-derivative corrections to the 11-d supergravity action, which are expected to correspond to $1/N$ corrections in the dual Chern-Simons gauge theory.

When a string or M-theory background does not preserve any supersymmetry, one should be concerned about various potential instabilities. We have shown that there is no low-temperature condensation of charged objects, but one should also check the perturbative stability of neutral fluctuations. We have carried out some preliminary checks for neutral scalars, but clearly more should be done. Finally, there may be some non-perturbative gravitational instabilities which were not studied here.

If our zero-temperature solution is completely stable, we should try to explain the microscopic origin of its large Bekenstein-Hawking entropy. One approach may be to study a dual quiver Chern-Simons gauge theory with a constant background magnetic field $d\mathcal{A}$ which produces a uniform $U(1)$ global charge density. Our membrane solution implies that this gauge theory develops IR quantum criticality corresponding to the appearance of the AdS_2 throat. We would like to gain some understanding of this phenomenon. It would also be interesting to study the apparent fractional statistics of the wrapped M2-branes in the $AdS_4 \times M^{1,1,1}/\mathbb{Z}_k$ background.

Another possible microscopic approach to the IR theory is motivated by the type IIB background (4.74) which is related by string dualities to the M-theory exact solution (4.31) with different large r asymptotics. In this type IIB background, the near-horizon AdS_2 region arises from a reduction of the extremal BTZ black hole on a circle. The extremal BTZ times squashed Y^7 background should be dual to the $(1+1)$-dimensional CFT on D3-branes partially wrapped over the squashed Y^7. Calculating the central charge of this CFT would provide a way of explaining the charged black hole entropy via (4.82).

4.A Metrics for the regular Sasaki-Einstein spaces

In this section we give the explicit metrics for the regular Sasaki-Einstein manifolds described in section 4.2.2. We also discuss the non-trivial cycles found in the bases of these manifolds that are useful for the probe brane computations in section 4.4.2 and Appendix 4.C.

Let us first describe the general approach to computing the range of the coordinate ψ appearing in (4.2). When the scalars χ and η_i vanish identically, (4.7) solves the eleven-dimensional SUGRA equations with $M = AdS_4$. Insisting that the radius of AdS_4 should be L, one finds that the metric on V_i, $i = 1, 2$, should be normalized so that the curvature two-form is related to the Kähler form through $R_i = 8\omega_i$. By definition, the first Chern class of V_i is $c_1(V_i) \equiv \frac{1}{2\pi} R_i$, so

$$c_1(V_i) = \frac{4}{\pi}\omega_i = \frac{2}{\pi} d\sigma_i \,, \tag{4.86}$$

where in the second equality we used (4.3). Note that $c_1(V_i)$ does not depend on the overall normalization of the metric on V_i, but of course the proportionality constant between $c_1(V_i)$ and ω_i does. By definition, the first Chern class of the fiber bundle $Y^7 \to V_1 \times V_2$ is

$$c_1 = \frac{1}{L_f} d(\sigma_1 + \sigma_2) \,, \tag{4.87}$$

where L_f is the length of the fiber, i.e. the range of ψ. From comparing (4.87) to (4.86) we see that in order to compute the length of the fiber we need to know the relation between the first Chern class c_1 of the fiber bundle and the first Chern class $c_1(V_1) + c_1(V_2)$ of the base. Using a Thom-Gysin sequence, one can show [97, 121] that the only requirement is that $c_1(V_1) + c_1(V_2)$ should be an integer multiple of c_1. Recalling that c_1 and $c_1(V_i)$ represent cohomology classes with integer coefficients, we denote by a_i the largest integers so that $\frac{1}{a_i} c_1(V_i) \in H^2(V_i; \mathbb{Z})$. Since $c_1(V_1) + c_1(V_2)$ must be an integer multiple of c_1, one can take

$$c_1 = \frac{1}{a} \left(c_1(V_1) + c_1(V_2) \right) \,, \tag{4.88}$$

where a can be any common divisor of a_1 and a_2. The length of the fiber is then

$$L_f = \frac{\pi}{2} a \,. \tag{4.89}$$

Seven-dimensional Sasaki-Einstein spaces like the ones above have $\mathcal{N}=2$ supersymmetry. The two Killing spinors are proportional to $e^{\pm 2i\psi}$ [121], and they are well-defined as long as the range of ψ is an integer multiple of $\pi/2$. Equation (4.89) shows this is indeed the case.

4.A.1 $M^{1,1,1}$

The manifold $M^{1,1,1}$ is the homogeneous space $\frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)\times U(1)}$ and by construction its isometry group is that of the standard model, $SU(3)\times SU(2)\times U(1)$ [99]. The cone over $M^{1,1,1}$ is a Calabi-Yau four-fold that can be described as a Kähler quotient $\mathbb{C}^5//\mathbb{C}^*$ as follows. One starts with $\mathbb{C}^5$ parameterized by the complex coordinates (u^1,u^2,u^3,v^1,v^2) and endowed with the Kähler potential

$$K=\left(2u^i\bar{u}_i\right)^{\frac{3}{4}}\left(3v^j\bar{v}_j\right)^{\frac{1}{2}}\ . \tag{4.90}$$

One then takes the Kähler quotient of this space with charges $(2,2,2,-3,-3)$, meaning that we restrict our attention to a submanifold of $\mathbb{C}^5$ defined by

$$2\left(\left|u^1\right|^2+\left|u^2\right|^2+\left|u^3\right|^2\right)=3\left(\left|v^1\right|^2+\left|v^2\right|^2\right)\ , \tag{4.91}$$

which we further mod out by the equivalence relation

$$u^i\sim e^{2i\delta}u^i\,,\qquad v^j\sim e^{-3i\delta}v^j\,. \tag{4.92}$$

The space described by equations (4.91) and (4.92) is precisely the cone over $M^{1,1,1}$. This space is a cone because both of these equations are invariant under $u^i\to\lambda u^i$ and $v^j\to\lambda v^j$ with $\lambda\in\mathbb{R}_+$. One can check that the induced metric coming from the Kähler potential (4.90) is Ricci flat, so the cone over $M^{1,1,1}$ is indeed Calabi-Yau. One can check that the holomorphic four-form Ω_4 on the cone is given by

$$\Omega_4 \sim d\Omega_3\,, \qquad \Omega_3 \equiv \left(\epsilon_{i_1 i_2 i_3} u^{i_1} du^{i_2} \wedge du^{i_3}\right) \wedge \left(\epsilon_{j_1 j_2} v^{j_1} dv^{j_2}\right)\,. \tag{4.93}$$

The space $M^{1,1,1}$ can be obtained by fixing the overall magnitude of u^i and v^j:

$$2\left(\left|u^1\right|^2 + \left|u^2\right|^2 + \left|u^3\right|^2\right) = 3\left(\left|v^1\right|^2 + \left|v^2\right|^2\right) = 1\,. \tag{4.94}$$

An explicit Sasaki-Einstein metric can be found from (4.90) by using the parameterization

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{2}}\sin\mu\cos\frac{\theta_1}{2}e^{\frac{i}{2}(\phi_1+\psi_1+R_u\psi)}\,, & v_1 &= \frac{1}{\sqrt{3}}\cos\frac{\theta_2}{2}e^{\frac{i}{2}(\phi_2+R_v\psi)}\,,\\ u_2 &= \frac{1}{\sqrt{2}}\sin\mu\sin\frac{\theta_1}{2}e^{\frac{i}{2}(-\phi_1+\psi_1+R_u\psi)}\,, & v_2 &= \frac{1}{\sqrt{3}}\sin\frac{\theta_2}{2}e^{\frac{i}{2}(-\phi_2+R_v\psi)}\,,\\ u_3 &= \frac{1}{\sqrt{2}}\cos\mu\, e^{\frac{i}{2}R_u\psi}\,. \end{aligned} \tag{4.95}$$

for any R_u and R_v satisfying $3R_u + 2R_v = 1$. This metric has the form (4.2) with[5]

$$\begin{aligned} ds^2_{V_1} &= \frac{3}{4}\left[d\mu^2 + \frac{1}{4}\sin^2\mu\left(s_1^2 + s_2^2 + \cos^2\mu\, s_3^2\right)\right]\,,\\ ds^2_{V_2} &= \frac{1}{8}\left[d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right]\,, \end{aligned} \tag{4.96}$$

and

$$\sigma_1 = \frac{3}{8}\sin^2\mu\, s_3\,, \qquad \sigma_2 = \frac{1}{4}\cos\theta_2 d\phi_2\,. \tag{4.97}$$

[5]The metric obtained from (4.90) does not depend on the angle δ appearing in (4.92). One way to see this is to promote (4.92) to $u^i \to \lambda^2 u^i$, $v^j \to \lambda^{-3} v^j$ with $\lambda \in \mathbb{C}^*$ and think of (4.94) as a gauge fixing condition for this transformation. Since the Kähler potential is independent of λ, which can be regarded as a complex coordinate in $\mathbb{C}^5$, the metric on $\mathbb{C}^5$ following from (4.90) is degenerate in the λ direction.

In the above equations we have defined

$$s_1 \equiv d\theta_1\,, \qquad s_2 \equiv \sin\theta_1 d\phi_1\,, \qquad s_3 \equiv d\psi_1 + \cos\theta_1 d\phi_1\,. \tag{4.98}$$

The metrics (4.96) describe $V_1 = \mathbb{CP}^2$ and $V_2 = \mathbb{CP}^1$.

Let the hyperplane divisor H be the generator of $H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$. H is the homology class of a $\mathbb{CP}^1 \subset \mathbb{CP}^2$, so in homogeneous coordinates H can represented by the two-cycle $\{[0, z_1, z_2] : z_1, z_2 \in \mathbb{C}\}$. Let us denote by D the generator of $H_2(\mathbb{CP}^1;\mathbb{Z}) \cong \mathbb{Z}$. From (4.86), one can compute

$$\int_H c_1(V_1) = 3\,, \qquad \int_D c_1(V_2) = 2\,, \tag{4.99}$$

so in this case $a_1 = 3$ and $a_2 = 2$. There is only one possibility for $a|\gcd(a_1, a_2)$, namely $a = 1$. The length of the fiber is $\pi/2$.

4.A.2 $Q^{1,1,1}$ and $Q^{2,2,2}$

The space $Q^{1,1,1}$ is also a homogeneous space, $\frac{SU(2)\times SU(2)\times SU(2)}{U(1)\times U(1)}$ [99]. The cone over it is Calabi-Yau and can be constructed from taking a Kähler quotient $\mathbb{C}^6//\mathbb{C}^{*2}$. If the coordinates on $\mathbb{C}^6$ are $(a^1, a^2, b^1, b^2, c^1, c^2)$, the Kähler quotient can be thought of as the level sets

$$\left|a^1\right|^2 + \left|a^2\right|^2 = \left|b^1\right|^2 + \left|b^2\right|^2 = \left|c^1\right|^2 + \left|c^2\right|^2\,, \tag{4.100}$$

and the following identifications

$$\begin{aligned} a^i &\sim e^{i\delta} a^i\,, \qquad b^j \sim e^{-i\delta} b^j\,, \qquad c^k \sim c^k\,, \\ a^i &\sim e^{i\delta} a^i\,, \qquad b^j \sim b^j\,, \qquad c^k \sim e^{-i\delta} c^k\,. \end{aligned} \tag{4.101}$$

With the Kähler potential

$$K = \left(a^i\bar{a}_i\right)^{\frac{1}{2}}\left(b^j\bar{b}_j\right)^{\frac{1}{2}}\left(c^k\bar{c}_k\right)^{\frac{1}{2}}\ , \tag{4.102}$$

the cone over $Q^{1,1,1}$ is Calabi-Yau. The holomorphic four-form Ω_4 is in this case

$$\Omega_4 \sim d\Omega_3\,, \qquad \Omega_3 \equiv \left(\epsilon_{i_1 i_2} a^{i_1} da^{i_2}\right) \wedge \left(\epsilon_{j_1 j_2} b^{j_1} db^{j_2}\right) \wedge \left(\epsilon_{k_1 k_2} c^{k_1} dc^{k_2}\right)\ . \tag{4.103}$$

In order to find a metric on $Q^{1,1,1}$ itself, one needs to restrict to the base of the cone by fixing the overall magnitude of a^i, b^j, and c^k:

$$\left|a^1\right|^2 + \left|a^2\right|^2 = \left|b^1\right|^2 + \left|b^2\right|^2 = \left|c^1\right|^2 + \left|c^2\right|^2 = 1\,. \tag{4.104}$$

From (4.102) one can obtain an explicit metric on $Q^{1,1,1}$ using the parameterization

$$\begin{aligned} a^1 &= \cos\frac{\theta_1}{2} e^{\frac{i}{2}(\phi_1 + R_a\psi)}\,, & a^2 &= \sin\frac{\theta_1}{2} e^{\frac{i}{2}(-\phi_1 + R_a\psi)}\,, \\ b^1 &= \cos\frac{\theta_2}{2} e^{\frac{i}{2}(\phi_2 + R_b\psi)}\,, & b^2 &= \sin\frac{\theta_2}{2} e^{\frac{i}{2}(-\phi_2 + R_b\psi)}\,, \\ c^1 &= \cos\frac{\theta_3}{2} e^{\frac{i}{2}(\phi_3 + R_c\psi)}\,, & c^2 &= \sin\frac{\theta_3}{2} e^{\frac{i}{2}(-\phi_3 + R_c\psi)}\,, \end{aligned} \tag{4.105}$$

for any R_a, R_b, and R_c such that $R_a + R_b + R_c = 1$. The metric takes the form (4.2) with

$$ds^2_{V_1} = \frac{1}{8}\sum_{i=1}^{2}\left[d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right]\,, \qquad ds^2_{V_2} = \frac{1}{8}\left[d\theta_3^2 + \sin^2\theta_3 d\phi_3^2\right]\,, \tag{4.106}$$

and

$$\sigma_1 = \frac{1}{4}\left(\cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)\,, \qquad \sigma_2 = \frac{1}{4}\cos\theta_3 d\phi_3\,. \tag{4.107}$$

The spaces V_1 are V_2 are in this case $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^1$, respectively.

Let us denote the two generators of $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}^2$ by C_1 and C_2 where C_1 is the homology class of the first $\mathbb{CP}^1$ factor and C_2 of the second. As in the case of $M^{1,1,1}$, let us denote the generator of $H_2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$ by D. Starting from (4.86) it is easy to see that

$$\int_{C_1} c_1(V_1) = \int_{C_2} c_1(V_1) = \int_D c_1(V_2) = 2\,, \tag{4.108}$$

so in this case $a_1 = a_2 = 2$. Therefore there are two possibilities for the integer $a|\gcd(a_1, a_2)$: taking $a = 1$ we obtain the space $Q^{1,1,1}$, and taking $a = 2$ we obtain $Q^{2,2,2}$. From (4.89) we see that the circle fibers of $Q^{1,1,1}$ have length π, while those of $Q^{2,2,2}$ have length $\pi/2$.

4.A.3 Circle bundles over $dP_n \times \mathbb{CP}^1$

The last class of seven-dimensional regular Sasaki-Einstein manifolds comes from principal $U(1)$ fiber bundles over $dP_n \times \mathbb{CP}^1$, dP_n being the nth del Pezzo surface, and $3 \leq n \leq 8$. Topologically, dP_n can be constructed from $\mathbb{CP}^2$ blown up at n generic points. (The points being generic means that no three points should be collinear and no six points should lie on a conic.) The del Pezzo's are known to admit Kähler-Einstein metrics with positive Ricci curvature [122, 123], but unfortunately these metrics are not known analytically.[6] Despite this fact, we can still describe some of the properties of the corresponding Sasaki-Einstein spaces.

We take $V_1 = dP_n$ and $V_2 = \mathbb{CP}^1$. The metric on V_1 is not known, but the metric on V_2 is given by

$$ds^2_{V_2} = \frac{1}{8}\left[d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right], \tag{4.109}$$

[6]See [124] where a Kähler-Einstein metric on dP_3 was computed numerically.

and

$$\sigma_2 = \frac{1}{4} \cos\theta_2 d\phi_2 \,, \tag{4.110}$$

as in the previous two cases.

The second homology group of dP_n is $H_2(dP_n;\mathbb{Z}) \cong \mathbb{Z}^{n+1}$, so there are $n+1$ generators which we will denote by H and E_i, $1 \le i \le n$. In algebraic geometry language, H is a hyperplane divisor and E_i are the exceptional divisors of the blown-up points. As in the previous two sections, we denote by D the generator of $H_2(\mathbb{CP}^1;\mathbb{Z})$. Using algebraic geometry, one can show

$$\int_H c_1(V_1) = 3 \,, \qquad \int_{E_i} c_1(V_1) = 1 \,, \qquad \int_D c_1(V_2) = 2 \,. \tag{4.111}$$

It follows that $a_1 = 1$ and $a_2 = 2$, so again the only possible value of a is $a = 1$, giving fibers of length $\pi/2$.

4.B Other supergravity fluctuations around the exact solution

It might be interesting to consider supergravity fluctuations around the extremal solution found in section 4.3.1 and see whether these fluctuations cause a run-away instability. Let us focus on fluctuations depending only on the radial variable r. They typically satisfy second order differential equations whose solutions near the extremal horizon behave as $(r-1)^\alpha$. The exponent α can be either real or complex. When it is real, the corresponding fluctuations correspond to either a source or a VEV of an operator in the effective quantum mechanics. When it is complex, the corresponding fluctuations are oscillatory as a function of r and typically cause an instability.

Investigating the behavior of supergravity fluctuations near the extremal horizon is a hard task, because these fluctuations depend on the details of the Sasaki-Einstein spaces Y^7. We will only examine a particularly simple fluctuation in the case $Y^7 = Q^{1,1,1}$. For $Q^{1,1,1}$, $V_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$, and the background (4.7)–(4.11) is symmetric under interchanging the two $\mathbb{CP}^1$ factors. The mode that we will look at is the leading $\mathbb{Z}_2$-odd mode that changes the sizes of the two $\mathbb{CP}^1$'s. We call this mode λ.

There is a non-linear consistent truncation that includes this additional mode λ. The eleven-dimensional metric is

$$\begin{aligned} ds^2 = e^{-7\chi/2} ds_M^2 + \frac{1}{2} L^2 e^{\chi} \Big[e^{\eta_1+\lambda}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + e^{\eta_1-\lambda}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ + e^{\eta_2}(d\theta_3^2 + \sin^2\theta_3 d\phi_3^2)\Big] + \frac{1}{4} L^2 e^{\chi - 4\eta_1 - 2\eta_2}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 + \cos\theta_3 d\phi_3)^2 \end{aligned} \tag{4.112}$$

and the four-form is

$$\begin{aligned} F_4 = -\frac{3}{L} e^{-\frac{21}{2}\chi} \mathrm{vol}_M + Q L^3 \frac{e^{-\frac{w}{2}-\frac{3}{2}\chi}}{r^2} dt \wedge dr \wedge \Big[e^{2\eta_1+2\lambda} \sin\theta_1 d\theta_1 \wedge d\phi_1 \\ + e^{2\eta_1-2\lambda} \sin\theta_2 d\theta_2 \wedge d\phi_2 - 2e^{2\eta_2} \sin\theta_3 d\theta_3 \wedge d\phi_3 \Big] . \end{aligned} \tag{4.113}$$

When $\lambda = 0$, equations (4.112) and (4.113) reduce to equations (4.7) and (4.9), respectively.

The linearized equation for λ following from the eleven-dimensional supergravity equations of motion is (we set $L = 1$)

$$\lambda'' + \lambda' \left(\frac{2}{r} + \frac{g'}{g} - \frac{w'}{2} \right) + \lambda \frac{2e^{-6\eta_1 - 2\eta_2 - \frac{9}{2}\chi}}{r^4 g} \left[2e^{5\eta_1 + 2\eta_2} r^4 - r^4 - 4e^{8\eta_1 + 2\eta_2 + 3\chi} Q^2 \right] = 0 . \tag{4.114}$$

When evaluated on the extremal solution (4.32), equation (4.114) has analytical solutions:

$$\lambda = c_+(r^4-1)^{-\frac{1}{2}\left(1+\sqrt{\frac{17}{3}}\right)} + c_-(r^4-1)^{-\frac{1}{2}\left(1-\sqrt{\frac{17}{3}}\right)} . \tag{4.115}$$

In the effective quantum mechanics, the solutions multiplying c_+ and c_- correspond respectively to a source and a VEV of an operator of dimension $\Delta = \frac{1}{2}\left(1+\sqrt{\frac{17}{3}}\right) \approx 1.69$. Since the exponents of r^4-1 are real, we conclude that these fluctuations do not cause an instability.

4.C Comments on wrapped branes

In this section we tie up some loose ends from our discussion in section 4.4.2 of M2-branes wrapping an internal two-cycle in $\tilde{Y}^7$. We first discuss in section 4.C.1 some topological properties of two-cycles in a general Sasaki-Einstein manifold Y^7 whose Kähler-Einstein base is $V_1 \times V_2$. In section 4.C.2 we give a proof of the bound (4.52) on the volumes of the two-cycles of $\tilde{Y}^7$.

4.C.1 The second homology of Y^7

Topologically, two-cycles in Y^7 are classified by the second homology of Y^7 with integer coefficients, $H_2(Y^7;\mathbb{Z})$. The homology of Y^7 can be calculated from the homology of the base of the fibration, the product manifold $V_1 \times V_2$. In turn, the homology of $V_1 \times V_2$ can be computed from the homology of V_1 and that of V_2. For all of the regular Sasaki-Einstein spaces we are interested in, $V_2 = \mathbb{CP}^1$ and the generator of $H_2(V_2;\mathbb{Z}) \cong \mathbb{Z}$ is represented by V_2 itself. Let us call this generator D. The homology of the Kähler-Einstein spaces V_1 is in all cases of interest $H_2(V_1;\mathbb{Z}) \cong \mathbb{Z}^k$ and let us denote its generators by C_i with $1 \le i \le k$. We have $k=2$ for $V_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$; $k=1$ for $V_1 = \mathbb{CP}^2$; and $k = n+1$ for $V_1 = dP_n$. We pick the orientations of C_i and D so that they can be represented by holomorphic surfaces as opposed to antiholomorphic ones. The second homology of $V_1 \times V_2$ is then $H_2(V_1 \times V_2; \mathbb{Z}) \cong \mathbb{Z}^{k+1}$, and its generators are constructed as follows. Given a surface that represents C_i in V_1 we can take the direct product between this surface and a point in V_2; this product is a closed surface in $V_1 \times V_2$ and represents a generator of $H_2(V_1 \times V_2; \mathbb{Z})$. Similarly, the direct product between V_2 and a point in V_1 is also a closed surface in $V_1 \times V_2$ representing a generator of $H_2(V_1 \times V_2; \mathbb{Z})$. By abuse of notation we will denote the first k generators of $H_2(V_1 \times V_2; \mathbb{Z})$ by C_i and the $(k+1)$th one by D, as they are constructed from the corresponding generators of $H_2(V_1; \mathbb{Z})$ and $H_2(V_2; \mathbb{Z})$ in a straightforward way.

It turns out that if $H_2(V; \mathbb{Z}) \cong \mathbb{Z}^{k+1}$ then $H_2(Y^7; \mathbb{Z}) \cong \mathbb{Z}^k$. The reason why $H_2(Y^7; \mathbb{Z})$ is smaller than $H_2(V; \mathbb{Z})$ is that whereas all topologically non-trivial closed surfaces in Y^7 project down to topologically non-trivial closed surfaces in V, not every closed surface in V can be lifted to a closed surface in Y^7. In fact, any two-dimensional surface S in V can be lifted to a *three*-dimensional surface $\tilde{S}$ in Y^7 by restricting the circle fibration over V to a circle fibration over S. In order for a two-dimensional closed surface S in V to be liftable to a *two*-dimensional closed surface in Y^7, one has to specify what the fiber coordinate ψ should be at each point in S. There is a topological restriction on the types of closed surfaces S one can lift precisely because it may be impossible to specify consistently what ψ is at all points of S. In algebraic topology language, a consistent assignment of ψ to every point in S gives a global section of the pull-back bundle $\tilde{S}$, and it is known that any circle bundle, in particular $\tilde{S}$, admits a global section if and only if it is trivial. Since circle bundles are completely classified by their first Chern class (the cohomology class of the curvature of the $U(1)$ fibration), it follows that a closed surface S in V is liftable to Y^7 if and only if the first Chern class of the circle bundle $\tilde{S}$ (which is nothing but the pull-back of the first

Chern class of Y^7 to S) is zero in cohomology. In particular,

$$S \text{ is liftable} \Longleftrightarrow \int_S c_1 = 0\,, \tag{4.116}$$

where c_1 is the first Chern class of Y^7. The above argument works only in the case where the surface S is connected—if S is not connected, then the condition (4.116) should be satisfied for each connected component separately.

Equation (4.116) suggests[7] how to construct $H_2(Y^7;\mathbb{Z})$ given $H_2(V;\mathbb{Z})$: $H_2(Y^7;\mathbb{Z})$ is isomorphic to the kernel of the map that assigns to each element C in $H_2(V;\mathbb{Z})$ the integer $\int_C c_1$. In other words, if we parameterize the homology classes in $H_2(V;\mathbb{Z})$ by

$$C = \sum_{i=1}^{k} \alpha_i C_i + \beta D\,, \tag{4.117}$$

with $\alpha_i, \beta \in \mathbb{Z}$, then there is a one-to-one correspondence between elements of the homology $H_2(Y^7;\mathbb{Z})$ of the total space Y^7 and classes C in the homology $H_2(V;\mathbb{Z})$ of the base V satisfying

$$\sum_{i=1}^{k} \alpha_i \int_{C_i} c_1 + \beta \int_D c_1 = 0\,. \tag{4.118}$$

Such classes form a $\mathbb{Z}^k$ subspace of $H_2(V;\mathbb{Z}) \cong \mathbb{Z}^{k+1}$, so indeed $H_2(Y^7;\mathbb{Z}) \cong \mathbb{Z}^k$. Note that only connected surfaces representing C can be lifted to Y^7 as embedded closed surfaces, as discussed above.

The first Chern class of the fibration, c_1, is by definition the cohomology class of the curvature of the connection one-form $\sigma_1 + \sigma_2$ appearing in the metric (4.2). By equation (4.3), c_1 is proportional to the sum $\omega_1 + \omega_2$ of the Kähler forms on V_1 and

[7]The following argument is not intended to be a proof. One can prove the result (4.118) using a Gysin sequence. See [99] for the cases $Y^7 = Q^{1,1,1}$ and $Y^7 = M^{1,1,1}$.

V_2, so equation (4.118) becomes

$$\sum_{i=1}^{k} \alpha_i \int_{C_i} \omega_1 + \beta \int_D \omega_2 = 0\,. \tag{4.119}$$

What's nice about this equation is that since V_1 and V_2 are Einstein spaces, the integrals of the Kähler forms over the cycles C_i and D are topological invariants that are known even when an explicit Einstein metric on V_1 or V_2 is not known.

As an example, for $Y^7 = M^{1,1,1}$, $V_1 = \mathbb{CP}^2$, $V_2 = \mathbb{CP}^1$, and the dimension of $H_2(V_1;\mathbb{Z})$ is $k=1$. Algebraic geometry arguments combined with the condition for an Einstein metric (see also Appendix 4.A.1) give $\int_{C_1}\omega_1 = \frac{3\pi}{4}$ and $\int_D \omega_2 = \frac{\pi}{2}$. Equation (4.119) shows that the generator of the homology of Y^7 has $\alpha_1 = 2$ and $\beta = -3$. An explicit cycle representing this homology class is given in (4.57).

As another example, for $Y^7 = Q^{1,1,1}$, $V_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$, $V_2 = \mathbb{CP}^1$, and $k=2$. In this case, $\int_{C_1}\omega_1 = \int_{C_2}\omega_1 = \int_D \omega_2 = \frac{\pi}{2}$. The second homology of Y^7 is therefore generated by $(\alpha_1, \alpha_2, \beta) = (1,-1,0)$ and $(\alpha_1,\alpha_2,\beta) = (1,0,-1)$. Explicit cycles representing these homology classes are given in (4.60)–(4.61).

As a last comment, note that the above discussion does not change if we replace Y^7 by $\tilde{Y}^7$ because the curvature of the $U(1)$ fibration stays unchanged. Moreover, any cycle $\mathcal{C}$ in $\tilde{Y}^7$ should satisfy (4.50) because $\mathcal{C}$ is in the same homology class as a two-cycle $\mathcal{C}'$ constructed by lifting a closed surface S in V, and for $\mathcal{C}'$ equation (4.50) is equivalent to (4.116).

4.C.2 A lower bound on the volumes of closed two-surfaces in $\tilde{Y}^7$

The bound (4.52) can be proven by finding a calibration. A calibration (for two-dimensional surfaces) is a closed two-form Ω with the property that for any orthonormal tangent vectors u and v

$$\Omega(u,v) \leq 1\,. \tag{4.120}$$

Consequently, the volume of any closed two-dimensional surface $\mathcal{C}$ in Y^7 satisfies

$$\mathrm{Vol}(\mathcal{C}) \geq \int_{\mathcal{C}} \Omega\,. \tag{4.121}$$

Since Ω is closed, the right-hand side of (4.121) depends only on the homology class of $\mathcal{C}$. In the space $\tilde{Y}^7$ with the metric (4.48) we will show that

$$\Omega = se^{\chi+\eta_1}\omega_1 + te^{\chi+\eta_2}\omega_2 \tag{4.122}$$

is a calibration for any $-1 \leq s,t \leq 1$. Here, by ω_1 and ω_2 we mean, as usual, the pull-backs of the Kähler forms on V_1 and V_2, respectively. Clearly, since we fix r, Ω is a closed two-from. To understand why Ω is in fact a calibration, let us pick a point p in $\tilde{Y}^7$ and define the orthonormal basis f_i, $i = 1, 2, \ldots, 7$, for the tangent space $T_p\tilde{Y}^7$ and the dual basis e_j, $j = 1, 2, \ldots, 7$ for $T^*_p\tilde{Y}^7$. Since ω_i are the pull-backs of the Kähler forms on V_i, we can require

$$\begin{aligned} e^{\chi+\eta_1}\omega_1 &= e_1 \wedge e_2 + e_3 \wedge e_4\,, \\ e^{\chi+\eta_2}\omega_2 &= e_5 \wedge e_6\,, \\ e^{\frac{1}{2}\chi-\frac{1}{2}\eta_1-\eta_2}(d\psi+\sigma_1+\sigma_2) &= e_7\,, \end{aligned} \tag{4.123}$$

and thus the metric on $\tilde{Y}^7$ is $ds^2_{\tilde{Y}} = \sum_{i=1}^7 (e_i)^2$. Now for any two arbitrary orthonormal tangent vectors $u = \sum_{i=1}^7 u_i f_i$ and $v = \sum_{i=1}^7 v_i f_i$ in $T_p\tilde{Y}^7$ we have

$$\begin{aligned}\Omega(u,v) &= s(u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3) + t(u_5v_6 - u_6v_5) \\ &\le \left[s^2(u_1^2+u_2^2+u_3^2+u_4^2) + t^2(u_5^2+u_6^2)\right]^{\frac{1}{2}} \left[v_1^2+v_2^2+v_3^2+v_4^2+v_5^2+v_6^2\right]^{\frac{1}{2}} \\ &\le \|u\|\,\|v\| = 1\,,\end{aligned} \tag{4.124}$$

where in the second line we used the Cauchy-Schwarz inequality and in the last line we made use of the fact that $-1 \le s, t \le 1$. Equation (4.124) holds for any orthonormal vectors u, v at any point p, so Ω is indeed a calibration. For a surface $\mathcal{C}$ in $\tilde{Y}^7$ we therefore have

$$\mathrm{Vol}(\mathcal{C}) \ge e^{\chi+\eta_1}\left|\int_{\mathcal{C}} \omega_1\right| + e^{\chi+\eta_2}\left|\int_{\mathcal{C}} \omega_2\right|\,. \tag{4.125}$$

In obtaining (4.125) we chose s and t to be ± 1 in such a way that the bound we got would be as restrictive as possible.

Combining (4.125) with (4.50), we obtain[8]

$$\mathrm{Vol}(\mathcal{C}) \ge e^{\chi}\left(e^{\eta_1}+e^{\eta_2}\right)\left|\beta\int_D \omega_2\right| = e^{\chi}\left(e^{\eta_1}+e^{\eta_2}\right)\left|\int_{\mathcal{C}} \omega_2\right|\,. \tag{4.126}$$

This inequality is saturated when both inequalities in (4.124) are saturated at every point p of $\mathcal{C}$, u and v being an orthonormal basis for the tangent space to $\mathcal{C}$ at p. The first inequality in (4.124) is saturated when the projection of $\mathcal{C}$ to V_1 is given by a holomorphic ($s = 1$) or anti-holomorphic ($s = -1$) surface and the projection to V_2 is also given by a holomorphic ($t = 1$) or anti-holomorphic ($t = -1$) surface. The second inequality in (4.124) is satisfied when the tangent space to $\mathcal{C}$ is "horizontal,"

[8] In the case of $AdS_5 \times T^{1,1}$ a similar inequality was proven in [98] using an explicit parameterization of two-cycles.

meaning intuitively that $\mathcal{C}$ does not "move" in the fiber direction. Only very special surfaces satisfy these two conditions. That said, two such surfaces are the one given in (4.57) in the case of $M^{1,1,1}$ and the one given in (4.61) in the case of $Q^{1,1,1}$; a direct computation of the volumes of these surfaces shows that they indeed saturate (4.52).

4.D Reduction to type IIA and T-duality

In this section we reduce the M-theory background (4.7)–(4.9) to type IIA along the x^2 direction and then T-dualize to type IIB along the x^1 direction. The type IIA string frame metric is

$$\begin{aligned} ds^2_{\mathrm{IIA}} = e^{-\frac{21}{4}\chi}\frac{r}{L}\left[-ge^{-w}dt^2 + \frac{r^2}{L^2}(dx^1)^2 + \frac{dr^2}{g}\right] \\ + 4L^2 e^{-\frac{3}{4}\chi}\frac{r}{L}\left[e^{\eta_1}ds^2_{V_1} + e^{\eta_2}ds^2_{V_2} + e^{-4\eta_1-2\eta_2}(d\psi+\sigma_1+\sigma_2)^2\right] . \end{aligned} \tag{4.127}$$

The dilaton is given by

$$\Phi_{\mathrm{IIA}} = -\frac{21\chi}{8} + \frac{3}{2}\log\frac{r}{L} . \tag{4.128}$$

The NS-NS three-form flux is

$$H_3^{\mathrm{IIA}} = 3e^{-\frac{1}{2}w-\frac{21}{2}\chi}\frac{r^2}{L^3}dt\wedge dx^1\wedge dr . \tag{4.129}$$

Out of the R-R forms, only F_4 is non-vanishing:

$$F_4^{\mathrm{IIA}} = -8Qe^{-\frac{1}{2}w-\frac{3}{2}\chi}\frac{L^3}{r^2}dt\wedge dr\wedge\left(e^{2\eta_1}\omega_1 - 2e^{2\eta_2}\omega_2\right) . \tag{4.130}$$

The type IIB background we obtain has only F_5 flux. In string frame, the metric is

$$\begin{aligned} ds^2_{\rm IIB} &= e^{-\frac{21}{4}\chi}\frac{r}{L}\left[-ge^{-w}dt^2+\frac{dr^2}{g}\right]+e^{\frac{21}{4}\chi}\frac{L^3}{r^3}\left(dx^1-P(r)dt\right)^2 \\ &+4L^2e^{-\frac{3}{4}\chi}\frac{r}{L}\left[e^{\eta_1}ds^2_{V_1}+e^{\eta_2}ds^2_{V_2}+e^{-4\eta_1-2\eta_2}(d\psi+\sigma_1+\sigma_2)^2\right] , \end{aligned} \tag{4.131}$$

where the function $P(r)$ satisfies

$$P'(r)=3e^{-\frac{1}{2}w-\frac{21}{2}\chi}\frac{r^2}{L^3} . \tag{4.132}$$

The self-dual five-form can be written as

$$F_5^{\rm IIB}=8Qe^{-\frac{1}{2}w-\frac{3}{2}\chi}\frac{L^3}{r^2}dt\wedge dx^1\wedge dr\wedge\left(e^{2\eta_1}\omega_1-2e^{2\eta_2}\omega_2\right)-64QL^4*_Y\omega\,, \tag{4.133}$$

where $*_Y\omega$ was defined in (4.5).

Chapter 5

Conclusions

Since its formulation by Maldacena in 1997 [1], the AdS/CFT duality has seen tremendous amounts of activity. The duality is now well-enough understood that a number of attempts exist at using it to derive results about empirically relevant theories. This thesis focused on some applications of AdS/CFT with relevance in condensed matter physics.

The first of these applications showed that a large class of $(3+1)$-dimensional gauge theories with gravitational duals exhibit a phase transition in which an operator $\mathcal{O}$ develops a nonzero expectation value. This is reminiscent of superconductivity, and echoes similar results obtained for M theory [39], raising hopes that string theoretic methods can provide insights into superconductivity at strong coupling. Compared to the M theory results, this work went further, by embedding one of the unstable modes in a non-linear truncation that allows to understand the phase diagram of the corresponding field theory operator. Using the universality of the instability, a suggestion was made about the identity of the universal condensing operator.

Working in a different direction, in search of a theory that is stable at arbitrarily low temperatures, a black hole deformation of the conifold was considered. In the string theory picture, D3 branes were wrapped around the topologically non-trivial cycle of $T^{1,1}$; in the dual picture, the conifold gauge theory was studied at nonzero baryonic chemical potential. It was found that the IR limit of the geometry was nearly AdS_2, warped by slowly-varying logarithmic factors of the radial direction. This can be interpreted as a new type of emergent quantum near criticality. The solution seems stable under scalar fluctuations, but unfortunately suffers from 'Fermi seasickness', a term coined in [78], referring to nucleation of spacetime filling D-branes at the AdS boundary.

A similar construction was considered in M theory, where several scenarios of black holes with topological charges were studied by using a universal consistent truncation. Zero-temperature solutions were analyzed, including an analytical solution that is ill-behaved in the UV, and a numerical solution that has the same IR behavior, but is asymptotically AdS in the UV. Dimensionally reducing the former solution to type IIA string theory, and T-dualizing to type IIB yields a well-behaved solution that is a product of a squashed Sasaki-Einstein manifold and an extremal BTZ black hole. Some checks of stability were performed, raising hopes that the backgrounds we found, though not supersymmetric, are stable.

Appendix A

Differential forms, conventions and some identities

I briefly review the conventions used for differential forms in this thesis, as well as some useful identities.

The components of a differential form are defined by

$$\omega_p = \frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{A.1}$$

In components, the wedge product between two forms is given by

$$(\zeta_p\wedge\eta_q)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\,\zeta_{[\mu_1\ldots\mu_p}\eta_{\mu_{p+1}\ldots\mu_{p+q}]}\,, \tag{A.2}$$

where antisymmetrization is defined with a factor of $1/k!$,

$$T_{[\mu_1\ldots\mu_k]} = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\,T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(k)}}\,. \tag{A.3}$$

Note that using this definition, the antisymmetrization of a fully antisymmetric tensor is equal to itself, *i.e.* for a p-form ω_p,

$$\omega_{[\mu_1 \ldots \mu_p]} = \omega_{\mu_1 \ldots \mu_p} \,. \tag{A.4}$$

The components of the exterior derivative are given by

$$(d\omega_p)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \nabla_{[\mu_1} \omega_{\mu_2 \ldots \mu_{p+1}]} \,. \tag{A.5}$$

Here ∇ denotes any torsion-free connection on the manifold.

We have

$$d(\zeta_p \wedge \eta_q) = d\zeta_p \wedge \eta_q + (-1)^p\, \zeta_p \wedge d\eta_q \,. \tag{A.6}$$

The *Hodge dual* maps p-forms to $(n-p)$-forms, where n is the dimensionality of the space. Its action can be described in component notation as

$$(*\omega_p)_{\mu_1 \ldots \mu_{n-p}} := \frac{1}{p!}\, \epsilon^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_{n-p}}\, \omega_{\nu_1 \ldots \nu_p} \,, \tag{A.7}$$

where ϵ is the Levi-Civita tensor. In an orthonormal basis $\{e_i\}$, this is equivalent to

$$* \,(e_1 \wedge \ldots \wedge e_p) = \pm e_{p+1} \wedge \ldots \wedge e_n \,, \tag{A.8}$$

where the sign is given by the product of $g(e_i, e_i)$ for $i = 1, \ldots, p$ (since this is an orthonormal basis, $g(e_i, e_i) = \pm 1$).

The Levi-Civita tensor obeys

$$\epsilon^{\mu_1 \ldots \mu_n} \epsilon_{\mu_1 \ldots \mu_n} = s\, n! \,. \tag{A.9}$$

Here s is the sign of the determinant of the metric, *e.g.* $s = 1$ for Riemannian manifolds and $s = -1$ for Minkowskian manifolds. This fixes the tensor ϵ up to a sign, which can be chosen by deciding on a "right-handed" coordinate frame. We can then write

$$\epsilon = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n \,, \tag{A.10}$$

where g is the determinant of the metric tensor.

Contractions of the Levi-Civita tensor are given by

$$\epsilon^{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_n} \epsilon_{\mu_1 \ldots \mu_k \nu_{k+1} \ldots \nu_n} = s\,(n-k)!\, k!\, \delta^{[\mu_{k+1}}{}_{\nu_{k+1}} \cdots \delta^{\mu_n]}{}_{\nu_n} \,. \tag{A.11}$$

The Levi-Civita tensor is the same as the volume form on the manifold,

$$\epsilon = \mathrm{vol} \,. \tag{A.12}$$

The normalization of the Hodge dual has been chosen so that

$$\omega_p \wedge * \omega_p = |\omega_p|^2 \;\mathrm{vol} = \langle \omega_p, \omega_p \rangle \,\mathrm{vol} \,. \tag{A.13}$$

Here we have used the inner product on p-forms induced by the metric,

$$\langle \eta_p, \zeta_p \rangle := \frac{1}{p!}\, g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p}\, \eta_{\mu_1 \ldots \mu_p} \zeta_{\nu_1 \ldots \nu_p} \,. \tag{A.14}$$

More directly, eq. (A.9) can also be written as

$$* \,\mathrm{vol} = s \,. \tag{A.15}$$

The Hodge dual is an involution up to a sign,

$$* * \omega_p = (-1)^{p(n-p)} s\, \omega_p \,. \tag{A.16}$$

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, `hep-th/9802109`.

[3] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[4] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges," *Phys.Rev.* **D58** (1998) 046004, `hep-th/9802042`.

[5] S. Gubser, I. R. Klebanov, and A. Peet, "Entropy and temperature of black 3-branes," *Phys.Rev.* **D54** (1996) 3915–3919, `hep-th/9602135`.

[6] I. R. Klebanov, "World volume approach to absorption by nondilatonic branes," *Nucl.Phys.* **B496** (1997) 231–242, `hep-th/9702076`.

[7] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, "String theory and classical absorption by three-branes," *Nucl.Phys.* **B499** (1997) 217–240, `hep-th/9703040`.

[8] S. S. Gubser and I. R. Klebanov, "Absorption by branes and Schwinger terms in the world volume theory," *Phys.Lett.* **B413** (1997) 41–48, hep-th/9708005.

[9] S. S. Gubser, D. R. Gulotta, S. S. Pufu, and F. D. Rocha, "Gluon energy loss in the gauge-string duality," *JHEP* **0810** (2008) 052, 0803.1470.

[10] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. A. Wiedemann, "Gauge/String Duality, Hot QCD and Heavy Ion Collisions," 1101.0618.

[11] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge, UK: Univ. Pr. (1998) 531 p.

[12] E. D'Hoker, D. Z. Freedman, and W. Skiba, "Field theory tests for correlators in the AdS / CFT correspondence," *Phys.Rev.* **D59** (1999) 045008, hep-th/9807098.

[13] A. Nunez and A. O. Starinets, "AdS / CFT correspondence, quasinormal modes, and thermal correlators in $\mathcal{N}=4$ SYM," *Phys.Rev.* **D67** (2003) 124013, hep-th/0302026.

[14] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," *Commun. Math. Phys.* **262** (2006) 51–89, hep-th/0411238.

[15] M. Benna, S. Benvenuti, I. Klebanov, and A. Scardicchio, "A Test of the AdS/CFT correspondence using high-spin operators," *Phys.Rev.Lett.* **98** (2007) 131603, hep-th/0611135.

[16] D. Berenstein, R. Cotta, and R. Leonardi, "Numerical tests of AdS/CFT at strong coupling," *Phys.Rev.* **D78** (2008) 025008, 0801.2739.

[17] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, "Multi-Matrix Models and Tri-Sasaki Einstein Spaces," *Phys.Rev.* **D83** (2011) 046001, 1011.5487.

[18] S. Cheon, D. Gang, S. Kim, and J. Park, "Refined test of AdS_4/CFT_3 correspondence for $\mathcal{N} = 2, 3$ theories," *JHEP* **1105** (2011) 027, 1102.4273.

[19] C. R. Stephens, G. 't Hooft, and B. F. Whiting, "Black hole evaporation without information loss," *Class.Quant.Grav.* **11** (1994) 621–648, gr-qc/9310006.

[20] L. Susskind, "The World as a hologram," *J.Math.Phys.* **36** (1995) 6377–6396, hep-th/9409089.

[21] E. Witten, "Bound states of strings and p-branes," *Nucl.Phys.* **B460** (1996) 335–350, hep-th/9510135.

[22] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," *Nucl. Phys.* **B536** (1998) 199–218, hep-th/9807080.

[23] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, 0806.1218.

[24] A. Ceresole, G. Dall'Agata, R. D'Auria, and S. Ferrara, "Spectrum of type IIB supergravity on $AdS\_5 \times T^{1,1}$: Predictions on $\mathcal{N} = 1$ SCFT's," *Phys. Rev.* **D61** (2000) 066001, hep-th/9905226.

[25] S. Hawking and G. Ellis, *The Large scale structure of space-time*. 1973.

[26] I. R. Klebanov, "TASI lectures: Introduction to the AdS / CFT correspondence," hep-th/0009139.

[27] I. Kanitscheider, K. Skenderis, and M. Taylor, "Precision holography for non-conformal branes," *JHEP* **0809** (2008) 094, `0807.3324`.

[28] C. Herzog and D. Son, "Schwinger-Keldysh propagators from AdS/CFT correspondence," *JHEP* **0303** (2003) 046, `hep-th/0212072`.

[29] I. Papadimitriou and K. Skenderis, "Thermodynamics of asymptotically locally AdS spacetimes," *JHEP* **0508** (2005) 004, `hep-th/0505190`.

[30] I. R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," *Nucl. Phys.* **B556** (1999) 89–114, `hep-th/9905104`.

[31] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," `0903.3246`.

[32] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," *J. Phys.* **A42** (2009) 343001, `0904.1975`.

[33] A. Kehagias, "New type IIB vacua and their F-theory interpretation," *Phys. Lett.* **B435** (1998) 337–342, `hep-th/9805131`.

[34] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. J. Spence, "Branes at conical singularities and holography," *Adv. Theor. Math. Phys.* **2** (1999) 1249–1286, `hep-th/9808014`.

[35] M. Cvetic and S. S. Gubser, "Phases of R-charged black holes, spinning branes and strongly coupled gauge theories," *JHEP* **04** (1999) 024, `hep-th/9902195`.

[36] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Charged AdS black holes and catastrophic holography," *Phys. Rev.* **D60** (1999) 064018, `hep-th/9902170`.

[37] M. Cvetic and S. S. Gubser, "Thermodynamic Stability and Phases of General Spinning Branes," *JHEP* **07** (1999) 010, `hep-th/9903132`.

[38] S. S. Gubser, "Breaking an Abelian gauge symmetry near a black hole horizon," arXiv:0801.2977 [hep-th].

[39] F. Denef and S. A. Hartnoll, "Landscape of superconducting membranes," 0901.1160.

[40] S. S. Gubser, C. P. Herzog, S. S. Pufu, and T. Tesileanu, "Superconductors from Superstrings," *Phys.Rev.Lett.* **103** (2009) 141601, 0907.3510.

[41] J. P. Gauntlett, J. Sonner, and T. Wiseman, "Holographic superconductivity in M-Theory," *Phys.Rev.Lett.* **103** (2009) 151601, 0907.3796.

[42] J. P. Gauntlett, J. Sonner, and T. Wiseman, "Quantum Criticality and Holographic Superconductors in M-theory," *JHEP* **02** (2010) 060, 0912.0512.

[43] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, "Building an AdS/CFT superconductor," 0803.3295.

[44] C. P. Herzog, P. K. Kovtun, and D. T. Son, "Holographic model of superfluidity," 0809.4870.

[45] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, "The string landscape, black holes and gravity as the weakest force," *JHEP* **06** (2007) 060, hep-th/0601001.

[46] L. Fritz and L. Heinz, "The Electromagnetic Equations of the Supraconductor," *Proc.Roy.Soc.Lond.* **A149** (1935) 71–99.

[47] V. Ginzburg and L. Landau, "On the Theory of Superconductivity," *Zh.Eksp.Teor.Fiz.* **20** (1950) 1064–1082.

[48] M. Tinkham, *Introduction to superconductivity.* Dover books on physics and chemistry. Dover Publications, 2004.

[49] G. Mahan, *Many-particle physics*. Physics of solids and liquids. Kluwer Academic/Plenum Publishers, 2000.

[50] X. G. Wen, *Quantum field theory of many-body systems: From the origin of sound to an origin of light and electrons*. Oxford Univ. Press, 2004.

[51] N. Ashcroft and N. Mermin, *Solid state physics*. Holt-Saunders International Editions: Science : Physics. Holt, Rinehart and Winston, 1976.

[52] S. Sachdev, *Quantum Phase Transitions*. John Wiley & Sons, Ltd, 2007.

[53] S. Sachdev and B. Keimer, "Quantum Criticality," *Phys.Today* **64N2** (2011) 29, 1102.4628.

[54] T. Valla, A. V. Fedorov, P. D. Johnson, B. O. Wells, S. L. Hulbert, Q. Li, G. D. Gu, and N. Koshizuka, "Evidence for Quantum Critical Behavior in the Optimally Doped Cuprate Bi_2Sr_2CaCu_2O_8+?," *Science* **285** (1999) 2110–2113.

[55] S. S. Gubser and F. D. Rocha, "The gravity dual to a quantum critical point with spontaneous symmetry breaking," *Phys. Rev. Lett.* **102** (2009) 061601, 0807.1737.

[56] S. S. Gubser, S. S. Pufu, and F. D. Rocha, "Quantum critical superconductors in string theory and M- theory," 0908.0011.

[57] T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, "Emergent quantum criticality, Fermi surfaces, and AdS_2," *Phys.Rev.* **D83** (2011) 125002, 0907.2694.

[58] C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, "Quantum critical transport, duality, and M-theory," *Phys.Rev.* **D75** (2007) 085020, hep-th/0701036.

[59] S. A. Hartnoll, "Quantum Critical Dynamics from Black Holes," 0909.3553.

[60] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, "Holographic Superconductors," 0810.1563.

[61] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," *Adv. Theor. Math. Phys.* **3** (1999) 1–81, hep-th/9810201.

[62] S. Franco, A. Garcia-Garcia, and D. Rodriguez-Gomez, "A General class of holographic superconductors," *JHEP* **1004** (2010) 092, 0906.1214.

[63] D. Berenstein, C. P. Herzog, and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," *JHEP* **06** (2002) 047, hep-th/0202150.

[64] J. P. Gauntlett, S. Kim, O. Varela, and D. Waldram, "Consistent supersymmetric Kaluza–Klein truncations with massive modes," *JHEP* **04** (2009) 102, 0901.0676.

[65] L. J. Romans, "New compactifications of chiral $\mathcal{N} = 2$ $d = 10$ supergravity," *Phys. Lett.* **B153** (1985) 392.

[66] G. T. Horowitz and M. M. Roberts, "Holographic Superconductors with Various Condensates," *Phys. Rev.* **D78** (2008) 126008, 0810.1077.

[67] G. Mack, "All Unitary Ray Representations of the Conformal Group SU(2, 2) with Positive Energy," *Commun.Math.Phys.* **55** (1977) 1.

[68] V. Dobrev and V. Petkova, "All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry," *Phys.Lett.* **B162** (1985) 127–132.

[69] O. Lunin and J. M. Maldacena, "Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals," *JHEP* **0505** (2005) 033, hep-th/0502086.

[70] C. P. Herzog, I. R. Klebanov, S. S. Pufu, and T. Tesileanu, "Emergent Quantum Near-Criticality from Baryonic Black Branes," *JHEP* **1003** (2010) 093, `0911.0400`.

[71] S.-S. Lee, "A Non-Fermi Liquid from a Charged Black Hole: A Critical Fermi Ball," `0809.3402`.

[72] H. Liu, J. McGreevy, and D. Vegh, "Non-Fermi liquids from holography," `0903.2477`.

[73] M. Cubrovic, J. Zaanen, and K. Schalm, "String Theory, Quantum Phase Transitions and the Emergent Fermi-Liquid," *Science* **325** (2009) 439–444, `0904.1993`.

[74] D. Yamada and L. G. Yaffe, "Phase diagram of $\mathcal{N} = 4$ super-Yang-Mills theory with R-symmetry chemical potentials," *JHEP* **09** (2006) 027, `hep-th/0602074`.

[75] D. Yamada, "Fragmentation of Spinning Branes," *Class. Quant. Grav.* **25** (2008) 145006, `0802.3508`.

[76] D. Yamada, "Metastability of R-charged black holes," *Class. Quant. Grav.* **24** (2007) 3347–3376, `hep-th/0701254`.

[77] R.-G. Cai and K.-S. Soh, "Critical behavior in the rotating D-branes," *Mod. Phys. Lett.* **A14** (1999) 1895–1908, `hep-th/9812121`.

[78] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, "Towards strange metallic holography," `0912.1061`.

[79] N. J. Evans and J. Hockings, "$\mathcal{N} = 4$ super Yang Mills at finite density: The naked truth," *JHEP* **07** (2002) 070, `hep-th/0205082`.

[80] S. S. Gubser and I. R. Klebanov, "Baryons and domain walls in an $\mathcal{N} = 1$ superconformal gauge theory," *Phys. Rev.* **D58** (1998) 125025, hep-th/9808075.

[81] C. P. Herzog, I. R. Klebanov, and P. Ouyang, "Remarks on the warped deformed conifold," hep-th/0108101.

[82] I. R. Klebanov and A. A. Tseytlin, "Gravity Duals of Supersymmetric $SU(N) \times SU(N+M)$ Gauge Theories," *Nucl. Phys.* **B578** (2000) 123–138, hep-th/0002159.

[83] G. T. Horowitz and M. M. Roberts, "Zero Temperature Limit of Holographic Superconductors," 0908.3677.

[84] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and χ_SB-resolution of naked singularities," *JHEP* **08** (2000) 052, hep-th/0007191.

[85] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, "Holography from Conformal Field Theory," 0907.0151.

[86] K. Goldstein, S. Kachru, S. Prakash, and S. P. Trivedi, "Holography of Charged Dilaton Black Holes," 0911.3586.

[87] G. Gibbons and S. A. Hartnoll, "A gravitational instability in higher dimensions," *Phys. Rev.* **D66** (2002) 064024, hep-th/0206202.

[88] I. R. Klebanov, S. S. Pufu, and T. Tesileanu, "Membranes with Topological Charge and AdS_4/CFT_3 Correspondence," *Phys.Rev.* **D81** (2010) 125011, 1004.0413.

[89] S.-J. Rey, "String theory on thin semiconductors: Holographic realization of Fermi points and surfaces," *Prog. Theor. Phys. Suppl.* **177** (2009) 128–142, 0911.5295.

[90] R. D'Auria and P. Fre, "On the spectrum of the $\mathcal{N} = 2$ $SU(3) \times SU(2) \times U(1)$ gauge theory from $D = 11$ supergravity," *Class. Quant. Grav.* **1** (1984) 447.

[91] E. Witten, "$SL(2, \mathbb{Z})$ action on three-dimensional conformal field theories with Abelian symmetry," hep-th/0307041.

[92] Y. Imamura and S. Yokoyama, "$\mathcal{N} = 4$ Chern-Simons theories and wrapped M5-branes in their gravity duals," *Prog. Theor. Phys.* **121** (2009) 915–940, 0812.1331.

[93] S. A. Hartnoll and C. P. Herzog, "Ohm's Law at strong coupling: S duality and the cyclotron resonance," *Phys. Rev.* **D76** (2007) 106012, 0706.3228.

[94] M. Banados, C. Teitelboim, and J. Zanelli, "The black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, hep-th/9204099.

[95] M. Banados, "Three-dimensional quantum geometry and black holes," hep-th/9901148.

[96] C. hyun Ahn and S.-J. Rey, "More CFTs and RG flows from deforming M2/M5-brane horizon," *Nucl. Phys.* **B572** (2000) 188–207, hep-th/9911199.

[97] T. Friedrich and I. Kath, "Seven-dimensional compact Riemannian manifolds with Killing spinors," *Commun. Math. Phys.* **133** (1990) 543–561.

[98] D. Arean, D. E. Crooks, and A. V. Ramallo, "Supersymmetric probes on the conifold," *JHEP* **11** (2004) 035, hep-th/0408210.

[99] D. Fabbri *et. al.*, "3D superconformal theories from Sasakian seven-manifolds: New nontrivial evidences for AdS_4/CFT_3," *Nucl. Phys.* **B577** (2000) 547–608, `hep-th/9907219`.

[100] J. Bagger and N. Lambert, "Modeling multiple M2's," *Phys. Rev.* **D75** (2007) 045020, `hep-th/0611108`.

[101] J. Bagger and N. Lambert, "Gauge Symmetry and Supersymmetry of Multiple M2-Branes," *Phys. Rev.* **D77** (2008) 065008, `0711.0955`.

[102] J. Bagger and N. Lambert, "Comments On Multiple M2-branes," *JHEP* **02** (2008) 105, `0712.3738`.

[103] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nucl. Phys.* **B811** (2009) 66–76, `0709.1260`.

[104] D. Martelli and J. Sparks, "Moduli spaces of Chern-Simons quiver gauge theories and AdS_4/CFT_3," *Phys. Rev.* **D78** (2008) 126005, `0808.0912`.

[105] A. Hanany and A. Zaffaroni, "Tilings, Chern-Simons Theories and M2 Branes," *JHEP* **10** (2008) 111, `0808.1244`.

[106] S. Franco, A. Hanany, J. Park, and D. Rodriguez-Gomez, "Towards M2-brane Theories for Generic Toric Singularities," *JHEP* **12** (2008) 110, `0809.3237`.

[107] S. Franco, I. R. Klebanov, and D. Rodriguez-Gomez, "M2-branes on Orbifolds of the Cone over $Q^{1,1,1}$," *JHEP* **08** (2009) 033, `0903.3231`.

[108] J. Davey, A. Hanany, N. Mekareeya, and G. Torri, "Phases of M2-brane Theories," *JHEP* **06** (2009) 025, `0903.3234`.

[109] D. P. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, "Statistical Mechanics of Anyons," *Nucl. Phys.* **B251** (1985) 117–126.

[110] T. Faulkner and J. Polchinski, "Semi-Holographic Fermi Liquids," 1001.5049.

[111] Y. Imamura, "Monopole operators in $\mathcal{N} = 4$ Chern-Simons theories and wrapped M2-branes," *Prog. Theor. Phys.* **121** (2009) 1173–1187, 0902.4173.

[112] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In $\mathcal{N} = 4$ Super Yang-Mills Theory," 0807.3720.

[113] D. Gaiotto and D. L. Jafferis, "Notes on adding D6-branes wrapping $\mathbb{RP}^3$ in $AdS\_4 \times \mathbb{CP}^3$," 0903.2175.

[114] M. K. Benna, I. R. Klebanov, and T. Klose, "Charges of Monopole Operators in Chern-Simons Yang-Mills Theory," *JHEP* **01** (2010) 110, 0906.3008.

[115] V. Borokhov, A. Kapustin, and X. kai Wu, "Monopole operators and mirror symmetry in three dimensions," *JHEP* **12** (2002) 044, hep-th/0207074.

[116] J. Davey, A. Hanany, N. Mekareeya, and G. Torri, "Higgsing M2-brane Theories," *JHEP* **11** (2009) 028, 0908.4033.

[117] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," *Phys. Lett.* **B379** (1996) 99–104, hep-th/9601029.

[118] C. G. Callan and J. M. Maldacena, "D-brane Approach to Black Hole Quantum Mechanics," *Nucl. Phys.* **B472** (1996) 591–610, hep-th/9602043.

[119] I. R. Klebanov and A. A. Tseytlin, "Intersecting M-branes as four-dimensional black holes," *Nucl. Phys.* **B475** (1996) 179–192, hep-th/9604166.

[120] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," *JHEP* **07** (1998) 023, hep-th/9806087.

[121] D. Martelli, J. Sparks, and S.-T. Yau, "Sasaki-Einstein manifolds and volume minimisation," *Commun. Math. Phys.* **280** (2008) 611–673, hep-th/0603021.

[122] G. Tian and S.-T. Yau, “Kähler-Einstein metrics on complex surfaces with $c_1 > 0$,” *Commun. Math. Phys.* **112** (1987) 175–203.

[123] G. Tian, “On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$,” *Invent. Math* **89** (1987) 225–246.

[124] C. Doran, M. Headrick, C. P. Herzog, J. Kantor, and T. Wiseman, “Numerical Kaehler-Einstein metric on the third del Pezzo,” *Commun. Math. Phys.* **282** (2008) 357–393, `hep-th/0703057`.